Spectral Analysis of High-Frequency Continuous-Time ARMA Models

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Zusammenfassung

In dieser Arbeit werden statistische Fragen für hochfrequent beobachtete zeitstetige autoregressive Moving-Average (CARMA) Prozesse untersucht. Zunächst wird für invertierbare CARMA Modelle mit endlicher Varianz ein L^2-konsistenter Schätzer für die Zuwächse des zugrundeliegenden Lévy-Prozesses konstruiert. Dieser ist unabhängig von der Ordnung des Prozesses. Darüber hinaus wird das Hochfrequenzverhalten von approximierenden Riemann-Summen analysiert und deren Autokovarianzstruktur mit der von gesampelten CARMA Prozessen verglichen.

Im zweiten Teil der Arbeit werden auch α-stabile CARMA Modelle analysiert. Im asymptotischen Kontext hochfrequenter Daten auf langen Zeitintervallen, werden Konvergenzresultate für verschieden normalisierte Periodogramm-Versionen gezeigt. Je nachdem, ob die ausgewählten Frequenzwerte über $\mathbb{Z}$ linear abhängig oder unabhängig sind, variieren die Grenzverteilungen. Die Beweise dazu verwenden Methoden aus der Geometrie der Zahlen. Über eine Glättung des Periodogramms wird ein konsistenter Schätzer für die normalisierte rationale Transferfunktion hergeleitet. Darauf basierend wird ein Schätzverfahren für die Parameter eines CARMA Prozesses vorgeschlagen.

Abstract

We consider high-frequency sampled continuous-time autoregressive moving average (CARMA) models driven by finite-variance zero-mean Lévy processes. An L^2-consistent estimator for the increments of the driving Lévy process without order selection is proposed if the CARMA model is invertible. The high-frequency behavior of approximating Riemann sum processes is analyzed and their autocovariance structure is compared to the one of sampled CARMA processes.

In the second part, the underlying process of the CARMA model is chosen to be either a symmetric α-stable Lévy process or a symmetric Lévy process with finite second moments. In the doubly asymptotic framework of high-frequency data within a long time interval, convergence of normalized and self-normalized versions of the periodogram to functions of stable distributions is shown. The limit distributions differ depending on whether the frequency values are linearly dependent or independent over $\mathbb{Z}$. For the proofs we require methods from the geometry of numbers. Moreover, a consistent estimate for the normalized power transfer function is established by applying a smoothing filter to the periodogram. This result is used to propose an estimator for the parameters of the CARMA process.

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Contents

1 Introduction

Being the continuous-time analog of the well-known ARMA processes (see, e.g., [17]), continuous-time autoregressive moving average (CARMA) processes have been extensively studied over the recent years. They provide, on the one hand, a mathematically tractable class of linear stochastic models in continuous time. On the other hand, as shown in [80], CARMA processes are equivalent to the class of continuous-time linear state space models if the variance of the underlying driving Lévy process is supposed to be finite. It is therefore not astonishing that this rich class is widely used in various areas of application as, e.g., signal processing and control (cf. [46, 61]), econometrics (cf. [7, 72]), high-frequency financial econometrics (cf. [87]), financial mathematics (cf. [6, 13, 49, 85]) and stochastic modeling of energy markets (cf. [5, 45]).

The question whether a time series should be modeled by a continuous-time or a discrete-time process is naturally of fundamental importance. One obvious advantage of continuous-time modeling is that it allows handling of irregularly spaced time series and in particular high-frequency data. The constantly increasing availability of the latter in finance and science in general has sparked a great deal of attention about the asymptotic behavior of high-frequency sampled processes in the last decade (see, e.g., [51]). The substantial part of this thesis will deal with statistical inference for the underlying Lévy process and the parameters of CARMA models in a high-frequency asymptotic framework.

The financial and economic crisis that started in 2008 with the Lehman default and the ongoing sovereign crisis demonstrate impressively that there is a necessity for new models incorporating more of the so-called stylized facts (for instance heavy tails, i.e. very high/low values are far more likely than in the normal distribution) which one can observe in real financial observation data. In order to illustrate the basic difference between a Gaussian distribution and a distribution with a long tail, Montroll and Shlesinger [66] proposed to compare the distribution of heights with the distribution of annual incomes for American adult males. An average individual who seeks a friend twice his height would fail. On the other hand, one who has an average income will have no trouble to discover a richer person, who, with a little diligence, may locate a third person with twice his income, etc. The income distribution in its upper range has a Pareto inverse power tail; however, most of the income distributions follow a log-normal curve, but the last few percent have a stable tail with exponent $\alpha = 1.6$ (cf. [3]), i.e., the mean is finite but the variance of the corresponding 1.6-stable distribution diverges. In the case of financial data these models are extreme in the sense that stable distributions (excluding the Gaussian) do not have a finite variance. But in contrast to "classical" finance (stocks, bonds, currencies, storable commodities, etc.) there are many other fields of application where it seems to be reasonable to assume infinite variance for the data. In [45], for instance, a stable CARMA(2, 1) model is fitted to spot prices from the Singapore New Electricity Market. An early example of α-stable stochastic modeling can be found already in Mandelbrot [62] and Fama [35], who proposed the stable distribution for stock returns. Internet traffic models are just one other possible application area for heavy-tailed models (cf. [27, 30, 64]), other examples of α-stable stochastic modeling are given in [52, Chapter 7].

Also from a theoretical point of view stable distributions are an interesting class to work with. If $X_1, X_2, \ldots$ are i.i.d. nondegenerate random variables and there are sequences of constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$a_n^{-1} \sum_{k=1}^{n} X_k - b_n \xrightarrow{\mathcal{D}} Z$$

as $n \to \infty$, then Z has a stable distribution (cf. [12, Proposition 9.25]).

In classical time series analysis second-order stationarity is a basic concept. Therefore, a distinctive feature when dealing with finite-variance models is the extensive use of Hilbert space methods (see, for instance, [17, Chapter 2]). However, if we allow for possibly infinite-variance processes, in particular stable models, these techniques cannot be used for a theoretical analysis of statistics of the corresponding models. Although there is a concept of orthogonality even in Banach spaces, called James orthogonality (see, e.g., [77, Section 2.9]), or a "covariance alternative" for infinitely divisible processes, called codifference (see, e.g., [77, Section 2.10]), it is not immediate to carry over results for finite-variance processes to possibly heavy-tailed models.

One possible way out, however, is to use spectral methods. Very often a significant tool for the analysis of statistical and probabilistic problems in conjunction with various stochastic processes is provided by spectral representations of these models. For instance, the spectral representations of symmetric stable processes have been used successfully to solve prediction and interpolation problems (see, e.g., [26, 50]) and to study structural and path properties for certain subclasses of these models (see, e.g., [25, 75]). In [44] it has been shown that also multivariate regularly varying CARMA processes, which in particular include stable CARMA models, possess spectral representations in the summability sense. Inspired by this work, we decided to address a deeper spectral analysis of (stable) CARMA processes in this thesis.

Outline of the thesis

Every of the following chapters of this thesis is based on a paper and hence, they are basically self-contained. In the following we present brief abstracts of the contents for each chapter. Detailed content information follows in the introductory sections of the individual chapters.

In this thesis we deal with statistical inference for the underlying Lévy process and the parameters of CARMA models in a high-frequency asymptotic framework. The common ground for the following three chapters is given, on the one hand, by the continuous-time model we consider, namely Lévy-driven CARMA processes. On the other hand, each chapter will either use spectral methods in proofs or study directly probabilistic properties of these models in the frequency domain.

More precisely, Chapter 2 is based on [40]. High-frequency sampled CARMA models driven by finite-variance zero-mean Lévy processes are considered. An L^2-consistent estimator for the increments of the driving Lévy process without order selection in advance is proposed if the CARMA model is invertible. In the second part of this chapter we analyze the high-frequency behavior of approximating Riemann sum processes, which represent a natural way to simulate continuous-time moving average processes on a discrete grid. We shall compare their autocovariance structure with the one of sampled CARMA processes, where the rule of integration plays a crucial role. Moreover, new insight into the kernel estimation procedure of [20] is given.

Chapter 3 is based on [39]. Again a CARMA process $(Y_t)_{t\in\mathbb{R}}$ is considered, this time driven by a symmetric α-stable Lévy process with $\alpha \in (0, 2]$. It is sampled at a high-frequency time-grid $\{0, \Delta_n, 2\Delta_n, \ldots, n\Delta_n\}$, where the observation grid gets finer and the last observation tends to infinity as $n \to \infty$. We investigate the normalized periodogram $I_{n,Y^{\Delta_n}}(\omega) = |n^{-1/\alpha} \sum_{k=1}^{n} Y_{k\Delta_n} \mathrm{e}^{-i\omega k}|^2$ of the sampled sequence. Under suitable conditions on Δ_n we show the convergence of the finite-dimensional distribution of both

$$\Delta_n^{2-\frac{2}{\alpha}} \left[I_{n,Y^{\Delta_n}}(\omega_1\Delta_n), \ldots, I_{n,Y^{\Delta_n}}(\omega_m\Delta_n) \right]$$

for $(\omega_1, \ldots, \omega_m) \in (\mathbb{R}\backslash\{0\})^m$ and of self-normalized versions of it to functions of stable distributions. The limit distributions differ depending on whether $\omega_1, \ldots, \omega_m$ are linearly dependent or independent over $\mathbb{Z}$. For the proofs we require methods from the geometry of numbers.

The last chapter is taken from [38]. Once more a CARMA process driven by either a symmetric α-stable Lévy process with $\alpha \in (0,2)$ or a symmetric Lévy process with finite second moments is studied. In the asymptotic framework of high-frequency data within a long time interval, we establish a consistent estimate for the normalized power transfer function by applying a smoothing filter to the periodogram of the CARMA process. Thereafter, this result is used in order to propose an estimator for the parameters of the CARMA process. The estimation procedure is exemplified by a simulation study.

2 Noise recovery and high-frequency behavior of approximating Riemann sums[1]

2.1 Introduction

The constantly increasing availability of high-frequency data in finance and science in general has sparked a great deal of attention about the asymptotic behavior of high-frequency sampled processes in the last decade, especially concerning the estimation of the multi-power variations of Itō semimartingales (see, e.g., [2, 4]), employing their realized counterparts. These quantities are of primary importance to practitioners, since they embody the deviation of data from a Brownian motion. Such methods are summarized in the book of [51] and references therein, which represents the most recent review on the subject.

In many areas of application Lévy-driven processes are used for modeling time series. An ample class within this group are continuous-time moving average (CMA) processes

$$Y_t = \int_{-\infty}^{\infty} g(t-s)\,\mathrm{d}L_s, \quad t \in \mathbb{R}, \tag{2.1}$$

where g is a so-called kernel function and $L = (L_t)_{t\in\mathbb{R}}$ is said to be the driving Lévy process (see, e.g., [78] for a detailed introduction). They cover, for instance, Ornstein-Uhlenbeck and continuous-time autoregressive moving average (CARMA) processes. The latter are the continuous-time analog of the well-known ARMA processes (see, e.g., [17]) and have been extensively studied over the recent years (cf. [14, 15, 22, 85]). Originally, driving processes of CARMA models were restricted to Brownian motion (see [32], and also [33]); however, [14] allowed for Lévy processes with a finite rth moment for some $r > 0$.

[1]The contents of this chapter are based on Ferrazzano, V. and Fuchs, F. (2012), Noise recovery for Lévy-driven CARMA processes and high-frequency behaviour of approximating Riemann sums, *submitted for publication.*

Lévy-driven CARMA processes are widely used in various areas of application like signal processing and control (cf. [46, 61]), high-frequency financial econometrics (cf. [87]) and financial mathematics (cf. [6, 13, 49, 85]). Stable CARMA processes can be relevant in modeling energy markets (cf. [5, 45]). Very often, a correct specification of the driving Lévy process is of primary importance in all these applications.

In this chapter we will be concerned with a high-frequency sampled CARMA process driven by a second-order zero-mean Lévy process. Under the assumption of *invertibility* of the CARMA model, we shall present an L^2-consistent estimator for the increments of the driving Lévy process, employing standard time series techniques. It is remarkable that the proposed procedure works for arbitrary autoregressive and moving average orders, i.e. there is no need for *order selection* in advance. In the light of the results in [20] and the flexibility of CARMA processes, the method might apply to a wider class of CMA processes, too. Moreover, since the proof employees only the fact that the increments of the Lévy process are orthogonal rather than independent, the result holds for a much ampler class of driving processes. Notable examples are the COGARCH processes ([13, 55]) or time-changed Lévy processes ([28]), which are often employed to model volatility clustering in finance and intermittency in turbulence.

This noise recovery result will give rise to the conjecture that the sampled CARMA process behaves on a high-frequency time grid approximately like a suitable MA(∞) process, which we shall call *approximating Riemann sum process*. By comparing the asymptotic properties of the autocovariance structure of high-frequency sampled CARMA models with the one of their approximating Riemann sum processes, it will turn out that the so-called *rule* of the Riemann sum plays a crucial role if the difference between the autoregressive and moving average order is greater than 1. On the one hand, this gives new insight for the kernel estimation procedure studied in [20] and explains at which points the kernel is indeed estimated. On the other hand, this has obvious consequences for simulation purposes. Riemann sum approximations are an easy tool to simulate CMA processes; however, our results show that one has to be careful with the chosen rule of integration in the context of certain CARMA processes.

The outline of the remaining chapter is as follows. In Section 2.2 we are going to recall the definition of finite-variance CARMA models and summarize important properties of high-frequency sampled CARMA processes. In particular, we fix a global assumption that guarantees causality and invertibility for the sampled sequence. In the third section we then derive an L^2-consistent estimator for the increments of the driving Lévy process starting from the Wold representation of the sampled process. It will turn out that *invertibility* of the original continuous-time process is sufficient and necessary for the recovery result to hold. Section 2.3 is completed by an illustrating example for CAR(2) and CARMA$(2,1)$ processes. Thereafter, the high-frequency behavior of approximating Riemann sum processes is studied in Section 2.4. First, an ARMA representation for the Riemann sum approximation is established in general and then the role of the rule of integration is analyzed by matching the asymptotic autocovariance structure of sampled CARMA processes and their Riemann sum approximations in the cases where the autoregressive order is less or equal to 3. The connection between the Wold representation and the approximating Riemann sum yields a deeper insight into the kernel estimation procedure introduced in [20]. A short conclusion can be found in Section 2.5 and finally, some auxiliary results are established in the last section.

2.2 Preliminaries to Chapter 2

2.2.1 Finite-variance CARMA processes

Throughout this chapter we shall be concerned with a CARMA process driven by a second-order zero-mean Lévy process $L = (L_t)_{t\in\mathbb{R}}$ with $E[L_1] = 0$ and $E[L_1^2] = 1$. It is defined as follows.

For non-negative integers p and q such that $p > q$, a CARMA(p,q) process $Y = (Y_t)_{t\in\mathbb{R}}$ with coefficients $a_1, \dots, a_p$, $c_0, \dots, c_q \in \mathbb{R}$ and driving Lévy process L is defined to be a strictly stationary solution of the suitably interpreted formal equation

$$a(\mathrm{D})\, Y_t = \sigma c(\mathrm{D})\, \mathrm{D} L_t, \quad t \in \mathbb{R}, \tag{2.2}$$

where D denotes differentiation with respect to t, $a(\cdot)$ and $c(\cdot)$ are the characteristic polynomials,

$$a(z) := z^p + a_1 z^{p-1} + \cdots + a_p \quad \text{and} \quad c(z) := c_0 + c_1 z + \cdots + c_{p-1} z^{p-1},$$

the coefficients c_j satisfy $c_q = 1$ and $c_j = 0$ for $q < j < p$, and $\sigma > 0$ is a constant. The polynomials $a(\cdot)$ and $c(\cdot)$ are assumed to have no common zeros. Throughout this chapter we shall denote by λ_i, $i = 1, \dots, p$, and $-\mu_i$, $i = 1, \dots, q$, the roots of $a(\cdot)$ and $c(\cdot)$ respectively, such that these polynomials can be written as $a(z) = \prod_{i=1}^{p}(z - \lambda_i)$ and $c(z) = \prod_{i=1}^{q}(z + \mu_i)$. Throughout this chapter we will further assume

Assumption 2.1.

(i) The zeros of the polynomial $a(\cdot)$ satisfy $\Re(\lambda_j) < 0$ for every $j = 1, \dots, p$,

(ii) the roots of $c(\cdot)$ have non-vanishing real part, i.e. $\Re(\mu_j) \neq 0$ for all $j = 1, \ldots, q$.

Since the derivative $\mathrm{D}L_t$ does not exist in the usual sense, we interpret (2.2) as being equivalent to the observation and state equations

$$Y_t = c^T X_t, \tag{2.3}$$

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + e_p\,\mathrm{d}L_t, \tag{2.4}$$

where

$$X_t = \begin{pmatrix} X(t) \\ X^{(1)}(t) \\ \vdots \\ X^{(p-2)}(t) \\ X^{(p-1)}(t) \end{pmatrix}, \quad c = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{p-2} \\ c_{p-1} \end{pmatrix}, \quad e_p = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \ldots & -a_1 \end{pmatrix} \quad \text{and } A = -a_1 \text{ for } p = 1.$$

It is easy to check that the eigenvalues of the matrix A are the same as the zeros of the autoregressive polynomial $a(\cdot)$.

Under Assumption 2.1 (i) it has been shown in [22, Theorem 3.3] that Eqs. (2.3)-(2.4) have the unique strictly stationary solution,

$$Y_t = \int_{-\infty}^{\infty} g(t-u)\,\mathrm{d}L_u, \quad t \in \mathbb{R}, \tag{2.5}$$

where

$$g(t) = \begin{cases} \dfrac{\sigma}{2\pi i}\displaystyle\int_\rho \frac{c(z)}{a(z)}\mathrm{e}^{tz}\,\mathrm{d}z = \sigma \sum_\lambda \mathrm{Res}_{z=\lambda}\left(\mathrm{e}^{zt}\frac{c(z)}{a(z)}\right), & \text{if } t > 0, \\ 0, & \text{if } t \leq 0, \end{cases} \tag{2.6}$$

and ρ is any simple closed curve in the open left half of the complex plane encircling the zeros of $a(\cdot)$. The sum is over the distinct zeros λ of $a(\cdot)$ and $\mathrm{Res}_{z=\lambda}(\cdot)$ denotes the residue at λ of the function in brackets. The kernel g can also be expressed (cf. [22], Eqs. (2.10) and (3.7)) as

$$g(t) = \sigma\, c^T \mathrm{e}^{tA}\, e_p\, \mathbb{1}_{(0,\infty)}(t), \quad t \in \mathbb{R}, \tag{2.7}$$

and its Fourier transform is

$$\mathcal{F}\left\{g(\cdot)\right\}(\omega) := \int_{\mathbb{R}} g(s)\, \mathrm{e}^{i\omega s}\, \mathrm{d}s = \sigma \frac{c(-i\omega)}{a(-i\omega)}, \quad \omega \in \mathbb{R}. \tag{2.8}$$

In the light of Eqs. (2.5)-(2.8), we can interpret a CARMA process as a continuous-time filtered white noise, whose transfer function has a finite number of poles and zeros. We outline that the request on the roots of $a(\cdot)$ to lie in the interior of the left half of the complex plane in order to have causality arises from Theorem V, p. 8, [71], which is intrinsically connected with the theorems in [86], pp. 125-129, on the Hilbert transform. A similar request on the roots of $c(\cdot)$ will turn out to be necessary for recovering the driving Lévy process.

2.2.2 Properties of high-frequency sampled CARMA processes

We now recall some properties of the sampled process $Y^\Delta := (Y_{n\Delta})_{n\in\mathbb{Z}}$ of a CARMA(p,q) model where $\Delta > 0$; cf. [20, 21] and references therein. It is known that the sampled sequence Y^Δ satisfies the ARMA$(p, p-1)$ equations

$$\Phi^\Delta(\mathrm{B})\, Y_n^\Delta = \Theta^\Delta(\mathrm{B})\, Z_n^\Delta, \quad n \in \mathbb{Z}, \quad (Z_n^\Delta) \sim \mathrm{WN}(0, \sigma_\Delta^2), \tag{2.9}$$

with the AR part $\Phi^\Delta(\mathrm{B}) := \prod_{i=1}^{p}(1 - \mathrm{e}^{\Delta\lambda_i}\mathrm{B})$, where B is the discrete-time backshift operator, $\mathrm{B}Y_n^\Delta := Y_{n-1}^\Delta$. Finally, the MA part $\Theta^\Delta(\cdot)$ is a polynomial of order $p-1$, chosen in such a way that it has no roots inside the unit circle. For $p > 3$ and fixed $\Delta > 0$ there is no explicit expression for the coefficients of $\Theta^\Delta(\cdot)$ nor the white noise process Z^Δ. Nonetheless, asymptotic expressions for $\Theta^\Delta(\cdot)$ and $\sigma_\Delta^2 = \mathrm{Var}(Z_n^\Delta)$ as $\Delta \downarrow 0$ were obtained in [20, 21]. Namely we have that the polynomial $\Theta^\Delta(z)$ can be written as (see [20, Theorem 2.1])

$$\Theta^\Delta(z) = \prod_{i=1}^{p-q-1}(1 + \eta(\xi_i)\, z) \prod_{k=1}^{q}(1 - \zeta_k\, z), \quad z \in \mathbb{C}, \tag{2.10}$$

$$\sigma_\Delta^2 = \frac{\sigma^2 \Delta^{2(p-q)-1}}{(2(p-q)-1)! \prod_{i=1}^{p-q-1}\eta(\xi_i)}(1 + o(1)), \quad \text{as } \Delta \downarrow 0, \tag{2.11}$$

where, again as $\Delta \downarrow 0$,

$$\begin{aligned} \zeta_k &= 1 \pm \mu_k\Delta + o(\Delta), & k &= 1, \ldots, q, \\ \eta(\xi_i) &= \xi_i - 1 \pm \sqrt{(\xi_i - 1)^2 - 1} + o(1), & i &= 1, \ldots, p-q-1. \end{aligned} \tag{2.12}$$

The signs $\pm$ in (2.12) are chosen respectively such that, for sufficiently small Δ, the coefficients ζ_k and $\eta(\xi_i)$ are less than 1 in absolute value. This ensures that Eq. (2.9) is invertible. Moreover, ξ_i, $i = 1, \dots, p-q-1$, are the zeros of $\alpha_{p-q-1}(\cdot)$, which is defined as the $(p-q-1)$-th coefficient in the series expansion

$$\frac{\sinh(z)}{\cosh(z) - 1 + x} = \sum_{k=0}^{\infty} \alpha_k(x) z^{2k+1}, \quad z \in \mathbb{C},\ x \in \mathbb{R}\backslash\{0\}, \tag{2.13}$$

where the LHS of Eq. (2.13) is a power transfer function arising from the sampling procedure (cf. [21, Eq. (11)]). Therefore the coefficients $\eta(\xi_i)$, $i = 1, \dots, p-q-1$, can be regarded as spurious, as they do not depend on the parameters of the underlying continuous-time process Y, but just on $p-q$.

Remark 2.2.1. Our notion of sampled process is a weak one, since we only require that the sampled process has the same autocovariance structure as the continuous-time process, when observed on a discrete grid. We know that the filtered process on the LHS of (2.9) (see [22, Lemma 2.1]) is a $(p-1)$-dependent discrete-time process. Therefore there exist 2^{p-1} possible representations for the RHS of (2.9) yielding the same autocovariance function of the filtered process, but only one having its roots outside the unit circle, which is called minimum-phase spectral factor (see [79] for a review on the topic). Since it is not possible to discriminate between them, we always take the minimum-phase spectral factor without any further question. This will be crucial for our main result.

Moreover, the rationale behind Assumption 2.1 (ii) becomes clear now: if $\Re(\mu_k) = 0$ for some k, then the corresponding $|\zeta_k|^2$ is equal to $1 + \Delta^2|\mu_k|^2 + o(\Delta^2)$, for either sign choice. Then the MA$(p-1)$ polynomial in Eq. (2.10) will never be invertible for small Δ. □

In order to ensure the invertibility of the sampled CARMA process, we need to verify that $|\eta(\xi_i)|$ is strictly less than 1 for all $i = 1, \dots, p-q-1$ and sufficiently small Δ.

Proposition 2.2.2. *The coefficients* $\eta(\xi_i)$ *in Eq.* (2.12) *are uniquely determined by*

$$\eta(\xi_i) = \xi_i - 1 - \sqrt{(\xi_i - 1)^2 - 1} + o(1), \quad i = 1, \dots, p-q-1,$$

and we have that $\xi_i - 1 - \sqrt{(\xi_i - 1)^2 - 1} \in (0, 1)$ *for all* $i = 1, \dots, p-q-1$.

Proof. It follows from Proposition 2.6.1 that $\xi_i \in (2, \infty)$ for all $i = 1, \dots, p-q-1$.

This yields $\xi_i - 1 + \sqrt{(\xi_i-1)^2-1} > 1$ for all i and hence, we have that

$$\eta(\xi_i) = \xi_i - 1 - \sqrt{(\xi_i-1)^2-1} + o(1), \quad i = 1, \dots, p-q-1.$$

Since the first-order term of $\eta(\xi_i)$ is positive and monotonously decreasing in ξ_i, the additional claim follows. □

2.3 Noise recovery

In this section we are going to prove the first main statement of the chapter, which is a recovery result for the driving Lévy process. We start with some motivation for our approach.

We know that the sampled CARMA process $Y^\Delta = (Y_{n\Delta})_{n\in\mathbb{Z}}$ has the Wold representation (cf. [17, p. 187])

$$Y_n^\Delta = \sum_{j=0}^\infty \psi_j^\Delta Z_{n-j}^\Delta = \sum_{j=0}^\infty \left(\frac{\sigma_\Delta}{\sqrt{\Delta}}\psi_j^\Delta\right)\left(\frac{\sqrt{\Delta}}{\sigma_\Delta} Z_{n-j}^\Delta\right), \quad n \in \mathbb{Z}, \tag{2.14}$$

where $\sum_{j=0}^\infty (\psi_j^\Delta)^2 < \infty$. Moreover, Eq. (2.14) is the causal representation of Eq. (2.9), and it has been shown in [20] that for every causal and invertible CARMA(p,q) process, as $\Delta \downarrow 0$,

$$\frac{\sigma_\Delta}{\sqrt{\Delta}}\psi_{\lfloor t/\Delta\rfloor}^\Delta \to g(t), \quad t \geq 0, \tag{2.15}$$

where g is the kernel in the moving average representation (2.5). Given the availability of classical time series methods to estimate $(\psi_j^\Delta)_{j\in\mathbb{N}}$ and σ_Δ^2, and the flexibility of CARMA processes, we argue that this result can be applied to more general CMA processes.

Given Eqs. (2.14) and (2.15) it is natural to investigate, whether the quantity

$$\bar{L}_n^\Delta := \frac{\sqrt{\Delta}}{\sigma_\Delta} Z_n^\Delta, \quad n \in \mathbb{Z},$$

approximates the increments of the driving Lévy process in the sense that for every fixed $t \in (0,\infty)$,

$$\sum_{i=1}^{\lfloor t/\Delta\rfloor} \bar{L}_i^\Delta \overset{L^2}{\to} L_t, \quad \Delta \downarrow 0. \tag{2.16}$$

As usual, convergence in L^2 implies also convergence in probability and distribution.

The first results on retrieving the increments of L were given in [19], and furthermore generalized to the multivariate case by [24]. The essential limitation of this parametric method is that it might not be robust with respect to model misspecification. More precisely, the fact that a CARMA(p,q) process is $(p-q-1)$-times differentiable (see Proposition 3.32 of [63]) is crucial for the procedure to work (cf. [24, Theorem 4.3]). However, if the underlying process is instead CARMA(p',q') with $p'-q'<p-q$, then some of the necessary derivatives do not exist anymore. In contrast to the aforementioned procedure, in the method we propose there is no need to specify the autoregressive and the moving average orders p and q in advance.

Before we start to prove the recovery result in Eq. (2.16), let us, in analogy to the discrete-time case, establish the notion of invertibility.

Definition 2.3.1.
A CARMA(p,q) process is said to be invertible *if the roots of the moving average polynomial $c(\cdot)$ have negative real parts, i.e. $\Re(\mu_i)>0$ for all $i=1,\dots,q$.*

Our main theorem is the following:

Theorem 2.3.2. *Let Y be a finite-variance CARMA(p,q) process and Z^Δ the noise on the RHS of the sampled Eq. (2.9). Moreover, let Assumption 2.1 hold and define $\bar{L}^\Delta := \sqrt{\Delta}/\sigma_\Delta Z^\Delta$. Then, as $\Delta \downarrow 0$,*

$$\sum_{i=1}^{\lfloor t/\Delta \rfloor} \bar{L}_i^\Delta \xrightarrow{L^2} L_t, \quad t\in(0,\infty), \tag{2.17}$$

if and only if the roots of the moving average polynomial $c(\cdot)$ on the RHS of the CARMA Eq. (2.2) have negative real parts, i.e. if and only if the CARMA process is invertible.

Proof. Under Assumption 2.1 (ii) and using Proposition 2.2.2, the sampled ARMA equation (2.9) is invertible. The noise on the RHS of (2.9) is then obtained using the classical inversion formula

$$Z_n^\Delta = \frac{\Phi^\Delta(\mathrm{B})}{\Theta^\Delta(\mathrm{B})} Y_n^\Delta, \quad n\in\mathbb{Z},$$

where B is the usual backshift operator. Let us consider the stationary *continuous-time* process

$$\mathcal{Z}_t^\Delta := \frac{\Phi^\Delta(\mathrm{B}_\Delta)}{\Theta^\Delta(\mathrm{B}_\Delta)} Y_t = \sum_{i=0}^{\infty} a_i^\Delta \int_{-\infty}^{t-i\Delta} g(t-i\Delta-s)\,\mathrm{d}L_s, \quad t\in\mathbb{R}, \tag{2.18}$$

where the continuous-time backshift operator B_Δ is defined such that $\mathrm{B}_\Delta Y_t := Y_{t-\Delta}$ for every $t\in\mathbb{R}$ and the series in the RHS of Eq. (2.18) is the Laurent expansion of the rational function $\Phi^\Delta(\cdot)/\Theta^\Delta(\cdot)$. Moreover, $\mathcal{Z}^\Delta_{n\Delta} = Z^\Delta_n$ for every $n \in \mathbb{N}$; as a consequence, the random variables $\mathcal{Z}^\Delta_s$, $\mathcal{Z}^\Delta_t$ are uncorrelated for $|t-s| \geq \Delta$ and $\operatorname{Var}(\mathcal{Z}^\Delta_t) = \operatorname{Var}(Z^\Delta_n)$. Exchanging the sum and the integral signs in Eq. (2.18), and the fact that $g(\cdot) = 0$ for negative arguments, we have that $\mathcal{Z}^\Delta$ is a continuous-time moving average process

$$\mathcal{Z}^\Delta_t = \int_{-\infty}^{t} g^\Delta(t-s)\,\mathrm{d}L_s, \quad t \in \mathbb{R},$$

whose kernel function g^Δ has Fourier transform (cf. Eq. (2.8))

$$\mathcal{F}\{g^\Delta(\cdot)\}(\omega) = \frac{\Phi^\Delta(\mathrm{e}^{i\omega\Delta})}{\Theta^\Delta(\mathrm{e}^{i\omega\Delta})}\,\mathcal{F}\{g(\cdot)\}(\omega) = \sigma\frac{\Phi^\Delta(\mathrm{e}^{i\omega\Delta})}{\Theta^\Delta(\mathrm{e}^{i\omega\Delta})}\frac{c(-i\omega)}{a(-i\omega)}, \quad \omega \in \mathbb{R}, \quad \Delta > 0.$$

Since we can write $L_t - L_{t-\Delta} = \int_{-\infty}^{t} \mathbb{1}_{(0,\Delta)}(t-s)\,\mathrm{d}L_s$, $t \in \mathbb{R}$, the sum of the differences between the rescaled sampled noise terms and the increments of the Lévy process is given by

$$\begin{aligned}\sum_{j=1}^{n} \bar{L}^\Delta_j - L_{n\Delta} &= \int_{-\infty}^{n\Delta} \sum_{j=1}^{n}\left[\frac{\sqrt{\Delta}}{\sigma_\Delta} g^\Delta(j\Delta - s) - \mathbb{1}_{(0,\Delta)}(j\Delta - s)\right]\mathrm{d}L_s \\ &= \int_{-\infty}^{n\Delta} h^\Delta_n(n\Delta - s)\,\mathrm{d}L_s, \qquad n \in \mathbb{N},\end{aligned} \tag{2.19}$$

where, for every $n \in \mathbb{N}$,

$$h^\Delta_n(s) := \sum_{j=1}^{n}\left[\frac{\sqrt{\Delta}}{\sigma_\Delta} g^\Delta(s + (j-n)\Delta) - \mathbb{1}_{(0,\Delta)}(s + (j-n)\Delta)\right], \quad s \in \mathbb{R},$$

and the stochastic integral in Eq. (2.19) w.r.t. L is still in the L^2-sense. It is a standard result, cf. [47, Ch. IV, §4], that the variance of the moving average process in Eq. (2.19) is given by

$$\mathbb{E}\left[\sum_{j=1}^{n} \bar{L}^\Delta_j - L_{n\Delta}\right]^2 = \int_{-\infty}^{n\Delta} \left(h^\Delta_n(n\Delta - s)\right)^2 \mathrm{d}s = \|h^\Delta_n(\cdot)\|^2_{L^2}, \tag{2.20}$$

where the latter equality is true since $h^\Delta_n(s) = 0$ for any $s \leq 0$.

Furthermore, the Fourier transform of $h^\Delta_n(\cdot)$ can be readily calculated, invoking the linearity and the shift property of the Fourier transform. We thus obtain

$$\begin{aligned}\mathcal{F}\{h_n^\Delta(\cdot)\}(\omega) &= \left[\frac{\sqrt{\Delta}}{\sigma_\Delta}\mathcal{F}\{g^\Delta(\cdot)\}(\omega) - \mathcal{F}\{\mathbb{1}_{(0,\Delta)}(\cdot)\}(\omega)\right]\sum_{j=1}^{n} \mathrm{e}^{i\omega(n-j)\Delta} \\ &= \left[\sigma\frac{\sqrt{\Delta}}{\sigma_\Delta}\frac{\prod_{j=1}^{p}(1-\mathrm{e}^{\Delta(\lambda_j+i\omega)})}{\Theta^\Delta(\mathrm{e}^{i\omega\Delta})}\frac{c(-i\omega)}{a(-i\omega)} - \frac{\mathrm{e}^{i\omega\Delta}-1}{i\omega}\right]\frac{1-\mathrm{e}^{i\omega\Delta n}}{1-\mathrm{e}^{i\omega\Delta}} \\ &=: \left[h^{\Delta,1}(\omega) - h^{\Delta,2}(\omega)\right]\cdot h_n^{\Delta,3}(\omega), \quad \omega\in\mathbb{R}.\end{aligned}$$

Due to Plancherel's Theorem, we deduce

$$\begin{aligned}&\mathrm{Var}\left[\sum_{i=1}^{n}\bar{L}_j^\Delta - L_{n\Delta}\right] = \|h_n^\Delta(\cdot)\|_{L^2}^2 = \frac{1}{2\pi}\int_{\mathbb{R}}|\mathcal{F}\{h_n^\Delta(\cdot)\}|^2(\omega)\,\mathrm{d}\omega, \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\left[\left|h^{\Delta,1}\cdot h_n^{\Delta,3}(\omega)\right|^2 + \left|h^{\Delta,2}\cdot h_n^{\Delta,3}(\omega)\right|^2 - 2\Re\left(h^{\Delta,1}\cdot\overline{h^{\Delta,2}}(\omega)\right)\left|h_n^{\Delta,3}(\omega)\right|^2\right]\mathrm{d}\omega.\end{aligned} \tag{2.21}$$

It is easy to see that the first two integrals in Eq. (2.21) are, respectively, the variances of $\sum_{i=1}^{n}\bar{L}_j^\Delta$ and $L_{n\Delta}$, both equal to $n\Delta$. Setting $n := \lfloor t/\Delta\rfloor$ yields for fixed positive t, as $\Delta\downarrow 0$,

$$\begin{aligned}\mathrm{Var}\left[\sum_{i=1}^{\lfloor t/\Delta\rfloor}\bar{L}_j^\Delta - L_{\lfloor t/\Delta\rfloor\Delta}\right] &= 2\lfloor t/\Delta\rfloor\Delta - \frac{1}{\pi}\int_{\mathbb{R}}\Re\left(h^{\Delta,1}\cdot\overline{h^{\Delta,2}}(\omega)\right)\left|h_{\lfloor t/\Delta\rfloor}^{\Delta,3}(\omega)\right|^2\mathrm{d}\omega \\ &= 2t(1+o(1)) - \frac{1}{\pi}\int_{\mathbb{R}}\Re\left(h^{\Delta,1}\cdot\overline{h^{\Delta,2}}(\omega)\right)\left|h_{\lfloor t/\Delta\rfloor}^{\Delta,3}(\omega)\right|^2\mathrm{d}\omega.\end{aligned}$$

Thus, in order to show Eq. (2.17), it remains to prove that

$$\frac{1}{\pi}\int_{\mathbb{R}}\Re\left(h^{\Delta,1}\cdot\overline{h^{\Delta,2}}(\omega)\right)\left|h_{\lfloor t/\Delta\rfloor}^{\Delta,3}(\omega)\right|^2\mathrm{d}\omega = 2t(1+o(1)) \quad \text{as } \Delta\downarrow 0,$$

which in turn is equivalent to

$$\begin{aligned}\frac{1}{2\pi t}\int_{\mathbb{R}}\sigma\frac{\sqrt{\Delta}}{\sigma_\Delta}\frac{1-\cos(\omega\lfloor t/\Delta\rfloor\Delta)}{1-\cos(\omega\Delta)}&\left[\frac{\sin(\omega\Delta)}{\omega}\Re\left(\frac{\prod_{j=1}^{p}(1-\mathrm{e}^{\Delta(\lambda_j+i\omega)})}{\Theta^\Delta(\mathrm{e}^{i\omega\Delta})}\frac{c(-i\omega)}{a(-i\omega)}\right)\right. \\ &\left.+\frac{1-\cos(\omega\Delta)}{\omega}\Im\left(\frac{\prod_{j=1}^{p}(1-\mathrm{e}^{\Delta(\lambda_j+i\omega)})}{\Theta^\Delta(\mathrm{e}^{i\omega\Delta})}\frac{c(-i\omega)}{a(-i\omega)}\right)\right]\mathrm{d}\omega = 1+o(1) \quad \text{as } \Delta\downarrow 0.\end{aligned} \tag{2.22}$$

Now, Lemma 2.6.2 shows that the integrand in Eq. (2.22) converges pointwise, for every $\omega \in \mathbb{R}\backslash\{0\}$, to $2(1-\cos(\omega t))/\omega^2$ as $\Delta \downarrow 0$. Since, for sufficiently small Δ, the integrand is dominated by an integrable function (see Lemma 2.6.3), we can apply Lebesgue's Dominated Convergence Theorem and deduce that the LHS of Eq. (2.22) converges, as $\Delta \downarrow 0$, to

$$\frac{1}{\pi t}\int_{\mathbb{R}} \frac{1-\cos(\omega t)}{\omega^2}\,\mathrm{d}\omega = \frac{2}{\pi}\int_0^\infty \frac{1-\cos(\omega)}{\omega^2}\,\mathrm{d}\omega = 1.$$

This proves Eq. (2.22) and concludes the proof of the "if"-statement.

As to the "only if"-statement, let $J := \{j = 1,\ldots,q : \Re(\mu_j) < 0\}$ and suppose that $J \neq \emptyset$. Then we have, by Eq. (2.10) for $\Delta \downarrow 0$,

$$\begin{aligned}
\frac{c(-i\omega)}{\Theta^\Delta(e^{i\omega\Delta})} &= \prod_{j=1}^{p-q-1} (1+\eta(\xi_j))^{-1} \prod_{j=1}^{q} \frac{\mu_j - i\omega}{1-\zeta_j\,\mathrm{e}^{i\omega\Delta}}(1+o(1)) \\
&= \prod_{j=1}^{p-q-1} (1+\eta(\xi_j))^{-1}\,\Delta^{-q} \prod_{j\in J} \frac{\mu_j - i\omega}{-\mu_j - i\omega}(1+o(1)) \\
&= \prod_{j=1}^{p-q-1} (1+\eta(\xi_j))^{-1}\,\Delta^{-q}(1+D(\omega))(1+o(1)), \qquad \omega\in\mathbb{R},
\end{aligned}$$

where $D(\omega) := -1 + \prod_{j\in J}(\mu_j - i\omega)/(-\mu_j - i\omega)$. By virtue of Lemmata 2.6.2 and 2.6.3, we then obtain that the LHS of Eq. (2.22) converges, as $\Delta \downarrow 0$, to

$$\frac{1}{\pi t}\int_{\mathbb{R}} \frac{1-\cos(\omega t)}{\omega^2}\big(1+\Re(D(\omega))\big)\,\mathrm{d}\omega = 1 + \frac{1}{\pi}\int_{\mathbb{R}} \frac{1-\cos(\omega)}{\omega^2}\Re(D(\omega/t))\,\mathrm{d}\omega.$$

Since $|\prod_{j\in J}(\mu_j - i\omega)/(-\mu_j - i\omega)| = 1$, we further deduce that $\Re(D(\omega)) \leq 0$ for any $\omega \in \mathbb{R}$. Obviously, $\Re(D(\omega)) \not\equiv 0$ and hence,

$$\frac{1}{\pi t}\int_{\mathbb{R}} \frac{1-\cos(\omega t)}{\omega^2}\big(1+\Re(D(\omega))\big)\,\mathrm{d}\omega < 1,$$

which, in turn, shows that the convergence result (2.17) cannot hold. □

Remark 2.3.3. (i) It is an easy consequence of the triangle and Hölder's inequality that, if the recovery result (2.17) holds, then also

$$\sum_{i=1}^{\lfloor t/\Delta\rfloor} \bar{L}_i^\Delta \sum_{j=\lfloor t/\Delta\rfloor+1}^{\lfloor s/\Delta\rfloor} \bar{L}_j^\Delta \xrightarrow{L^1} L_t\,(L_s - L_t), \quad t,s\in(0,\infty),\ t\leq s,$$

is valid.

(ii) Minor modifications of the proof above show that the recovery result remains valid if we drop the assumption of causality (i.e. Assumption 2.1 (i)) and suppose instead only $\Re(\lambda_j) \neq 0$ for every $j = 1, \ldots, p$. Hence, invertibility of the CARMA process is necessary for the noise recovery result to hold, whereas causality is not. In the non-causal case, the obtained white noise process will not be the same as in the Wold representation (2.14).

(iii) The necessity and sufficiency of the invertibility assumption descends directly from the fact that we choose always the minimum-phase spectral factor, as pointed out in Remark 2.2.1.

(iv) The proof suggests that this procedure should work in a much more general framework. Let $I^{\Delta}(\cdot)$ denote the inversion filter in Eq. (2.18) and $\psi^{\Delta} := \left\{\psi_i^{\Delta}\right\}_{i\in\mathbb{N}}$ the coefficients in the Wold representation (2.14). Then the proof of Theorem 2.3.2 essentially needs, apart from the rather technical Lemma 2.6.3, that, as $\Delta \downarrow 0$,

$$I^{\Delta}(\mathrm{e}^{i\omega\Delta})\,\mathcal{F}\{g(\cdot)\}(\omega) = \frac{\int_0^\infty g(s)\,\mathrm{e}^{i\omega s}\,\mathrm{d}s}{\sum_{k=0}^\infty \psi_k^{\Delta}\,\mathrm{e}^{ik\omega\Delta}} \to 1, \quad \omega \in \mathbb{R}, \tag{2.23}$$

provided that the function $\sum_{k=0}^\infty \psi_k^{\Delta} z^k$ does not have any zero inside the unit circle. In other words, we need that the discrete Fourier transform in the denominator of (2.23) converges to the Fourier transform in the numerator; this can be easily related to the kernel estimation result in Eq. (2.15). Given the peculiar structure of CARMA processes, this relationship can be calculated explicitly, but the results should hold true for continuous-time moving average processes with more general kernels, too. □

We illustrate Theorem 2.3.2 and the necessity of the invertibility assumption by an example where the convergence result is established using a time domain approach. That gives an explicit result also when the invertibility assumption is violated.

Unfortunately this strategy is not viable for a general CARMA process, due to the complexity of calculations involved when $p > 2$.

Example 2.3.4 (CARMA$(2, q)$ process)**.** The causal CARMA$(2, q)$ process is the strictly stationary solution to the formal stochastic differential equation

$$(\mathrm{D} - \lambda_2)\,(\mathrm{D} - \lambda_1)\,Y_t = \sigma\,\mathrm{D}L_t, \qquad q = 0,$$
$$(\mathrm{D} - \lambda_2)\,(\mathrm{D} - \lambda_1)\,Y_t = \sigma\,(c + \mathrm{D})\,\mathrm{D}L_t, \qquad q = 1,$$

and it can be represented as a continuous-time moving average process, as in Eq. (2.5), for $\lambda_1, \lambda_2 < 0$, $\lambda_1 \neq \lambda_2$ and $c \in \mathbb{R}\backslash\{0\}$, with kernel

$$g(t) = \sigma \frac{e^{t\lambda_1} - e^{t\lambda_2}}{\lambda_1 - \lambda_2}, \qquad q = 0,$$
$$g(t) = \sigma \frac{c+\lambda_1}{\lambda_1 - \lambda_2} e^{t\lambda_1} + \sigma \frac{c+\lambda_2}{\lambda_2 - \lambda_1} e^{t\lambda_2}, \qquad q = 1,$$

for $t > 0$ and 0 elsewhere. The corresponding sampled process $Y_n^\Delta = Y_{n\Delta}$, $n \in \mathbb{Z}$, satisfies the causal and invertible ARMA$(2,1)$ equations as in (2.9) where, from Eq. (27) of [21] and for $n \in \mathbb{Z}$,

$$\Phi^\Delta(\mathrm{B})\, Y_n^\Delta = \int_{(n-1)\Delta}^{n\Delta} g(n\Delta - u)\, \mathrm{d}L_u + \int_{(n-2)\Delta}^{(n-1)\Delta} [g(n\Delta - u) - (e^{\lambda_1\Delta} + e^{\lambda_2\Delta})\, g((n-1)\Delta - u)]\, \mathrm{d}L_u. \tag{2.24}$$

The corresponding MA(1) polynomial in Eq. (2.9) is $\Theta^\Delta(\mathrm{B}) = 1 - \theta_\Delta \mathrm{B}$, with asymptotic parameters

$$\theta_\Delta = \sqrt{3} - 2 + o(1), \quad \sigma_\Delta^2 = \sigma^2 \Delta^3 (2+\sqrt{3})/6 + o(\Delta^3), \qquad q = 0,$$
$$\theta_\Delta = 1 - \mathrm{sgn}(b)\, b\, \Delta + o(\Delta), \quad \sigma_\Delta^2 = \sigma^2 \Delta + o(\Delta), \qquad q = 1.$$

The inversion of Eqs. (2.9) and (2.24) gives, for every $\Delta > 0$,

$$\begin{aligned} Z_n^\Delta &= \frac{\Phi^\Delta(\mathrm{B})}{\Theta^\Delta(\mathrm{B})} Y_n^\Delta = \sum_{i=0}^\infty (\theta_\Delta \mathrm{B})^i \prod_{i=1}^2 (1 - e^{\lambda_i \Delta}\mathrm{B})\, Y_n^\Delta \\ &= \int_{(n-1)\Delta}^{n\Delta} g(n\Delta - u)\, \mathrm{d}L_u \\ &\quad + \sum_{i=0}^\infty \theta_\Delta^i \int_{(n-i-2)\Delta}^{(n-i-1)\Delta} [g((n-i)\Delta - u) - (e^{\lambda_1\Delta} + e^{\lambda_2\Delta} - \theta_\Delta) g((n-i-1)\Delta - u)]\, \mathrm{d}L_u. \end{aligned}$$

Then $Z^\Delta = (Z_n^\Delta)_{n\in\mathbb{Z}}$ is a weak white-noise process. Moreover, using $\Delta L_n = \int_{(n-1)\Delta}^{n\Delta} \mathrm{d}L_s$, we have

$$\mathbb{E}[Z_n^\Delta\, \Delta L_{n-j}] = \begin{cases} 0, & j < 0, \\ \int_0^\Delta g(s)\, \mathrm{d}s, & j = 0, \\ \theta_\Delta^{j-1} \int_0^\Delta [g(\Delta + s) - (e^{\lambda_1\Delta} + e^{\lambda_2\Delta} - \theta_\Delta)\, g(s)]\, \mathrm{d}s, & j > 0. \end{cases} \tag{2.25}$$

For a fixed $t \in (0, \infty)$, using the fact that ΔL and $\bar{L}^\Delta$ are both second-order stationary white noises with variance Δ, we have that

$$\mathbb{E}\left[\sum_{i=1}^{\lfloor t/\Delta \rfloor} (\bar{L}_i^\Delta - \Delta L_i)\right]^2 = 2\lfloor t/\Delta \rfloor \Delta - 2\sum_{i=1}^{\lfloor t/\Delta \rfloor} \mathbb{E}[\bar{L}_i^\Delta \Delta L_i] - 2\sum_{i \neq j} \mathbb{E}[\bar{L}_i^\Delta \Delta L_j]$$
$$= 2\lfloor t/\Delta \rfloor \Delta - \frac{2\sqrt{\Delta}}{\sigma_\Delta} \lfloor t/\Delta \rfloor \int_0^\Delta g(s)\,\mathrm{d}s$$
$$- \frac{2\sqrt{\Delta}}{\sigma_\Delta} \int_0^\Delta [g(\Delta + s) - (\mathrm{e}^{\lambda_1 \Delta} + \mathrm{e}^{\lambda_2 \Delta} - \theta_\Delta)\, g(s)]\,\mathrm{d}s \sum_{i=1}^{\lfloor t/\Delta \rfloor} \sum_{j=1}^{i-1} \theta_\Delta^{j-1}$$

where the last equality is obtained using Eq. (2.25). Moreover, for every $a \neq 1$,

$$\sum_{i=1}^{n} \sum_{j=1}^{i-1} a^{j-1} = \frac{a^n + (1-a)n - 1}{(1-a)^2}, \quad n \in \mathbb{N}.$$

Then the variance of the error can be explicitly calculated as

$$\mathbb{E}\left[\sum_{i=1}^{\lfloor t/\Delta \rfloor} (\bar{L}_i^\Delta - \Delta L_i)\right]^2$$
$$= 2\lfloor t/\Delta \rfloor \Delta - \frac{2\sqrt{\Delta}}{\sigma_\Delta} \lfloor t/\Delta \rfloor \int_0^\Delta g(s)\,\mathrm{d}s - \frac{2\sqrt{\Delta}}{\sigma_\Delta} \frac{{\theta_\Delta}^{\lfloor t/\Delta \rfloor} + \lfloor t/\Delta \rfloor (1 - \theta_\Delta) - 1}{(1 - \theta_\Delta)^2}$$
$$\times \int_0^\Delta [g(\Delta + s) - (\mathrm{e}^{\lambda_1 \Delta} + \mathrm{e}^{\lambda_2 \Delta} - \theta_\Delta)\, g(s)]\,\mathrm{d}s.$$

We now compute the asymptotic expansion for $\Delta \downarrow 0$ of the equation above. We obviously have that $2\lfloor t/\Delta \rfloor \Delta = 2t(1+o(1))$ and, using the explicit formulas for the kernel functions g,

$$\left.\begin{array}{r} \frac{2\sqrt{\Delta}}{\sigma_\Delta} \lfloor t/\Delta \rfloor \int_0^\Delta g(s)\,\mathrm{d}s \\ \frac{2\sqrt{\Delta}}{\sigma_\Delta} \int_0^\Delta [g(\Delta + s) - (\mathrm{e}^{\lambda_1 \Delta} + \mathrm{e}^{\lambda_2 \Delta} - \theta_\Delta) g(s)]\,\mathrm{d}s \\ ({\theta_\Delta}^{\lfloor t/\Delta \rfloor} + \lfloor t/\Delta \rfloor (1 - \theta_\Delta) - 1)/(1 - \theta_\Delta)^2 \end{array}\right\} = \left\{\begin{array}{l} \underline{q = 0} \\ \left(3 - \sqrt{3}\right) t + o(1) \\ \left(4\sqrt{3} - 6\right) \Delta (1 + o(1)) \\ \frac{1}{6}\left(3 + \sqrt{3}\right) t/\Delta (1 + o(1)) \\ \\ \underline{q = 1} \\ 2t + o(1) \\ 2(c - \operatorname{sgn}(c)\, c)\Delta^2 + o(\Delta^2) \\ (\mathrm{e}^{-\operatorname{sgn}(c) ct} + \operatorname{sgn}(c) ct - 1)/(c\Delta)^2 + o(\Delta^{-2}). \end{array}\right.$$

Hence, for a fixed $t \in (0,\infty)$ and $\Delta \downarrow 0$, we get

$$\mathbb{E}\left[\sum_{i=1}^{\lfloor t/\Delta \rfloor} (\bar{L}_i^\Delta - \Delta L_i)\right]^2 = \begin{cases} o(1), & q=0, \\ -2(\mathrm{e}^{-\mathrm{sgn}(c)ct} + \mathrm{sgn}(c)ct - 1)(1-\mathrm{sgn}(c))/c + o(1), & q=1, \end{cases}$$

i.e. (2.17) holds always for $q=0$, whereas for $q=1$ if and only if $c>0$. If $c<0$, the error approximating the driving Lévy by inversion of the discretized process grows as $4t$ for large t. □

2.4 High-frequency behavior of approximating Riemann sums

The fact that, in the sense of Eq. (2.16), $\bar{L}_n^\Delta \approx \Delta L_n = L_{n\Delta} - L_{(n-1)\Delta}$ for small Δ, along with Eq. (2.15), gives rise to the conjecture that the Wold representation (2.14) for Y^Δ behaves on a high-frequency time grid approximately like the MA(∞) process

$$\tilde{Y}_n^{\Delta,h} := \sum_{j=0}^{\infty} g(\Delta(j+h))\,\Delta L_{n-j}, \quad n \in \mathbb{Z}, \tag{2.26}$$

for some $h \in [0,1]$ and g is the kernel function as in (2.7). In other words, we have for a CARMA process, under the assumption of invertibility and causality, that the discrete-time quantities appearing in the Wold representation approximate the quantities in Eq. (2.26) when $\Delta \downarrow 0$. The transfer function of (2.26) is then defined as

$$\psi_h^\Delta(\omega) := \sum_{j=0}^{\infty} g(\Delta(j+h))\,\mathrm{e}^{-i\omega j}, \quad -\pi \le \omega \le \pi, \tag{2.27}$$

and its spectral density can be written as

$$f_h^\Delta(\omega) = \frac{1}{2\pi}|\psi_h^\Delta|^2(\omega), \quad -\pi \le \omega \le \pi.$$

It is well known that a CMA process can be defined (for a fixed time point t) as the L^2-limit of Eq. (2.26); this fact is naturally employed to simulate a CMA process easily, when all the relevant quantities are known a priori. Therefore we will name the process $\tilde{Y}^{\Delta,h}$ *approximating Riemann sum* of Eq. (2.5), and h is the so-called *rule* of the approximating sum; e.g. if $h=1/2$, we have the popular *mid-point rule.*

In order to give an answer to our conjecture, we will investigate properties of the approximating Riemann sum $\tilde{Y}^{\Delta,h}$ of a CARMA process and compare its asymptotic autocovariance structure with the one of the sampled CARMA process Y^{Δ}. This will yield more insight into the role of h for the behavior of $\tilde{Y}^{\Delta,h}$ as a process.

We start with a well known property of approximating sums.

Proposition 2.4.1. *Let g be in L^2 and Riemann-integrable. Then, for every $h \in [0,1]$, as $\Delta \downarrow 0$:*

(i) $\tilde{Y}_k^{\Delta,h} - Y_k^{\Delta} \xrightarrow{L^2} 0$, *for every* $k \in \mathbb{Z}$.

(ii) $\tilde{Y}_{\lfloor t/\Delta \rfloor}^{\Delta,h} \xrightarrow{L^2} Y_t$, *for every* $t \in \mathbb{R}$.

Proof. This follows immediately from the hypotheses made on g and the definition of L^2-integrals. □

This result essentially says only that approximating sums converge to Y_t for every fixed t. However, for a CARMA(p,q) process we have that the approximating Riemann sum process satisfies for every h and Δ an ARMA$(p,p-1)$ equation (see Proposition 2.4.2 below), meaning that there might exist a process, whose autocorrelation structure is the same as the one of the approximating sum. Given that the AR filter in this representation is the same as in Eq. (2.9), it is reasonable to investigate whether $\Phi^{\Delta}(\mathrm{B})\, Y^{\Delta}$ and $\Phi^{\Delta}(\mathrm{B})\, \tilde{Y}^{\Delta,h}$ have, as $\Delta \downarrow 0$, the same asymptotic autocovariance structure, which can be expected but is not granted by Proposition 2.4.1.

The following proposition gives the ARMA$(p,p-1)$ representation for the approximating Riemann sum.

Proposition 2.4.2. *Let Y be a CARMA(p,q) process, satisfying Assumption 2.1, and furthermore suppose that the roots of the autoregressive polynomial $a(\cdot)$ are distinct. Then the approximating Riemann sum process $\tilde{Y}^{\Delta,h}$ of Y defined by Eq.* (2.26) *satisfies, for every $h \in [0,1]$, the ARMA$(p,p-1)$ equation*

$$\Phi^{\Delta}(\mathrm{B})\, \tilde{Y}_n^{\Delta,h} = \sigma\, \tilde{\Theta}^{\Delta,h}(\mathrm{B})\, \Delta L_n, \quad n \in \mathbb{Z}, \tag{2.28}$$

where

$$\tilde{\Theta}^{\Delta,h}(z) := \tilde{\theta}_0^{\Delta,h} - \tilde{\theta}_1^{\Delta,h} z + - \ldots + (-1)^{p-1} \tilde{\theta}_{p-1}^{\Delta,h}\, z^{p-1} \tag{2.29}$$

and

$$\tilde{\theta}_k^{\Delta,h} := \sum_{l=1}^{p} \frac{c(\lambda_l)}{a'(\lambda_l)} \mathrm{e}^{h\Delta\lambda_l} \sum \mathrm{e}^{\Delta(\lambda_{j_1}+\lambda_{j_2}+\ldots+\lambda_{j_k})}, \quad k = 0, \ldots, p-1,$$

where the right-hand sum is defined to be 1 *for* $k = 0$ *and it is evaluated over all possible subsets* $\{j_1, \ldots, j_k\}$ *of* $\{1, \ldots, p\}\backslash\{l\}$*, having cardinality* k*, for* $k > 0$.

Proof. Write $\Phi^{\Delta}(z) = \prod_{j=1}^{p}(1 - \mathrm{e}^{\Delta\lambda_j} z) = -\sum_{j=0}^{p} \phi_j^{\Delta} z^j$ and observe that

$$\begin{aligned}\Phi^{\Delta}(B)\,\tilde{Y}_n^{\Delta,h} &= -\sum_{j=0}^{p} \phi_j^{\Delta}\, Y_{n-j}^{\Delta,h}\\ &= -\sigma c^T \sum_{k=0}^{p-1}\left(\sum_{j=0}^{k} \phi_j^{\Delta} \mathrm{e}^{(k-j)\Delta A}\right) \mathrm{e}^{h\Delta A}\, e_p \cdot \Delta L_{n-k}\\ &\qquad\qquad - \sigma c^T \sum_{j=0}^{p}\sum_{k=p-j}^{\infty} \phi_j^{\Delta} \mathrm{e}^{(h+k)\Delta A}\, e_p \cdot \Delta L_{n-j-k}\\ &= -\sigma c^T \sum_{k=0}^{p-1}\left(\sum_{j=0}^{k} \phi_j^{\Delta} \mathrm{e}^{(k-j)\Delta A}\right) \mathrm{e}^{h\Delta A}\, e_p \cdot \Delta L_{n-k}\\ &\qquad\qquad + \sigma c^T \sum_{k=p}^{\infty}\left(-\sum_{j=0}^{p} \phi_j^{\Delta} \mathrm{e}^{-j\Delta A}\right) \mathrm{e}^{(h+k)\Delta A}\, e_p \cdot \Delta L_{n-k}.\end{aligned}$$

By virtue of the Cayley-Hamilton Theorem (cf. also [22, proof of Lemma 2.1]), we have that

$$-\sum_{j=0}^{p} \phi_j^{\Delta} \mathrm{e}^{-j\Delta A} = 0,$$

and hence, $\Phi^{\Delta}(\mathrm{B})\,\tilde{Y}_n^{\Delta,h} = -\sigma c^T \sum_{k=0}^{p-1}\left(\sum_{j=0}^{k} \phi_j^{\Delta} \mathrm{e}^{(k-j)\Delta A}\right) \mathrm{e}^{h\Delta A}\, e_p \cdot \Delta L_{n-k}$. We conclude with [39, Lemma 2.1(i) and Eq. (4.4)]. □

Remark 2.4.3. (i) The approximating Riemann sum of a causal CARMA process is automatically a causal ARMA process. On the other hand, even if the CARMA process is invertible in the sense of Definition 2.3.1, the roots of $\tilde{\Theta}^{\Delta,h}(\cdot)$ may lie inside the unit circle, causing $\tilde{Y}^{\Delta,h}$ to be non-invertible.

(ii) It is easy to see that $\tilde{\theta}_0^{\Delta,h} = g(h\Delta)$. Then for $p - q \geq 2$, if $h = 0$, we have that $\tilde{\theta}_0^{\Delta,0} = 0$, giving that $\tilde{\Theta}^{\Delta,0}(0) = 0$. This is never the case for $\Theta^{\Delta}(\cdot)$, as one can see from Eq. (2.10) and Proposition 2.2.2. Moreover, it is possible to show that for $h = 1$ and $p - q \geq 2$, the coefficient $\tilde{\theta}_{p-1}^{\Delta,1}$ is 0, implying that Eq. (2.28) is actually an ARMA$(p, p-2)$ equation. For those values of h, the ARMA equations solved by the approximating Riemann sums will never have the same asymptotic form as Eq. (2.9): therefore we shall restrict ourselves to the case $h \in (0, 1)$ from now on.

(iii) The assumption of distinct autoregressive roots might seem restrictive, but the omitted cases can be obtained by letting distinct roots tend to each other. This would, of course, change the coefficients of the MA polynomial in Eq. (2.29). Moreover, as shown in [20, 21], the multiplicity of the zeros does not matter when L^2-asymptotic relationships as $\Delta \downarrow 0$ are considered. □

Due to the complexity of retrieving the roots of a polynomial of arbitrary order from its coefficients, finding the asymptotic expression of $\tilde{\Theta}^{\Delta,h}(\cdot)$ for arbitrary p is a daunting task. Nonetheless, using Proposition 2.4.2, it is not difficult to give an answer for processes with $p \leq 3$, which are the most used in practice.

Proposition 2.4.4. *Let $\tilde{Y}^{\Delta,h}$ be the approximating Riemann sum for a CARMA(p,q) process, with $p \leq 3$, let Assumption 2.1 hold and the roots of $a(\cdot)$ be distinct.*

If $p = 1$, the process $\tilde{Y}^{\Delta,h}$ is an AR(1) process driven by $Z_n^\Delta = \sigma e^{\Delta h \lambda_1} \Delta L_n$, for every $\Delta > 0$. If $p = 2, 3$, we have

$$\Phi^\Delta(\mathrm{B})\, \tilde{Y}_n^{\Delta,h} = \prod_{i=1}^{q} (1 - (1 - \Delta\mu_i + o(\Delta))\mathrm{B}) \prod_{i=1}^{p-q-1} (1 - \chi_{p-q,i}(h)\mathrm{B}) \left(\sigma \frac{(h\Delta)^{p-q-1}}{(p-q-1)!} \Delta L_n \right) \tag{2.30}$$

where, for $h \in (0,1)$ and $\Delta \downarrow 0$, $\chi_{2,1}(h) = (h-1)/h + o(1)$ and

$$\chi_{3,j}(h) = \frac{2(h-1)^2}{2(h-1)h - 1 - (-1)^j \sqrt{1 - 4(h-1)h}} + o(1), \quad j = 1, 2.$$

Proof. Since $p \leq 3$, $\tilde{\Theta}^{\Delta,h}(z)$ is a polynomial of order $p-1$ with real coefficients; therefore the roots, if any, can be calculated from the coefficients, and asymptotic expressions can be obtained by computing the Taylor expansions of the roots around $\Delta = 0$.

If $p = 1$, the statement follows directly from Eq. (2.28). For $p = 2, 3$, the roots of Eq. (2.29) are $\{1 + \Delta\mu_i + o(\Delta)\}_{i=1,\dots,q}$ and $\{1/\chi_{p-q,i}(h)\}_{i=1,\dots p-q-1}$, giving that

$$\tilde{\Theta}^{\Delta,h}(z) = \tilde{\theta}_{p-1}^{\Delta,h} \prod_{i=1}^{q} (1 + \Delta\mu_i + o(\Delta) - z) \prod_{i=1}^{p-q-1} (1/\chi_{p-q,i}(h) - z), \quad z \in \mathbb{C}.$$

Vieta's Theorem gives that the product of the roots of Eq. (2.29) is equal to $\tilde{\theta}_0^{\Delta,h}/\tilde{\theta}_{p-1}^{\Delta,h}$, which yields

$$\tilde{\Theta}^{\Delta,h}(z) = \tilde{\theta}_0^{\Delta,h} \prod_{i=1}^{q} (1 - (1 - \Delta\mu_i + o(\Delta))z) \prod_{i=1}^{p-q-1} (1 - \chi_{p-q,i}(h)z).$$

Since $\tilde{\theta}_0^{\Delta,h} = g(h\Delta) = \sigma(h\Delta)^{p-q-1}/(p-q-1)!(1+o(1))$, we have established the result. □

In general the autocorrelation structure will depend on h through the parameters $\chi_{p-q,i}(h)$, $i = 1, \dots, p-q-1$. In a time series context, it is reasonable to require that the approximating Riemann sum process has the same asymptotic autocorrelation structure as the CARMA process we want to approximate.

Corollary 2.4.5. *Let the assumptions of Proposition 2.4.4 hold. Then* $\Phi^\Delta(\mathrm{B})\, Y^\Delta$ *and* $\Phi^\Delta(\mathrm{B})\, \tilde{Y}^{\Delta,h}$ *have the same asymptotic autocovariance structure as* $\Delta \downarrow 0$

$$\begin{aligned}
&\text{for every } h \in (0,1), && \text{if } p-q=1,\\
&\text{for } h = (3 \pm \sqrt{3})/6, && \text{if } p-q=2,\\
&\text{and for } h = \left(15 \pm \sqrt{225 - 30\sqrt{30}}\right)/30, && \text{if } p-q=3.
\end{aligned}$$

Moreover, the MA polynomials in Eqs. (2.10) *and* (2.30) *coincide if and only if the CARMA process is invertible and* $|\chi_{p-q,i}(h)| < 1$, *that is*

$$\begin{aligned}
&\text{for every } h \in (0,1), && \text{if } p-q=1,\\
&\text{for } h = (3 + \sqrt{3})/6, && \text{if } p-q=2.
\end{aligned}$$

For $p-q=3$, *such* h *does not exist.*

Proof. The claim for $p-q=1$ follows immediately from Proposition 2.4.4 and Eqs. (2.10)-(2.11). For $p=2$ and $q=0$, we have to solve the spectral factorization problem

$$\begin{aligned}
\sigma_\Delta^2(1+\eta(\xi_1)^2) &= \sigma^2\Delta^3(1+\chi_{2,1}(h)^2)h^2\\
\sigma_\Delta^2\eta(\xi_1) &= -\sigma^2\Delta^3\chi_{2,1}(h)h^2
\end{aligned}$$

with $\eta(\xi_1) = 2-\sqrt{3}+o(1)$ and $\chi_{2,1}(h) = (h-1)/h + o(1)$. Eq. (2.11) then yields the two solutions $h = (3 \pm \sqrt{3})/6$. For $p=3$ and $q=1$, analogous calculations lead to the same solutions. Finally, consider the case $p=3$ and $q=0$. We have to solve asymptotically the following system of equations

$$\begin{aligned}
\sigma_\Delta^2(1+(\eta(\xi_1)+\eta(\xi_2))^2+\eta(\xi_1)^2\eta(\xi_2)^2) &= \frac{\sigma^2\Delta^5}{4}(1+(\chi_{3,1}(h)+\chi_{3,2}(h))^2+\chi_{3,1}(h)^2\chi_{3,2}(h)^2)h^4\\
\sigma_\Delta^2(\eta(\xi_1)+\eta(\xi_2))(1+\eta(\xi_1)\eta(\xi_2)) &= -\frac{\sigma^2\Delta^5}{4}(\chi_{3,1}(h)+\chi_{3,2}(h))(1+\chi_{3,1}(h)\chi_{3,2}(h))h^4
\end{aligned}$$

$$\sigma_\Delta^2 \eta(\xi_1)\eta(\xi_2) = \frac{\sigma^2\Delta^5}{4}\chi_{3,1}(h)\chi_{3,2}(h)h^4$$

where $\eta(\xi_{1,2}) = \big(13 \pm \sqrt{105} - \sqrt{270 \pm 26\sqrt{105}}\big)/2 + o(1)$ and $\chi_{3,1}(h)$ and $\chi_{3,2}(h)$ are as in Proposition 2.4.4. Solving that system for h gives the claimed values.

To prove the second part of the corollary, we start observing that, under the assumption of an invertible CARMA process, the coefficients depending on μ_i will coincide automatically, if any. Then it remains to check whether the coefficients depending on h can be smaller than 1 in absolute value. The cases $p - q = 1, 2$ follow immediately. Moreover, to see that there is no such h for $p - q = 3$, it is enough to notice that, for every $h \in (0, 1)$, $|\chi_{3,1}(h)| > 1$ and $0 < |\chi_{3,2}(h)| < 1$, i.e. they do never satisfy the sought requirement for $h \in (0, 1)$. □

Corollary 2.4.5 can be interpreted as a criterion to choose an h such that the Riemann sum approximates the continuous-time process Y in a stronger sense than the simple convergence as a random variable for every fixed t. The second part of the corollary says that there is an even more restrictive way to choose h such that Eqs. (2.10) and (2.30) will coincide. If the two processes satisfy asymptotically the same causal and invertible ARMA equation, they will have the same coefficients in their Wold representations as $\Delta \downarrow 0$, which are given in the case of the approximating Riemann sum explicitly by definition in (2.26).

In the light of (2.15) and Theorem 2.3.2, the sampled CARMA process will asymptotically behave like its approximating Riemann sum process for some specific $h = \bar{h}$, which might not even exist, as in the case $p = 3$, $q = 0$. However, if such an $\bar{h}$ exists, the kernel estimators (2.15) can be improved to

$$\frac{\sigma_\Delta}{\sqrt{\Delta}}\psi^\Delta_{\lfloor t/\Delta\rfloor} = g(\Delta(\lfloor t/\Delta\rfloor + \bar{h})) + o(\Delta), \quad t \in \mathbb{R}.$$

For invertible CARMA(p, q) processes with $p-q = 1$, any choice of h would accomplish that. In principle an $\bar{h}$ can be found matching a higher order expansion in Δ, where higher order terms will depend on h.

For $p - q = 2$, there is only a specific value $h = \bar{h} := (3 + \sqrt{3})/6$, such that $\tilde{Y}^{\Delta,\bar{h}}$ behaves as Y^Δ in this particular sense, and therefore it gives a recommendation for a unique, optimal value for, e.g., simulation purposes.

Finally, for $p - q = 3$, a similar value does not exist, meaning that it is not possible to mimic Y^Δ in this sense with any approximating Riemann sum.

In order to confirm this, we now give a small numerical study. We consider three different causal and invertible processes, a CARMA$(2,1)$, a CAR(2) and a CAR(3) process, with parameters $\lambda_1 = -0.7$, $\lambda_2 = -1.2$, $\lambda_3 = -2.6$ and $\mu_1 = 3$. Of course, for the CARMA$(2,1)$ we use only λ_1, λ_2 and μ_1, while for the CAR processes there is no need for μ_1. Then we estimate the kernel functions from the theoretical autocorrelation functions using (2.15) as in [20], for moderately high sampling rates, namely $2^2 = 4$ (Figure 2.1) and $2^6 = 64$ samplings per unit of time (Figure 2.2). In order to see where the kernel is being estimated, we plot the kernel estimations on different grids. The small circles denote the extremal cases $h = 0$ and $h = 1$, the vertical sign the mid-point rule $h = 0.5$, and the diamond and the square are the values given in Corollary 2.4.5, if any. The true kernel function is then plotted with a solid, continuous line. For the sake of clarity, only the first 8 estimates are plotted.

For the CARMA$(2,1)$ process, the kernel estimation seems to follow a mid-point rule. For the CAR(2) process, the predicted value $\bar{h} = (3+\sqrt{3})/6$ (denoted with squares) is definitely the correct one, and for the CAR(3) for every $h \in [0,1]$ the estimation is close but constantly biased. Of course in the limit $\Delta \downarrow 0$, the slightly weaker results given by Eq. (2.15) still hold, giving that the bias vanishes in the limit. The conclusion expressed above is true for both considered sampling rates, which is remarkable since they are only moderately high, in comparison with the chosen parameters.

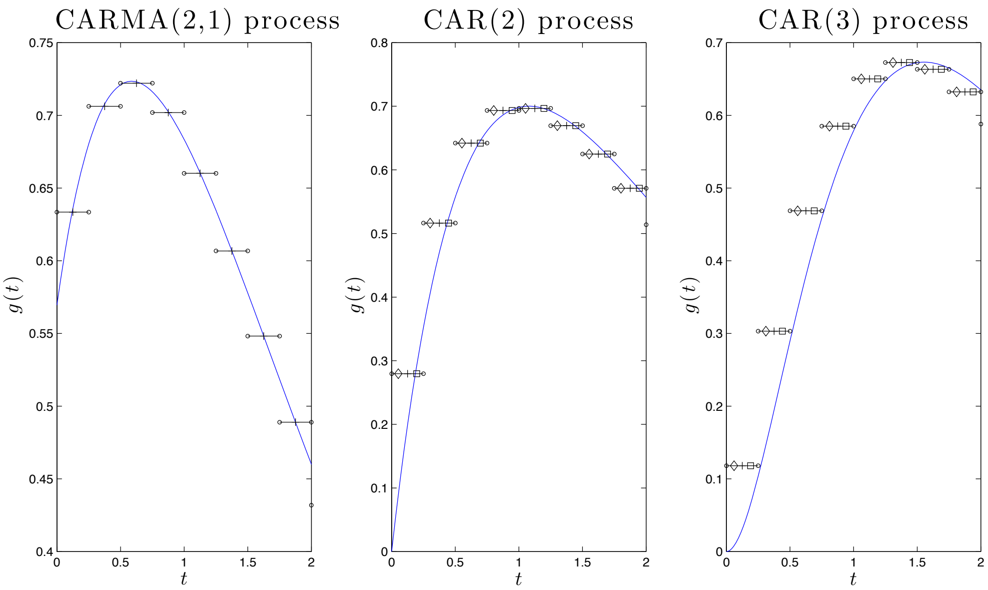

Figure 2.1: Kernel estimation for a sampling frequency of 2^2 samplings per unit of time, i.e. $\Delta = 0.25$. The diamond and the square symbols denote, where available, the values of h suggested by Corollary 2.4.5.

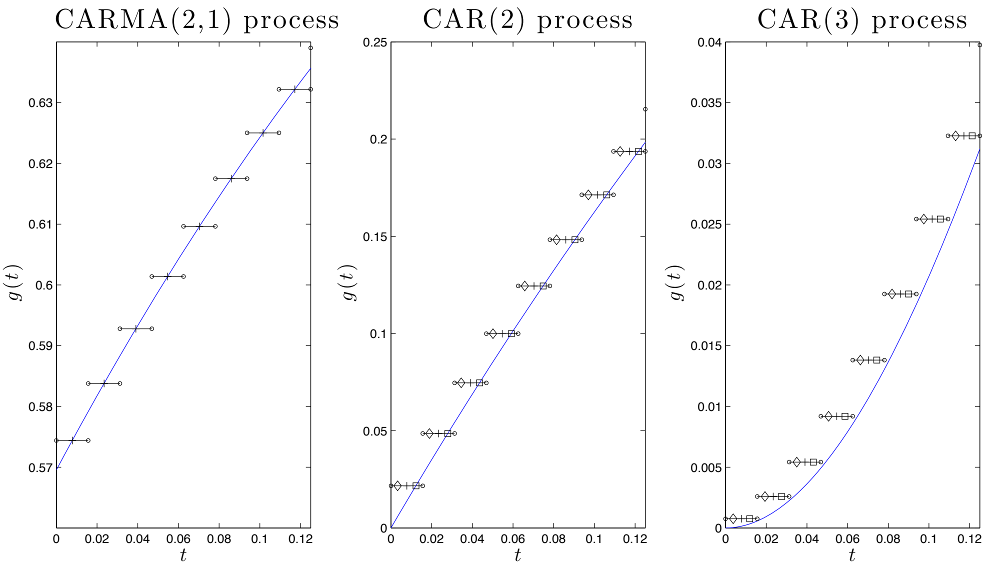

Figure 2.2: Kernel estimation for a sampling frequency of 2^6 samplings per unit of time, i.e. $\Delta \approx 0.016$. The diamond and the square symbols denote, where available, the values of h suggested by Corollary 2.4.5.

2.5 Conclusion

In this chapter we have dealt with the asymptotic behavior of two classes of discrete-time processes closely related to the one of CARMA processes, when the sampling frequency tends to infinity.

First, we studied the behavior of the white noise appearing in the ARMA equation solved by the sampled sequence of a CARMA process of arbitrary order. Then we showed, under a necessary and sufficient identifiability condition, that the aforementioned white noise approximates the increments of the driving Lévy process of the continuous-time model. The proposed procedure is non-parametric in nature, and it can be applied without assuming any knowledge on the order of the process. Moreover, it is argued that such results should be extendable to CMA processes with more general kernel functions. This is left for future research.

The results in the first part of this chapter, considered jointly with those in [20], show that the Wold representation of a sampled causal and invertible CARMA process behaves somehow like a Riemann sum as $\Delta \downarrow 0$. Then in the second part of this chapter the converse is investigated, i.e. whether a Riemann sum approximating a causal CARMA process satisfies the same ARMA$(p, p-1)$ equation of the process we want to approximate, as the spacing of the grid tends to zero. This is in particular important for simulations, where such approximating Riemann sums constitute a natural choice.

It has been shown that these processes satisfy an ARMA$(p, p-1)$ equation, but in general they are not invertible. For $p \leq 3$, the roots of the moving average polynomial of such discrete-time processes depend, apart from the roots of $c(\cdot)$, also on the rule h. Moreover, in the case $p = 3$, $q = 0$, no choice of h makes the Riemann sum invertible, implying that the Riemann sum and the sampled process will never satisfy asymptotically the same causal and invertible ARMA equation. Although only a finite number of cases has been considered, it shows that in general a causal and invertible CARMA process cannot be approximated by a Riemann sum, at least not in the sense of the second part of Corollary 2.4.5. Further investigations on this matter are left for future research.

2.6 Appendix to Chapter 2

Throughout this appendix, we shall use the same notation as in the preceding sections.

Proposition 2.6.1. *The function $\alpha_n(x)$, as defined in Eq.* (2.13), *has the form*

$$\alpha_n(x) = \frac{1}{(2n+1)!} \frac{P_n(x)}{x^{n+1}}, \quad x \neq 0, \ n \in \mathbb{N}, \tag{2.31}$$

where $P_n(x)$ is a polynomial of order n in x, namely

$$\begin{aligned} P_n(x) = &\sum_{j=0}^{n} x^{n-j} \sum_{k=j+1}^{n} (2k)! \left\{ {2n+1 \atop 2k} \right\} \sum_{i=j}^{k} \left[\binom{i+1}{j+1}\binom{2k}{2i+1} - \binom{i}{j+1}\binom{2k}{2i} \right] (-2)^{j+1-2k} \\ &+ \sum_{j=0}^{n} x^{n-j} \sum_{k=j}^{n} (2k+1)! \left\{ {2n+1 \atop 2k+1} \right\} \sum_{i=j}^{k} \left[\binom{i+1}{j+1}\binom{2k+1}{2i+1} - \binom{i}{j+1}\binom{2k+1}{2i} \right] (-2)^{j-2k}, \end{aligned} \tag{2.32}$$

with $\left\{ {\cdot \atop \cdot} \right\}$ being the Stirling number of the second kind. Moreover, all the zeros of $\alpha_n(x)$ are real, distinct and greater than 2.

Proof. Using the definition of the hyperbolic functions, we can write

$$\frac{\sinh(z)}{\cosh(z) - 1 + x} = \frac{\mathrm{e}^{2z} - 1}{\mathrm{e}^{2z} + 1 + 2(x-1)\mathrm{e}^{z}} =: f(z, x), \quad z \in \mathbb{C}, \quad x \neq 0,$$

i.e. $f(z, x) = g(\mathrm{e}^z, x)$, where $g(y, x) := (y^2 - 1)/(y^2 + 1 + 2(x-1)y)$. Then the n-th derivative of the function $f(\cdot, x)$ is, by virtue of the Faà di Bruno formula

$$\frac{\partial^n}{\partial z^n} f(z,x) = \frac{\partial^n}{\partial z^n} g(\mathrm{e}^z, x) = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} \frac{\partial^k}{\partial y^k} g(y,x) \bigg|_{y=\mathrm{e}^z} \mathrm{e}^{kz}, \quad z \in \mathbb{C}, \quad x \neq 0,$$

where the coefficients $\left\{ {n \atop k} \right\}$ are the Stirling numbers of the second kind. Using the previous formula and the definition of $\alpha_n(x)$, for $x \neq 0$,

$$\begin{aligned}(2n+1)!\,\alpha_n(x) &= \frac{\partial^{2n+1}}{\partial z^{2n+1}} f(z,x) \bigg|_{z=0} = \sum_{k=1}^{2n+1} \left\{ {2n+1 \atop k} \right\} \frac{\partial^k}{\partial y^k} g(y,x) \bigg|_{y=1} \\ &= \sum_{k=1}^{2n+1} k! \left\{ {2n+1 \atop k} \right\} \frac{1}{2\pi i} \int_\rho \frac{(z-1)(z+1)}{(z-a_2(x))(z-a_1(x))(z-1)^{k+1}} \mathrm{d}z \\ &= \sum_{k=1}^{2n+1} k! \left\{ {2n+1 \atop k} \right\} \frac{1}{2\pi i} \int_\rho \frac{z+1}{(z-a_2(x))(z-a_1(x))(z-1)^{k}} \mathrm{d}z \qquad (2.33)\end{aligned}$$

where the latter equality comes from the Cauchy differentiation formula, $a_1(x) = 1-x-\sqrt{x(x-2)}$, $a_2(x) = 1-x+\sqrt{x(x-2)}$, i.e. they are the roots (in y) of the polynomial $y^2+1+2(x-1)y$, and ρ is a counter-clockwise oriented closed curve in the complex plane encircling $z=1$. The case $x=0$ has been excluded because $f(\cdot,0)$ is not defined in $z=0$, and $\lim_{z\to 0} |f(z,0)| = \infty$.

Let us denote the integrand in Eq. (2.33) by $f_k(z,x)$; it is a rational function having one pole of order k in $z=1$. Moreover, there are two simple poles in $z=a_1(x)$ and $z=a_2(x)$ if $x\neq 2$, or just a simple one in $z=-1$ if $x=2$, due to cancellation with the zero at $z=-1$ in the numerator. The case $x=2$ can be also obtained by letting the difference between $a_1(x)$ and $a_2(x)$ tend to zero in the upcoming calculations and is therefore not treated by itself.

Then by the Cauchy theorem of residues, Theorem 1, p. 24 of [65], we have that the integral in Eq. (2.33) is

$$\begin{aligned}\frac{1}{2\pi i}\int_\rho f_k(z,x)\mathrm{d}z &= \mathrm{Res}_{z=1} f_k(z,x) \\ &= -\mathrm{Res}_{z=a_1(x)} f_k(z,x) - \mathrm{Res}_{z=a_2(x)} f_k(z,x) - \mathrm{Res}_{z=\infty} f_k(z,x),\ \ x\neq 0,2,\end{aligned}$$

where the latter identity is obtained using Theorem 2, p. 25 of [65]. Moreover, since the difference of orders between the polynomials in the denominator and in the numerator of $f_k(z,x)$ is $k+1>1$, the residue at infinity is always zero (Section 3.1.2.3, pp. 27-28,

[65]). For $x \neq 0, 2$, we have that $z = a_1(x)$ and $z = a_2(x)$ are simple poles, yielding

$$\begin{aligned}
\frac{1}{2\pi i}\int_\rho f_k(z,x)\,\mathrm{d}z &= \frac{1+a_1(x)}{(a_1(x)-1)^k\,(a_2(x)-a_1(x))} - \frac{1+a_2(x)}{(a_2(x)-1)^k\,(a_2(x)-a_1(x))} \\
&= (2x)^{-k}\sum_{i=0}^{\lfloor k/2\rfloor}\left[2\binom{k}{2i+1}(-x)^{-1} + \binom{k}{2i+1} - \binom{k}{2i}\right](-x)^{k-2i}\,(x(x-2))^i \\
&= \sum_{j=0}^{\lfloor k/2\rfloor-1}\sum_{i=j}^{\lfloor k/2\rfloor}\left[\binom{i+1}{j+1}\binom{k}{2i+1} - \binom{i}{j+1}\binom{k}{2i}\right](-2)^{j+1-k}\,x^{-j-1} \\
&\qquad + (k \bmod 2)\,(-2)^{\lfloor k/2\rfloor+1-k}\,x^{-\lfloor k/2\rfloor-1}; \qquad (2.34)
\end{aligned}$$

where the last equality is obtained by using the Binomial theorem. Plugging Eq. (2.34) in Eq. (2.33), and separating the outermost sum in odd and even k, we get, still for $x \neq 0, 2$,

$$\begin{aligned}
&(2n+1)!\,\alpha_n(x) \\
&= \sum_{k=1}^{n}\sum_{j=0}^{k-1}\sum_{i=j}^{k}(2k)!\left\{\begin{matrix}2n+1\\2k\end{matrix}\right\}\left[\binom{i+1}{j+1}\binom{2k}{2i+1} - \binom{i}{j+1}\binom{2k}{2i}\right](-2)^{j+1-2k}\,x^{-j-1} \\
&+ \sum_{k=0}^{n}\sum_{j=0}^{k-1}\sum_{i=j}^{k}(2k+1)!\left\{\begin{matrix}2n+1\\2k+1\end{matrix}\right\}\left[\binom{i+1}{j+1}\binom{2k+1}{2i+1} - \binom{i}{j+1}\binom{2k+1}{2i}\right](-2)^{j-2k}\,x^{-j-1} \\
&+ \sum_{k=0}^{n}(2k+1)!\left\{\begin{matrix}2n+1\\2k+1\end{matrix}\right\}(-2)^{-k}\,x^{-k-1}.
\end{aligned}$$

Then Eq. (2.31) can be obtained by merging the last two lines and rearranging the indexes.

Using (2.32), we easily see that, for $P_n(x) = p_0 + p_1 x + \ldots + p_n x^n$,

$$p_0 = (-2)^{-n}\,(2n+1)!, \quad p_n = 1, \qquad (2.35)$$

i.e. $P_n(x)$ will have n, potentially complex, roots, and they can not be zero. Moreover, it is easy to verify that $f(z,x)$ solves the mixed partial differential equation

$$\frac{\partial^2}{\partial z^2}f(z,x) = \left[(x-1)\frac{\partial}{\partial x} + x(x-2)\frac{\partial^2}{\partial x^2}\right]f(z,x), \quad z \in \mathbb{C},\ x \neq 0. \qquad (2.36)$$

Then we take $2n-1$, $n \in \mathbb{N}$, derivatives in z on both sides of Eq. (2.36); invoking the Schwarz theorem, the product rule for derivatives and evaluating the resulting expression at $z = 0$, we obtain that the function $\alpha_n(x)$ is given by recursion, for $x \notin (0, 2)$,

$$(2n+3)\,(2n+1)\,\alpha_{n+1}(x) = \sqrt{x(x-2)}\frac{\partial}{\partial x}\left[\sqrt{x(x-2)}\frac{\partial}{\partial x}\alpha_n(x)\right], \qquad (2.37)$$
$$\alpha_0(x) = 1/x.$$

We now prove by induction the claim regarding the roots being real, distinct and greater than 2. The cases $\alpha_0(x) = 1/x$ and $6\,\alpha_1(x) = (x-3)/x^2$ have respectively no and one zero, so the claim can be verified directly; then we start from $\alpha_2(x) = (30-15x+x^2)/(120x^3)$, whose zeros are $\xi_{2,1} = 1/2\left(15-\sqrt{105}\right) \approx 2.37652$, $\xi_{2,2} = 1/2\left(15+\sqrt{105}\right) \approx 12.6235$, and they satisfy the claim, too. We now assume that the claim holds for $\alpha_n(x)$, $n \geq 2$, and its zeros are $2 < \xi_{n,1} < \xi_{n,2} < \ldots < \xi_{n,n}$.

The derivative of $\alpha_n(x)$ is of the form $Q_n(x)/x^{n+2}$, where $(2n+1)!\,Q_n(x) = x\frac{\partial}{\partial x}P_n(x) - (1+n)\,P_n(x)$. By Rolle's theorem, $Q_n(x)$ has $n-1$ real roots $\chi_{n,i}$, $i = 1, \ldots, n-1$, such that $2 < \xi_{n,1} < \chi_{n,1} < \xi_{n,2} < \chi_{n,2} < \ldots < \chi_{n,n-1} < \xi_{n,n}$. Using the product rule and the value of the coefficients in Eq. (2.35), we get

$$\frac{\partial}{\partial x}\alpha_n(x) \sim -x^{-2}/(2n+1)! \to 0, \quad x \to \infty.$$

Again by Rolle's theorem, since $\frac{\partial}{\partial x}\alpha_n(x) \to 0$ and $\alpha_n(x) \to 0$ as $x \to \infty$, $Q_n(x)$ has a root at some point $\xi_{n,n} < \chi_{n,n} < \infty$. For $x \geq 2$, the function $\sqrt{x(x-2)}\frac{\partial}{\partial x}\alpha_n(x)$ is well defined and it is zero for $x = 2$ and $x = \chi_{n,i}$, $i = 1, \ldots, n$. With the same arguments as before, we then obtain that $\frac{\partial}{\partial x}[\sqrt{x(x-2)}\frac{\partial}{\partial x}\alpha_n(x)]$ is zero for $x = \xi_{n+1,i}$, $i = 1, \ldots, n+1$, where $2 < \xi_{n+1,1} < \chi_{n,1} < \xi_{n+1,2} < \chi_{n,2} < \ldots < \chi_{n,n} < \xi_{n+1,n+1} < \infty$. Then by Eq. (2.37), those zeros are also roots of, respectively, $\alpha_{n+1}(x)$ and $P_{n+1}(x)$. Since $P_{n+1}(x)$ is a polynomial of order $n+1$, it can have only $n+1$ roots, which have been found already. Moreover, they are all real, distinct and strictly greater than 2, and the claim is proven. □

Lemma 2.6.2. *Suppose that $\Re(\mu_j) \neq 0$ for all $j = 1, \ldots, q$. Then we have, for any $t \in (0, \infty)$ and $\omega \in \mathbb{R}\backslash\{0\}$,*

$$\lim_{\Delta \downarrow 0} \sigma \frac{\sqrt{\Delta}}{\sigma_\Delta} \frac{1-\cos(\omega\lfloor t/\Delta\rfloor\Delta)}{\omega} \frac{\sin(\omega\Delta)}{1-\cos(\omega\Delta)} \Re\left(\frac{\prod_{j=1}^{p}(1-\mathrm{e}^{\Delta(\lambda_j+i\omega)})}{\Theta^\Delta(\mathrm{e}^{i\omega\Delta})}\frac{c(-i\omega)}{a(-i\omega)}\right)$$
$$= \frac{2-2\cos(\omega t)}{\omega^2}\left(1+\Re(D(\omega))\right)$$

and

$$\lim_{\Delta\downarrow 0}\sigma\frac{\sqrt{\Delta}}{\sigma_\Delta}\frac{1-\cos(\omega\lfloor t/\Delta\rfloor\Delta)}{\omega}\Im\left(\frac{\prod_{j=1}^p(1-\mathrm{e}^{\Delta(\lambda_j+i\omega)})}{\Theta^\Delta(\mathrm{e}^{i\omega\Delta})}\frac{c(-i\omega)}{a(-i\omega)}\right)=0,$$

where $D(\omega):=-1+\prod_{j\in J}(\mu_j-i\omega)/(-\mu_j-i\omega)$ *and* $J:=\{j=1,\dots,q:\Re(\mu_j)<0\}$. *Obviously, if* $\Re(\mu_j)>0$ *for all* $j=1,\dots,q$, *then* $D(\omega)=0$ *for every* $\omega\in\mathbb{R}$.

Proof. By Proposition 2.2.2, we have that $\eta(\xi_j)\in(0,1)$ for sufficiently small Δ. Hence, for any $\omega\in\mathbb{R}$,

$$\begin{aligned}\frac{\prod_{j=1}^p(1-\mathrm{e}^{\Delta(\lambda_j+i\omega)})}{\Theta^\Delta(\mathrm{e}^{i\omega\Delta})}\frac{c(-i\omega)}{a(-i\omega)}&=\frac{1}{\prod_{j=1}^{p-q-1}(1+\eta(\xi_j)\mathrm{e}^{i\omega\Delta})}\prod_{j=1}^p\frac{\mathrm{e}^{\Delta(\lambda_j+i\omega)}-1}{i\omega+\lambda_j}\prod_{j=1}^q\frac{\mu_j-i\omega}{1-\zeta_j\mathrm{e}^{i\omega\Delta}}\\&=\Delta^{p-q}(1+D(\omega))\prod_{j=1}^{p-q-1}(1+\eta(\xi_j))^{-1}(1+o(1))\quad\text{as }\Delta\downarrow 0.\end{aligned}$$

Moreover, using Eq. (2.11), we obtain

$$\sigma\frac{\sqrt{\Delta}}{\sigma_\Delta}=\frac{\sqrt{[2(p-q)-1]!\cdot\prod_{j=1}^{p-q-1}\eta(\xi_j)}}{\Delta^{p-q-1}}(1+o(1))\qquad\text{as }\Delta\downarrow 0.$$

Since $\cos(\omega\lfloor t/\Delta\rfloor\Delta)\to\cos(\omega t)$ and $\Delta\sin(\omega\Delta)/(1-\cos(\omega\Delta))\to 2/\omega$ as $\Delta\downarrow 0$ for any $\omega\in\mathbb{R}\backslash\{0\}$, we can use the equality (cf. [20, proof of Theorem 3.2])

$$\frac{\sqrt{[2(p-q)-1]!\cdot\prod_{j=1}^{p-q-1}\eta(\xi_j)}}{\prod_{j=1}^{p-q-1}(1+\eta(\xi_j))}=\frac{\prod_{j=1}^{p-q-1}|1+\eta(\xi_j)|}{\prod_{j=1}^{p-q-1}(1+\eta(\xi_j))}(1+o(1))=1+o(1)\qquad\text{as }\Delta\downarrow 0$$

to conclude the proof. □

Lemma 2.6.3. *Suppose that* $t\in(0,\infty)$ *and* $\Re(\mu_j)\neq 0$ *for all* $j=1,\dots,q$, *and let the functions* $h^{\Delta,1}(\cdot),h^{\Delta,2}(\cdot)$ *and* $h^{\Delta,3}_{\lfloor t/\Delta\rfloor}(\cdot)$ *be defined as in the proof of Theorem 2.3.2. Then there is a* $C>0$ *such that, for any* $\omega\in\mathbb{R}$ *and any sufficiently small* Δ,

$$\left|2\Re\left(h^{\Delta,1}\cdot h^{\Delta,3}_{\lfloor t/\Delta\rfloor}(\omega)\cdot\overline{h^{\Delta,2}\cdot h^{\Delta,3}_{\lfloor t/\Delta\rfloor}(\omega)}\right)\right|\leq h(\omega),$$

where $h(\omega):=\left(7^{2p}/2^{2p+q}+1\right)t^2\mathbb{1}_{(-1,1)}(\omega)+C\omega^{-2}\mathbb{1}_{\mathbb{R}\backslash(-1,1)}(\omega)$. *Moreover,* h *is integrable over the real line.*

Proof. We obviously have

$$\left|2\Re\left(h^{\Delta,1}\,h^{\Delta,3}_{\lfloor t/\Delta\rfloor}(\omega)\cdot\overline{h^{\Delta,2}\,h^{\Delta,3}_{\lfloor t/\Delta\rfloor}(\omega)}\right)\right|\leq\left|h^{\Delta,1}\cdot h^{\Delta,3}_{\lfloor t/\Delta\rfloor}(\omega)\right|^2+\left|h^{\Delta,2}\cdot h^{\Delta,3}_{\lfloor t/\Delta\rfloor}(\omega)\right|^2\qquad(2.38)$$

for any $\omega \in \mathbb{R}$ and any Δ. Let us first consider the second addend on the RHS of Eq. (2.38).

We obtain $|h^{\Delta,2} \cdot h^{\Delta,3}_{\lfloor t/\Delta \rfloor}(\omega)|^2 = 2\,(1-\cos(\omega\lfloor t/\Delta\rfloor\Delta))/\omega^2$ and since $\lfloor t/\Delta\rfloor\Delta \leq t$ holds, we can bound, for any Δ, the latter function by t^2 on the interval $(-1,1)$ and by $4/\omega^2$ on $\mathbb{R}\backslash(-1,1)$.

As to the first addend on the RHS of Eq. (2.38), we calculate

$$\left|h^{\Delta,1} \cdot h^{\Delta,3}_{\lfloor t/\Delta \rfloor}(\omega)\right|^2 = \sigma^2 \frac{\Delta}{\sigma^2_\Delta} \frac{\prod_{j=1}^p \left|1-\mathrm{e}^{\Delta(\lambda_j+i\omega)}\right|^2}{|\Theta^\Delta(\mathrm{e}^{i\omega\Delta})|^2} \frac{|c(-i\omega)|^2}{|a(-i\omega)|^2} \frac{1-\cos(\omega\lfloor t/\Delta\rfloor\Delta)}{1-\cos(\omega\Delta)}. \tag{2.39}$$

Let now $|\omega|<1$ and suppose that Δ is sufficiently small, i.e. the following inequalities will be true for any $|\omega|<1$ whenever Δ is sufficiently small. Using $|1-\mathrm{e}^z| \leq 7/4|z|$ for $|z|<1$ (see, e.g., [1, 4.2.38]) yields

$$\frac{\prod_{j=1}^p \left|1-\mathrm{e}^{\Delta(\lambda_j+i\omega)}\right|^2}{|a(-i\omega)|^2} \leq \left(\frac{7}{4}\Delta\right)^{2p}.$$

Then $(1-\cos(\omega\Delta))/(\omega\Delta)^2 \geq 1/4$ together with $4\,(1-\cos(\omega\lfloor t/\Delta\rfloor\Delta))/\omega^2 \leq 2t^2$ (see above) implies

$$\frac{1-\cos(\omega\lfloor t/\Delta\rfloor\Delta)}{1-\cos(\omega\Delta)} \leq 2\left(\frac{t}{\Delta}\right)^2.$$

As in the proof of Lemma 2.6.2, write $\Theta^\Delta(z) = \prod_{j=1}^{p-q-1}(1+\eta(\xi_j)z)\prod_{j=1}^q(1-\zeta_j z)$, where $\zeta_j = 1-\mathrm{sgn}(\Re(\mu_j))\,\mu_j\,\Delta + o(\Delta)$ (see [20, Theorem 2.1]). Since $\prod_{j=1}^q \left(\left|1-\zeta_j\mathrm{e}^{i\omega\Delta}\right|/\Delta\right)^2 \geq \prod_{j=1}^q 1/2\,|\mathrm{sgn}(\Re(\mu_j))\,\mu_j - i\omega|^2$, we further deduce

$$\frac{|c(-i\omega)|^2}{\prod_{j=1}^q |1-\zeta_j\mathrm{e}^{i\omega\Delta}|^2} \leq \frac{2^q}{\Delta^{2q}}.$$

By virtue of Eq. (2.11), we then obtain

$$\sigma^2\frac{\Delta}{\sigma^2_\Delta}\prod_{j=1}^{p-q-1}\left|1+\eta(\xi_j)\mathrm{e}^{i\omega\Delta}\right|^{-2} \leq \frac{2\cdot[2(p-q)-1]!}{\Delta^{2(p-q-1)}}\prod_{j=1}^{p-q-1}\frac{|\eta(\xi_j)|}{|1+\eta(\xi_j)\mathrm{e}^{i\omega\Delta}|^2}$$

and since $|\eta(\xi_j)|<1$ for all j (see Proposition 2.2.2) we also have that $|1+\eta(\xi_j)\mathrm{e}^{i\omega\Delta}| \geq \frac{1}{2}|1+\eta(\xi_j)|$ for all j, resulting in

$$\sigma^2\frac{\Delta}{\sigma^2_\Delta}\prod_{j=1}^{p-q-1}\left|1+\eta(\xi_j)\mathrm{e}^{i\omega\Delta}\right|^{-2} \leq \frac{2^{2(p-q)-1}}{\Delta^{2(p-q-1)}}\,[2(p-q)-1]!\prod_{j=1}^{p-q-1}\frac{|\eta(\xi_j)|}{|1+\eta(\xi_j)|^2} = \frac{2^{2(p-q)-1}}{\Delta^{2(p-q-1)}}$$

where the latter equality follows from [20, proof of Theorem 3.2]. All together the RHS of Eq. (2.39) can be bounded for any $|\omega| < 1$ and any sufficiently small Δ by $(7/2)^{2p}\, 2^{-q} t^2$.

It remains to bound the RHS of (2.39) also for $|\omega| \geq 1$. Hence, for the rest of the proof let us suppose $|\omega| \geq 1$ and in addition we assume again that Δ is sufficiently small in order that all the following inequalities hold. We are going to show that

$$\sigma^2 \frac{\Delta}{\sigma_\Delta^2} \frac{\prod_{j=1}^p \left|1 - \mathrm{e}^{\Delta(\lambda_j + i\omega)}\right|^2}{|\Theta^\Delta(\mathrm{e}^{i\omega\Delta})|^2} \frac{|c(-i\omega)|^2}{|a(-i\omega)|^2} \frac{1 - \cos(\omega \lfloor t/\Delta \rfloor \Delta)}{1 - \cos(\omega\Delta)} \leq \frac{C}{\omega^2}$$

for some $C > 0$. Since $\left|\sigma^2 \Delta / \sigma_\Delta^2\right| \leq \text{const.} \cdot \Delta^{-2(p-q-1)}$ (see (2.11)) and since $\prod_{j=1}^{p-q-1} |1 + \eta(\xi_j) \mathrm{e}^{i\omega\Delta}|^{-2} \leq \prod_{j=1}^{p-q-1} (1 - |\eta(\xi_j)|)^{-2} \leq \text{const.}$ (cf. Proposition 2.2.2), it is sufficient to prove

$$\frac{(\omega\Delta)^2}{\Delta^{2(p-q)}} \frac{\prod_{j=1}^p \left|1 - \mathrm{e}^{\Delta(\lambda_j + i\omega)}\right|^2}{\prod_{j=1}^q |1 - \zeta_j \mathrm{e}^{i\omega\Delta}|^2} \frac{|c(-i\omega)|^2}{|a(-i\omega)|^2} \frac{1 - \cos(\omega \lfloor t/\Delta \rfloor \Delta)}{1 - \cos(\omega\Delta)} \leq C \tag{2.40}$$

for some $C > 0$. For any $\omega \in \mathbb{R}$, the power transfer function satisfies $|c(-i\omega)/a(-i\omega)|^2 \leq \text{const.}/(\omega^{2(p-q)} + 1)$. Thus, Eq. (2.40) will follow from

$$\frac{(\omega\Delta)^2}{(\omega\Delta)^{2(p-q)} + \Delta^{2(p-q)}} \frac{\prod_{j=1}^p \left|1 - \mathrm{e}^{\Delta(\lambda_j + i\omega)}\right|^2}{\prod_{j=1}^q |1 - \zeta_j \mathrm{e}^{i\omega\Delta}|^2} \frac{1 - \cos(\omega \lfloor t/\Delta \rfloor \Delta)}{1 - \cos(\omega\Delta)} \leq C. \tag{2.41}$$

We even show that (2.41) is true for any $\omega \in \mathbb{R}$. However, using symmetry and periodicity arguments it is sufficient to prove Eq. (2.41) on the interval $[0, \frac{2\pi}{\Delta}]$. We split that interval into the following six subintervals

$$I_1 := \left[0, \min_{j=1,\dots,q} \frac{|\mu_j|}{2}\right],\ I_2 := \left[\min_{j=1,\dots,q} \frac{|\mu_j|}{2}, \max_{j=1,\dots,q} 2|\mu_j|\right],\ I_3 := \left[\max_{j=1,\dots,q} 2|\mu_j|, \frac{\pi}{\Delta}\right],$$

$$I_4 := \left[\frac{\pi}{\Delta}, \frac{2\pi}{\Delta} - \max_{j=1,\dots,q} 2|\mu_j|\right],\ I_5 := \left[\frac{2\pi}{\Delta} - \max_{j=1,\dots,q} 2|\mu_j|, \frac{2\pi}{\Delta} - \min_{j=1,\dots,q} \frac{|\mu_j|}{2}\right] \text{ and}$$

$$I_6 := \left[\frac{2\pi}{\Delta} - \min_{j=1,\dots,q} \frac{|\mu_j|}{2}, \frac{2\pi}{\Delta}\right].$$

For any $\omega \in I_1 \cup I_6$, the fraction $\frac{1-\cos(\omega \lfloor t/\Delta \rfloor \Delta)}{1-\cos(\omega\Delta)}$ can be bounded by $\lfloor t/\Delta \rfloor^2$. In the other intervals we have the obvious bound $2/(1 - \cos(\omega\Delta))$ for that term.

Now, for any $j = 1, \dots, p$, we have, as $\Delta \downarrow 0$,

$$\left|1 - \mathrm{e}^{\Delta\lambda_j} \cdot \mathrm{e}^{i\omega\Delta}\right|^2 \leq 2\left|1 - \mathrm{e}^{i\omega\Delta}\right|^2 + 4\Delta^2 |\lambda_j|^2 = 8 \sin^2\left(\frac{\omega\Delta}{2}\right) + 4\Delta^2 |\lambda_j|^2$$

$$\leq 4\Delta^2\left(\omega^2 + |\lambda_j|^2\right)$$

if $\omega \in I_1 \cup I_2 \cup I_3$, and $\left|1 - \mathrm{e}^{\Delta\lambda_j} \cdot \mathrm{e}^{i\omega\Delta}\right|^2 \leq 4\Delta^2\left((2\pi/\Delta - \omega)^2 + |\lambda_j|^2\right)$ if $\omega \in I_4 \cup I_5 \cup I_6$.

The first fraction on the LHS of Eq. (2.41) satisfies

$$\frac{(\omega\Delta)^2}{(\omega\Delta)^{2(p-q)} + \Delta^{2(p-q)}} \leq \begin{cases} \min_{j=1,\ldots,q} \frac{|\mu_j|}{2} \cdot \frac{\Delta^2}{\Delta^{2(p-q)}}, & \text{if } \omega \in I_1, \\ \frac{(\omega\Delta)^2}{(\omega\Delta)^{2(p-q)}}, & \text{if } \omega \in I_2 \cup I_3, \\ \frac{(2\pi)^2}{\pi^{2(p-q)}}, & \text{if } \omega \in I_4 \cup I_5 \cup I_6. \end{cases}$$

Then, for any $j = 1, \ldots, q$ and $\omega \in I_1 \cup I_6$, we obtain

$$\begin{aligned} \left|1 - \zeta_j \mathrm{e}^{i\omega\Delta}\right|^2 &= \left|1 - (1 - \operatorname{sgn}(\Re(\mu_j))\,\mu_j\Delta + o(\Delta))\mathrm{e}^{i\omega\Delta}\right|^2 \geq \frac{1}{2}\Delta^2\left|\operatorname{sgn}(\Re(\mu_j))\,\mu_j - i\omega\right|^2 \\ &\geq \frac{1}{8}\Delta^2|\mu_j|^2. \end{aligned}$$

If $\omega \in I_3$, then we have

$$\begin{aligned} \left|1 - \zeta_j \mathrm{e}^{i\omega\Delta}\right|^2 &\geq \left(\left|1 - \mathrm{e}^{i\omega\Delta}\right| - |\mu_j + o(1)|\,\Delta\right)^2 = \left(2\sin\left(\frac{\omega\Delta}{2}\right) - |\mu_j + o(1)|\,\Delta\right)^2 \\ &\geq \Delta^2\left(\frac{3}{5}\omega - |\mu_j + o(1)|\right)^2 \end{aligned}$$

and likewise, for $\omega \in I_4$, we deduce $\left|1 - \zeta_j \mathrm{e}^{i\omega\Delta}\right|^2 \geq \Delta^2\left(\frac{3}{5}(\frac{2\pi}{\Delta} - \omega) - |\mu_j + o(1)|\right)^2$. On I_2 we get for arbitrary $\varepsilon > 0$

$$\begin{aligned} \left|1 - \zeta_j \mathrm{e}^{i\omega\Delta}\right|^2 &= 2(1 - \cos(\omega\Delta)) \cdot (1 - \Delta\operatorname{sgn}(\Re(\mu_j))\,\Re(\mu_j) + o(\Delta)) \\ &\quad + 2\sin(\omega\Delta) \cdot (-\Delta\operatorname{sgn}(\Re(\mu_j))\,\Im(\mu_j) + o(\Delta)) + \Delta^2|\mu_j|^2 + o(\Delta^2) \\ &\geq (\omega\Delta)^2 \cdot (1 - \varepsilon) - 2(\omega\Delta) \cdot \Delta\,|\Im(\mu_j)| \cdot (1 + \varepsilon) + \Delta^2\left(|\mu_j|^2 + o(1)\right) \\ &=: f_\varepsilon^\Delta(\omega\Delta). \end{aligned}$$

Since $f_\varepsilon^\Delta(\omega)/\omega^2 \to 1 - \varepsilon$ $(\omega \to \infty)$ and $f_\varepsilon^\Delta(\omega)/\omega^2 \to \infty$ $(\omega \to 0)$, a (global) minimum of $f_\varepsilon^\Delta(\omega)/\omega^2$ on $(0, \infty)$ could be achieved in any ω^* with $\left(\frac{\mathrm{d}}{\mathrm{d}\omega}\frac{f_\varepsilon^\Delta(\omega)}{\omega^2}\right)(\omega^*) = 0$. The only such value is $\omega^* = \frac{\Delta(|\mu_j|^2 + o(1))}{(1+\varepsilon)|\Im(\mu_j)|}$. Now

$$\frac{f_\varepsilon^\Delta(\omega^*)}{(\omega^*)^2} = 1 - \varepsilon - (1 + \varepsilon)^2\frac{|\Im(\mu_j)|^2}{|\mu_j|^2 + o(1)} \geq (1 + \varepsilon)\frac{\Re(\mu_j)^2}{|\mu_j|^2} - 3\varepsilon - \varepsilon^2 \geq \frac{1}{2}\frac{\Re(\mu_j)^2}{|\mu_j|^2};$$

if we choose $\varepsilon = \frac{1}{6}\frac{\Re(\mu_j)^2}{|\mu_j|^2}$, we obtain $\frac{f_\varepsilon^\Delta(\omega)}{\omega^2} \geq \frac{1}{2}\frac{\Re(\mu_j)^2}{|\mu_j|^2}$ for any $\omega \in (0, \infty)$. Hence,

$$\left|1 - \zeta_j \mathrm{e}^{i\omega\Delta}\right|^2 \geq f_\varepsilon^\Delta(\omega\Delta) \geq \frac{1}{2}\frac{\Re(\mu_j)^2}{|\mu_j|^2}(\omega\Delta)^2 \quad \text{for all } \omega \in I_2.$$

Using periodic properties of the sine and cosine terms, we likewise get

$$\left|1 - \zeta_j \mathrm{e}^{i\omega\Delta}\right|^2 \geq \frac{1}{2}\frac{\Re(\mu_j)^2}{|\mu_j|^2}\Delta^2 \left(\frac{2\pi}{\Delta} - \omega\right)^2 \quad \text{for any } \omega \in I_5.$$

Putting all together, we can bound the LHS of Eq. (2.41) in I_1 by

$$\begin{aligned}\min_{j=1,\ldots,q} \frac{|\mu_j|}{2} \cdot \frac{(\lfloor t/\Delta \rfloor \Delta)^2}{\Delta^{2(p-q)}} \cdot \frac{4^p \Delta^{2p} \cdot \prod_{j=1}^p \left(\min_{k=1,\ldots,q} |\mu_k|^2/4 + |\lambda_j|^2\right)}{8^{-q}\Delta^{2q}\prod_{j=1}^q |\mu_j|^2} \\ \leq \min_{j=1,\ldots,q} \frac{|\mu_j|}{2} \cdot t^2 \cdot \frac{4^{p+q} \cdot \prod_{j=1}^p \left(\min_{k=1,\ldots,q} |\mu_k|^2/4 + |\lambda_j|^2\right)}{\prod_{j=1}^q \frac{1}{2}|\mu_j|^2} = C,\end{aligned}$$

in I_2 by

$$\begin{aligned}\frac{2(\omega\Delta)^2}{1-\cos(\omega\Delta)} \frac{4^p \Delta^{2p} \cdot \prod_{j=1}^p (4\max_{k=1,\ldots,q} |\mu_k|^2 + |\lambda_j|^2)}{(\omega\Delta)^{2p} \cdot \prod_{j=1}^q \frac{1}{2}\frac{\Re(\mu_j)^2}{|\mu_j|^2}} \\ \leq \frac{5 \cdot 4^{2p} \cdot \prod_{j=1}^p (4\max_{k=1,\ldots,q} |\mu_k|^2 + |\lambda_j|^2)}{\min_{j=1,\ldots,q} |\mu_j|^{2p} \cdot \prod_{j=1}^q \frac{1}{2}\frac{\Re(\mu_j)^2}{|\mu_j|^2}} = C,\end{aligned}$$

in I_3 by

$$\begin{aligned}\frac{2(\omega\Delta)^2}{1-\cos(\omega\Delta)} \frac{4^p (\omega\Delta)^{2p} \cdot \prod_{j=1}^p \left(1 + \frac{|\lambda_j|^2}{4\max_{k=1,\ldots,q} |\mu_k|^2}\right)}{(\omega\Delta)^{2(p-q)} \cdot (\frac{1}{20}\omega\Delta)^{2q}} \\ \leq \pi^2\, 4^p\, 20^{2q} \prod_{j=1}^p \left(1 + \frac{|\lambda_j|^2}{4\max_{k=1,\ldots,q} |\mu_k|^2}\right) = C,\end{aligned}$$

in I_4 by

$$\begin{aligned}\frac{(2\pi)^2}{\pi^{2(p-q)}} \frac{2}{1-\cos(\omega\Delta)} \frac{4^p (2\pi - \omega\Delta)^{2p} \cdot \prod_{j=1}^p \left(1 + \frac{|\lambda_j|^2}{4\max_{k=1,\ldots,q} |\mu_k|^2}\right)}{20^{-2q}\,(2\pi - \omega\Delta)^{2q}} \\ \leq 4^{p+1}\, 20^{2q} \prod_{j=1}^p \left(1 + \frac{|\lambda_j|^2}{4\max_{k=1,\ldots,q} |\mu_k|^2}\right) \frac{2 \cdot (2\pi - \omega\Delta)^2}{1 - \cos(2\pi - \omega\Delta)} \\ \leq \pi^2\, 4^{p+1}\, 20^{2q} \prod_{j=1}^p \left(1 + \frac{|\lambda_j|^2}{4\max_{k=1,\ldots,q} |\mu_k|^2}\right) = C,\end{aligned}$$

in I_5 by

$$\begin{aligned}\frac{(2\pi)^2}{\pi^{2(p-q)}} \frac{2}{1-\cos(\omega\Delta)} \frac{4^p \Delta^{2p} \cdot \prod_{j=1}^{p} (4 \max_{k=1,\dots,q} |\mu_k|^2 + |\lambda_j|^2)}{\Delta^{2q} \prod_{j=1}^{q} \frac{1}{8} \min_{k=1,\dots,q} |\mu_k|^2 (\Re(\mu_j)/|\mu_j|)^2} \\ \le \frac{(2\pi)^2}{\pi^{2(p-q)}} \frac{4^p \cdot \prod_{j=1}^{p} (4 \max_{k=1,\dots,q} |\mu_k|^2 + |\lambda_j|^2)}{\prod_{j=1}^{q} \frac{1}{8} \min_{k=1,\dots,q} |\mu_k|^2 (\Re(\mu_j)/|\mu_j|)^2} \frac{2\Delta^2}{1-\cos(2\pi-\omega\Delta)} \\ \le \frac{(2\pi)^2}{\pi^{2(p-q)}} \frac{4^p \cdot \prod_{j=1}^{p} (4 \max_{k=1,\dots,q} |\mu_k|^2 + |\lambda_j|^2)}{\prod_{j=1}^{q} \frac{1}{8} \min_{k=1,\dots,q} |\mu_k|^2 (\Re(\mu_j)/|\mu_j|)^2} \frac{5 \cdot 4}{\min_{j=1,\dots,q} |\mu_j|^2} = C,\end{aligned}$$

and, finally, in I_6 by

$$\begin{aligned}\frac{(2\pi \lfloor t/\Delta \rfloor)^2}{\pi^{2(p-q)}} \frac{4^p \Delta^{2p} \cdot \prod_{j=1}^{p} \left(\min_{k=1,\dots,q} |\mu_k|^2/4 + |\lambda_j|^2\right)}{8^{-q} \Delta^{2q} \prod_{j=1}^{q} |\mu_j|^2} \\ \le \frac{(2\pi t)^2}{\pi^{2(p-q)}} \frac{4^{p+q} \cdot \prod_{j=1}^{p} (\min_{k=1,\dots,q} |\mu_k|^2/4 + |\lambda_j|^2)}{\prod_{j=1}^{q} \frac{1}{2} |\mu_j|^2} = C.\end{aligned}$$

This shows Eq. (2.41) and thus concludes the proof. □

3 On the limit behavior of the periodogram of high-frequency sampled stable CARMA processes[2]

3.1 Introduction

Continuous-time ARMA (CARMA) processes are the continuous-time versions of the well known ARMA processes in discrete time having short memory. The advantage of continuous-time modeling is that it allows handling of irregularly spaced time series and in particular of high-frequency data often appearing in turbulence and finance. In this chapter we consider a CARMA process $Y = (Y_t)_{t\in\mathbb{R}}$ driven by a symmetric α-stable Lévy process $(L_t)_{t\in\mathbb{R}}$. Before we start with its definition, we recall that a real-valued random variable X is called symmetric α-stable ($S\alpha S$) with index of stability $\alpha \in (0, 2]$, if its characteristic function is of the form

$$\Phi_X(z) = \mathbb{E}\left[\exp\{i\, z\, X\}\right] = \exp\{-\sigma^\alpha\, |z|^\alpha\}, \quad z \in \mathbb{R},$$

for some $\sigma \geq 0$, and a real random vector $X = (X_1, \ldots, X_d)^T$ is $S\alpha S$, if all linear combinations $\sum_{i=1}^d a_i\, X_i$, $(a_1, \ldots, a_d)^T \in \mathbb{R}^d$ are $S\alpha S$; see the monograph of Samorodnitsky and Taqqu [77] for details on stable distributions. Then a symmetric α-stable Lévy process $(L_t)_{t\in\mathbb{R}}$ is a stochastic process with $L_0 = 0$ almost surely, independent and stationary increments which are $S\alpha S$ distributed with characteristic function

$$\Phi_{L_t}(z) = \mathbb{E}\left[\exp\{i\, z\, L_t\}\right] = \exp\{-|t|\sigma_L^\alpha\, |z|^\alpha\}, \quad z,\, t \in \mathbb{R},$$

for some $\sigma_L \geq 0$ and almost surely càdlàg sample paths (cf. the book of Sato [78] on Lévy processes). A symmetric α-stable CARMA process is then defined as follows. Let $(L_t)_{t\in\mathbb{R}}$ be a symmetric α-stable Lévy process. Assume that we have given $p,\, q \in \mathbb{N}$, $p > q$, and

[2]The contents of this chapter appeared in Fasen, V. and Fuchs, F. (2013), On the limit behavior of the periodogram of high-frequency sampled stable CARMA processes, *Stochastic Process. Appl.*, 123 no. 1, pp. 229-273

$a_1, \dots, a_p, c_0, \dots, c_q \in \mathbb{R}$, $a_p, c_0 \neq 0$, set

$$A := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ -a_p & -a_{p-1} & \dots & \dots & -a_1 \end{pmatrix} \in \mathbb{R}^{p \times p}$$

and let $(X_t)_{t\in\mathbb{R}}$ be a strictly stationary solution to the stochastic differential equation

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + e_p\,\mathrm{d}L_t, \quad t \in \mathbb{R}, \tag{3.1a}$$

where e_p denotes the p-th unit vector in $\mathbb{R}^p$. Then the process

$$Y_t := c^T X_t, \quad t \in \mathbb{R}, \tag{3.1b}$$

with $c = (c_q, c_{q-1}, \dots, c_{q-p+1})^T$ (where we use the convention $c_j = 0$ for $j < 0$) is said to be a *symmetric α-stable CARMA process* of order (p, q). Necessary and sufficient conditions for the existence of a strictly stationary CARMA process are given in [22]. A CARMA process can be interpreted as a solution to the formal p-th order stochastic differential equation

$$a(\mathrm{D})Y_t = c(\mathrm{D})\mathrm{D}L_t, \quad t \in \mathbb{R},$$

where D denotes the differential operator with respect to t and

$$a(z) := z^p + a_1 z^{p-1} + \dots + a_p \qquad \text{and} \qquad c(z) := c_0 z^q + c_1 z^{q-1} + \dots + c_q$$

are the autoregressive and the moving average polynomial, respectively. Hence, $S\alpha S$ CARMA processes can be seen as the continuous-time analog of $S\alpha S$ (discrete-time) ARMA processes. The representation (3.1) of a CARMA process is the *controller canonical state space representation* going back to [14]. Alternatively there exists also the *observer canonical form* of a CARMA process (see (3.14) below) as derived in [63] for multivariate CARMA models. For an overview and a comprehensive list of references on CARMA processes we refer to [16, 23]. CARMA processes are important for stochastic modeling in many areas of application as, e.g., signal processing and control (cf. [46, 61]), econometrics (cf. [7, 72]), high-frequency financial econometrics (cf. [87]) and financial mathematics (cf. [6]). Stable CARMA processes are particularly relevant in modeling energy markets (cf. [5, 45]).

The aim of this chapter is to investigate the sampled sequence $Y^\Delta := (Y_{k\Delta})_{k\in\mathbb{Z}}$ of a *causal* (i.e., current values of the process only depend on *past* values of the driving process) stable CARMA process, meaning we only observe the underlying CARMA process $(Y_t)_{t\in\mathbb{R}}$ at equidistant time points $0, \Delta, 2\Delta, \ldots$ with $\Delta > 0$ small as used for modeling high-frequency data (cf. [21, 36]), and to study the asymptotic behavior of the sampled process Y^Δ in the frequency domain. In the time domain the autocovariance function

$$\gamma_Y(h) = \frac{\sigma_L^2}{\pi} \int_{-\infty}^{\infty} \mathrm{e}^{ih\omega} \frac{|c(i\omega)|^2}{|a(i\omega)|^2} \,\mathrm{d}\omega = c^T \mathrm{e}^{|h|A}\, \gamma_X(0)\, c, \quad h \in \mathbb{R}, \tag{3.2}$$

with $\gamma_X(0) = 2\sigma_L^2 \int_0^\infty \mathrm{e}^{sA} e_p e_p^T \mathrm{e}^{sA^T} \,\mathrm{d}s$, gives information about the dependence structure, whereas in the frequency domain the spectral density

$$f_Y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma_Y(h)\, \mathrm{e}^{-ih\omega} \,\mathrm{d}h = \frac{\sigma_L^2}{\pi} \cdot \frac{|c(i\omega)|^2}{|a(i\omega)|^2}, \quad \omega \in \mathbb{R}, \tag{3.3}$$

gives information about the periodicities of the CARMA process. Both the spectral density and the autocovariance function exist only for $\alpha = 2$. The spectral density of the sampled process Y^Δ is

$$f_\Delta(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_Y(k\Delta)\, \mathrm{e}^{-ik\omega} = \frac{1}{\Delta} \sum_{k=-\infty}^{\infty} f_Y\Big(\frac{\omega + 2k\pi}{\Delta}\Big), \quad -\pi \le \omega \le \pi, \tag{3.4}$$

where the second equality follows from [11, p. 206]. It is related to f_Y by

$$\lim_{\Delta \to 0} \Delta\, f_\Delta(\omega\Delta)\, \mathbb{1}_{[-\frac{\pi}{\Delta}, \frac{\pi}{\Delta}]}(\omega) = f_Y(\omega), \quad \omega \in \mathbb{R}, \tag{3.5}$$

(see p. 55 for a proof). Loosely spoken, this means that in the limit $\Delta \to 0$ we can identify every CARMA process from its equidistantly sampled observations. The question arises whether this is also true if we study the empirical version of the spectral density, the *periodogram*. We investigate normalized and self-normalized versions. The normalized periodogram of Y^Δ at frequency $\omega \in [-\pi, \pi]$ is given by

$$I_{n,Y^\Delta}(\omega) = \Big| n^{-1/\alpha} \sum_{k=1}^{n} Y_{k\Delta}\, \mathrm{e}^{-i\omega k} \Big|^2.$$

Equation (3.5) suggests that we obtain a non-trivial limit by studying the behavior of the properly rescaled periodogram I_{n,Y^Δ} of the sampled CARMA process at point $\omega\Delta$.

More precisely, we will show that the finite-dimensional distribution of the periodogram $\Delta^{2-2/\alpha}[I_{n,Y^\Delta}(\omega_1\Delta), \ldots, I_{n,Y^\Delta}(\omega_m\Delta)]$ for $(\omega_1, \ldots, \omega_m) \in (\mathbb{R}\backslash\{0\})^m$ converges weakly to a function of stable distributions, if simultaneously the grid distance Δ goes to 0 with a suitable rate and the number of observations n goes to infinity (see Theorem 3.2.6). A small grid distance and a huge number of observations reflect the behavior of high-frequency data. A consequence of this is the fact that the normalized periodogram is not a consistent estimator of the so-called power transfer function $|c(i\,\cdot)/a(i\,\cdot)|^2$. Moreover, if $(L_t)_{t\in\mathbb{R}}$ is a Brownian motion then the limit distribution has independent components. In contrast, if $(L_t)_{t\in\mathbb{R}}$ is a $S\alpha S$-stable Lévy process with $\alpha \in (0,2)$ then the components are dependent. In both cases the limit distributions differ depending on whether $\omega_1, \ldots, \omega_m$ are linearly dependent or independent over $\mathbb{Z}$. However, the one-dimensional distributions do not depend on ω. Our result is comparable to Brockwell and Davis [17, Chapter 10.3] for the finite variance and Klüppelberg and Mikosch [57, Theorem 2.4] for the stable case, respectively, of an ARMA process in discrete time; although the α-stable limit distributions are different in the discrete-time and the continuous-time model.

Since the normalized periodogram depends on α, which is in general an unknown parameter, we also analyze different normalizations. So-called *self-normalized periodogram* versions are given by

$$\widetilde{I}_{n,Y^\Delta}(\omega) = \frac{\left|\sum_{k=1}^n Y_{k\Delta}\,\mathrm{e}^{-i\omega k}\right|^2}{(\sum_{k=1}^n Y_{k\Delta})^2} \quad \text{and} \quad \widehat{I}_{n,Y^\Delta}(\omega) = \frac{\left|\sum_{k=1}^n Y_{k\Delta}\,\mathrm{e}^{-i\omega k}\right|^2}{\sum_{k=1}^n Y_{k\Delta}^2}, \quad -\pi \le \omega \le \pi, \tag{3.6}$$

having the obvious benefit that they only depend on the data and not on the index of stability α. Again the finite-dimensional distributions of $\widetilde{I}_{n,Y^\Delta}(\Delta\,\cdot)$ converge to functions of stable distributions and do not provide consistent estimators (cf. Theorem 3.2.10). The limit distribution has similar properties as for the normalized periodogram. The second version $\widehat{I}_{n,Y^\Delta}$ has to be rescaled with Δ as in (3.5) to derive a limit result (see Theorem 3.2.11). Our conclusions for the self-normalized periodogram are in analogy to those for ARMA models in discrete time obtained by Klüppelberg and Mikosch [58].

The chapter is structured in the following way. We start with our main results in Section 3.2. The sampled CARMA process Y^Δ has a representation as an MA process in discrete time where the noise sequence is p-dependent. In Section 3.2.1 we investigate this moving average structure in detail. Then the asymptotic behavior of the normalized and the self-normalized periodogram is topic of Sections 3.2.2 and 3.2.3. Finally, in Section 3.3 we derive results for the characterization of the limit distributions of the normalized and the self-normalized periodogram versions. These are based on the geometry of numbers and on manifolds. The proofs of the results are presented in Section 3.4.

Notation

We use $\mathbb{N}^*$ and $\mathbb{R}^*$ for the natural and real numbers, respectively, excluding zero and $\mathbb{Z}$ for the integers. For the minimum of two real numbers $a, b \in \mathbb{R}$ we write shortly $a \wedge b$ and for the maximum $a \vee b$. The real and imaginary part of a complex number $z \in \mathbb{C}$ is written as $\Re(z)$ and $\Im(z)$, respectively, and its complex conjugate as $\overline{z}$. For two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ we say $a_n \sim b_n$ as $n \to \infty$ if $\lim_{n\to\infty} a_n/b_n = 1$. The transpose of a matrix M is written as M^T and the m-dimensional identity matrix shall be denoted by I_m.

For a subset $S \subseteq \mathbb{N}$ and $k \in \mathbb{N}$ we set

$$\binom{S}{k} := \{B \subseteq S : |B| = k\}.$$

The orthogonal complement of $S \subseteq \mathbb{R}^m$ is denoted by $S^\perp$.

On $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ the Euclidean norm is denoted by $|\cdot|$ whereas on $\mathbb{K}^m$ it will be usually written as $\|\cdot\|$. A scalar product on a linear space is written as $\langle \cdot, \cdot \rangle$; in $\mathbb{R}^m$ and $\mathbb{C}^m$, we usually take the Euclidean one. If X and Y are normed linear spaces, let $B(X, Y)$ be the set of bounded linear operators from X into Y. On $B(X, Y)$ we will usually use the operator norm which, in the case of Y being a Banach space, turns $B(X, Y)$ itself into a Banach space. In particular we always equip $B(\mathbb{K}^m, \mathbb{K}^n)$ with the corresponding operator norm if not stated otherwise.

For two random variables X and Y the notation $X \overset{\mathcal{D}}{=} Y$ means equality in distribution. If we consider a sequence of random variables $(X_n)_{n\in\mathbb{N}}$, we denote convergence in probability to some random variable X by $X_n \overset{\mathbb{P}}{\to} X$ as $n \to \infty$ and convergence in distribution by $X_n \overset{\mathcal{D}}{\to} X$ as $n \to \infty$.

3.2 Main results

Before stating the main results, we establish the moving average structure of the sampled sequence together with two auxiliary lemmata.

3.2.1 Moving average structure of the sampled process

The aim of this section is to better understand the structure of the discrete-time sampled process Y^{Δ}. Let $\lambda_1, \ldots, \lambda_p$ denote the eigenvalues of A. By defining the filter $\Phi^{\Delta}(\mathrm{B}) := \prod_{j=1}^{p}\big(1-\mathrm{e}^{\lambda_j\Delta}\mathrm{B}\big)$ where, as usual, B denotes the backward shift operator and applying it to the sampled sequence Y^{Δ}, we obtain (cf. [22, Lemma 2.1]), for any $k \in \mathbb{Z}$,

$$\widetilde{Z}_{k,\Delta} := \Phi^{\Delta}(\mathrm{B})\, Y_k^{\Delta} = \sum_{r=1}^{p} Z^r_{k-r+1,\Delta}, \tag{3.7}$$

where

$$Z^r_{k,\Delta} := \int_{(k-1)\Delta}^{k\Delta} c^T\Big(-\sum_{j=0}^{r-1}\Phi_j^{\Delta}\,\mathrm{e}^{(r-1-j)\Delta A}\Big)\mathrm{e}^{(k\Delta-s)A}\, e_p\,\mathrm{d}L_s, \quad r=1,\ldots,p, \tag{3.8a}$$

and

$$\Phi_j^{\Delta} := (-1)^{j+1}\cdot \sum_{\{i_1,\ldots,i_j\}\in\binom{\{1,\,\ldots,\,p\}}{j}} \mathrm{e}^{\Delta\cdot\sum_{m=1}^{j}\lambda_{i_m}}, \qquad j=0,1,\ldots,p. \tag{3.8b}$$

We can rewrite the filter as $\Phi^{\Delta}(z) = \prod_{j=1}^{p}\big(1-\mathrm{e}^{\lambda_j\Delta}z\big) = -\sum_{j=0}^{p}\Phi_j^{\Delta} z^j$ for any $z \in \mathbb{C}$. In this chapter we will suppose that the eigenvalues $\lambda_1, \ldots, \lambda_p$ of A have strictly negative real parts (see Assumption 3.1 below). Under this assumption we observe that $\Phi^{\Delta}(z) \neq 0$ for all $|z| \leq 1$ and thus deduce, for any $|z| \leq 1$,

$$\Psi^{\Delta}(z) := (\Phi^{\Delta}(z))^{-1} = \sum_{j=0}^{\infty}\Psi_j^{\Delta} z^j \quad \text{with} \quad \Psi_j^{\Delta} = \sum_{\substack{j_1,\ldots,j_p\in\{0,1,\ldots,j\}\\ \sum_{m=1}^{p} j_m = j}} \mathrm{e}^{\Delta\cdot\sum_{m=1}^{p}\lambda_m j_m}, \quad j \in \mathbb{N}.$$

We can hence rewrite Eq. (3.7) as

$$Y_k^{\Delta} = \Psi^{\Delta}(\mathrm{B})\,\widetilde{Z}_{k,\Delta}, \quad k \in \mathbb{Z}, \tag{3.9}$$

showing that the sampled CARMA process Y^{Δ} is a (discrete-time) moving average process of the noise sequence $\widetilde{Z}^{\Delta} := (\widetilde{Z}_{k,\Delta})_{k\in\mathbb{Z}}$. A challenge is that $\widetilde{Z}^{\Delta}$ is not an i.i.d. sequence; it is p-dependent. For this reason we define, for any $k \in \mathbb{Z}$, $\omega \in \mathbb{R}$ and $m \in \{1, \ldots, p\}$, the auxiliary (random) functions

$$\tilde{\widetilde{Z}}_{k,\Delta}(\omega) := \sum_{r=1}^{p} Z^r_{k,\Delta}\,\mathrm{e}^{-i\omega(r-1)} \quad \text{and} \quad f_{\Delta}^{(m)}(\omega) := \sum_{r=1}^{p}\mathrm{e}^{-i\omega(r-1)}\Big(-\sum_{j=0}^{r-1}\Phi_j^{\Delta}\,\mathrm{e}^{(r-1-j)\Delta\lambda_m}\Big). \tag{3.10}$$

In contrast to $\widetilde{Z}^{\Delta}$ we have now that $\tilde{\tilde{Z}}^{\Delta}(\omega) := (\tilde{\tilde{Z}}_{k,\Delta})_{k\in\mathbb{Z}}(\omega)$ is an i.i.d. sequence, and the idea is to rewrite the periodogram essentially by means of $\tilde{\tilde{Z}}^{\Delta}(\omega)$. Then the next auxiliary lemma holds.

Lemma 3.2.1.

(i) Under the assumption that the eigenvalues $\lambda_1, \ldots, \lambda_p$ of A are distinct, we have, for any $\Delta > 0$, $r \in \{1, \ldots, p\}$, $k \in \mathbb{Z}$ and $s \in \mathbb{R}$,

$$c^T\left(-\sum_{j=0}^{r-1}\Phi_j^{\Delta}\,\mathrm{e}^{(r-1-j)\Delta A}\right)\mathrm{e}^{(k\Delta-s)A}e_p = \sum_{m=1}^{p}\frac{c(\lambda_m)}{a'(\lambda_m)}\left(-\sum_{j=0}^{r-1}\Phi_j^{\Delta}\,\mathrm{e}^{(r-1-j)\Delta\lambda_m}\right)\mathrm{e}^{(k\Delta-s)\lambda_m}.$$

(ii) We have, for any $\lambda \in \mathbb{C}$,

$$\frac{1}{\Delta}\int_0^{\Delta}\left|\mathrm{e}^{(\Delta-s)\lambda}-1\right|^{\alpha}\mathrm{d}s \to 0 \qquad \text{as } \Delta \to 0.$$

(iii) Assume that the eigenvalues $\lambda_1, \ldots, \lambda_p$ of A possess non-vanishing real parts. We then have, for any $m \in \{1, \ldots, p\}$ and any $\omega \in \mathbb{R}$,

$$f_{\Delta}^{(m)}(\omega\Delta) \sim \Delta^{p-1}\,a(i\omega)\,\frac{1}{i\omega-\lambda_m} \qquad \text{as } \Delta \to 0.$$

(iv) Assume that the eigenvalues $\lambda_1, \ldots, \lambda_p$ of A are distinct and possess non-vanishing real parts. Then we have, for any $\omega \in \mathbb{R}$,

$$\sum_{m=1}^{p}\frac{c(\lambda_m)}{a'(\lambda_m)}\cdot\frac{1}{i\omega-\lambda_m} = \frac{c(i\omega)}{a(i\omega)}.$$

By virtue of Lemma 3.2.1(i), Eqs. (3.8a) and (3.10) we obtain that

$$\begin{aligned}\left(\tilde{\tilde{Z}}_{k,\Delta}\right)_{k\in\mathbb{Z}}(\omega) &= \left(\int_{(k-1)\Delta}^{k\Delta}\sum_{m=1}^{p}\frac{c(\lambda_m)}{a'(\lambda_m)}f_{\Delta}^{(m)}(\omega)\,\mathrm{e}^{(k\Delta-s)\lambda_m}\,\mathrm{d}L_s\right)_{k\in\mathbb{Z}}\\ &=: \left(\int_{(k-1)\Delta}^{k\Delta} g_{\Delta,\omega}^{(k)}(s)\,\mathrm{d}L_s\right)_{k\in\mathbb{Z}} \qquad (3.11)\end{aligned}$$

is an i.i.d. sequence of complex $S\alpha S$ random variables since $g_{\Delta,\omega}^{(k)} : \mathbb{R} \to \mathbb{C}$ is complex-valued. Recall that integration of complex-valued deterministic functions with respect to a $S\alpha S$ Lévy process is well defined as a limit in probability for all functions in

$L^\alpha(\mathbb{C}) := \{f : \mathbb{R} \to \mathbb{C}$ measurable, $\int_\mathbb{R} |f(x)|^\alpha \, \mathrm{d}x < \infty\}$ (for further details, see [77, Section 3.4 and Section 6.2]). The characteristic function of the stable integral $\int_\mathbb{R} g \, \mathrm{d}L$ is given by

$$\mathbb{E}\left[\exp\left\{i\, z_1 \int_\mathbb{R} \Re\big(g(s)\big)\mathrm{d}L_s + i\, z_2 \int_\mathbb{R} \Im\big(g(s)\big)\mathrm{d}L_s\right\}\right]$$
$$= \exp\left\{ -\sigma_L^\alpha \int_\mathbb{R} |z_1\, \Re(g(x)) + z_2\, \Im(g(x))|^\alpha \, \mathrm{d}x\right\} \tag{3.12}$$

for any $z_1,\ z_2 \in \mathbb{R}$ (cf. [77, Example 6.1.5 and Proposition 6.2.1(i)]) such that the random vector $\big(\Re\left(\int_\mathbb{R} g\,\mathrm{d}L\right), \Im\left(\int_\mathbb{R} g\,\mathrm{d}L\right)\big)$ is $S\alpha S$.

Finally, we require the following conclusions for $(\Psi_j^\Delta)_{j\in\mathbb{N}}$ for the proofs of our results.

Lemma 3.2.2. *Suppose $\Delta = \Delta_n \to 0$ as $n \to \infty$ and that the eigenvalues $\lambda_1, \dots, \lambda_p$ of A possess strictly negative real parts. Then we have:*

(i) There is a constant $C(p) > 0$ such that

$$\left|\Psi_j^{\Delta_n}\right| \le C(p)\, \Delta_n^{-(p-1)}\, \mathrm{e}^{\Delta_n \lambda_{\max} j} \quad \forall j \in \mathbb{N},$$

where $\lambda_{\max} := \max_{k\in\{1,\dots,p\}} \Re(\lambda_k) \in (-\infty, 0)$.

(ii) If $n\Delta_n^{1+\delta} \overset{n\to\infty}{\to} \infty$ for some $\delta > 0$, then we have

$$\sum_{j=n+1}^{\infty} \left|\Psi_j^{\Delta_n}\right| \overset{n\to\infty}{\to} 0 \quad \text{and} \quad \frac{\Delta_n^\alpha}{n} \sum_{k=-\infty}^{-n-1} \left(\sum_{j=1-k}^{n-k} \left|\Psi_j^{\Delta_n}\right|\right)^\alpha \overset{n\to\infty}{\to} 0.$$

(iii) If $n\Delta_n \overset{n\to\infty}{\to} \infty$, then $\Delta_n^{\alpha p}\, n^{-1} \sum_{k=1-n}^{1-p} \left(\sum_{j=1-k}^{n} \left|\Psi_j^{\Delta_n}\right|\right)^\alpha \overset{n\to\infty}{\to} 0$.

(iv) If $n\Delta_n^{\alpha(p-1)+1-\alpha} \overset{n\to\infty}{\to} \infty$, then

$$\frac{\Delta_n^\alpha}{n} \sum_{k=2-p-n}^{-1} \left(\sum_{j=1\vee(2-p-k)}^{n\wedge(-k)} \left|\Psi_j^{\Delta_n}\right|\right)^\alpha \overset{n\to\infty}{\to} 0.$$

(v) If $n\Delta_n^{\alpha(p-1)} \overset{n\to\infty}{\to} \infty$, then $\Delta_n^\alpha\, n^{-1} \sum_{k=2-p}^{0} \left(\sum_{j=1}^{n} \left|\Psi_j^{\Delta_n}\right|\right)^\alpha \overset{n\to\infty}{\to} 0$.

3.2.2 Normalized periodogram

Before we formulate the main limit results for the normalized and the self-normalized periodogram, we introduce a random vector that will show up in the limits.

Let $m \in \mathbb{N}^*$, $\omega_1, \ldots, \omega_m \in \mathbb{R}^*$ and set $\underset{\sim}{\omega} = (\omega_1, \ldots, \omega_m)^T$. We define the $(2m+1)$-dimensional (stable) random vector $((S_j^{\Re}(\underset{\sim}{\omega}), S_j^{\Im}(\underset{\sim}{\omega}))_{j \in \{1, \ldots, m\}}, S_{m+1}(\underset{\sim}{\omega}))$ via its joint characteristic function

$$\mathbb{E}\left[\exp\left\{i\left(\sum_{j=1}^{m} \theta_j S_j^{\Re}(\underset{\sim}{\omega}) + \nu_j S_j^{\Im}(\underset{\sim}{\omega}) + \tau S_{m+1}(\underset{\sim}{\omega})\right)\right\}\right] = \exp\{-\sigma_L^\alpha \cdot K_{\underset{\sim}{\omega}}(\underset{\sim}{\theta}, \underset{\sim}{\nu}, \tau)\}, \tag{3.13a}$$

with $\underset{\sim}{\theta}, \underset{\sim}{\nu} \in \mathbb{R}^m$, $\tau \in \mathbb{R}$ and $K_{\underset{\sim}{\omega}}(\underset{\sim}{\theta}, \underset{\sim}{\nu}, \tau)$ given as follows:

(i) If $\omega_1, \ldots, \omega_m$ are linearly independent over $\mathbb{Z}$ (i.e. there is no $h \in \mathbb{Z}^m$, $h \neq 0$, such that $\langle h, \underset{\sim}{\omega} \rangle = 0$), then

$$K_{\underset{\sim}{\omega}}(\underset{\sim}{\theta}, \underset{\sim}{\nu}, \tau) = \int_{[0,1)^m} \left| \sum_{j=1}^{m} \theta_j \cos(2\pi x_j) + \nu_j \sin(2\pi x_j) + \tau \right|^\alpha \mathrm{d}(x_1, \ldots, x_m). \tag{3.13b}$$

(ii) If $\omega_1, \ldots, \omega_m$ are linearly dependent over $\mathbb{Z}$, then there is an $s \in \{1, \ldots, m-1\}$ such that

$$K_{\underset{\sim}{\omega}}(\underset{\sim}{\theta}, \underset{\sim}{\nu}, \tau) = \frac{1}{\mathscr{H}^{m-s}(\mathscr{M})} \int_{\mathscr{M}} \left| \sum_{j=1}^{m} \theta_j \cos(2\pi x_j) + \nu_j \sin(2\pi x_j) + \tau \right|^\alpha \mathrm{d}\mathscr{H}^{m-s}(x_1, \ldots, x_m), \tag{3.13c}$$

where $\mathscr{M} = \mathscr{M}(\omega_1, \ldots, \omega_m)$ is the $(m-s)$-dimensional linear manifold in $[0,1)^m$ defined in Eq. (3.17) below and $\mathscr{H}^{m-s}$ is the $(m-s)$-dimensional Lebesgue (Hausdorff) measure on $\mathscr{M}(\omega_1, \ldots, \omega_m)$ (for a definition of manifolds, see, e.g., [67, pp. 200-201]).

We start to investigate the normalized periodogram in analogy to [17, 57]. Since we use Lemmata 3.2.1 and 3.2.2 for the proofs of the asymptotic behavior of the normalized periodogram we require the following.

Assumption 3.1.
The eigenvalues $\lambda_1, \ldots, \lambda_p$ of A are distinct and possess strictly negative real parts.

Moreover, we establish our limit results for the different periodogram versions in the asymptotic framework of high-frequency data within a long time interval using Lemma 3.2.2. Thus we need

Assumption 3.2.
There is some $\delta > 0$ such that, with $\beta = \max\{1+\delta,\, \alpha(p-1)+\max\{0,\, 1-\alpha\}\}$, we have $\Delta = \Delta_n \to 0$ whereas $n\Delta_n^{\beta} \to \infty$ as $n \to \infty$.

Remark 3.2.3.

(i) Note that in the case of a symmetric α-stable Ornstein-Uhlenbeck process (i.e. $p = 1$), Assumption 3.2 becomes $\Delta_n \to 0$ and $n\Delta_n^{1+\delta} \to \infty$ as $n \to \infty$ for some $\delta > 0$ and does not depend on α.

(ii) Conversely, if $p \geq 2$, the convergence rate of Δ_n depends on α. However, one easily verifies that $\beta \leq 2p-1$ is always true and thus, if $\Delta_n \to 0$ and $n\Delta_n^{2p-1} \to \infty$ as $n \to \infty$ hold, Assumption 3.2 is satisfied as well. □

The following is an analog result to the discrete-time ones [17, Theorem 10.3.1] and [57, Proposition 2.1], respectively.

Proposition 3.2.4. *Let $\Delta = \Delta_n$ and $Y^{\Delta_n} = (Y_{k\Delta_n})_{k\in\mathbb{Z}}$ be the sampled SαS CARMA process. Under Assumption 3.1 the periodogram $I_{n,Y^{\Delta_n}}$ satisfies, for any $\omega \in [-\pi,\pi]$,*

$$I_{n,Y^{\Delta_n}}(\omega) = \left|\Psi^{\Delta_n}(\mathrm{e}^{-i\omega})\right|^2 I_{n,\widetilde{Z}^{\Delta_n}}(\omega) + R_{n,\Delta_n}(\omega)$$

with $\widetilde{Z}^{\Delta_n} := (\widetilde{Z}_{k,\Delta_n})_{k\in\mathbb{Z}}$ as given in Eq. (3.7). If in addition Assumption 3.2 holds, then we have for any $\omega \in \mathbb{R}^*$

$$\lim_{n\to\infty} \mathbb{P}\Big(\Delta_n^{2-\frac{2}{\alpha}} |R_{n,\Delta_n}(\omega\Delta_n)| > \varepsilon\Big) = 0 \quad \textit{for every } \varepsilon > 0.$$

This shows that we have to study the limit behavior of the periodogram of $\widetilde{Z}^{\Delta_n}$ in order to get insight into the asymptotic properties of $I_{n,Y^{\Delta_n}}$. The next theorem provides the key result therefore. Note that in terms of the discrete Fourier transform of $\widetilde{Z}^{\Delta_n}$,

$$J_{n,\widetilde{Z}^{\Delta_n}}(\omega) := n^{-1/\alpha} \sum_{k=1}^{n} \widetilde{Z}_{k,\Delta_n}\, \mathrm{e}^{-i\omega k}, \quad -\pi \leq \omega \leq \pi,$$

we can write $I_{n,\widetilde{Z}^{\Delta_n}}(\omega) = |J_{n,\widetilde{Z}^{\Delta_n}}(\omega)|^2$.

Theorem 3.2.5. *If Assumption 3.1 holds, $\Delta = \Delta_n \to 0$ and $n\Delta_n^{1\vee\alpha(p-1)} \to \infty$ as $n \to \infty$, then we have, for any $m \in \mathbb{N}^*$ and $\underset{\sim}{\omega} = (\omega_1, \ldots, \omega_m)^T \in (\mathbb{R}^*)^m$,*

$$\Delta_n^{1-p-\frac{1}{\alpha}} \Big[J_{n,\widetilde{Z}^{\Delta_n}}(\omega_j\Delta_n)\Big]_{j=1,\ldots,m} \overset{\mathcal{D}}{\to} \Big[c(i\omega_j)\cdot\big(S_j^{\Re}(\underset{\sim}{\omega}) - iS_j^{\Im}(\underset{\sim}{\omega})\big)\Big]_{j=1,\ldots,m} \quad \textit{as } n \to \infty.$$

The joint characteristic function of the 2m-dimensional stable random vector $\left(S_j^{\Re}(\underset{\sim}{\omega}),\, S_j^{\Im}(\underset{\sim}{\omega})\right)_{j\in\{1,\,\dots,\,m\}}$ *is given in Eq.* (3.13) *(with* $\tau = 0$*).*

Combining now Proposition 3.2.4 and Theorem 3.2.5 together with the fact that

$$\left|\Psi^{\Delta_n}(\mathrm{e}^{-i\omega\Delta_n})\right|^2 \sim \Delta_n^{-2p}\,|a(i\omega)|^{-2} \qquad \text{as } n \to \infty,$$

where the latter can be easily derived from the definition of Ψ^{Δ_n} together with the convergence of Δ_n to 0, we deduce the following main result for the limit behavior of the normalized periodogram.

Theorem 3.2.6. *Suppose* $\alpha \in (0,2]$ *and let* $Y^{\Delta_n} = (Y_{k\Delta_n})_{k\in\mathbb{Z}}$ *denote the sampled SαS CARMA(p, q) process. If Assumptions 3.1 and 3.2 hold, then* $I_{n,Y^{\Delta_n}}$ *satisfies for any* $m \in \mathbb{N}^*$ *and* $\underset{\sim}{\omega} = (\omega_1,\,\dots,\,\omega_m)^T \in (\mathbb{R}^*)^m$

$$\Delta_n^{2-\frac{2}{\alpha}}\left[I_{n,Y^{\Delta_n}}(\omega_j\Delta_n)\right]_{j=1,\dots,m} \xrightarrow{\mathcal{D}} \left[\frac{|c(i\omega_j)|^2}{|a(i\omega_j)|^2}\left(\left[S_j^{\Re}(\underset{\sim}{\omega})\right]^2 + \left[S_j^{\Im}(\underset{\sim}{\omega})\right]^2\right)\right]_{j=1,\dots,m} \quad \text{as } n \to \infty,$$

where the stable random vector $\left(S_j^{\Re}(\underset{\sim}{\omega}),\, S_j^{\Im}(\underset{\sim}{\omega})\right)_{j\in\{1,\,\dots,\,m\}}$ *has joint characteristic function as given in Eq.* (3.13) *(with* $\tau = 0$*).*

Remark 3.2.7.

(i) We highlight two important differences of our limit result to the one in [57] for ARMA models in discrete time. First, we do not have to distinguish between rational and irrational multiples of 2π in the frequency vector $\underset{\sim}{\omega}$ as it has been the case in discrete time (see, e.g., [57, Theorem 2.4]). The reason therefore is our asymptotic framework $\Delta_n \to 0$ as $n \to \infty$ which yields that in the proof of Proposition 3.3.4 the crucial Eq. (3.52) holds for any $h \in \mathbb{Z}^m$, $h \neq 0$, whereas with $\Delta_n := \Delta$ constant and one frequency component being a rational multiple of 2π, (3.52) could not hold for all $h \in \mathbb{Z}^m$, $h \neq 0$. Second, the same equation explains why in our framework the limit distributions differ depending on whether or not the frequencies $\omega_1,\,\dots,\,\omega_m$ are linearly dependent over $\mathbb{Z}$ (cf. Eq. (3.13)). In discrete time they depend on whether or not 2π, $\omega_1,\,\dots,\,\omega_m$ (with $\omega_1,\,\dots,\,\omega_m$ being irrational multiples of 2π) are linearly dependent over $\mathbb{Z}$ (see again [57, Theorem 2.4]). Note that the latter is also the reason why the manifold $\mathscr{M}(\omega_1,\dots,\omega_m)$ in (3.17) is different from the manifold that appears in the discrete-time result.

(ii) Moreover, for linearly independent $\omega_1,\,\dots,\,\omega_m$ the distribution of the random vector $(S_j^{\Re}(\underset{\sim}{\omega}),\, S_j^{\Im}(\underset{\sim}{\omega}))_{j\in\{1,\,\dots,\,m\}}$ does not depend on $\underset{\sim}{\omega}$ anymore. In the dependent case, $\underset{\sim}{\omega}$ determines the manifold, and hence, has an influence on the limit distribution. The sequence of random variables $(S_j^{\Re}(\underset{\sim}{\omega}),\, S_j^{\Im}(\underset{\sim}{\omega}))_{j\in\{1,\dots,m\}}$ is independent in the case $\alpha = 2$, whereas for $\alpha < 2$ it is dependent; in particular for $m = 1$ and $\underset{\sim}{\omega} = \omega \in \mathbb{R}^*$, the random variables $S_1^{\Re}(\omega)$ and $S_1^{\Im}(\omega)$ are dependent.

(iii) Investigating the special case $m = 1$, Theorem 3.2.6 gives for any $\omega \in \mathbb{R}^*$

$$\Delta_n^{2-\frac{2}{\alpha}}\, I_{n,Y^{\Delta_n}}(\omega\Delta_n) \xrightarrow{\mathcal{D}} \frac{|c(i\omega)|^2}{|a(i\omega)|^2} \cdot \left| \int_{[0,1)} \mathrm{e}^{2\pi i\, s}\, \mathrm{d}L_s \right|^2$$

as $n \to \infty$. Hence, the limit distribution factorizes in a parametric factor depending on ω (the so-called power transfer function) and a random factor, which does not depend on ω anymore. The limit distribution coincides with the limit distribution of the normalized periodogram of ARMA models if ω is an irrational multiple of 2π.

(iv) Let $\alpha = 2$. Then with $\omega \in \mathbb{R}^*$ as $n \to \infty$,

$$\Delta_n\, I_{n,Y^{\Delta_n}}(\omega\Delta_n) \xrightarrow{\mathcal{D}} 2\pi f_Y(\omega) \left(\frac{N_1^2}{2} + \frac{N_2^2}{2} \right) \stackrel{\mathcal{D}}{=} 2\pi f_Y(\omega)\, E,$$

where N_1 and N_2 are i.i.d. standard normal random variables and E is a standard exponential random variable. This limit result is the empirical counterpart to (3.5) with scaling factor Δ_n and in analogy to the results for ARMA models (cf. [17, Theorem 10.3.2]). It confirms, that $\Delta_n\, I_{n,Y^{\Delta_n}}(\omega\Delta_n)$ is not a consistent estimator for the spectral density.

(v) For any $h \in \mathbb{R}^*$, $(S_j^{\Re}(h\underset{\sim}{\omega}),\, S_j^{\Im}(h\underset{\sim}{\omega}))_{j\in\{1,\dots,m\}} \stackrel{\mathcal{D}}{=} (S_j^{\Re}(\underset{\sim}{\omega}),\, S_j^{\Im}(\underset{\sim}{\omega}))_{j\in\{1,\dots,m\}}$, such that as $n \to \infty$,

$$\Delta_n^{2-\frac{2}{\alpha}} \left[I_{n,Y^{\Delta_n}}(h\omega_j\Delta_n) \right]_{j=1,\dots,m} \xrightarrow{\mathcal{D}} \left[\frac{|c(ih\omega_j)|^2}{|a(ih\omega_j)|^2} \left(\left[S_j^{\Re}(\underset{\sim}{\omega}) \right]^2 + \left[S_j^{\Im}(\underset{\sim}{\omega}) \right]^2 \right) \right]_{j=1,\dots,m}.$$

On the other hand, if $\omega_1, \dots, \omega_m$ are linearly independent over $\mathbb{Z}$, then there exists an $h \in \mathbb{R}$ with $h + \omega_1,\, \dots,\, h + \omega_m$ linearly dependent over $\mathbb{Z}$ such that the limit distributions $(S_j^{\Re}(\underset{\sim}{\omega}),\, S_j^{\Im}(\underset{\sim}{\omega}))_{j\in\{1,\dots,m\}}$ and $(S_j^{\Re}(h\underset{\sim}{1} + \underset{\sim}{\omega}),\, S_j^{\Im}(h\underset{\sim}{1} + \underset{\sim}{\omega}))_{j\in\{1,\dots,m\}}$ are different. Consequently, there is no general result how a frequency shift influences the limit distribution. □

Remark 3.2.8. We conjecture that Assumption 3.2 is in this formulation not a necessary assumption for Theorem 3.2.6. However, it seems to be (close to) necessary for

Proposition 3.2.4, but Proposition 3.2.4 is not necessary for Theorem 3.2.6. □

3.2.3 Self-normalized periodogram

Next we derive the limit behavior of the self-normalized periodogram $\widetilde{I}_{n,Y^{\Delta_n}}$ and $\widehat{I}_{n,Y^{\Delta_n}}$, respectively, as given in (3.6), which is comparable to those in [58, Section 3] for ARMA processes. As in the normalized case they converge to functions of stable distributions as the following two theorems show.

First, we have to state some notation. The observer canonical form of a CARMA process (cf. [63]) is given under Assumption 3.1 by the stationary and causal multivariate Ornstein-Uhlenbeck process

$$V_t = \int_{-\infty}^{t} \mathrm{e}^{(t-s)A}\beta \, \mathrm{d}L_s, \quad t \in \mathbb{R}, \tag{3.14a}$$

where the vector $\beta = (\beta_1, \dots, \beta_p)^T \in \mathbb{R}^p$ is defined recursively by

$$\beta_{p-j} = -\sum_{i=1}^{p-1-j} a_i \beta_{p-j-i} + c_{q-j}, \quad j = 0,\, 1,\, \dots,\, p-1,$$

(with the convention $c_j = 0$ for $j < 0$). Then

$$Y_t = e_1^T V_t, \quad t \in \mathbb{R}, \tag{3.14b}$$

where $e_1 = (1,\, 0,\, \dots,\, 0)^T \in \mathbb{R}^p$. Hence, every $S\alpha S$ CARMA process can also be written as a Lévy-driven *moving average process* $Y_t = \int_{-\infty}^{\infty} g(t-s)\, \mathrm{d}L_s$, $t \in \mathbb{R}$, with kernel function

$$g(t) = e_1^T \mathrm{e}^{tA}\beta \, \mathbb{1}_{[0,\infty)}(t). \tag{3.15}$$

The following proposition is crucial for the asymptotic behavior of the different self-normalized periodogram versions.

Proposition 3.2.9. *Assume $\alpha \in (0, 2]$ and let $Y^{\Delta_n} = (Y_{k\Delta_n})_{k\in\mathbb{Z}}$ denote the sampled $S\alpha S$ CARMA(p, q) process. Moreover, define $\Delta L(k\Delta_n) := L_{k\Delta_n} - L_{(k-1)\Delta_n}$ for $k \in \mathbb{Z}$, $n \in \mathbb{N}^*$. Suppose Assumption 3.1, $\Delta_n \to 0$ and $n\Delta_n \to \infty$ as $n \to \infty$ hold. Then*

(i) $\sum_{k=1}^{n} Y_{k\Delta_n} = \sum_{j=0}^{\infty} g(j\Delta_n) \cdot \sum_{k=1}^{n} \Delta L(k\Delta_n) + o_P\left(\Delta_n^{-1}(n\Delta_n)^{\frac{1}{\alpha}}\right)$ *as* $n \to \infty$,

(ii) $\sum_{k=1}^{n} Y_{k\Delta_n}^2 = \sum_{j=0}^{\infty} g^2(j\Delta_n) \cdot \sum_{k=1}^{n} \Delta L(k\Delta_n)^2 + o_P\left(\Delta_n^{-1}(n\Delta_n)^{\frac{2}{\alpha}}\right)$ *as* $n \to \infty$.

The main limit results are then as follows.

Theorem 3.2.10. *Suppose* $\alpha \in (0,2]$ *and let* $Y^{\Delta_n} = (Y_{k\Delta_n})_{k\in\mathbb{Z}}$ *denote the sampled SαS CARMA(p, q) process. The self-normalized periodogram* $\widetilde{I}_{n,Y^{\Delta_n}}$ *is as in* (3.6)*. If Assumptions 3.1 and 3.2 hold, and in addition* $c_q \neq 0$*, then we have for any* $m \in \mathbb{N}^*$ *and* $\underset{\sim}{\omega} = (\omega_1, \dots, \omega_m)^T \in (\mathbb{R}^*)^m$*, as* $n \to \infty$*,*

$$\left[\widetilde{I}_{n,Y^{\Delta_n}}(\omega_j\Delta_n)\right]_{j=1,\dots,m} \xrightarrow{\mathcal{D}} \left[\frac{|c(i\omega_j)|^2}{(\int_0^\infty g(s)\,\mathrm{d}s)^2 \cdot |a(i\omega_j)|^2} \cdot \frac{\left[S_j^{\Re}(\underset{\sim}{\omega})\right]^2 + \left[S_j^{\Im}(\underset{\sim}{\omega})\right]^2}{S_{m+1}^2(\underset{\sim}{\omega})}\right]_{j=1,\dots,m},$$

where g *is the kernel function of the CARMA process as given in Eq.* (3.15) *and the* $(2m+1)$*-dimensional stable random vector* $\left(\left(S_j^{\Re}(\underset{\sim}{\omega}), S_j^{\Im}(\underset{\sim}{\omega})\right)_{j\in\{1,\dots,m\}}, S_{m+1}(\underset{\sim}{\omega})\right)$ *has joint characteristic function given by Eq.* (3.13)*.*

Theorem 3.2.11. *Suppose* $\alpha \in (0,2]$ *and let* $Y^{\Delta_n} = (Y_{k\Delta_n})_{k\in\mathbb{Z}}$ *denote the sampled SαS CARMA(p, q) process. The self-normalized periodogram* $\widehat{I}_{n,Y^{\Delta_n}}$ *is as in* (3.6)*. If Assumptions 3.1 and 3.2 hold, then we have for any* $m \in \mathbb{N}^*$ *and* $\underset{\sim}{\omega} = (\omega_1, \dots, \omega_m)^T \in (\mathbb{R}^*)^m$*, as* $n \to \infty$*,*

$$\Delta_n\left[\widehat{I}_{n,Y^{\Delta_n}}(\omega_j\Delta_n)\right]_{j=1,\dots,m} \xrightarrow{\mathcal{D}} \left[\frac{|c(i\omega_j)|^2}{\int_0^\infty g^2(s)\,\mathrm{d}s \cdot |a(i\omega_j)|^2} \cdot \frac{\left[S_j^{\Re}(\underset{\sim}{\omega})\right]^2 + \left[S_j^{\Im}(\underset{\sim}{\omega})\right]^2}{S^2}\right]_{j=1,\dots,m},$$

where g *is again the kernel function of the CARMA process as given in Eq.* (3.15)*, the* $(2m)$*-dimensional stable random vector* $\left(S_j^{\Re}(\underset{\sim}{\omega}), S_j^{\Im}(\underset{\sim}{\omega})\right)_{j\in\{1,\dots,m\}}$ *has joint characteristic function as given in Eq.* (3.13) *(with* $\tau = 0$*) and* S^2 *is a positive* $\alpha/2$*-stable random variable.*

Remark 3.2.12.

(i) Theorems 3.2.10 and 3.2.11 show that also the self-normalized periodogram versions do not yield consistent estimators for the (normalized) power transfer function. However, based on these results we will show in [38] that applying a smoothing filter to the self-normalized periodogram gives such a consistent estimate. Since the model parameters influence the power transfer function and, causality and invertibility of the CARMA process preconditioned, the latter uniquely determines those parameters, it is possible to use that consistent estimator of the normalized power transfer function for statistical inference on the CARMA parameters.

(ii) We have not specified explicitly the joint characteristic function of the random vector that determines the limit in Theorem 3.2.11. However, it is uniquely identifiable from the calculated Laplace transform in Eq. (3.50). Note that the limit distributions in Theorems 3.2.10 and 3.2.11 are not the same.

(iii) Moreover, we have to multiply $(\widehat{I}_{n,Y^{\Delta_n}}(\omega_j\Delta_n))_{j\in\{1,\ldots,m\}}$ in Theorem 3.2.11 by Δ_n to obtain an asymptotic limit result. This normalization is not necessary for $(\widetilde{I}_{n,Y^{\Delta_n}}(\omega_j\Delta_n))_{j\in\{1,\ldots,m\}}$ in Theorem 3.2.10. Observing (3.5), the rescaling with Δ_n seems to be natural in some way. The point is that with Proposition 3.2.9 we have for the different normalizations

$$\frac{\Delta_n\left(\sum_{k=1}^n Y_{k\Delta_n}\right)^2}{\sum_{k=1}^n Y_{k\Delta_n}^2} = \frac{\left(\Delta_n \sum_{j=0}^\infty g(j\Delta_n)\right)^2}{\Delta_n \sum_{j=0}^\infty g(j\Delta_n)^2} \cdot \frac{\left(\sum_{k=1}^n \Delta L(k\Delta_n)\right)^2}{\sum_{k=1}^n \Delta L(k\Delta_n)^2} + o_P(1)$$
$$\xrightarrow{\mathcal{D}} \frac{\left(\int_0^\infty g(s)\,\mathrm{d}s\right)^2}{\int_0^\infty g(s)^2\,\mathrm{d}s} \cdot \frac{L_1^2}{[L,L]_1}$$

as $n\to\infty$, where $([L,\,L]_t)_{t\geq 0}$ is the quadratic variation process of $(L_t)_{t\geq 0}$. For this reason Δ_n appears in Theorem 3.2.11. □

3.3 Lattices in $\mathbb{R}^m$ and the manifolds $\mathscr{M}(\omega_1, \ldots, \omega_m)$

In this section we recall some basic facts about lattices in $\mathbb{R}^m$ and use them to construct the manifolds $\mathscr{M}(\omega_1, \ldots, \omega_m)$ in Eq. (3.13c). For more details concerning the theory of lattices we refer the reader to [29, 48].

Definition 3.3.1 (Lattice)**.** *For $S \subseteq \mathbb{R}^m$ let $\operatorname{span}^{\mathbb{Z}}(S)$ and $\operatorname{span}^{\mathbb{R}}(S)$, respectively, denote the integer and linear hull of S. For any linearly independent vectors $b_1, \ldots, b_d \in \mathbb{R}^m$ the additive subgroup of $\mathbb{R}^m$*

$$\mathscr{L} := \mathscr{L}(b_1, \ldots, b_d) := \operatorname{span}^{\mathbb{Z}}(\{b_1, \ldots, b_d\})$$

is said to be a lattice *and $b_1, \ldots, b_d$ is called a* basis *of $\mathscr{L}$. The* dimension *of the lattice $\mathscr{L}$ is given by*

$$\dim(\mathscr{L}) := \dim\left(\operatorname{span}^{\mathbb{R}}(\mathscr{L})\right) = d.$$

We call a subset S in $\mathbb{R}^m$ *discrete* if S has no accumulation point in $\mathbb{R}^m$. It is a classical result that discreteness characterizes lattices among additive subgroups in $\mathbb{R}^m$.

Theorem 3.3.2 (cf. [48], § 3.2)**.** *A subset $S \subseteq \mathbb{R}^m$ is a lattice if and only if it is a discrete, additive subgroup of $\mathbb{R}^m$. In either case the dimension of the lattice is equal to the maximal number of linearly independent vectors in S.*

Suppose that we have given $\omega_1, \ldots, \omega_m \in \mathbb{R}^*$ which are linearly dependent over $\mathbb{Z}$. Let $\underset{\sim}{\omega} = (\omega_1, \ldots, \omega_m)^T = 2\pi\underset{\sim}{\eta}$. Note that all lattices as well as the manifolds $\mathscr{M}(\omega_1, \ldots, \omega_m)$ in this chapter depend on the frequency vector $\underset{\sim}{\omega}$ and $\underset{\sim}{\eta}$, respectively. We neglect, however, that dependency for ease of notation. We define

$$\widetilde{\mathscr{L}} := \{\underset{\sim}{\eta}\}^\perp \cap \mathbb{Z}^m.$$

Then $\widetilde{\mathscr{L}}$ constitutes a discrete, additive subgroup of $\mathbb{R}^m$ and since the maximal possible number of linearly independent vectors in $\widetilde{\mathscr{L}}$ is $m-1$, we apply Theorem 3.3.2 and obtain an $s \in \{1, \ldots, m-1\}$ and a basis $b_{m-s+1}, \ldots, b_m \in \mathbb{Z}^m$ of the lattice $\widetilde{\mathscr{L}}$. Now

$$\mathscr{L} := \widetilde{\mathscr{L}}^\perp \cap \mathbb{Z}^m \tag{3.16}$$

is a discrete, additive subgroup in $\mathbb{R}^m$ as well and hence, again due to Theorem 3.3.2, it is a lattice generated by a basis $b_1, \ldots, b_{m-s} \in \mathbb{Z}^m$. That the dimension of $\mathscr{L}$ is indeed $m-s$ (i.e. the maximal possible dimension of the orthogonal complement of $\widetilde{\mathscr{L}}$) can be seen from the following fact: let

$$H := \begin{pmatrix} b_{m-s+1}^T \\ \vdots \\ b_m^T \end{pmatrix} \in \mathbb{Z}^{s\times m}$$

and note that there has to be an $s \times s$-block with non-vanishing determinant. W.l.o.g. this block is given by the first s columns of H, denoted by $H^{[s]}$. We can solve, for any $j \in \{s+1, \ldots, m\}$, the linear systems $H^{[s]}x_j = -h_j$ where h_j is the j-th column of H and obtain, using Cramer's rule, solutions $x_j \in \mathbb{Q}^s$ with common denominator $\det\left(H^{[s]}\right) \in \mathbb{Z}$. Hence, the vectors

$$v_j := \det\left(H^{[s]}\right) \cdot \left[\begin{pmatrix} x_j \\ 0 \\ \vdots \\ 0 \end{pmatrix} + e_j\right] \in \mathbb{Z}^m, \qquad j \in \{s+1, \ldots, m\},$$

with e_j being the j-th unit vector in $\mathbb{R}^m$, are linearly independent and $Hv_j = 0$ for all $j \in \{s+1, \ldots, m\}$. This shows that $v_j \in \{b_{m-s+1}, \ldots, b_m\}^\perp \cap \mathbb{Z}^m = \mathscr{L}$ for any $j \in \{s+1, \ldots, m\}$, and hence, the dimension of the lattice $\mathscr{L}$ has to be $m-s$ as claimed above. Let

$$B := \begin{pmatrix} b_1 & b_2 & \ldots & b_{m-s} \end{pmatrix} \in \mathbb{Z}^{m\times(m-s)}$$

and

$$\begin{aligned} T : (\mathbb{R} \bmod 1)^{m-s} &\to (\mathbb{R} \bmod 1)^m \\ x = (x_1, \ldots, x_{m-s})^T &\mapsto B\,x \bmod 1 = \left(\sum_{j=1}^{m-s} x_j\, b_j \right) \bmod 1, \end{aligned}$$

where the mod-operator has to be applied componentwise. We then define

$$\mathscr{M} := T\big((\mathbb{R} \bmod 1)^{m-s}\big), \tag{3.17}$$

the Gram matrix $G := B^T B$ and the set of functions on $\mathscr{M}$

$$\mathscr{T} := \left\{ f_h : \mathscr{M} \to \mathbb{C} : \; f_h = \mathrm{e}^{2\pi i \langle h, \cdot \rangle} \circ T \circ G^{-1} \circ T^{-1} \text{ for an } h \in \mathscr{L} \right\}. \tag{3.18}$$

$\mathscr{T}$ is well-defined due to the injectivity of T (see the proof of the upcoming Theorem 3.3.3(i)). Moreover, it can be shown that all the functions in $\mathscr{T}$ are continuous (mod 1) on $\mathscr{M}$. The following theorem holds.

Theorem 3.3.3.

(i) $\mathscr{M}$ *is an* $(m-s)$*-dimensional* C^1*-manifold in* $[0,1)^m$.

(ii) Let $\underset{\sim}{\mu} \in \mathbb{R}^{m-s}$ *be the coordinates of* $\underset{\sim}{\eta}$ *in the basis* B*, i.e.* $\underset{\sim}{\eta} = B\underset{\sim}{\mu}$*. Then* $\langle z, \underset{\sim}{\mu} \rangle \neq 0$ *for all* $z \in \mathbb{Z}^{m-s}$*,* $z \neq 0$.

(iii) For any $f_h \in \mathscr{T}$ *with* $h \in \mathscr{L}$*,* $h \neq 0$*, we have*

$$\frac{1}{\mathscr{H}^{m-s}(\mathscr{M})} \int_{\mathscr{M}} f_h(x)\, \mathscr{H}^{m-s}(\mathrm{d}x) = 0,$$

where $\mathscr{H}^{m-s}$ *is the* $(m-s)$*-dimensional Lebesgue measure on* $\mathscr{M}$.

(iv) For any $x, y \in \mathscr{M}$*,* $x \neq y$*, there is an* $h \in \mathscr{L}$ *such that* $f_h(x) \neq f_h(y)$.

Since $(\mathbb{R} \bmod 1)^m$ and $(\mathbb{R} \bmod 1)^{m-s}$ are compact Hausdorff spaces, one immediately obtains that also $\mathscr{M}$ is a compact Hausdorff space. Note that the subalgebra $\mathrm{span}^{\mathbb{C}}(\mathscr{T})$ of the algebra $C(\mathscr{M})$ of all continuous complex-valued functions on $\mathscr{M}$ contains the constant function 1 (take $h = 0$). Moreover, $\mathrm{span}^{\mathbb{C}}(\mathscr{T})$ is closed under complex conjugation and separates points (see Theorem 3.3.3(iv)). Applying the Stone-Weierstraß Theorem (cf. [76, p. 122] or [82, p. 161]), this yields that $\mathrm{span}^{\mathbb{C}}(\mathscr{T})$ is dense in $C(\mathscr{M})$ with respect to the topology of uniform convergence.

An application of Theorem 3.3.3 as given in the next proposition characterizes the limit distributions of the normalized and the first version of the self-normalized periodogram, respectively, by random vectors with characteristic functions as given in (3.13).

Proposition 3.3.4.
Suppose $\Delta = \Delta_n \to 0$ *and* $n\Delta_n \to \infty$ *as* $n \to \infty$. *Moreover, define for any* $z_1, z_2 \in \mathbb{R}$ *the function* $\Xi_{z_1, z_2} : \mathbb{C} \to \mathbb{R}$ *by* $\Xi_{z_1, z_2}(x) := z_1 \Re(x) + z_2 \Im(x)$. *Then, for any* $m \in \mathbb{N}^*$, $\omega_1, \ldots, \omega_m \in \mathbb{R}^*$ *and* $\underset{\sim}{\theta}, \underset{\sim}{\nu} \in \mathbb{R}^m$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-p+1} \left| \sum_{j=1}^{m} \Xi_{\theta_j, \nu_j} \left(\mathrm{e}^{-i\omega_j \Delta_n k} \, c(i\omega_j) \right) \right|^{\alpha}$$
$$= K_{\underset{\sim}{\omega}} \Big(\big(\Xi_{\theta_j, \nu_j}\big(c(i\omega_j)\big)\big)_{j \in \{1, \ldots, m\}}, \, \big(\Xi_{-\nu_j, \theta_j}\big(c(i\omega_j)\big)\big)_{j \in \{1, \ldots, m\}}, \, 0 \Big),$$

where $K_{\underset{\sim}{\omega}}$ *is given by eqs.* (3.13b) *and* (3.13c), *respectively.*

For $\omega_1, \ldots, \omega_m$ linearly independent over $\mathbb{Z}$ a similar result has been derived in [59, Corollary 4].

Finally, we shall require Proposition 3.3.5 from below for the limit result of the second version of the self-normalized periodogram. The proof of this proposition is based on Theorem 3.3.3 as well.

Proposition 3.3.5. *Suppose* $\Delta = \Delta_n \to 0$ *and* $n\Delta_n \to \infty$ *as* $n \to \infty$. *Let* $m \in \mathbb{N}^*$, $\omega_1, \ldots, \omega_m \in \mathbb{R}^*$ *and write* $\underset{\sim}{\omega} = (\omega_1, \ldots, \omega_m)^T = 2\pi(\eta_1, \ldots, \eta_m)^T = 2\pi\underset{\sim}{\eta}$. *Moreover, suppose that* $(N_k)_{k\in\mathbb{N}^*}$ *are i.i.d. standard normal random variables.*

(i) If $\omega_1, \ldots, \omega_m$ *are linearly independent over* $\mathbb{Z}$, *we assume that we have given a random variable* $\underset{\sim}{U}$, *uniformly distributed on* $[0, 1)^m$ *and independent of* $(N_k)_{k\in\mathbb{N}^*}$, *and a function* $f : (\mathbb{R} \bmod 1)^m \times \mathbb{R} \to \mathbb{R}$ *such that* $\mathbb{E}[f^2(\underset{\sim}{U}, N_1)] < \infty$ *and* $g^{(k)}(x) := \mathbb{E}[f^k(x, N_1)]$, $k = 1, 2$, *is continuous on* $(\mathbb{R} \bmod 1)^m$.

(ii) If $\omega_1, \ldots, \omega_m$ *are linearly dependent over* $\mathbb{Z}$, *we assume that we have given a random variable* $\underset{\sim}{V}$, *uniformly distributed on* $[0, 1)^{m-s}$ *and independent of* $(N_k)_{k\in\mathbb{N}^*}$, *and a function* $f : \mathscr{M} \times \mathbb{R} \to \mathbb{R}$ *such that* $\mathbb{E}[f^2(\underset{\sim}{U}, N_1)] < \infty$ *and* $g^{(k)}(x) := \mathbb{E}[f^k(x, N_1)]$, $k = 1, 2$, *is continuous on* $\mathscr{M}$, *where* $\underset{\sim}{U} := T(\underset{\sim}{V})$ *and* T *is the parametrization of* $\mathscr{M}$.

Then in either case

$$\frac{1}{n} \sum_{k=1}^{n} f(k\Delta_n \underset{\sim}{\eta} \bmod 1, \, N_k) \xrightarrow{\mathbb{P}} \mathbb{E}\left[f(\underset{\sim}{U}, N_1)\right] \qquad \text{as } n \to \infty. \tag{3.19}$$

3.4 Proofs of Chapter 3

3.4.1 Proofs of Section 3.1

Proof of Equation (3.5)***.***
Fix an arbitrary $\omega \in \mathbb{R}$ and assume that Δ is sufficiently small such that $\omega\Delta \in [-\pi, \pi]$. Then

$$\Delta f_\Delta(\omega\Delta) \overset{(3.4)}{=} \frac{\Delta}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_Y(k\Delta)\, \mathrm{e}^{-ik\omega\Delta} \overset{(3.2)}{=} \frac{1}{2\pi} c^T \Big(\Delta \sum_{k=-\infty}^{\infty} \mathrm{e}^{|k|\Delta A} \mathrm{e}^{-ik\omega\Delta} \Big) \gamma_X(0)\, c. \quad (3.20)$$

For any $\varepsilon > 0$, there exist an $N_0 \in \mathbb{N}$ and $\Delta_0 > 0$ such that

$$\begin{aligned}
&\Big\| \int_{-\infty}^{\infty} \mathrm{e}^{|h|A} \mathrm{e}^{-ih\omega}\, \mathrm{d}h - \Delta \sum_{k=-\infty}^{\infty} \mathrm{e}^{|k|\Delta A} \mathrm{e}^{-ik\omega\Delta} \Big\| \\
&\leq \int_{|h|\geq N_0} \big\| \mathrm{e}^{|h|A} \big\|\, \mathrm{d}h + \Big\| \int_{-N_0}^{N_0} \mathrm{e}^{|h|A} \mathrm{e}^{-ih\omega}\, \mathrm{d}h - \Delta \sum_{|k|\leq \lfloor N_0/\Delta \rfloor} \mathrm{e}^{|k|\Delta A} \mathrm{e}^{-ik\omega\Delta} \Big\| \\
&\qquad + \Delta \sum_{|k|\geq \lfloor N_0/\Delta \rfloor + 1} \big\| \mathrm{e}^{|k|\Delta A} \big\| \\
&\leq \frac{\varepsilon}{3} + \Big\| \int_{-N_0}^{N_0} \mathrm{e}^{|h|A} \mathrm{e}^{-ih\omega}\, \mathrm{d}h - \Delta \sum_{|k|\leq \lfloor N_0/\Delta \rfloor} \mathrm{e}^{|k|\Delta A} \mathrm{e}^{-ik\omega\Delta} \Big\| + \frac{\varepsilon}{3} \qquad (3.21)
\end{aligned}$$

for all $0 < \Delta \leq \Delta_0$. The second addend on the right-hand side converges to 0 as $\Delta \to 0$ (Riemann sums!), i.e. there is a $\Delta_1 > 0$ such that (3.21) is less or equal to ε for any $\Delta \leq \Delta_1$. Hence, the right-hand side of Eq. (3.20) converges, as $\Delta \to 0$, to

$$\frac{1}{2\pi} c^T \Big(\int_{-\infty}^{\infty} \mathrm{e}^{|h|A} \mathrm{e}^{-ih\omega}\, \mathrm{d}h \Big) \gamma_X(0)\, c = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{c^T \mathrm{e}^{|h|A} \gamma_X(0)\, c}_{\overset{(3.2)}{=} \gamma_Y(h)} \cdot \mathrm{e}^{-ih\omega}\, \mathrm{d}h \overset{(3.3)}{=} f_Y(\omega).$$

□

3.4.2 Proofs of Section 3.2.1

Proof of Lemma 3.2.1. *(i)* By virtue of [8, Proposition 11.2.1] we have, for any $t \in \mathbb{R}$,

$$\mathrm{e}^{tA} = \frac{1}{2\pi i} \int_\rho (z\mathrm{I}_p - A)^{-1} \mathrm{e}^{tz}\, \mathrm{d}z,$$

where ρ is a simple closed curve in the complex plane enclosing the spectrum of A. Moreover, from [24, Lemma 3.1] we immediately obtain

$$c^T(z\mathrm{I}_p - A)^{-1}e_p = \frac{c(z)}{a(z)}$$

for any $z \in \mathbb{C}\backslash\{\lambda_1, \ldots, \lambda_p\}$. Hence,

$$\begin{aligned}
&c^T\Bigg(-\sum_{j=0}^{r-1}\Phi_j^\Delta\, \mathrm{e}^{(r-1-j)\Delta A}\Bigg)\mathrm{e}^{(k\Delta-s)A}e_p \\
&\quad= -\sum_{j=0}^{r-1}\Phi_j^\Delta\, c^T\left(\frac{1}{2\pi i}\int_\rho (z\mathrm{I}_p - A)^{-1}\mathrm{e}^{(r-1-j)\Delta z+(k\Delta-s)z}\,\mathrm{d}z\right)e_p \\
&\quad= -\sum_{j=0}^{r-1}\Phi_j^\Delta\cdot\frac{1}{2\pi i}\int_\rho \frac{c(z)}{a(z)}\,\mathrm{e}^{(r-1-j)\Delta z+(k\Delta-s)z}\,\mathrm{d}z \\
&\quad= \sum_{m=1}^{p}\frac{c(\lambda_m)}{a'(\lambda_m)}\Bigg(-\sum_{j=0}^{r-1}\Phi_j^\Delta\,\mathrm{e}^{(r-1-j)\Delta\lambda_m}\Bigg)\mathrm{e}^{(k\Delta-s)\lambda_m},
\end{aligned}$$

where the last equality follows from the Residue Formula (see, e.g., [60, Chapter VI, Theorem 1.2 and Lemma 1.3] or [43, Theorem III.6.3 and Remark III.6.4]) and the fact that the eigenvalues $\lambda_1, \ldots, \lambda_p$ of A are supposed to be distinct.

(ii) We obviously have

$$\begin{aligned}
\frac{1}{\Delta}\int_0^\Delta \left|\mathrm{e}^{(\Delta-s)\lambda} - 1\right|^\alpha \mathrm{d}s &= \frac{1}{\Delta}\int_0^\Delta \left|\mathrm{e}^{s\lambda} - 1\right|^\alpha \mathrm{d}s \\
&\leq \frac{2^\alpha}{\Delta}\int_0^\Delta \left|\mathrm{e}^{s\,\Re(\lambda)}\cos\big(s\,\Im(\lambda)\big) - 1\right|^\alpha + \left|\mathrm{e}^{s\,\Re(\lambda)}\sin\big(s\,\Im(\lambda)\big)\right|^\alpha \mathrm{d}s.
\end{aligned}$$

Due to the Mean Value Theorem there exists an $\varepsilon(\Delta) \in [0, \Delta]$ such that

$$\frac{1}{\Delta}\int_0^\Delta \left|\mathrm{e}^{s\,\Re(\lambda)}\cos\big(s\,\Im(\lambda)\big) - 1\right|^\alpha \mathrm{d}s = \left|\mathrm{e}^{\varepsilon(\Delta)\,\Re(\lambda)}\cos\big(\varepsilon(\Delta)\,\Im(\lambda)\big) - 1\right|^\alpha. \tag{3.22}$$

Since $\varepsilon(\Delta) \to 0$ as $\Delta \to 0$, we immediately obtain that the right-hand side of (3.22) converges to 0 as $\Delta \to 0$. Likewise we deduce that

$$\frac{1}{\Delta}\int_0^\Delta \left|\mathrm{e}^{s\,\Re(\lambda)}\sin\big(s\,\Im(\lambda)\big)\right|^\alpha \mathrm{d}s \to 0 \quad \text{as } \Delta \to 0$$

and hence, *(ii)* follows.

(iii) By virtue of Eq. (3.8b) we have, for any $r \in \{1, \ldots, p\}$,

$$
\begin{aligned}
-\sum_{j=0}^{r-1} \Phi_j^\Delta \, e^{(r-1-j)\Delta\lambda_m} \\
&= -e^{(r-1)\Delta\lambda_m} \, \Phi_0^\Delta - e^{(r-2)\Delta\lambda_m} \, \Phi_1^\Delta - e^{(r-3)\Delta\lambda_m} \, \Phi_2^\Delta - \ldots - \Phi_{r-1}^\Delta \\
&= (-1)^2 \, e^{(r-1)\Delta\lambda_m} - (-1)^2 \, e^{(r-2)\Delta\lambda_m} \sum_{\{i_1\} \in \binom{\{1, \ldots, p\}}{1}} e^{\Delta\lambda_{i_1}} - e^{(r-3)\Delta\lambda_m} \, \Phi_2^\Delta - \ldots - \Phi_{r-1}^\Delta \\
&= (-1)^3 \, e^{(r-2)\Delta\lambda_m} \sum_{\{i_1\} \in \binom{\{1, \ldots, p\} \setminus \{m\}}{1}} e^{\Delta\lambda_{i_1}} - (-1)^3 \, e^{(r-3)\Delta\lambda_m} \sum_{\{i_1, i_2\} \in \binom{\{1, \ldots, p\}}{2}} e^{\Delta(\lambda_{i_1} + \lambda_{i_2})} \\
&\qquad - \ldots - \Phi_{r-1}^\Delta \\
&= (-1)^4 \, e^{(r-3)\Delta\lambda_m} \sum_{\{i_1, i_2\} \in \binom{\{1, \ldots, p\} \setminus \{m\}}{2}} e^{\Delta(\lambda_{i_1} + \lambda_{i_2})} \\
&\qquad - \ldots - (-1)^r \sum_{\{i_1, \ldots, i_{r-1}\} \in \binom{\{1, \ldots, p\}}{r-1}} e^{\Delta \sum_{s=1}^{r-1} \lambda_{i_s}} \\
&= \ldots = (-1)^{r+1} \sum_{\{i_1, \ldots, i_{r-1}\} \in \binom{\{1, \ldots, p\} \setminus \{m\}}{r-1}} e^{\Delta \sum_{s=1}^{r-1} \lambda_{i_s}} \qquad (3.23)
\end{aligned}
$$

and hence, due to Eqs. (3.10) and (3.23),

$$
\begin{aligned}
f_\Delta^{(m)}(\omega\Delta) &= \sum_{r=1}^{p} e^{-i\omega\Delta(r-1)} \left(-\sum_{j=0}^{r-1} \Phi_j^\Delta \, e^{(r-1-j)\Delta\lambda_m} \right) \\
&= \sum_{r=0}^{p-1} (-1)^r \, e^{-i\omega\Delta r} \sum_{\{i_1, \ldots, i_r\} \in \binom{\{1, \ldots, p\} \setminus \{m\}}{r}} e^{\Delta \sum_{s=1}^{r} \lambda_{i_s}} \\
&= \sum_{r=0}^{p-1} (-1)^r \sum_{\{i_1, \ldots, i_r\} \in \binom{\{1, \ldots, p\} \setminus \{m\}}{r}} e^{\Delta \left(\sum_{s=1}^{r} \lambda_{i_s} - i\omega r \right)} \\
&= \sum_{j=0}^{p-1} \frac{\Delta^j}{j!} \sum_{r=0}^{p-1} (-1)^r \sum_{\{i_1, \ldots, i_r\} \in \binom{\{1, \ldots, p\} \setminus \{m\}}{r}} \left(\sum_{s=1}^{r} \lambda_{i_s} - i\omega r \right)^j + o(\Delta^{p-1})
\end{aligned}
\qquad (3.24)
$$

as $\Delta \to 0$. Now, since the eigenvalues of A are also the zeros of the autoregressive polynomial $a(z)$, we observe that in order to show Lemma 3.2.1(iii) it remains to prove the following

$$\sum_{r=0}^{p-1}(-1)^r \sum_{\{i_1,\dots,i_r\}\in\binom{\{1,\,\dots\,,\,p\}\setminus\{m\}}{r}} \left(\sum_{s=1}^{r}\lambda_{i_s} - i\omega r\right)^j = (p-1)!\prod_{\substack{s=1\\ s\neq m}}^{p}(i\omega-\lambda_s)\,\delta_{\{p-1\}}(j), \tag{3.25}$$

where $\delta_{\{p-1\}}(j) = 1$ if $j = p-1$ and 0 otherwise.

If $p = 1$, one immediately verifies that (3.25) holds since both sides are equal to 1. Hence, we assume $p > 1$ in the following.

For $j = 0$, due to the Binomial Theorem, the left-hand side of (3.25) is equal to

$$\sum_{r=0}^{p-1}(-1)^r\binom{p-1}{r} = \big(1+(-1)\big)^{p-1} = 0.$$

For $j \in \{1,\,\dots\,,\,p-1\}$ we obtain

$$\begin{aligned}
&\sum_{r=0}^{p-1}(-1)^r \sum_{\{i_1,\dots,i_r\}\in\binom{\{1,\,\dots\,,\,p\}\setminus\{m\}}{r}} \left(\sum_{s=1}^{r}\lambda_{i_s} - i\omega r\right)^j \\
&= \sum_{r=1}^{p-1}(-1)^r \sum_{\{i_1,\dots,i_r\}\in\binom{\{1,\,\dots\,,\,p\}\setminus\{m\}}{r}} \left(\sum_{s=1}^{r}(\lambda_{i_s} - i\omega)\right)^j \\
&= \sum_{r=1}^{p-1}(-1)^r \sum_{t=1}^{j}\binom{p-1-t}{r-t} \sum_{k_1=1}^{p-1-(t-1)}\;\sum_{k_2=1}^{p-1-(t-2)-k_1}\cdots \sum_{k_{t-1}=1}^{p-1-(t-(t-1))-\sum_{h=1}^{t-2}k_h}\binom{j}{k_1} \\
&\quad\times\binom{j-k_1}{k_2}\cdots\binom{j-\sum_{h=1}^{t-2}k_h}{k_{t-1}} \sum_{\substack{u_1,\dots,u_t\in\{1,\dots,p\}\setminus\{m\}\\ u_1<u_2<\dots<u_t}} (\lambda_{u_1}-i\omega)^{j-\sum_{h=1}^{t-1}k_h} \\
&\quad\times\prod_{s=2}^{t}(\lambda_{u_s}-i\omega)^{k_{t+1-s}} \\
&= \sum_{t=1}^{j}\;\sum_{k_1=1}^{p-1-(t-1)}\;\sum_{k_2=1}^{p-1-(t-2)-k_1}\cdots\sum_{k_{t-1}=1}^{p-2-\sum_{h=1}^{t-2}k_h}\binom{j}{k_1}\binom{j-k_1}{k_2}\cdots\binom{j-\sum_{h=1}^{t-2}k_h}{k_{t-1}} \\
&\quad\times\sum_{\substack{u_1,\dots,u_t\in\{1,\dots,p\}\setminus\{m\}\\ u_1<u_2<\dots<u_t}} (\lambda_{u_1}-i\omega)^{j-\sum_{h=1}^{t-1}k_h}\prod_{s=2}^{t}(\lambda_{u_s}-i\omega)^{k_{t+1-s}}\sum_{r=1}^{p-1}(-1)^r\binom{p-1-t}{r-t}.
\end{aligned} \tag{3.26}$$

Since $\binom{n}{j} = 0$ for all $n \in \mathbb{N}$ and $j < 0$, we get

$$\sum_{r=1}^{p-1} (-1)^r \binom{p-1-t}{r-t} = (-1)^t \sum_{r=0}^{p-1-t} (-1)^r \binom{p-1-t}{r} = (-1)^t \cdot \big(1 + (-1)\big)^{p-1-t}$$
$$= \begin{cases} 0 & \text{if } t = 1, \ldots, p-2, \\ (-1)^{p-1} & \text{if } t = p-1, \end{cases}$$

where we used again the Binomial Theorem. Consequently, for any $j \in \{1, \ldots, p-2\}$, the right-hand side of (3.26) vanishes, whereas for $j = p-1$ it becomes

$$(-1)^{p-1} \binom{p-1}{1} \binom{p-2}{1} \cdots \binom{2}{1} \prod_{\substack{s=1 \\ s \neq m}}^{p} (\lambda_s - i\omega) = (p-1)! \prod_{\substack{s=1 \\ s \neq m}}^{p} (i\omega - \lambda_s),$$

which completes the proof of Eq. (3.25) and hence, *(iii)* is shown.

(iv) It is a simple consequence of Liouville's Theorem (see, for instance, [60, Chapter III, Theorem 7.5]) that any rational function $f(z) = \frac{q(z)}{p(z)}$ with $\deg(q) < \deg(p)$ can be written as

$$f(z) = h_f(z;\, \lambda_1) + \ldots + h_f(z;\, \lambda_r)$$

where $\lambda_1, \ldots, \lambda_r$ are the distinct zeros of $p(z)$ and $h_f(z;\, \lambda_m)$ is the principal part of the Laurent series expansion of f at the point λ_m.

Again, the eigenvalues of A are also the zeros of the autoregressive polynomial $a(z)$. Consequently, we can apply the above result to the rational function $c(z)/a(z)$ (note that $\deg(a) = p > q = \deg(c)$) and obtain

$$\frac{c(z)}{a(z)} = h_{c/a}(z;\, \lambda_1) + \ldots + h_{c/a}(z;\, \lambda_p).$$

Since $\lambda_1, \ldots, \lambda_p$ are distinct, every λ_m, $m \in \{1, \ldots, p\}$, is a pole of order 1 of the rational function c/a. In this case, it is well known (see, e.g., [60, p. 174]) that the principal part of the Laurent series expansion of c/a at the point λ_m reduces to

$$\frac{c(\lambda_m)}{a'(\lambda_m)} \cdot \frac{1}{z - \lambda_m}.$$

Since $\lambda_1, \ldots, \lambda_p$ are supposed to have non-vanishing real parts, we have $a(i\omega) \neq 0$ for any $\omega \in \mathbb{R}$. Hence, Lemma 3.2.1(iv) holds for any $\omega \in \mathbb{R}$. □

Proof of Lemma 3.2.2. *(i)* This statement follows easily by induction over p from the definition of the $\Psi_j^{\Delta_n}$.

(ii) We deduce from *(i)* that

$$\sum_{j=n+1}^{\infty} \left|\Psi_j^{\Delta_n}\right| \le C(p)\, \Delta_n^{-(p-1)} \sum_{j=n+1}^{\infty} \mathrm{e}^{\Delta_n \lambda_{\max} j} = C(p)\, \Delta_n^{-(p-1)} \frac{\mathrm{e}^{(n+1)\Delta_n \lambda_{\max}}}{1-\mathrm{e}^{\Delta_n \lambda_{\max}}}$$

$$\sim -\frac{C(p)}{\lambda_{\max}}\, \mathrm{e}^{n\Delta_n\left(\lambda_{\max} - p\frac{\log(\Delta_n)\cdot\Delta_n^{\delta}}{n\Delta_n^{1+\delta}}\right)} \overset{n\to\infty}{\to} 0, \tag{3.27}$$

since $\Delta_n \to 0$ and $n\Delta_n^{1+\delta} \to \infty$ as $n \to \infty$.

If $0 < \alpha \le 1$, we have (cf. also [57, Proof of Proposition 2.1])

$$\frac{\Delta_n^{\alpha}}{n} \sum_{k=-\infty}^{-n-1} \Big(\sum_{j=1-k}^{n-k} \left|\Psi_j^{\Delta_n}\right| \Big)^{\alpha} \le \Delta_n^{\alpha} \sum_{j=n+2}^{\infty} \left|\Psi_j^{\Delta_n}\right|^{\alpha},$$

and analogously to (3.27) it can be shown that the right-hand side converges to 0 as $n \to \infty$. Otherwise, if $1 < \alpha \le 2$, we set $\widetilde{\Psi}_j^{\Delta_n} := \Psi_j^{\Delta_n} / \sum_{j=n+2}^{\infty} \left|\Psi_j^{\Delta_n}\right|$ and obtain

$$\frac{\Delta_n^{\alpha}}{n} \sum_{k=-\infty}^{-n-1} \Big(\sum_{j=1-k}^{n-k} \left|\Psi_j^{\Delta_n}\right| \Big)^{\alpha} = \Big(\sum_{j=n+2}^{\infty} \left|\Psi_j^{\Delta_n}\right| \Big)^{\alpha} \frac{\Delta_n^{\alpha}}{n} \sum_{k=-\infty}^{-n-1} \Big(\sum_{j=1-k}^{n-k} \left|\widetilde{\Psi}_j^{\Delta_n}\right| \Big)^{\alpha}$$

$$\le \Big(\sum_{j=n+2}^{\infty} \left|\Psi_j^{\Delta_n}\right| \Big)^{\alpha-1} \Delta_n^{\alpha} \sum_{j=n+2}^{\infty} \left|\Psi_j^{\Delta_n}\right| \overset{n\to\infty}{\to} 0$$

due to Eq. (3.27).

(iii) We use again *(i)* to derive

$$\frac{\Delta_n^{\alpha p}}{n} \sum_{k=1-n}^{1-p} \Big(\sum_{j=1-k}^{n} \left|\Psi_j^{\Delta_n}\right| \Big)^{\alpha} \le \frac{C(p)^{\alpha}\, \Delta_n^{\alpha}}{n} \sum_{k=1}^{n} \Big(\sum_{j=k}^{n} \mathrm{e}^{\Delta_n \lambda_{\max} j} \Big)^{\alpha}$$

$$\le \frac{C(p)^{\alpha}\, \Delta_n^{\alpha}}{n\,(1-\mathrm{e}^{\Delta_n\lambda_{\max}})^{\alpha}} \sum_{k=1}^{n} \mathrm{e}^{\alpha\Delta_n\lambda_{\max} k} \le \frac{C(p)^{\alpha}\, \Delta_n^{\alpha}}{n\,(1-\mathrm{e}^{\Delta_n\lambda_{\max}})^{\alpha}} \cdot \frac{1}{1-\mathrm{e}^{\alpha\Delta_n\lambda_{\max}}}$$

$$\sim \frac{C(p)^{\alpha}}{(-\lambda_{\max})^{\alpha}} \cdot \frac{1}{-\alpha\lambda_{\max} n\Delta_n} \to 0$$

as $n \to \infty$, since we suppose $n\Delta_n \to \infty$.

(iv) We have, once again due to *(i)*,

$$\frac{\Delta_n^{\alpha}}{n} \sum_{k=2-p-n}^{-1} \Big(\sum_{j=1\vee(2-p-k)}^{n\wedge(-k)} \left|\Psi_j^{\Delta_n}\right| \Big)^{\alpha} \le \frac{\Delta_n^{\alpha}}{n} \left[\sum_{k=1}^{p-2} \Big(\sum_{j=1}^{k} \left|\Psi_j^{\Delta_n}\right| \Big)^{\alpha} + \sum_{k=p-1}^{n+p-2} \Big(\sum_{j=k+2-p}^{k} \left|\Psi_j^{\Delta_n}\right| \Big)^{\alpha} \right]$$

$$\leq \frac{\Delta_n^\alpha}{n}\left[(p-2)\,(p-1)^{\alpha p} + \left(C(p)\,(p-1)\,\Delta_n^{-p+1}\right)^\alpha \sum_{k=p-1}^{n+p-2} \mathrm{e}^{\alpha\Delta_n\lambda_{\max}(k+2-p)}\right]$$
$$\leq \frac{\Delta_n^\alpha}{n}\left[(p-2)\,(p-1)^{\alpha p} + \left(C(p)\,(p-1)\,\Delta_n^{-p+1}\right)^\alpha \frac{1}{1-\mathrm{e}^{\alpha\Delta_n\lambda_{\max}}}\right],$$

where the first summand obviously vanishes as $n \to \infty$. The second term is asymptotically equivalent to

$$\frac{(C(p)\,(p-1))^\alpha}{-\alpha\lambda_{\max}} \cdot \frac{1}{n\Delta_n^{\alpha(p-2)+1}} \to 0$$

as $n \to \infty$ by assumption.

(v) It is once more *(i)* that gives

$$\frac{\Delta_n^\alpha}{n}\sum_{k=2-p}^{0}\left(\sum_{j=1}^{n}\left|\Psi_j^{\Delta_n}\right|\right)^\alpha \leq (p-1)\frac{\Delta_n^\alpha}{n}\left(\sum_{j=1}^{\infty}\left|\Psi_j^{\Delta_n}\right|\right)^\alpha \leq (p-1)\frac{\Delta_n^\alpha}{n}\left(\frac{C(p)\,\Delta_n^{-p+1}}{1-\mathrm{e}^{\Delta_n\lambda_{\max}}}\right)^\alpha$$
$$\sim \frac{C(p)^\alpha\,(p-1)}{(-\lambda_{\max})^\alpha} \cdot \frac{1}{n\Delta_n^{\alpha(p-1)}} \to 0$$

as $n \to \infty$, since we assume that $n\Delta_n^{\alpha(p-1)} \to \infty$ as $n \to \infty$. □

3.4.3 Proofs of Section 3.2.2

Since the proof of Proposition 3.2.4 is based on Theorem 3.2.5, we prove first Theorem 3.2.5 and then Proposition 3.2.4. For the proof of Theorem 3.2.5 we need the following additional result:

Proposition 3.4.1. *If Assumption 3.1 holds, $\Delta = \Delta_n \to 0$ and $n\Delta_n^{\alpha(p-1)} \to \infty$ as $n \to \infty$, then, for any $\omega \in \mathbb{R}$,*

$$J_{n,\widetilde{Z}^{\Delta_n}}(\omega\Delta_n) = J^{(2)}_{n,\Delta_n}(\omega\Delta_n) + o_P\left(\Delta_n^{\frac{1}{\alpha}+p-1}\right) \quad \text{as } n \to \infty$$

with $J^{(2)}_{n,\Delta_n}(\omega\Delta_n) := n^{-1/\alpha}\sum_{k=1}^{n-p+1}\tilde{\tilde{Z}}_{k,\Delta_n}(\omega\Delta_n)\,\mathrm{e}^{-i\omega\Delta_n k}$ and $(\tilde{\tilde{Z}}_{k,\Delta_n})_{k\in\mathbb{Z}}$ as in (3.10).

Proof. We first observe that

$$J_{n,\widetilde{Z}^{\Delta_n}}(\omega\Delta_n) = n^{-\frac{1}{\alpha}}\sum_{k=1}^{n}\widetilde{Z}_{k,\Delta_n}\,\mathrm{e}^{-i\omega\Delta_n k} \overset{(3.7)}{=} n^{-\frac{1}{\alpha}}\sum_{k=1}^{n}\left(\sum_{r=1}^{p} Z^r_{k-r+1,\Delta_n}\right)\mathrm{e}^{-i\omega\Delta_n k}$$
$$= n^{-\frac{1}{\alpha}}\sum_{k=2-p}^{n}\sum_{r=1\vee(2-k)}^{p\wedge(n+1-k)} Z^r_{k,\Delta_n}\,\mathrm{e}^{-i\omega\Delta_n(k+r-1)}$$

$$= J^{(1)}_{n,\Delta_n}(\omega\Delta_n) + J^{(2)}_{n,\Delta_n}(\omega\Delta_n) + J^{(3)}_{n,\Delta_n}(\omega\Delta_n) \tag{3.28}$$

with

$$J^{(1)}_{n,\Delta_n}(\omega\Delta_n) := n^{-\frac{1}{\alpha}} \sum_{k=2-p}^{0} \sum_{r=2-k}^{p} Z^r_{k,\Delta_n}\, \mathrm{e}^{-i\omega\Delta_n(k+r-1)},$$

$$J^{(2)}_{n,\Delta_n}(\omega\Delta_n) := n^{-\frac{1}{\alpha}} \sum_{k=1}^{n-p+1} \sum_{r=1}^{p} Z^r_{k,\Delta_n} \mathrm{e}^{-i\omega\Delta_n(k+r-1)} \overset{(3.10)}{=} n^{-\frac{1}{\alpha}} \sum_{k=1}^{n-p+1} \mathrm{e}^{-i\omega\Delta_n k}\, \tilde{\tilde{Z}}_{k,\Delta_n}(\omega\Delta_n),$$

$$J^{(3)}_{n,\Delta_n}(\omega\Delta_n) := n^{-\frac{1}{\alpha}} \sum_{k=n-p+2}^{n} \sum_{r=1}^{n+1-k} Z^r_{k,\Delta_n}\, \mathrm{e}^{-i\omega\Delta_n(k+r-1)}.$$

Moreover, we define, for any $z_1, z_2 \in \mathbb{R}$, the function $\Xi_{z_1,z_2} : \mathbb{C} \to \mathbb{R}$, $\Xi_{z_1,z_2}(x) := z_1 \Re(x) + z_2 \Im(x)$. Then we have, due to Eq. (3.8a) and Lemma 3.2.1(i),

$$\begin{aligned}
&J^{(1)}_{n,\Delta_n}(\omega\Delta_n) \\
&= n^{-\frac{1}{\alpha}} \sum_{k=2-p}^{0} \sum_{r=2-k}^{p} \mathrm{e}^{-i\omega\Delta_n(k+r-1)} \int_{(k-1)\Delta_n}^{k\Delta_n} c^T \Big(-\sum_{j=0}^{r-1} \Phi_j^{\Delta_n}\, \mathrm{e}^{(r-1-j)\Delta_n A} \Big) \mathrm{e}^{(k\Delta_n - s)A} e_p \,\mathrm{d}L_s \\
&= n^{-\frac{1}{\alpha}} \sum_{k=2-p}^{0} \sum_{m=1}^{p} \frac{c(\lambda_m)}{a'(\lambda_m)} \sum_{r=2-k}^{p} \mathrm{e}^{-i\omega\Delta_n(k+r-1)} \Big(-\sum_{j=0}^{r-1} \Phi_j^{\Delta_n}\, \mathrm{e}^{(r-1-j)\Delta_n\lambda_m} \Big) \\
&\qquad\qquad \times \int_{(k-1)\Delta_n}^{k\Delta_n} \mathrm{e}^{(k\Delta_n - s)\lambda_m}\,\mathrm{d}L_s \\
&= n^{-\frac{1}{\alpha}} \sum_{k=2-p}^{0} \int_{(k-1)\Delta_n}^{k\Delta_n} \mathrm{e}^{-i\omega\Delta_n k}\, \zeta^{(k)}_{\Delta_n,\omega\Delta_n}(s)\,\mathrm{d}L_s,
\end{aligned} \tag{3.29}$$

where, for any $\omega \in \mathbb{R}$ and $\Delta > 0$,

$$\zeta^{(k)}_{\Delta,\omega}(s) := \sum_{m=1}^{p} \frac{c(\lambda_m)}{a'(\lambda_m)} f_{\Delta}^{(m;\,2-k)}(\omega)\, \mathrm{e}^{(k\Delta - s)\lambda_m} \qquad \text{and}$$

$$f_{\Delta}^{(m;\,2-k)}(\omega) := \sum_{r=2-k}^{p} \mathrm{e}^{-i\omega\Delta(r-1)} \Big(-\sum_{j=0}^{r-1} \Phi_j^{\Delta}\, \mathrm{e}^{(r-1-j)\Delta\lambda_m} \Big).$$

Hence, the complex $S\alpha S$ random variable $\Delta_n^{1-p-1/\alpha}\, J^{(1)}_{n,\Delta_n}(\omega\Delta_n)$ has joint characteristic function (cf. (3.12))

$$\Phi_{J^{(1)}_{n,\Delta_n}}(z_1,\, z_2) = \exp\Bigg\{ -\frac{\sigma_L^\alpha}{n\Delta_n^{1+\alpha(p-1)}} \sum_{k=2-p}^{0} \int_{(k-1)\Delta_n}^{k\Delta_n} \Big| \Xi_{z_1,z_2}\Big(\mathrm{e}^{-i\omega\Delta_n k}\, \zeta^{(k)}_{\Delta_n,\omega\Delta_n}(s) \Big) \Big|^\alpha \mathrm{d}s \Bigg\},$$

$z_1, z_2 \in \mathbb{R}$. With the same arguments as in Eqs. (3.23) and (3.24) we further obtain, as $n \to \infty$,

$$f_{\Delta_n}^{(m;\,2-k)}(\omega\Delta_n) = \sum_{r=1-k}^{p-1} (-1)^r \binom{p-1}{r} + O(\Delta_n) \tag{3.30}$$

and hence, $\left|f_{\Delta_n}^{(m;\,2-k)}(\omega\Delta_n)\right| \le 2^{p-1}$ for any $m \in \{1, \ldots, p\}$ and $k = 2-p, 3-p, \ldots, 0$, if only n is sufficiently large. Thus,

$$\begin{aligned}
&\frac{1}{n\Delta_n^{1+\alpha(p-1)}} \sum_{k=2-p}^{0} \int_{(k-1)\Delta_n}^{k\Delta_n} \left|\Xi_{z_1,\,z_2}\left(\mathrm{e}^{-i\omega\Delta_n k}\, \zeta_{\Delta_n,\,\omega\Delta_n}^{(k)}(s)\right)\right|^\alpha \mathrm{d}s \\
&\le \frac{(|z_1| + |z_2|)^\alpha}{n\Delta_n^{1+\alpha(p-1)}} \sum_{k=2-p}^{0} \int_{(k-1)\Delta_n}^{k\Delta_n} \left|\zeta_{\Delta_n,\,\omega\Delta_n}^{(k)}(s)\right|^\alpha \mathrm{d}s \\
&\le (p-1)\frac{(|z_1| + |z_2|)^\alpha}{n\Delta_n^{\alpha(p-1)}} \left(2^{p-1} \sum_{m=1}^{p} \frac{|c(\lambda_m)|}{|a'(\lambda_m)|}\right)^\alpha
\end{aligned}$$

and the right-hand side converges to 0 as $n \to \infty$, since we suppose $n\Delta_n^{\alpha(p-1)} \to \infty$. This obviously yields $J_{n,\,\Delta_n}^{(1)}(\omega\Delta_n) = o_P\big(\Delta_n^{1/\alpha+p-1}\big)$ as $n \to \infty$.

Likewise we obtain $J_{n,\,\Delta_n}^{(3)}(\omega\Delta_n) = o_P\big(\Delta_n^{1/\alpha+p-1}\big)$ as $n \to \infty$ which completes the proof of Proposition 3.4.1. □

Proof of Theorem 3.2.5. We prove that $\Delta_n^{1-p-1/\alpha}\big[J_{n,\,\Delta_n}^{(2)}(\omega_j\Delta_n)\big]_{j=1,\ldots,m}$ converges weakly to $\big[c(i\omega_j) \cdot \big(S_j^{\Re}(\underset{\sim}{\omega}) - iS_j^{\Im}(\underset{\sim}{\omega})\big)\big]_{j=1,\ldots,m}$ as $n \to \infty$ and then conclude with Proposition 3.4.1. By virtue of (3.11) we have

$$J_{n,\,\Delta_n}^{(2)}(\omega_j\Delta_n) = n^{-\frac{1}{\alpha}} \sum_{k=1}^{n-p+1} \int_{(k-1)\Delta_n}^{k\Delta_n} \mathrm{e}^{-i\omega_j\Delta_n k}\, g_{\Delta_n,\,\omega_j\Delta_n}^{(k)}(s)\,\mathrm{d}L_s \tag{3.31}$$

for any $j \in \{1, \ldots, m\}$ and the joint characteristic function of the complex $S\alpha S$ random vector $\Delta_n^{1-p-1/\alpha}\big[J_{n,\,\Delta_n}^{(2)}(\omega_j\Delta_n)\big]_{j=1,\ldots,m}$ is given by

$$\Phi_{J_{n,\Delta_n}^{(2)}}\big(\underset{\sim}{\theta}, \underset{\sim}{\nu}\big) = \exp\left\{-\frac{\sigma_L^\alpha}{n\Delta_n^{1+\alpha(p-1)}} \sum_{k=1}^{n-p+1} \int_{(k-1)\Delta_n}^{k\Delta_n} \left|\sum_{j=1}^{m} \Xi_{\theta_j,\nu_j}\left(\mathrm{e}^{-i\omega_j\Delta_n k} g_{\Delta_n,\omega_j\Delta_n}^{(k)}(s)\right)\right|^\alpha \mathrm{d}s\right\} \tag{3.32}$$

with arbitrary $\underset{\sim}{\theta}, \underset{\sim}{\nu} \in \mathbb{R}^m$. Hence, due to Lévy's Continuity Theorem, we have to show for any $\underset{\sim}{\theta}, \underset{\sim}{\nu} \in \mathbb{R}^m$

$$\lim_{n\to\infty} \frac{1}{n\Delta_n^{1+\alpha(p-1)}} \sum_{k=1}^{n-p+1} \int_{(k-1)\Delta_n}^{k\Delta_n} \left|\sum_{j=1}^{m} \Xi_{\theta_j,\,\nu_j}\left(\mathrm{e}^{-i\omega_j\Delta_n k}\, g_{\Delta_n,\,\omega_j\Delta_n}^{(k)}(s)\right)\right|^\alpha \mathrm{d}s$$

$$= K_{\underset{\sim}{\omega}}\Big(\big(\Xi_{\theta_j,\nu_j}\big(c(i\omega_j)\big)\big)_{j\in\{1,\dots,m\}}, \big(\Xi_{-\nu_j,\theta_j}\big(c(i\omega_j)\big)\big)_{j\in\{1,\dots,m\}}, 0\Big), \tag{3.33}$$

where $K_{\underset{\sim}{\omega}}$ has been defined in (3.13b) and (3.13c), respectively.

We first claim

$$\left|\frac{1}{n}\sum_{k=1}^{n-p+1}\left(\frac{1}{\Delta_n}\int_{(k-1)\Delta_n}^{k\Delta_n}\left|\sum_{j=1}^{m}\Xi_{\theta_j,\nu_j}\left(\mathrm{e}^{-i\omega_j\Delta_n k}\,\frac{g^{(k)}_{\Delta_n,\omega_j\Delta_n}(s)}{\Delta_n^{p-1}}\right)\right|^{\alpha}\mathrm{d}s\right.\right.$$
$$\left.\left.-\left|\sum_{j=1}^{m}\Xi_{\theta_j,\nu_j}\left(\mathrm{e}^{-i\omega_j\Delta_n k}\,c(i\omega_j)\right)\right|^{\alpha}\right)\right|\overset{n\to\infty}{\to}0. \tag{3.34}$$

To this end, we use $\big||x|^{\alpha}-|y|^{\alpha}\big|\le(|x|^{\alpha/2}+|y|^{\alpha/2})\cdot|x-y|^{\alpha/2}$ for $\alpha\in(0,2]$ together with the Cauchy-Schwarz inequality and obtain

$$\begin{aligned}
&\left|\frac{1}{n\Delta_n}\sum_{k=1}^{n-p+1}\int_{(k-1)\Delta_n}^{k\Delta_n}\left|\sum_{j=1}^{m}\Xi_{\theta_j,\nu_j}\left(\mathrm{e}^{-i\omega_j\Delta_n k}\,\frac{g^{(k)}_{\Delta_n,\omega_j\Delta_n}(s)}{\Delta_n^{p-1}}\right)\right|^{\alpha}\right.\\
&\qquad\left.-\left|\sum_{j=1}^{m}\Xi_{\theta_j,\nu_j}\left(\mathrm{e}^{-i\omega_j\Delta_n k}\,c(i\omega_j)\right)\right|^{\alpha}\mathrm{d}s\right|\\
&\le\frac{1}{n\Delta_n}\sum_{k=1}^{n-p+1}\int_{(k-1)\Delta_n}^{k\Delta_n}\left(\left|\sum_{j=1}^{m}\Xi_{\theta_j,\nu_j}\left(\mathrm{e}^{-i\omega_j\Delta_n k}\,\frac{g^{(k)}_{\Delta_n,\omega_j\Delta_n}(s)}{\Delta_n^{p-1}}\right)\right|^{\frac{\alpha}{2}}\right.\\
&\qquad\left.+\left|\sum_{j=1}^{m}\Xi_{\theta_j,\nu_j}\left(\mathrm{e}^{-i\omega_j\Delta_n k}\,c(i\omega_j)\right)\right|^{\frac{\alpha}{2}}\right)\\
&\qquad\times\left|\sum_{j=1}^{m}\Xi_{\theta_j,\nu_j}\left(\mathrm{e}^{-i\omega_j\Delta_n k}\left(\frac{g^{(k)}_{\Delta_n,\omega_j\Delta_n}(s)}{\Delta_n^{p-1}}-c(i\omega_j)\right)\right)\right|^{\frac{\alpha}{2}}\mathrm{d}s\\
&\le\left[\frac{1}{n}\sum_{k=1}^{n-p+1}\frac{1}{\Delta_n}\int_{(k-1)\Delta_n}^{k\Delta_n}\left(\sum_{j=1}^{m}(|\theta_j|+|\nu_j|)^{\frac{\alpha}{2}}\cdot\left(\left|\frac{g^{(k)}_{\Delta_n,\omega_j\Delta_n}(s)}{\Delta_n^{p-1}}\right|^{\frac{\alpha}{2}}+|c(i\omega_j)|^{\frac{\alpha}{2}}\right)\right)^{2}\mathrm{d}s\right]^{\frac{1}{2}}\\
&\qquad\times\left[\frac{1}{n}\sum_{k=1}^{n-p+1}\frac{1}{\Delta_n}\int_{(k-1)\Delta_n}^{k\Delta_n}\left|\sum_{j=1}^{m}\Xi_{\theta_j,\nu_j}\left(\mathrm{e}^{-i\omega_j\Delta_n k}\left(\frac{g^{(k)}_{\Delta_n,\omega_j\Delta_n}(s)}{\Delta_n^{p-1}}-c(i\omega_j)\right)\right)\right|^{\alpha}\mathrm{d}s\right]^{\frac{1}{2}}\\
&=:I_1\times I_2,
\end{aligned}$$

where, due to Assumption 3.1, Eq. (3.11) and Lemma 3.2.1(iii), there are constants $C(\omega_j)>0$ such that for all sufficiently large n

$$I_1^2\le 2m^2\sum_{j=1}^{m}(|\theta_j|+|\nu_j|)^{\alpha}\cdot\frac{1}{n}\sum_{k=1}^{n-p+1}\frac{1}{\Delta_n}\int_{(k-1)\Delta_n}^{k\Delta_n}\left|\frac{g^{(k)}_{\Delta_n,\omega_j\Delta_n}(s)}{\Delta_n^{p-1}}\right|^{\alpha}+|c(i\omega_j)|^{\alpha}\,\mathrm{d}s$$

$$\leq 2m^2 \sum_{j=1}^{m} (|\theta_j| + |\nu_j|)^\alpha \cdot \left(\left(C(\omega_j) \sum_{l=1}^{p} \frac{|c(\lambda_l)|}{|a'(\lambda_l)|} \right)^\alpha + |c(i\omega_j)|^\alpha \right) < \infty$$

and hence, I_1 is bounded. Setting

$$h^{(k)}_{\Delta_n,\omega}(s) := \sum_{l=1}^{p} \frac{c(\lambda_l)}{a'(\lambda_l)} \frac{a(i\omega)}{i\omega - \lambda_l} \mathrm{e}^{(k\Delta_n - s)\lambda_l}, \quad k \in \{1, \ldots, p\},$$

we obtain for the second term

$$\begin{aligned} I_2^2 &\leq m^\alpha \sum_{j=1}^{m} (|\theta_j| + |\nu_j|)^\alpha \frac{1}{n} \sum_{k=1}^{n-p+1} \frac{1}{\Delta_n} \int_{(k-1)\Delta_n}^{k\Delta_n} \left| \frac{g^{(k)}_{\Delta_n,\omega_j\Delta_n}(s)}{\Delta_n^{p-1}} - c(i\omega_j) \right|^\alpha \mathrm{d}s \\ &\leq (2m)^\alpha \sum_{j=1}^{m} (|\theta_j| + |\nu_j|)^\alpha \frac{1}{n} \sum_{k=1}^{n-p+1} \left[\frac{1}{\Delta_n} \int_{(k-1)\Delta_n}^{k\Delta_n} \left| \frac{g^{(k)}_{\Delta_n,\omega_j\Delta_n}(s)}{\Delta_n^{p-1}} - h^{(k)}_{\Delta_n,\omega_j}(s) \right|^\alpha \right. \\ &\qquad \left. + \left| h^{(k)}_{\Delta_n,\omega_j}(s) - c(i\omega_j) \right|^\alpha \mathrm{d}s \right]. \end{aligned} \tag{3.35}$$

Then, for any $j \in \{1, \ldots, m\}$,

$$\begin{aligned} &\frac{1}{n} \sum_{k=1}^{n-p+1} \frac{1}{\Delta_n} \int_{(k-1)\Delta_n}^{k\Delta_n} \left| \frac{g^{(k)}_{\Delta_n,\omega_j\Delta_n}(s)}{\Delta_n^{p-1}} - h^{(k)}_{\Delta_n,\omega_j}(s) \right|^\alpha \mathrm{d}s \\ &\qquad \leq \left(\sum_{l=1}^{p} \frac{|c(\lambda_l)|}{|a'(\lambda_l)|} \cdot \left| \frac{f^{(l)}_{\Delta_n}(\omega_j\Delta_n)}{\Delta_n^{p-1}} - \frac{a(i\omega_j)}{i\omega_j - \lambda_l} \right| \right)^\alpha \overset{n\to\infty}{\to} 0 \end{aligned} \tag{3.36}$$

by virtue of Lemma 3.2.1(iii). Moreover,

$$\begin{aligned} &\frac{1}{n} \sum_{k=1}^{n-p+1} \frac{1}{\Delta_n} \int_{(k-1)\Delta_n}^{k\Delta_n} \left| h^{(k)}_{\Delta_n,\omega_j}(s) - c(i\omega_j) \right|^\alpha \mathrm{d}s \\ &\quad = \frac{1}{n} \sum_{k=1}^{n-p+1} \frac{1}{\Delta_n} \int_{(k-1)\Delta_n}^{k\Delta_n} \left| \sum_{l=1}^{p} \frac{c(\lambda_l)}{a'(\lambda_l)} \frac{a(i\omega_j)}{i\omega_j - \lambda_l} \left(\mathrm{e}^{(k\Delta_n - s)\lambda_l} - 1 \right) \right|^\alpha \mathrm{d}s \\ &\quad \leq p^\alpha \sum_{l=1}^{p} \left(\frac{|c(\lambda_l)|}{|a'(\lambda_l)|} \cdot \frac{|a(i\omega_j)|}{|i\omega_j - \lambda_l|} \right)^\alpha \frac{1}{\Delta_n} \int_0^{\Delta_n} \left| \mathrm{e}^{(\Delta_n - s)\lambda_l} - 1 \right|^\alpha \mathrm{d}s \overset{n\to\infty}{\to} 0, \end{aligned} \tag{3.37}$$

where we used Lemma 3.2.1(ii) and (iv). Hence, by Eqs. (3.36) and (3.37) the right-hand side of (3.35) converges to 0 as $n \to \infty$ and thus, (3.34) is shown, as well.

In order to obtain (3.33) and hence,

$$\Delta_n^{1-p-\frac{1}{\alpha}} \left[J^{(2)}_{n,\Delta_n}(\omega_j\Delta_n) \right]_{j=1,\ldots,m} \overset{\mathcal{D}}{\to} \left[c(i\omega_j) \cdot \left(S_j^{\Re}(\underset{\sim}{\omega}) - iS_j^{\Im}(\underset{\sim}{\omega}) \right) \right]_{j=1,\ldots,m}$$

as $n \to \infty$, it remains to prove that

$$\frac{1}{n}\sum_{k=1}^{n-p+1}\left|\sum_{j=1}^{m}\Xi_{\theta_j,\nu_j}\left(\mathrm{e}^{-i\omega_j\Delta_n k}\,c(i\omega_j)\right)\right|^{\alpha} \overset{n\to\infty}{\to} K_{\underset{\sim}{\omega}}\left(\left(\Xi_{\theta_j,\nu_j}\left(c(i\omega_j)\right)\right)_{j\in\{1,\dots,m\}},\ \left(\Xi_{-\nu_j,\theta_j}\left(c(i\omega_j)\right)\right)_{j\in\{1,\dots,m\}},\ 0\right).$$

Since we suppose in particular $n\Delta_n \to \infty$ as $n \to \infty$, this follows from Proposition 3.3.4.

Finally, since also $n\Delta_n^{\alpha(p-1)} \to \infty$ as $n \to \infty$ holds, Proposition 3.4.1 yields for any $\omega \in \mathbb{R}$

$$J^{(1)}_{n,\,\Delta_n}(\omega\Delta_n) + J^{(3)}_{n,\,\Delta_n}(\omega\Delta_n) = o_P\left(\Delta_n^{1/\alpha+p-1}\right)$$

and hence, $\Delta_n^{1-p-1/\alpha}\left[J_{n,\,\widetilde{Z}^{\Delta_n}}(\omega_j\Delta_n)\right]_{j=1,\dots,m} \overset{\mathcal{D}}{\to} \left[c(i\omega_j)\cdot\left(S_j^{\Re}(\underset{\sim}{\omega}) - iS_j^{\Im}(\underset{\sim}{\omega})\right)\right]_{j=1,\dots,m}$ as $n \to \infty$. This completes the proof. □

Proof of Proposition 3.2.4. We immediately obtain

$$\begin{aligned} J_{n,\,Y^{\Delta_n}}(\omega) &= n^{-\frac{1}{\alpha}}\sum_{k=1}^{n} Y_{k\Delta_n}\,\mathrm{e}^{-i\omega k} \overset{(3.9)}{=} n^{-\frac{1}{\alpha}}\sum_{k=1}^{n}\left(\sum_{j=0}^{\infty}\Psi_j^{\Delta_n}\,\widetilde{Z}_{k-j,\,\Delta_n}\right)\mathrm{e}^{-i\omega k} \\ &= n^{-\frac{1}{\alpha}}\sum_{j=0}^{\infty}\Psi_j^{\Delta_n}\,\mathrm{e}^{-i\omega j}\left(\sum_{k=1}^{n}\widetilde{Z}_{k,\,\Delta_n}\,\mathrm{e}^{-i\omega k} + U_{n,\,j,\,\Delta_n}(\omega)\right) \\ &= \Psi^{\Delta_n}(\mathrm{e}^{-i\omega})\,J_{n,\,\widetilde{Z}^{\Delta_n}}(\omega) + W_{n,\,\Delta_n}(\omega), \end{aligned}$$

where

$$U_{n,\,j,\,\Delta_n}(\omega) = \sum_{k=1-j}^{n-j}\widetilde{Z}_{k,\,\Delta_n}\,\mathrm{e}^{-i\omega k} - \sum_{k=1}^{n}\widetilde{Z}_{k,\,\Delta_n}\,\mathrm{e}^{-i\omega k} \qquad \text{and}$$

$$W_{n,\,\Delta_n}(\omega) = n^{-\frac{1}{\alpha}}\sum_{j=0}^{\infty}\Psi_j^{\Delta_n}\,\mathrm{e}^{-i\omega j}\,U_{n,\,j,\,\Delta_n}(\omega).$$

Hence,

$$I_{n,\,Y^{\Delta_n}}(\omega) = \left|\Psi^{\Delta_n}(\mathrm{e}^{-i\omega})\right|^2 I_{n,\,\widetilde{Z}^{\Delta_n}}(\omega) + R_{n,\,\Delta_n}(\omega),$$

with

$$R_{n,\,\Delta_n}(\omega) = \Psi^{\Delta_n}(\mathrm{e}^{-i\omega})\,J_{n,\,\widetilde{Z}^{\Delta_n}}(\omega)\,\overline{W_{n,\,\Delta_n}(\omega)} + \overline{\Psi^{\Delta_n}(\mathrm{e}^{-i\omega})\,J_{n,\,\widetilde{Z}^{\Delta_n}}(\omega)}\,W_{n,\,\Delta_n}(\omega) + |W_{n,\,\Delta_n}(\omega)|^2.$$

For the rest of the proof suppose that Assumption 3.2 holds and fix an arbitrary $\omega \in \mathbb{R}^*$. We have to show that $\Delta_n^{2-2/\alpha}\,|R_{n,\,\Delta_n}(\omega\Delta_n)| \overset{\mathbb{P}}{\to} 0$ as $n \to \infty$.

Since $\Psi^{\Delta_n}(\mathrm{e}^{-i\omega\Delta_n}) \sim \Delta_n^{-p}\, a(i\omega)^{-1}$ as $n \to \infty$ and since in particular $n\Delta_n^{1\vee\alpha(p-1)} \to \infty$ if Assumption 3.2 holds, it follows from Theorem 3.2.5 that

$$\Delta_n^{1-\frac{1}{\alpha}}\, \Psi^{\Delta_n}(\mathrm{e}^{-i\omega\Delta_n})\, J_{n,\widetilde{Z}^{\Delta_n}}(\omega\Delta_n) \xrightarrow{\mathcal{D}} \frac{c(i\omega)}{a(i\omega)}\big(S_1^{\Re}(\omega) - iS_1^{\Im}(\omega)\big) \qquad \text{as } n \to \infty,$$

where the joint characteristic function of $\big(S_1^{\Re}(\omega),\, S_1^{\Im}(\omega)\big)$ is given by Eq. (3.13) (with $m = 1$ and $\tau = 0$). Hence, in order to show $\Delta_n^{2-2/\alpha}|R_{n,\Delta_n}(\omega\Delta_n)| \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$, it is sufficient to prove that

$$\Delta_n^{1-\frac{1}{\alpha}}\, W_{n,\Delta_n}(\omega\Delta_n) \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n \to \infty. \tag{3.38}$$

We shall prove (3.38) by an appropriate decomposition of the sum $W_{n,\Delta_n}(\omega\Delta_n)$, analogously to the one in [57, Proof of Proposition 2.1]. We write

$$\begin{aligned} W_{n,\Delta_n}(\omega\Delta_n) &= n^{-\frac{1}{\alpha}} \sum_{j=n+1}^{\infty} \Psi_j^{\Delta_n} \mathrm{e}^{-i\omega\Delta_n j} U_{n,j,\Delta_n}(\omega\Delta_n) + n^{-\frac{1}{\alpha}} \sum_{j=0}^{n} \Psi_j^{\Delta_n} \mathrm{e}^{-i\omega\Delta_n j} U_{n,j,\Delta_n}(\omega\Delta_n) \\ &=: W_{n,\Delta_n}^{(1)}(\omega\Delta_n) + W_{n,\Delta_n}^{(2)}(\omega\Delta_n) \end{aligned}$$

and

$$\begin{aligned} W_{n,\Delta_n}^{(1)}(\omega\Delta_n) &= n^{-\frac{1}{\alpha}} \sum_{j=n+1}^{\infty} \Psi_j^{\Delta_n}\, \mathrm{e}^{-i\omega\Delta_n j} \Bigg(- \sum_{k=1}^{n} \widetilde{Z}_{k,\Delta_n}\, \mathrm{e}^{-i\omega\Delta_n k} \Bigg) \\ &\qquad + n^{-\frac{1}{\alpha}} \sum_{j=n+1}^{\infty} \Psi_j^{\Delta_n}\, \mathrm{e}^{-i\omega\Delta_n j} \sum_{k=1-j}^{n-j} \widetilde{Z}_{k,\Delta_n}\, \mathrm{e}^{-i\omega\Delta_n k} \\ &=: W_{n,\Delta_n}^{(11)}(\omega\Delta_n) + W_{n,\Delta_n}^{(12)}(\omega\Delta_n). \end{aligned}$$

We have

$$\Delta_n^{1-\frac{1}{\alpha}} \Big|W_{n,\Delta_n}^{(11)}(\omega\Delta_n)\Big| \le \Delta_n^{1-p-\frac{1}{\alpha}}\, \big|J_{n,\widetilde{Z}^{\Delta_n}}(\omega\Delta_n)\big|\, \Delta_n^{p} \sum_{j=n+1}^{\infty} \big|\Psi_j^{\Delta_n}\big|$$

and it is again Theorem 3.2.5 together with the Continuous Mapping Theorem (see, e.g., [54, Theorem 13.25]) showing $\Delta_n^{1-p-1/\alpha}\, |J_{n,\widetilde{Z}^{\Delta_n}}(\omega\Delta_n)| \xrightarrow{\mathcal{D}} |c(i\omega)| \cdot \big|S_1^{\Re}(\omega) - iS_1^{\Im}(\omega)\big|$ as $n \to \infty$. Since we have $\sum_{j=n+1}^{\infty} \big|\Psi_j^{\Delta_n}\big| \to 0$ by virtue of Lemma 3.2.2(ii), we immediately deduce $\Delta_n^{1-1/\alpha}\, W_{n,\Delta_n}^{(11)}(\omega\Delta_n) \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$.

Concerning the term $W_{n,\Delta_n}^{(12)}(\omega\Delta_n)$ we write

$$W_{n,\Delta_n}^{(12)}(\omega\Delta_n) = n^{-\frac{1}{\alpha}} \sum_{j=n+1}^{\infty} \Psi_j^{\Delta_n}\, \mathrm{e}^{-i\omega\Delta_n j} \sum_{k=1-j}^{n-j} \widetilde{Z}_{k,\Delta_n}\, \mathrm{e}^{-i\omega\Delta_n k}$$

$$
\begin{aligned}
&= n^{-\frac{1}{\alpha}} \sum_{k=-n}^{-1} \widetilde{Z}_{k,\,\Delta_n}\, \mathrm{e}^{-i\omega\Delta_n k} \sum_{j=n+1}^{n-k} \Psi_j^{\Delta_n}\, \mathrm{e}^{-i\omega\Delta_n j} \\
&\qquad\qquad + n^{-\frac{1}{\alpha}} \sum_{k=-\infty}^{-n-1} \widetilde{Z}_{k,\,\Delta_n}\, \mathrm{e}^{-i\omega\Delta_n k} \sum_{j=1-k}^{n-k} \Psi_j^{\Delta_n}\, \mathrm{e}^{-i\omega\Delta_n j} \\
&=: W_{n,\,\Delta_n}^{(121)}(\omega\Delta_n) + W_{n,\,\Delta_n}^{(122)}(\omega\Delta_n)
\end{aligned}
$$

and obtain for arbitrary $\varepsilon > 0$

$$
\begin{aligned}
&\mathbb{P}\Big(\Delta_n^{1-\frac{1}{\alpha}} \big|W_{n,\,\Delta_n}^{(121)}(\omega\Delta_n)\big| > \varepsilon\Big) \\
&\quad \le \sum_{r=1}^{p} \mathbb{P}\left(\Delta_n^{1-\frac{1}{\alpha}}\, n^{-\frac{1}{\alpha}} \left| \sum_{k=-n}^{-1} Z_{k-r+1,\,\Delta_n}^{r} \sum_{j=n+1}^{n-k} \Psi_j^{\Delta_n}\, \mathrm{e}^{-i\omega\Delta_n(k+j)} \right| > \frac{\varepsilon}{p}\right) \\
&\quad \le \sum_{r=1}^{p} \left[\mathbb{P}\left(\Delta_n^{1-\frac{1}{\alpha}}\, n^{-\frac{1}{\alpha}} \left| \sum_{k=-n}^{-1} Z_{k-r+1,\,\Delta_n}^{r} \cdot \Re\left(\sum_{j=n+1}^{n-k} \Psi_j^{\Delta_n}\, \mathrm{e}^{-i\omega\Delta_n(k+j)}\right) \right| > \frac{\varepsilon}{2p}\right)\right. \\
&\qquad\qquad \left. + \mathbb{P}\left(\Delta_n^{1-\frac{1}{\alpha}}\, n^{-\frac{1}{\alpha}} \left| \sum_{k=-n}^{-1} Z_{k-r+1,\,\Delta_n}^{r} \cdot \Im\left(\sum_{j=n+1}^{n-k} \Psi_j^{\Delta_n}\, \mathrm{e}^{-i\omega\Delta_n(k+j)}\right) \right| > \frac{\varepsilon}{2p}\right)\right]. \qquad (3.39)
\end{aligned}
$$

Since, for every $r \in \{1, \ldots, p\}$ and $n \in \mathbb{N}^*$, the random variables $Z_{k-r+1,\,\Delta_n}^{r}$, $k \in \{-n, -n+1, \ldots, -1\}$, are independent and symmetric we apply [84, Theorem 1.2] and the right-hand side of (3.39) can be bounded by

$$
4 \sum_{r=1}^{p} \mathbb{P}\left(\Delta_n^{1-\frac{1}{\alpha}}\, n^{-\frac{1}{\alpha}} \sum_{j=n+1}^{2n} \big|\Psi_j^{\Delta_n}\big| \cdot \left| \sum_{k=-n}^{-1} Z_{k-r+1,\,\Delta_n}^{r} \right| > \frac{\varepsilon}{2p}\right). \qquad (3.40)
$$

By virtue of (3.8a), (3.23) and Lemma 3.2.1(i), the characteristic function of $\Delta_n^{1-1/\alpha}\, n^{-1/\alpha} \sum_{j=n+1}^{2n} \big|\Psi_j^{\Delta_n}\big| \cdot \sum_{k=-n}^{-1} Z_{k-r+1,\,\Delta_n}^{r}$ is given by

$$
\begin{aligned}
\Phi(z_1,\, z_2) = \exp\Bigg\{ &- \frac{\sigma_L^{\alpha}\, \Delta_n^{\alpha}}{n\Delta_n} \left(\sum_{j=n+1}^{2n} \big|\Psi_j^{\Delta_n}\big|\right)^{\alpha} \sum_{k=-n}^{-1} \int_{(k-r)\Delta_n}^{(k-r+1)\Delta_n} \Bigg| \Xi_{z_1,\, z_2}\Bigg(\sum_{m=1}^{p} \frac{c(\lambda_m)}{a'(\lambda_m)} \\
&\times \sum_{\{i_1,\ldots,i_{r-1}\} \in \binom{\{1,\,\ldots,\,p\}\setminus\{m\}}{r-1}} \mathrm{e}^{\Delta_n \sum_{h=1}^{r-1} \lambda_{i_h}}\, \mathrm{e}^{((k-r+1)\Delta_n - s)\lambda_m} \Bigg) \Bigg|^{\alpha} \mathrm{d}s \Bigg\}
\end{aligned}
$$

for any $z_1,\, z_2 \in \mathbb{R}$ (see proof of Proposition 3.4.1 for the definition of $\Xi_{z_1,\, z_2}$). We then obtain with $\lambda_{\max} := \max_{k\in\{1,\ldots,p\}} \Re(\lambda_k) < 0$

$$\left| -\frac{\log \Phi(z_1,\, z_2)}{\sigma_L^\alpha} \right| \leq \left(\Delta_n \sum_{j=n+1}^{2n} \left|\Psi_j^{\Delta_n}\right| (|z_1| + |z_2|) \binom{p-1}{r-1} \mathrm{e}^{\Delta_n \lambda_{\max}(r-1)} \sum_{m=1}^{p} \frac{|c(\lambda_m)|}{|a'(\lambda_m)|} \right)^\alpha$$

and the right-hand side converges to 0 as $n \to \infty$ due to Lemma 3.2.2(ii). Thus, (3.40) converges to 0 as well and $\Delta_n^{1-1/\alpha} W_{n,\Delta_n}^{(121)}(\omega\Delta_n) \xrightarrow{\mathbb{P}} 0$ is shown.

In order to get $\Delta_n^{1-1/\alpha} W_{n,\Delta_n}^{(122)}(\omega\Delta_n) \xrightarrow{\mathbb{P}} 0$, we prove, for any $r \in \{1,\,\ldots,\,p\}$,

$$\Delta_n^{1-\frac{1}{\alpha}} n^{-\frac{1}{\alpha}} \sum_{k=-\infty}^{-n-1} Z_{k-r+1,\,\Delta_n}^r \sum_{j=1-k}^{n-k} \Psi_j^{\Delta_n} \mathrm{e}^{-i\omega\Delta_n(k+j)} \xrightarrow{\mathbb{P}} 0.$$

Therefore it is sufficient (using the same arguments as above via characteristic functions) to show that

$$\frac{\Delta_n^\alpha}{n} \sum_{k=-\infty}^{-n-1} \left(\sum_{j=1-k}^{n-k} \left|\Psi_j^{\Delta_n}\right| \right)^\alpha \to 0$$

as $n \to \infty$. This can be found in Lemma 3.2.2(ii) and hence, $\Delta_n^{1-1/\alpha} W_{n,\Delta_n}^{(122)}(\omega\Delta_n) \xrightarrow{\mathbb{P}} 0$. All together we have shown that $\Delta_n^{1-1/\alpha} W_{n,\Delta_n}^{(1)}(\omega\Delta_n)$ converges to 0 in probability.

It remains to prove that also $\Delta_n^{1-1/\alpha} W_{n,\Delta_n}^{(2)}(\omega\Delta_n) \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$. To this end, we define

$$\begin{aligned} W_{n,\Delta_n}^{(21)}(\omega\Delta_n) &:= n^{-\frac{1}{\alpha}} \sum_{j=1}^{n} \Psi_j^{\Delta_n} \mathrm{e}^{-i\omega\Delta_n j} \Bigg[\Bigg(\sum_{k=2-p-j}^{-j} \sum_{r=2-j-k}^{p} + \sum_{k=2-p}^{0} \sum_{r=1}^{1-k} \\ &\qquad - \sum_{k=n+2-p-j}^{n-j} \sum_{r=n+2-j-k}^{p} - \sum_{k=n-p+2}^{n} \sum_{r=1}^{n+1-k} \Bigg) Z_{k,\,\Delta_n}^r \mathrm{e}^{-i\omega\Delta_n(k+r-1)} \Bigg] \\ &=: W_{n,\Delta_n}^{(211)}(\omega\Delta_n) + W_{n,\Delta_n}^{(212)}(\omega\Delta_n) - W_{n,\Delta_n}^{(213)}(\omega\Delta_n) - W_{n,\Delta_n}^{(214)}(\omega\Delta_n) \end{aligned}$$

and write

$$\begin{aligned} &W_{n,\Delta_n}^{(2)}(\omega\Delta_n) \\ &= n^{-\frac{1}{\alpha}} \sum_{j=1}^{n} \Psi_j^{\Delta_n} \mathrm{e}^{-i\omega\Delta_n j} \left(\sum_{k=1-j}^{0} \sum_{r=1}^{p} Z_{k-r+1,\Delta_n}^r \mathrm{e}^{-i\omega\Delta_n k} - \sum_{k=n-j+1}^{n} \sum_{r=1}^{p} Z_{k-r+1,\Delta_n}^r \mathrm{e}^{-i\omega\Delta_n k} \right) \\ &= n^{-\frac{1}{\alpha}} \sum_{j=1}^{n} \Psi_j^{\Delta_n} \mathrm{e}^{-i\omega\Delta_n j} \Bigg(\sum_{k=2-p-j}^{0} \sum_{r=1\vee(2-j-k)}^{p\wedge(1-k)} Z_{k,\,\Delta_n}^r \mathrm{e}^{-i\omega\Delta_n(k+r-1)} \end{aligned}$$

$$- \sum_{k=n+2-p-j}^{n} \sum_{r=1\vee(n+2-j-k)}^{p\wedge(n+1-k)} Z^r_{k,\,\Delta_n} \,\mathrm{e}^{-i\omega\Delta_n(k+r-1)}\Big)$$

$$= W^{(21)}_{n,\,\Delta_n}(\omega\Delta_n) + n^{-\frac{1}{\alpha}} \sum_{j=1}^{n} \Psi^{\Delta_n}_j \,\mathrm{e}^{-i\omega\Delta_n j}\Big[\Big(\sum_{k=1-j}^{1-p} - \sum_{k=n-j+1}^{n-p+1}\Big)\tilde{\tilde{Z}}_{k,\,\Delta_n}(\omega\Delta_n)\,\mathrm{e}^{-i\omega\Delta_n k}\Big]$$

$$=: W^{(21)}_{n,\,\Delta_n}(\omega\Delta_n) + W^{(22)}_{n,\,\Delta_n}(\omega\Delta_n) - W^{(23)}_{n,\,\Delta_n}(\omega\Delta_n).$$

By virtue of Eq. (3.11) we have

$$\Delta_n^{1-\frac{1}{\alpha}} W^{(22)}_{n,\,\Delta_n}(\omega\Delta_n) = \Delta_n^{1-\frac{1}{\alpha}} n^{-\frac{1}{\alpha}} \sum_{k=1-n}^{1-p} \tilde{\tilde{Z}}_{k,\,\Delta_n}(\omega\Delta_n)\,\mathrm{e}^{-i\omega\Delta_n k} \sum_{j=1-k}^{n} \Psi^{\Delta_n}_j \,\mathrm{e}^{-i\omega\Delta_n j}$$

$$= \frac{\Delta_n}{(n\Delta_n)^{1/\alpha}} \sum_{k=1-n}^{1-p} \sum_{j=1-k}^{n} \Psi^{\Delta_n}_j \,\mathrm{e}^{-i\omega\Delta_n(k+j)} \int_{(k-1)\Delta_n}^{k\Delta_n} \sum_{m=1}^{p} \frac{c(\lambda_m)}{a'(\lambda_m)} f^{(m)}_{\Delta_n}(\omega\Delta_n)\,\mathrm{e}^{(k\Delta_n-s)\lambda_m}\,\mathrm{d}L_s.$$

Since, due to Lemma 3.2.1(iii), $f^{(m)}_{\Delta_n}(\omega\Delta_n) \sim \Delta_n^{p-1} a(i\omega)\frac{1}{i\omega-\lambda_m}$ as $n\to\infty$ for every $m \in \{1, \ldots, p\}$, one observes by calculating the characteristic function of $\Delta_n^{1-1/\alpha} W^{(22)}_{n,\Delta_n}(\omega\Delta_n)$ that it is enough to show that

$$\frac{\Delta_n^{\alpha p}}{n} \sum_{k=1-n}^{1-p} \Big(\sum_{j=1-k}^{n} \big|\Psi^{\Delta_n}_j\big|\Big)^{\alpha} \overset{n\to\infty}{\to} 0.$$

This follows immediately from Lemma 3.2.2(iii) and hence also $\Delta_n^{1-1/\alpha} W^{(22)}_{n,\,\Delta_n}(\omega\Delta_n) \overset{\mathbb{P}}{\to} 0$ as $n\to\infty$ holds.

Since the complex $S\alpha S$ random variables $\big(\tilde{\tilde{Z}}_{k,\,\Delta_n}\big)_{k\in\mathbb{Z}}(\omega\Delta_n)$ are i.i.d. (cf. Eq. (3.11)), we easily derive

$$\Delta_n^{1-\frac{1}{\alpha}} W^{(23)}_{n,\,\Delta_n}(\omega\Delta_n) = \mathrm{e}^{-i\omega\Delta_n n} \cdot \Delta_n^{1-\frac{1}{\alpha}} n^{-\frac{1}{\alpha}} \sum_{j=1}^{n} \Psi^{\Delta_n}_j \,\mathrm{e}^{-i\omega\Delta_n j} \sum_{k=1-j}^{1-p} \tilde{\tilde{Z}}_{k+n,\,\Delta_n}(\omega\Delta_n)\,\mathrm{e}^{-i\omega\Delta_n k}$$
$$\overset{\mathcal{D}}{=} \mathrm{e}^{-i\omega\Delta_n n} \cdot \Delta_n^{1-\frac{1}{\alpha}} W^{(22)}_{n,\,\Delta_n}(\omega\Delta_n)$$

and thus $\Delta_n^{1-1/\alpha} W^{(23)}_{n,\,\Delta_n}(\omega\Delta_n) \overset{\mathbb{P}}{\to} 0$ as $n\to\infty$, as well.

Finally, we have to prove that $\Delta_n^{1-1/\alpha} W^{(21)}_{n,\,\Delta_n}(\omega\Delta_n) \overset{\mathbb{P}}{\to} 0$. Therefore, observe that

$$\Delta_n^{1-\frac{1}{\alpha}} W^{(211)}_{n,\,\Delta_n}(\omega\Delta_n)$$
$$= \frac{\Delta_n}{(n\Delta_n)^{1/\alpha}} \sum_{k=2-p-n}^{-1} \mathrm{e}^{-i\omega\Delta_n k} \sum_{j=1\vee(2-p-k)}^{n\wedge(-k)} \Psi^{\Delta_n}_j \,\mathrm{e}^{-i\omega\Delta_n j} \sum_{r=2-j-k}^{p} \mathrm{e}^{-i\omega\Delta_n(r-1)}\, Z^r_{k,\,\Delta_n}$$

$$= \frac{\Delta_n}{(n\Delta_n)^{1/\alpha}} \sum_{k=2-p-n}^{-1} \sum_{j=1\vee(2-p-k)}^{n\wedge(-k)} \Psi_j^{\Delta_n} \, \mathrm{e}^{-i\omega\Delta_n(k+j)}$$
$$\times \int_{(k-1)\Delta_n}^{k\Delta_n} \sum_{m=1}^{p} \frac{c(\lambda_m)}{a'(\lambda_m)} f_{\Delta_n}^{(m;\,2-j-k)}(\omega\Delta_n) \, \mathrm{e}^{(k\Delta_n - s)\lambda_m} \, \mathrm{d}L_s \tag{3.41}$$

(cf. Eq. (3.29)). Using Eq. (3.30) and its upper bound (see proof of Proposition 3.4.1), the joint characteristic function of the right-hand side of (3.41), denoted once more by Φ, satisfies

$$\left| -\frac{\log \Phi(z_1,\, z_2)}{\sigma_L^\alpha} \right| \leq \left(2^{p-1} \left(|z_1| + |z_2| \right) \sum_{m=1}^{p} \frac{|c(\lambda_m)|}{|a'(\lambda_m)|} \right)^{\alpha} \frac{\Delta_n^\alpha}{n} \sum_{k=2-p-n}^{-1} \left(\sum_{j=1\vee(2-p-k)}^{n\wedge(-k)} \left| \Psi_j^{\Delta_n} \right| \right)^{\alpha}.$$

By virtue of Lemma 3.2.2(iv) we then have

$$\frac{\Delta_n^\alpha}{n} \sum_{k=2-p-n}^{-1} \left(\sum_{j=1\vee(2-p-k)}^{n\wedge(-k)} \left| \Psi_j^{\Delta_n} \right| \right)^{\alpha} \overset{n\to\infty}{\to} 0,$$

and hence, $\Delta_n^{1-1/\alpha} \, W_{n,\,\Delta_n}^{(211)}(\omega\Delta_n) \overset{\mathbb{P}}{\to} 0$ as $n \to \infty$.

Likewise, we get

$$\Delta_n^{1-\frac{1}{\alpha}} \, W_{n,\,\Delta_n}^{(212)}(\omega\Delta_n) = \frac{\Delta_n}{(n\Delta_n)^{1/\alpha}} \sum_{k=2-p}^{0} \mathrm{e}^{-i\omega\Delta_n k} \sum_{j=1}^{n} \Psi_j^{\Delta_n} \, \mathrm{e}^{-i\omega\Delta_n j} \sum_{r=1}^{1-k} \mathrm{e}^{-i\omega\Delta_n(r-1)} \, Z_{k,\,\Delta_n}^{r}$$

and, as before, one derives that it is sufficient to show that $\frac{\Delta_n^\alpha}{n} \sum_{k=2-p}^{0} \left(\sum_{j=1}^{n} \left| \Psi_j^{\Delta_n} \right| \right)^{\alpha}$ converges to 0 as $n \to \infty$. This has been done in Lemma 3.2.2(v).

One can show analogously to $W_{n,\,\Delta_n}^{(211)}$ that also $\Delta_n^{1-1/\alpha} \, W_{n,\,\Delta_n}^{(213)}(\omega\Delta_n) \overset{\mathbb{P}}{\to} 0$ and analogously to $W_{n,\,\Delta_n}^{(212)}$ it follows that $\Delta_n^{1-1/\alpha} \, W_{n,\,\Delta_n}^{(214)}(\omega\Delta_n) \overset{\mathbb{P}}{\to} 0$ as $n \to \infty$. This implies $\Delta_n^{1-1/\alpha} \, W_{n,\,\Delta_n}^{(21)}(\omega\Delta_n) \overset{\mathbb{P}}{\to} 0$ and $\Delta_n^{1-1/\alpha} \, W_{n,\,\Delta_n}^{(2)}(\omega\Delta_n) \overset{\mathbb{P}}{\to} 0$ as $n \to \infty$, as well, and the proof is completed. □

3.4.4 Proofs of Section 3.2.3

Proof of Proposition 3.2.9. *(i)* We first observe that the state vector in Eq. (3.14a) can be written as

$$V_{k\Delta_n} = \sum_{j=0}^{\infty} \mathrm{e}^{j\Delta_n A} \xi_{n,\,k-j} \qquad \forall n \in \mathbb{N}^*,\; k \in \mathbb{Z},$$

where $\xi_{n,k} := \int_{(k-1)\Delta_n}^{k\Delta_n} \mathrm{e}^{(k\Delta_n - s)A}\beta \,\mathrm{d}L_s$ (cf. [36, Proof of Lemma 5.4]). Its Beveridge-Nelson decomposition (cf. [9]) is then given by

$$V_{k\Delta_n} = \left(\sum_{j=0}^{\infty} \mathrm{e}^{j\Delta_n A}\right) \xi_{n,k} + \widetilde{V}_{n,k-1} - \widetilde{V}_{n,k} \qquad \forall n \in \mathbb{N}^*,\ k \in \mathbb{Z},$$

with $\widetilde{V}_{n,k} := \sum_{j=0}^{\infty} \big(\sum_{l=j+1}^{\infty} \mathrm{e}^{l\Delta_n A}\big)\xi_{n,k-j}$ (see also [36, Proof of Theorem 4.2]). Hence,

$$\sum_{k=1}^{n} V_{k\Delta_n} = \big(\mathrm{I}_p - \mathrm{e}^{\Delta_n A}\big)^{-1} \sum_{k=1}^{n} \xi_{n,k} + \widetilde{V}_{n,0} - \widetilde{V}_{n,n},$$

where $\widetilde{V}_{n,0} - \widetilde{V}_{n,n} = \big(\mathrm{I}_p - \mathrm{e}^{\Delta_n A}\big)^{-1}\mathrm{e}^{\Delta_n A}(V_0 - V_{n\Delta_n})$. Since $\Delta_n\big(\mathrm{I}_p - \mathrm{e}^{\Delta_n A}\big)^{-1} \overset{n\to\infty}{\to} -A^{-1}$ and $V_0 \overset{\mathcal{D}}{=} V_{n\Delta_n}$ for any $n \in \mathbb{N}^*$, we obviously get $\widetilde{V}_{n,0} - \widetilde{V}_{n,n} = o_P(\Delta_n^{-1}(n\Delta_n)^{1/\alpha})$ as $n \to \infty$. By analog calculations via characteristic functions (as used in the proofs of Theorem 3.2.5 and Proposition 3.4.1), we further obtain $\sum_{k=1}^{n} \xi_{n,k} = \beta \sum_{k=1}^{n} \Delta L(k\Delta_n) + o_P((n\Delta_n)^{1/\alpha})$ as $n \to \infty$. Putting all this together, we have

$$\begin{aligned}
\sum_{k=1}^{n} Y_{k\Delta_n} &\overset{(3.14b)}{=} e_1^T \sum_{k=1}^{n} V_{k\Delta_n} \\
&= e_1^T \big(\mathrm{I}_p - \mathrm{e}^{\Delta_n A}\big)^{-1} \left(\beta \sum_{k=1}^{n} \Delta L(k\Delta_n) + o_P\left((n\Delta_n)^{\frac{1}{\alpha}}\right)\right) + o_P\left(\Delta_n^{-1}(n\Delta_n)^{\frac{1}{\alpha}}\right) \\
&= \sum_{j=0}^{\infty} g(j\Delta_n) \cdot \sum_{k=1}^{n} \Delta L(k\Delta_n) + o_P\left(\Delta_n^{-1}(n\Delta_n)^{\frac{1}{\alpha}}\right) \quad \text{as } n \to \infty
\end{aligned}$$

and *(i)* is shown.

(ii) Let $(0, \Sigma_L, \nu_L)$ denote the characteristic triplet of the underlying Lévy process L. As in the proof of [36, Proposition A.1(c)]), we first factorize the Lévy measure ν_L into two Lévy measures

$$\nu_{L^{(1)}}(A) := \nu_L(A \backslash \{x \in \mathbb{R} : |x| \le 1\}) \quad \text{and} \quad \nu_{L^{(2)}}(A) := \nu_L(A \cap \{x \in \mathbb{R} : |x| \le 1\}),$$

for any Borel set $A \subseteq \mathbb{R}^*$, such that $\nu_L = \nu_{L^{(1)}} + \nu_{L^{(2)}}$. We decompose L into two independent Lévy processes $L = L^{(1)} + L^{(2)}$ where $L^{(1)}$ has characteristic triplet $(0, 0, \nu_{L^{(1)}})$ and $L^{(2)}$ has characteristic triplet $(0, \Sigma_L, \nu_{L^{(2)}})$.

Then one can show, as in the proof of [36, Lemma 5.6], that

$$\sum_{k=1}^{n} V_{k\Delta_n} V_{k\Delta_n}^T = \sum_{j=0}^{\infty} \mathrm{e}^{j\Delta_n A} \left(\sum_{k=1}^{n} \xi_{n,k}^{(1)} \left(\xi_{n,k}^{(1)}\right)^T\right) \mathrm{e}^{j\Delta_n A^T} + o_P\left(\Delta_n^{-1}(n\Delta_n)^{\frac{2}{\alpha}}\right)$$

as $n \to \infty$, where $V_{k\Delta_n}$ is the state vector in Eq. (3.14a), $\xi^{(1)}_{n,k} := \int_{(k-1)\Delta_n}^{k\Delta_n} \mathrm{e}^{(k\Delta_n - s)A} \beta \, \mathrm{d}L^{(1)}_s$ if $\alpha \in (0,2)$ and $\xi^{(1)}_{n,k} := \xi_{n,k}$ if $\alpha = 2$ where $\xi_{n,k} := \int_{(k-1)\Delta_n}^{k\Delta_n} \mathrm{e}^{(k\Delta_n - s)A} \beta \, \mathrm{d}L_s$. Next we claim that, also for $\alpha \in (0,2)$,

$$\sum_{k=1}^{n} \xi^{(1)}_{n,k} \left(\xi^{(1)}_{n,k}\right)^T = \sum_{k=1}^{n} \xi_{n,k} \xi^T_{n,k} + o_P\left((n\Delta_n)^{\frac{2}{\alpha}}\right) \tag{3.42}$$

as $n \to \infty$. Together with $\lim_{n\to\infty} \Delta_n \sum_{j=0}^{\infty} \mathrm{e}^{j\Delta_n A} B_n \, \mathrm{e}^{j\Delta_n A^T} = \int_0^\infty \mathrm{e}^{sA} B \, \mathrm{e}^{sA^T} \mathrm{d}s$ for all matrices B_n, $B \in \mathbb{R}^{p\times p}$ with $\lim_{n\to\infty} B_n = B$, this yields

$$\sum_{k=1}^{n} V_{k\Delta_n} V^T_{k\Delta_n} = \sum_{j=0}^{\infty} \mathrm{e}^{j\Delta_n A} \left(\sum_{k=1}^{n} \xi_{n,k} \xi^T_{n,k} \right) \mathrm{e}^{j\Delta_n A^T} + o_P\left(\Delta_n^{-1} (n\Delta_n)^{\frac{2}{\alpha}}\right) \tag{3.43}$$

as $n \to \infty$. As to (3.42), we observe with $\xi^{(2)}_{n,k} := \xi_{n,k} - \xi^{(1)}_{n,k} = \int_{(k-1)\Delta_n}^{k\Delta_n} \mathrm{e}^{(k\Delta_n - s)A} \beta \, \mathrm{d}L^{(2)}_s$ that

$$\sum_{k=1}^{n} \xi_{n,k} \xi^T_{n,k} = \sum_{k=1}^{n} \xi^{(1)}_{n,k} \left(\xi^{(1)}_{n,k}\right)^T + \sum_{k=1}^{n} \xi^{(1)}_{n,k} \left(\xi^{(2)}_{n,k}\right)^T + \sum_{k=1}^{n} \xi^{(2)}_{n,k} \left(\xi^{(1)}_{n,k}\right)^T + \sum_{k=1}^{n} \xi^{(2)}_{n,k} \left(\xi^{(2)}_{n,k}\right)^T$$

and thus, by virtue of Hölder's Inequality and taking the norm $\|M\| := \|\mathrm{vec}(M)\|$, we obtain

$$\left\| \sum_{k=1}^{n} \xi_{n,k} \xi^T_{n,k} - \sum_{k=1}^{n} \xi^{(1)}_{n,k} \left(\xi^{(1)}_{n,k}\right)^T \right\| \le 2 \left(\sum_{k=1}^{n} \left\| \xi^{(1)}_{n,k} \right\|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^{n} \left\| \xi^{(2)}_{n,k} \right\|^2 \right)^{\frac{1}{2}} + \sum_{k=1}^{n} \left\| \xi^{(2)}_{n,k} \right\|^2 .$$

Note that the second Lévy component $L^{(2)}$ has finite moments of any order (cf. [78, Corollary 25.8]) and hence, we can apply [36, Proposition A.1(a)] and deduce for some $C > 0$ and all sufficiently large n

$$\mathbb{E}\left[(n\Delta_n)^{-\frac{2}{\alpha}} \sum_{k=1}^{n} \left\| \xi^{(2)}_{n,k} \right\|^2 \right] = (n\Delta_n)^{-\frac{2}{\alpha}} \sum_{k=1}^{n} \mathbb{E}\left[\left\| \xi^{(2)}_{n,k} \right\|^2 \right] \le C \cdot (n\Delta_n)^{1-\frac{2}{\alpha}},$$

where the right-hand side converges to 0, since we suppose $n\Delta_n \to \infty$ and $1 - 2/\alpha < 0$ for any $\alpha \in (0,2)$. We further obtain by combining [36, Proposition A.2(a,c)] and [74, Theorem 7.1] that $(n\Delta_n)^{-2/\alpha} \sum_{k=1}^{n} \left\| \xi^{(1)}_{n,k} \right\|^2$ converges weakly as $n \to \infty$ (note that $L^{(1)}$ is a compound Poisson process). This completes the proof of (3.42) and hence also Eq. (3.43) is shown.

Now also

$$\sum_{k=1}^{n} \xi_{n,k}\,\xi_{n,k}^T = \beta \sum_{k=1}^{n} \Delta L(k\Delta_n)^2\,\beta^T + o_P\Big((n\Delta_n)^{\frac{2}{\alpha}}\Big) \quad \text{as } n\to\infty \tag{3.44}$$

holds. For, the (i,j)-th component of $\sum_{k=1}^n \xi_{n,k}\,\xi_{n,k}^T - \beta\sum_{k=1}^n \Delta L(k\Delta_n)^2\,\beta^T$ can be bounded, again due to Hölder's Inequality, by

$$\begin{aligned}\left|\left[\sum_{k=1}^{n} \xi_{n,k}\,\xi_{n,k}^T - \beta \sum_{k=1}^{n} \Delta L(k\Delta_n)^2\,\beta^T\right]_{i,j}\right| &\le \left(\sum_{k=1}^{n} [\xi_{n,k}]_i^2 \sum_{k=1}^{n}\Big([\xi_{n,k}]_j - \beta_j\,\Delta L(k\Delta_n)\Big)^2\right)^{\frac{1}{2}}\\ &+ \left(\sum_{k=1}^{n}\Big(\beta_j\,\Delta L(k\Delta_n)\Big)^2 \sum_{k=1}^{n}\Big([\xi_{n,k}]_i - \beta_i\,\Delta L(k\Delta_n)\Big)^2\right)^{\frac{1}{2}}\end{aligned}$$

with $\big[\xi_{n,k}\big]_i$ and β_j being the i-th and the j-th component of $\xi_{n,k}$ and β, respectively. Similar arguments as used above for $\sum_{k=1}^n \big\|\xi_{n,k}^{(1)}\big\|^2$ yield that $(n\Delta_n)^{-2/\alpha}\sum_{k=1}^n [\xi_{n,k}]_i^2$ as well as $(n\Delta_n)^{-2/\alpha}\sum_{k=1}^n \big(\beta_j\,\Delta L(k\Delta_n)\big)^2$ converge weakly to positive $\alpha/2$-stable random variables. In order to obtain Eq. (3.44), it hence remains to prove that, for any $i\in\{1,\ldots,p\}$, the sum $(n\Delta_n)^{-2/\alpha}\sum_{k=1}^n \big([\xi_{n,k}]_i - \beta_i\,\Delta L(k\Delta_n)\big)^2$ converges to 0 in probability. This is indeed true, since the random variables $[\xi_{n,k}]_i - \beta_i\,\Delta L(k\Delta_n)$, $k\in\{1,\ldots,n\}$, are i.i.d. symmetric α-stable with scale parameter $\sigma_L\big(\int_0^{\Delta_n}\big|e_i^T(\mathrm{e}^{(\Delta_n-s)A}-\mathrm{I}_p)\beta\big|^\alpha\,\mathrm{d}s\big)^{1/\alpha}$ and $\Delta_n^{-1}\int_0^{\Delta_n}\big|e_i^T(\mathrm{e}^{(\Delta_n-s)A}-\mathrm{I}_p)\beta\big|^\alpha\,\mathrm{d}s \to 0$ as $n\to\infty$ (cf. Lemma 3.2.1(ii)). We thus deduce

$$\begin{aligned}\sum_{k=1}^{n} Y_{k\Delta_n}^2 &\overset{(3.14b)}{=} e_1^T\left(\sum_{k=1}^{n} V_{k\Delta_n}V_{k\Delta_n}^T\right)e_1\\ &\overset{(3.43)}{=} e_1^T\left(\sum_{j=0}^{\infty}\mathrm{e}^{j\Delta_n A}\left(\sum_{k=1}^{n}\xi_{n,k}\xi_{n,k}^T\right)\mathrm{e}^{j\Delta_n A^T}\right)e_1 + o_P\Big(\Delta_n^{-1}(n\Delta_n)^{\frac{2}{\alpha}}\Big)\\ &\overset{(3.44)}{=} e_1^T\left(\sum_{j=0}^{\infty}\mathrm{e}^{j\Delta_n A}\left(\beta\sum_{k=1}^{n}\Delta L(k\Delta_n)^2\,\beta^T + o_P\Big((n\Delta_n)^{\frac{2}{\alpha}}\Big)\right)\mathrm{e}^{j\Delta_n A^T}\right)e_1\\ &\qquad + o_P\Big(\Delta_n^{-1}(n\Delta_n)^{\frac{2}{\alpha}}\Big)\\ &= \sum_{j=0}^{\infty} g^2(j\Delta_n)\cdot\sum_{k=1}^{n}\Delta L(k\Delta_n)^2 + o_P\Big(\Delta_n^{-1}(n\Delta_n)^{\frac{2}{\alpha}}\Big) \quad \text{as } n\to\infty\end{aligned}$$

and *(ii)* is shown. □

Proof of Theorem 3.2.10. Assume that $c_q \neq 0$. By virtue of [24, Lemma 3.1], the integrated kernel function $\int_0^\infty g(s)\,\mathrm{d}s$ is equal to $\int_0^\infty e_1^T\mathrm{e}^{sA}\beta\,\mathrm{d}s = -e_1^T A^{-1}\beta = c_q\,a_p^{-1}$.

Due to Proposition 3.2.4 we immediately obtain, for any $\omega \in \mathbb{R}^*$ and n sufficiently large

$$\widetilde{I}_{n,\,Y^{\Delta_n}}(\omega\Delta_n) = \left|\Psi^{\Delta_n}(\mathrm{e}^{-i\omega\Delta_n})\right|^2 \frac{I_{n,\,\widetilde{Z}^{\Delta_n}}(\omega\Delta_n)}{\left(n^{-1/\alpha}\sum_{k=1}^n Y_{k\Delta_n}\right)^2} + \widetilde{R}_{n,\,\Delta_n}(\omega\Delta_n)$$

with $\widetilde{R}_{n,\Delta_n}(\omega\Delta_n) = R_{n,\Delta_n}(\omega\Delta_n)\left(n^{-1/\alpha}\sum_{k=1}^n Y_{k\Delta_n}\right)^{-2}$. Since $R_{n,\Delta_n}(\omega\Delta_n) = o_P\left(\Delta_n^{2/\alpha-2}\right)$ as $n \to \infty$ (see again Proposition 3.2.4) and since

$$\left(\frac{\Delta_n}{(n\Delta_n)^{1/\alpha}}\sum_{k=1}^n Y_{k\Delta_n}\right)^2 \xrightarrow{\mathcal{D}} \left(\int_0^\infty g(s)\,\mathrm{d}s\right)^2 \cdot S^2 = \frac{c_q^2}{a_p^2}\cdot S^2$$

as $n \to \infty$ with S being a $S\alpha S$ random variable with scale parameter σ_L (cf. [36, Theorem 3.1(a)]), we have

$$\widetilde{R}_{n,\,\Delta_n}(\omega\Delta_n) = o_P(1) \quad \text{as } n \to \infty. \tag{3.45}$$

Since $\left|\Psi^{\Delta_n}(\mathrm{e}^{-i\omega\Delta_n})\right|^2 \sim \Delta_n^{-2p}\,|a(i\omega)|^{-2}$ and $\Delta_n\sum_{j=0}^\infty g(j\Delta_n) \to \int_0^\infty g(s)\,\mathrm{d}s$ as $n \to \infty$, we combine Eq. (3.45), Proposition 3.4.1 and Proposition 3.2.9(i), and observe that, in order to show Theorem 3.2.10, it remains to prove

$$\left(\Delta_n^{1-p-\frac{1}{\alpha}}\left[J^{(2)}_{n,\,\widetilde{Z}^{\Delta_n}}(\omega_j\Delta_n)\right]_{j\in\{1,\,\dots,\,m\}},\ (n\Delta_n)^{-\frac{1}{\alpha}}\sum_{k=1}^n \Delta L(k\Delta_n)\right)$$
$$\xrightarrow{\mathcal{D}} \left(\left[c(i\omega_j)\cdot\left(S_j^{\Re}(\underset{\sim}{\omega}) - iS_j^{\Im}(\underset{\sim}{\omega})\right)\right]_{j\in\{1,\,\dots,\,m\}},\ S_{m+1}(\underset{\sim}{\omega})\right)$$

as $n \to \infty$ and to apply the Continuous Mapping Theorem (see, e.g., [54, Theorem 13.25]). However, this weak convergence result can be shown along the lines of the proof of Theorem 3.2.5. □

Proof of Theorem 3.2.11. Assume w.l.o.g. that $\int_0^\infty g^2(s)\,\mathrm{d}s \neq 0$ (the CARMA process would be trivial otherwise). Furthermore, we obtain as in the proof of Theorem 3.2.10 for all sufficiently large n

$$\widehat{I}_{n,\,Y^{\Delta_n}}(\omega\Delta_n) = \left|\Psi^{\Delta_n}(\mathrm{e}^{-i\omega\Delta_n})\right|^2 \frac{I_{n,\,\widetilde{Z}^{\Delta_n}}(\omega\Delta_n)}{n^{-2/\alpha}\sum_{k=1}^n Y_{k\Delta_n}^2} + \widehat{R}_{n,\,\Delta_n}(\omega\Delta_n)$$

with $\widehat{R}_{n,\Delta_n}(\omega\Delta_n) = R_{n,\Delta_n}(\omega\Delta_n)\left(n^{-2/\alpha}\sum_{k=1}^n Y_{k\Delta_n}^2\right)^{-1}$. Since $R_{n,\Delta_n}(\omega\Delta_n) = o_P\left(\Delta_n^{2/\alpha-2}\right)$ as $n \to \infty$ (see Proposition 3.2.4) and since $\Delta_n\,(n\Delta_n)^{-2/\alpha}\sum_{k=1}^n Y_{k\Delta_n}^2 \xrightarrow{\mathcal{D}} \int_0^\infty g^2(s)\,\mathrm{d}s \cdot [L,\,L]_1$ as $n \to \infty$ with $([L,\,L]_t)_{t\geq 0}$ being the quadratic variation process of $(L_t)_{t\geq 0}$ (cf.

[36, Theorem 3.6(a)]), we get

$$\Delta_n \, \widehat{R}_{n,\Delta_n}(\omega\Delta_n) = o_P(1) \quad \text{as } n \to \infty. \tag{3.46}$$

Since $\left|\Psi^{\Delta_n}(\mathrm{e}^{-i\omega\Delta_n})\right|^2 \sim \Delta_n^{-2p} \, |a(i\omega)|^{-2}$ and $\Delta_n \sum_{j=0}^{\infty} g^2(j\Delta_n) \to \int_0^\infty g^2(s)\,\mathrm{d}s$ as $n \to \infty$, we combine (3.46), Proposition 3.4.1 and Proposition 3.2.9(ii), and observe that

$$\Delta_n \, \widehat{I}_{n,Y^{\Delta_n}}(\omega\Delta_n) = |a(i\omega)|^{-2} \left(\int_0^\infty g^2(s)\,\mathrm{d}s \right)^{-1} \cdot \frac{\Delta_n^{2-2p-\frac{2}{\alpha}} \left| J^{(2)}_{n,\Delta_n}(\omega\Delta_n) \right|^2}{(n\Delta_n)^{-2/\alpha} \sum_{k=1}^n \Delta L(k\Delta_n)^2} \cdot \big(1 + o_P(1)\big) \tag{3.47}$$

as $n \to \infty$. In the proof of Theorem 3.2.5 it has been shown that, for any $\omega \in \mathbb{R}^*$,

$$\Delta_n^{1-p-\frac{1}{\alpha}} \, J^{(2)}_{n,\Delta_n}(\omega\Delta_n) - \frac{c(i\omega)}{(n\Delta_n)^{1/\alpha}} \sum_{k=1}^n \Delta L(k\Delta_n)\, \mathrm{e}^{-i\omega\Delta_n k} \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty$$

(cf. Eqs. (3.31), (3.32) and (3.35)-(3.37)). Hence, (3.47) becomes

$$\Delta_n \, \widehat{I}_{n,Y^{\Delta_n}}(\omega\Delta_n) = \frac{|c(i\omega)|^2}{\int_0^\infty g^2(s)\,\mathrm{d}s \cdot |a(i\omega)|^2} \cdot \frac{\left| \sum_{k=1}^n \Delta L(k\Delta_n)\, \mathrm{e}^{-i\omega\Delta_n k} \right|^2}{\sum_{k=1}^n \Delta L(k\Delta_n)^2} \cdot \big(1 + o_P(1)\big)$$

as $n \to \infty$. We introduce an i.i.d. sequence $(Z_k)_{k\in\mathbb{N}^*}$ of symmetric α-stable random variables with scale parameter σ_L and observe that $\big(\Delta L(k\Delta_n)\big)_{k\in\mathbb{N}^*} \overset{\mathcal{D}}{=} \Delta_n^{1/\alpha} \, (Z_k)_{k\in\mathbb{N}^*}$. Consequently, to finish the proof of Theorem 3.2.11, it is sufficient to show that

$$\left[\frac{\left| \sum_{k=1}^n Z_k \, \mathrm{e}^{-i\omega_j \Delta_n k} \right|^2}{\sum_{k=1}^n Z_k^2} \right]_{j\in\{1,\ldots,m\}} \xrightarrow{\mathcal{D}} \left[\frac{\left[S_j^{\Re}(\underset{\sim}{\omega}) \right]^2 + \left[S_j^{\Im}(\underset{\sim}{\omega}) \right]^2}{S^2} \right]_{j\in\{1,\ldots,m\}} \quad \text{as } n \to \infty. \tag{3.48}$$

Since $n^{-2/\alpha} \left| \sum_{k=1}^n Z_k \mathrm{e}^{-i\omega_j\Delta_n k} \right|^2 \xrightarrow{\mathcal{D}} \left[S_j^{\Re}(\underset{\sim}{\omega}) \right]^2 + \left[S_j^{\Im}(\underset{\sim}{\omega}) \right]^2$ as $n \to \infty$, which follows implicitly from the proofs of Proposition 3.3.4 and Theorem 3.2.5, and since $n^{-2/\alpha} \sum_{k=1}^n Z_k^2 \xrightarrow{\mathcal{D}} S^2$ as $n \to \infty$ with S^2 being a positive $\alpha/2$-stable random variable, which can be easily derived from, e.g., [74, Theorem 7.1], we will show that also the random vector

$$\big(\gamma^2_{n,Z}, \, \alpha^2_{n,Z}(\omega_j\Delta_n), \, \beta^2_{n,Z}(\omega_j\Delta_n) \big)_{j\in\{1,\ldots,m\}}, \tag{3.49}$$

with

$$\gamma^2_{n,Z} := n^{-\frac{2}{\alpha}} \sum_{k=1}^n Z_k^2, \quad \alpha_{n,Z}(\omega_j\Delta_n) := n^{-\frac{1}{\alpha}} \sum_{k=1}^n Z_k \, \cos(\omega_j\Delta_n k) \quad \text{and}$$

$$\beta_{n,Z}(\omega_j\Delta_n) := n^{-\frac{1}{\alpha}} \sum_{k=1}^n Z_k \, \sin(\omega_j\Delta_n k),$$

converges weakly. Note that this implies Eq. (3.48).

We take the same approach as in the proof of [58, Proposition 2.2] (which can be found in [56]). Let $(N_k)_{k\in\mathbb{N}^*}$, $P_1, P_2, \dots, P_m, M_1, M_2, \dots, M_m$ be i.i.d. standard normal random variables, independent of $(Z_k)_{k\in\mathbb{N}^*}$. Then, with $\varphi \geq 0$ and $\underset{\sim}{\theta}, \underset{\sim}{\nu} \in [0,\infty)^m$, the Laplace transform of the random vector in (3.49) is given by

$$\begin{aligned}
&f_{n,\Delta_n}(\varphi, \underset{\sim}{\theta}, \underset{\sim}{\nu}) \\
&= \mathbb{E}\left[\exp\left\{-\frac{\varphi^2}{2}\gamma^2_{n,Z} - \sum_{j=1}^{m}\left(\frac{\theta_j^2}{2}\alpha^2_{n,Z}(\omega_j\Delta_n) + \frac{\nu_j^2}{2}\beta^2_{n,Z}(\omega_j\Delta_n)\right)\right\}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\exp\left\{i\varphi n^{-\frac{1}{\alpha}}\sum_{k=1}^{n} Z_k N_k + i\sum_{j=1}^{m}\left(\theta_j P_j \alpha_{n,Z}(\omega_j\Delta_n) + \nu_j M_j \beta_{n,Z}(\omega_j\Delta_n)\right)\right\}\middle|(Z_k)_{k\in\mathbb{N}^*}\right]\right] \\
&= \mathbb{E}\left[\exp\left\{i\varphi n^{-\frac{1}{\alpha}}\sum_{k=1}^{n} Z_k\, N_k + i\sum_{j=1}^{m}\left(\theta_j\, P_j\, \alpha_{n,Z}(\omega_j\Delta_n) + \nu_j\, M_j\, \beta_{n,Z}(\omega_j\Delta_n)\right)\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{in^{-\frac{1}{\alpha}}\sum_{k=1}^{n} Z_k\left(\varphi\, N_k + \sum_{j=1}^{m}\left(\theta_j\, P_j\, \cos(\omega_j\Delta_n k) + \nu_j\, M_j\, \sin(\omega_j\Delta_n k)\right)\right)\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{in^{-\frac{1}{\alpha}}\, Z_1\left(\sum_{k=1}^{n}\left|\varphi\, N_k + \sum_{j=1}^{m}\left(\theta_j\, P_j\, \cos(\omega_j\Delta_n k) + \nu_j\, M_j\, \sin(\omega_j\Delta_n k)\right)\right|^{\alpha}\right)^{\frac{1}{\alpha}}\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{-\frac{\sigma_L^\alpha}{n}\sum_{k=1}^{n}\left|\varphi\, N_k + \sum_{j=1}^{m}\left(\theta_j\, P_j\, \cos(\omega_j\Delta_n k) + \nu_j\, M_j\, \sin(\omega_j\Delta_n k)\right)\right|^{\alpha}\right\}\right] \\
&=: \mathbb{E}\left[\exp\left\{-\sigma_L^\alpha \cdot K_{n,\Delta_n}(\varphi, \underset{\sim}{\theta}, \underset{\sim}{\nu})\right\}\right]
\end{aligned}$$

with $K_{n,\Delta_n}(\varphi, \underset{\sim}{\theta}, \underset{\sim}{\nu}) := \frac{1}{n}\sum_{k=1}^{n}\left|\varphi\, N_k + \sum_{j=1}^{m}\left(\theta_j\, P_j\, \cos(\omega_j\Delta_n k) + \nu_j\, M_j\, \sin(\omega_j\Delta_n k)\right)\right|^{\alpha}$. We define the function $h(x, y) := \left|\varphi\, y + \sum_{j=1}^{m}\left(\theta_j\, P_j\, \cos(2\pi x_j) + \nu_j\, M_j\, \sin(2\pi x_j)\right)\right|^{\alpha}$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}$. Note that h satisfies the assumptions of Proposition 3.3.5 for every realization of $\underset{\sim}{P} = (P_1, \dots, P_m)^T$ and $\underset{\sim}{M} = (M_1, \dots, M_m)^T$.

Now, if $\omega_1, \dots, \omega_m$ are linearly independent over $\mathbb{Z}$ we obtain by virtue of Proposition 3.3.5

$$\begin{aligned}
&f_{n,\Delta_n}(\varphi, \underset{\sim}{\theta}, \underset{\sim}{\nu}) \overset{n\to\infty}{\to} \mathbb{E}\left[\exp\left\{-\sigma_L^\alpha \cdot \mathbb{E}\left[h(\underset{\sim}{U}, N_1) \mid \underset{\sim}{P}, \underset{\sim}{M}\right]\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{-\sigma_L^\alpha \cdot \mathbb{E}\left[\left|\varphi\, N_1 + \sum_{j=1}^{m}\left(\theta_j\, P_j\, \cos(2\pi U_j) + \nu_j\, M_j\, \sin(2\pi U_j)\right)\right|^{\alpha}\middle|\, \underset{\sim}{P}, \underset{\sim}{M}\right]\right\}\right] \\
&=: f(\varphi, \underset{\sim}{\theta}, \underset{\sim}{\nu}). \qquad (3.50)
\end{aligned}$$

Here $U_1, \dots, U_m$ are i.i.d. $[0,1)$-uniform random variables independent of $P_1, \dots, P_m$, $M_1, \dots, M_m$ and N_1.

If $\omega_1, \dots, \omega_m$ are linearly dependent over $\mathbb{Z}$, then also by virtue of Proposition 3.3.5 $f_{n,\Delta_n}(\varphi, \underset{\sim}{\theta}, \underset{\sim}{\nu}) \to f(\varphi, \underset{\sim}{\theta}, \underset{\sim}{\nu})$ as $n \to \infty$ but now $\underset{\sim}{U} = T(V_1, \dots, V_{m-s})$ with T being the parametrization of the $(m-s)$-dimensional manifold $\mathscr{M}(\omega_1, \dots, \omega_m)$ (cf. (3.17)) and $V_1, \dots, V_{m-s}$ are i.i.d. $[0,1)$-uniform random variables independent of $\underset{\sim}{P}$, $\underset{\sim}{M}$ and N_1.

Hence, in both cases the Laplace transform $f_{n,\Delta_n}(\varphi, \underset{\sim}{\theta}, \underset{\sim}{\nu})$ of the random vector (3.49) converges to a function that is continuous in the origin. This implies that the random vector $\big(\gamma^2_{n,Z}, \alpha^2_{n,Z}(\omega_j\Delta_n), \beta^2_{n,Z}(\omega_j\Delta_n)\big)_{j\in\{1,\dots,m\}}$ converges weakly and completes the proof. $\square$

3.4.5 Proofs of Section 3.3

Proof of Theorem 3.3.3. We identify the equivalence classes in $(\mathbb{R} \bmod 1)^{m-s}$ and $(\mathbb{R} \bmod 1)^m$, respectively, by their representatives in $[0,1)^{m-s}$ and $[0,1)^m$.

(i) Define

$$N := \big\{x = (x_1, \dots, x_{m-s})^T \in [0,1)^{m-s} : \exists j \in \{1, \dots, m-s\},\, i \in \{1, \dots, m\} \\ \text{such that } x_j = k \cdot \big|b_j^{(i)}\big|^{-1} \text{ for some } k \in \{0, 1, \dots, \big|b_j^{(i)}\big| - 1\}\big\},$$

where $b_j^{(i)}$ denotes the i-th component of the vector b_j. Clearly $\mathscr{H}^{m-s}(T(N)) = 0$ and $T|_{[0,1)^{m-s}\setminus N}$ is continuously differentiable with $\operatorname{rank}(D\,T|_{[0,1)^{m-s}\setminus N}(x)) = \operatorname{rank}(B) = m-s$ for all $x \in [0,1)^{m-s}\setminus N$. Moreover, T is injective. The reason is the following. Suppose that $T(x_1, \dots, x_{m-s}) = T(y_1, \dots, y_{m-s})$ for some $(x_1, \dots, x_{m-s})^T$, $(y_1, \dots, y_{m-s})^T \in [0,1)^{m-s}$. Then

$$\left(\sum_{j=1}^{m-s} x_j b_j\right) \bmod 1 = \left(\sum_{j=1}^{m-s} y_j b_j\right) \bmod 1 \quad \iff \quad \sum_{j=1}^{m-s} (x_j - y_j) b_j \in \mathbb{Z}^m.$$

Since $\sum_{j=1}^{m-s}(x_j - y_j)b_j \in \operatorname{span}^{\mathbb{R}}(\{b_1, \dots, b_{m-s}\}) \cap \mathbb{Z}^m \subseteq \widetilde{\mathscr{L}^\perp} \cap \mathbb{Z}^m = \mathscr{L}$, there exist integers z_j, $j \in \{1, \dots, m-s\}$, such that $\sum_{j=1}^{m-s}(x_j - y_j - z_j)b_j = 0$ and hence, $(x_j - y_j) = z_j \in \mathbb{Z}$ for all $j \in \{1, \dots, m-s\}$. Since $x_j - y_j \in (-1,1)$ we must have $x_j = y_j$ for all $j \in \{1, \dots, m-s\}$. This shows that T is indeed injective. Note that T^{-1} is continuous (mod 1) on $\mathscr{M}$ and thus, $T([0,1)^{m-s}\setminus N)$ is an $(m-s)$-dimensional C^1-manifold in $[0,1)^m$ (for a definition of manifolds, see, e.g., [67, pp. 200-201]). Since $\mathscr{H}^{m-s}(T(N)) = 0$, also $\mathscr{M}$ is an $(m-s)$-dimensional C^1-manifold and integration over $\mathscr{M}$ is the same as integration over $T([0,1)^{m-s}\backslash N) = \mathscr{M}\backslash T(N)$ (note that $T(N)$ itself is a manifold in $[0,1)^m$ from lower dimension than $m-s$).

(ii) Suppose there is a $z = (z_1, \ldots, z_{m-s})^T \in \mathbb{Z}^{m-s}$, $z \neq 0$, such that $\langle z, \underset{\sim}{\mu}\rangle = 0$. W.l.o.g. $z_1 \neq 0$. Then

$$\mu_1 = -\sum_{i=2}^{m-s} \frac{z_i}{z_1}\mu_i \qquad \text{and} \qquad \underset{\sim}{\eta} = \sum_{i=2}^{m-s} \mu_i \cdot \left(-\frac{z_i}{z_1} b_1 + b_i\right).$$

The vectors $\widetilde{b}_i := -\frac{z_i}{z_1} b_1 + b_i \in \mathbb{Q}^m$, $i = 2, \ldots, m-s$, are obviously linearly independent. Thus,

$$\left(\operatorname{span}^{\mathbb{R}}\left\{\widetilde{b}_2, \ldots, \widetilde{b}_{m-s}\right\}\right)^{\perp} \subseteq \{\underset{\sim}{\eta}\}^{\perp} \Rightarrow \left(\operatorname{span}^{\mathbb{R}}\left\{\widetilde{b}_2, \ldots, \widetilde{b}_{m-s}\right\}\right)^{\perp} \cap \mathbb{Z}^m \subseteq \{\underset{\sim}{\eta}\}^{\perp} \cap \mathbb{Z}^m = \widetilde{\mathscr{L}},$$

and since the dimension of $\widetilde{\mathscr{L}}$ is s whereas the dimension of $\operatorname{span}^{\mathbb{R}}\{\widetilde{b}_2, \ldots, \widetilde{b}_{m-s}\}^{\perp} \cap \mathbb{Z}^m$ is $s+1$ (the latter can be obtained as in the proof of $\dim(\mathscr{L}) = m-s$ on p. 52), we have a contradiction. Hence, $\langle z, \underset{\sim}{\mu}\rangle \neq 0$ for all $z \in \mathbb{Z}^{m-s}$, $z \neq 0$.

(iii) We have, with $h = B\,z$ and $z \in \mathbb{Z}^{m-s}$, $z \neq 0$,

$$\begin{aligned}\frac{1}{\mathscr{H}^{m-s}(\mathscr{M})} \int_{\mathscr{M}} f_h(x)\, \mathscr{H}^{m-s}(\mathrm{d}x) &= \int_{[0,1)^{m-s}} f_h(T(x))\, \mathrm{d}x = \int_{[0,1)^{m-s}} \mathrm{e}^{2\pi i \langle h,\, T(G^{-1}x)\rangle}\, \mathrm{d}x \\ &= \int_{[0,1)^{m-s}} \mathrm{e}^{2\pi i \langle h,\, BG^{-1}x \mod 1\rangle}\, \mathrm{d}x = \int_{[0,1)^{m-s}} \mathrm{e}^{2\pi i \langle z,\, B^T BG^{-1}x\rangle}\, \mathrm{d}x \\ &= \prod_{j=1}^{m-s} \int_0^1 \mathrm{e}^{2\pi i\, z_j x_j}\, \mathrm{d}x_j. \end{aligned} \tag{3.51}$$

Since $z \neq 0$ there is a $j \in \{1, \ldots, m-s\}$ with $z_j \in \mathbb{Z}\backslash\{0\}$, and the right-hand side of (3.51) has to be zero.

(iv) Let $T(x), T(y) \in \mathscr{M}$, $T(x) \neq T(y)$. Since T is injective, there is some $j_0 \in \{1, \ldots, m-s\}$ such that $x_{j_0} \neq y_{j_0}$. For $h = B\,e_{j_0} = b_{j_0}$ we have

$$f_h(T(x)) \cdot f_h(T(y))^{-1} = \mathrm{e}^{2\pi i \langle b_{j_0},\, T(G^{-1}x) - T(G^{-1}y)\rangle} = \mathrm{e}^{2\pi i \langle Be_{j_0},\, BG^{-1}(x-y)\rangle} = \mathrm{e}^{2\pi i\,(x_{j_0} - y_{j_0})} \neq 1,$$

since $x_{j_0} - y_{j_0} \in (-1,1)\backslash\{0\}$. □

Proof of Proposition 3.3.4. Letting $\underset{\sim}{\omega} = (\omega_1, \ldots, \omega_m)^T = 2\pi(\eta_1, \ldots, \eta_m)^T = 2\pi\underset{\sim}{\eta}$, we immediately get

$$\frac{1}{n} \sum_{k=1}^{n-p+1} \left| \sum_{j=1}^{m} \Xi_{\theta_j, \nu_j}\left(\mathrm{e}^{-i\omega_j \Delta_n k}\, c(i\omega_j)\right) \right|^{\alpha}$$

$$\overset{n\to\infty}{\sim} \frac{1}{n}\sum_{k=1}^{n}\left|\sum_{j=1}^{m}\cos\left(2\pi\{\eta_j\Delta_n k\}\right)\Xi_{\theta_j,\nu_j}\big(c(i\omega_j)\big)+\sin\left(2\pi\{\eta_j\Delta_n k\}\right)\Xi_{-\nu_j,\theta_j}\big(c(i\omega_j)\big)\right|^{\alpha}.$$

Let us first consider the case where $\omega_1, \ldots, \omega_m$ are linearly independent over $\mathbb{Z}$. We claim that, for any $h \in \mathbb{Z}^m$, $h \neq 0$,

$$\frac{1}{n}\sum_{k=1}^{n} \mathrm{e}^{2\pi i\langle h,\, \underset{\sim}{\eta}\rangle \Delta_n k} \to 0 \quad \text{as } n \to \infty. \tag{3.52}$$

To this end, note that for n sufficiently large

$$\left|\frac{1}{n}\sum_{k=1}^{n} \mathrm{e}^{2\pi i\langle h,\, \underset{\sim}{\eta}\rangle \Delta_n k}\right| = \frac{1}{n}\cdot\frac{\left|\mathrm{e}^{2\pi i\langle h,\, \underset{\sim}{\eta}\rangle \Delta_n n}-1\right|}{\left|\mathrm{e}^{2\pi i\langle h,\, \underset{\sim}{\eta}\rangle \Delta_n}-1\right|} \leq \frac{1}{\left|\langle h,\, \underset{\sim}{\eta}\rangle\right|}\cdot\frac{1}{n\Delta_n}$$

and the right-hand side converges to 0 as $n \to \infty$ since $n\Delta_n \to \infty$ by assumption and since $\omega_1, \ldots, \omega_m$ are supposed to be linearly independent over $\mathbb{Z}$.

However, (3.52) already implies that

$$\frac{1}{n}\sum_{k=1}^{n} f(\Delta_n k\, \underset{\sim}{\eta}) \overset{n\to\infty}{\to} \int_{[0,1)^m} f(x)\,\mathrm{d}x \tag{3.53}$$

for any continuous function $f : \mathbb{R}^m \to \mathbb{C}$ with period 1 in each component variable (more precisely, f should be seen as a function, mapping from the compact Hausdorff space $(\mathbb{R} \bmod 1)^m$ to the complex numbers). An explanation is the following. If we fix $\varepsilon > 0$, we know from the Weierstrass Approximation Theorem (cf. [83, Theorem 17]) that there exists a trigonometrical polynomial Ψ_ϵ, i.e. a finite linear combination of functions of the type $\mathrm{e}^{2\pi i\langle h,\,\cdot\,\rangle}$, $h \in \mathbb{Z}^m$, such that $\sup_{x\in\mathbb{R}^m}|f(x) - \Psi_\epsilon(x)| \leq \varepsilon$. This yields

$$\begin{aligned}
&\left|\int_{[0,1)^m} f(x)\,\mathrm{d}x - \frac{1}{n}\sum_{k=1}^{n} f(\Delta_n k\, \underset{\sim}{\eta})\right| \\
&\quad\leq \left|\int_{[0,1)^m} (f(x)-\Psi_\epsilon(x))\,\mathrm{d}x\right| + \left|\int_{[0,1)^m} \Psi_\epsilon(x)\,\mathrm{d}x - \frac{1}{n}\sum_{k=1}^{n}\Psi_\epsilon(\Delta_n k\, \underset{\sim}{\eta})\right| \\
&\qquad\qquad + \left|\frac{1}{n}\sum_{k=1}^{n}\Psi_\epsilon(\Delta_n k\, \underset{\sim}{\eta}) - f(\Delta_n k\, \underset{\sim}{\eta})\right| \\
&\quad\leq 2\varepsilon + \left|\int_{[0,1)^m} \Psi_\epsilon(x)\,\mathrm{d}x - \frac{1}{n}\sum_{k=1}^{n}\Psi_\epsilon(\Delta_n k\, \underset{\sim}{\eta})\right|.
\end{aligned} \tag{3.54}$$

Since $\int_{[0,1)^m} \mathrm{e}^{2\pi i\langle h,\, x\rangle}\,\mathrm{d}x = 0$ for any $h \in \mathbb{Z}^m$, $h \neq 0$, Eq. (3.52) implies that the second term on the right-hand side of (3.54) converges to 0 as $n \to \infty$. This shows that (3.52) already implies (3.53).

We conclude the first part of the proof by applying (3.53) to the function

$$f(x_1, \ldots, x_m) := \left| \sum_{j=1}^{m} \cos(2\pi x_j)\, \Xi_{\theta_j,\nu_j}\big(c(i\omega_j)\big) + \sin(2\pi x_j)\, \Xi_{-\nu_j,\theta_j}\big(c(i\omega_j)\big) \right|^{\alpha}. \quad (3.55)$$

In the case where $\omega_1, \ldots, \omega_m$ are linearly dependent over $\mathbb{Z}$, we first observe that for any $f_h \in \mathscr{T}$ with $h \in \mathscr{L}$, $h \neq 0$,

$$\frac{1}{n} \sum_{k=1}^{n} f_h(\Delta_n k \underset{\sim}{\eta} \bmod 1) \to 0 \quad \text{as } n \to \infty \quad (3.56)$$

(where the mod-operator is defined componentwise; for the definition of $\mathscr{T}$ and $\mathscr{L}$ see (3.16) and (3.18), respectively). Therefore note that $\Delta_n k \underset{\sim}{\eta} \bmod 1 \in \mathscr{M}$ for any $n \in \mathbb{N}^*$, $k \in \{1, \ldots, n\}$, since (cf. Theorem 3.3.3)

$$\Delta_n k \underset{\sim}{\eta} \bmod 1 = B(\Delta_n k \underset{\sim}{\mu}) \bmod 1 = B(\underbrace{\Delta_n k \underset{\sim}{\mu} \bmod 1}_{\in[0,1)^{m-s}}) \bmod 1 = T(\Delta_n k \underset{\sim}{\mu} \bmod 1) \in \mathscr{M}. \quad (3.57)$$

Then, with $h = Bz \in \mathscr{L}$, $z \in \mathbb{Z}^{m-s} \backslash \{0\}$,

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n} f_h(\Delta_n k \underset{\sim}{\eta} \bmod 1) &= \frac{1}{n} \sum_{k=1}^{n} \mathrm{e}^{2\pi i \langle Bz,\, BG^{-1}T^{-1}(\Delta_n k \underset{\sim}{\eta} \bmod 1)\rangle} \\ &= \frac{1}{n} \sum_{k=1}^{n} \mathrm{e}^{2\pi i \langle z,\, T^{-1}(\Delta_n k \underset{\sim}{\eta} \bmod 1)\rangle} \overset{(3.57)}{=} \frac{1}{n} \sum_{k=1}^{n} \mathrm{e}^{2\pi i \langle z,\, \underset{\sim}{\mu}\rangle \Delta_n k}, \end{aligned}$$

and since $\langle z, \underset{\sim}{\mu}\rangle \neq 0$ for all $z \in \mathbb{Z}^{m-s} \backslash \{0\}$ (see Theorem 3.3.3(ii)), we obtain Eq. (3.56) in the same way as we have shown (3.52) in the linearly independent case.

Now, in the linearly dependent case (3.56) already implies

$$\frac{1}{n} \sum_{k=1}^{n} f(\Delta_n k \underset{\sim}{\eta} \bmod 1) \overset{n\to\infty}{\to} \frac{1}{\mathscr{H}^{m-s}(\mathscr{M})} \int_{\mathscr{M}} f(x)\, \mathscr{H}^{m-s}(\mathrm{d}x) \quad (3.58)$$

for any continuous function $f : \mathscr{M} \to \mathbb{C}$. Indeed, $\operatorname{span}^{\mathbb{C}}(\mathscr{T})$ is a dense subalgebra in $C(\mathscr{M})$, the algebra of all continuous complex-valued functions on the compact Hausdorff space $\mathscr{M}$, with respect to the topology of uniform convergence (cf. also comments after Theorem 3.3.3). Hence, for any continuous function $f : \mathscr{M} \to \mathbb{C}$ and any fixed $\varepsilon > 0$ there is a finite linear combination Ψ_ϵ of functions in $\mathscr{T}$ such that $\sup_{x\in\mathscr{M}} |f(x) - \Psi_\epsilon(x)| \le \varepsilon$. This yields, analogously to (3.54),

$$\begin{aligned}&\left|\frac{1}{\mathscr{H}^{m-s}(\mathscr{M})}\int_{\mathscr{M}} f(x)\,\mathscr{H}^{m-s}(\mathrm{d}x) - \frac{1}{n}\sum_{k=1}^{n} f(\Delta_n k\,\underset{\sim}{\eta} \bmod 1)\right| \\ &\qquad \le 2\varepsilon + \left|\frac{1}{\mathscr{H}^{m-s}(\mathscr{M})}\int_{\mathscr{M}} \Psi_\epsilon(x)\,\mathscr{H}^{m-s}(\mathrm{d}x) - \frac{1}{n}\sum_{k=1}^{n} \Psi_\epsilon(\Delta_n k\,\underset{\sim}{\eta} \bmod 1)\right|,\end{aligned}$$

and the second term on the right-hand side converges to 0 as $n \to \infty$ by virtue of Theorem 3.3.3(iii) and Eq. (3.56). This shows (3.58).

We conclude the linearly dependent case by applying Eq. (3.58) to the function $f|_{\mathscr{M}}$ with the same f as in the linearly independent case in (3.55). □

Proof of Proposition 3.3.5. We have

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{n} f(k\Delta_n\underset{\sim}{\eta} \bmod 1,\, N_k) - \mathbb{E}\left[f(\underset{\sim}{U},\, N_1)\right] \\ &\quad = \frac{1}{n}\sum_{k=1}^{n}\Big(f(k\Delta_n\underset{\sim}{\eta} \bmod 1,\, N_k) - \mathbb{E}\Big[f(k\Delta_n\underset{\sim}{\eta} \bmod 1,\, N_1)\Big]\Big) \\ &\qquad\qquad + \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\Big[f(k\Delta_n\underset{\sim}{\eta} \bmod 1,\, N_1)\Big] - \mathbb{E}\left[f(\underset{\sim}{U},\, N_1)\right] \\ &\quad =: I_1 + I_2.\end{aligned}$$

We consider first the case where $\omega_1, \ldots, \omega_m$ are linearly independent over $\mathbb{Z}$. Then, by virtue of Eq. (3.53) and the assumption that $g^{(1)}$ is continuous on $(\mathbb{R} \bmod 1)^m$, we have

$$\begin{aligned}I_2 &= \frac{1}{n}\sum_{k=1}^{n} g^{(1)}(k\Delta_n\underset{\sim}{\eta} \bmod 1) - \mathbb{E}\left[f(\underset{\sim}{U},\, N_1)\right] \overset{n\to\infty}{\to} \int_{[0,1)^m} g^{(1)}(x)\,\mathrm{d}x - \mathbb{E}\left[f(\underset{\sim}{U},\, N_1)\right] \\ &= \int_{[0,1)^m} \mathbb{E}\left[f(x,\, N_1)\right]\mathrm{d}x - \mathbb{E}\left[f(\underset{\sim}{U},\, N_1)\right] = 0.\end{aligned}$$

With Chebyshev's Inequality and the assumption that $g^{(2)}$ is continuous on $(\mathbb{R} \bmod 1)^m$, we further obtain

$$\mathbb{P}(|I_1| > \varepsilon) \le \frac{1}{\varepsilon^2\, n^2}\sum_{k=1}^{n}\mathbb{E}\left[\Big(f(k\Delta_n\underset{\sim}{\eta} \bmod 1,\, N_1) - \mathbb{E}\Big[f(k\Delta_n\underset{\sim}{\eta} \bmod 1,\, N_1)\Big]\Big)^2\right]$$

$$\leq \frac{1}{\varepsilon^2 n^2} \sum_{k=1}^{n} \mathbb{E}\left[f^2(k\Delta_n \underset{\sim}{\eta} \bmod 1,\, N_1)\right] = \frac{1}{\varepsilon^2 n^2} \sum_{k=1}^{n} g^{(2)}(k\Delta_n \underset{\sim}{\eta} \bmod 1)$$
$$= \frac{1}{\varepsilon^2 n} \int_{[0,1)^m} g^{(2)}(x)\,\mathrm{d}x \cdot (1 + o(1)) = \frac{1}{\varepsilon^2 n} \mathbb{E}[f^2(\underset{\sim}{U},\, N_1)] \cdot (1 + o(1))$$
$$\overset{n\to\infty}{\to} 0,$$

where we used once more (3.53). Hence, Eq. (3.19) is shown in the linearly independent case.

Suppose now that $\omega_1, \ldots, \omega_m$ are linearly dependent over $\mathbb{Z}$. As above, now due to Eq. (3.58),

$$I_2 \overset{n\to\infty}{\to} \frac{1}{\mathscr{H}^{m-s}(\mathscr{M})} \int_{\mathscr{M}} g^{(1)}(x)\,\mathscr{H}^{m-s}(\mathrm{d}x) - \mathbb{E}\left[f(\underset{\sim}{U},\, N_1)\right]$$
$$= \int_{[0,1)^{m-s}} g^{(1)}(T(x))\,\mathrm{d}x - \mathbb{E}\left[f(T(\underset{\sim}{V}),\, N_1)\right]$$
$$= \int_{[0,1)^{m-s}} \mathbb{E}\left[f(T(x),\, N_1)\right]\mathrm{d}x - \mathbb{E}\left[f(T(\underset{\sim}{V}),\, N_1)\right] = 0$$

and

$$\mathbb{P}(|I_1| > \varepsilon) \leq \frac{1}{\varepsilon^2 n^2} \sum_{k=1}^{n} g^{(2)}(k\Delta_n \underset{\sim}{\eta} \bmod 1)$$
$$= \frac{1}{\varepsilon^2 n} \cdot \underbrace{\frac{1}{\mathscr{H}^{m-s}(\mathscr{M})} \int_{\mathscr{M}} g^{(2)}(x)\,\mathscr{H}^{m-s}(\mathrm{d}x)}_{=\mathbb{E}[f^2(T(\underset{\sim}{V}),\, N_1)]} \cdot (1 + o(1)) \overset{n\to\infty}{\to} 0.$$

Thus, also in the linearly dependent case (3.19) holds. $\square$

4 Spectral estimates for high-frequency sampled CARMA processes[3]

4.1 Introduction

In this chapter we investigate continuous-time ARMA (CARMA) processes $Y = (Y_t)_{t\in\mathbb{R}}$ in the spectral domain and propose an estimator for the model parameters. For an overview and a comprehensive list of references on CARMA processes and their applications in several fields such as signal processing and control, econometrics and financial mathematics, we refer to [16, 23, 36]. The driving force of a CARMA process is a Lévy process $(L_t)_{t\in\mathbb{R}}$. A Lévy process $(L_t)_{t\geq 0}$ is defined (cf. [78]) to satisfy $L_0 = 0$ a.s., $(L_t)_{t\geq 0}$ has independent and stationary increments and the paths of $(L_t)_{t\geq 0}$ are stochastically continuous. An extension of a Lévy process $(L_t)_{t\geq 0}$ from the positive to the whole real line is given by $L_t := L_t \mathbb{1}_{\{t\geq 0\}} - \widetilde{L}_{-t-}\mathbb{1}_{\{t<0\}}$ for $t \in \mathbb{R}$, where $(\widetilde{L}_t)_{t\geq 0}$ is an independent copy of $(L_t)_{t\geq 0}$. Prominent examples are Brownian motions and stable Lévy processes. In this chapter we restrict our attention to *symmetric* stable Lévy processes and *symmetric* Lévy processes with finite second moments. Then a CARMA process can be interpreted (its formal definition is given in Section 4.2) as a solution to the p-th order stochastic differential equation

$$a(\mathrm{D})\, Y_t = c(\mathrm{D})\, \mathrm{D}L_t, \quad t \in \mathbb{R}, \tag{4.1}$$

where D denotes the differential operator with respect to t and

$$a(z) := z^p + a_1 z^{p-1} + \ldots + a_p \qquad \text{and} \qquad c(z) := c_0 z^q + c_1 z^{q-1} + \ldots + c_q$$

[3]The contents of this chapter are based on Fasen, V. and Fuchs, F. (2012), Spectral Estimates for High-Frequency Sampled CARMA Processes, *submitted for publication.*

are the autoregressive and the moving average polynomial, respectively. Hence, CARMA processes can be seen as the continuous-time analog of (discrete-time) ARMA processes. From a statistical point of view, the so-called *power transfer function*

$$\Psi(\omega) := \frac{|c(i\omega)|^2}{|a(i\omega)|^2}, \quad \omega \in \mathbb{R}, \tag{4.2}$$

which corresponds (up to a constant) to the classical spectral density in the finite-variance case, is of central interest since it determines the model completely. The zeros of Ψ contain the zeros of $c(i\,\cdot)$, and hence, provided that the sign of the real part of any zero of $c(\cdot)$ is supposed to be known, one can identify uniquely the coefficients of the moving average polynomial from the power transfer function Ψ. Likewise the zeros of Ψ^{-1} characterize completely the coefficients of the autoregressive polynomial if one assumes to know the sign of the real parts of the zeros of $a(\cdot)$. From this it is obvious that, under *causality* and *invertibility* assumptions on the CARMA process, estimators for the power transfer function can be used to construct estimators for the coefficients of $a(\cdot)$ and $c(\cdot)$.

The empirical version of the power transfer function (spectral density) is in the finite second moment case the *periodogram*. In [39] we have investigated the limit behavior of normalized and self-normalized versions of the periodogram of high-frequency sampled symmetric α-stable CARMA processes. Here we assume again that we observe the CARMA process Y only at equidistant time points $\{0, \Delta_n, 2\Delta_n, \ldots, n\Delta_n\}$ where $\Delta_n > 0$ is small, as used for modeling high-frequency data appearing in turbulence and finance (cf. [21, 36]), and $n \in \mathbb{N}$ is the total number of observations. More precisely, our asymptotic results hold under

Assumption 4.1. *We suppose that simultaneously* $\Delta_n \to 0$ *and* $n\Delta_n \to \infty$ *as* $n \to \infty$.

The normalized periodogram of the sampled sequence $Y^{\Delta_n} := (Y_{k\Delta_n})_{k\in\mathbb{Z}}$ at frequency $\omega \in [-\pi, \pi]$ is given by

$$I_{n,Y^{\Delta_n}}(\omega) = \left| n^{-1/\alpha} \sum_{k=1}^{n} Y_{k\Delta_n} \, \mathrm{e}^{-i\omega k} \right|^2, \tag{4.3}$$

where for finite-variance CARMA processes we have $\alpha = 2$ and for α-stable CARMA processes α is the index of stability. A self-normalized alternative, no longer depending on α, is given for $\omega \in [-\pi, \pi]$ by

$$\widehat{I}_{n,Y^{\Delta_n}}(\omega) = \frac{I_{n,Y^{\Delta_n}}(\omega)}{n^{-2/\alpha} \sum_{k=1}^{n} Y_{k\Delta_n}^2} = \frac{\left| \sum_{k=1}^{n} Y_{k\Delta_n} \, \mathrm{e}^{-i\omega k} \right|^2}{\sum_{k=1}^{n} Y_{k\Delta_n}^2}. \tag{4.4}$$

As stated in [39, Theorems 3.5 and 3.10], both the normalized as well as the self-normalized periodogram are not consistent estimators for the power transfer function if the Lévy process is α-stable, $\alpha \in (0,2]$. The limit distribution is a function of an α-stable random vector which reduces in the finite-variance case to an exponential distribution. We will generalize these results to finite-variance CARMA processes and to a very general high-frequency grid distance Δ_n. The limit results for high-frequency sampled finite-variance CARMA processes are analog to the results for finite-variance CARMA processes sampled at an equidistant time grid as derived in [37].

However, by applying linear smoothing filters to the periodogram consistent estimators for the (normalized) power transfer function can be constructed which is the main topic of this chapter. We will consider the class of estimators of the form

$$T_{n,Y^{\Delta_n}}(\omega) = \sum_{|k|\leq m_n} W_n(k)\, I_{n,Y^{\Delta_n}}(\omega_k) \tag{4.5}$$

and

$$\widehat{T}_{n,Y^{\Delta_n}}(\omega) = \sum_{|k|\leq m_n} W_n(k)\, \widehat{I}_{n,Y^{\Delta_n}}(\omega_k) \tag{4.6}$$

where

$$\omega_k = \omega + \frac{k}{n}, \qquad |k| \leq m_n, \tag{4.7}$$

and $(m_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{N}$ satisfying

Assumption 4.2. *We suppose that simultaneously $m_n \to \infty$ and $\frac{m_n}{n\Delta_n} \to 0$ as $n \to \infty$.*

The sequence of weight functions $W_n : \mathbb{Z} \to \mathbb{R}$ is specified by

$$W_n(k) = W_n(-k), \qquad W_n(k) \geq 0, \qquad \forall\, k \in \mathbb{N}, \tag{4.8a}$$

$$\sum_{|k|\leq m_n} W_n(k) = 1, \qquad \forall\, n \in \mathbb{N}, \tag{4.8b}$$

$$\max_{|k|\leq m_n} W_n^2(k) = o\left(\frac{1}{m_n}\right) \quad \text{as } n \to \infty. \tag{4.8c}$$

On the one hand, we will show that the sequence of smoothed self-normalized periodograms $\Delta_n\, \widehat{T}_{n,Y^{\Delta_n}}(\omega\Delta_n)$ is a consistent estimator for the normalized power transfer function. This result is in analogy to the one for ARMA models in discrete time obtained by Klüppelberg and Mikosch in [58]. On the other hand, for finite-variance CARMA processes the smoothed normalized periodograms $\Delta_n\, T_{n,Y^{\Delta_n}}(\omega\Delta_n)$ provide consistent estimators for the 2π-multiple of the spectral density, as well.

Thereafter, these results are used to develop an estimator for the parameters of the CARMA process. Our heuristic will basically consist of a constrained nonlinear least squares problem where the constraints come from the (necessary) additional assumption of causality and invertibility of the CARMA process. The estimator is then given as the best, in terms of least squares, (normalized) rational approximation for the smoothed periodogram values. It is an alternative to the ones presented in [18, 19, 45] working for both finite-variance and stable CARMA processes with infinite second moments. The Gaussian quasi-maximum-likelihood estimation has been derived in [19, 81] for Lévy-driven (multivariate) CARMA processes with finite second moments. In [45] a heuristic study of the estimation of stable CARMA$(2,1)$ processes is presented. A nonparametric estimator for the kernel function of a CARMA process is proposed in [20], and for Ornstein-Uhlenbeck processes, which are CARMA$(1,0)$ processes, an efficient estimator for the mean reversion parameter of the Ornstein-Uhlenbeck model has been obtained in [18, 53] by using methods of [31]. Compared to the other estimators the new contribution of this chapter is that the estimator performs well for both finite-variance and infinite-variance models and we are able to estimate both the autoregressive and the moving average polynomial.

The chapter is structured in the following way. In Section 4.2 we recall the formal definition of a Lévy-driven CARMA process and present some assumptions and notations of the chapter. The main results are stated in Section 4.3. These include the asymptotic behavior of the different smoothed periodogram versions and of the periodogram itself. The topic of Section 4.4 is then the statistical inference for the model parameters of a CARMA process, illustrated by a simulation study for a CARMA$(2,1)$ process. Finally, Section 4.5 contains the proofs.

Notation

We use $\mathbb{N}^*$ and $\mathbb{R}^*$ for the natural and real numbers, respectively, excluding zero and $\mathbb{Z}$ for the integers. For the minimum of two real numbers $a, b \in \mathbb{R}$ we write shortly $a \wedge b$ and for the maximum $a \vee b$. The real and imaginary part of a complex number $z \in \mathbb{C}$ is written as $\Re(z)$ and $\Im(z)$, respectively, and its complex conjugate as $\overline{z}$. For two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ we say $a_n \sim b_n$ as $n \to \infty$ if $\lim_{n\to\infty} a_n/b_n = 1$. The transpose of a matrix M is written as M^T and the m-dimensional identity matrix shall be denoted by I_m. On $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ the Euclidean norm is denoted by $|\cdot|$ whereas on $\mathbb{K}^m$ it will be usually written as $\|\cdot\|$. For two random variables X and Y the notation $X \overset{\mathcal{D}}{=} Y$ means equality in distribution. If we consider a sequence of random variables

$(X_n)_{n\in\mathbb{N}}$, we denote convergence in probability to some random variable X by $X_n \xrightarrow{\mathbb{P}} X$ as $n \to \infty$ and convergence in distribution by $X_n \xrightarrow{\mathcal{D}} X$ as $n \to \infty$.

4.2 Preliminaries to Chapter 4

4.2.1 Lévy-driven CARMA processes

We recall the definition of an α-stable random variable and then present the notation which we use throughout the chapter for the underlying driving Lévy process.

Definition 4.2.1. *A real-valued random variable Z is called* symmetric α-stable *($S\alpha S$) with index of stability $\alpha \in (0, 2]$, if its characteristic function is of the form*

$$\Phi_Z(u) = \mathbb{E}\left[\exp\{iuZ\}\right] = \exp\{-\sigma^\alpha |u|^\alpha\}, \quad u \in \mathbb{R},$$

for some $\sigma \geq 0$. The parameter σ is called scale parameter*. A* symmetric α-stable Lévy process *$(L_t)_{t\in\mathbb{R}}$ with scale parameter σ_L is a Lévy process where L_1 is $S\alpha S$ with scale parameter σ_L.*

In particular a $S2S$ random variable is normally distributed and a 2-stable Lévy process is a Brownian motion. For the driving Lévy process we use the following notation.

Definition 4.2.2. *Let $\alpha \in (0, 2]$ and $\sigma_L \geq 0$. By $L(\alpha, \sigma_L)$ we denote a* symmetric *Lévy process that is either*

(i) α-stable with scale parameter σ_L if $\alpha \in (0, 2)$, or

(ii) satisfies $\mathbb{E}\left[L_1^2\right] = \sigma_L^2$ if $\alpha = 2$.

A CARMA process driven by $L(\alpha, \sigma_L)$ is then defined as follows. Assume that we have given $p, q \in \mathbb{N}$, $p > q$, and $a_1, \ldots, a_p, c_0, \ldots, c_q \in \mathbb{R}$, $a_p, c_0 \neq 0$, let

$$A := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ -a_p & -a_{p-1} & \ldots & \ldots & -a_1 \end{pmatrix} \in \mathbb{R}^{p\times p}$$

and let $(X_t)_{t\in\mathbb{R}}$ be a strictly stationary solution to the stochastic differential equation

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + e_p\,\mathrm{d}L_t, \quad t \in \mathbb{R}, \tag{4.9a}$$

where e_p denotes the p-th unit vector in $\mathbb{R}^p$. Then the process

$$Y_t := c^T X_t, \quad t \in \mathbb{R}, \tag{4.9b}$$

with $c = (c_q, c_{q-1}, \ldots, c_{q-p+1})^T$ (where we use the convention $c_j = 0$ for $j < 0$) is said to be a *CARMA process* of order (p, q). Necessary and sufficient conditions for the existence of a strictly stationary CARMA process are given in [22]. In this chapter we will suppose

Assumption 4.3.
The eigenvalues $\lambda_1, \ldots, \lambda_p$ of A are distinct and possess strictly negative real parts.

Under this assumption, the solution for the state process in (4.9a) is unique, strictly stationary, causal and can be written as

$$X_t = \int_{-\infty}^{t} \mathrm{e}^{(t-s)A} e_p \,\mathrm{d}L_s, \quad t \in \mathbb{R}. \tag{4.10a}$$

Hence, the CARMA process Y can also be expressed as a Lévy-driven *moving average process* $Y_t = \int_{-\infty}^{\infty} g(t-s)\,\mathrm{d}L_s$, $t \in \mathbb{R}$, with kernel function

$$g(t) = c^T \mathrm{e}^{tA} e_p \,\mathbb{1}_{[0,\infty)}(t). \tag{4.10b}$$

Notably the CARMA process can be interpreted as solution of the stochastic differential equation given in (4.1).

4.2.2 Decomposition of the smoothed (self-normalized) periodogram

Before stating the main results, we derive a series representation of the sampled sequence Y^{Δ_n} driven by a Lévy process $L(\alpha, \sigma_L)$ as in Definition 4.2.2. We use this representation for a suitable decomposition of the Fourier transform of Y^{Δ_n} and its associated smoothed (self-normalized) periodogram. Recall that the discrete Fourier transform is given by $F_{n,Y^{\Delta_n}}(\omega) := n^{-1/\alpha} \sum_{k=1}^{n} Y_{k\Delta_n}\,\mathrm{e}^{-i\omega k}$ for any $\omega \in [-\pi, \pi]$.

It is well known that every solution to (4.9a) satisfies

$$X_t = \mathrm{e}^{(t-s)A} X_s + \int_s^t \mathrm{e}^{(t-u)A} e_p \,\mathrm{d}L_u, \qquad \forall s,\, t \in \mathbb{R},\ s < t.$$

Then, under Assumption 4.3, we have by iteration that the state process X at time point

$k\Delta_n$ can be written in the *series representation*

$$X_{k\Delta_n} = \sum_{j=0}^{\infty} \mathrm{e}^{j\Delta_n A} \xi^*_{n,\,k-j}\, e_p, \qquad k \in \mathbb{Z}, \tag{4.11}$$

with the $\mathbb{R}^{p\times p}$-valued noise sequence

$$\xi^*_{n,\,k} := \int_{(k-1)\Delta_n}^{k\Delta_n} \mathrm{e}^{(k\Delta_n - s)A} \mathrm{d}L_s, \qquad n \in \mathbb{N},\ k \in \mathbb{Z}. \tag{4.12}$$

We define, for any $\omega \in [-\pi, \pi]$,

$$\begin{aligned}
U_{n,\,j}(\omega) &:= \sum_{k=1-j}^{n-j} \xi^*_{n,\,k}\, \mathrm{e}^{-i\omega k} - \sum_{k=1}^{n} \xi^*_{n,\,k}\, \mathrm{e}^{-i\omega k}, \\
K_{n,\,\Delta_n}(\omega) &:= \sum_{j=0}^{\infty} \mathrm{e}^{j(\Delta_n A - i\omega \mathrm{I}_p)} U_{n,\,j}(\omega), \\
M_{n,\,\Delta_n}(\omega) &:= \left(\mathrm{I}_p - \mathrm{e}^{-i\omega} \cdot \mathrm{e}^{\Delta_n A}\right)^{-1} \sum_{k=1}^{n} \xi^*_{n,\,k}\, \mathrm{e}^{-i\omega k}, \qquad (4.13) \\
J_{n,\,\Delta_n}(\omega) &:= c^T M_{n,\,\Delta_n}(\omega)\, e_p \qquad \text{and} \\
R_{n,\,\Delta_n}(\omega) &:= J_{n,\,\Delta_n}(\omega) \cdot \overline{c^T K_{n,\,\Delta_n}(\omega)\, e_p} + c^T K_{n,\,\Delta_n}(\omega)\, e_p \cdot \overline{J_{n,\,\Delta_n}(\omega)} + \left|c^T K_{n,\,\Delta_n}(\omega)\, e_p\right|^2.
\end{aligned}$$

The series representation of the state process X in Eq. (4.11) then immediately yields the following decomposition for the Fourier transform of the sampled sequence Y^{Δ_n}

$$\begin{aligned}
n^{\frac{1}{\alpha}} F_{n,\,Y^{\Delta_n}}(\omega) &= \sum_{k=1}^{n} Y_{k\Delta_n}\, \mathrm{e}^{-i\omega k} = c^T \left(\sum_{k=1}^{n} \sum_{j=0}^{\infty} \mathrm{e}^{j\Delta_n A} \xi^*_{n,\,k-j}\, \mathrm{e}^{-i\omega k} \right) e_p \\
&= c^T \left(\sum_{j=0}^{\infty} \mathrm{e}^{j(\Delta_n A - i\omega \mathrm{I}_p)} \sum_{k=1}^{n} \xi^*_{n,\,k}\, \mathrm{e}^{-i\omega k} \right) e_p + c^T \left(\sum_{j=0}^{\infty} \mathrm{e}^{j(\Delta_n A - i\omega \mathrm{I}_p)}\, U_{n,\,j}(\omega) \right) e_p \\
&= c^T M_{n,\,\Delta_n}(\omega)\, e_p + c^T K_{n,\,\Delta_n}(\omega)\, e_p = J_{n,\,\Delta_n}(\omega) + c^T K_{n,\,\Delta_n}(\omega)\, e_p \qquad (4.14)
\end{aligned}$$

and hence, we may split the smoothed (self-normalized) periodogram in a main, limit-determining, part and a vanishing rest term (cf. upcoming Propositions 4.3.3 and 4.3.4):

$$\widehat{T}_{n,\,Y^{\Delta_n}}(\omega) \;=\; \sum_{|k| \le m_n} W_n(k)\, \frac{\left| \sum_{u=1}^{n} Y_{u\Delta_n}\, \mathrm{e}^{-i\omega_k u} \right|^2}{\sum_{u=1}^{n} Y^2_{u\Delta_n}}$$

$$\overset{(4.14)}{=} \sum_{|k|\le m_n} W_n(k)\, \frac{\left|J_{n,\Delta_n}(\omega_k) + c^T K_{n,\Delta_n}(\omega_k)\, e_p\right|^2}{\sum_{u=1}^n Y_{u\Delta_n}^2}$$

$$= \sum_{|k|\le m_n} W_n(k)\, \frac{\left|J_{n,\Delta_n}(\omega_k)\right|^2}{\sum_{u=1}^n Y_{u\Delta_n}^2} + \sum_{|k|\le m_n} W_n(k)\, \frac{R_{n,\Delta_n}(\omega_k)}{\sum_{u=1}^n Y_{u\Delta_n}^2}. \qquad (4.15)$$

4.3 Limit behavior of the smoothed periodogram

Our main limit theorem is the following:

Theorem 4.3.1. *Suppose $\alpha \in (0,2]$, $\sigma_L > 0$ and let $Y^{\Delta_n} = (Y_{k\Delta_n})_{k\in\mathbb{Z}}$ denote the sampled sequence of a non-trivial CARMA(p, q) process driven by the Lévy process $L(\alpha, \sigma_L)$ as in Definition 4.2.2. Moreover, Assumptions 4.1 to 4.3 may hold, and assume that the weight functions W_n satisfy* (4.8)*. Then the smoothed self-normalized periodogram as in Eq.* (4.6) *satisfies for any $\omega \in \mathbb{R}^*$,*

$$\Delta_n\, \widehat{T}_{n,Y^{\Delta_n}}(\omega\Delta_n) \xrightarrow{\mathbb{P}} \frac{|c(i\omega)|^2}{\int_0^\infty g^2(s)\,\mathrm{d}s \cdot |a(i\omega)|^2} \qquad \text{as } n\to\infty,$$

where g is the kernel function of the CARMA process (see (4.10b)*), i.e., the smoothed self-normalized periodogram is a weakly consistent estimator for the normalized power transfer function.*

For $\alpha = 2$ the normalization $n^{-1}\sum_{k=1}^n Y_{k\Delta_n}^2$ converges in probability, as $n\to\infty$, to $\int_0^\infty g^2(s)\,\mathrm{d}s \cdot \sigma_L^2$ (cf. [36, Theorem 5.5(a)]) such that a direct conclusion is

Corollary 4.3.2. *Under the same assumptions as in Theorem 4.3.1, suppose in addition that $\alpha = 2$. Then the smoothed normalized periodogram as in Eq.* (4.5) *satisfies for any $\omega \in \mathbb{R}^*$,*

$$\Delta_n\, T_{n,Y^{\Delta_n}}(\omega\Delta_n) \xrightarrow{\mathbb{P}} \sigma_L^2 \cdot \frac{|c(i\omega)|^2}{|a(i\omega)|^2} \qquad \text{as } n\to\infty,$$

i.e., the smoothed normalized periodogram is a weakly consistent estimator for the 2π-multiple of the spectral density.

The proof of Theorem 4.3.1 will be divided into two parts. The first one shows that the main part in (4.15) converges to the normalized power transfer function as $n\to\infty$.

Proposition 4.3.3. *Under the same assumptions as in Theorem 4.3.1 we have for any* $\omega \in \mathbb{R}^*$,

$$\Delta_n \sum_{|k|\leq m_n} W_n(k) \frac{\left|J_{n,\Delta_n}((\omega\Delta_n)_k)\right|^2}{\sum_{u=1}^n Y_{u\Delta_n}^2} \xrightarrow{\mathbb{P}} \frac{|c(i\omega)|^2}{\int_0^\infty g^2(s)\,\mathrm{d}s \cdot |a(i\omega)|^2} \qquad \text{as } n\to\infty,$$

where $J_{n,\Delta_n}(\,\cdot\,)$ *has been defined in* (4.13).

The second part shows that the rest term in (4.15) vanishes as $n \to \infty$.

Proposition 4.3.4. *Suppose* $\alpha \in (0,2]$, $\sigma_L > 0$ *and let* $Y^{\Delta_n} = (Y_{k\Delta_n})_{k\in\mathbb{Z}}$ *denote the sampled sequence of a non-trivial CARMA(p, q) process driven by the Lévy process* $L(\alpha, \sigma_L)$ *as in Definition 4.2.2. Moreover, Assumptions 4.1 to 4.3 may hold, and assume that the weight functions* W_n *satisfy* (4.8a) *and* (4.8b). *Then we have for any* $\omega \in \mathbb{R}^*$,

$$\Delta_n \sum_{|k|\leq m_n} W_n(k) \frac{R_{n,\Delta_n}((\omega\Delta_n)_k)}{\sum_{u=1}^n Y_{u\Delta_n}^2} \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n\to\infty,$$

where $R_{n,\Delta_n}(\,\cdot\,)$ *has been defined in* (4.13).

In Theorem 4.3.1 we have shown that the smoothed self-normalized periodogram provides consistent estimates for the (normalized) power transfer function of symmetric α-stable as well as finite-variance CARMA processes. Recall that normalized and self-normalized periodogram versions have been investigated in [39] under more restrictive assumptions on Δ_n than here. Moreover, in that paper only the Gaussian case has been studied but not the general finite-variance setting. Therefore, the following theorem should be seen as an extension of the results in [39]. It concerns the limit behavior of the normalized periodogram including finite-variance CARMA processes.

Theorem 4.3.5. *Suppose* $\alpha \in (0,2]$, $\sigma_L > 0$ *and let* $Y^{\Delta_n} = (Y_{k\Delta_n})_{k\in\mathbb{Z}}$ *denote the sampled sequence of a non-trivial CARMA(p, q) process driven by the Lévy process* $L(\alpha, \sigma_L)$ *as in Definition 4.2.2. Moreover, Assumptions 4.1 and 4.3 may hold. Then the periodogram as in* (4.3) *satisfies for any* $\omega \in \mathbb{R}^*$,

$$\Delta_n^{2-\frac{2}{\alpha}}\, I_{n,Y^{\Delta_n}}(\omega\Delta_n) \xrightarrow{\mathcal{D}} \frac{|c(i\omega)|^2}{|a(i\omega)|^2} \cdot \left|\int_{[0,1)} \mathrm{e}^{2\pi i s}\,\mathrm{d}L_s^*\right|^2 \qquad \text{as } n\to\infty,$$

where $(L_t^*)_{t\geq 0}$ *is a symmetric* α*-stable Lévy process with scale parameter* σ_L *if* $\alpha \in (0,2)$ *and for* $\alpha = 2$ *it is a symmetric Brownian motion with* $\mathrm{Var}(L_1^*) = \sigma_L^2$.

A direct conclusion is the asymptotic behavior of the periodogram for finite-variance CARMA processes.

Corollary 4.3.6. *Let the assumptions of Theorem 4.3.5 hold and suppose $\alpha = 2$. Then the normalized periodogram as in* (4.3) *satisfies for any $\omega \in \mathbb{R}^*$,*

$$\Delta_n \, I_{n,\, Y^{\Delta_n}}(\omega \Delta_n) \xrightarrow{\mathcal{D}} \sigma_L^2 \cdot \frac{|c(i\omega)|^2}{|a(i\omega)|^2} \cdot \left(\frac{N_1^2}{2} + \frac{N_2^2}{2} \right) \qquad \text{as } n \to \infty,$$

where N_1 and N_2 are i.i.d. standard normal random variables, and the self-normalized periodgram as in (4.4) *satisfies for any $\omega \in \mathbb{R}^*$,*

$$\Delta_n \, \widehat{I}_{n,\, Y^{\Delta_n}}(\omega \Delta_n) \xrightarrow{\mathcal{D}} \frac{|c(i\omega)|^2}{\int_0^\infty g^2(s)\, \mathrm{d}s \cdot |a(i\omega)|^2} \cdot \left(\frac{N_1^2}{2} + \frac{N_2^2}{2} \right) \qquad \text{as } n \to \infty,$$

where g is the kernel function of the CARMA process (see (4.10b)*).*

From Proposition 4.3.4 we know already that the rest term in (4.15) with $W_n(0) = 1$ and $W_n(k) = W_n(-k) = 0$ for $k \in \mathbb{N}^*$ vanishes. These weights do not satisfy (4.8c), but obviously (4.8a) and (4.8b). The next proposition investigates the main part.

Proposition 4.3.7. *Under the same assumptions as in Theorem 4.3.5 we have for any $\omega \in \mathbb{R}^*$,*

$$\Delta_n^{2-\frac{2}{\alpha}} \, n^{-\frac{2}{\alpha}} \left| J_{n,\, \Delta_n}(\omega \Delta_n) \right|^2 \xrightarrow{\mathbb{P}} \frac{|c(i\omega)|^2}{|a(i\omega)|^2} \cdot \left| \int_{[0,1)} \mathrm{e}^{2\pi i\, s} \, \mathrm{d}L_s^* \right|^2 \qquad \text{as } n \to \infty,$$

where $J_{n,\, \Delta_n}(\,\cdot\,)$ has been defined in (4.13) *and $(L_t^*)_{t \geq 0}$ is as in Theorem 4.3.5.*

Remark 4.3.8.

(i) As already mentioned above, in [39] the general finite-variance case has not been considered. In this spirit, Theorem 4.3.5 and Corollary 4.3.6 should be seen as an extension of [39, Theorems 3.5 and 3.10], although we have stated only the univariate limit distributions for the normalized and self-normalized periodogram here. However, it seems to be possible to derive also the limit behavior for different frequencies. In this case, the limit depends again on the dependence structure of those frequencies if $\alpha < 2$, cf. [39, Section 2.2]. As in the Gaussian case (cf. [39, Remark 3.6(ii)]) different periodogram ordinates of finite-variance CARMA models are asymptotically independent.

(ii) Theorem 4.3.5 (and its proof) confirms our conjecture in [39, Remark 3.7], namely that the assumption $n\Delta_n^\beta \to \infty$ with $\beta = \max\{1+\delta, \alpha(p-1) + \max\{0, 1-\alpha\}\}$ for some $\delta > 0$ is not necessary for the limit results of normalized and self-normalized periodogram versions of symmetric α-stable CARMA processes. Instead, supposing $n\Delta_n \to \infty$ as $n \to \infty$ is already sufficient. Note that, anyway, the partition of the periodogram used in [39] provides deeper insight into structural properties of CARMA processes in the frequency domain and therein lay the necessity for the stronger condition on the observation grid (cf. also [39, proof of Proposition 3.2 and Remark 3.7]).

(iii) We want to compare the limit results for the high-frequency sampled finite-variance CARMA process Y^{Δ_n} with the results for a finite-variance CARMA process sampled at an equidistant time grid $Y^\Delta = (Y_{k\Delta})_{k\in\mathbb{Z}}$ for some $\Delta > 0$ fixed. For that, let $f_{Y^{\Delta_n}}$ denote the spectral density of Y^{Δ_n}, f_{Y^Δ} the spectral density of Y^Δ and finally

$$f_Y(\omega) = \frac{\sigma_L^2}{2\pi} \frac{|c(i\omega)|^2}{|a(i\omega)|^2}, \qquad \omega \in \mathbb{R},$$

the spectral density of the continuous-time process Y. Moreover, the periodogram of the sampled sequence Y^Δ is denoted by $I_{n,Y^\Delta}(\omega) = \left|n^{-1/2}\sum_{k=1}^n Y_{k\Delta}\,\mathrm{e}^{-i\omega k}\right|^2$ for $\omega \in [-\pi, \pi]$. A conclusion of [37, Theorem 3.1] for the equidistant sampling is that for any $\omega \in (-\pi/\Delta, 0) \cup (0, \pi/\Delta)$,

$$\frac{I_{n,Y^\Delta}(\omega\Delta)}{f_{Y^\Delta}(\omega\Delta)} \xrightarrow{\mathcal{D}} 2\pi\left(\frac{N_1^2}{2} + \frac{N_2^2}{2}\right) \qquad \text{as } n \to \infty,$$

and of Corollary 4.3.6 and [39, Eq. (1.5)] for the high-frequency time sampling that for any $\omega \in \mathbb{R}^*$,

$$\frac{I_{n,Y^{\Delta_n}}(\omega\Delta_n)}{f_{Y^{\Delta_n}}(\omega\Delta_n)} \xrightarrow{\mathcal{D}} 2\pi\left(\frac{N_1^2}{2} + \frac{N_2^2}{2}\right) \qquad \text{as } n \to \infty.$$

Surprisingly the structure of the limit results is the same and will be of advantage for statistical inference. The similarities suggest that the rate of convergence of Δ_n has no influence on the asymptotic behavior. □

4.4 Estimation of the CARMA parameters

In this section we propose a (spectral) estimation procedure for the autoregressive (AR) and moving average (MA) parameters of a CARMA process, based on Theorem 4.3.1 and Corollary 4.3.2. We will exemplify our method by a simulation study for the CARMA$(2,1)$ case.

Let $\alpha \in (0,2]$, $\sigma_L > 0$ and $Y^{\Delta_n} = (Y_{k\Delta_n})_{k\in\mathbb{Z}}$ be the sampled sequence of a non-trivial CARMA(p,q) process driven by the Lévy process $L(\alpha,\, \sigma_L)$ as in Definition 4.2.2. W.l.o.g. we assume in the following that $c_0 = 1$ (note that multiplying the MA polynomial by constants is equivalent to multiplying the scale parameter σ_L of the underlying Lévy process by the same factor). Thus its MA and AR polynomials are given by

$$c(z) = z^q + c_1 z^{q-1} + \ldots + c_q = \prod_{k=1}^{q} (z-\mu_k)\,, \quad a(z) = z^p + a_1 z^{p-1} + \ldots + a_p = \prod_{j=1}^{p} (z-\lambda_j)\,,$$

where $\mu_1,\ldots,\mu_q$ denote the zeros of c and $\lambda_1,\ldots,\lambda_p$, as usual, the zeros of a. The corresponding normalized power transfer function (cf. (4.2)) can be written as

$$C\cdot\Psi(\omega) = C\cdot \frac{\prod_{k=1}^{q}(\omega^2 - 2\,\Im(\mu_k)\,\omega + |\mu_k|^2)}{\prod_{j=1}^{p}(\omega^2 - 2\,\Im(\lambda_j)\,\omega + |\lambda_j|^2)} = C\cdot \frac{\prod_{k=1}^{q}(\omega + i\,\mu_k)\,(\omega - i\,\overline{\mu_k})}{\prod_{j=1}^{p}(\omega + i\,\lambda_j)\,(\omega - i\,\overline{\lambda_j})},\ \ \omega\in\mathbb{R},$$

with $C^{-1} = \int_0^\infty g^2(s)\,\mathrm{d}s$ (where g is as in (4.10b)).

The following example illustrates this relationship in the case of a CARMA$(2,1)$ process.

Example 4.4.1 (CARMA$(2,1)$ process)**.** Consider a CARMA$(2,1)$ process which is the strictly stationary solution to

$$\left(\mathrm{D}^2 + a_1\mathrm{D} + a_2\right) Y_t = (\mathrm{D}+\mu)\,\mathrm{D}L_t, \quad t\in\mathbb{R},$$

i.e. $c(z) = z+\mu$ and $a(z) = z^2 + a_1 z + a_2 = (z-\lambda_1)(z-\lambda_2)$. In this case the kernel g in (4.10b) is given by

$$g(t) = \frac{\lambda_1+\mu}{\lambda_1-\lambda_2}\,\mathrm{e}^{t\lambda_1} + \frac{\lambda_2+\mu}{\lambda_2-\lambda_1}\,\mathrm{e}^{t\lambda_2}, \quad t\ge 0,$$

and the normalized power transfer function can be written as

$$\frac{\Psi(\omega)}{\int_0^\infty g^2(s)\,\mathrm{d}s} = \frac{|c(i\omega)|^2}{\int_0^\infty g^2(s)\,\mathrm{d}s\cdot|a(i\omega)|^2} = C(a_1,a_2,\mu)\cdot\frac{\omega^2+\mu^2}{\omega^4 + (a_1^2 - 2a_2)\omega^2 + a_2^2}, \quad \omega\in\mathbb{R},$$

with $C(a_1, a_2, \mu) = \left(\int_0^\infty g^2(s)\,\mathrm{d}s\right)^{-1} = -2\lambda_1\lambda_2 \frac{\lambda_1+\lambda_2}{\mu^2+\lambda_1\lambda_2} = 2\frac{a_1 a_2}{\mu^2+a_2}$. □

Hence, we observe that the zeros of

$$\widetilde{\Psi}(\omega) := C \cdot \Psi(-i\omega) = C \cdot (-1)^{p-q} \frac{\prod_{k=1}^{q} (\omega - \mu_k)(\omega + \overline{\mu_k})}{\prod_{j=1}^{p} (\omega - \lambda_j)(\omega + \overline{\lambda_j})}$$

are given by μ_k and $-\overline{\mu_k}$, $k \in \{1, \ldots, q\}$, and the poles of $\widetilde{\Psi}$ (i.e. the zeros of $\widetilde{\Psi}^{-1}$) are λ_j and $-\overline{\lambda_j}$, $j \in \{1, \ldots, p\}$. Consequently, we will have to suppose

Assumption 4.4. *The zeros $\mu_1, \ldots, \mu_q$ of the moving average and the zeros $\lambda_1, \ldots, \lambda_p$ of the autoregressive polynomial are all distinct and possess strictly negative real parts*

in order to be able to identify the parameters of the CARMA process from its normalized power transfer function.

Remark 4.4.2. Note that Assumption 4.3 is included in Assumption 4.4. Moreover, requiring that the AR zeros $\lambda_1, \ldots, \lambda_p$ possess strictly negative real parts is a standard assumption that ensures causality of the CARMA process. The analog condition on the MA zeros guarantees invertibility. □

It is clear that Assumption 4.4 will lead to a constraint for the parameter vector $\theta := (a_1, \ldots, a_p, c_1, \ldots, c_q)^T$, i.e. θ has to be an element of some subset $\Theta \subseteq \mathbb{R}^{p+q}$. The power transfer function is henceforth denoted by Ψ_θ and its normalization by C_θ.

Our estimation heuristic is the following. Suppose we have observed the CARMA(p, q) process on the time grid $\{\Delta_n, \ldots, n\Delta_n\}$ and let $m \in \mathbb{N}^*$. Then we choose m different frequencies $\omega_j \in (0, \pi/\Delta_n)$, $j = 1, \ldots, m$, and solve the *constrained nonlinear least squares problem*

$$\widehat{\theta} := \underset{\theta\in\Theta}{\operatorname{argmin}} \sum_{j=1}^{m} \left| \log\left(C_\theta \cdot \Psi_\theta(\omega_j)\right) - \log\left(\Delta_n \widehat{T}_{n, Y^{\Delta_n}}(\omega_j \Delta_n)\right)\right|^2. \tag{4.16}$$

Remark 4.4.3. Under the additional assumption of a finite fourth moment of the driving Lévy process, the asymptotic behavior of the variance of the smoothed periodogram for ARMA models in discrete time [17, Theorem 10.4.1] and the proof of Theorem 4.3.5 suggest that for $\omega \in \mathbb{R}^*$,

$$\lim_{n\to\infty} \mathbb{E}\left[\Delta_n I_{n, Y^{\Delta_n}}(\omega\Delta_n)\right] = \sigma_L^2\, \Psi(\omega) \qquad \text{and} \tag{4.17}$$

$$\lim_{n\to\infty} \left(\sum_{|k|\le m_n} W_n(k)^2\right)^{-1} \Delta_n^2 \operatorname{Var}(T_{n, Y^{\Delta_n}}(\omega\Delta_n)) = \sigma_L^4\, \Psi^2(\omega). \tag{4.18}$$

Eq. (4.18) implies that the variance of the smoothed periodogram is higher for frequencies with a high and lower for frequencies with a low power transfer function, respectively. Together with (4.17) this suggests to use the logarithmic transformation as a *variance stabilizing technique* (see also [73, Sections 2.9.1 and 6.2.4]). We have observed in our simulation study that also in the α-stable case, this transformation made the results more reliable. □

Methods for constrained optimization and (non)linear least squares problems are discussed, for instance, in the monographs [10, 41, 70]. We have decided to use the solver MINOS and as interface the modeling language AMPL (see [42, 69] for the MINOS user's guide and a general introduction to AMPL, respectively). In the presence of linear constraints (which will be the case in our setting) MINOS solves (4.16) using a *reduced-gradient* algorithm combined with a *quasi-Newton* algorithm that is described in [68].

In our example of a CARMA$(2,1)$ process, the optimization problem (4.16) becomes the following.

Example 4.4.4 (CARMA$(2,1)$ process)**.** We consider again the CARMA$(2,1)$ process as in Example 4.4.1. Assumption 4.4 yields immediately that $a_1,\, a_2,\, \mu > 0$ must hold. Hence, the (unknown) parameter vector $\theta = (a_1,\, a_2,\, \mu)^T$ is an element of $\Theta := (0,\infty)^3$. The optimization problem in (4.16) then becomes

$$\begin{pmatrix} \widehat{a_1} \\ \widehat{a_2} \\ \widehat{\mu} \end{pmatrix} = \operatorname*{argmin}_{a_1,a_2,\mu>0} \sum_{j=1}^{m} \left| \log\left(\frac{2a_1a_2}{\mu^2+a_2} \cdot \frac{\omega_j^2+\mu^2}{\omega_j^4+(a_1^2-2a_2)\omega_j^2+a_2^2} \right) - \log\left(\Delta_n \widehat{T}_{n,Y^{\Delta_n}}(\omega_j\Delta_n) \right) \right|^2 . \tag{4.19}$$

□

Simulation Study

As announced at the beginning of this section, we will carry out a simulation study for a CARMA$(2,1)$ process in order to show how the estimation heuristic (4.19) performs in the finite-variance as well as in the stable case. Our simulation study should be compared to the one in [45, Chapter 4]. Therefore, we have chosen not only similar values of α but also comparable CARMA parameters.

For each α taking on the values 2, 1.8, 1.6, 1.4 and 1.25, we have simulated 250 different sample paths of an α-stable CARMA$(2,1)$ process with parameters $a_1 = 2$, $a_2 = 0.1$ and $\mu = 0.2$. In the Gaussian case (i.e. $\alpha = 2$) we have chosen the standard deviation of the underlying Lévy process to be $\sigma_L = 1.5$ and in the other scenarios we have fixed the same value as the scale parameter for the driving process. Every CARMA sample path is simulated by means of an Euler approximation of the corresponding SDE in its state space representation (cf. (4.9)). The mesh of the simulation time grid has been set to 0.01 and the number of total time steps is equal to 150000. The observed CARMA sample, however, is chosen to be only every 10th simulated value, i.e. the CARMA process has been observed at time points $\{\Delta_n, 2\Delta_n, \ldots, n\Delta_n\}$ with $\Delta_n = 0.1$ and $n = 15000$.

Note that in the Gaussian case, we can easily reformulate (4.19) as

$$\begin{pmatrix} \widehat{\sigma_L} \\ \widehat{a_1} \\ \widehat{a_2} \\ \widehat{\mu} \end{pmatrix} = \underset{\sigma_L, a_1, a_2, \mu > 0}{\operatorname{argmin}} \sum_{j=1}^{m} \left| \log\left(\sigma_L^2\right) + \log\left(\frac{\omega_j^2 + \mu^2}{\omega_j^4 + (a_1^2 - 2a_2)\omega_j^2 + a_2^2} \right) - \log\left(\Delta_n T_{n,Y^{\Delta_n}}(\omega_j \Delta_n)\right) \right|^2$$

by virtue of Corollary 4.3.2. Thus, by using the normalized smoothed periodogram in the Gaussian case, we shall get an estimate for the standard deviation σ_L of the underlying Lévy process on top.

For each realized time series, we computed then smoothed periodogram values for 300 equidistant frequencies ω_j in the interval $[0.005, 2\pi]$, i.e. $\omega_j = 0.005 + (j-1)/299 \cdot (2\pi - 0.005)$, $j = 1, \ldots, 300$. Our smoothing filter has $m_n = \lfloor \sqrt{n\Delta_n} \rfloor = 38$ nodes with equal weights $W_n(k) = 1/(2m_n + 1) = 1/77$ for any $|k| \leq m_n = 38$. Concerning several aspects of these (necessary) specifications in practice, we refer the reader, for instance, to [73, Chapter 7].

Our results are reported in Table 4.1. As in [45], we observe that the estimates of the CARMA parameters become better in terms of the standard deviation when α decreases. However, in terms of the bias no evident relationship is visible. In Figure 4.1, we plotted smoothed periodogram values for some selected time series in order to show the effect of the logarithmic transformation we have used (cf. also Remark 4.4.3).

4.5 Proofs of Chapter 4

We start with three lemmata that we will need for the proofs of our main results. The third one is the “Ornstein-Uhlenbeck version” of Proposition 4.3.4.

Lemma 4.5.1. *Under the same assumptions as in Theorem 4.3.1 we have for any*

		σ_L	a_1	a_2	μ
	True	1.5	2.0	0.1	0.2
	Mean	1.5127	2.0859	0.1182	0.2159
$\alpha = 2$	Bias	0.0127	0.0859	0.0182	0.0159
	Std. dev.	0.0392	0.1204	0.0358	0.0366
	Mean	-	2.0580	0.1108	0.2185
$\alpha = 1.8$	Bias	-	0.0580	0.0108	0.0185
	Std. dev.	-	0.1240	0.0372	0.0378
	Mean	-	2.0626	0.1079	0.2127
$\alpha = 1.6$	Bias	-	0.0626	0.0079	0.0127
	Std. dev.	-	0.1130	0.0315	0.0361
	Mean	-	2.0659	0.1101	0.2129
$\alpha = 1.4$	Bias	-	0.0659	0.0101	0.0129
	Std. dev.	-	0.1151	0.0311	0.0329
	Mean	-	2.0776	0.1140	0.2149
$\alpha = 1.25$	Bias	-	0.0776	0.0140	0.0149
	Std. dev.	-	0.0928	0.0286	0.0307

Table 4.1: Simulation study for different values of α, based on 250 sample paths each: mean, bias and standard deviation of the estimates for the CARMA parameters.

$\omega \in \mathbb{R}^*$,

$$\frac{1}{\Delta_n} \sum_{|k| \leq m_n} W_n(k) \frac{\left| J^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k) \right|^2}{\sum_{u=1}^n Y^2_{u\Delta_n}} \xrightarrow{\mathbb{P}} \frac{|c(i\omega)|^2}{\int_0^\infty g^2(s)\,\mathrm{d}s \cdot |a(i\omega)|^2} \qquad \text{as } n \to \infty,$$

where $J^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k) := c^T(i\omega \mathrm{I}_p - A)^{-1} e_p \left(\sum_{u=1}^n \Delta L(u\Delta_n)\, \mathrm{e}^{-i(\omega\Delta_n)_k u} \right)$ with $\Delta L(u\Delta_n) := L_{u\Delta_n} - L_{(u-1)\Delta_n}$ for any $u \in \{1, \dots, n\}$ and $n \in \mathbb{N}$.

Proof. First, we note from [24, Lemma 3.1] that $c^T(i\omega \mathrm{I}_p - A)^{-1} e_p = c(i\omega)\, a(i\omega)^{-1}$ for any $\omega \in \mathbb{R}$ and from [39, Proposition 3.8(ii)] we obtain

$$\sum_{u=1}^n Y^2_{u\Delta_n} = \sum_{j=0}^\infty g^2(j\Delta_n) \cdot \sum_{u=1}^n \Delta L(u\Delta_n)^2 + o_P\left(\Delta_n^{-1} (n\Delta_n)^{\frac{2}{\alpha}} \right) \qquad \text{as } n \to \infty.$$

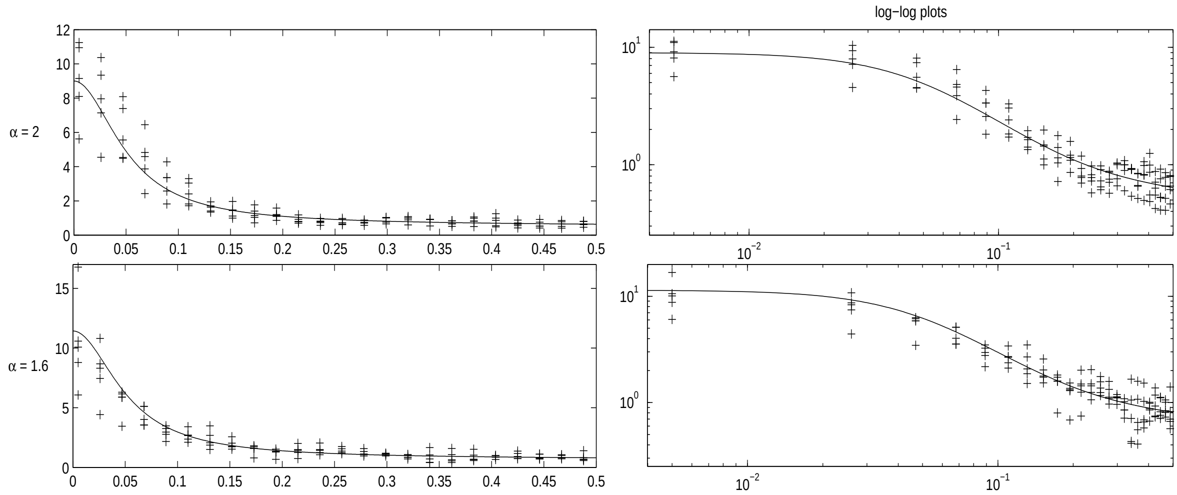

Figure 4.1: Smoothed periodogram values plotted against frequencies for five selected time series (pluses) in the Gaussian case (on top) and the 1.6-stable case (below). The true spectral density and normalized power transfer function is plotted as a solid line, respectively. The two graphs on the RHS are the left ones on a log-log scale.

Thus, we deduce

$$\frac{1}{\Delta_n}\sum_{|k|\le m_n} W_n(k)\frac{\left|J^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k)\right|^2}{\sum_{u=1}^n Y^2_{u\Delta_n}}$$
$$= \frac{|c(i\omega)|^2}{\int_0^\infty g^2(s)\,\mathrm{d}s\cdot|a(i\omega)|^2}\sum_{|k|\le m_n} W_n(k)\frac{\left|\sum_{u=1}^n \Delta L(u\Delta_n)\,\mathrm{e}^{-i(\omega\Delta_n)_k u}\right|^2}{\sum_{u=1}^n \Delta L(u\Delta_n)^2}\cdot(1+o_P(1))$$

and it is sufficient to show that

$$\sum_{|k|\le m_n} W_n(k)\frac{\left|\sum_{u=1}^n \Delta L(u\Delta_n)\,\mathrm{e}^{-i(\omega\Delta_n)_k u}\right|^2}{\sum_{u=1}^n \Delta L(u\Delta_n)^2}\xrightarrow{\mathbb{P}} 1 \quad \text{as } n\to\infty. \tag{4.20}$$

Define $Z_{n,u} := \Delta_n^{-1/\alpha}\Delta L(u\Delta_n)$ for any $n\in\mathbb{N}$, $u\in\mathbb{Z}$. If $\alpha\in(0,2)$, $(Z_{n,u})_{u\in\mathbb{Z}}$ is a sequence of i.i.d. symmetric α-stable random variables with scale parameter σ_L, and in the case $\alpha=2$ they are symmetric satisfying $\mathbb{E}[Z^2_{n,u}]=\sigma_L^2$ for any $n\in\mathbb{N}$, $u\in\mathbb{Z}$. Then we write as in the proof of [56, Lemma 6.1]

$$\sum_{|k|\le m_n} W_n(k)\frac{\left|\sum_{u=1}^n \Delta L(u\Delta_n)\,\mathrm{e}^{-i(\omega\Delta_n)_k u}\right|^2}{\sum_{u=1}^n \Delta L(u\Delta_n)^2} = \sum_{|k|\le m_n} W_n(k)\frac{\left|\sum_{u=1}^n Z_{n,u}\,\mathrm{e}^{-i(\omega\Delta_n)_k u}\right|^2}{\sum_{u=1}^n Z^2_{n,u}}$$

$$\overset{(4.8b)}{=} 1 + \sum_{1\le u\neq s\le n} \frac{Z_{n,u}\, Z_{n,s}}{\sum_{r=1}^{n} Z_{n,r}^2} \sum_{|k|\le m_n} W_n(k)\, \cos((\omega\Delta_n)_k(u-s))$$
$$=: 1 + \sum_{1\le u\neq s\le n} a_{us}(\omega\Delta_n)\cdot \frac{Z_{n,u}\, Z_{n,s}}{\sum_{r=1}^{n} Z_{n,r}^2}.$$

Now,

$$\mathbb{E}\left[\left(\sum_{1\le u\neq s\le n} a_{us}(\omega\Delta_n)\cdot \frac{Z_{n,u}\, Z_{n,s}}{\sum_{r=1}^{n} Z_{n,r}^2}\right)^2\right] = 2\,\mathbb{E}\left[\frac{Z_{n,1}^2\, Z_{n,2}^2}{\left(\sum_{u=1}^{n} Z_{n,u}^2\right)^2}\right] \sum_{1\le u\neq s\le n} a_{us}^2(\omega\Delta_n)$$
$$= O\left(n^{-2} \sum_{1\le u\neq s\le n} a_{us}^2(\omega\Delta_n)\right)$$

as $n\to\infty$, where for the first inequality we used that $(Z_{n,u})_{u\in\mathbb{Z}}$ is a sequence of i.i.d. symmetric random variables. The second equality follows from [56, Lemma 5.8] if $\alpha\in(0,2)$ and from the SLLN together with the Dominated Convergence Theorem if $\alpha=2$, respectively. Hence, in order to show (4.20), it remains to prove that for any $\omega\in\mathbb{R}^*$,

$$\sum_{1\le u\neq s\le n} a_{us}^2(\omega\Delta_n) = o(n^2) \qquad \text{as } n\to\infty. \tag{4.21}$$

By virtue of [56, Lemma 5.9(iv)] and Eq. (4.7) we obtain for some $C>0$

$$\sum_{1\le u\neq s\le n} a_{us}^2(\omega\Delta_n)$$
$$= \sum_{|k_1|,|k_2|\le m_n} W_n(k_1)W_n(k_2) \sum_{1\le u\neq s\le n} \cos\big((\omega\Delta_n)_{k_1}(u-s)\big)\cdot \cos\big((\omega\Delta_n)_{k_2}(u-s)\big)$$
$$= \frac{1}{2}\sum_{-m_n\le k_1\neq k_2\le m_n} W_n(k_1)\,W_n(k_2)\left\{\frac{\sin^2\left(\frac{k_1-k_2}{2}\right)}{\sin^2\left(\frac{k_1-k_2}{2n}\right)} + \frac{\sin^2\left(\frac{k_1+k_2}{2}+\omega n\Delta_n\right)}{\sin^2\left(\frac{k_1+k_2}{2n}+\omega\Delta_n\right)} - 2n\right\}$$
$$+ \frac{1}{2}\sum_{k=-m_n}^{m_n} W_n^2(k)\left\{n^2 + \frac{\sin^2(\omega n\Delta_n + k)}{\sin^2(\omega\Delta_n + \frac{k}{n})} - 2n\right\}$$
$$\le C\cdot n^2\left\{\sum_{-m_n\le k_1\neq k_2\le m_n} W_n(k_1)\,W_n(k_2)\left[(k_1-k_2)^{-2} + (2\omega n\Delta_n + k_1+k_2)^{-2}\right]\right.$$
$$\left. + \sum_{k=-m_n}^{m_n} W_n^2(k)\left[1 + (\omega n\Delta_n + k)^{-2}\right]\right\},$$

if n is only sufficiently large. Since $m_n\to\infty$ and $n\Delta_n\, m_n^{-1}\to\infty$ as $n\to\infty$ (see Assumption 4.2), we deduce that, for any $k_1\neq k_2\in\{-m_n,\,\dots,\,m_n\}$ and $\omega\in\mathbb{R}^*$, the term

$(2\omega n\Delta_n + k_1 + k_2)^{-2}$ can be bounded by $(k_1 - k_2)^{-2}$ and $(\omega n\Delta_n + k)^{-2}$, $|k| \le m_n$, can be bounded by 1, respectively, for all sufficiently large n. Hence, we have

$$n^{-2} \sum_{1 \le u \ne s \le n} a_{us}^2(\omega\Delta_n) \le 2\,C \cdot \max_{|k| \le m_n} W_n^2(k) \cdot \left[2 \sum_{j=1}^{2m_n} (2m_n - j + 1) \frac{1}{j^2} + O(m_n) \right]$$
$$= 2\,C \cdot \max_{|k| \le m_n} W_n^2(k) \cdot O(m_n) \overset{(4.8c)}{=} o(1) \quad \text{as } n \to \infty,$$

which completes the proof of the lemma. □

Remark 4.5.2. If we assume only (4.8a) and (4.8b) on the weight functions W_n, Eq. (4.21) is no longer valid. However, a slight modification of the proof above shows that $\sum_{1 \le u \ne s \le n} a_{us}^2(\omega\Delta_n) = O(n^2)$ as $n \to \infty$ in this case. Hence, we still have, as $n \to \infty$,

$$\sum_{|k| \le m_n} W_n(k) \frac{\left| \sum_{u=1}^n \Delta L(u\Delta_n)\, \mathrm{e}^{-i(\omega\Delta_n)_k u} \right|^2}{\sum_{u=1}^n \Delta L(u\Delta_n)^2} = 1 + O_P(1) \tag{4.22}$$

and consequently also

$$\frac{1}{\Delta_n} \sum_{|k| \le m_n} W_n(k) \frac{\left| J_{n,\Delta_n}^{(1)}((\omega\Delta_n)_k) \right|^2}{\sum_{u=1}^n Y_{u\Delta_n}^2} = \frac{|c(i\omega)|^2}{\int_0^\infty g^2(s)\,\mathrm{d}s \cdot |a(i\omega)|^2} \cdot (1 + O_P(1)) \tag{4.23}$$

as $n \to \infty$, if we drop assumption (4.8c) on the weight functions. We will use these facts in the upcoming proofs of Lemmata 4.5.3 and 4.5.4 and Proposition 4.3.4. □

Lemma 4.5.3. *Under the same assumptions as in Proposition 4.3.4 we have for any* $\omega \in \mathbb{R}^*$,

$$\Delta_n^{-\frac{2}{\alpha}} \sum_{|k| \le m_n} W_n(k) \left| \frac{\Delta_n}{n^{1/\alpha}} J_{n,\Delta_n}((\omega\Delta_n)_k) - \frac{1}{n^{1/\alpha}} J_{n,\Delta_n}^{(1)}((\omega\Delta_n)_k) \right|^2 \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty,$$

where $J_{n,\Delta_n}^{(1)}(\,\cdot\,)$ *is as in Lemma 4.5.1.*

Proof. We split the proof in two parts. First, we will establish

$$(n\Delta_n)^{-\frac{2}{\alpha}} \sum_{|k| \le m_n} W_n(k) \left| J_{n,\Delta_n}^{(2)}((\omega\Delta_n)_k) - J_{n,\Delta_n}^{(1)}((\omega\Delta_n)_k) \right|^2 \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty, \tag{4.24}$$

where $J_{n,\Delta_n}^{(2)}((\omega\Delta_n)_k) := c^T(i\omega \mathrm{I}_p - A)^{-1} \left(\sum_{u=1}^n \xi_{n,u}^* \,\mathrm{e}^{-i(\omega\Delta_n)_k u} \right) e_p$ and $\xi_{n,u}^*$ is as in (4.12).

Thereafter, we will show that also

$$\Delta_n^{-\frac{2}{\alpha}} \sum_{|k|\leq m_n} W_n(k) \left| \frac{\Delta_n}{n^{1/\alpha}} J_{n,\,\Delta_n}((\omega\Delta_n)_k) - \frac{1}{n^{1/\alpha}} J^{(2)}_{n,\,\Delta_n}((\omega\Delta_n)_k) \right|^2 \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty. \tag{4.25}$$

Note that Eqs. (4.24) and (4.25) together imply the claim of the lemma.

As to (4.24), we observe first that, due to Assumption 4.3, the eigenvalues of A are supposed to be distinct. Hence, there exists an invertible matrix $D \in \mathbb{C}^{p\times p}$ such that $A = D\,\mathrm{diag}(\lambda_1, \ldots, \lambda_p)\,D^{-1}$ and thus,

$$\mathrm{e}^A = D\,\mathrm{diag}\left(\mathrm{e}^{\lambda_1}, \ldots, \mathrm{e}^{\lambda_p}\right) D^{-1}. \tag{4.26}$$

Setting

$$\widehat{\xi}_{n,\,u} := D^{-1}\,\xi^*_{n,\,u}\,D \overset{(4.26)}{=} \mathrm{diag}\left(\int_{(u-1)\Delta_n}^{u\Delta_n} \mathrm{e}^{(u\Delta_n - s)\lambda_1}\mathrm{d}L_s, \ldots, \int_{(u-1)\Delta_n}^{u\Delta_n} \mathrm{e}^{(u\Delta_n - s)\lambda_p}\mathrm{d}L_s\right), \tag{4.27}$$

we obtain

$$\begin{aligned} J^{(2)}_{n,\,\Delta_n}((\omega\Delta_n)_k) - J^{(1)}_{n,\,\Delta_n}((\omega\Delta_n)_k) &= c^T(i\omega\mathrm{I}_p - A)^{-1} \sum_{u=1}^{n} \left(\xi^*_{n,u} - \Delta L(u\Delta_n)\mathrm{I}_p\right) \mathrm{e}^{-i(\omega\Delta_n)_k u} e_p \\ &\overset{(4.27)}{=} c^T(i\omega\mathrm{I}_p - A)^{-1}\,D\left[\sum_{u=1}^{n}\left(\widehat{\xi}_{n,\,u} - \Delta L(u\Delta_n)\,\mathrm{I}_p\right)\mathrm{e}^{-i(\omega\Delta_n)_k u}\right] D^{-1}\,e_p \end{aligned}$$

and hence, for some $C > 0$,

$$\begin{aligned} &(n\Delta_n)^{-\frac{2}{\alpha}} \sum_{|k|\leq m_n} W_n(k) \left| J^{(2)}_{n,\,\Delta_n}((\omega\Delta_n)_k) - J^{(1)}_{n,\,\Delta_n}((\omega\Delta_n)_k) \right|^2 \\ &\quad\leq C\,(n\Delta_n)^{-\frac{2}{\alpha}} \sum_{j=1}^{p} \sum_{|k|\leq m_n} W_n(k) \left| \sum_{u=1}^{n} \mathrm{e}^{-i(\omega\Delta_n)_k u} \int_{(u-1)\Delta_n}^{u\Delta_n} \left(\mathrm{e}^{(u\Delta_n - s)\lambda_j} - 1\right) \mathrm{d}L_s \right|^2 \\ &\quad\leq 2C\,(n\Delta_n)^{-\frac{2}{\alpha}} \sum_{j=1}^{p} \sum_{|k|\leq m_n} W_n(k) \\ &\qquad\times \Bigg[\left| \sum_{u=1}^{n} \mathrm{e}^{-i(\omega\Delta_n)_k u} \int_{(u-1)\Delta_n}^{u\Delta_n} \left(\mathrm{e}^{(u\Delta_n - s)\Re(\lambda_j)} \cos((u\Delta_n - s)\Im(\lambda_j)) - 1\right) \mathrm{d}L_s \right|^2 \\ &\qquad\qquad + \left| \sum_{u=1}^{n} \mathrm{e}^{-i(\omega\Delta_n)_k u} \int_{(u-1)\Delta_n}^{u\Delta_n} \mathrm{e}^{(u\Delta_n - s)\Re(\lambda_j)} \sin((u\Delta_n - s)\Im(\lambda_j))\,\mathrm{d}L_s \right|^2 \Bigg] \end{aligned}$$

$$=: 2C\,(n\Delta_n)^{-\frac{2}{\alpha}} \sum_{j=1}^{p} I_1^{(j)} + I_2^{(j)}. \tag{4.28}$$

Again we define $Z_{n,u} := \Delta_n^{-1/\alpha} \Delta L(u\Delta_n)$ for any $n \in \mathbb{N}$, $u \in \mathbb{Z}$, and for $j \in \{1, \ldots, p\}$, we set

$$d_{\Delta_n}^{(j)} := \left(\int_0^{\Delta_n} \left| \mathrm{e}^{s\,\Re(\lambda_j)} \cos(s\,\Im(\lambda_j)) - 1 \right|^{\alpha} \mathrm{d}s \right)^{1/\alpha} \quad \text{and}$$
$$f_{\Delta_n}^{(j)} := \left(\int_0^{\Delta_n} \left| \mathrm{e}^{s\,\Re(\lambda_j)} \sin(s\,\Im(\lambda_j)) \right|^{\alpha} \mathrm{d}s \right)^{1/\alpha}.$$

We will use that $\lim_{n\to\infty} \Delta_n^{-1/\alpha} d_{\Delta_n}^{(j)} = \lim_{n\to\infty} \Delta_n^{-1/\alpha} f_{\Delta_n}^{(j)} = 0$ for any $j \in \{1, \ldots, p\}$ (cf. the proof of [39, Lemma 2.1(ii)]). Now, for any $j \in \{1, \ldots, p\}$,

$$(n\Delta_n)^{-\frac{2}{\alpha}} I_1^{(j)} \overset{\mathcal{D}}{=} \Delta_n^{-\frac{2}{\alpha}} \left(d_{\Delta_n}^{(j)} \right)^2 \cdot \sum_{|k|\le m_n} W_n(k) \left| n^{-\frac{1}{\alpha}} \sum_{u=1}^{n} Z_{n,u}\, \mathrm{e}^{-i(\omega\Delta_n)_k u} \right|^2 \quad \text{and}$$

$$(n\Delta_n)^{-\frac{2}{\alpha}} I_2^{(j)} \overset{\mathcal{D}}{=} \Delta_n^{-\frac{2}{\alpha}} \left(f_{\Delta_n}^{(j)} \right)^2 \cdot \sum_{|k|\le m_n} W_n(k) \left| n^{-\frac{1}{\alpha}} \sum_{u=1}^{n} Z_{n,u}\, \mathrm{e}^{-i(\omega\Delta_n)_k u} \right|^2.$$

Since $n^{-2/\alpha} \sum_{u=1}^{n} Z_{n,u}^2$ converges weakly as $n \to \infty$, respectively, to a (positive) $\alpha/2$-stable random variable if $\alpha \in (0, 2)$ and to σ_L^2 if $\alpha = 2$, we deduce from (4.22) that both $(n\Delta_n)^{-2/\alpha} I_1^{(j)}$ and $(n\Delta_n)^{-2/\alpha} I_2^{(j)}$ converge to 0 in probability as $n \to \infty$ for any $j \in \{1, \ldots, p\}$. This implies that the right-hand side of (4.28) converges to 0 in probability and completes the proof of Eq. (4.24).

As to (4.25), for any $k \in \{-m_n, \ldots, m_n\}$ and n sufficiently large, the inequality

$$\begin{aligned}
&\left\| \Delta_n \left(\mathrm{I}_p - \mathrm{e}^{\Delta_n \left(A - i\left(\omega + \frac{k}{n\Delta_n}\right)\mathrm{I}_p\right)} \right)^{-1} - (i\omega \mathrm{I}_p - A)^{-1} \right\| \\
&\quad \le \left\| \Delta_n \left(\mathrm{I}_p - \mathrm{e}^{\Delta_n \left(A - i\left(\omega + \frac{k}{n\Delta_n}\right)\mathrm{I}_p\right)} \right)^{-1} \right\| \cdot \left\| (i\omega \mathrm{I}_p - A)^{-1} \right\| \\
&\qquad\qquad \times \left\| i\omega \mathrm{I}_p - A - \Delta_n^{-1} \left(\mathrm{I}_p - \mathrm{e}^{\Delta_n \left(A - i\left(\omega + \frac{k}{n\Delta_n}\right)\mathrm{I}_p\right)} \right) \right\| \\
&\quad \overset{(4.26)}{\le} \text{const.} \sum_{j=1}^{p} \Delta_n \left| 1 - \mathrm{e}^{\Delta_n \left(\lambda_j - i\left(\omega + \frac{k}{n\Delta_n}\right)\right)} \right|^{-1} \cdot \Bigg[\left\| i\omega \mathrm{I}_p - A - \Delta_n^{-1} \left(\mathrm{I}_p - \mathrm{e}^{\Delta_n (A - i\omega \mathrm{I}_p)} \right) \right\| \\
&\qquad\qquad + \left\| \Delta_n^{-1} \left(\mathrm{I}_p - \mathrm{e}^{\Delta_n (A - i\omega \mathrm{I}_p)} \right) \right\| \cdot \left| 1 - \mathrm{e}^{-i\frac{k}{n}} \right| \Bigg]
\end{aligned}$$

$$
\begin{aligned}
&\leq \text{const.} \sum_{j=1}^{p} 2 \left| \lambda_j - i \left(\omega + \frac{k}{n\Delta_n} \right) \right|^{-1} \cdot \mathrm{e}^{\Delta_n \|A - i\omega \mathrm{I}_p\|} \\
&\qquad\qquad\qquad \times \left[\frac{\Delta_n}{2} \|A - i\omega \mathrm{I}_p\|^2 + \|A - i\omega \mathrm{I}_p\| \cdot \left| 1 - \mathrm{e}^{-i\frac{k}{n}} \right| \right] \\
&\leq \text{const.} \sum_{j=1}^{p} \left(|\lambda_j - i\omega| - \frac{m_n}{n\Delta_n} \right)^{-1} \cdot \mathrm{e}^{\Delta_n \|A - i\omega \mathrm{I}_p\|} \\
&\qquad\qquad\qquad \times \left[\frac{\Delta_n}{2} \|A - i\omega \mathrm{I}_p\|^2 + \|A - i\omega \mathrm{I}_p\| \cdot \frac{m_n}{n} \right] \\
&\overset{n\to\infty}{\to} 0
\end{aligned}
$$

holds, where the last convergence result follows from Assumptions 4.1 to 4.3. Thus, define

$$
\epsilon_n := \max_{|k| \leq m_n} \left\| \Delta_n \left(\mathrm{I}_p - \mathrm{e}^{\Delta_n \left(A - i\left(\omega + \frac{k}{n\Delta_n}\right)\mathrm{I}_p \right)} \right)^{-1} - (i\omega \mathrm{I}_p - A)^{-1} \right\|,
$$

where, for any $\omega \in \mathbb{R}^*$, we have $\epsilon_n \searrow 0$ as $n \to \infty$. Then, for some $C > 0$,

$$
\begin{aligned}
&\Delta_n^{-\frac{2}{\alpha}} \sum_{|k| \leq m_n} W_n(k) \left| \frac{\Delta_n}{n^{1/\alpha}} J_{n,\Delta_n}((\omega\Delta_n)_k) - \frac{1}{n^{1/\alpha}} J^{(2)}_{n,\Delta_n}((\omega\Delta_n)_k) \right|^2 \\
&= \Delta_n^{-\frac{2}{\alpha}} \sum_{|k| \leq m_n} W_n(k) \left| c^T \left[\Delta_n \left(\mathrm{I}_p - \mathrm{e}^{\Delta_n \left(A - i\left(\omega + \frac{k}{n\Delta_n}\right)\mathrm{I}_p \right)} \right)^{-1} - (i\omega \mathrm{I}_p - A)^{-1} \right] \right. \\
&\qquad\qquad\qquad \left. \times n^{-\frac{1}{\alpha}} \left(\sum_{u=1}^{n} \xi^*_{n,u}\, \mathrm{e}^{-i(\omega\Delta_n)_k u} \right) e_p \right|^2 \\
&\overset{(4.27)}{\leq} C\, \epsilon_n \sum_{j=1}^{p} \Delta_n^{-\frac{2}{\alpha}} \sum_{|k| \leq m_n} W_n(k) \left| n^{-\frac{1}{\alpha}} \sum_{u=1}^{n} \mathrm{e}^{-i(\omega\Delta_n)_k u} \int_{(u-1)\Delta_n}^{u\Delta_n} \mathrm{e}^{(u\Delta_n - s)\lambda_j}\, \mathrm{d}L_s \right|^2 \\
&\leq 2C\, \epsilon_n \sum_{j=1}^{p} \Delta_n^{-\frac{2}{\alpha}} \sum_{|k| \leq m_n} W_n(k) \left[\left| n^{-\frac{1}{\alpha}} \sum_{u=1}^{n} \mathrm{e}^{-i(\omega\Delta_n)_k u} \int_{(u-1)\Delta_n}^{u\Delta_n} \left(\mathrm{e}^{(u\Delta_n - s)\lambda_j} - 1 \right) \mathrm{d}L_s \right|^2 \right. \\
&\qquad\qquad\qquad \left. + \left| n^{-\frac{1}{\alpha}} \sum_{u=1}^{n} \Delta L(u\Delta_n)\, \mathrm{e}^{-i(\omega\Delta_n)_k u} \right|^2 \right].
\end{aligned}
\tag{4.29}
$$

Now, having in mind that $\epsilon_n \searrow 0$ as $n \to \infty$, the same arguments as used above give that the right-hand side of (4.29) converges to 0 in probability as $n \to \infty$. This completes the proof of Eq. (4.25) and hence, Lemma 4.5.3 is shown. □

Lemma 4.5.4. *Suppose $\alpha \in (0,2]$, $\sigma_L > 0$ and define a family of sequences of i.i.d. random variables $(Z_{n,u})_{u\in\mathbb{Z}}$ such that, if $\alpha \in (0,2)$, $(Z_{n,u})_{u\in\mathbb{Z}} = (S_u)_{u\in\mathbb{Z}}$ for all $n \in \mathbb{N}$ with independent symmetric α-stable random variables S_u each with scale parameter σ_L and in the case $\alpha = 2$ the random variables $Z_{n,u}$ are symmetric and satisfy $\mathbb{E}\left[Z_{n,u}^2\right] = \sigma_L^2$ for any $n \in \mathbb{N}$ and $u \in \mathbb{Z}$. Moreover, Assumptions 4.1 to 4.3 may hold, and assume that the weight functions W_n satisfy* (4.8a) *and* (4.8b)*. Then we have for any $\omega \in \mathbb{R}^*$ and $r \in \{1,\ldots,p\}$,*

$$\frac{\Delta_n^2}{n^{2/\alpha}} \sum_{|k|\le m_n} W_n(k) \left| \sum_{j=0}^{\infty} \mathrm{e}^{j(\Delta_n\lambda_r - i(\omega\Delta_n)_k)} \left[\sum_{u=1-j}^{n-j} - \sum_{u=1}^{n} \right] Z_{n,u}\, \mathrm{e}^{-i(\omega\Delta_n)_k u} \right|^2 \xrightarrow{\mathbb{P}} 0 \quad \text{as } n\to\infty.$$

Proof. We follow along the lines of [56, Lemmata 6.2 and 6.3]. Setting

$$U_{n,j}^Z(\omega) := \sum_{u=1-j}^{n-j} Z_{n,u}\, \mathrm{e}^{-i\omega u} - \sum_{u=1}^{n} Z_{n,u}\, \mathrm{e}^{-i\omega u},$$

we observe that

$$\begin{aligned}
&n^{-\frac{2}{\alpha}} \left| \sum_{j=0}^{\infty} \mathrm{e}^{j(\Delta_n\lambda_r - i(\omega\Delta_n)_k)} \left[\sum_{u=1-j}^{n-j} - \sum_{u=1}^{n} \right] Z_{n,u}\, \mathrm{e}^{-i(\omega\Delta_n)_k u} \right|^2 \\
&\le 2 \left[\left| n^{-\frac{1}{\alpha}} \sum_{j=n+1}^{\infty} \mathrm{e}^{j(\Delta_n\lambda_r - i(\omega\Delta_n)_k)} U_{n,j}^Z((\omega\Delta_n)_k) \right|^2 + \left| n^{-\frac{1}{\alpha}} \sum_{j=0}^{n} \mathrm{e}^{j(\Delta_n\lambda_r - i(\omega\Delta_n)_k)} U_{n,j}^Z((\omega\Delta_n)_k) \right|^2 \right] \\
&=: 2\left(A_{n,\Delta_n}^{(1)}((\omega\Delta_n)_k) + A_{n,\Delta_n}^{(2)}((\omega\Delta_n)_k) \right).
\end{aligned}$$

We start with the proof of

$$\Delta_n^2 \sum_{|k|\le m_n} W_n(k)\, A_{n,\Delta_n}^{(1)}((\omega\Delta_n)_k) \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n\to\infty. \tag{4.30}$$

We have

$$\begin{aligned}
&\sum_{|k|\le m_n} W_n(k)\, A_{n,\Delta_n}^{(1)}((\omega\Delta_n)_k) \\
&\quad\le 2\, n^{-\frac{2}{\alpha}} \Bigg\{ \sum_{|k|\le m_n} W_n(k) \left| \sum_{j=n+1}^{\infty} \mathrm{e}^{j\Delta_n\lambda_r - ij(\omega\Delta_n)_k} \sum_{u=1-j}^{n-j} Z_{n,u}\, \mathrm{e}^{-i(\omega\Delta_n)_k u} \right|^2 \\
&\qquad\qquad + \sum_{|k|\le m_n} W_n(k) \left| \sum_{j=n+1}^{\infty} \mathrm{e}^{j\Delta_n\lambda_r - ij(\omega\Delta_n)_k} \right|^2 \cdot \left| \sum_{u=1}^{n} Z_{n,u}\, \mathrm{e}^{-i(\omega\Delta_n)_k u} \right|^2 \Bigg\}
\end{aligned}$$

$$=: 2\, n^{-\frac{2}{\alpha}} (V_1 + V_2)$$

and

$$\Delta_n^2\, n^{-\frac{2}{\alpha}}\, V_2 \le \Delta_n^2 \left(\sum_{j=n+1}^{\infty} \mathrm{e}^{j\Delta_n \Re(\lambda_r)} \right)^2 \cdot \sum_{|k|\le m_n} W_n(k) \frac{\left|\sum_{u=1}^{n} Z_{n,\,u}\, \mathrm{e}^{-i(\omega\Delta_n)_k u}\right|^2}{\sum_{u=1}^{n} Z_{n,\,u}^2} \cdot n^{-\frac{2}{\alpha}} \sum_{u=1}^{n} Z_{n,\,u}^2$$

where the second term is equal to $1 + O_P(1)$ as $n \to \infty$ (this is a simple consequence of Eq. (4.22)). The third term converges, if $\alpha \in (0, 2)$, weakly to a positive $\alpha/2$-stable random variable (see, for instance, [74, Theorem 7.1]) and for $\alpha = 2$ we know due to the WLLN that $n^{-1} \sum_{u=1}^{n} Z_{n,\,u}^2 \xrightarrow{\mathbb{P}} \sigma_L^2$ as $n \to \infty$. The first term satisfies

$$\Delta_n \sum_{j=n+1}^{\infty} \mathrm{e}^{j\Delta_n \Re(\lambda_r)} = \Delta_n \frac{\mathrm{e}^{(n+1)\Delta_n \Re(\lambda_r)}}{1 - \mathrm{e}^{\Delta_n \Re(\lambda_r)}} \overset{n\to\infty}{\sim} -\frac{1}{\Re(\lambda_r)} \mathrm{e}^{n\Delta_n \Re(\lambda_r)} \to 0 \qquad \text{as } n \to \infty \quad (4.31)$$

by virtue of Assumptions 4.1 and 4.3 and hence, $\Delta_n^2\, n^{-2/\alpha}\, V_2 \xrightarrow{\mathbb{P}} 0$.

As to V_1, we get

$$\begin{aligned} V_1 \le 2 \Bigg\{ &\sum_{|k|\le m_n} W_n(k) \left| \sum_{u=-n}^{-1} Z_{n,\,u}\, \mathrm{e}^{-i(\omega\Delta_n)_k u} \sum_{j=n+1}^{n-u} \mathrm{e}^{j\Delta_n\lambda_r - ij(\omega\Delta_n)_k} \right|^2 \\ &+ \sum_{|k|\le m_n} W_n(k) \left| \sum_{u=-\infty}^{-n-1} Z_{n,\,u}\, \mathrm{e}^{-i(\omega\Delta_n)_k u} \sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n\lambda_r - ij(\omega\Delta_n)_k} \right|^2 \Bigg\} \\ =:\, & 2(V_{11} + V_{12}) \end{aligned}$$

and

$$\begin{aligned} V_{11} = &\sum_{u=-n}^{-1} Z_{n,\,u}^2 \sum_{|k|\le m_n} W_n(k) \left| \sum_{j=n+1}^{n-u} \mathrm{e}^{j\Delta_n\lambda_r - ij(\omega\Delta_n)_k} \right|^2 + \sum_{-n\le u_1 \ne u_2 \le -1} Z_{n,\,u_1} Z_{n,\,u_2} \\ &\times \sum_{|k|\le m_n} W_n(k) \sum_{j_1=n+1}^{n-u_1} \sum_{j_2=n+1}^{n-u_2} \mathrm{e}^{j_1\Delta_n\lambda_r + j_2\Delta_n\overline{\lambda_r} - i(\omega\Delta_n)_k\cdot(u_1-u_2+j_1-j_2)} \\ =:\, & V_{111} + V_{112}. \end{aligned}$$

As above, we know that $n^{-2/\alpha} \sum_{u=-n}^{-1} Z_{n,\,u}^2$ converges in distribution as $n \to \infty$. Together with Eq. (4.31) this yields

$$\Delta_n^2\, n^{-\frac{2}{\alpha}}\, V_{111} \le n^{-\frac{2}{\alpha}} \sum_{u=-n}^{-1} Z_{n,u}^2 \cdot \underbrace{\sum_{|k|\le m_n} W_n(k)}_{\overset{(4.8b)}{=}1} \cdot \left(\Delta_n \sum_{j=n+1}^{\infty} \mathrm{e}^{j\Delta_n \Re(\lambda_r)}\right)^2 \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n\to\infty. \tag{4.32}$$

For any $\varepsilon > 0$ a conditional application of Bonami's inequality (cf. [56, Section 5.2]) yields a $C(\varepsilon) > 0$ such that

$$\begin{aligned}
&\mathbb{P}\left(\Delta_n^2\, n^{-\frac{2}{\alpha}} |V_{112}| > \varepsilon\right)\\
&\quad\le \mathbb{E}\Bigg[\exp\Bigg\{-C(\varepsilon)\,\varepsilon\,\Delta_n^{-2}\, n^{\frac{2}{\alpha}}\Bigg[\sum_{-n\le u_1\ne u_2\le -1} Z_{n,u_1}^2\, Z_{n,u_2}^2 \Bigg(\sum_{|k|\le m_n} W_n(k)\\
&\qquad\qquad \times \sum_{j_1=n+1}^{n-u_1}\sum_{j_2=n+1}^{n-u_2} \Re\left(\mathrm{e}^{j_1\Delta_n\lambda_r + j_2\Delta_n\overline{\lambda_r} - i(\omega\Delta_n)_k\cdot(u_1-u_2+j_1-j_2)}\right)\Bigg)^2\Bigg]^{-1/2}\Bigg\}\Bigg]\\
&\quad\overset{(4.8b)}{\le} \mathbb{E}\Bigg[\exp\Bigg\{-C(\varepsilon)\,\varepsilon\,\Delta_n^{-2}\, n^{\frac{2}{\alpha}}\Bigg[\sum_{-n\le u_1\ne u_2\le -1} Z_{n,u_1}^2\, Z_{n,u_2}^2 \left(\sum_{j=n+1}^{\infty} \mathrm{e}^{j\Delta_n\Re(\lambda_r)}\right)^4\Bigg]^{-1/2}\Bigg\}\Bigg]\\
&\quad\le \mathbb{E}\Bigg[\exp\Bigg\{-C(\varepsilon)\,\varepsilon\left(n^{-\frac{2}{\alpha}}\sum_{u=-n}^{-1} Z_{n,u}^2\right)^{-1}\left(\Delta_n\sum_{j=n+1}^{\infty}\mathrm{e}^{j\Delta_n\Re(\lambda_r)}\right)^{-2}\Bigg\}\Bigg]
\end{aligned}$$

and the right-hand side converges to 0 as $n\to\infty$ by virtue of Eq. (4.32) and Lebesgue dominated convergence.

Hence, $\Delta_n^2\, n^{-2/\alpha}\, V_{11} \xrightarrow{\mathbb{P}} 0$ as $n\to\infty$ is shown. Concerning V_{12} we proceed similarly. We write

$$\begin{aligned}
V_{12} = \sum_{u=-\infty}^{-n-1} Z_{n,u}^2 \sum_{|k|\le m_n} W_n(k) \left|\sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n\lambda_r - ij(\omega\Delta_n)_k}\right|^2 + \sum_{-\infty\le u_1\ne u_2\le -n-1} Z_{n,u_1}\, Z_{n,u_2}\\
\times \sum_{|k|\le m_n} W_n(k) \sum_{j_1=1-u_1}^{n-u_1}\sum_{j_2=1-u_2}^{n-u_2} \mathrm{e}^{j_1\Delta_n\lambda_r + j_2\Delta_n\overline{\lambda_r} - i(\omega\Delta_n)_k\cdot(u_1-u_2+j_1-j_2)}\\
=: V_{121} + V_{122}
\end{aligned}$$

and observe that $V_{121} \le \sum_{u=-\infty}^{-n-1} Z_{n,u}^2 \left(\sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n\Re(\lambda_r)}\right)^2$. We prove that, for any $\delta \ge 0$,

$$f_n(\delta) := \mathbb{E}\left[\exp\left\{-\frac{\delta^2}{2}\Delta_n^2\, n^{-\frac{2}{\alpha}} \sum_{u=-\infty}^{-n-1} Z_{n,u}^2 \left(\sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n\Re(\lambda_r)}\right)^2\right\}\right] \to 1 \qquad \text{as } n\to\infty. \tag{4.33}$$

Note that this implies $\Delta_n^2\, n^{-2/\alpha}\, V_{121} \overset{\mathbb{P}}{\to} 0$ as $n\to\infty$. Let $(N_u)_{u\in\mathbb{Z}}$ be i.i.d. $N(0,1)$-random variables with characteristic function $\mathbb{E}[\exp(i\theta N_u)] = \exp(-\theta^2/2)$ independent of $(Z_{n,u})_{u\in\mathbb{Z}}$ for any $n\in\mathbb{N}$. Then we have for $\alpha\in(0,2)$

$$\begin{aligned}
f_n(\delta) &= \mathbb{E}\left[\exp\left\{-\frac{\delta^2}{2}\Delta_n^2\, n^{-\frac{2}{\alpha}} \sum_{u=-\infty}^{-n-1} S_u^2 \left(\sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n \Re(\lambda_r)}\right)^2\right\}\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[\exp\left\{i\delta\Delta_n\, n^{-\frac{1}{\alpha}} \sum_{u=-\infty}^{-n-1} S_u\, N_u \sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n \Re(\lambda_r)}\right\}\middle| (S_u)_{u\in\mathbb{Z}}\right]\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[\exp\left\{i\delta\Delta_n\, n^{-\frac{1}{\alpha}} \sum_{u=-\infty}^{-n-1} S_u\, N_u \sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n \Re(\lambda_r)}\right\}\middle| (N_u)_{u\in\mathbb{Z}}\right]\right]\\
&= \mathbb{E}\left[\exp\left\{-\sigma_L^\alpha \delta^\alpha \frac{\Delta_n^\alpha}{n} \sum_{u=-\infty}^{-n-1} |N_u|^\alpha \left(\sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n \Re(\lambda_r)}\right)^\alpha\right\}\right]
\end{aligned}$$

and

$$\mathbb{E}\left[\frac{\Delta_n^\alpha}{n} \sum_{u=-\infty}^{-n-1} |N_u|^\alpha \left(\sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n \Re(\lambda_r)}\right)^\alpha\right] = \mathbb{E}[|N_1|^\alpha]\cdot \frac{\Delta_n^\alpha}{n} \sum_{u=-\infty}^{-n-1} \left(\sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n \Re(\lambda_r)}\right)^\alpha \to 0$$

as $n\to\infty$, since $\mathbb{E}[|N_1|^\alpha]<\infty$ and

$$\begin{aligned}
\frac{\Delta_n^\alpha}{n} \sum_{u=-\infty}^{-n-1} \left(\sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n \Re(\lambda_r)}\right)^\alpha &= \frac{\Delta_n^\alpha}{n}\cdot \left(\frac{\mathrm{e}^{\Delta_n\Re(\lambda_r)}\left(1-\mathrm{e}^{n\Delta_n\Re(\lambda_r)}\right)}{1-\mathrm{e}^{\Delta_n\Re(\lambda_r)}}\right)^\alpha \cdot \frac{\mathrm{e}^{(n+1)\Delta_n\Re(\lambda_r)\alpha}}{1-\mathrm{e}^{\Delta_n\Re(\lambda_r)\alpha}}\\
&\overset{n\to\infty}{\sim} \left(-\frac{1}{\Re(\lambda_r)}\right)^\alpha \cdot \frac{\mathrm{e}^{n\Delta_n\Re(\lambda_r)\alpha}}{-n\Delta_n\Re(\lambda_r)\alpha} \overset{n\to\infty}{\to} 0 \qquad (4.34)
\end{aligned}$$

due to Assumptions 4.1 and 4.3. Lebesgue dominated convergence then obviously gives $f_n(\delta)\to 1$ for any $\delta\geq 0$, i.e. Eq. (4.33) is shown for $\alpha\in(0,2)$. If $\alpha=2$, we first write as above

$$f_n(\delta) = \mathbb{E}\left[\exp\left\{i\delta\Delta_n\, n^{-\frac{1}{2}} \sum_{u=-\infty}^{-n-1} Z_{n,u}\, N_u \sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n \Re(\lambda_r)}\right\}\right].$$

Then, using the independence of $(N_u)_{u\in\mathbb{Z}}$ and $(Z_{n,u})_{u\in\mathbb{Z}}$, we obtain

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{\Delta_n}{n^{1/2}} \sum_{u=-\infty}^{-n-1} Z_{n,u}\, N_u \sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n \Re(\lambda_r)}\right)^2\right] &= \frac{\Delta_n^2}{n} \sum_{u=-\infty}^{-n-1} \underbrace{\mathbb{E}\left[Z_{n,u}^2\, N_u^2\right]}_{=\sigma_L^2} \left(\sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n \Re(\lambda_r)}\right)^2\\
&\overset{n\to\infty}{\to} 0,
\end{aligned}$$

where the latter can be shown as in the case $\alpha \in (0,2)$ above (cf. (4.34)). We can apply again the Dominated Convergence Theorem and deduce that $f_n(\delta) \to 1$ for any $\delta \geq 0$ also in the case $\alpha = 2$. Hence, (4.33) and $\Delta_n^2\, n^{-2/\alpha}\, V_{121} \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$ is shown.

Analogously to V_{112} above, we obtain for V_{122} with $\varepsilon > 0$

$$\begin{aligned}
&\mathbb{P}\Big(\Delta_n^2\; n^{-\frac{2}{\alpha}} |V_{122}| > \varepsilon\Big) \\
&\quad \leq \mathbb{E}\Bigg[\exp\Bigg\{ - C(\varepsilon)\,\varepsilon\,\Delta_n^{-2}\; n^{\frac{2}{\alpha}} \Bigg[\sum_{-\infty \leq u_1 \neq u_2 \leq -n-1} Z_{n,\,u_1}^2\, Z_{n,\,u_2}^2 \Bigg(\sum_{j_1=1-u_1}^{n-u_1} \mathrm{e}^{j\Delta_n \Re(\lambda_r)} \Bigg)^2 \\
&\qquad\qquad \times \Bigg(\sum_{j_2=1-u_2}^{n-u_2} \mathrm{e}^{j\Delta_n \Re(\lambda_r)} \Bigg)^2 \Bigg]^{-1/2} \Bigg\}\Bigg] \\
&\quad \leq \mathbb{E}\Bigg[\exp\Bigg\{ -C(\varepsilon)\,\varepsilon \Bigg(\Delta_n^2\; n^{-\frac{2}{\alpha}} \sum_{u=-\infty}^{-n-1} Z_{n,\,u}^2 \Bigg(\sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n \Re(\lambda_r)} \Bigg)^2 \Bigg)^{-1} \Bigg\}\Bigg] \to 0 \quad \text{as } n \to \infty
\end{aligned}$$

due to (4.33) and, once more, Lebesgue dominated convergence.

Hence, also $\Delta_n^2\, n^{-2/\alpha}\, V_{122} \xrightarrow{\mathbb{P}} 0$ and $\Delta_n^2\, n^{-2/\alpha}\, V_{12} \xrightarrow{\mathbb{P}} 0$ holds as $n \to \infty$. Note at this point that Eq. (4.30) has been shown.

Thus, it remains to prove that also

$$\Delta_n^2 \sum_{|k| \leq m_n} W_n(k)\, A_{n,\,\Delta_n}^{(2)}((\omega\Delta_n)_k) \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n \to \infty. \tag{4.35}$$

First,

$$\begin{aligned}
&\sum_{|k| \leq m_n} W_n(k)\, A_{n,\,\Delta_n}^{(2)}((\omega\Delta_n)_k) \\
&\qquad \leq 2\, n^{-\frac{2}{\alpha}} \Bigg\{ \sum_{|k| \leq m_n} W_n(k) \Bigg| \sum_{j=1}^{n} \mathrm{e}^{j\Delta_n \lambda_r - ij(\omega\Delta_n)_k} \sum_{u=1-j}^{0} Z_{n,\,u}\, \mathrm{e}^{-i(\omega\Delta_n)_k u} \Bigg|^2 \\
&\qquad\qquad + \sum_{|k| \leq m_n} W_n(k) \Bigg| \sum_{j=1}^{n} \mathrm{e}^{j\Delta_n \lambda_r - ij(\omega\Delta_n)_k} \sum_{u=n-j+1}^{n} Z_{n,\,u}\, \mathrm{e}^{-i(\omega\Delta_n)_k u} \Bigg|^2 \Bigg\} \\
&\qquad =: 2\, n^{-\frac{2}{\alpha}} (\widetilde{V}_1 + \widetilde{V}_2)
\end{aligned}$$

and

$$\widetilde{V}_1 = \sum_{|k| \leq m_n} W_n(k) \Bigg| \sum_{u=1-n}^{0} Z_{n,\,u}\, \mathrm{e}^{-i(\omega\Delta_n)_k u} \sum_{j=1-u}^{n} \mathrm{e}^{j\Delta_n \lambda_r - ij(\omega\Delta_n)_k} \Bigg|^2$$

$$\stackrel{(4.8b)}{\leq} \sum_{u=1-n}^{0} Z_{n,u}^2 \left(\sum_{j=1-u}^{n} \mathrm{e}^{j\Delta_n \Re(\lambda_r)} \right)^2 + \sum_{1-n\leq u_1\neq u_2 \leq 0} Z_{n,u_1} Z_{n,u_2}$$
$$\times \sum_{|k|\leq m_n} W_n(k) \sum_{j_1=1-u_1}^{n} \sum_{j_2=1-u_2}^{n} \mathrm{e}^{j_1\Delta_n\lambda_r + j_2\Delta_n\overline{\lambda_r} - i(\omega\Delta_n)_k(u_1-u_2+j_1-j_2)}$$
$$=: \widetilde{V}_{11} + \widetilde{V}_{12}.$$

Now, $\widetilde{V}_{11}$ can be dealt with like V_{121} above and one observes that in order to show $\Delta_n^2\, n^{-2/\alpha}\, \widetilde{V}_{11} \stackrel{\mathbb{P}}{\to} 0$ as $n\to\infty$, it is sufficient to prove

$$\lim_{n\to\infty} \frac{\Delta_n^\alpha}{n} \sum_{u=1-n}^{0} \left(\sum_{j=1-u}^{n} \mathrm{e}^{j\Delta_n\Re(\lambda_r)} \right)^\alpha = 0. \tag{4.36}$$

This follows from [39, Lemma 2.2(iii)] by setting $p=1$ (note that in this case $\Psi_j^{\Delta_n} = \mathrm{e}^{j\Delta_n\lambda_r}$). The proof of $\Delta_n^2\, n^{-2/\alpha}\, \widetilde{V}_{12} \stackrel{\mathbb{P}}{\to} 0$ as $n\to\infty$ is then completely analog to the one of $\Delta_n^2\, n^{-2/\alpha}\, V_{122} \stackrel{\mathbb{P}}{\to} 0$ above.

Finally,

$$\widetilde{V}_2 \leq \sum_{u=1}^{n} Z_{n,u}^2 \left(\sum_{j=n+1-u}^{n} \mathrm{e}^{j\Delta_n\Re(\lambda_r)} \right)^2 + \sum_{1\leq u_1\neq u_2\leq n} Z_{n,u_1} Z_{n,u_2}$$
$$\times \sum_{|k|\leq m_n} W_n(k) \sum_{j_1=n+1-u_1}^{n} \sum_{j_2=n+1-u_2}^{n} \mathrm{e}^{j_1\Delta_n\lambda_r + j_2\Delta_n\overline{\lambda_r} - i(\omega\Delta_n)_k(u_1-u_2+j_1-j_2)}$$
$$=: \widetilde{V}_{21} + \widetilde{V}_{22}$$

and in order to show $\Delta_n^2\, n^{-2/\alpha}\, \widetilde{V}_{21} \stackrel{\mathbb{P}}{\to} 0$ as $n\to\infty$, it is, as for $\widetilde{V}_{11}$, sufficient to prove

$$\lim_{n\to\infty} \frac{\Delta_n^\alpha}{n} \sum_{u=1}^{n} \left(\sum_{j=n+1-u}^{n} \mathrm{e}^{j\Delta_n\Re(\lambda_r)} \right)^\alpha = 0.$$

However, this is exactly Eq. (4.36). As for V_{122} and $\widetilde{V}_{12}$, one obtains that $\Delta_n^2\, n^{-2/\alpha}\, \widetilde{V}_{22} \stackrel{\mathbb{P}}{\to} 0$ as $n\to\infty$, as well. This completes the proof of Eq. (4.35).

Equations (4.30) and (4.35) together yield the statement of the lemma. □

Proof of Proposition 4.3.3. Lemma 4.5.1 has shown that, for any $\omega\in\mathbb{R}^*$,

$$\frac{1}{\Delta_n} \sum_{|k|\leq m_n} W_n(k) \frac{\left| J_{n,\Delta_n}^{(1)}((\omega\Delta_n)_k) \right|^2}{\sum_{u=1}^{n} Y_{u\Delta_n}^2} \stackrel{\mathbb{P}}{\to} \frac{|c(i\omega)|^2}{\int_0^\infty g^2(s)\,\mathrm{d}s \cdot |a(i\omega)|^2} \quad \text{as } n\to\infty,$$

with $J^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k) = c^T(i\omega\mathrm{I}_p - A)^{-1}e_p \left(\sum_{u=1}^n \Delta L(u\Delta_n)\,\mathrm{e}^{-i(\omega\Delta_n)_k u}\right)$. Since the random variable $\Delta_n\,(n\Delta_n)^{-2/\alpha}\sum_{u=1}^n Y^2_{u\Delta_n}$ converges in distribution as $n\to\infty$, respectively, to $\int_0^\infty g^2(s)\,\mathrm{d}s\cdot[L,\,L]_1$ with $([L,\,L]_t)_{t\geq 0}$ being the quadratic variation process of $(L_t)_{t\geq 0}$ if $\alpha\in(0,2)$ and to $\int_0^\infty g^2(s)\,\mathrm{d}s\cdot\sigma_L^2$ if $\alpha = 2$ (cf. [36, Theorem 5.5(a)]), a straightforward application of the Cauchy-Schwarz inequality shows that it is sufficient to prove

$$\Delta_n^{-\frac{2}{\alpha}}\sum_{|k|\leq m_n} W_n(k)\left|\frac{\Delta_n}{n^{1/\alpha}}\,J_{n,\Delta_n}((\omega\Delta_n)_k) - \frac{1}{n^{1/\alpha}}\,J^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k)\right|^2 \xrightarrow{\mathbb{P}} 0 \quad \text{as } n\to\infty. \tag{4.37}$$

However, (4.37) is a consequence of Lemma 4.5.3. □

Proof of Proposition 4.3.4. First, by virtue of the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\Delta_n\sum_{|k|\leq m_n} W_n(k)\,\frac{|R_{n,\Delta_n}((\omega\Delta_n)_k)|}{\sum_{u=1}^n Y^2_{u\Delta_n}}\\
&\leq 2\Bigg(2\Delta_n\sum_{|k|\leq m_n} W_n(k)\frac{\left|J_{n,\Delta_n}((\omega\Delta_n)_k) - \frac{1}{\Delta_n}J^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k)\right|^2 + \left|\frac{1}{\Delta_n}J^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k)\right|^2}{\sum_{u=1}^n Y^2_{u\Delta_n}}\\
&\times\Delta_n\sum_{|k|\leq m_n} W_n(k)\frac{\left|c^T K_{n,\Delta_n}((\omega\Delta_n)_k)e_p\right|^2}{\sum_{u=1}^n Y^2_{u\Delta_n}}\Bigg)^{1/2} + \Delta_n\sum_{|k|\leq m_n} W_n(k)\frac{\left|c^T K_{n,\Delta_n}((\omega\Delta_n)_k)e_p\right|^2}{\sum_{u=1}^n Y^2_{u\Delta_n}},
\end{aligned}$$

where $K_{n,\Delta_n}(\cdot)$ and $J_{n,\Delta_n}(\cdot)$ are as in Eq. (4.13) and $J^{(1)}_{n,\Delta_n}(\cdot)$ has been defined in Lemma 4.5.1.

Since $\Delta_n\,(n\Delta_n)^{-2/\alpha}\sum_{u=1}^n Y^2_{u\Delta_n}$ converges in distribution, respectively, to $\int_0^\infty g^2(s)\,\mathrm{d}s\cdot[L,\,L]_1$ as $n\to\infty$ if $\alpha\in(0,2)$ with $([L,\,L]_t)_{t\geq 0}$ being the quadratic variation process of $(L_t)_{t\geq 0}$ and g the kernel function in (4.10b) and to $\int_0^\infty g^2(s)\,\mathrm{d}s\cdot\sigma_L^2$ if $\alpha = 2$ (cf. [36, Theorem 5.5(a)]), we can combine Lemma 4.5.3 and (4.23) in order to deduce that

$$\begin{aligned}
&\Delta_n\sum_{|k|\leq m_n} W_n(k)\,\frac{|R_{n,\Delta_n}((\omega\Delta_n)_k)|}{\sum_{u=1}^n Y^2_{u\Delta_n}}\\
&\qquad\leq O_P(1)\left(\Delta_n\sum_{|k|\leq m_n} W_n(k)\frac{\left|c^T K_{n,\Delta_n}((\omega\Delta_n)_k)\,e_p\right|^2}{\sum_{u=1}^n Y^2_{u\Delta_n}}\right)^{1/2}\\
&\qquad\qquad + \Delta_n\sum_{|k|\leq m_n} W_n(k)\frac{\left|c^T K_{n,\Delta_n}((\omega\Delta_n)_k)\,e_p\right|^2}{\sum_{u=1}^n Y^2_{u\Delta_n}}
\end{aligned}$$

as $n \to \infty$. Therefore, it is sufficient to prove the following:

$$\Delta_n^{2-\frac{2}{\alpha}} n^{-\frac{2}{\alpha}} \sum_{|k|\leq m_n} W_n(k) \left|c^T K_{n,\Delta_n}((\omega\Delta_n)_k)\, e_p\right|^2 \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty. \tag{4.38}$$

To this end, we define

$$\widehat{U}_{n,j}(\omega) := \sum_{u=1-j}^{n-j} \widehat{\xi}_{n,u}\, \mathrm{e}^{-i\omega u} - \sum_{u=1}^{n} \widehat{\xi}_{n,u}\, \mathrm{e}^{-i\omega u} \qquad \text{and}$$

$$\widehat{K}_{n,\Delta_n}(\omega) := \sum_{j=0}^{\infty} \mathrm{e}^{j(\Delta_n \mathrm{diag}(\lambda_1,\dots,\lambda_p) - i\omega \mathrm{I}_p)}\, \widehat{U}_{n,j}(\omega), \quad -\pi \leq \omega \leq \pi,$$

where $\widehat{\xi}_{n,u}$ is given by (4.27). Then

$$K_{n,\Delta_n}(\omega) = D \sum_{j=0}^{\infty} \mathrm{e}^{j(\Delta_n \mathrm{diag}(\lambda_1,\dots,\lambda_p) - i\omega \mathrm{I}_p)}\, \widehat{U}_{n,j}(\omega)\, D^{-1} = D\, \widehat{K}_{n,\Delta_n}(\omega)\, D^{-1},$$

which implies

$$\begin{aligned}
&\Delta_n^{2-\frac{2}{\alpha}} n^{-\frac{2}{\alpha}} \sum_{|k|\leq m_n} W_n(k) \left|c^T K_{n,\Delta_n}((\omega\Delta_n)_k)\, e_p\right|^2 \\
&\qquad \leq \text{const.} \cdot \Delta_n^{2-\frac{2}{\alpha}} n^{-\frac{2}{\alpha}} \sum_{|k|\leq m_n} W_n(k) \left\|\mathrm{vec}\big(\widehat{K}_{n,\Delta_n}((\omega\Delta_n)_k)\big)\right\|^2 \\
&\qquad = \text{const.} \cdot \sum_{r,s=1}^{p} \Delta_n^{2-\frac{2}{\alpha}} n^{-\frac{2}{\alpha}} \sum_{|k|\leq m_n} W_n(k) \left|\widehat{K}_{n,\Delta_n}^{(r,s)}((\omega\Delta_n)_k)\right|^2 \\
&\qquad = \text{const.} \cdot \sum_{r=1}^{p} \Delta_n^{2-\frac{2}{\alpha}} n^{-\frac{2}{\alpha}} \sum_{|k|\leq m_n} W_n(k) \left|\widehat{K}_{n,\Delta_n}^{(r,r)}((\omega\Delta_n)_k)\right|^2,
\end{aligned} \tag{4.39}$$

since $\widehat{K}_{n,\Delta_n}(\,\cdot\,) = \left(\widehat{K}_{n,\Delta_n}^{(r,s)}(\,\cdot\,)\right)_{r,s\in\{1,\dots,p\}}$ is diagonal.

Now, for any $r \in \{1,\dots,p\}$,

$$\begin{aligned}
&\widehat{K}_{n,\Delta_n}^{(r,r)}((\omega\Delta_n)_k) = e_r^T \widehat{K}_{n,\Delta_n}((\omega\Delta_n)_k)\, e_r \\
&= \sum_{j=0}^{\infty} \mathrm{e}^{j(\Delta_n\lambda_r - i(\omega\Delta_n)_k)} \left[\sum_{u=1-j}^{n-j} - \sum_{u=1}^{n}\right] \mathrm{e}^{-i(\omega\Delta_n)_k u} \int_{(u-1)\Delta_n}^{u\Delta_n} \mathrm{e}^{(u\Delta_n - s)\lambda_r} \mathrm{d}L_s \\
&= \sum_{j=0}^{\infty} \mathrm{e}^{j(\Delta_n\lambda_r - i(\omega\Delta_n)_k)} \left[\sum_{u=1-j}^{n-j} - \sum_{u=1}^{n}\right] \mathrm{e}^{-i(\omega\Delta_n)_k u} \int_{(u-1)\Delta_n}^{u\Delta_n} \Re\left(\mathrm{e}^{(u\Delta_n - s)\lambda_r}\right) \mathrm{d}L_s
\end{aligned}$$

$$+\, i \sum_{j=0}^{\infty} \mathrm{e}^{j(\Delta_n \lambda_r - i(\omega\Delta_n)_k)} \left[\sum_{u=1-j}^{n-j} - \sum_{u=1}^{n} \right] \mathrm{e}^{-i(\omega\Delta_n)_k u} \int_{(u-1)\Delta_n}^{u\Delta_n} \Im\left(\mathrm{e}^{(u\Delta_n - s)\lambda_r}\right) \mathrm{d}L_s.$$

We define $(Z_{n,u})_{u\in\mathbb{Z}} := \Delta_n^{-1/\alpha}(\Delta L(u\Delta_n))_{u\in\mathbb{Z}}$ such that, if $\alpha \in (0,2)$, $(Z_{n,u})_{u\in\mathbb{Z}}$ are i.i.d. symmetric α-stable random variables with scale parameter σ_L, and in the case $\alpha = 2$ it is an i.i.d. symmetric sequence satisfying $\mathbb{E}[Z_{n,u}^2] = \sigma_L^2$ for any $n \in \mathbb{N}$, $u \in \mathbb{Z}$.

Note that

$$\left(\int_{(u-1)\Delta_n}^{u\Delta_n} \Re\left(\mathrm{e}^{(u\Delta_n - s)\lambda_r}\right) \mathrm{d}L_s\right)_{u\in\mathbb{Z}} \overset{\mathcal{D}}{=} \left(\int_0^{\Delta_n} \left|\Re\left(\mathrm{e}^{s\lambda_r}\right)\right|^\alpha \mathrm{d}s\right)^{\frac{1}{\alpha}} (Z_{n,u})_{u\in\mathbb{Z}} =: C_n^{(r)}\, (Z_{n,u})_{u\in\mathbb{Z}}$$

and likewise

$$\left(\int_{(u-1)\Delta_n}^{u\Delta_n} \Im\left(\mathrm{e}^{(u\Delta_n - s)\lambda_r}\right) \mathrm{d}L_s\right)_{u\in\mathbb{Z}} \overset{\mathcal{D}}{=} \left(\int_0^{\Delta_n} \left|\Im\left(\mathrm{e}^{s\lambda_r}\right)\right|^\alpha \mathrm{d}s\right)^{\frac{1}{\alpha}} (Z_{n,u})_{u\in\mathbb{Z}} =: \widetilde{C}_n^{(r)}\, (Z_{n,u})_{u\in\mathbb{Z}}.$$

Since $C_n^{(r)} \sim \Delta_n^{1/\alpha}$ and $\Delta_n^{-1/\alpha}\, \widetilde{C}_n^{(r)} \to 0$ as $n \to \infty$ for any $r \in \{1,\ldots,p\}$ (cf. [39, Lemma 2.1(ii) and its proof]) and since, for any $r \in \{1,\ldots,p\}$,

$$\Delta_n^2\, n^{-\frac{2}{\alpha}} \sum_{|k|\le m_n} W_n(k) \left| \sum_{j=0}^{\infty} \mathrm{e}^{j(\Delta_n\lambda_r - i(\omega\Delta_n)_k)} \left[\sum_{u=1-j}^{n-j} - \sum_{u=1}^{n} \right] Z_{n,u}\, \mathrm{e}^{-i(\omega\Delta_n)_k u} \right|^2 \xrightarrow{\mathbb{P}} 0$$

as $n \to \infty$ (see Lemma 4.5.4), we obtain that the right-hand side of Eq. (4.39) converges to 0 in probability as $n \to \infty$ which in turn yields (4.38) and hence, completes the proof of the proposition. □

Proof of Proposition 4.3.7. Note first that we can understand the self-normalized periodogram as a special case of the smoothed one by choosing the weight functions W_n as $W_n(0) = 1$ and $W_n(k) = 0$ for any $k \neq 0$. These weights do not satisfy (4.8c), but obviously (4.8a) and (4.8b). With that "degenerated" choice of weight functions and Lemma 4.5.3, we deduce immediately that it is sufficient to prove the following:

$$(n\Delta_n)^{-\frac{2}{\alpha}} \left| J_{n,\Delta_n}^{(1)}(\omega\Delta_n) \right|^2 \xrightarrow{\mathcal{D}} \frac{|c(i\omega)|^2}{|a(i\omega)|^2} \cdot \left| \int_{[0,1)} \mathrm{e}^{2\pi i x}\, \mathrm{d}L_x^* \right|^2 \qquad \text{as } n \to \infty,$$

for any $\omega \in \mathbb{R}^*$, where $J_{n,\Delta_n}^{(1)}(\cdot)$ has been defined in Lemma 4.5.1. Now, it is clearly enough to show that

$$(n\Delta_n)^{-\frac{2}{\alpha}} \left| \sum_{u=1}^{n} \Delta L(u\Delta_n)\, \mathrm{e}^{-i\omega\Delta_n u} \right|^2 \xrightarrow{\mathcal{D}} \left| \int_{[0,1)} \mathrm{e}^{2\pi i x}\, \mathrm{d}L_x^* \right|^2 \qquad \text{as } n \to \infty. \tag{4.40}$$

Let $(Z_{n,u})_{u\in\mathbb{Z}} := \Delta_n^{-1/\alpha}(\Delta L(u\Delta_n))_{u\in\mathbb{Z}}$ for $n \in \mathbb{N}$. Then (4.40) follows, by virtue of the Continuous Mapping Theorem (see, for instance, [54, Theorem 13.25]), from

$$n^{-\frac{1}{\alpha}}\left(\sum_{u=1}^{n} Z_{n,u}\cos(\omega\Delta_n u), \sum_{u=1}^{n} Z_{n,u}\sin(\omega\Delta_n u)\right) \xrightarrow{\mathcal{D}} \left(\int_0^1 \cos(2\pi x)\,\mathrm{d}L_x^*, \int_0^1 \sin(2\pi x)\,\mathrm{d}L_x^*\right),$$

as $n \to \infty$, which in turn is equivalent to

$$n^{-\frac{1}{\alpha}}\sum_{u=1}^{n} \underbrace{Z_{n,u}\left(b_1\cos(\omega\Delta_n u) + b_2\sin(\omega\Delta_n u)\right)}_{=:X_{n,u}} \xrightarrow{\mathcal{D}} \int_0^1 \left[b_1\cos(2\pi x) + b_2\sin(2\pi x)\right]\,\mathrm{d}L_x^*, \tag{4.41}$$

as $n \to \infty$ for any $(b_1, b_2)^T \in \mathbb{R}^2$.

First, we prove (4.41) for $\alpha \in (0,2)$. Since $(Z_{n,u})_{u\in\mathbb{Z}}$ is an i.i.d. sequence of symmetric α-stable random variables with scale paramater σ_L, the random variable $n^{-\frac{1}{\alpha}}\sum_{u=1}^{n} X_{n,u}$ is again symmetric α-stable with scale parameter $\sigma_{n,L}$ where

$$\sigma_{n,L}^{\alpha} = \frac{\sigma_L^{\alpha}}{n}\sum_{u=1}^{n} \left|b_1\cos(\omega\Delta_n u) + b_2\sin(\omega\Delta_n u)\right|^{\alpha}.$$

Moreover, writing $\omega = 2\pi\eta$, we have

$$\begin{aligned}\sigma_{n,L}^{\alpha} &= \frac{\sigma_L^{\alpha}}{n}\sum_{u=1}^{n} \left|b_1\cos(2\pi\{\eta\Delta_n u\}) + b_2\sin(2\pi\{\eta\Delta_n u\})\right|^{\alpha} \\ &\overset{n\to\infty}{\to} \sigma_L^{\alpha}\int_0^1 \left|b_1\cos(2\pi x) + b_2\sin(2\pi x)\right|^{\alpha}\mathrm{d}x\end{aligned}$$

where the convergence can be shown as in the proof of [39, Proposition 2.6, Eq. (4.11)]. This results in (4.41) for $\alpha \in (0,2)$.

For $\alpha = 2$ we prove (4.41) with the Lindeberg-Feller Theorem (see, e.g., [34, p. 114]). Obviously, for each n, the random variables $X_{n,u}$, $1 \le u \le n$, are independent with $\mathbb{E}[X_{n,u}] = 0$ since $Z_{n,u}$ are supposed to be symmetric. Moreover, writing again $\omega = 2\pi\eta$, we have

$$\begin{aligned}\frac{1}{n}\sum_{u=1}^{n}\mathrm{Var}(X_{n,u}) &= \frac{\sigma_L^2}{n}\sum_{u=1}^{n}\left(b_1\cos(2\pi\{\eta\Delta_n u\}) + b_2\sin(2\pi\{\eta\Delta_n u\})\right)^2 \\ &\overset{n\to\infty}{\to} \sigma_L^2\int_0^1 \left(b_1\cos(2\pi x) + b_2\sin(2\pi x)\right)^2\mathrm{d}x = \sigma_L^2\left(\frac{b_1^2}{2} + \frac{b_2^2}{2}\right),\end{aligned}$$

where the convergence can be shown again as in the proof of [39, Proposition 2.6, Eq.

(4.11)]. Since, for any $\varepsilon > 0$,

$$\frac{1}{n}\sum_{u=1}^{n}\mathbb{E}\left[X_{n,u}^2\,\mathbb{1}_{\{|X_{n,u}|>\varepsilon\sqrt{n}\}}\right] \leq (|b_1|+|b_2|)^2\cdot\mathbb{E}\left[Z_{n,1}^2\,\mathbb{1}_{\left\{|Z_{n,1}|>\frac{\varepsilon\sqrt{n}}{|b_1|+|b_2|}\right\}}\right] \overset{n\to\infty}{\to} 0,$$

we can apply the Lindeberg-Feller Theorem and deduce

$$\begin{aligned} n^{-\frac{1}{2}}\sum_{u=1}^{n} Z_{n,u}\left(b_1\,\cos(\omega\Delta_n u)+b_2\,\sin(\omega\Delta_n u)\right) &\overset{\mathcal{D}}{\to} \sqrt{\sigma_L^2\left(\frac{b_1^2}{2}+\frac{b_2^2}{2}\right)}\cdot N(0,1) \\ &\overset{\mathcal{D}}{=} \sigma_L\left(\frac{b_1}{\sqrt{2}}\,N_1+\frac{b_2}{\sqrt{2}}\,N_2\right) \qquad (4.42) \\ &\overset{\mathcal{D}}{=} \int_0^1 \left[b_1\cos(2\pi x)+b_2\sin(2\pi x)\right]\,\mathrm{d}L_x^*, \end{aligned}$$

where N_1, N_2 are i.i.d. $N(0,1)$ random variables. This shows (4.41) and completes the proof. □

Proof of Theorem 4.3.5. We can understand the (self-)normalized periodogram as a special case of the smoothed one by choosing the weight functions W_n as $W_n(0)=1$ and $W_n(k)=0$ for any $k\neq 0$, which do not satisfy (4.8c), but obviously (4.8a) and (4.8b). Then we can use the same partition as in Eq. (4.15) and apply Proposition 4.3.4 together with Proposition 4.3.7 to obtain the statement. □

Proof of Corollary 4.3.6. Follows from Theorem 4.3.5, (4.42) and $\frac{1}{n}\sum_{u=1}^{n} Y_{u\Delta_n}^2 \overset{\mathbb{P}}{\to} \int_0^\infty g^2(s)\,\mathrm{d}s\cdot\sigma_L^2$ if $\alpha=2$ (cf. [36, Theorem 5.5(a)]). □

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