Technische Universität München
Fakultät für Mathematik

Biofilm Models with Various Nonlinear Effects: Long-time Behavior of Solutions

Anna Zhigun

Vollständiger Abdruck der von der Fakultät für Mathematik der Technischen Universität München zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigten Dissertation.

Vorsitzender:		Univ.-Prof. Dr. Martin Brokate
Prüfer der Dissertation:	1.	Univ.-Prof. Dr. Rupert Lasser
	2.	Prof. Dr. Dariusz Wrzosek (University of Warsaw/ Polen)
	3.	Prof. Dr. Atsushi Yagi (Osaka University/ Japan) nur schriftliche Beurteilung

Die Dissertation wurde am 29.01.2013 bei der Technischen Universität München eingereicht und durch die Fakultät für Mathematik am 17.06.2013 angenommen.

Contents

Preface

This study investigates the well-posedness and long-time behavior of two mathematical models for a biofilm formation in the presence of a chemoattractant.

Biofilms are accumulations of microorganisms that grow on solid surfaces or interfaces between solid surfaces and liquids. They are frequently embedded in a film of their own creation: a matrix of extracellular polymeric substance (EPS). The physiology and behavior of the microbial cells of an organism growing inside a biofilm, especially if being inside an EPS matrix, and of a free-swimming organism of the same species differ considerably. Organized and protected by EPS, biofilm populations became ubiquitous. They can be found virtually everywhere where environmental conditions allow microbial growth. Whereas many biofilms can cause negative effects like biofouling, biocorrosion and microbial infections, some of them are active in useful technologies, like self-purification of water and soil remediation.

A clear understanding of biofilm processes is highly relevant in environmental, industrial and medical engineering applications, as well as in medicine. Examples include [32]: developing better control mechanism over microbially induced effects: corrosion, surface contamination (on food production surfaces, natural or implanted surfaces in the body, teeth, contact lenses, etc.) and fouling of drinking water and food; designing more efficient and stable self-purification technologies in soil remediation and waste- and groundwater treatment.

Over the last 25 years, mathematical modeling, analysis and simulation of biofilms have greatly contributed to a better understanding of biofilm processes by explaining experimental findings and gaining insight into biofilm structure, function, dynamics, population dynamics and the stability of their processes.

Most of the proposed biofilm formation models are discrete. They describe the local cell motion on a chosen lattice in terms of a cellular automaton (a microscopic approach). These models are able to capture different effects which can be observed in connection with biofilms, but they are rather difficult to analyze. In the present study, we treat the microbial cells and the exopolysaccharide molecules surrounding them as single biomass entity described in terms of its density in time and space. Such (mesoscopic) descriptions of biofilms lead to continuous models which are much easier to handle analytically.

Based on the experimental evidence, several typical assumptions on a biofilm are usually included into a spatial biofilm model:

(i) The EPS matrix is a porous medium. The microbial cells defuse with a non-constant, density-dependent diffusion coefficient. The diffusion coefficient is a monotonically increasing function. It is zero where there is no biofilm, that is, for zero biomass density, and it tends to infinity as the biofilm density tends to its maximum possible value;

(ii) The cells cannot accumulate without bound. This means that the biomass density at any point cannot exceed the density of the tight packing state - a so-called volume filling effect.

Since our goal is to study the role of positive chemotaxis, we assume the presence of a chemoattractant, a chemical which controls the cell motion by attracting them to the areas of its higher availability. In addition to assumptions (i)-(ii), we make yet another one - also based on experimental studies. It concerns the way a biofilm responds to the chemoattractant:

(iii) In the areas with low biomass density there is little response to the chemoattractant.

Apart from actually attracting the cells, the chemoattractant may have diverse additional effects. For example, it may be a nutrient or a poison, thus having an impact on the biofilm growth, or it may be produced by the cells themselves.

The majority of the chemotaxis models that were developed and analyzed in the recent years originate from the Keller-Segel model for chemotaxis [18]. It is a system of two strongly coupled parabolic partial differential equations for two functions: the biomass density and the concentration of the chemoattractant.

Although, in its most general formulation, the Keller-Segel model allows a variety of diffusion and chemotaxis scenarios and may include growth, death and volume filling effects, it was extensively studied only for the case of free-swimming microorganism colonies. In this research, we deal with two generalizations of the Keller-Segel model for the case of a biofilm colony. Both enjoy, under certain conditions on the parameter values, the basic assumptions (i)-(iii). *Chapter 2* of this work gives a generalization of a well-posed prototype proposed in [12] (see also references therein), which merges together the classical porous medium equation [31] and the Keller-Segel model with a source term. In this part, we present the first study of the long-time dynamics for this system. In *Chapter 3*, we perform, for the first time, the full study of the well-posedness and the long-time dynamics of a nonautonomous version of the model from *Chapter 2*. An alternative model is developed in *Chapter 4*. This new model extends the prototype proposed in [32] and analyzed in [9] for a biofilm growing in the presence of a nutrient, allowing the nutrient to be a chemoattractant as well. We analyze its well-posedness and long-time dynamics and illustrate possible model behavior in numerical simulations.

Chapter 1 is a preliminary one. It deals with the functional spaces that we use throughout this work and with the notions of the global and the pull-back attractors.

Acknowledgment

I would like to express my profound gratitude to my PhD adviser Prof. Messoud Efendiev for his generous help and advice through every phase of this study. He introduced me to many beautiful applications of nonlinear analysis in partial differential equations, dynamical systems, and mathematical modeling. He has shaped me as a scientist and has always believed in me.

I wish to thank my PhD adviser Prof. Rupert Lasser who supported me with his competent and kind advice both throughout my studies at the Technical University of Munich and the PhD research.

I also very much appreciate the opportunity for the scientific and scholarly communication which I had during the years of my PhD-studies. Thus, the intense collaboration with Prof. Hermann Eberl, Prof. Takashi Senba and Prof. Dariusz Wrzosek and, as a result, a valuable feedback gave rise to the important parts of my thesis.

I am grateful to Prof. Amy Novick-Cohen for inviting me to the Technion - Israel Institute of Technology, Haifa. During my three-month stay there, she supported me with her advice and arranged my contacts with other mathematicians.

Throughout my PhD-studies, I received a full scholarship from the Elite Network of Bavaria, which enabled me to accomplish the research in a short space of time. My deepest thanks go to the Universität Bayern e.V., who deemed my work worthy of the support.

The Institute of Biomathematics and Biometry, the Helmholtz Centre Munich provided me with ideal working conditions and facilities which I could use during the years of my research.

I would like to extend my warm thanks to my colleagues at the Institute of Biomathematics and Biometry, Helmholtz Centre Munich, at the Ludwig Maximilian and the Technical Universities of Munich. The time and work shared meant a lot for me.

Lastly, I thank my mother, Stella, and my brother, Michael, most sincerely for their support. Without their encouragement, truly, nothing could have been accomplished.

Chapter 1

Preliminaries

In this chapter, we present notation and generally known facts (mostly without proofs) that we use to state and derive the results of the subsequent chapters. For the sake of convenience, we introduce the following conventions:

- $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ are sets of natural, non-negative integer, integer, real, positive real and non-negative real numbers respectively;

- $x^+ := \max\{x, 0\}$ returns the positive part of a number $x \in \mathbb{R}$;

$$\operatorname{sign}(x) := \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -1 & \text{for } x < 0 \end{cases}$$

  returns its sign;

- The integer and fractional parts of a number $x \in \mathbb{R}$ are the numbers $[x] := \max\{q \in \mathbb{Z} |\ x \geqslant q\}$ and $\{x\} := x - [x]$ respectively;

- By $|\cdot|$ we denote:

  - for a number x its absolute value $|x| = \max\{M, -M\}$;

  - for a vector $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ its Euclidean norm $|x| := \left(\sum_{i=1}^d |x_i|^2\right)^{\frac{1}{2}}$;

  - for a multiindex $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ for $d \in \mathbb{N}$ its absolute value $|\alpha| = \sum_{j=1}^d \alpha_j$;

  - for a Lebesgue measurable set its Lebesgue measure;

- By measure we always understand the Lebesgue measure;

- In a topological space X, we denote by $cl_X(A)$ the closure of a set A in X. In $\mathbb{R}^d$ we use the notation $\overline{A}$ instead. ∂A denotes the topological boundary of A.
- In a linear space X, we define $x + A := \{x + a|\ a \in A\}$ for $x \in X$, $A \subset X$.

1.1 Functional spaces and their properties

This section is devoted to the classical Lebesgue and Sobolev spaces and to some of their modifications. We refer to [1, 30] for a detailed analysis of the classical spaces.

Most of the presented spaces are normable (e.g., $L^p(\Omega)$ and $W^{s,p}(\Omega)$), some are metriziable, but not normable (e.g., $L^p_{loc}(\Omega)$ and $W^{s,p}_{loc}(\Omega)$), and some (e.g., $L^\infty_{w-*}(\Omega)$) are not even metriziable, though locally convex. Let us, therefore, before looking at concrete examples, briefly recall several facts originating from the general framework in locally convex and normed spaces. We refer to [26] (or some other standard textbook) for these as well as for the other facts from functional analysis that we use in this work.

Let X and Y be two locally convex spaces. The (continuous) dual space of X is denoted by X', its weak and weak-$*$ topologies by $\sigma(X, X')$ and $\sigma(X', X)$ respectively. $X \cong Y$ denotes the topological equivalence of X and Y, which means that they are homeomorphic.

Let F be an operator, not necessarily linear, between X and Y. F is said to be compact if it maps bounded subsets of X onto relatively compact subsets of Y. It is said to be closed if its graph $\Gamma(F) := \{(x, F(x))|\ x \in X\}$ is closed in $X \times Y$.

We will often consider embeddings of a 'smaller' locally convex space into a 'larger' one:

Definition 1.1 (Embedding)**.** *Let X and Y be two locally convex spaces. An injective linear operator $\iota : X \to Y$ is called an embedding. If such an operator ι exists between X and Y, then X is said to be embedded in Y. Further, we say that*

(1) X is continuously embedded in Y (and write $X \hookrightarrow Y$) if ι is a continuous operator;

(2) X is compactly embedded in Y (and write: $X \hookrightarrow\hookrightarrow Y$) if ι is a compact operator;

(3) X is densely and continuously embedded in Y (and write: $X \overset{d}{\hookrightarrow} Y$) if ι is a continuous operator and $\iota(X)$ is dense in Y;

(4) X is densely and compactly embedded in Y (and write: $X \overset{d}{\hookrightarrow\hookrightarrow} Y$) if ι is a compact operator and $\iota(X)$ is dense in Y.

An important property of the dense and continuous embeddings concerns duality:

Theorem 1.1 (Embedding of dual spaces)**.** *Let X and Y be two locally convex spaces. Then*

$$X \overset{d}{\hookrightarrow} Y \Rightarrow Y' \overset{d}{\hookrightarrow} X'. \tag{1.1}$$

It is well known that every compact linear operator between Banach spaces is continuous. It is even weak-to-norm continuous, which means that it is continuous between $(X, \sigma(X, X'))$ and Y. A similar property holds for the weak-$*$-to-norm continuity:

Theorem 1.2 (Weak-$*$-to-norm continuity of a compact linear operator)**.** *Let X be a normed space, Y a Banach space and F a compact linear operator between X' and Y. Then F is a continuous operator between $(X', \sigma(X', X))$ and Y.*

L^p spaces

We assume in this subsection that Ω is a nonempty measurable subset of $\mathbb{R}^d$, $d \in \mathbb{N}$. Let us denote by $L^0(\Omega)$ the (linear) space of all equivalence classes of measurable functions on Ω. Each such class consists of functions that are equal almost everywhere in Ω. As usual, we identify the functions from one equivalence class and write u instead of $[u]$.
For $p \in [1, \infty]$ the function

$$\begin{aligned} &||\cdot||_{L^p(\Omega)} : L^0(\Omega) \to [0, \infty], \\ &||u||_{L^p(\Omega)} := \begin{cases} \left(\int_\Omega |u(x)|\, dx\right)^{\frac{1}{p}} & \text{for } p \in [1, \infty), \\ \operatorname{ess\,sup}_{x \in \Omega} |u(x)| & \text{for } p = \infty \end{cases} \end{aligned}$$

is called the L^p norm. It is a well defined norm on the space

$$L^p(\Omega) := \left\{ u \in L^0(\Omega) |\ ||u||_{L^p(\Omega)} < \infty \right\}.$$

Each L^p space equipped with the L^p norm is a Banach space. The space $L^2(\Omega)$ is a Hilbert space with the scalar product

$$(u, v)_{L^2(\Omega)} := \int_\Omega u(x)v(x)\, dx \text{ for } u, v \in L^2(\Omega).$$

We write $||\cdot||_p$ instead of $||\cdot||_{L^p(\Omega)}$ and even $||\cdot||$ instead of $||\cdot||_2$ and $(\cdot,\cdot)$ instead of $(\cdot,\cdot)_{L^2(\Omega)}$ to shorten the notation.
Some of the most important results about L^p spaces are:

Theorem 1.3 (Hölder inequality)**.** *Let $p, q \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, we have $uv \in L^1(\Omega)$, and the following inequality holds*

$$||uv||_1 \leqslant ||u||_p ||v||_q.$$

Theorem 1.4 (Interpolation inequality for $L^p(\Omega)$)**.** *Let $1 \leqslant p_1 < p < p_2 \leqslant \infty$. Then for all $u \in L^{p_1}(\Omega) \cap L^{p_2}(\Omega)$ we have $u \in L^p(\Omega)$, and the following inequality holds*

$$||u||_p \leqslant ||u||_{p_1}^{1-\theta} ||u||_{p_2}^{\theta},$$

where

$$\theta := \frac{\frac{1}{p_1} - \frac{1}{p}}{\frac{1}{p_1} - \frac{1}{p_2}}.$$

Theorem 1.5 (Dual representation for $L^p(\Omega)$)**.** *Let $p \in [1, \infty)$. Put*

$$p' := \begin{cases} \frac{p}{p-1} & \text{for } p \in (1, \infty), \\ \infty & \text{for } p = 1. \end{cases}$$

Then it holds that

$$(L^p(\Omega))' \cong L^{p'}(\Omega).$$

In particular, consider the linear operator

$$\iota_p : L^{p'}(\Omega) \to (L^p(\Omega))', \ \ \iota_p(u)(v) := \int_\Omega u(x)v(x)\, dx \text{ for all } v \in L^p(\Omega).$$

ι_p is an isometric isomorphism between $L^{p'}(\Omega)$ and $(L^p(\Omega))'$.

Up to this point, we have considered the $L^\infty(\Omega)$ space, as well as other L^p spaces, equipped with the topology produced by the corresponding norm. For some of our applications (see below), the original topology appears too restrictive. We are forced to pass to weaker topologies where it is easier to prove compactness. We start with the following

Definition 1.2 ($L^\infty_{w-*}(\Omega)$ space)**.** *We define $L^\infty_{w-*}(\Omega)$ to be the set of all $L^\infty(\Omega)$-functions equipped with the topology*

$$\left\{ \iota_1^{-1}(O) \middle| \ O \in \sigma((L^1(\Omega))', L^1(\Omega)) \right\},$$

where ι_1 is the isometric isomorphism defined in Theorem 1.5.

Some properties of $L^\infty_{w-*}(\Omega)$ are collected in *Theorem 1.6* and *Remark 1.1* below.

Theorem 1.6 (Properties of $L^\infty_{w-*}(\Omega)$)**.** *(1) The space $L^\infty_{w-*}(\Omega)$ is a locally convex space;*

(2) A subset of $L^\infty(\Omega)$ is bounded in $L^\infty_{w-*}(\Omega)$ if and only if it is norm bounded;

(3) The topology of $L^\infty_{w-*}(\Omega)$, if restricted to an $L^\infty(\Omega)$ ball, is completely metrizable;

(4) $L^\infty(\Omega)$ *balls are compact in* $L^\infty_{w-*}(\Omega)$.

Sketch of the proof. Observe that ι_1 is not only an isometric isomorphism between $(L^1(\Omega))'$ and $L^\infty(\Omega)$, it is, due to *Definition 1.2*, also a linear homeomorphism between $((L^1(\Omega))', \sigma((L^1(\Omega))', L^1(\Omega)))$ and $L^\infty_{w-*}(\Omega)$. But every homeomorphism preserves metrizability and compactness properties, and every linear homeomorphism preserves locally convex structure and boundedness of subsets. Therefore, the properties (i)-(iv) are consequences of the corresponding properties of the space $((L^1(\Omega))', \sigma((L^1(\Omega))', L^1(\Omega)))$, which is the dual space of the infinitely dimensional separable Banach space $L^1(\Omega)$, equipped with the weak-$*$ topology, and the fact that compact metric spaces are complete.

□

Remark 1.1 (Further properties of $L^\infty_{w-*}(\Omega)$)**.**

(1) The weak-$*$ topology is the topology of pointwise convergence, so that the topology of $L^\infty_{w-*}(\Omega)$ can be also obtained by means of the following convergence notion: A sequence $\{v_n\}_{n\in\mathbb{N}}$ converges in $L^\infty_{w-*}(\Omega)$ to a v if and only if

$$\int_\Omega u(x)v_n(x)dx \underset{n\to\infty}{\to} \int_\Omega u(x)v(x)\, dx \text{ for all } u \in L^1(\Omega);$$

(2) A metric for the restriction of the $L^\infty_{w-*}(\Omega)$ topology to a ball of radius R centered at 0 can be defined in the following way. Let $\{u_n\}_{n\in\mathbb{N}}$ be a dense subset of $L^1(\Omega)$ and let $\{B_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers, such that

$$\sum_{n\in\mathbb{N}} B_n ||u_n||_{L^1(\Omega)} < \infty.$$

Then the function defined by

$$d_*^{(\infty)}(v_1, v_2) := \sum_{n\in\mathbb{N}} B_n \left| \int_\Omega (v_1(x) - v_2(x))u_n(x)\, dx \right| \tag{1.2}$$

for all $v_1, v_2 \in L^\infty(\Omega)$, $||v_1||_\infty, ||v_2||_\infty \leqslant R$, is an example of a metric which produces the relative topology;

(3) The property 4. from *Theorem 1.6* is equivalent to $L^\infty(\Omega) \hookrightarrow\hookrightarrow L^\infty_{w-*}(\Omega)$. This is due to the definition of compact embedding.

(4) For more information on compactness and metrizability in the weak-$*$ topology see [26].

Thus the $L^\infty(\Omega)$ balls are metrizable compact subsets of $L^\infty_{w-*}(\Omega)$. This property is used in *Section 2.3* to prove the existence of a compact absorbing (see *Section 1.3*) set. In *Sections 3.5* and *4.4*, we make use of intersections of the $L^\infty_{w-*}(\Omega)$ topology with the L^p norm topology for $p \in [1, \infty)$. The definition is as follows:

Definition 1.3 (The space $\mathcal{H}^p(\Omega)$)**.** *Let $p \in [1, \infty]$. We define $\mathcal{H}^p(\Omega)$ to be the set of all $L^\infty(\Omega)$-functions equipped, for $p = \infty$, with the topology of $L^\infty_{w-*}(\Omega)$ and, for $p \in [1, \infty)$, with the intersection of the topologies of the spaces $L^\infty_{w-*}(\Omega)$ and $L^p(\Omega)$.*

Next theorem contains the properties of the $\mathcal{H}^p(\Omega)$ spaces that are used in this work.

Theorem 1.7 (Properties of $\mathcal{H}^p(\Omega)$)**.** *Let $p \in [1, \infty)$. Then:*

(1) The space $\mathcal{H}^p(\Omega)$ is a locally convex space;

(2) A subset of $L^\infty(\Omega)$ is bounded in $\mathcal{H}^p(\Omega)$ if and only if it is norm bounded;

(3) The topology of $\mathcal{H}^p(\Omega)$, if restricted to an $L^\infty(\Omega)$ ball, is completely metrizable;

(4) The topologies of $\mathcal{H}^p(\Omega)$ and $L^2(\Omega)$, if restricted to an $L^\infty(\Omega)$ ball, coincide.

Sketch of the proof. Let $p \in [1, \infty)$.

(1) We observe that the set $\{l_u |\ u \in L^1(\Omega)\}$, where $l_u(u') := |u'(u)|$ for $u' \in (L^1(\Omega))'$, is an example of a system of seminorms on $(L^1(\Omega))'$ that generates $\sigma((L^1(\Omega))', L^1(\Omega))$ topology on $(L^1(\Omega))'$. Hence the locally convex structure of the space $L^\infty_{w-*}(\Omega)$ is given by the family $\{\omega_u |\ u \in L^1(\Omega)\}$, where $\omega_u(v) := \left|\int_\Omega u(x)v(x)\, dx\right|$ for all $u \in L^1(\Omega)$ and $v \in L^\infty(\Omega)$. Consequently, the family $\{\omega_u + ||\cdot||_p |\ u \in L^1(\Omega)\}$ is an example of a system of norms that generates the locally convex structure of $\mathcal{H}^p(\Omega)$;

(2) A set is bounded in $\mathcal{H}^p(\Omega)$ if and only if it is bounded in each of the seminorms that defines it locally convex structure. In the particular case of $\mathcal{H}^p(\Omega)$ it follows with the proof of the property (1) that a set is bounded in $\mathcal{H}^p(\Omega)$ if and only if it is bounded in each of the norms ω_u and in the L^p norm, which is equivalent to the boundedness in both $L^\infty_{w-*}(\Omega)$ and $L^p(\Omega)$. The statement now follows with the property (2) from *Theorem 1.6* and the fact that, for Ω bounded, the L^∞ norm is stronger then any other L^p norm;

(3) It is a consequence of (4);

(4) Observe first that, due to the Hölder inequality and *Theorems 1.4*, we have

$$\frac{1}{||1||_{\frac{p}{p-1}}}||v||_1 \leqslant ||v||_p \leqslant ||v||_\infty^{1-\frac{1}{p}}||v||_1^{\frac{1}{p}} \text{ for all } v \in L^\infty(\Omega).$$

This shows that the topologies of $L^p(\Omega)$ and $L^1(\Omega)$, if restricted to an $L^\infty(\Omega)$ ball, coincide. To show the property (4) it then suffices to check that the restriction of the $L^\infty_{w-*}(\Omega)$ topology is weaker than, for example, the $L^2(\Omega)$ topology.
Since the space $L^2(\Omega)$ is dense in $L^1(\Omega)$, we may assume that $\{u_n\}_{n\in\mathbb{N}} \subset$

$L^2(\Omega)$ in the definition of the metric $d_*^{(\infty)}$ from (1.2) and choose the sequence $\{B_n\}_{n\in\mathbb{N}}$ to be such that

$$\sum_{n\in\mathbb{N}} B_n ||u_n||_{L^2(\Omega)} < \infty$$

holds. Consequently, we obtain with the Hölder inequality that

$$\begin{aligned} d_*^{(\infty)}(v_1, v_2) &= \sum_{n\in\mathbb{N}} B_n \left| \int_\Omega (v_1(x) - v_2(x)) u_n(x)\, dx \right| \\ &\leqslant \sum_{n\in\mathbb{N}} B_n ||u_n||_{L^2(\Omega)} ||v_1 - v_2||_2. \end{aligned}$$

This shows that, if restricted to an $L^\infty(\Omega)$ ball, the $L^\infty_{w-*}(\Omega)$ topology is weaker then the topology of $L^2(\Omega)$.

□

Thus the $L^\infty(\Omega)$ balls are metrizable subsets of $\mathcal{H}^p(\Omega)$, and, for $p \in [1, \infty)$, a subset of an $L^\infty(\Omega)$ ball is compact if an only if it is compact in $L^p(\Omega)$. These properties we use in *Sections 3.5* and *4.4* to prove the existence of a compact absorbing (see *Sections 1.4* and *1.3*, respectively) set.

Sometimes, especially in case when Ω is unbounded, it is useful (see [5, 10]) to consider the local version of an L^p space, the space

$$L^p_{loc}(\Omega) := \left\{ u \in L^0(\Omega) |\ u \in L^p(K) \text{ for all compact sets } K \subset \Omega \right\}.$$

This space is not normable, though metrizable. Define the function

$$\begin{aligned} &|| \cdot ||_{L^p_b(\Omega)} : L^p_{loc}(\Omega) \to [0, \infty], \\ &||u||_{L^p_b(\Omega)} := \sup_{x_0 \in \mathbb{R}^d} ||u||_{L^p\left(\Omega \cap B_{x_0}(1)\right)}, \end{aligned}$$

where $B_{x_0}(1)$ is a unit ball in $\mathbb{R}^d$ centered at x_0. $|| \cdot ||_{L^p_b(\Omega)}$ is a norm on a subspace of $L^p_{loc}(\Omega)$, namely on the space

$$L^p_b(\Omega) := \left\{ u \in L^p_{loc}(\Omega) |\ ||u||_{L^p_b(\Omega)} < \infty \right\}.$$

Note that $L^1_{loc}(\Omega)$ is the largest of the presented spaces of the L^p type.

Sobolev spaces

From now on we assume Ω to be a nonempty domain (i.e. a nonempty open connected set) in $\mathbb{R}^d$. We denote by $D(\Omega)$ the (locally convex) space of all test functions over Ω. As a set, $D(\Omega)$ coincides with $C_0^\infty(\Omega)$, the set of all infinitely differentiable functions with compact support in Ω. The dual space of $D(\Omega)$, the space $D'(\Omega)$, is the space of distributions over Ω.
Distributions of the form $v \to \int_\Omega u(x)v(x)\, dx$ for $u \in L^1_{loc}(\Omega)$ are called regular.

In case of a regular distribution, we identify the distribution with the L^1_{loc} function that produces it.
For $u \in D'(\Omega)$ and $v \in D(\Omega)$ we denote by (u, v) the value of u on v. In case when $u \in L^2(\Omega)$ we recover the scalar product in $L^2(\Omega)$.
For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_d)$ we define the differential operator of the order $|\alpha|$: $D^{(\alpha)} := \left(\partial^{\alpha_1}_{x_1}, \ldots, \partial^{\alpha_d}_{x_d}\right)$, where ∂_{x_k} is the partial distributional derivative along the variable x_k and $\partial^{\alpha_k}_{x_k} = (\partial_{x_k})^{\alpha_k}$. Recall that any distribution is infinitely differentiable in the distributional sense.

For $s \in \mathbb{N}_0$ and $p \in [1, \infty]$ the function

$$||u||_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \leqslant k} \left\|D^{(\alpha)} u\right\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} & \text{for } p \in [1, \infty), \\ \max\limits_{|\alpha| \leqslant k} \left\|D^{(\alpha)} u\right\|_\infty & \text{for } p = \infty \end{cases}$$

is a well defined norm on the space

$$W^{k,p}(\Omega) := \left\{u \in L^p(\Omega) |\ D^{(\alpha)} u \in L^p(\Omega) \text{ for } |\alpha| \leqslant k\right\},$$

Equipped with the $||\cdot||_{W^{k,p}(\Omega)}$ norm, the space $W^{k,p}(\Omega)$ is the classical Sobolev space of order k. All $W^{k,p}(\Omega)$ spaces are Banach spaces. The space $H^k(\Omega) := W^{k,2}(\Omega)$ is a Hilbert space with the scalar product

$$(u, v)_{H^k(\Omega)} := \sum_{|\alpha| \leqslant k} \left(D^{(\alpha)} u, D^{(\alpha)} v\right) \text{ for } u, v \in H^k(\Omega).$$

With $k = 0$ we recover the definitions of the corresponding L^p spaces. One of those subspaces of $W^{k,p}(\Omega)$ that play an important role in partial differential equations is the space $W^{k,p}_0(\Omega)$ for Ω bounded. It consists of functions that 'vanish on the boundary' in the sense of trace (see [1]). One of the equivalent ways to define these spaces is:

$$W^{k,p}_0(\Omega) := cl_{W^{k,p}(\Omega)} (D(\Omega))$$

for $k \in \mathbb{N}$ and $p \in [1, \infty]$. For Ω bounded the seminorm

$$||\cdot||_{W^{k,p}_0(\Omega)} : W^{k,p}(\Omega) \to [0, \infty),$$

$$||u||_{W^{k,p}_0(\Omega)} := \left(\sum_{|\alpha| = k} \left\|D^{(\alpha)} u\right\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}$$

is an equivalent norm on $W^{k,p}_0(\Omega)$. This is a consequence of the Poincaré inequality:

Theorem 1.8 (Poincaré inequality)**.** *Let $p \in [1, \infty]$ and let Ω be a smooth bounded domain in $\mathbb{R}^d$. Then there exists a positive constant $P(\Omega, p)$ that depends only on Ω and p and such that*

$$||u||_p \leqslant P(\Omega, p) ||Du||_p$$

holds for all $u \in W^{1,p}_0(\Omega)$.

The norm $||\cdot||_{W_0^{k,p}(\Omega)}$ is called the energy norm. On the space $H_0^k(\Omega) := W_0^{k,2}(\Omega)$ the bilinear form defined via

$$(u,v)_{H_0^k(\Omega)} := \sum_{|\alpha|=k} \left(D^{(\alpha)}u, D^{(\alpha)}v \right) \text{ for } u,v \in H_0^k(\Omega)$$

is a scalar product. The space $W_0^{k,p}(\Omega)$ is a closed subspace of $W^{k,p}(\Omega)$, thus it is a Banach space, while the space $H_0^k(\Omega)$ is a Hilbert space.

It is often useful to consider a class of 'in-between' spaces, that is, to extend the notion of classical Sobolev spaces of non-negative integer order k to the case $s \in \mathbb{R}_0^+ \backslash \mathbb{N}_0$. One of the possible contractions uses the Slobodeckij seminorm

$$[u]_{\theta,p} := \begin{cases} \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{p\theta+d}} \, dx\, dy \right)^{\frac{1}{p}} & \text{for } p \in [1,\infty), \\ \operatorname*{ess\,sup}_{x,y\in\Omega, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\theta} & \text{for } p = \infty \end{cases}$$

defined for $\theta \in (0,1)$ and $p \in [1,\infty]$. For $s \in \mathbb{R}_0^+ \backslash \mathbb{N}_0$ and $p \in [1,\infty]$ the function

$$|| \cdot ||_{W^{s,p}(\Omega)} : W^{[s],p}(\Omega) \to [0,\infty],$$

$$||u||_{W^{s,p}(\Omega)} := \begin{cases} \left(||u||^p_{W^{[s],p}(\Omega)} + \sum_{|\alpha|=s} \left[D^{(\alpha)}u \right]^p_{\{s\},p} \right)^{\frac{1}{p}} & \text{for } p \in [1,\infty), \\ ||u||_{W^{[s],\infty}(\Omega)} + \max_{|\alpha|=s} \left[D^{(\alpha)}u \right]_{\{s\},\infty} & \text{for } p = \infty \end{cases}$$

is a well defined norm on the space

$$W^{s,p}(\Omega) := \left\{ u \in W^{[s],p}(\Omega) | \ \left[D^{(\alpha)}u \right]_{\{s\},p} < \infty \text{ for } |\alpha| = [s] \right\}.$$

The spaces $W^{s,p}(\Omega)$ for $s \in \mathbb{R}_0^+ \backslash \mathbb{N}_0$ are called Sobolev-Slobodeckij spaces. These spaces are Banach spaces. The space $H^s(\Omega) := W^{s,2}(\Omega)$ is a Hilbert space with the scalar product

$$\begin{aligned}(u,v)_{H^s(\Omega)} :=& (u,v)_{H^{[s]}(\Omega)} \\ &+ \sum_{|\alpha|=s} \int_\Omega \int_\Omega \frac{\left(D^{(\alpha)}u(x) - D^{(\alpha)}u(y) \right) \left(D^{(\alpha)}v(x) - D^{(\alpha)}v(y) \right)}{|x-y|^{2\theta+d}} \, dx\, dy.\end{aligned}$$

If the domain Ω is suitably regular then, indeed, $W^{s_2,p}$ is a subset of $W^{s_1,p}$ for all $0 \leqslant s_1 < s_2 < \infty$.
Just as in case of integer order Sobolev spaces, we can define for $s \in \mathbb{R}_0^+ \backslash \mathbb{N}_0$ and $p \in [1,\infty]$ the space

$$W_0^{s,p}(\Omega) := cl_{W^{s,p}(\Omega)} \left(D(\Omega) \right).$$

Observe that for all $s \in \mathbb{R}^+$ and $p \in [1,\infty]$ the space $D(\Omega)$ is densely and continuously embedded in the space $W_0^{s,p}(\Omega)$ by means of the identity operator.

This is because the convergence in $D(\Omega)$ is stronger than the convergence in $W_0^{s,p}(\Omega)$ and because $D(\Omega)$ is dense in $W_0^{s,p}(\Omega)$ (by definition). With (1.1) it follows that $(W_0^{s,p}(\Omega))' \overset{d}{\hookrightarrow} D'(\Omega)$. Now, for $s \in \mathbb{R}^+$ and $p \in (1,\infty]$ set

$$W^{-s,p}(\Omega) := \left(W_0^{s,p'}(\Omega)\right)', \; p' := \begin{cases} \frac{p}{p-1} & \text{for } p \in (1,\infty), \\ 1 & \text{for } p = \infty, \end{cases}$$
$$H^{-s}(\Omega) := W^{-s,2}(\Omega).$$

This is the way to define the Sobolev spaces of negative order. For $p \in (1,\infty)$ it also holds

$$\left(W^{-s,p}(\Omega)\right)' \cong W_0^{s,p'}(\Omega).$$

This is a consequence of

Theorem 1.9 (Reflexivity of $W^{s,p}(\Omega)$)**.** *Let $s \in \mathbb{R}$ and $p \in (1,\infty)$. The space $W^{s,p}(\Omega)$ is reflexive.*

For all $s \in \mathbb{R}$ and $p \in [1,\infty]$ the number $\gamma = s - \frac{d}{p}$ is called the Sobolev number (corresponding to the pair s,p). The numbers s,p and γ can be used to compare a Sobolev space with another Sobolev space or with a Hölder space. This is the subject of

Theorem 1.10 (Sobolev embedding theorem)**.** *Let Ω be smooth and bounded. Let $-\infty < s_1 < s_2 < \infty$, $1 \leqslant p_2 \leqslant p_1 \leqslant \infty$ and let γ_1 and γ_2 be the Sobolev numbers corresponding to the pairs s_1, p_1 and s_2, p_2 respectively. Then:*
(Part I)

$$\gamma_2 > \gamma_1 \Rightarrow W^{s_2,p_2}(\Omega) \hookrightarrow\hookrightarrow W^{s_1,p_1}(\Omega),$$
$$\gamma_2 = \gamma_1 \text{ and } p_1 < \infty \Rightarrow W^{s_2,p_2}(\Omega) \hookrightarrow W^{s_1,p_1}(\Omega),$$

the embedding being the identity operator. In both cases the Sobolev inequality

$$||u||_{W^{s_1,p_1}(\Omega)} \leqslant C_0(s_1,s_2,p_1,p_2)||u||_{W^{s_2,p_2}(\Omega)} \text{ for all } u \in W^{s_2,p_2}(\Omega) \tag{1.3}$$

holds. The embedding constant $C_0(s_1,s_2,p_1,p_2)$ depends only on s_1,s_2,p_1,p_2 and the domain Ω.
(Part II)

$$\gamma_2 > \gamma_1 \text{ and } p_1 = \infty,\ s_1 > 0 \Rightarrow W^{s_2,p_2}(\Omega) \hookrightarrow\hookrightarrow C^{[s_1],\{s_1\}}(\overline{\Omega}),$$

the embedding being the identity operator and the Sobolev inequality

$$||u||_{C^{[s_1],\{s_1\}}(\overline{\Omega})} \leqslant C_1(s_1,s_2,p_2)||u||_{W^{s_2,p_2}(\Omega)}$$

holds. The embedding constant $C_1(s_1,s_2,p_2)$ depends only on s_1,s_2,p_2 and Ω.

Remark 1.2. In part II of the Sobolev embedding theorem, the Sobolev spaces are compared with the spaces of continuously differentiable functions $C^k(\overline{\Omega})$ and the Hölder spaces $C^{k,\theta}(\overline{\Omega})$. They are continuous versions of the Sobolev spaces $W^{k,\infty}(\Omega)$ and the Sobolev-Slobodeckij spaces $W^{k+\theta,\infty}(\Omega)$, respectively:

$$
\begin{aligned}
&C^0(\overline{\Omega}) := C(\overline{\Omega}) := \{u : \overline{\Omega} \to \mathbb{R} |\ u \text{ continuous on } \overline{\Omega}\},\\
&C^k(\overline{\Omega}) := \left\{u \in C(\overline{\Omega}) | D^{(\alpha)}u \in C(\overline{\Omega}) \text{ for } |\alpha| \leqslant k\right\},\ k \in \mathbb{N},\\
&C^{k,\theta}(\overline{\Omega}) := \left\{u \in C^k(\overline{\Omega}) | \sup_{x,y\in\overline{\Omega},x\neq y} \frac{|D^{(\alpha)}u(x) - D^{(\alpha)}u(y)|}{|x-y|^\theta} < \infty\right\},\\
&||\cdot||_{C^{k,\theta}(\overline{\Omega})} := ||\cdot||_{W^{k+\theta,\infty}(\Omega)}, k \in \mathbb{N}_0,\ \theta \in [0,1).
\end{aligned}
$$

As in case of L^p spaces, we have an interpolation inequality for a space 'in-between':

Theorem 1.11 (Interpolation inequality for $W^{s,p}(\Omega)$)**.** *Let Ω be smooth and bounded. Let $s_1, s, s_2 \in (0,\infty)$ and $p_1, p, p_2 \in [1,\infty]$ be such that*

$$
\begin{aligned}
&s_2 > s \geqslant s_1,\\
&\gamma_2 > \gamma > \gamma_1,\\
&\theta := \frac{\gamma-\gamma_1}{\gamma_2-\gamma_1} \in \left(\frac{s-s_1}{s_2-s_1}, 1\right),
\end{aligned}
$$

where γ_1, γ and γ_2 are the Sobolev numbers corresponding to the pairs s_1, p_1, s, p and s_2, p_2 respectively. Then the following interpolation inequality holds for all $u \in W^{s_2,p_2}(\Omega)$:

$$
||u||_{W^{s,p}(\Omega)} \leqslant I(s_1,s,s_2,p_1,p,p_2)||u||^{1-\theta}_{W^{s_1,p_1}(\Omega)}||u||^{\theta}_{W^{s_2,p_2}(\Omega)}. \tag{1.4}
$$

The constant $I(s_1,s,s_2,p_1,p,p_2)$ depends only on s_1,s,s_2,p_1,p,p_2 and the domain Ω.

The following useful nonlinear version of the Sobolev inequality (1.3) is a consequence of Lemma 1.2 from [8].

Lemma 1.1. *Let $s \in (0,1)$, $p \in [1,\infty)$ and $q \in (1,\infty)$. Then there exists a constant $N(q)$ that depends only on q and such that it holds*

$$
||u||_{W^{\frac{s}{q},sq}(\Omega)} \leqslant N(q) \left\||u|^{q-1}u\right\|^{\frac{1}{q}}_{W^{s,p}(\Omega)} \text{ for all } u \in W^{s,p}(\Omega).
$$

We conclude this subsection with the definition of a local Sobolev space:

$$
W^{k,p}_{loc}(\Omega) := \left\{u \in L^p_{loc}(\Omega)|\ D^{(\alpha)}u \in L^p_{loc}(\Omega) \text{ for } |\alpha| \leqslant k\right\}.
$$

The space $W^{1,(p_1,p_0)}((a,b);E_1,E_0)$

The treatment of the parabolic problems dealt with in this work requires vector-valued analogs of the functional spaces already introduced above. Let E_1 be a Banach space. Now we focus on the functions which are defined on an interval (a,b), $-\infty \leqslant a < b \leqslant \infty$, with values in E_1:

$$u : (a,b) \to E_1.$$

Similarly to the scalar case, we define vector-valued: Lebesgue spaces $L^p((a,b);E_1)$, spaces of test functions $D((a,b);E_1)$ and distributions $D'((a,b);E_1)$, Sobolev spaces $W^{s,p}((a,b);E_1)$ and $H^s((a,b);E_1)$. Another important example of a vector-valued Sobolev space is the space in which the time derivative $\partial_t u$ maps from (a,b) to a 'larger' space E_0 such that $E_1 \hookrightarrow E_0$: we introduce for $p_1, p_0 \in [1,\infty]$ the space

$$W^{1,(p_1,p_0)}((a,b);E_1,E_0) := \{u \in L^{p_1}((a,b);E_1) |\ \partial_t u \in L^{p_0}((a,b);E_0)\}$$

equipped with the norm

$$||u||_{W^{1,(p_1,p_0)}((a,b);E_1,E_0)} := ||u||_{L^{p_1}((a,b);E_1)} + \|\partial_t u\|_{L^{p_0}((a,b);E_0)}\,.$$

Let E be an in-between Banach space:

$$E_1 \hookrightarrow E \hookrightarrow E_0.$$

Clearly we have the following continuous embedding

$$W^{1,(p_1,p_0)}((a,b);E_1,E_0) \hookrightarrow L^{p_1}((a,b);E).$$

The important compactness result for this kind of spaces is given by the following theorem [5, 22].

Theorem 1.12 (Compact embeddings in vector-valued Sobolev spaces)**.** *Let $-\infty < a < b < \infty$, $p_1 \in [1,\infty]$, $p_0 \in (1,\infty]$ and let E_1, E and E_0 be Banach spaces. Then*

$$E_1 \overset{d}{\hookrightarrow\!\!\!\to} E \overset{d}{\hookrightarrow} E_0 \Rightarrow W^{1,(p_1,p_0)}((a,b);E_1,E_0) \overset{d}{\hookrightarrow\!\!\!\to} \begin{cases} L^{p_1}((a,b);E) & \textit{for } p_1 < \infty, \\ C((a,b);E) & \textit{for } p_1 = \infty. \end{cases}$$

Remark 1.3 (Weak continuity)**.** A similar result holds in case of a reflexive space E equipped with its weak topology. In particular, the following useful weak continuity property holds: let $-\infty < a < b < \infty$, $p_0 \in (1,\infty]$ and let E and E_0 be Banach spaces, then

$$E \overset{d}{\hookrightarrow\!\!\!\to} E_0 \Rightarrow W^{1,(\infty,p_0)}((a,b);E,E_0) \overset{d}{\hookrightarrow\!\!\!\to} C((a,b);(E,\sigma(E,E'))), \qquad (1.5)$$

see [5, 22].

Positivity

We conclude the *Section 1.1* with two remarks on notation connected with positivity. The first one concerns the fact that in this work we deal with mathematical models that can describe evolution of two non-negative physical quantities: density and concentration. Such models are naturally expected to preserve positivity in the following sense:

Definition 1.4 (Positivity preserving property)**.** *We say that an initial problem is positivity preserving if its solutions are non-negative for non-negative initial data.*

The second point is that we sometimes consider non-negative functions coming from a function space. To shorten the notation, let us introduce the positive cone notion:

Definition 1.5 (Positive cone)**.** *Let X be a subset of $L^0(\Omega)$. We define the positive cone of X to be the subset X^+ of X that consists of all $u \in X$ such that $u \geqslant 0$ almost everywhere in Ω.*

1.2 Laplace operator

In this section, we consider the Laplace operator Δ from two different points of view: as product of divergence and gradient and as infinitesimal generator of an analytic semigroup.

The domain Ω is assumed to be smooth and bounded throughout the section.

Let us start with the following realization of the gradient operator:

$$\nabla : H_0^1(\Omega) \to \left(L^2(\Omega)\right)^d, \ \nabla u := (\partial_{x_1} u, \dots, \partial_{x_d} u)^t,$$

where ^{t} denotes the transposition operator. In this realization, ∇ is clearly an isometry (see the definition of the $H_0^1(\Omega)$ norm), hence its range is closed. As a consequence, it possesses the Moore-Penrose pseudoinverse [16]. It is given by the operator

$$\nabla^+ : \left(L^2(\Omega)\right)^d \to H_0^1(\Omega), \ \nabla^+ := (\nabla)^{-1} P_{Ran(\nabla)},$$

where $P_{Ran(\nabla)}$ means the orthogonal projection on the range of ∇.

Further, since $\left(L^2(\Omega)\right)^d$ is a Hilbert space, we can identify it with its dual. Then the corresponding adjoint operator coincides with a realization of the minus divergence operator $\nabla^* = -(\nabla\cdot)$, where

$$\nabla\cdot : \left(L^2(\Omega)\right)^d \to H^{-1}(\Omega), \ \nabla \cdot (u_1, \dots, u_d) := \sum_{j=1}^{d} \partial_{x_j} u_j.$$

For the minus Laplace operator, we then obtain

$$-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega), \ -\Delta = \nabla^* \nabla.$$

It follows directly from the properties of a pseudoinverse that

$$(-\Delta)^{-1} = \nabla^+\nabla^{+*},$$

$$\nabla(-\Delta)^{-1} = \nabla^{+*}, \tag{1.6}$$

$$\nabla^+\nabla = id. \tag{1.7}$$

For an isometric operator the adjoint of its pseudoinverse is also isometric, therefore

$$||\nabla^{+*}v^*||_{(L^2(\Omega))^d} = ||v^*||_{H^{-1}(\Omega)} \text{ for all } v^* \in H^{-1}(\Omega). \tag{1.8}$$

The other way to handle the Laplace operator is to apply to this operator the general theory of abstract parabolic evolution equations [34]. Having a $p \in (1, \infty)$ fixed, consider this time the Laplace operator as an unbounded operator

$$\Delta : \ L^p(\Omega) \to L^p(\Omega)$$

equipped with the domain

$$D(\Delta) := \left\{ u \in W_0^{1,p}(\Omega) | \ \Delta u \in L^p(\Omega) \right\}.$$

It is known that this operator generates an analytic semigroup $e^{t\Delta}$ and that its spectrum lies entirely in $\{\lambda \in \mathbb{R} : \lambda \leqslant -\beta\}$ for some $\beta > 0$ depending on Ω. As such, it has the following properties:

$$(-\Delta)^\mu e^{t\Delta} = e^{t\Delta}(-\Delta)^\mu, \tag{1.9}$$

$$||e^{t\Delta}(-\Delta)^\mu||_p \leqslant A(\mu, p)e^{-\beta t}t^{-\mu} \tag{1.10}$$

for all $t > 0$ and $\mu > 0$. The constant $A(\mu, p)$ depends only on μ, p and the domain Ω.

1.3 Global attractor

It is well known that the long time behavior of an autonomous dynamical system can be described in terms of its global attractor. Let us recall several definitions and facts from the general theory of attractors (for details we refer to [4, 5, 10]). For our purpose, it is enough to make the presentation for locally convex spaces.

Consider an arbitrary locally convex space $\mathcal{T}$.

Definition 1.6 (Semigroup). *A (one-parametric) family $\{S(t)\}_{t\geqslant 0}$ of operators in $\mathcal{T}$ is called a semigroup on $\mathcal{T}$ if it satisfies two conditions:*

$$S(0) = id_{\mathcal{T}},$$

$$S(t+s) = S(t) \circ S(s) \text{ for all } t, s \geqslant 0,$$

where $id_{\mathcal{T}}$ denotes the identity operator.

Remark 1.4. We call a semigroup $\{S(t)\}_{t\geqslant 0}$ continuous (closed, compact) if the semigroup operators $S(t)$ are continuous (closed, compact) operators in $\mathcal{T}$ for all $t \geqslant 0$.

Definition 1.7 (Global attractor)**.** *Let $\{S(t)\}_{t\geqslant 0}$ be a semigroup on $\mathcal{T}$. A set $\mathcal{A} \subset \mathcal{T}$ is called a global attractor for $\{S(t)\}_{t\geqslant 0}$ if*

(i) $\mathcal{A}$ is compact in $\mathcal{T}$;

(ii) $\mathcal{A}$ attracts bounded subsets of $\mathcal{T}$: for every B bounded and every neighborhood V of $\mathcal{A}$ there exists a $T(B,V) > 0$ such that $S(t)B \subset V$ for all $t \geqslant T(B,V)$;

(iii) $\mathcal{A}$ is invariant with respect to $\{S(t)\}_{t\geqslant 0}$:

$$S(t)\mathcal{A} = \mathcal{A} \text{ for all } t \geqslant 0.$$

Remark 1.5 (Uniqueness of global attractor)**.** If a global attractor exists, it is also unique. In fact, it is the maximal (with respect to inclusion) bounded invariant (with respect to $\{S(t)\}_{t\geqslant 0}$) subset of $\mathcal{T}$.

Definition 1.8 (Absorbing set)**.** *A set $C \subset \mathcal{T}$ is called absorbing for a semigroup $\{S(t)\}_{t\geqslant 0}$ if for every B bounded there exists a $T(B) > 0$ such that $S(t)B \subset C$ for all $t \geqslant T(B)$.*

Remark 1.6. Every absorbing set is an attracting set.

Remark 1.7 (Positively invariant absorbing set)**.** If $C \subset \mathcal{T}$ is an absorbing set for a semigroup $\{S(t)\}_{t\geqslant 0}$, then, clearly,

$$\bigcup_{s\in[0,T(C)]} S(s)C$$

is a positively invariant absorbing set for $\{S(t)\}_{t\geqslant 0}$. The latter means that

$$S(t)C \subset C \text{ for all } t \geqslant 0.$$

In this work, we use the following general result on the existence of global attractors in complete metric spaces (see [24]):

Theorem 1.13 (Existence of global attractor)**.** *Let $\{S(t)\}_{t\geqslant 0}$ be a closed semigroup in a complete metric space $\mathcal{E}$ having a compact absorbing set $K \subset \mathcal{E}$. Then the semigroup $\{S(t)\}_{t\geqslant 0}$ possesses the global attractor. It is given by*

$$\mathcal{A} := \bigcap_{t\geqslant 0} cl_{\mathcal{E}} \left(\bigcup_{s\geqslant t} S(s)K \right).$$

1.4 Pullback attractor

It is well known that the way to extend the attractor theory for autonomous dynamical systems to nonautonomous dynamical systems is not unique. The two essential approaches here are the uniform and the pullback attractors. Introduced in [17], further studied and developed in [5, 6], the uniform attractor is a time-independent set. Its existence for a nonautonomous system is usually obtained by means of the skew-product technique (see [5]). This object is quite useful in periodic and quasiperiodic settings and usually leads to finite dimensional dynamics. However, in more general (translation-compact, see [5]) settings, the uniform attractor often turns out to be infinite-dimensional, even if the underling dynamics are actually very simple. Such is the case, e.g., with the inhomogeneous heat equation in a bounded domain

$$\partial_t u = \Delta u + f(t), \ u|_{\partial\Omega} = 0$$

This equation produces very simple dynamics, namely, a single exponentially attracting trajectory. At the same time, the corresponding uniform attractor has an infinite dimension and infinite topological entropy [5]. To avoid such artificial effects, the pullback attractor concept was initiated in [19, 28]. We follow this approach in the present work.

Let us recall several definitions and facts from the general theory of pullback attractors (for details we refer to [19, 28] and, for further development, to [10, 13, 29] and the references therein). For our purpose, it is enough to make the presentation for locally convex spaces.

Consider an arbitrary locally convex space $\mathcal{T}$.

Definition 1.9 (Process)**.** *A (two-parametric) family $\{U(t,\tau)\}_{t\geqslant\tau}$ of operators in $\mathcal{T}$ is called a process on $\mathcal{T}$ if it satisfies two conditions:*

$$U(\tau,\tau) = id_{\mathcal{T}} \text{ for all } \tau \in \mathbb{R},$$
$$U(t,\tau) = U(t,s) \circ U(s,\tau) \text{ for all } t \geqslant s \geqslant \tau, \ \tau, s, t \in \mathbb{R},$$

where $id_{\mathcal{T}}$ denotes the identity operator.

Remark 1.8. We call a process $\{U(t,\tau)\}_{t\geqslant\tau}$ continuous (closed, compact) if the process operators $U(t,\tau)$ are continuous (closed, compact) operators in $\mathcal{T}$ for all $t \geqslant \tau$.

Definition 1.10 (Pullback attractor)**.** *Let $\{U(t,\tau)\}_{t\geqslant\tau}$ be a process on $\mathcal{T}$. A (one-parametric) family $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is called a pullback attractor for $\{U(t,\tau)\}_{t\geqslant\tau}$ if*

(i) The sets $\mathcal{A}(t)$ are compact in $\mathcal{T}$ for all $t \in \mathbb{R}$;

(ii) The pullback attracting property holds: for all $t \in \mathbb{R}$, every B bounded and every neighborhood V of $\mathcal{A}(t)$ there exists a $T(B,V,t) > 0$ such that $U(t,t-s)B \subset V$ for all $s \geqslant T(B,V,t)$;

(iii) *The invariance property holds:*

$$U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t) \text{ for all } t \geqslant \tau. \tag{1.11}$$

(iv) *The minimality property holds: for all families $\{\mathcal{A}'(t)\}_{t\in\mathbb{R}}$ of closed sets that enjoy the pullback attracting, property it holds*

$$\mathcal{A}(t) \subset \mathcal{A}'(t) \text{ for all } t \in \mathbb{R}.$$

Remark 1.9 (Uniqueness of pullback attractor)**.** If a pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ exists for a process $\{U(t,\tau)\}_{t\geqslant\tau}$ on $\mathcal{T}$, it is also unique. In fact, it is the maximal (with respect to inclusion) family of bounded subsets of $\mathcal{T}$ with invariance property (1.11): for all $\{\mathcal{A}'(t)\}_{t\in\mathbb{R}}$ that enjoy

(i) The sets $\mathcal{A}'(t)$ are bounded in $\mathcal{T}$ for all $t \in \mathbb{R}$;

(ii) $U(t,\tau)\mathcal{A}'(\tau) = \mathcal{A}'(t)$ for all $t \geqslant \tau$;

it holds

$$\mathcal{A}'(t) \subset \mathcal{A}(t) \text{ for all } t \in \mathbb{R}.$$

Definition 1.11 (Uniformly absorbing set)**.** *A set $C \subset \mathcal{T}$ is called uniformly absorbing for a process $\{U(t,\tau)\}_{t\geqslant\tau}$ in $\mathcal{T}$ if for every B bounded there exists a $T(B) > 0$ such that $U(t,t-s)B \subset C$ for all $t \in \mathbb{R}$ and $s \geqslant T(B)$.*

Remark 1.10. Every uniformly absorbing set is a pullback attracting set.

Remark 1.11 (Positively invariant uniformly absorbing set)**.** If $C \subset \mathcal{T}$ is a uniformly absorbing set for a process $\{U(t,\tau)\}_{t\geqslant\tau}$, then, clearly,

$$\bigcup_{s\in[0,T(C)]} \bigcup_{t\in\mathbb{R}} U(t,t-s)C$$

is a positively invariant uniformly absorbing set for $\{U(t,\tau)\}_{t\geqslant\tau}$. The latter means that

$$U(t,t-s)C \subset C \text{ for all } t \geqslant s.$$

It is not difficult to extend the *Theorem 1.13* to the following existence criterion for the pullback attractor in a metric space:

Theorem 1.14 (Existence of pullback attractor)**.** *Let $\{U(t,\tau)\}_{t\geqslant\tau}$ be a closed process in a complete metric space $\mathcal{E}$ having a compact uniformly absorbing set $K \subset \mathcal{E}$. Then the process $\{U(t,\tau)\}_{t\geqslant\tau}$ possesses the pullback attractor. It is given by*

$$\mathcal{A}(t) := \bigcap_{r\geqslant 0} cl_{\mathcal{E}} \left(\bigcup_{s\geqslant r} U(t,t-s)K \right) \text{ for all } t \in \mathbb{R}.$$

Remark 1.12 (Forward attractor)**.** Under the assumptions of *Theorem 1.14* the pullback attractor $\mathcal{A}(t)$ is at the same time the (uniform) forward attractor for the process $\{U(t,\tau)\}_{t\geqslant\tau}$, i.e., the following (uniform) forward attracting property (compare with the property (ii) from the *Definition 1.10*) holds: for all $\varepsilon > 0$ and B bounded there exists a $T(B,\varepsilon) > 0$ such that $U(t+s,t)B \subset O_\varepsilon(\mathcal{A}(t+s))$ for all $t \in \mathbb{R}$ and $s \geqslant T(B,\varepsilon)$. Here $O_\varepsilon(C)$ denotes the ε-neighborhood of a set $C \subset \mathcal{E}$.

Chapter 2

A biofilm model with chemotaxis effect: autonomous case

2.1 The Model

In this chapter, we consider the following model:

$$
\begin{aligned}
&\partial_t M = \nabla \cdot (|M|^\alpha \nabla M) - \nabla \cdot (|M|^\gamma \nabla \rho) + f(M, \rho) && \text{in } (0, \infty) \times \Omega, && (2.1)\\
&\partial_t \rho = \Delta \rho - g(M, \rho) && \text{in } (0, \infty) \times \Omega, && (2.2)\\
&M = 0, \quad \rho = 1 && \text{in } (0, \infty) \times \partial\Omega, && (2.3)\\
&M(0, \cdot) = M_0, \quad \rho(0, \cdot) = \rho_0 && \text{in } \Omega, && (2.4)
\end{aligned}
$$

where α and γ are given positive constants satisfying

$$\frac{\alpha}{2} + 1 \leqslant \gamma < \alpha. \tag{2.5}$$

Remark 2.1 (On condition (2.5))**.**

(1) In this study, we call conditions of this type for α and γ 'balance' conditions since they establish a balance between the diffusion and transport terms, that is, between the porous medium and the chemotaxis effects.

(2) It is clear from (2.5) that $\alpha, \gamma > 2$ should hold.

$\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ is a nonempty smooth bounded domain and $M_0 \in L^\infty(\Omega)$, $\rho_0 \in W^{1,\infty}(\Omega)$. We assume that the functions f and g satisfy the following assumptions: for all $M, \rho \in \mathbb{R}$ let

$$|f(M,\rho)| \leqslant F_1(1+|M|^{\xi})^{\frac{1}{2}} \text{ for some } \xi \in [0, \alpha-\gamma+2)\,,\ F_1 \in \mathbb{R}_0^+, \tag{2.6}$$

$$f(M,\rho)\,\mathrm{sign}(M) \leqslant -F_2|M| + F_3 \text{ for some } F_2 \in \mathbb{R}^+,\ F_3 \in \mathbb{R}_0^+, \tag{2.7}$$

$$g(M,\rho) = G_1\rho + g_2(\rho)M \text{ for some } G_1 \in \mathbb{R}_0^+, \tag{2.8}$$

$$|g_2(\rho)| \leqslant G_3 \text{ for some } G_3 \in \mathbb{R}_0^+ \tag{2.9}$$

and, in order to ensure uniqueness and non-negativity of solutions for non-negative initial data, let

$$\widetilde{f}(M,\rho) := f\left(M|M|^{\frac{2}{2+\alpha}-1}, \rho\right) - F_4 M|M|^{\frac{2}{2+\alpha}-1} \text{ for some } F_4 \in \mathbb{R}, \tag{2.10}$$

$$\widetilde{f} \in W^{1,\infty}_{loc}(\mathbb{R}^2),\ g_2 \in W^{1,\infty}_{loc}(\mathbb{R}),\ f(0,\rho) \geqslant 0 \text{ for all } \rho \in \mathbb{R},\ g_2(0) \leqslant 0. \tag{2.11}$$

The following example of functions f and g satisfies the conditions (2.6)-(2.11):

Example 2.1.

$$\begin{aligned} f(M,\rho) &= -M + \frac{(M^+)^{\frac{2+\alpha}{2}}}{(M^+)^{\frac{2+\alpha}{2}}+1}\arctan\rho, \\ g(M,\rho) &= \rho + M\frac{\rho}{\rho+1}. \end{aligned}$$

The system (2.1)-(2.4) was first introduced in [12]. It can model different formation scenarios of a biofilm population described in terms of its density M in the presence of a chemoattractant, a chemical described in terms of its concentration ρ.

Let us compare our model with the classical chemotaxis models - the models for the free-swimming populations.

As in the case of a free-swimming population with possibility of growth, the evolution equation (2.2) for the chemoattractant includes the standard linear diffusion term and a nonlinear reaction interaction term. Depending on the particular choice of the reaction interaction term, the chemoattractant can be subject to abiotic decay, be produced or degraded by the population.

The governing evolution equation, the equation (2.1) for the biomass density, differs considerably from the classical one. It includes two nonlinear spatial movement effects: a degenerate diffusion term and a chemotaxis transport term. Both diffusion and chemotaxis coefficients are density-dependent - following a power law for positive exponents α and $\gamma - 1$, respectively. Hence, both motion effects disappear in the regions with zero biomass density, and they intensify as the population grows.

The way diffusion and chemotaxis work together is responsible for the local changes in the biomass density. Extensive studies made for the chemotaxis models for the free-swimming populations show that in high dimensions the diffusion is not strong enough to dominate over the positive chemotaxis effect. This leads to local aggregation of cells and even to blow-up effect. The latter means that the biomass density becomes infinite in finite or infinite time.

In our case of a biofilm population, we can impose a 'balance' condition on the exponents α and γ so as to balance the diffusion and transport terms. As a result, we obtain a well-posed system, and its solutions are uniformly bounded in time and space.

The study of the model (2.1)-(2.4) expands [12] (see also references therein). In our research, we propose a less restrictive 'balance' condition. The condition $\alpha > \gamma$ (an improvement over the condition $\alpha \geqslant \gamma + 1$ imposed in [12]) reads: the density-dependent diffusion coefficient 'dominates' the intensity of response to the chemical signal as the population grows. This, together with the homogenous Dirichlet boundary conditions, results in the uniform boundedness of M and ρ and in their dissipation with time. On the other hand, in the areas with low biomass density the porous medium effect is due to the condition $\frac{\alpha}{2} + 1 \leqslant \gamma$ strong enough to keep the population spreading without vanishing locally, which means that the support of $M(t,\cdot)$ - the set $\overline{\{x \in \Omega : \ M(t,x) > 0\}}$ - is not shrinking in t (see *Section 2.4*).

Finally, the equation (2.1) includes a 'source' term: a nonlinear reaction interaction term, which in our case allows a more general dependence upon the biomass density than in the original model (see [12]). As usual, it corresponds to the sink/source density: net number of particles created/lost per unit time and per unit volume. The source term provides the possibility to model the impact which the chemoattractant and external forces, such as predation or intoxication, can have on the population growth and death. In particular, apart from actually attracting the biofilm cells, the chemoattractant can be a nutrient or a poison, or a product of the cells themselves.

We emphasize the fact that even the analysis for the models that include either the degenerate diffusion or the chemotaxis is rather challenging (see [31] and [18], respectively) so that, in our case of a joint model, we face significant difficulties. In this work, we consider weak solutions of the system (2.1)-(2.4). The definition is as follows:

Definition 2.1 (Weak solution)**.** *A pair of functions* (M,ρ) *defined in* $[0,\infty)\times\overline{\Omega}$ *is said to be a weak solution of (2.1)-(2.4) for* $M_0 \in L^\infty(\Omega)$, $\rho_0 \in W^{1,\infty}(\Omega)$, *if for all* $T > 0$

(i) $M \in L^\infty\left((0,T)\times\Omega\right)$, $|M|^\alpha M \in L^2\left((0,T);H_0^1(\Omega)\right)$, $\partial_t M \in L^2\left((0,T);H^{-1}(\Omega)\right)$;

(ii) $\rho - 1 \in C((0,T);W_0^{1,\infty}(\Omega))$;

(iii) (M,ρ) *satisfies the equation (2.1) in* $L^2\left((0,T);H^{-1}(\Omega)\right)$, $M(0) = M_0$ *in* $C((0,T);(L^2(\Omega),\sigma(L^2(\Omega),(L^2(\Omega))')))$*-sense and*

$$\rho(t) - 1 = e^{t\Delta}(\rho_0 - 1) - \int_0^t e^{(t-s)\Delta} g(M(s),\rho(s))\, ds$$

in $W_0^{1,\infty}(\Omega)$.

Remark 2.2 (Initial condition)**.** From $M \in L^\infty((0,T);L^2(\Omega))$ and $\partial_t M \in L^2\left((0,T);H^{-1}(\Omega)\right)$, it follows with (1.5) for $p_0 = 2$, $E = L^2(\Omega)$ and $E_1 = H^{-1}(\Omega)$ and the compact embedding (see *Theorem 1.10*) $L^2(\Omega) \overset{d}{\hookrightarrow\hookrightarrow} H^{-1}(\Omega)$ that $M \in C((0,T);(L^2(\Omega),\sigma(L^2(\Omega),(L^2(\Omega))')))$. Therefore, the initial condition for M makes sense.

For the convenience of the reader, we formulate the following

Theorem 2.1 (Well-posedness and boundedness)**.** *Let the functions f and g satisfy the assumptions (2.6)-(2.11) and the constants α and γ satisfy $\gamma \in \left[\frac{\alpha}{2}+1,\alpha\right)$. Then the initial boundary-value problem (2.1)-(2.4) is uniquely solvable (in the sense of Definition 2.1) for each pair of starting values $(M_0,\rho_0) \in L^\infty(\Omega)\times W^{1,\infty}(\Omega)$ and positivity preserving (in the sense of Definition 1.4). The solution is uniformly bounded in time in the phase space $L^\infty(\Omega)\times W^{1,\infty}(\Omega)$.*

The proof of *Theorem 2.1* for a more general (nonautonomous) setting is given in *Chapter 3* (see also [12] for the first result on the well-posedness for this model in the autonomous case).

In *Theorem 2.2* of *Section 2.2* we prove a dissipative estimate for the problem (2.1)-(2.4). As a consequence of *Theorem 2.2*, we derive in *Section 2.3* the existence of a weak global attractor for (2.1)-(2.4).

Remark 2.3 (Notation)**.** For the sake of convenience, we assume throughout this chapter that the constants B_i (appear below) for all indices i are only dependent upon the parameters of the problem, that is, upon the constants α and γ, the functions f and g and the domain Ω, and **not** upon the initial data M_0,ρ_0 or t, or, unless stated otherwise, any other parameters.

2.2 Dissipative estimate

In this section, we use the condition $\alpha > \gamma$ to establish a dissipative estimate for our model, which is necessary to show the existence of the global attractor (see *Section 2.3*). Our result reads:

Theorem 2.2 (Dissipative estimate)**.** *Let the functions f and g satisfy the assumptions (2.6)-(2.11) and let the constants α and γ satisfy $\gamma \in \left[\frac{\alpha}{2}+1,\alpha\right)$. Then the following dissipative estimate holds for the initial boundary-value problem (2.1)-(2.4):*

$$||M(t)||_{L^\infty(\Omega)} + ||\rho(t)||_{W^{1,\infty}(\Omega)} \leqslant C_\infty \left(||M_0||_{L^\infty(\Omega)} + ||\rho_0||_{W^{1,\infty}(\Omega)}\right)^{r_\infty} \cdot e^{-\omega_\infty t} + D_\infty \text{ for all } t \geqslant 0, \tag{2.12}$$

where the positive constants $C_\infty, r_\infty, \omega_\infty, D_\infty$ depend only on α,γ,f and g, and are independent of M_0,ρ_0 or t.

Remark 2.4. As will become clear from the proof below, we do not actually need the condition $\gamma \geqslant \frac{\alpha}{2}+1$ to obtain the dissipative estimate (2.12). However, this condition is crucial for uniqueness of solutions (see the proof of *Theorem 3.1* or [12]).

Proof. The main idea of the proof is to derive a collection of coupled dissipative estimates for M and $\nabla\rho$ in various L^δ norms, with $\delta < \infty$ for the M component, and then apply a bootstrap argument in order to obtain the desired dissipative estimate in the L^∞ norm for both components. The estimates are done formally, they can be justified by passing to an appropriate sequence of regularization problems (e.g., (3.11)-(3.14) from *Chapter 3*), performing the estimates in the same manner for the solutions of these problems and then passing to the limit.

We start with rewriting the equation (2.1) in the following way:

$$\partial_t M = \nabla\cdot\left(|M|^\gamma\nabla\left(\frac{1}{\alpha-\gamma+1}M|M|^{\alpha-\gamma}-\rho\right)\right)+f(M,\rho)$$

In order to derive our first a priori estimate, we multiply this equation by $\left(\frac{1}{\alpha-\gamma+1}M|M|^{\alpha-\gamma}-\rho\right)$ and integrate by parts over Ω to obtain

$$\begin{aligned}
&\left(\partial_t M,\frac{1}{\alpha-\gamma+1}M|M|^{\alpha-\gamma}-\rho\right)\\
&=-\left(|M|^\gamma,\left|\nabla\left(\frac{1}{\alpha-\gamma+1}M|M|^{\alpha-\gamma}-\rho\right)\right|^2\right)\\
&\quad+\left(f(M,\rho),\frac{1}{\alpha-\gamma+1}M|M|^{\alpha-\gamma}-\rho\right)\\
&\leqslant\left(f(M,\rho),\frac{1}{\alpha-\gamma+1}M|M|^{\alpha-\gamma}-\rho\right)\\
\Leftrightarrow\quad&\frac{d}{dt}\left(\frac{1}{(\alpha-\gamma+1)(\alpha-\gamma+2)}\left\||M|^{\frac{\alpha-\gamma+2}{2}}\right\|^2-(M,\rho)\right)\\
&\leqslant\left(f(M,\rho),\frac{1}{\alpha-\gamma+1}M|M|^{\alpha-\gamma}-\rho\right)-(\partial_t\rho,M)\,. \qquad (2.13)
\end{aligned}$$

Next, we multiply the equation (2.2) by $(\partial_t\rho+\rho-1)$ in the same sense as above, in order to obtain

$$\begin{aligned}
&\|\partial_t\rho\|^2+\frac{1}{2}\frac{d}{dt}||\rho-1||^2=-\frac{1}{2}\frac{d}{dt}||\nabla\rho||^2-||\nabla\rho||^2-(g(M,\rho),\partial_t\rho+\rho-1)\Leftrightarrow\\
&\frac{1}{2}\frac{d}{dt}\left(||\nabla\rho||^2+||\rho-1||^2\right)=-||\nabla\rho||^2-\|\partial_t\rho\|^2-(g(M,\rho),\partial_t\rho+\rho-1)\,. \quad (2.14)
\end{aligned}$$

By adding the inequalities (2.13) and (2.14) together, we obtain that

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{(\alpha-\gamma+1)(\alpha-\gamma+2)}\left\||M|^{\frac{\alpha-\gamma+2}{2}}\right\|^2-(M,\rho)+\frac{1}{2}||\nabla\rho||^2+\frac{1}{2}||\rho-1||^2\right)\\
&\leqslant\left(f(M,\rho),\frac{1}{\alpha-\gamma+1}M|M|^{\alpha-\gamma}-\rho\right)-||\nabla\rho||^2-(\partial_t\rho,M)-\|\partial_t\rho\|^2\\
&\quad-(g(M,\rho),\partial_t\rho+\rho-1)\,. \qquad (2.15)
\end{aligned}$$

We consider first the term containing $g(M,\rho) = G_1\rho + g_2(\rho)M$. Then

$$\begin{aligned} -(G_1\rho, \partial_t\rho + \rho - 1) = & -\frac{1}{2}\frac{d}{dt}\left(G_1||\rho||^2\right) - G_1\left(||\rho||^2 - (1,\rho)\right) \\ \leqslant & -\frac{1}{2}\frac{d}{dt}\left(G_1||\rho||^2\right) - (1-\varepsilon)G_1||\rho||^2 + \frac{1}{4\varepsilon}G_1|\Omega| \end{aligned} \quad (2.16)$$

and

$$\begin{aligned} -(g_2(\rho)M, \partial_t\rho + \rho - 1) \leqslant & \ \varepsilon\left\|\partial_t\rho\right\|^2 + \varepsilon||\rho - 1||^2 + \frac{1}{2\varepsilon}||g_2(\rho)M||^2 \\ \underset{(2.9)}{\leqslant} & \ \varepsilon\left\|\partial_t\rho\right\|^2 + \varepsilon||\rho - 1||^2 + \frac{1}{2\varepsilon}G_3^2||M||^2. \end{aligned} \quad (2.17)$$

By combining (2.16) and (2.17) with the inequality

$$-(\partial_t\rho, M) - \left\|\partial_t\rho\right\|^2 \leqslant \frac{1}{2}||M||^2 - \frac{1}{2}\left\|\partial_t\rho\right\|^2$$

and choosing $\varepsilon \leqslant \frac{1}{2}$, we have

$$\begin{aligned} & -(\partial_t\rho, M) - \left\|\partial_t\rho\right\|^2 - (g(M,\rho), \partial_t\rho + \rho - 1) \\ \leqslant & -\frac{1}{2}\frac{d}{dt}\left(G_1||\rho||^2\right) - (1-\varepsilon)G_1||\rho||^2 + \varepsilon||\rho-1||^2 + \frac{1}{4\varepsilon}G_1|\Omega| - \left(\frac{1}{2} - \varepsilon\right)\left\|\partial_t\rho\right\|^2 \\ & + \left(\frac{1}{2} + \frac{1}{2\varepsilon}G_3^2\right)||M||^2 \\ \underset{\varepsilon\leqslant\frac{1}{2}}{\leqslant} & -\frac{1}{2}\frac{d}{dt}\left(G_1||\rho||^2\right) - (1-\varepsilon)G_1||\rho||^2 + \varepsilon||\rho-1||^2 + \frac{1}{4\varepsilon}G_1|\Omega| \\ & + \left(\frac{1}{2} + \frac{1}{2\varepsilon}G_3^2\right)||M||^2. \end{aligned} \quad (2.18)$$

Further, we can estimate the terms with f from (2.15) in the following way:

$$\begin{aligned} \left(f(M,\rho), M|M|^{\alpha-\gamma}\right) \underset{(2.7)}{\leqslant} & \left(-F_2M^2 + F_3|M|, |M|^{(\alpha-\gamma+1)-1}\right) \\ = & -F_2\left\||M|^{\frac{\alpha-\gamma+2}{2}}\right\|^2 + F_3\left\||M|^{\frac{\alpha-\gamma+1}{2}}\right\|^2, \end{aligned} \quad (2.19)$$

$$\begin{aligned} -(f(M,\rho),\rho) \underset{(2.6)}{\leqslant} & \ \varepsilon||\rho||^2 + \frac{1}{4\varepsilon}F_1^2\left(|\Omega| + \left\||M|^{\frac{\xi}{2}}\right\|^2\right) \\ \leqslant & \ 2\varepsilon||\rho-1||^2 + \left(2\varepsilon + \frac{1}{4\varepsilon}F_1^2\right)|\Omega| + \frac{1}{4\varepsilon}F_1^2\left\||M|^{\frac{\xi}{2}}\right\|^2. \end{aligned} \quad (2.20)$$

By using the inequalities (2.18)-(2.20) we conclude from (2.15) that

$$\begin{aligned}
&\frac{d}{dt}\Bigg(\frac{1}{(\alpha-\gamma+1)(\alpha-\gamma+2)}\left\||M|^{\frac{\alpha-\gamma+2}{2}}\right\|^2-(M,\rho)\\
&\qquad+\frac{1}{2}||\nabla\rho||^2+\frac{1}{2}||\rho-1||^2+\frac{1}{2}G_1||\rho||^2\Bigg)\\
&\leqslant -F_2\left\||M|^{\frac{\alpha-\gamma+2}{2}}\right\|^2+F_3\left\||M|^{\frac{\alpha-\gamma+1}{2}}\right\|^2+\frac{1}{4\varepsilon}F_1^2\left\||M|^{\frac{\xi}{2}}\right\|^2+\left(\frac{1}{2}+\frac{1}{2\varepsilon}G_3^2\right)||M||^2\\
&\quad-||\nabla\rho||^2-(1-\varepsilon)G_1||\rho||^2+3\varepsilon||\rho-1||^2+\left(2\varepsilon+\frac{1}{4\varepsilon}G_1+\frac{1}{4\varepsilon}F_1^2\right)|\Omega|.
\end{aligned}\tag{2.21}$$

In order to shorten the formulas, we introduce a new variable:

$$\begin{aligned}
\varphi:=&\frac{1}{(\alpha-\gamma+1)(\alpha-\gamma+2)}\left\||M|^{\frac{\alpha-\gamma+2}{2}}\right\|^2-(M,\rho)\\
&+\frac{1}{2}||\nabla\rho||^2+\frac{1}{2}||\rho-1||^2+\frac{1}{2}G_1||\rho||^2.
\end{aligned}\tag{2.22}$$

Observe that $|M|^{\frac{\alpha-\gamma+2}{2}}$ is the leading M power in (2.21)-(2.22) due to the assumptions made on α, γ and ξ. Further, due to the Cauchy-Schwarz and Young inequalities it holds that

$$(M,\rho)\leqslant\varepsilon||\rho||^2+\frac{1}{4\varepsilon}||M||^2,\tag{2.23}$$

so that the term (M,ρ) is negligible in (2.22). Altogether, applying the Poincaré and the Hölder inequalities and adjusting the constant ε, we can deduce from (2.21)-(2.23) the inequality

$$\frac{d}{dt}\varphi\leqslant -B_1\varphi+B_2$$

for some $B_1\in\mathbb{R}^+$, $B_2\in\mathbb{R}_0^+$. Gronwall's lemma yields

$$\begin{aligned}
\varphi(t)&\leqslant e^{-B_1t}\varphi(0)+\frac{B_2}{B_1}\left(1-e^{-B_1t}\right),\\
&\leqslant e^{-B_1t}\varphi(0)+\frac{B_2}{B_1}.
\end{aligned}\tag{2.24}$$

We finally obtain our first dissipative estimate: set

$$y_{\delta_0}:=||M||_{\delta_0}^{\delta_0}+1+||\nabla\rho||^2,\ \delta_0:=\alpha-\gamma+2,\tag{2.25}$$

it follows from (2.24) with the help of (2.22)-(2.23) and the Poincaré inequality that

$$y_{\delta_0}(t)\leqslant C_{y_{\delta_0}}y_{\delta_0}(0)e^{-\omega_{y_{\delta_0}}t}+D_{y_{\delta_0}}$$

for some $C_{y_{\delta_0}}, \omega_{y_{\delta_0}}, D_{y_{\delta_0}}$ that depend only upon the parameters of the problem.

Now, the equation (2.2) can be rewritten in the following way:

$$\partial_t(\rho - 1) = \Delta(\rho - 1) - g(M, \rho)$$

and can thus be regarded as an abstract parabolic evolution equation with respect to $\rho - 1$. Therefore, for all $t > 0$ it holds (see [34]):

$$\rho(t) - 1 = e^{t\Delta}(\rho_0 - 1) - \int_0^t e^{(t-s)\Delta} g(M(s), \rho(s))\, ds \tag{2.26}$$

and by applying operator ∇ to both sides of (2.26), we obtain that

$$\nabla\rho(t) = e^{t\Delta}\nabla\rho_0 - \int_0^t \nabla\left(e^{(t-s)\Delta} g(M(s), \rho(s))\right) ds. \tag{2.27}$$

The initial value ρ_0 is assumed to be sufficiently smooth, so that the following holds:

$$||\nabla\rho_0||_\delta < \infty. \tag{2.28}$$

What remains is to estimate the δ norm of the integral from (2.27) with the help of (1.9)-(1.10) and assumptions on g. By choosing $\mu \in \left(\frac{1}{2}, 1\right)$ and $\hat{\delta} \geqslant 1$ such that $W^{2\mu,\hat{\delta}}(\Omega) \hookrightarrow W^{1,\delta}(\Omega)$, we arrive at the estimate

$$\begin{aligned}
&\left\|\int_0^t \nabla\left(e^{(t-s)\Delta} g(M(s), \rho(s))\right) ds\right\|_\delta \\
\leqslant &\int_0^t \left\|(-\Delta)^\mu \left(e^{(t-s)\Delta} g(M(s), \rho(s))\right)\right\|_{\hat{\delta}} ds \\
\leqslant &A(\mu, \hat{\delta}) \int_0^t e^{-\beta(t-s)}(t-s)^{-\mu}\left(G_1||\rho(s)||_{\hat{\delta}} + G_3||M(s)||_{\hat{\delta}}\right) ds.
\end{aligned} \tag{2.29}$$

Altogether, we obtain from (2.27)-(2.29) the following estimate:

$$\begin{aligned}
||\nabla\rho(t)||_\delta \leqslant &e^{-\beta t}||\nabla\rho_0||_\delta + A(\mu, \hat{\delta})(G_1 + G_3)\cdot \\
&\cdot \int_0^t e^{-\beta(t-s)}(t-s)^{-\mu}\left(||\rho(s)||_{\hat{\delta}} + ||M(s)||_{\hat{\delta}}\right) ds.
\end{aligned} \tag{2.30}$$

Leaving this result for a moment and returning to the equation (2.1) we multiply this equation by $M|M|^{\delta-1}$ for an arbitrary $\delta \geqslant \alpha - \gamma + 1$, so that all occurring powers remain non-negative, and (formally) integrate over Ω:

$$\left(\partial_t M, M|M|^{\delta-1}\right) = \left(\nabla\cdot(|M|^\alpha\nabla M) - \nabla\cdot(|M|^\gamma\nabla\rho) + f(M, \rho), M|M|^{\delta-1}\right).$$

It follows that

$$\begin{aligned}
\frac{1}{\delta+1}\frac{d}{dt}\left\||M|^{\frac{\delta+1}{2}}\right\|^2 = &-\frac{4\delta}{(\alpha+\delta+1)^2}\left\|\nabla|M|^{\frac{\alpha+\delta+1}{2}}\right\|^2 \\
&+\frac{2\delta}{\alpha+\delta+1}\left(\nabla|M|^{\frac{\alpha+\delta+1}{2}}, |M|^{\gamma-\frac{\alpha}{2}+\frac{\delta-1}{2}}\nabla\rho\right) \\
&+(f(M, \rho), M|M|^{\delta-1}).
\end{aligned} \tag{2.31}$$

Set $\vartheta(\delta) := \frac{\gamma - \frac{\alpha}{2} + \frac{\delta-1}{2}}{\frac{\alpha+\delta+1}{2}}$. Then $\vartheta(\delta) < 1$ holds due to the assumption $\alpha > \gamma$. Applying Hölder's inequality, we obtain that

$$\begin{aligned}\left(\nabla |M|^{\frac{\alpha+\delta+1}{2}}, |M|^{\gamma-\frac{\alpha}{2}+\frac{\delta-1}{2}}\nabla\rho\right) &= \left(\nabla |M|^{\frac{\alpha+\delta+1}{2}}, |M|^{\vartheta(\delta)\frac{\alpha+\delta+1}{2}}\nabla\rho\right)\\ &\leqslant \|1\|_{\frac{6}{1-\theta(\delta)}} \left\|\nabla |M|^{\frac{\alpha+\delta+1}{2}}\right\| \left\| |M|^{\frac{\alpha+\delta+1}{2}}\right\|_6^{\vartheta(\delta)} \|\nabla\rho\|_3\\ &\leqslant B_3 \left\|\nabla |M|^{\frac{\alpha+\delta+1}{2}}\right\|^{1+\vartheta(\delta)} \|\nabla\rho\|_3 . \end{aligned} \tag{2.32}$$

For the last inequality, the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ (recall that $d \leqslant 3$) was used. Further, we apply once more the Hölder inequality and assumptions on the function f and thus derive:

$$(f(M,\rho), M|M|^{\delta-1}) \leqslant - F_2 \left\| |M|^{\frac{\delta+1}{2}}\right\|^2 + F_3 \left\| |M|^{\frac{\delta}{2}}\right\|^2 \tag{2.33}$$

$$\leqslant - F_2 \left\| |M|^{\frac{\delta+1}{2}}\right\|^2 + F_3 \|1\|_{\delta+1} \left(\left\| |M|^{\frac{\delta+1}{2}}\right\|^2\right)^{\frac{\delta}{\delta+1}} . \tag{2.34}$$

We can conclude from (2.31) using (2.32) and (2.34) that

$$\begin{aligned}\frac{1}{\delta+1}\frac{d}{dt}\left\| |M|^{\frac{\delta+1}{2}}\right\|^2 \leqslant &- \frac{4\delta}{(\alpha+\delta+1)^2}\left\|\nabla |M|^{\frac{\alpha+\delta+1}{2}}\right\|^2\\ &+ \frac{2\delta}{\alpha+\delta+1} B_3 \left\|\nabla |M|^{\frac{\alpha+\delta+1}{2}}\right\|^{1+\vartheta(\delta)} \|\nabla\rho\|_3\\ &- F_2 \left\| |M|^{\frac{\delta+1}{2}}\right\|^2 + F_3 \|1\|_{\delta+1} \left(\left\| |M|^{\frac{\delta+1}{2}}\right\|^2\right)^{\frac{\delta}{\delta+1}} .\end{aligned}$$

Since $1 + \vartheta(\delta) < 2$, it follows with the Young inequality that

$$\begin{aligned}\frac{1}{\delta+1}\frac{d}{dt}\left\| |M|^{\frac{\delta+1}{2}}\right\|^2 \leqslant &\; - F_2 \left\| |M|^{\frac{\delta+1}{2}}\right\|^2 + F_3 \|1\|_{\delta+1} \left(\left\| |M|^{\frac{\delta+1}{2}}\right\|^2\right)^{\frac{\delta}{\delta+1}}\\ &+ B_4(\delta) \|\nabla\rho\|_3^{\frac{2}{1-\vartheta(\delta)}} ,\end{aligned} \tag{2.35}$$

where $B_4(\delta) = \frac{1-\vartheta(\delta)}{2}\left(\frac{2\delta}{\alpha+\delta+1}B_3\right)^{\frac{2}{1-\vartheta(\delta)}}\left(\frac{4\delta}{(\alpha+\delta+1)^2}\frac{2}{1+\vartheta(\delta)}\right)^{-\frac{1+\vartheta(\delta)}{1-\vartheta(\delta)}}$, therefore this constant depends only on δ and the parameters of the problem.

Next, we return to the equality (2.31) in order to repeat the whole procedure once more, but this time we will be more precise about the estimates made, and will use the regularity achieved up to this point. First, due to (2.33) and two obvious inequalities we have

$$\begin{aligned}\frac{d}{dt}\left\||M|^{\frac{\delta+1}{2}}\right\|^2 =& -\frac{4\delta(\delta+1)}{(\alpha+\delta+1)^2}\left\|\nabla|M|^{\frac{\alpha+\delta+1}{2}}\right\|^2\\ &+\frac{2\delta(\delta+1)}{\alpha+\delta+1}\left(\nabla|M|^{\frac{\alpha+\delta+1}{2}},|M|^{\gamma-\frac{\alpha}{2}+\frac{\delta-1}{2}}\nabla\rho\right)\\ &+(\delta+1)(f(M,\rho),M|M|^{\delta-1}).\\ \leqslant& -B_5\left\|\nabla|M|^{\frac{\alpha+\delta+1}{2}}\right\|^2\\ &+(\delta+1)B_6||\nabla\rho||_\infty\left\|\nabla|M|^{\frac{\alpha+\delta+1}{2}}\right\|\left\||M|^{\frac{\alpha+\delta+1}{2}}\right\|^{\vartheta(\delta)}\\ &-(\delta+1)F_2\left\||M|^{\frac{\delta+1}{2}}\right\|^2+(\delta+1)B_7F_3\left\||M|^{\frac{\alpha+\delta+1}{2}}\right\|^{2\zeta}\end{aligned} \tag{2.36}$$

for $\delta \geqslant \alpha-\gamma+1$ with $\zeta = \frac{\delta}{\alpha+\delta+1}$. Taking into account a special case of the interpolation inequality (1.4), the inequality

$$||v|| \leqslant B_8||\nabla v||^{\frac{3}{5}}||v||_1^{\frac{2}{5}},$$

we obtain with the help of the Young inequality that

$$\begin{aligned}&(\delta+1)||\nabla v||||v||^{\vartheta(\delta)}\\ \leqslant&(\delta+1)B_8^{\vartheta(\delta)}||\nabla v||^{1+\vartheta(\delta)\frac{3}{5}}||v||_1^{\vartheta(\delta)\frac{2}{5}}\\ \leqslant&B_8^{\vartheta(\delta)}\left(\varepsilon||\nabla v||^2+B_9(\varepsilon)(\delta+1)^{\frac{2}{1-\vartheta(\delta)\frac{3}{5}}}||v||_1^{\frac{2\vartheta(\delta)\frac{2}{5}}{1-\vartheta(\delta)\frac{3}{5}}}\right)\end{aligned} \tag{2.37}$$

and

$$\begin{aligned}(\delta+1)F_3||v||^{2\zeta} &\leqslant (\delta+1)F_3B_8^{2\zeta}||\nabla v||^{2\zeta\frac{3}{5}}||v||_1^{2\zeta\frac{2}{5}}\\ &\leqslant B_8^{2\zeta}\left(\varepsilon||\nabla v||^2+B_{10}(\varepsilon)\left(F_3(\delta+1)\right)^{\frac{1}{1-\zeta\frac{3}{5}}}||v||_1^{\frac{2\zeta\frac{2}{5}}{1-\zeta\frac{3}{5}}}\right),\end{aligned} \tag{2.38}$$

where $B_9(\varepsilon)$ and $B_{10}(\varepsilon)$ depend only on ε and the parameters of the problem. With the Hölder inequality, we also have

$$\left\||M|^{\frac{\alpha+\delta+1}{2}}\right\|_1 \leqslant \left\||M|^{\frac{\alpha}{2}}\right\|_{\frac{q}{q-1}}\left\||M|^{\frac{\delta+1}{2}}\right\|_q \tag{2.39}$$

for $q \in (1,2)$ independent of δ. By combining (2.37)-(2.38) for $v := |M|^{\frac{\alpha+\delta+1}{2}}$ with (2.39) and choosing ε small enough depending only on the parameters of the problem, we can conclude from (2.36):

$$\frac{d}{dt}\left\||M|^{\frac{\delta+1}{2}}\right\|^2 \leqslant B_{11}\left(\|\nabla\rho\|_\infty(\delta+1)\right)^{\frac{2}{1-\vartheta(\delta)\frac{3}{5}}}\left(\left\||M|^{\frac{\alpha}{2}}\right\|_{\frac{q}{q-1}}\left\||M|^{\frac{\delta+1}{2}}\right\|_q\right)^{\frac{2\vartheta(\delta)\frac{2}{5}}{1-\vartheta(\delta)\frac{3}{5}}}$$
$$+ B_{11}\left(F_3(\delta+1)\right)^{\frac{1}{1-\zeta\frac{3}{5}}}\left(\left\||M|^{\frac{\alpha}{2}}\right\|_{\frac{q}{q-1}}\left\||M|^{\frac{\delta+1}{2}}\right\|_q\right)^{\frac{2\zeta\frac{2}{5}}{1-\zeta\frac{3}{5}}}$$
$$- F_2(\delta+1)\left\||M|^{\frac{\delta+1}{2}}\right\|^2$$

for $\delta \geqslant \alpha-\gamma+1$. Since $\vartheta(\delta), \zeta \in (0,1)$ it follows for all $\delta \geqslant \alpha-\gamma+2$ that

$$\frac{d}{dt}\left(\|M\|_\delta^\delta+1\right) \leqslant B_{11}\delta^5\left(\|\nabla\rho(s)\|_\infty+1\right)^5\|M\|^{\alpha}_{\frac{\alpha}{2}\frac{q}{q-1}}\left(\|M\|^{q\delta/2}_{q\delta/2}+1\right)^{\frac{2}{q}}$$
$$- F_2\delta\left(\|M\|_\delta^\delta+1\right).$$

Once more we get an integral inequality for $\|M(t)\|_\delta^\delta+1$:

$$\|M(t)\|_\delta^\delta+1 \leqslant B_{11}\int_0^t e^{-\delta F_2(t-s)}\delta^5\left(\|\nabla\rho(s)\|_\infty+1\right)^5\|M(s)\|^{\alpha}_{\frac{\alpha}{2}\frac{q}{q-1}}$$
$$\cdot\left(\|M(s)\|^{q\delta/2}_{q\delta/2}+1\right)^{\frac{2}{q}}\,ds + e^{-\delta F_2 t}\left(\|M_0\|_\delta^\delta+1\right). \tag{2.40}$$

Now we are ready to derive some more dissipative estimates for the problem (2.1)-(2.4). We will extensively use *Lemma A.1* from the *Appendix*. This lemma appears to be very useful in our situation. It actually shows that the 'dissipative property' is preserved under standard operations (addition, multiplication, raising to a power and integration). To shorten the formulas, let us set

$$h_1 := ||\nabla\rho||_3+1,$$
$$h_2 := ||\nabla\rho||_\infty+1,$$
$$u_\delta := ||M||_\delta^\delta+1, \quad \delta\in[1,\infty).$$

Observe that particular powers of y_{δ_0} and h_1, h_2 and u_δ (for sufficiently large δ), u_7 and h_2 can be connected with one another by the inequalities of the type (A.1) in the same manner as z_1 and z_3 from *Lemma A.1* are. From the *Lemma A.1* we can conclude that all of them dissipate exponentially with t:

$$h_1(t) \leqslant C_{h_1}(h_1+y_{\delta_0})^{r_{h_1}}(0)e^{-\omega_{h_1}t}+D_{h_1}, \tag{2.41}$$
$$h_2(t) \leqslant C_{h_2}(h_2+u_7)^{r_{h_2}}(0)e^{-\omega_{h_2}t}+D_{h_2}, \tag{2.42}$$
$$u_\delta(t) \leqslant U\left(u_\delta(0)+C_{u_\delta}(h_1+y_{\delta_0})^{r\delta}(0)\right)e^{-\frac{F_2}{2}\delta t}+D_{u_\delta} =: \widetilde{u}_\delta(t), \tag{2.43}$$

where the appearing coefficients depend on the parameters of the problem, and only the coefficients C_{u_δ} and D_{u_δ} depend on δ as well. We especially emphasize that r is independent from δ (it will be crucial for the existence of the uniform dissipative estimate). Indeed, from (2.30) and the definition of y_{δ_0} ($y_{\delta_0} > 1$, see (2.25)), we obtain that

$$\begin{aligned}||\nabla\rho(t)||_3 \leqslant & e^{-\beta t}||\nabla\rho_0||_3 + A\left(\frac{3}{4},2\right)(G_1+G_3)\cdot \\ & \cdot\int_0^t e^{-\beta(t-s)}(t-s)^{-\frac{3}{4}}\left(||\rho(s)||_2 + ||M(s)||_2\right)ds \\ \leqslant & e^{-\beta t}||\nabla\rho_0||_3 + C(1,2,2)A\left(\frac{3}{4},2\right)(G_1+G_3)\cdot \\ & \cdot\int_0^t e^{-\beta(t-s)}(t-s)^{-\frac{3}{4}}y_{\delta_0}(0)\,ds \end{aligned} \tag{2.44}$$

since $\alpha-\gamma+2>2$, $W^{2\cdot\frac{3}{4},2}\hookrightarrow W^{1,3}$ and $W^{1,2}\hookrightarrow L^2(\Omega)$ (with the embedding constant $C(1,2,2)$). Next, using (2.30) one more time, we obtain that

$$\begin{aligned}||\nabla\rho(t)||_\infty \leqslant & e^{-\beta t}||\nabla\rho_0||_\infty + A\left(\frac{3}{4},7\right)(G_1+G_3)\cdot \\ & \cdot\int_0^t e^{-\beta(t-s)}(t-s)^{-\frac{3}{4}}\left(||\rho(s)||_7 + ||M(s)||_7\right)ds \\ \leqslant & e^{-\beta t}||\nabla\rho_0||_\infty + C(1,7,3)A\left(\frac{3}{4},7\right)(G_1+G_3)\cdot \\ & \cdot\int_0^t e^{-\beta(t-s)}(t-s)^{-\frac{3}{4}}\left(||\nabla\rho(s)||_3 + 1 + ||M(s)||_7\right)ds \end{aligned} \tag{2.45}$$

since $W^{2\cdot\frac{3}{4},7}\hookrightarrow W^{1,\infty}$ and $W^{1,3}(\Omega)\hookrightarrow L^7(\Omega)$ (with the embedding constant $C(1,7,3)$). The estimates for h_1 and h_2 now follow with (2.44)-(2.45) and *Lemma A.1* due to the fact that for the function $d(t,s):=(t-s)_+^{-\frac{3}{4}}$ the condition $||d||_{L^\infty(\mathbb{R}_0^+,L_b^1(\mathbb{R}_0^+))}<\infty$ is satisfied.

Let us now check the dissipative estimate (2.43). With (2.35) we have:

$$\frac{1}{\delta}\frac{d}{dt}u_\delta \leqslant -F_2u_\delta + F_3|\Omega|u_\delta^{\frac{\delta-1}{\delta}} + B_4(\delta)h_1^{\frac{2}{1-\vartheta(\delta)}}. \tag{2.46}$$

Recall that $\vartheta(\delta) = \frac{\gamma-\frac{\alpha}{2}+\frac{\delta-2}{2}}{\frac{\alpha+\delta}{2}}$ and, consequently, $\frac{2}{1-\vartheta(\delta)} = \frac{\alpha+\delta}{\alpha-\gamma+1} \leqslant B_{12}\delta$ for some B_{12} and $\delta\geqslant\delta_*$ sufficiently large. Now, the Young inequality yields:

$$u_\delta^{\frac{\delta-1}{\delta}} = (\varepsilon u_\delta)^{\frac{\delta-1}{\delta}}\varepsilon^{-\frac{\delta-1}{\delta}} \leqslant \frac{\delta-1}{\delta}\varepsilon u_\delta + \frac{1}{\delta}\varepsilon^{-(\delta-1)},$$

therefore it follows from (2.46) that

$$\begin{aligned}\frac{d}{dt}u_\delta \leqslant & -\delta\left(F_2 - \varepsilon F_3|\Omega|\frac{\delta-1}{\delta}\right)u_\delta + \varepsilon^{-(\delta-1)}F_3|\Omega| + \delta B_4(\delta)h_1^{B_{12}\delta} \\ \leqslant & -\delta\frac{F_2}{2}u_\delta + \varepsilon^{-(\delta-1)}F_3|\Omega| + \delta B_4(\delta)h_1^{B_{12}\delta}\end{aligned}$$

for ε small (depends only on the parameters of the problem). Gronwall's lemma then yields

$$u_\delta(t) \leqslant \int_0^t e^{-\delta\frac{F_2}{2}(t-s)} \Big(\varepsilon^{-(\delta-1)}F_3|\Omega| + \delta B_4(\delta)h_1^{B_{12}\delta}(s)\Big)\, ds \\ + e^{-\delta\frac{F_2}{2}t}u_\delta(0). \tag{2.47}$$

The dissipate estimate (2.43) follows now with the estimate (A.2) of *Lemma A.1* and the dissipate estimate (2.41) for h_1.

Now, from the inequality (2.40) we can conclude that

$$u_\delta(t) \leqslant e^{-\delta F_2 t}u_\delta(0) + B_{11}\delta^5 \int_0^t e^{-\delta F_2(t-s)}H_1(s)\widetilde{u}_{\frac{q}{2}\delta}^{\frac{2}{q}}(s)\, ds, \tag{2.48}$$

where

$$H_1(t) := h_2^5(t)\widetilde{u}_{\frac{\alpha}{2}\frac{q}{q-1}}^{\frac{2(q-1)}{q}}(t).$$

Taking into account that $u_{\frac{q}{2}\delta}^{\frac{2}{q}}$ dissipates with $e^{-\delta\frac{F_2}{2}t}$ and that H_1 dissipates with an exponent independent of δ, we consecutively apply (A.3) to (2.48) and get

$$\begin{aligned} u_\delta(t) \leqslant& e^{-\delta\frac{F_2}{2}t}u_\delta(0) + B_{11}\widetilde{u}_{\frac{q}{2}\delta}^{\frac{2}{q}}(t)\int_0^t e^{-\delta\frac{F_2}{2}(t-s)}\delta^5 H_1(s)\, ds \\ \leqslant& e^{-\delta\frac{F_2}{2}t}u_\delta(0) + \frac{2}{F_2}B_{11}\delta^4 H_1(t)\widetilde{u}_{\frac{q}{2}\delta}^{\frac{2}{q}}(t) \end{aligned}$$

for $\delta \geqslant \delta_*$ sufficiently large. The bound δ_* depends only on the parameters of the problem. Therefore, we may assume that

$$\widetilde{u}_\delta(t) = e^{-\delta\frac{F_2}{2}t}u_\delta(0) + B_{13}\delta^4 H_1(t)\widetilde{u}_{\frac{q}{2}\delta}^{\frac{2}{q}}(t). \tag{2.49}$$

Since

$$u_\delta(0) = ||M_0||_\delta^\delta + 1 \leqslant ||M_0||_\infty^\delta|\Omega| + 1$$

we conclude from (2.49) that for

$$A_\delta(t) := \widetilde{u}_\delta(t)\left(\frac{e^{\frac{F_2}{2}t}}{||M_0||_\infty + 1}\right)^\delta + 1 \tag{2.50}$$

it holds that

$$A_\delta(t) \leqslant B_{14}\delta^4 H_1(t) A_{\frac{q}{2}\delta}^{\frac{2}{q}}(t).$$

One can show by induction then that

$$\begin{aligned} A_{\left(\frac{2}{q}\right)^n\delta_*}^{\left(\frac{q}{2}\right)^n}(t) \leqslant\ & \left(B_{14}\delta_*^4 H_1(t)\right)^{\sum_{k=1}^n \left(\frac{q}{2}\right)^k} \left(\frac{q}{2}\right)^{4\sum_{k=1}^n k\left(\frac{q}{2}\right)^k} A_{\delta_*}(t) \\ \underset{n\to\infty}{\to}\ & \left(B_{14}\delta_*^4 H_1(t)\right)^{\frac{\frac{q}{2}}{1-\frac{q}{2}}} \left(\frac{q}{2}\right)^{2q\left(\frac{1}{1-\frac{q}{2}}\right)^2} A_{\delta_*}(t) \\ =:\ & H^{\delta_*}(t) A_{\delta_*}(t). \end{aligned}$$

Therefore, we get

$$\limsup_{\delta\to\infty} A_\delta^{\frac{1}{\delta}}(t) \leqslant H(t) A_{\delta_*}^{\frac{1}{\delta_*}}(t). \tag{2.51}$$

By combining (2.51) with (2.50), we finally arrive at an estimate for $||M(t)||_\infty$:

$$\begin{aligned} ||M(t)||_\infty + 1 =\ & \lim_{\delta\to\infty} u_\delta^{\frac{1}{\delta}}(t) \\ \leqslant\ & \limsup_{\delta\to\infty} \tilde{u}_\delta^{\frac{1}{\delta}}(t) \\ \leqslant\ & H(t)\left(\tilde{u}_{\delta_*}^{\frac{1}{\delta_*}}(t) + (||M_0||_\infty + 1)\, e^{-\frac{F_2}{2}t}\right). \end{aligned} \tag{2.52}$$

Now, since the functions H and $\tilde{u}_{\delta_*}$ dissipate exponentially (recall (2.42)-(2.43) and the definition of H and H_1), we apply *Lemma A.1* to (2.52) and conclude that $||M||_\infty$ dissipates exponentially as well. Moreover, it follows from the proof that there exists a dissipative estimate for $||M||_\infty$ of the form given in (2.12). The dissipative estimate for $||\nabla\rho||_\infty + 1 = h_2$ is given in (2.42) and the *Theorem 2.2* is thus proven.

□

2.3 Global attractor

The aim of this section is to apply the general theory from *Section 1.3* to the problem (2.1)-(2.4). We prove

Theorem 2.3 (Global Attractor)**.** *Let the functions f and g satisfy the assumptions (2.6)-(2.11) and let the constants α and γ satisfy $\gamma \in \left[\frac{\alpha}{2}+1, \alpha\right)$. Then the solutions of the problem (2.1)-(2.4) can be described by a semigroup $\{S(t)\}_{t\geqslant 0}$ that acts on the space $L^\infty_{w-*}(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$. The semigroup $\{S(t)\}_{t\geqslant 0}$ possesses the global attractor in $L^\infty_{w-*}(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$.*

Proof. We observed in *Section 2.1* that the problem (2.1)-(2.4), if considered as an equation with respect to (M, ρ) in the space $L^\infty(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$, is well-posed: for each pair of initial values $(M_0, \rho_0) \in L^\infty(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$ there exists a unique solution (M, ρ) in terms of *Definition 2.1*.

Since $L^\infty(\Omega)$ and $L^\infty_{w-*}(\Omega)$ coincide as sets, we can define the solving semigroup $\{S(t)\}_{t\geqslant 0}$ of the problem (2.1)-(2.4) on the phase space $L^\infty_{w-*}(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$ as follows: for all $t \geqslant 0$ let

$$S(t) : L^\infty_{w-*}(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right) \to L^\infty_{w-*}(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right),$$
$$S(t)(M_0, \rho_0) := (M(t), \rho(t)) \text{ for all } (M_0, \rho_0) \in L^\infty_{w-*}(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right).$$

Clearly, the space $L^\infty_{w-*}(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$ inherits all the properties of the space $L^\infty_{w-*}(\Omega)$ that are listed in *Theorem 1.6*. In particular, it is a locally convex space, so that the general theory from *Section 1.3* is applicable.

Another consequence is that, due to *Theorem 1.6(2)*, there is no difference between the spaces $L^\infty(\Omega)$ and $L^\infty_{w-*}(\Omega)$ with concern to boundedness, the same thus holds for $L^\infty(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$ and $L^\infty_{w-*}(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$.

The dissipative estimate (2.12) provides for the semigroup $\{S(t)\}_{t\geqslant 0}$ the existence of an absorbing $L^\infty(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$ ball B_* centered at $(0, 1)$ of a radius $2D_\infty$. According to *Remark 1.7*, the set

$$C_* := \bigcup_{s \in [0, T(B_*)]} S(s)B_*,$$

where $T(B_*)$ is such that $S(s)B_* \subset B_*$ for all $s \geqslant T(B_*)$, is a positively invariant absorbing set. With the dissipative estimate (2.12), it follows also that C_* is actually contained in an $L^\infty(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$ ball B_{**} of a radius R_{**}, $R_{**} \geqslant 2D_\infty$, centered at $(0, 1)$, thus it is a bounded positively invariant absorbing set for $\{S(t)\}_{t\geqslant 0}$.

Since C_* is bounded, its closure, the set

$$\bar{C}_* := cl_{L^\infty_{w-*}(\Omega)\times\left(1+W_0^{1,\infty}(\Omega)\right)}(C_*),$$

is completely metrizable in its relative topology (see *Theorem 1.6(3)*).

$$m_*^{(\infty)}((M_1, \rho_1), (M_2, \rho_2)) := \max\left\{d_*^{(\infty)}(M_1, M_2), ||\rho_1 - \rho_2||_{W_0^{1,\infty}(\Omega)}\right\}$$

is an example of a complete metric which induces the relative topology. Here $d_*^{(\infty)}$ is the metric defined in (1.2) for $R := R_{**}$.

Let us assume for a moment that $\{S(t)\}_{t\geqslant 0}$ is a closed semigroup. As a closure of a positively invariant set, $\bar{C}_*$ is then also positively invariant under $\{S(t)\}_{t\geqslant 0}$. Let us assume further that the set $S(1)\bar{C}_*$ is relatively compact in $L^\infty_{w-*}(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$. As the closure of a time shift of an absorbing set, $cl_{L^\infty_{w-*}(\Omega)\times\left(1+W_0^{1,\infty}(\Omega)\right)}(S(1)\bar{C}_*)$ is absorbing.

All assumptions of *Theorem 1.13* are then fulfilled since we are dealing with a closed semigroup (the semigroup $\{S(t)\}_{t\geqslant 0}$) in a complete metric space (the space $\left(\bar{C}_*, m_*^{(\infty)}\right)$), and this semigroup possesses a compact absorbing set (the set $cl_{L^\infty_{w-*}(\Omega)\times\left(1+W_0^{1,\infty}(\Omega)\right)}(S(1)\bar{C}_*)$). *Theorem 1.13* yields the existence of the global attractor.

To finish the proof, it remains to check the closedness of $\{S(t)\}_{t\geqslant 0}$ and the relative compactness of $S(1)\bar{C}_*$.

The projection of $\bar{C}_*$ on the M component is a bounded norm closed set in $L^\infty(\Omega)$, therefore it is compact in $L^\infty_{w-*}(\Omega)$ due to *Theorem 1.6*. Let us now show the relative compactness for the ρ component.

Since

$$(-\Delta)^{\frac{11}{12}}\rho(1) = (-\Delta)^{\frac{11}{12}} e^{\Delta}\rho_0 - \int_0^1 (-\Delta)^{\frac{11}{12}} e^{(1-\omega)\Delta} g(M(\omega),\rho(\omega))\,d\omega$$

we conclude with (1.9)-(1.10) and assumptions on g that for all $(M_0,\rho_0)\in\bar{C}_*$ it holds

$$\begin{aligned}\left\|(-\Delta)^{\frac{11}{12}}\rho(1)\right\|_6 &\leqslant \left\|(-\Delta)^{\frac{11}{12}} e^{\Delta}\rho_0 - \int_0^1 (-\Delta)^{\frac{11}{12}} e^{(1-\omega)\Delta} g(M(\omega),\rho(\omega))\,d\omega\right\|_6\\ &\leqslant A\left(\frac{5}{12},6\right)\|\nabla\rho_0\|_6 + A\left(\frac{11}{12},6\right)\int_0^1 (1-\omega)^{-\frac{11}{12}}\|g(M(\omega),\rho(\omega))\,d\omega\|_6\\ &\leqslant B_\rho(R_{**}),\end{aligned} \tag{2.53}$$

where the constant $B_\rho(R_{**})$ depends only on R_{**} and the parameters of the problem. With (2.53) and the compact embedding (see *Theorem 1.10*)

$$W^{\frac{11}{6},6}(\Omega) \hookrightarrow\hookrightarrow W^{1,\infty}(\Omega)$$

it follows that the projection of the set $S(1)\bar{C}_*$ on the ρ component is relatively compact.

It remains to show the closedness of the semigroup operators. In the proof of *Theorem 3.1* we will encounter the local Lipschitz-type continuity property (3.39) for the solutions of (2.1)-(2.4). It can be translated into the following form:

$$\begin{aligned}&\left\|S(t)\left(M_0^{(1)},\rho_0^{(1)}\right) - S(t)\left(M_0^{(2)},\rho_0^{(2)}\right)\right\|_{H^{-1}(\Omega)\times L^2(\Omega)}\\ \leqslant &L(t,R_{**})\left\|\left(M_0^{(1)},\rho_0^{(1)}\right) - \left(M_0^{(2)},\rho_0^{(2)}\right)\right\|_{H^{-1}(\Omega)\times L^2(\Omega)}\end{aligned} \tag{2.54}$$

for $\left(M_0^{(1)},\rho_0^{(1)}\right),\left(M_0^{(2)},\rho_0^{(2)}\right)\in\bar{C}_*$. The constant $L(t,R_{**})$ depends only on t, R_{**} and the parameters of the problem.

Recall that due to *Theorem 1.10* we have

$$W^{1,\infty}(\Omega) \hookrightarrow L^2(\Omega), \tag{2.55}$$

$$L^\infty(\Omega) \hookrightarrow\hookrightarrow H^{-1}(\Omega). \tag{2.56}$$

Since due to *Theorem 1.2* compact operators are weak-$*$-to-norm continuous, the property (2.56) allows the interpretation

$$L^\infty_{w-*}(\Omega) \hookrightarrow H^{-1}(\Omega). \tag{2.57}$$

By combining (2.55) with (2.57), we obtain that

$$L^\infty_{w-*}(\Omega) \times W^{1,\infty}(\Omega) \hookrightarrow H^{-1}(\Omega) \times L^2(\Omega). \tag{2.58}$$

Let $\left\{\left(M_0^{(n)}, \rho_0^{(n)}\right)\right\}_{n\in\mathbb{N}} \subset \bar{C}_*$ be a sequence of initial data convergent to some (M_0, ρ_0) in $L^\infty_{w-*}(\Omega) \times W^{1,\infty}(\Omega)$. Due to the continuous embedding (2.58), this sequence converges in $H^{-1}(\Omega) \times L^2(\Omega)$ to the same limit. From the property (2.54) we deduce that the sequence $\left\{S(t)\left(M_0^{(n)}, \rho_0^{(n)}\right)\right\}_{n\in\mathbb{N}}$ converges to $S(t)\left(M_0, \rho_0\right)$ in $H^{-1}(\Omega) \times L^2(\Omega)$ for all $t \geqslant 0$.
Let us further assume that for some $t \geqslant 0$ the sequence $\left\{S(t)\left(M_0^{(n)}, \rho_0^{(n)}\right)\right\}_{n\in\mathbb{N}}$ is convergent in $L^\infty_{w-*}(\Omega) \times W^{1,\infty}(\Omega)$. Again, due to the continuity of the embedding (2.58), the limit is the same. This proves closedness, and the proof of *Theorem 2.3* is thus finished.

□

2.4 Biofilm mass: local behavior

In this section, we study the local behavior of the biofilm mass. We prove

Theorem 2.4 (Local mass behavior)**.** *Let the functions f and g satisfy the assumptions (2.6)-(2.11) and the constants α and γ satisfy $\gamma \in \left[\frac{\alpha}{2} + 1, \alpha\right)$. Further, let $(M_0, \rho_0) \in L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ and the starting value M_0 be strictly separated from 0 in some open ball $B \subset \Omega$. Then*

$$\int\limits_{\widetilde{B}} M(t,x)\,dx > 0$$

for all $t > 0$ and all open balls $\widetilde{B}$ with $\overline{\widetilde{B}} \subset B$.

Proof. We observed in *Section 2.1* that the problem (2.1)-(2.4), if considered as an equation with respect to (M, ρ) in the space $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$, is well-posed: for each pair of initial values $(M_0, \rho_0) \in L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ there exists a unique solution (M, ρ) in terms of *Definition 2.1*.

Assume the starting value M_0 to be separated from 0 in a ball $B \subset \Omega$ by a constant $\varepsilon > 0$:

$$\operatorname*{ess\,inf}_B M_0 \geqslant \varepsilon > 0. \tag{2.59}$$

This time we multiply the equation (2.1) by $-\varphi M^{-1}$, where φ is a smooth cut-off function with the properties

$$\varphi \in C_0^{\infty}(B), \quad 0 \leqslant \varphi \leqslant 1, \quad \varphi = 1 \text{ in a ball } \widetilde{B} \subset B$$

for an arbitrary ball $\widetilde{B}$ with $\overline{\widetilde{B}} \subset B$, and integrate over Ω:

$$\left(\partial_t M, -\varphi M^{-1}\right) = \left(\nabla \cdot (M^{\alpha} \nabla M) - \nabla \cdot (M^{\gamma} \nabla \rho) + f(M,\rho), -\varphi M^{-1}\right) \quad (2.60)$$

Integrating (2.60) by parts we get:

$$\begin{aligned} \frac{d}{dt} \int_B -\varphi(x) \ln M(\cdot, x)\, dx = &- \frac{4}{\alpha^2} \left\| \varphi^{\frac{1}{2}} \nabla M^{\frac{\alpha}{2}} \right\|^2 + \frac{4}{\alpha} \left(\varphi^{\frac{1}{2}} \nabla M^{\frac{\alpha}{2}}, M^{\frac{\alpha}{2}} \nabla \varphi^{\frac{1}{2}} \right) \\ &+ \frac{2}{\alpha} \left(\varphi^{\frac{1}{2}} \nabla M^{\frac{\alpha}{2}}, \varphi^{\frac{1}{2}} M^{\gamma - \frac{\alpha}{2} - 1} \nabla \rho \right) - \left(M^{\gamma-1}, \nabla \varphi \cdot \nabla \rho \right) \\ &- \left(\varphi, \frac{f(M,\rho)}{M} \right). \end{aligned} \quad (2.61)$$

Let us now fix an arbitrary $T > 0$. The expression (2.61) is quite similar to what we had in (2.31). It can be shown in the same way that the right side of (2.61) is bounded from above by some positive constant $B_1(T)$ for all $t \in [0, T]$. This is due to the assumptions made on f, the uniform boundedness of $\nabla \rho$ and the cut-off function and also to the fact that $\frac{\alpha}{2} + 1 \leqslant \gamma$ (and $\gamma > 1$, of course). Thus we get

$$\frac{d}{dt} \int_B -\varphi(x) \ln M(t,x)\, dx \leqslant B_1(T) \quad (2.62)$$

for all $t \in [0, T]$. Integrating over $[0, T]$, we obtain from (2.62) with the help of (2.59) that

$$\begin{aligned} \int_B -\varphi(x) \ln M(T,x)\, dx \leqslant & T B_1(T) + \int_B -\varphi(x) \ln M_0(x)\, dx \\ \leqslant & T B_1(T) - \int_B \varphi(x)\, dx \ln \varepsilon \\ =: & B_2(T, \varepsilon). \end{aligned}$$

It follows with M being uniformly bounded that

$$\begin{aligned} \int\limits_{\widetilde{B} \cap \{M(T,\cdot) < 1\}} - \ln M(T,x)\, dx \leqslant & B_2(T,\varepsilon) + \int\limits_{B \cap \{M(T,\cdot) \geqslant 1\}} \ln M(T,x) \\ \leqslant & B_3(T, \varepsilon). \end{aligned} \quad (2.63)$$

If $|\widetilde{B} \cap \{M(T,\cdot) < 1\}| = 0$ holds, then $M(T,\cdot) \geqslant 1$ in $\widetilde{B}$, and nothing is left to prove. If $|\widetilde{B} \cap \{M(T,\cdot) < 1\}| > 0$ then we can use the Jensen's inequality [27] for the measure $\frac{1}{|\widetilde{B}\cap\{M(T,\cdot)<1\}|}\,dx$ and convex function $-\ln$ to estimate to the left side of (2.63) from below. We obtain:

$$-\ln\left(\frac{1}{|\widetilde{B}\cap\{M(T,\cdot)<1\}\,|}\int\limits_{\widetilde{B}\cap\{M(T,\cdot)<1\}} M(T,x)\,dx\right)\leqslant\frac{B_3(T,\varepsilon)}{|\widetilde{B}\cap\{M(T,\cdot)<1\}\,|}.$$

Hence it follows with obvious calculations that

$$\int\limits_{\widetilde{B}} M(T,x)\,dx\geqslant\int\limits_{\widetilde{B}\cap\{M(T,\cdot)<1\}} M(T,x)\,dx>0 \text{ for all } T>0.$$

Theorem 2.4 is thus proven.

□

Chapter 3

A biofilm model with chemotaxis effect: nonautonomous case

3.1 The Model

This chapter is devoted to the nonautonomous version of system (2.1)-(2.4): the system

$$\begin{aligned}
&\partial_t M = \nabla \cdot (|M|^\alpha \nabla M) - \nabla \cdot (|M|^\gamma \nabla \rho) + f(t, M, \rho) && \text{in } (\tau, \infty) \times \Omega, && (3.1)\\
&\partial_t \rho = \Delta \rho - g(t, M, \rho) && \text{in } (\tau, \infty) \times \Omega, && (3.2)\\
&M = 0, \quad \rho = 1 && \text{in } (\tau, \infty) \times \partial\Omega, && (3.3)\\
&M(\tau, \cdot) = M_\tau, \quad \rho(\tau, \cdot) = \rho_\tau && \text{in } \Omega, && (3.4)
\end{aligned}$$

where α and γ satisfy the 'balance' condition (2.5):

$$\frac{\alpha}{2} + 1 \leqslant \gamma < \alpha.$$

$\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ is a nonempty smooth bounded domain and $M_\tau \in L^\infty(\Omega)$, $\rho_\tau \in W^{1,\infty}(\Omega)$. We assume that the functions f and g satisfy the following nonautonomous version of the assumptions (2.6)-(2.11): for all $t, M, \rho \in \mathbb{R}$ let

$$|f(t,M,\rho)| \leqslant f_1(t)(1+|M|^{\xi})^{\frac{1}{2}} \text{ for some } \xi \in [0,\alpha-\gamma+2)\,,\ f_1 \in (L^2_b(\mathbb{R}))^+, \tag{3.5}$$

$$f(t,M,\rho)\,\mathrm{sign}(M) \leqslant -F_2|M| + f_3(t) \text{ for some } F_2 \in \mathbb{R}^+,\ f_3 \in (L^{\kappa}_b(\mathbb{R}))^+,\ \kappa > 1, \tag{3.6}$$

$$g(t,M,\rho) = g_1(t)\rho + g_2(t,\rho)M \text{ for some } g_1 \in (C^1(\mathbb{R}))^+ :$$

$$\frac{d}{dt}g_1(t) \leqslant 0,\ g_1(-\infty) < \infty, \tag{3.7}$$

$$|g_2(t,\rho)| \leqslant g_3(t),\ \text{for some } g_3 \in (L^{\eta}_b(\mathbb{R}))^+,\ \eta > 4 \tag{3.8}$$

and, in order to ensure uniqueness and non-negativity of solutions for non-negative initial data,

$$\widetilde{f}(t,M,\rho) := f\left(t, M|M|^{\frac{2}{2+\alpha}-1}, \rho\right) - F_4 M|M|^{\frac{2}{2+\alpha}-1},\ \text{for some } F_4 \in \mathbb{R}, \tag{3.9}$$

$$\widetilde{f} \in W^{1,\infty}_{loc}(\mathbb{R}^3),\ g_2 \in W^{1,\infty}_{loc}(\mathbb{R}^2),\ f(t,0,\rho) \geqslant 0,\ g_2(t,0) \leqslant 0 \text{ for all } t,\rho \in \mathbb{R}. \tag{3.10}$$

The following example of functions f and g satisfies the conditions (3.5)- (3.10):

Example 3.1.

$$f(t,M,\rho) = -M + \frac{(M^+)^{\frac{2+\alpha}{2}}}{(M^+)^{\frac{2+\alpha}{2}}+1}\arctan(\rho)\sin(t),$$
$$g(t,M,\rho) = \operatorname{arccot}(t)\rho + M\frac{\rho}{\rho+1}\cos(t).$$

We define weak solutions of (3.1)-(3.4) in the following way:

Definition 3.1 (Weak solution)**.** *A pair of functions (M,ρ) defined in $[\tau,\infty)\times\overline{\Omega}$ is said to be a weak solution of (3.1)-(3.4) for $M_\tau \in L^\infty(\Omega)$, $\rho_\tau \in W^{1,\infty}(\Omega)$, if for all $T > \tau$*

(i) $M \in L^\infty((\tau,T)\times\Omega)$, $|M|^\alpha M \in L^2\left((\tau,T);H^1_0(\Omega)\right)$, $\partial_t M \in L^2\left((\tau,T);H^{-1}(\Omega)\right)$;

(ii) $\rho - 1 \in C((\tau,T);W^{1,\infty}_0(\Omega))$;

(iii) (M,ρ) *satisfies the equation (3.1) in* $L^2\left((\tau,T);H^{-1}(\Omega)\right)$, $M(\tau) = M_\tau$ *in* $C_w((\tau,T);L^2(\Omega))$*-sense and*

$$\rho(t) - 1 = e^{(t-\tau)\Delta}(\rho_\tau - 1) - \int_\tau^t e^{(t-s)\Delta} g(s,M(s),\rho(s))\,ds$$

in $W^{1,\infty}_0(\Omega)$.

Remark 3.1 (Initial condition)**.** From $M \in L^\infty((\tau,T);L^2(\Omega))$ and $\partial_t M \in L^2\left((\tau,T);H^{-1}(\Omega)\right)$, it follows with (1.5) for $p_0 = 2$, $E = L^2(\Omega)$ and $E_1 = H^{-1}(\Omega)$ and the compact embedding (see *Theorem 1.10*) $L^2(\Omega) \overset{d}{\hookrightarrow} H^{-1}(\Omega)$ that $M \in C((\tau,T);(L^2(\Omega),\sigma(L^2(\Omega),(L^2(\Omega))')))$. Therefore, the initial condition for M makes sense.

In the present work, we study the well-posedness and the long-time behavior of the nonautonomous system (3.1)-(3.4). We prove the following nonautonomous version of *Theorem 2.1*:

Theorem 3.1. *Let the functions f and g satisfy the assumptions (3.5)-(3.10) and the constants α and γ satisfy $\gamma \in \left[\frac{\alpha}{2}+1,\alpha\right)$. Then the initial boundary-value problem (3.1)-(3.4) is uniquely solvable (in the sense of Definition 3.1) for each pair of starting values $(M_\tau,\rho_\tau) \in L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ and (in the sense of Definition 1.4). The solution is uniformly bounded in time in the phase space $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$.*

The proof of *Theorem 3.1* is divided between *Sections 3.2* (existence) and *3.3* (uniqueness). The positivity preserving property is a consequence of the positivity results from [11]. *Sections 3.4* (dissipative estimate) and *3.5* (pullback attractor) are devoted to the study of the long-time behavior for our system. The main result is *Theorem 3.2*. It states the existence of a dissipative estimate for the solutions of (3.1)-(3.4), and the dissipation proves to be with exponential speed. In *Section 3.5* we establish the existence of a global pullback attractor.

Remark 3.2 (Notation)**.** For the sake of convenience, we assume throughout this chapter that the constants B_i (appear below) for all indices i are only dependent upon the parameters of the problem, that is, upon the constants α and γ, the functions f and g and the domain Ω, and **not** upon the initial data M_τ, ρ_τ or the time variables τ and t, or, unless stated otherwise, any other parameters.

3.2 Existence of solutions

Proof of *Theorem 3.1* **(Existence)**. The main idea of the existence proof is to choose a suitable regularization sequence for the problem (3.1)-(3.4) and then apply the compactness method (see [21]).

Let us consider for arbitrary $T > 0$, $n \in \mathbb{N}$ a non-degenerate approximation of the problem (3.1)-(3.4), the system

$$\partial_t M_n = \nabla \cdot \left(\left(|M_n| + \frac{1}{n}\right)^\alpha \nabla M_n\right) - \nabla \cdot \left(\left(|M_n| + \frac{1}{n}\right)^\gamma \nabla \rho_n\right) + f(t, M_n, \rho_n) \quad \text{in } (\tau,T)\times\Omega, \tag{3.11}$$

$$\partial_t \rho_n = \Delta \rho_n - g(t, M_n, \rho_n) \quad \text{in } (\tau,T)\times\Omega \tag{3.12}$$

with the same initial and boundary conditions as before:

$$M_n = 0, \qquad \rho_n = 1 \qquad \text{in } (\tau, T) \times \partial\Omega, \tag{3.13}$$
$$M_n(\cdot, \tau) = M_\tau, \quad \rho_n(\cdot, \tau) = \rho_\tau \text{ in } \Omega. \tag{3.14}$$

As a consequence of the a priory uniform boundedness, the general theory from [20] may be extended to the non-degenerate problems (3.11)-(3.14) (for an alternative treatment via maximal regularity see [2]). It follows that they are uniquely solvable in the class of functions which is defined by *Definition 3.1*.

Further, in the same manner as the estimates for the solution itself are derived in *Section 3.4*, one can show for the approximating sequence $\{(M_n, \rho_n)\}_{n\in\mathbb{N}}$ the following a priory estimate:

$$\|M_n\|_{L^\infty((\tau,T)\times\Omega)}, \|\nabla\rho_n\|_{L^\infty((\tau,T)\times\Omega)}, \||M_n|^\alpha M_n\|_{L^2((\tau,T),H_0^1(\Omega))} \leqslant B_1. \tag{3.15}$$

With the help of (3.15) we can now estimate the right side of the equation (3.11) in the $L^2((\tau,T), H^{-1}(\Omega))$ norm uniformly in $n \in \mathbb{N}$. We obtain

$$\|\partial_t M_n\|_{L^2((\tau,T),H^{-1}(\Omega))} \leqslant B_2. \tag{3.16}$$

Moreover, applying *Lemma 1.1* for $q = \alpha + 1$ and (for example) $s = \frac{1}{2}$ together with the Sobolev embedding theorem yields

$$\begin{aligned} ||M_n||^{2(\alpha+1)}_{W^{\frac{1}{2(\alpha+1)},2(\alpha+1)}(\Omega)} \leqslant & N^{2(\alpha+1)}(\alpha+1) \||M_n|^\alpha M_n\|^2_{W^{\frac{1}{2},2}(\Omega)} \\ \leqslant & B_3 \||M_n|^\alpha M_n\|^2_{H_0^1(\Omega)}. \end{aligned} \tag{3.17}$$

Integrating (3.17) over (τ, T) and combining with (3.15)-(3.16) we conclude that

$$||M_n||_{W^{1,(2(\alpha+1),2)}\left((\tau,T);W^{\frac{1}{2(\alpha+1)},2(\alpha+1)}(\Omega),H^{-1}(\Omega)\right)} \leqslant B_4.$$

The spaces $E_1 := W^{\frac{1}{2(\alpha+1)},2(\alpha+1)}(\Omega)$, $E := L^{2(\alpha+1)}(\Omega)$, $E_0 := H^{-1}(\Omega)$ satisfy the assumptions of *Theorem 1.12*, consequently, it holds (set $p_1 := 2(\alpha+1)$, $p_0 := 2$)

$$W^{1,(2(\alpha+1),2)}\left((\tau,T); W^{\frac{1}{2(\alpha+1)},2(\alpha+1)}(\Omega), H^{-1}(\Omega)\right) \hookrightarrow\hookrightarrow L^{2(\alpha+1)}((\tau,T)\times\Omega),$$

and the set $\{M_n | \ n \in \mathbb{N}\}$ is thus compact in the space $L^{2(\alpha+1)}((\tau,T)\times\Omega)$.

For the second component, we now use the (3.15) to estimate the right side of the equation (3.12) in $L^\infty((\tau,T), W^{-1,\infty}(\Omega))$ and get

$$||\rho_n||_{W^{1,(\infty,\infty)}\left((\tau,T);W_0^{1,\infty}(\Omega),W^{-1,\infty}(\Omega)\right)} \leqslant B_5(||\nabla\rho_0||_\infty).$$

The spaces $E_1 := W_0^{1,\infty}(\Omega)$, $E := L^\infty(\Omega)$, $E_0 := W^{-1,\infty}(\Omega)$ satisfy the assumptions of *Theorem 1.12*, consequently, it holds (set $p_1 := p_0 := \infty$)

$$W^{1,(\infty,\infty)}\left((\tau,T); W_0^{1,\infty}(\Omega), W^{-1,\infty}(\Omega)\right) \hookrightarrow\hookrightarrow C([\tau,T], L^\infty(\Omega)).$$

The set $\{\rho_n|\ n \in \mathbb{N}\}$ is thus compact in the space $C([\tau, T], L^\infty(\Omega))$, hence also in the larger space $L^\infty((\tau, T) \times \Omega)$.

By combining these results, we obtain that there is a subsequence (n_m) such that

$$\begin{aligned} & M_{n_m} \underset{m\to\infty}{\rightarrow} M \text{ in } L^{2(\alpha+1)}((\tau, T) \times \Omega), \\ & \rho_{n_m} \underset{m\to\infty}{\rightarrow} \rho \text{ in } L^\infty((\tau, T) \times \Omega), \\ & \nabla\rho_{n_m} \underset{m\to\infty}{\rightarrow} \nabla\rho \text{ in } L^\infty_{w-*}((\tau, T) \times \Omega). \end{aligned} \tag{3.18}$$

for some $(M, \rho) \in L^{2(\alpha+1)}((\tau, T) \times \Omega) \times L^\infty((\tau, T) \times \Omega)$, and for a subsequence (not relabeled) the convergence is almost everywhere in the cylinder $(\tau, T) \times \Omega$.

It thus suffices to check that (M, ρ) is indeed a solution of the original problem (3.1)-(3.4) in the sense of distributions. Recall first that $f, g \in W^{1,\infty}(Q)$ for $Q := (\tau, T) \times (-B_1, B_1)^2$, so that, with the second part of the Sobolev embedding theorem, we have that $f, g \in C(\overline{Q})$.
With the continuity argument and the dominant convergence theorem, we obtain that

$$\begin{aligned} & f(\cdot, M_{n_m}, \rho_{n_m}) \underset{m\to\infty}{\rightarrow} f(\cdot, M, \rho) \text{ in } L^2((\tau, T) \times \Omega), \\ & g(\cdot, M_{n_m}, \rho_{n_m}) \underset{m\to\infty}{\rightarrow} g(\cdot, M, \rho) \text{ in } L^2((\tau, T) \times \Omega), \\ & \int_0^{M_{n_m}} \left(|M| + \frac{1}{n}\right)^\alpha dM \underset{m\to\infty}{\rightarrow} \frac{1}{\alpha+1}|M|^\alpha M \text{ in } L^2((\tau, T) \times \Omega), \\ & \left(|M_{n_m}| + \frac{1}{n_m}\right)^\gamma \underset{m\to\infty}{\rightarrow} |M|^\gamma \text{ almost everywhere in } (\tau, T) \times \Omega. \end{aligned} \tag{3.19}$$

Moreover, combining (3.19) with (3.18), we obtain with the dominant convergence theorem that

$$\left(|M_{n_m}| + \frac{1}{n_m}\right)^\gamma \nabla\rho_{n_m} \underset{m\to\infty}{\rightarrow} |M|^\gamma \nabla\rho \text{ in } L^\infty_{w-*}((\tau, T) \times \Omega).$$

Since the convergence in the distributional sense is weaker than the L^p convergence for any $p \in [1, \infty]$ or than the L^∞_{w-*} convergence and since differential operators are continuous in the space of distributions, it follows with the convergences we derived in this subsection that (M, ρ) solves the problem (3.1)-(3.4) in the sense of distributions. The existence part of *Theorem 3.1* is thus proven.

□

3.3 Uniqueness of solutions

Proof of *Theorem 3.1* **(Uniqueness)**. Let us assume that the problem (3.1)-

(3.4) has two different solutions (M_1, ρ_1), (M_2, ρ_2) with the same initial data:

$$M_1(\tau) = M_2(\tau), \quad \rho_1(\tau) = \rho_2(\tau).$$

Since both (M_1, ρ_1) and (M_2, ρ_2) are solutions of the equation (3.1), we get

$$\begin{aligned}
\partial_t(M_1 - M_2) =& \frac{1}{\alpha+1}\Delta\left(|M_1|^\alpha M_1 - |M_2|^\alpha M_2\right) - \nabla\cdot\left(|M_1|^\gamma \nabla\rho_1 - |M_2|^\gamma\nabla\rho_2\right) \\
&+ \left(f(t, M_1, \rho_1) - f(t, M_2, \rho_2)\right) \\
=& \frac{1}{\alpha+1}\Delta\left(|M_1|^\alpha M_1 - |M_2|^\alpha M_2\right) \\
&- \nabla\cdot\left(|M_1|^\gamma \nabla(\rho_1 - \rho_2)\right) - \nabla\cdot\left(\left(|M_1|^\gamma - |M_2|^\gamma\right)\nabla\rho_2\right) \\
&+ \left(f(t, M_1, \rho_1) - f(t, M_1, \rho_2)\right) \\
&+ \left(f(t, M_1, \rho_2) - f(t, M_2, \rho_2)\right).
\end{aligned} \tag{3.20}$$

We want to estimate the difference $M_1 - M_2$, and we choose to do so in the $||\nabla^{+*}(\cdot)||$ norm. For this purpose we multiply (3.20) by $(-\Delta)^{-1}(M_1 - M_2)$ and integrate over Ω:

$$\begin{aligned}
&\big(\partial_t(M_1 - M_2), (-\Delta)^{-1}(M_1 - M_2)\big) \\
=& \big(\Delta\left(|M_1|^\alpha M_1 - |M_2|^\alpha M_2\right), (-\Delta)^{-1}(M_1 - M_2)\big) \\
&+ \left(-\nabla\cdot\left(|M_1|^\gamma\nabla(\rho_1 - \rho_2)\right), (-\Delta)^{-1}(M_1 - M_2)\right) \\
&+ \left(-\nabla\cdot\left(\left(|M_1|^\gamma - |M_2|^\gamma\right)\nabla\rho_2\right), (-\Delta)^{-1}(M_1 - M_2)\right) \\
&+ \big(f(t, M_1, \rho_1) - f(t, M_1, \rho_2), (-\Delta)^{-1}(M_1 - M_2)\big) \\
&+ \big(f(t, M_1, \rho_2) - f(t, M_2, \rho_2), (-\Delta)^{-1}(M_1 - M_2)\big).
\end{aligned} \tag{3.21}$$

On the left side of the resulting equation, there appears:

$$\big(\partial_t(M_1 - M_2), (-\Delta)^{-1}(M_1 - M_2)\big) = \frac{1}{2}\frac{d}{dt}\left\|\nabla^{+*}(M_1 - M_2)\right\|^2. \tag{3.22}$$

Suitable estimates for the terms on the right side of (3.21) are required now. The operator Δ is self-adjoint, therefore

$$\begin{aligned}
&\big(\Delta\left(|M_1|^\alpha M_1 - |M_2|^\alpha M_2\right), (-\Delta)^{-1}(M_1 - M_2)\big) \\
=& -\left(|M_1|^\alpha M_1 - |M_2|^\alpha M_2, M_1 - M_2\right).
\end{aligned} \tag{3.23}$$

In the subsequent *Section 3.4*, we prove the uniform boundedness for solutions of the problem (3.1)-(3.4), so that

$$R := \max\left\{\|M_i\|_\infty, \|\nabla\rho_i\|_\infty, \|\rho_i\|_\infty\right\} < \infty \tag{3.24}$$

holds. This and (1.6) leads to

$$\begin{aligned}
&\left(-\nabla\cdot\left(|M_1|^\gamma\nabla(\rho_1 - \rho_2)\right), (-\Delta)^{-1}(M_1 - M_2)\right) \\
=& \left(\left(|M_1|^\gamma\nabla(\rho_1 - \rho_2)\right), \nabla^{+*}(M_1 - M_2)\right) \\
\leqslant& R^\gamma\left\|\nabla(\rho_1 - \rho_2)\right\|\left\|\nabla^{+*}(M_1 - M_2)\right\|,
\end{aligned} \tag{3.25}$$

$$\begin{aligned}&(-\nabla\cdot((|M_1|^\gamma-|M_2|^\gamma)\,\nabla\rho_2),(-\Delta)^{-1}(M_1-M_2))\\ =&((|M_1|^\gamma-|M_2|^\gamma)\,\nabla\rho_2,\nabla^{+*}(M_1-M_2))\\ \leqslant& R\,\||M_1|^\gamma-|M_2|^\gamma\|\,\|\nabla^{+*}(M_1-M_2)\|\,.\end{aligned}\tag{3.26}$$

Finally, combining (3.24) with the assumptions made on f and fixing an (arbitrary) $t>0$, we obtain that

$$\begin{aligned}|f(s,M_1,\rho_1)-f(s,M_1,\rho_2)| =& \left|\int_0^1\frac{\partial f}{\partial\rho}(s,M_1,\rho_1+z(\rho_2-\rho_1))\,dz\right||\rho_1-\rho_2|\\ \underset{(2.11)}{\leqslant}& B_1(t,R)\,|\rho_1-\rho_2|\end{aligned}\tag{3.27}$$

and

$$\begin{aligned}&|f(s,M_1,\rho_2)-f(s,M_2,\rho_2)|\\ \underset{(2.10)}{=}&\left|\widetilde f\left(s,|M_1|^{\frac{\alpha}{2}}M_1,\rho_2\right)-\widetilde f\left(s,|M_2|^{\frac{\alpha}{2}}M_2,\rho_2\right)+F_4\left(|M_1|^{\frac{\alpha}{2}}M_1-|M_2|^{\frac{\alpha}{2}}M_2\right)\right|\\ \leqslant&\left(\left|\int_0^1\frac{\partial\widetilde f}{\partial M}\left(s,|M_1|^{\frac{\alpha}{2}}M_1+z\left(|M_2|^{\frac{\alpha}{2}}M_2-|M_1|^{\frac{\alpha}{2}}M_1\right),\rho_2\right)\,dz\right|+|F_4|\right)\\ &\cdot\left||M_1|^{\frac{\alpha}{2}}M_1-|M_2|^{\frac{\alpha}{2}}M_2\right|\\ \underset{(2.10)}{\leqslant}& B_2(t,R)\left||M_1|^{\frac{\alpha}{2}}M_1-|M_2|^{\frac{\alpha}{2}}M_2\right|\end{aligned}\tag{3.28}$$

for all $s\in[0,t]$ and some constants $B_1(t,R),B_2(t,R)>0$ depending only on the parameters of the problem and on R and t. With (3.22)-(3.28) we conclude from (3.21) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\nabla^{+*}(M_1-M_2)\right\|^2\\ \leqslant&-(|M_1|^\alpha M_1-|M_2|^\alpha M_2,M_1-M_2)+B_3(t,R)\left\|\nabla^{+*}(M_1-M_2)\right\|\\ &\cdot\left(\||M_1|^\gamma-|M_2|^\gamma\|+\|\nabla(\rho_1-\rho_2)\|+\left\||M_1|^{\frac{\alpha}{2}}M_1-|M_2|^{\frac{\alpha}{2}}M_2\right\|\right)\end{aligned}\tag{3.29}$$

for some constant $B_3(t,R)>0$ depending only on the parameters of the problem and on R and t.

With $\lim\limits_{x\to1}\frac{\left(x^{\frac{\alpha}{2}+1}-1\right)^2}{(x^{\alpha+1}-1)(x-1)}=\frac{\left(\frac{\alpha}{2}+1\right)^2}{\alpha+1}<\infty$ it follows that

$$\left(|M_1|^{\frac{\alpha}{2}}M_1-|M_2|^{\frac{\alpha}{2}}M_2\right)^2\leqslant B_4\left(|M_1|^\alpha M_1-|M_2|^\alpha M_2\right)(M_1-M_2),\tag{3.30}$$

and with $\lim\limits_{x\to1}\frac{x^{\frac{\alpha}{2}+1}-1}{x^\gamma-1}=\frac{\frac{\alpha}{2}+1}{\gamma}<\infty$ and $\gamma\geqslant\frac{\alpha}{2}+1$ we have

$$\left||M_1|^\gamma-|M_2|^\gamma\right|\leqslant B_5R^{\gamma-\frac{\alpha}{2}+1}\left||M_1|^{\frac{\alpha}{2}}M_1-|M_2|^{\frac{\alpha}{2}}M_2\right|.\tag{3.31}$$

Applying (3.30)-(3.31) together with the Young's inequality to (3.29) yields finally

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla^{+*}(M_1-M_2)\right\|^2\leqslant B_6(t,R)\left\|\nabla^{+*}(M_1-M_2)\right\|^2+B_6(t,R)\left\|\nabla(\rho_1-\rho_2)\right\|^2\tag{3.32}$$

for some constant $B_6(t,R)>0$ depending only on the parameters of the problem and on R and t.

Now we turn to equation (3.2). Both (M_1,ρ_1) and (M_2,ρ_2) solve it, hence

$$\begin{aligned}\partial_t(\rho_1-\rho_2) =&\Delta(\rho_1-\rho_2)-(g(t,M_1,\rho_1)-g(t,M_2,\rho_2))\\ =&\Delta(\rho_1-\rho_2)-g_1(\rho_1-\rho_2)-(g_2(t,\rho_1)M_1-g_2(t,\rho_2)M_2)\\ =&\Delta(\rho_1-\rho_2)-g_1(\rho_1-\rho_2)-(g_2(t,\rho_1)-g_2(t,\rho_2))M_1\\ &-g_2(t,\rho_2)(M_1-M_2).\end{aligned} \tag{3.33}$$

As usual, we multiply (3.33) by $\rho_1-\rho_2$ and integrate over Ω, so that it comes out

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\rho_1-\rho_2\|^2 =&-\|\nabla(\rho_1-\rho_2)\|^2-g_1\|\rho_1-\rho_2\|^2\\ &-((g_2(t,\rho_1)-g_2(t,\rho_2))M_1,\rho_1-\rho_2)\\ &-(g_2(t,\rho_2)(M_1-M_2),\rho_1-\rho_2)\,.\end{aligned} \tag{3.34}$$

Due to the property (1.7) of ∇^+ and the assumptions made on g_2 together with the Poincaré inequality and uniform boundedness of $\nabla\rho_2$ and M_1, we obtain the estimates:

$$\begin{aligned}&(-g_2(s,\rho_2)(M_1-M_2),\rho_1-\rho_2)\\ =\;&-(g_2(s,\rho_2)(\rho_1-\rho_2),M_1-M_2)\\ =\;&\big(\nabla(g_2(s,\rho_2)(\rho_1-\rho_2)),\nabla^{+*}(M_1-M_2)\big)\\ =\;&\big(g_2(s,\rho_2)\nabla(\rho_1-\rho_2),\nabla^{+*}(M_1-M_2)\big)\\ &+\left(\frac{\partial g_2}{\partial\rho}(s,\rho_2)(\rho_1-\rho_2)\nabla\rho_2,\nabla^{+*}(M_1-M_2)\right)\\ \underset{(2.11)}{\leqslant}\;&g_3\|\nabla(\rho_1-\rho_2)\|\,\|\nabla^{+*}(M_1-M_2)\|+\|\nabla\rho_2\|_\infty\\ &\cdot\left\|\frac{\partial g_2}{\partial\rho}(s,\rho_2)|\rho_1-\rho_2|\right\|\,\|\nabla^{+*}(M_1-M_2)\|\\ \underset{(2.11)}{\leqslant}\;&(g_3+B_7(t,R))\,\|\nabla(\rho_1-\rho_2)\|\,\|\nabla^{+*}(M_1-M_2)\|\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}|(g_2(s,\rho_1)-g_2(s,\rho_2))M_1| \underset{(2.11)}{\leqslant}\;& R\left|\int_0^1\frac{\partial g_2}{\partial\rho}(s,\rho_1+z(\rho_2-\rho_1))\,dz\right||\rho_1-\rho_2|.\\ \leqslant\;& B_8(t,R)|\rho_1-\rho_2|\end{aligned} \tag{3.36}$$

for some constants $B_7(t,R),B_8(t,R)>0$ depending only on the parameters of the problem and on R and t. Applying the estimates (3.35) and (3.36) to (3.34) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\rho_1-\rho_2\|^2\leqslant&-\|\nabla(\rho_1-\rho_2)\|^2+B_9(t,R)\|\rho_1-\rho_2\|^2\\ &+(g_3+B_7(t,R))\,\|\nabla(\rho_1-\rho_2)\|\,\|\nabla^{+*}(M_1-M_2)\|\,.\end{aligned} \tag{3.37}$$

By adding (3.32) and (3.37) together and using the Young's inequality one more time, we finally come to the inequality

$$\begin{aligned} &\frac{d}{dt}\left(\left\|\nabla^{+*}(M_1-M_2)\right\|^2 + B_6(t,R)\left\|\rho_1-\rho_2\right\|^2\right) \\ \leqslant &B_{10}(t,R)(g_3+1)\left(\left\|\nabla^{+*}(M_1-M_2)\right\|^2 + B_6(t,R)\left\|\rho_1-\rho_2\right\|^2\right) \end{aligned} \tag{3.38}$$

for some constant $B_{10}(t,R) > 0$ depending only on the parameters of the problem and on R and t. Integrating (3.38), we conclude that

$$\begin{aligned} &\left\|\nabla^{+*}(M_1(t)-M_2(t))\right\|^2 + B_6(t,R)\left\|\rho_1(t)-\rho_2(t)\right\|^2 \\ \leqslant &\left(\left\|\nabla^{+*}(M_1(\tau)-M_2(\tau))\right\|^2 + B_6(t,R)\left\|\rho_1(\tau)-\rho_2(\tau)\right\|^2\right) e^{B_{10}(t,R)\int_0^t g_3(s)+1\, ds} \\ \leqslant &B_{11}(t,R)\left(\left\|\nabla^{+*}(M_1(\tau)-M_2(\tau))\right\|^2 + B_6(t,R)\left\|\rho_1(\tau)-\rho_2(\tau)\right\|^2\right) \end{aligned} \tag{3.39}$$

for some constant $B_{11}(t,R) > 0$ depending only on the parameters of the problem and on R and t. This proves uniqueness for the problem (3.1)-(3.4) since the solutions (M_1,ρ_1), (M_2,ρ_2) coincide at $t=\tau$. The uniqueness part of *Theorem 3.1* is thus proven.

□

3.4 Dissipative estimate

In this section, we use the condition $\alpha > \gamma$ to establish a dissipative estimate for our model, which is necessary to show the existence of the pullback attractor (see *Section 3.5*). Our result reads:

Theorem 3.2 (Dissipative estimate)**.** *Let the functions f and g satisfy the assumptions (3.5)-(3.10) and let the constants α and γ satisfy $\gamma \in \left[\frac{\alpha}{2}+1,\alpha\right)$. Then the following dissipative estimate holds for the initial boundary-value problem (3.1)-(3.4):*

$$\begin{aligned} ||M(t)||_\infty + ||\rho(t)||_{W^{1,\infty}(\Omega)} \leqslant &C_\infty\left(||M_\tau||_\infty + ||\rho_\tau||_{W^{1,\infty}(\Omega)}\right)^{r_\infty} \\ &\cdot e^{-\omega_\infty(t-\tau)} + D_\infty \text{ for all } t \geqslant \tau, \end{aligned} \tag{3.40}$$

where the positive constants $C_\infty, r_\infty, \omega_\infty, D_\infty$ depend only on α, γ, f and g, and are independent of M_τ, ρ_τ or t.

Remark 3.3. As will become clear from the proof below, we do not actually need the condition $\gamma \geqslant \frac{\alpha}{2}+1$ to obtain the dissipative estimate (3.40). However, this condition is crucial for uniqueness of solutions (see the proof of *Theorem 3.1*).

Proof. The main idea of the proof is to derive a collection of coupled dissipative estimates for M and $\nabla\rho$ in various L^δ norms, with $\delta < \infty$ for the M component, and then apply a bootstrap argument in order to obtain the desired dissipative estimate in the L^∞ norm for both components. The estimates are done formally, they can be justified by passing to an appropriate sequence of regularization problems (e.g., (3.11)-(3.14)), performing the estimates in the same manner for the solutions of these problems and then passing to the limit.

We start with rewriting the equation (3.1) in the following way:

$$\partial_t M = \nabla\cdot\left(|M|^\gamma\nabla\left(\frac{1}{\alpha-\gamma+1}M|M|^{\alpha-\gamma}-\rho\right)\right)+f(t,M,\rho). \tag{3.41}$$

In order to derive our first a priori estimate, we multiply this equation by $\left(\frac{1}{\alpha-\gamma+1}M|M|^{\alpha-\gamma}-\rho\right)$ and integrate by parts over Ω to obtain

$$\begin{aligned}
&\left(\partial_t M,\frac{1}{\alpha-\gamma+1}M|M|^{\alpha-\gamma}-\rho\right)\\
&=-\left(|M|^\gamma,\left|\nabla\left(\frac{1}{\alpha-\gamma+1}M|M|^{\alpha-\gamma}-\rho\right)\right|^2\right)\\
&\quad+\left(f(t,M,\rho),\frac{1}{\alpha-\gamma+1}M|M|^{\alpha-\gamma}-\rho\right)\\
&\leqslant\left(f(t,M,\rho),\frac{1}{\alpha-\gamma+1}M|M|^{\alpha-\gamma}-\rho\right)\\
\Leftrightarrow\quad&\frac{d}{dt}\left(\frac{1}{(\alpha-\gamma+1)(\alpha-\gamma+2)}\left\||M|^{\frac{\alpha-\gamma+2}{2}}\right\|^2-(M,\rho)\right)\\
&\leqslant\left(f(t,M,\rho),\frac{1}{\alpha-\gamma+1}M|M|^{\alpha-\gamma}-\rho\right)-(\partial_t\rho,M)\,.
\end{aligned}\tag{3.42}$$

Next, we multiply the equation (3.2) by $(\partial_t\rho+\rho-1)$ in the same sense as above, in order to obtain

$$\begin{aligned}
&\|\partial_t\rho\|^2+\frac{1}{2}\frac{d}{dt}||\rho-1||^2=-\frac{1}{2}\frac{d}{dt}||\nabla\rho||^2-||\nabla\rho||^2-(g(t,M,\rho),\partial_t\rho+\rho-1)\Leftrightarrow\\
&\frac{1}{2}\frac{d}{dt}\left(||\nabla\rho||^2+||\rho-1||^2\right)=-||\nabla\rho||^2-\|\partial_t\rho\|^2-(g(t,M,\rho),\partial_t\rho+\rho-1)\,.
\end{aligned}\tag{3.43}$$

By adding the inequalities (3.42) and (3.43) together, we obtain that

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{(\alpha-\gamma+1)(\alpha-\gamma+2)}\left\||M|^{\frac{\alpha-\gamma+2}{2}}\right\|^2-(M,\rho)+\frac{1}{2}||\nabla\rho||^2+\frac{1}{2}||\rho-1||^2\right)\\
&\leqslant\left(f(t,M,\rho),\frac{1}{\alpha-\gamma+1}M|M|^{\alpha-\gamma}-\rho\right)-||\nabla\rho||^2-(\partial_t\rho,M)-\|\partial_t\rho\|^2\\
&\quad-(g(t,M,\rho),\partial_t\rho+\rho-1)\,.
\end{aligned}\tag{3.44}$$

We consider first the term containing $g(t, M, \rho) = g_1(t)\rho + g_2(t, \rho)M$. It then holds:

$$\begin{aligned} -(g_1\rho, \partial_t\rho + \rho - 1) = & -\frac{1}{2}\frac{d}{dt}\left(g_1||\rho||^2\right) + \frac{1}{2}\frac{d}{dt}g_1||\rho||^2 - g_1\left(||\rho||^2 - (1,\rho)\right) \\ \underset{(3.7)}{\leqslant} & -\frac{1}{2}\frac{d}{dt}\left(g_1||\rho||^2\right) - g_1\left(||\rho||^2 - (1,\rho)\right) \\ \leqslant & -\frac{1}{2}\frac{d}{dt}\left(g_1||\rho||^2\right) - (1-\varepsilon)g_1||\rho||^2 + \frac{1}{4\varepsilon}g_1|\Omega| \\ \underset{(3.7)}{\leqslant} & -\frac{1}{2}\frac{d}{dt}\left(g_1||\rho||^2\right) - (1-\varepsilon)g_1||\rho||^2 + \frac{1}{4\varepsilon}g_1(\tau)|\Omega| \end{aligned} \tag{3.45}$$

and

$$\begin{aligned} -(g_2(t,\rho)M, \partial_t\rho + \rho - 1) \leqslant & \ \varepsilon\left\|\partial_t\rho\right\|^2 + \varepsilon||\rho - 1||^2 + \frac{1}{2\varepsilon}||g_2(t,\rho)M||^2 \\ \underset{(3.8)}{\leqslant} & \varepsilon\left\|\partial_t\rho\right\|^2 + \varepsilon||\rho - 1||^2 + \frac{1}{2\varepsilon}g_3^2||M||^2. \end{aligned} \tag{3.46}$$

By combining (3.45) and (3.46) with the inequality

$$-(\partial_t\rho, M) - \left\|\partial_t\rho\right\|^2 \leqslant \frac{1}{2}||M||^2 - \frac{1}{2}\left\|\partial_t\rho\right\|^2 \tag{3.47}$$

and choosing $\varepsilon \leqslant \frac{1}{2}$, we have

$$\begin{aligned} & -(\partial_t\rho, M) - \left\|\partial_t\rho\right\|^2 - (g(t,M,\rho), \partial_t\rho + \rho - 1) \\ \leqslant & -\frac{1}{2}\frac{d}{dt}\left(g_1||\rho||^2\right) - (1-\varepsilon)g_1||\rho||^2 + \varepsilon||\rho-1||^2 + \frac{1}{4\varepsilon}g_1(\tau)|\Omega| - \left(\frac{1}{2} - \varepsilon\right)\left\|\partial_t\rho\right\|^2 \\ & + \left(\frac{1}{2} + \frac{1}{2\varepsilon}g_3^2\right)||M||^2 \\ \underset{\varepsilon\leqslant\frac{1}{2}}{\leqslant} & -\frac{1}{2}\frac{d}{dt}\left(g_1||\rho||^2\right) - (1-\varepsilon)g_1||\rho||^2 + \varepsilon||\rho-1||^2 + \frac{1}{4\varepsilon}g_1(\tau)|\Omega| \\ & + \left(\frac{1}{2} + \frac{1}{2\varepsilon}g_3^2\right)||M||^2. \end{aligned} \tag{3.48}$$

Further, we can estimate the terms with f from (3.44) in the following way:

$$\begin{aligned} \left(f(t,M,\rho), M|M|^{\alpha-\gamma}\right) \underset{(3.6)}{\leqslant} & \left(-F_2M^2 + f_3|M|, |M|^{\alpha-\gamma}\right) \\ = & -F_2\left\||M|^{\frac{\alpha-\gamma+2}{2}}\right\|^2 + f_3\left\||M|^{\frac{\alpha-\gamma+1}{2}}\right\|^2, \end{aligned} \tag{3.49}$$

$$\begin{aligned} -(f(t,M,\rho),\rho) \underset{(3.5)}{\leqslant} & \varepsilon||\rho||^2 + \frac{1}{4\varepsilon}f_1^2\left(|\Omega| + \left\||M|^{\frac{\xi}{2}}\right\|^2\right) \\ \leqslant & 2\varepsilon||\rho-1||^2 + 2\varepsilon + \frac{1}{4\varepsilon}f_1^2 + \frac{1}{4\varepsilon}f_1^2\left\||M|^{\frac{\xi}{2}}\right\|^2. \end{aligned} \tag{3.50}$$

By using the inequalities (3.48)-(3.50) we conclude from (3.44) that

$$\begin{aligned}
&\frac{d}{dt}\Bigg(\frac{1}{(\alpha-\gamma+1)(\alpha-\gamma+2)}\left\||M|^{\frac{\alpha-\gamma+2}{2}}\right\|^2-(M,\rho)\\
&\qquad+\frac{1}{2}||\nabla\rho||^2+\frac{1}{2}||\rho-1||^2+\frac{1}{2}g_1||\rho||^2\Bigg)\\
&\leqslant -F_2\left\||M|^{\frac{\alpha-\gamma+2}{2}}\right\|^2+f_3\left\||M|^{\frac{\alpha-\gamma+1}{2}}\right\|^2+\frac{1}{4\varepsilon}f_1^2\left\||M|^{\frac{\xi}{2}}\right\|^2+\left(\frac{1}{2}+\frac{1}{2\varepsilon}g_3^2\right)||M||^2\\
&\quad-||\nabla\rho||^2-(1-\varepsilon)g_1||\rho||^2+3\varepsilon||\rho-1||^2+\left(2\varepsilon+\frac{1}{4\varepsilon}g_1(\tau)+\frac{1}{4\varepsilon}f_1^2\right)|\Omega|.
\end{aligned} \tag{3.51}$$

In order to shorten the formulas, we introduce a new variable:

$$\begin{aligned}
\varphi :=&\frac{1}{(\alpha-\gamma+1)(\alpha-\gamma+2)}\left\||M|^{\frac{\alpha-\gamma+2}{2}}\right\|^2-(M,\rho)\\
&+\frac{1}{2}||\nabla\rho||^2+\frac{1}{2}||\rho-1||^2+\frac{1}{2}g_1||\rho||^2+1.
\end{aligned} \tag{3.52}$$

$|M|^{\frac{\alpha-\gamma+2}{2}}$ is the leading M power present in the expressions (3.51)-(3.52) due to the assumptions made on α, γ and ξ, and we also have the estimate

$$(M,\rho)\leqslant \varepsilon||\rho||^2+\frac{1}{4\varepsilon}||M||^2 \tag{3.53}$$

for all $\varepsilon > 0$. Altogether, applying the Poincaré and the Hölder inequalities and adjusting the constant ε, we can deduce from (3.51) the inequality

$$\frac{d}{dt}\varphi \leqslant -A_1\varphi + a_2\varphi^{\theta} \tag{3.54}$$

for some $A_1 \in \mathbb{R}_+$ and $a_2 \in L^1_b(\mathbb{R})$, $a_2 \geqslant 0$ and

$$\theta := \frac{\max\left\{\frac{\alpha-\gamma+1}{2}, \frac{\xi}{2}\right\}}{\frac{\alpha-\gamma+2}{2}} \in (0,1).$$

A simple calculation shows that any solution φ of the inequality (3.54) satisfies the inequality

$$\varphi(t) \leqslant \left(\varphi^{1-\theta}(\tau)e^{-A_1(1-\theta)(t-\tau)}+(1-\theta)\int_{\tau}^{t} e^{-A_1(1-\theta)(t-s)}a_2(s)\,ds\right)^{\frac{1}{1-\theta}}. \tag{3.55}$$

Applying *Lemma A.1* from the *Appendix* to the inequality (3.55) and taking into account that $a_2 \in L^1_b(\mathbb{R})$ and the inequality (3.53) holds, we finally obtain our first dissipative estimate. Set

$$\begin{aligned}
y_{\delta_0} &:= ||M||^{\delta_0}_{\delta_0} + 1 + ||\nabla\rho||^2, \\
\delta_0 &:= \alpha-\gamma+2 \geqslant 2,
\end{aligned} \tag{3.56}$$

it holds then that

$$y_{\delta_0}(t) \leqslant C_{y_{\delta_0}} y_{\delta_0}(\tau) e^{-\omega_{y_{\delta_0}}(t-\tau)} + D_{y_{\delta_0}}$$

for some $C_{y_{\delta_0}}, \omega_{y_{\delta_0}}, D_{y_{\delta_0}}$ that depend only upon the parameters of the problem.

Now, the equation (3.2) can be rewritten in the following way:

$$\partial_t(\rho - 1) = \Delta(\rho - 1) - g(t, M, \rho)$$

and can thus be regarded as an abstract parabolic evolution equation with respect to $\rho - 1$. Therefore, for all $t > 0$ it holds (see [34]) that

$$\rho(t) - 1 = e^{(t-\tau)\Delta}(\rho(\tau) - 1) - \int_\tau^t e^{(t-s)\Delta} g(s, M(s), \rho(s))\, ds, \tag{3.57}$$

and by applying operator ∇ to both sides of (3.57), we obtain that

$$\nabla\rho(t) = e^{(t-\tau)\Delta}\nabla\rho(\tau) - \int_\tau^t \nabla\left(e^{(t-s)\Delta} g(s, M(s), \rho(s))\right) ds. \tag{3.58}$$

The initial value $\rho(\tau)$ is assumed to be sufficiently smooth, so that the following holds:

$$||\nabla\rho(\tau)||_\delta < \infty. \tag{3.59}$$

What remains is to estimate the δ norm of the integral from (3.58) with the help of (1.9)-(1.10) and assumptions on g. By choosing $\mu \in \left(\frac{1}{2}, 1\right)$ and $\hat{\delta} \geqslant 1$ such that $W^{2\mu,\hat{\delta}}(\Omega) \hookrightarrow W^{1,\delta}(\Omega)$, we arrive at the estimate

$$\begin{aligned}
&\left\| \int_\tau^t \nabla\left(e^{(t-s)\Delta} g(s, M(s), \rho(s))\right) ds \right\|_\delta \\
\leqslant &\int_\tau^t \left\|(-\Delta)^\mu \left(e^{(t-s)\Delta} g(s, M(s), \rho(s))\right)\right\|_{\hat{\delta}} ds \\
\leqslant &A(\mu, \hat{\delta}) \int_\tau^t e^{-\beta(t-s)}(t-s)^{-\mu}\left(|g_1(s)|||\rho(s)||_{\hat{\delta}} + g_3(s)||M(s)||_{\hat{\delta}}\right) ds.
\end{aligned} \tag{3.60}$$

Altogether, we obtain from (3.58)-(3.60) the following estimate:

$$\begin{aligned}
||\nabla\rho(t)||_\delta \leqslant &e^{-\beta t}||\nabla\rho(\tau)||_\delta + A(\mu, \hat{\delta})\cdot \\
&\cdot \int_\tau^t e^{-\beta(t-s)}(t-s)^{-\mu}(|g_1(s)| + g_3(s))\left(||\rho(s)||_{\hat{\delta}} + ||M(s)||_{\hat{\delta}}\right) ds.
\end{aligned} \tag{3.61}$$

Leaving this result for a moment and returning to the equation (3.1) we multiply this equation by $M|M|^{\delta-1}$ for an arbitrary $\delta \geqslant \alpha - \gamma + 1$, so that all occurring powers remain non-negative, and (formally) integrate over Ω:

$$\begin{aligned}
\left(\partial_t M, M|M|^{\delta-1}\right) = &\left(\nabla\cdot(|M|^\alpha\nabla M) - \nabla\cdot(|M|^\gamma\nabla\rho)\right. \\
&\left.+ f(t, M, \rho), M|M|^{\delta-1}\right).
\end{aligned}$$

It follows that

$$\frac{1}{\delta+1}\frac{d}{dt}\left\||M|^{\frac{\delta+1}{2}}\right\|^2 = -\frac{4\delta}{(\alpha+\delta+1)^2}\left\|\nabla|M|^{\frac{\alpha+\delta+1}{2}}\right\|^2 + \frac{2\delta}{\alpha+\delta+1}\left(\nabla|M|^{\frac{\alpha+\delta+1}{2}}, |M|^{\gamma-\frac{\alpha}{2}+\frac{\delta-1}{2}}\nabla\rho\right) + (f(t,M,\rho), M|M|^{\delta-1}). \tag{3.62}$$

Set $\vartheta(\delta) := \frac{\gamma-\frac{\alpha}{2}+\frac{\delta-1}{2}}{\frac{\alpha+\delta+1}{2}}$. Then $\vartheta(\delta) < 1$ holds due to the assumption $\alpha > \gamma$. Applying Hölder's inequality, we obtain that

$$\begin{aligned}\left(\nabla|M|^{\frac{\alpha+\delta+1}{2}}, |M|^{\gamma-\frac{\alpha}{2}+\frac{\delta-1}{2}}\nabla\rho\right) &= \left(\nabla|M|^{\frac{\alpha+\delta+1}{2}}, |M|^{\vartheta(\delta)\frac{\alpha+\delta+1}{2}}\nabla\rho\right)\\ &\leqslant \|1\|_{\frac{6}{1-\vartheta(\delta)}}\left\|\nabla|M|^{\frac{\alpha+\delta+1}{2}}\right\|\left\||M|^{\frac{\alpha+\delta+1}{2}}\right\|_6^{\vartheta(\delta)}\|\nabla\rho\|_3\\ &\leqslant B_1\left\|\nabla|M|^{\frac{\alpha+\delta+1}{2}}\right\|^{1+\vartheta(\delta)}\|\nabla\rho\|_3.\end{aligned} \tag{3.63}$$

For the last inequality, the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ (recall that $d \leqslant 3$) was used. Further, we apply once more the Hölder inequality and assumptions on the function f and thus derive:

$$(f(t,M,\rho), M|M|^{\delta-1}) \leqslant -F_2\left\||M|^{\frac{\delta+1}{2}}\right\|^2 + f_3\left\||M|^{\frac{\delta}{2}}\right\|^2 \tag{3.64}$$

$$\leqslant -F_2\left\||M|^{\frac{\delta+1}{2}}\right\|^2 + f_3\|1\|_{\delta+1}\left(\left\||M|^{\frac{\delta+1}{2}}\right\|^2\right)^{\frac{\delta}{\delta+1}}. \tag{3.65}$$

We can conclude from (3.62) using (3.63) and (3.65) that

$$\begin{aligned}\frac{1}{\delta+1}\frac{d}{dt}\left\||M|^{\frac{\delta+1}{2}}\right\|^2 \leqslant &-\frac{4\delta}{(\alpha+\delta+1)^2}\left\|\nabla|M|^{\frac{\alpha+\delta+1}{2}}\right\|^2\\ &+\frac{2\delta}{\alpha+\delta+1}B_3\left\|\nabla|M|^{\frac{\alpha+\delta+1}{2}}\right\|^{1+\vartheta(\delta)}\|\nabla\rho\|_3\\ &-F_2\left\||M|^{\frac{\delta+1}{2}}\right\|^2 + f_3\|1\|_{\delta+1}\left(\left\||M|^{\frac{\delta+1}{2}}\right\|^2\right)^{\frac{\delta}{\delta+1}}.\end{aligned}$$

Since $1+\vartheta(\delta) < 2$, it follows with the Young inequality that

$$\frac{1}{\delta+1}\frac{d}{dt}\left\||M|^{\frac{\delta+1}{2}}\right\|^2 \leqslant -F_2\left\||M|^{\frac{\delta+1}{2}}\right\|^2 + f_3\|1\|_{\delta+1}\left(\left\||M|^{\frac{\delta+1}{2}}\right\|^2\right)^{\frac{\delta}{\delta+1}} + B_2(\delta)\|\nabla\rho\|_3^{\frac{2}{1-\vartheta(\delta)}}, \tag{3.66}$$

where $B_4(\delta) = \frac{1-\vartheta(\delta)}{2}\left(\frac{2\delta}{\alpha+\delta+1}B_3\right)^{\frac{2}{1-\vartheta(\delta)}}\left(\frac{4\delta}{(\alpha+\delta+1)^2}\frac{2}{1+\vartheta(\delta)}\right)^{-\frac{1+\vartheta(\delta)}{1-\vartheta(\delta)}}$, therefore this constant depends only on δ and the parameters of the problem.

Next, we return to the equality (3.62) in order to repeat the whole procedure once more, but this time we will be more precise about the estimates made, and will use the regularity achieved up to this point. First, due to (3.64) and two obvious inequalities we have

$$\begin{aligned}\frac{d}{dt}\left\||M|^{\frac{\delta+1}{2}}\right\|^2 =& -\frac{4\delta(\delta+1)}{(\alpha+\delta+1)^2}\left\|\nabla|M|^{\frac{\alpha+\delta+1}{2}}\right\|^2 \\ &+\frac{2\delta(\delta+1)}{\alpha+\delta+1}\left(\nabla|M|^{\frac{\alpha+\delta+1}{2}}, |M|^{\gamma-\frac{\alpha}{2}+\frac{\delta-1}{2}}\nabla\rho\right) \\ &+(\delta+1)(f(t,M,\rho), M|M|^{\delta-1}). \\ \leqslant& -B_3\left\|\nabla|M|^{\frac{\alpha+\delta+1}{2}}\right\|^2 \\ &+(\delta+1)B_4||\nabla\rho||_\infty\left\|\nabla|M|^{\frac{\alpha+\delta+1}{2}}\right\|\left\||M|^{\frac{\alpha+\delta+1}{2}}\right\|^{\vartheta(\delta)} \\ &-(\delta+1)F_2\left\||M|^{\frac{\delta+1}{2}}\right\|^2+(\delta+1)B_5f_3\left\||M|^{\frac{\alpha+\delta+1}{2}}\right\|^{2\zeta} \end{aligned} \tag{3.67}$$

for $\delta \geqslant \alpha-\gamma+1$ with $\zeta = \frac{\delta}{\alpha+\delta+1}$.
Recall that $f_3 \in L^\kappa_b(\mathbb{R})$ and $\kappa > 1$. Taking into account a special case of the interpolation inequality (1.4), the inequality

$$||v|| \leqslant B_6||\nabla v||^{1-\frac{1}{\kappa}}||v||_p^{\frac{1}{\kappa}}, \quad p = \frac{6}{1+2\kappa},$$

where the constant B_8 depends only on κ and the domain Ω, we obtain with the help of the Young inequality that

$$\begin{aligned} &(\delta+1)||\nabla v||||v||^{\vartheta(\delta)} \\ \leqslant&(\delta+1)\left(B_6\right)^{\vartheta(\delta)}||\nabla v||^{1+\vartheta(\delta)\left(1-\frac{1}{\kappa}\right)}||v||_p^{\vartheta(\delta)\frac{1}{\kappa}} \\ \leqslant&\left(B_6\right)^{\vartheta(\delta)}\left(\varepsilon||\nabla v||^2+B_7(\varepsilon)(\delta+1)^{\frac{2}{1-\vartheta(\delta)\left(1-\frac{1}{\kappa}\right)}}||v||_p^{\frac{2\vartheta(\delta)\frac{1}{\kappa}}{1-\vartheta(\delta)\left(1-\frac{1}{\kappa}\right)}}\right) \end{aligned} \tag{3.68}$$

and

$$\begin{aligned}(\delta+1)f_3||v||^{2\zeta} \leqslant& (\delta+1)f_3\left(B_6\right)^{2\zeta}||\nabla v||^{2\zeta\left(1-\frac{1}{\kappa}\right)}||v||_p^{2\zeta\frac{1}{\kappa}} \\ \leqslant& \left(B_6\right)^{2\zeta}\left(\varepsilon||\nabla v||^2+B_8(\varepsilon)\left(f_3(\delta+1)\right)^{\frac{1}{1-\zeta\left(1-\frac{1}{\kappa}\right)}}||v||_p^{\frac{2\zeta\frac{1}{\kappa}}{1-\zeta\left(1-\frac{1}{\kappa}\right)}}\right), \end{aligned} \tag{3.69}$$

where $B_9(\varepsilon)$ and $B_{10}(\varepsilon)$ depend only on ε and the parameters of the problem. With the Hölder inequality, we also have

$$\left\||M|^{\frac{\alpha+\delta+1}{2}}\right\|_p \leqslant \left\||M|^{\frac{\alpha}{2}}\right\|_{\frac{qp}{q-p}}\left\||M|^{\frac{\delta+1}{2}}\right\|_q, \quad q > \frac{6}{1+2\kappa}. \tag{3.70}$$

Since $\frac{6}{1+2\kappa} < 2$ holds due to $\kappa > 1$, we can assume that $q < 2$ and that it is independent from δ. By combining (3.68)-(3.69) for $v := |M|^{\frac{\alpha+\delta+1}{2}}$ with (3.70) and choosing ε small enough depending only on B_5 and B_6 (thus it depends only on the parameters of the problem) we can conclude from (3.67):

$$\begin{aligned}
&\frac{d}{dt}\left\||M|^{\frac{\delta+1}{2}}\right\|^2\\
\leqslant &B_9\left(\|\nabla\rho\|_\infty(\delta+1)\right)^{\frac{2}{1-\vartheta(\delta)\left(1-\frac{1}{\kappa}\right)}}\left(\left\||M|^{\frac{\alpha}{2}}\right\|_{\frac{qp}{q-p}}\left\||M|^{\frac{\delta+1}{2}}\right\|_q\right)^{\frac{2\vartheta(\delta)\frac{1}{\kappa}}{1-\vartheta(\delta)\left(1-\frac{1}{\kappa}\right)}}\\
&+B_{11}\left(f_3(\delta+1)\right)^{\frac{1}{1-\zeta\left(1-\frac{1}{\kappa}\right)}}\left(\left\||M|^{\frac{\alpha}{2}}\right\|_{\frac{qp}{q-p}}\left\||M|^{\frac{\delta+1}{2}}\right\|_q\right)^{\frac{2\zeta\frac{1}{\kappa}}{1-\zeta\left(1-\frac{1}{\kappa}\right)}}\\
&-F_2(\delta+1)\left\||M|^{\frac{\delta+1}{2}}\right\|^2
\end{aligned}$$

for $\delta \geqslant \alpha-\gamma+1$. Since $\vartheta(\delta), \zeta \in (0,1)$ it follows for all $\delta \geqslant \alpha-\gamma+2$ that

$$\begin{aligned}
&\frac{d}{dt}\left(\|M\|_\delta^\delta+1\right)\\
\leqslant &B_{11}\delta^{2\kappa}\left(\|\nabla\rho\|_\infty^{2\kappa}+f_3^\kappa+1\right)\left(\|M\|^\alpha_{\frac{\alpha}{2}\frac{qp}{q-p}}+1\right)\left(\|M\|^{q\delta/2}_{q\delta/2}+1\right)^{\frac{2}{q}}\\
&-F_2\delta\left(\|M\|_\delta^\delta+1\right).
\end{aligned}$$

Once more we get an integral inequality for $\|M(t)\|_\delta^\delta+1$:

$$\begin{aligned}
&\|M(t)\|_\delta^\delta+1\\
\leqslant &B_{11}\delta^{2\kappa}\int_\tau^t e^{-\delta F_2(t-s)}\left(\|\nabla\rho(s)\|_\infty^{2\kappa}+f_3^\kappa(s)+1\right)\left(\|M(s)\|^\alpha_{\frac{\alpha}{2}\frac{qp}{q-p}}+1\right)\\
&\cdot\left(\|M(s)\|^{q\delta/2}_{q\delta/2}+1\right)^{\frac{2}{q}}\,ds+e^{-\delta F_2(t-\tau)}\left(\|M(\tau)\|_\delta^\delta+1\right). \qquad (3.71)
\end{aligned}$$

Now we are ready to derive some more dissipative estimates for the problem (3.1)-(3.4). We will extensively use *Lemma A.1* from the *Appendix*. This lemma appears to be very useful in our situation. It actually shows that the 'dissipative property' is preserved under standard operations (addition, multiplication, raising to a power and integration). To shorten the formulas, let us set:

$$\begin{aligned}
h_1 &:= ||\nabla\rho||_3+1,\\
h_2 &:= ||\nabla\rho||_\infty+1,\\
u_\delta &:= ||M||_\delta^\delta+1, \quad \delta\in[1,\infty).
\end{aligned}$$

Observe that particular powers of y_{δ_0} and h_1, h_2 and u_δ (for sufficiently large δ), u_7 and h_2 can be connected with one another by the inequalities of the type (A.1) in the same manner as z_1 and z_3 from *Lemma A.1* are. From the *Lemma A.1* we can conclude that all of them dissipate exponentially with t:

$$h_1(t) \leqslant C_{h_1}(h_1 + y_{\delta_0})^{r_{h_1}}(\tau)e^{-\omega_{h_1}(t-\tau)} + D_{h_1}, \tag{3.72}$$

$$h_2(t) \leqslant C_{h_2}(h_2 + u_7)^{r_{h_2}}(\tau)e^{-\omega_{h_2}(t-\tau)} + D_{h_2}, \tag{3.73}$$

$$u_\delta(t) \leqslant U\left(u_\delta(\tau) + C_{u_\delta}(h_1 + y_{\delta_0})^{r\delta}(\tau)\right)e^{-\frac{F_2}{2}\delta(t-\tau)} + D_{u_\delta} =: \widetilde{u}_\delta(t), \tag{3.74}$$

where the appearing coefficients depend on the parameters of the problem, and only the coefficients C_{u_δ} and D_{u_δ} depend on δ as well. We especially emphasize that r is independent from δ (it will be crucial for the existence of the uniform dissipative estimate). Indeed, from (3.61) and the definition of y_{δ_0} ($y_{\delta_0} > 1$, see (3.56)), we obtain that

$$\begin{aligned}
||\nabla\rho(t)||_3 \leqslant & e^{-\beta t}||\nabla\rho(\tau)||_3 + A\left(\frac{3}{4}, 2\right)\cdot \\
& \cdot\int_\tau^t e^{-\beta(t-s)}(t-s)^{-\frac{3}{4}}(|g_1(s)| + g_3(s))\left(||\rho(s)||_2 + ||M(s)||_2\right) ds \\
\leqslant & e^{-\beta t}||\nabla\rho(\tau)||_3 + C(1,2,2)A\left(\frac{3}{4}, 2\right)\cdot \\
& \cdot\int_\tau^t e^{-\beta(t-s)}(t-s)^{-\frac{3}{4}}(|g_1(s)| + g_3(s))y_{\delta_0}(0)\, ds
\end{aligned} \tag{3.75}$$

since $\alpha - \gamma + 2 > 2$, $W^{2\cdot\frac{3}{4},2} \hookrightarrow W^{1,3}$ and $W^{1,2} \hookrightarrow L^2(\Omega)$ (with the embedding constant $C(1,2,2)$). Next, using (3.61) one more time, we obtain that

$$\begin{aligned}
||\nabla\rho(t)||_\infty \leqslant & e^{-\beta t}||\nabla\rho(\tau)||_\infty + A\left(\frac{3}{4}, 7\right)\cdot \\
& \cdot\int_\tau^t e^{-\beta(t-s)}(t-s)^{-\frac{3}{4}}(|g_1(s)| + g_3(s))\left(||\rho(s)||_7 + ||M(s)||_7\right) ds \\
\leqslant & e^{-\beta t}||\nabla\rho(\tau)||_\infty + C(1,7,3)A\left(\frac{3}{4}, 7\right)\cdot \\
& \cdot\int_\tau^t e^{-\beta(t-s)}(t-s)^{-\frac{3}{4}}(|g_1(s)| + g_3(s))\left(||\nabla\rho(s)||_3 + 1 + ||M(s)||_7\right) ds
\end{aligned} \tag{3.76}$$

since $W^{2\cdot\frac{3}{4},7} \hookrightarrow W^{1,\infty}$ and $W^{1,3}(\Omega) \hookrightarrow L^7(\Omega)$ (with the embedding constant $C(1,7,3)$). The estimates for h_1 and h_2 now follow with (3.75)-(3.76) and *Lemma A.1* due to the fact that for the function $d(t,s) := (t-s)_+^{-\frac{3}{4}}(|g_1(s)| + g_3(s))$ the condition $\sup\limits_{t>0}||d(t,\cdot)||_{L^1_b(\mathbb{R})} < \infty$ is satisfied (recall that we assumed that $g_1 \in L^\infty(\mathbb{R})$ and $g_3 \in L^\eta_b(\mathbb{R})$, $\eta > 4$).

Let us now check the dissipative estimate (3.74). With (3.66)

$$\frac{1}{\delta}\frac{d}{dt}u_\delta \leqslant -F_2 u_\delta + |\Omega| f_3 u_\delta^{\frac{\delta-1}{\delta}} + B_4(\delta)h_1^{\frac{2}{1-\vartheta(\delta)}}. \tag{3.77}$$

Recall that $\vartheta(\delta) = \frac{\gamma-\frac{\alpha}{2}+\frac{\delta-2}{2}}{\frac{\alpha+\delta}{2}}$ and, consequently, $\frac{2}{1-\vartheta(\delta)} = \frac{\alpha+\delta}{\alpha-\gamma+1} \leqslant B_{10}\delta$ for some B_{12} and $\delta \geqslant \delta_*$ sufficiently large. Now, the Young inequality yields:

$$u_\delta^{\frac{\delta-1}{\delta}} = (\varepsilon u_\delta)^{\frac{\delta-1}{\delta}}\varepsilon^{-\frac{\delta-1}{\delta}} \leqslant \frac{\delta-1}{\delta}\varepsilon u_\delta + \frac{1}{\delta}\varepsilon^{-(\delta-1)},$$

therefore it follows from (3.77) that

$$\frac{d}{dt}u_\delta \leqslant -\,\delta\left(F_2 - \varepsilon|\Omega|f_3\frac{\delta-1}{\delta}\right)u_\delta + \varepsilon^{-(\delta-1)}|\Omega|f_3 + \delta B_4(\delta)h_1^{B_{12}\delta}.$$

Gronwall's lemma then yields

$$\begin{aligned} u_\delta(t) \leqslant & \int_\tau^t e^{-\delta\int_s^t F_2-\varepsilon|\Omega|f_3(s)\frac{\delta-1}{\delta}\,ds}\left(\varepsilon^{-(\delta-1)}|\Omega|f_3(s) + \delta B_4(\delta)h_1^{B_{12}\delta}(s)\right)ds \\ & + e^{-\delta\int_\tau^t F_2-\varepsilon|\Omega|f_3(s)\frac{\delta-1}{\delta}\,ds}u_\delta(\tau). \end{aligned} \tag{3.78}$$

Observe that it holds that

$$\begin{aligned} \int_\tau^t F_2 - \varepsilon|\Omega|f_3(s)\,ds \geqslant & F_2(t-\tau) - \varepsilon|\Omega|\int_{\lfloor\tau\rfloor}^{\lceil t\rceil} f_3(s)\,ds \\ \geqslant & F_2(t-\tau) - \varepsilon|\Omega|||f_3||_{L_b^1(\mathbb{R})}(\lceil t\rceil - \lfloor\tau\rfloor) \\ \geqslant & \left(F_2 - \varepsilon|\Omega|||f_3||_{L_b^1(\mathbb{R})}\right)(t-\tau) - 2\varepsilon|\Omega|||f_3||_{L_b^1(\mathbb{R})}. \end{aligned} \tag{3.79}$$

For $\varepsilon := \frac{F_2}{2|\Omega|||f_3||_{L_b^1(\mathbb{R})}}$, it follows with (3.78) and (3.79) that

$$\begin{aligned} u_\delta(t) \leqslant & e^{F_2}\left(\int_\tau^t e^{-(t-s)\delta\frac{F_2}{2}}\left(\varepsilon^{-(\delta-1)}|\Omega|f_3(s) + \delta B_4(\delta)h_1^{B_{12}\delta}(s)\right)ds \right. \\ & \left. + e^{-(t-\tau)\delta\frac{F_2}{2}}u_\delta(\tau)\right). \end{aligned}$$

The dissipate estimate (3.74) follows now with the estimate (A.2) of *Lemma A.1* and the dissipate estimate (3.72) for h_1.

Now, from the inequality (3.71) we can conclude that

$$u_\delta(t) \leqslant e^{-\delta F_2(t-\tau)}u_\delta(\tau) + B_{11}\delta^{2\kappa}\int_\tau^t e^{-\delta F_2(t-s)}H_1(s)\widetilde{u}_{\frac{q}{2}\delta}^{\frac{2}{q}}(s)\;ds, \tag{3.80}$$

where

$$H_1(t) := \left(h_2^{2\kappa}(t) + f_3^\kappa(t) + 1\right)\widetilde{u}_{\frac{\alpha}{2}\frac{qp}{q-p}}^{\frac{2(q-p)}{qp}}(t).$$

Taking into account that $u_{\frac{q}{2}\delta}^{\frac{2}{q}}$ dissipates with $e^{-\delta\frac{F_2}{2}(t-\tau)}$ and that H_1 dissipates with an exponent independent of δ, we consecutively apply (A.3) to (3.80) and get

$$\begin{aligned} u_\delta(t) \leqslant & e^{-\delta \frac{F_2}{2}(t-\tau)} u_\delta(\tau) + B_{11}\delta^{2\kappa} \widetilde{u}^{\frac{2}{q}}_{\frac{q}{2}\delta}(t) \int_\tau^t e^{-\delta \frac{F_2}{2}(t-s)} H_1(s)\, ds \\ \leqslant & e^{-\delta \frac{F_2}{2}(t-\tau)} u_\delta(\tau) + B_{11}\delta^{2\kappa-1} H_2(t) \widetilde{u}^{\frac{2}{q}}_{\frac{q}{2}\delta}(t), \end{aligned}$$

where

$$H_2(t) := \left(h_2^{2\kappa}(t) + ||f_3||^{\kappa}_{L^\kappa_b(\mathbb{R})} + 1\right) \widetilde{u}^{\frac{2(q-p)}{qp}}_{\frac{\alpha}{2}\frac{qp}{q-p}}(t)$$

and $\delta \geqslant \delta_*$ is sufficiently large. The bound δ_* depends only on the parameters of the problem. Therefore, we may assume that

$$\widetilde{u}(t) = e^{-\delta \frac{F_2}{2}(t-\tau)} u_\delta(\tau) + B_{13}\delta^{2\kappa-1} H_2(t) \widetilde{u}^{\frac{2}{q}}_{\frac{q}{2}\delta}(t), \tag{3.81}$$

Since

$$u_\delta(\tau) = ||M(\tau)||^\delta_\delta + 1 \leqslant ||M(\tau)||_\infty |\Omega| + 1$$

we conclude from (3.81) that for

$$A_\delta(t) := \widetilde{u}_\delta(t) \left(\frac{e^{\frac{F_2}{2}(t-\tau)}}{||M(\tau)||_\infty + 1} \right)^\delta + 1 \tag{3.82}$$

it holds that

$$A_\delta(t) \leqslant B_{12} H_2(t) \delta^{2\kappa-1} A^{\frac{2}{q}}_{\frac{q}{2}\delta}(t).$$

One can show by induction then that

$$\begin{aligned} A^{\left(\frac{q}{2}\right)^n}_{\left(\frac{2}{q}\right)^n \delta_*}(t) \leqslant & \left(B_{14} H_2(t) \delta_*^{2\kappa-1}\right)^{\sum_{k=1}^n \left(\frac{q}{2}\right)^k} \left(\frac{q}{2}\right)^{(2\kappa-1)\sum_{k=1}^n k\left(\frac{q}{2}\right)^k} A_{\delta_*}(t) \\ \underset{n\to\infty}{\to} & \left(B_{14} H_2(t) \delta_*^{2\kappa-1}\right)^{\frac{\frac{q}{2}}{1-\frac{q}{2}}} \left(\frac{q}{2}\right)^{(2\kappa-1)\frac{q}{2}\left(\frac{1}{1-\frac{q}{2}}\right)^2} A_{\delta_*}(t) \\ =: & \ H^{\delta_*}(t) A_{\delta_*}(t). \end{aligned}$$

Therefore, we get

$$\limsup_{\delta\to\infty} A^{\frac{1}{\delta}}_\delta(t) \leqslant H(t) A^{\frac{1}{\delta_*}}_{\delta_*}(t). \tag{3.83}$$

By combining (3.83) with (3.82), we finally arrive at an estimate for $||M(t)||_\infty$:

$$\begin{aligned} ||M(t)||_\infty + 1 = & \lim_{\delta\to\infty} u^{\frac{1}{\delta}}_\delta(t) \\ \leqslant & \limsup_{\delta\to\infty} \widetilde{u}^{\frac{1}{\delta}}_\delta(t) \\ \leqslant & H(t) \left(\widetilde{u}^{\frac{1}{\delta_*}}_{\delta_*}(t) + \left(||M_\tau||_\infty + 1\right) e^{-\frac{F_2}{2}(t-\tau)} \right). \end{aligned} \tag{3.84}$$

Now, since the functions H and $\widetilde{u}_{\delta_*}$ dissipate exponentially (recall (3.73)-(3.74) and the definition of H and H_2), we apply *Lemma A.1* to (3.84) and conclude that $||M||_\infty$ dissipates exponentially as well. Moreover, it follows from the proof that there exists a dissipative estimate for $||M||_\infty$ of the form given in (3.40). The dissipative estimate for $||\nabla\rho||_\infty + 1 = h_2$ is given in (3.73) and the *Theorem 3.2* is thus proven.

□

3.5 Pullback attractor

The aim of this section is to apply the general theory from *Section 1.4* to the problem (3.1)-(3.4). We prove

Theorem 3.3. *Let the functions f and g satisfy the assumptions (3.5)-(3.10) and let the constants α and γ satisfy $\gamma \in \left[\frac{\alpha}{2}+1, \alpha\right)$. Then for all $p \in [1,\infty]$ the solutions of the problem (3.1)-(3.4) can be described by a process $\{U(t,\tau)\}_{t\geqslant\tau}$ that acts on the space $\mathcal{H}^p(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$. The process $\{U(t,\tau)\}_{t\geqslant\tau}$ possesses the pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ in $\mathcal{H}^p(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$, which is independent of the concrete choice of p. Moreover, the set $\bigcup_{t\in\mathbb{R}} \mathcal{A}(t)$ is relative compact in $\mathcal{H}^p(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$ for all $p \in [1,\infty]$.*

Remark 3.4 (Rate of convergence to the pullback attractor)**.** The rate of convergence to the pullback attractor $\mathcal{A}(t)$ may, of course, depend on p and can be arbitrarily slow.

Proof of *Theorem 3.3***.** We showed in *Theorem 3.1* that the problem (3.1)-(3.4), if considered in the space $L^\infty(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$, is well-posed: for each pair of initial values $(M_\tau, \rho_\tau) \in L^\infty(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$ there exists a unique solution (M, ρ) in terms of *Definition 3.1*.
We define the solving process $\{U(t,\tau)\}_{t\geqslant\tau}$ of the problem (3.1)-(3.4) on the phase space $L^\infty(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$ as follows: for all $t \geqslant \tau$ let

$$U(t,\tau) : L^\infty(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right) \to L^\infty(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right),$$
$$U(t,\tau)(M_\tau, \rho_\tau) := (M(t), \rho(t)) \text{ for all } (M_\tau, \rho_\tau) \in L^\infty(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right).$$

The dissipative estimate (3.2) provides the existence of the ball B_* of radius $2D_\infty$ centered at $(0,1)$, which uniformly absorbs all bounded subsets of $L^\infty(\Omega) \times$

$\left(1 + W_0^{1,\infty}(\Omega)\right)$. According to *Remark 1.11*, the set

$$C_* := \bigcup_{s \in [0,T(B_*)]} \bigcup_{t \in \mathbb{R}} U(t, t-s) B_*,$$

where $T(B_*)$ is such that $U(t, t-s)B_* \subset B_*$ for all $s \geqslant T(B_*)$, is a positively invariant set that uniformly absorbs all bounded subsets of $L^\infty(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$. With the dissipative estimate (3.2), it follows also that C_* is actually contained in a ball B_{**} of a radius R_{**}, $R_{**} \geqslant 2D_\infty$, centered at $(0, 1)$.

Clearly, the spaces $\mathcal{H}^p(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$ inherit all the properties of the space $\mathcal{H}^p(\Omega)$ that are listed in *Theorems 1.6* and *1.7* for $p = \infty$ and $p \in [1, \infty)$, respectively. In particular, they are locally convex spaces, so that the general theory from *Section 1.4* is applicable.

The fact that $L^\infty(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$ and all $\mathcal{H}^p(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$ coincide as sets and have the same set of bounded sets (see *Theorems 1.6(2)* and *1.7(2)*) has as a consequence that, in the spaces $\mathcal{H}^p(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$, the family $\{U(t,\tau)\}_{t \geqslant \tau}$ remains to be a process, and the set C_* remains to be a bounded positively invariant uniformly absorbing set for $\{U(t,\tau)\}_{t \geqslant \tau}$.

For $p \in [1, \infty)$, the $\mathcal{H}^p(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$ topology is, per definition, stronger than the $H^\infty(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$ topology, and, if restricted to a bounded set, it coincides with the restriction of the $L^2(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$ topology (see *Theorem 1.7(4)*).

Altogether, we arrive at the following conclusion: it suffices to prove the existence of the family of compact sets with the invariance property (1.11) that pullback attracts C_* equipped with one of the metrics that generates the $L^2(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$ topology, for example, with the metric $m_*^{(2)}$ defined as follows:

$$m_*^{(2)}((M_1, \rho_1), (M_2, \rho_2)) := \left(||M_1 - M_2||^2 + ||\rho_1 - \rho_2||^2_{W_0^{1,\infty}(\Omega)} \right)^{\frac{1}{2}}$$

for all $(M_1, \rho_1), (M_2, \rho_2) \in L^2(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$. This pullback attracting family is then the pullback attractor for the whole $\mathcal{H}^p(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$ for each $p \in [1, \infty]$.

However, to be able to apply the existence criterion for metric spaces, *Theorem 1.14*, it is necessary for the process operators to be mapping in a complete metric space.

The underling space, $\left(L^2(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right), m_*^{(2)}\right)$, is a complete metric space, so that the set

$$\bar{C}_* := cl_{L^2(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)}(C_*),$$

equipped with the metric $m_*^{(2)}$, is complete too.

Every $L^\infty(\Omega)$ ball is a closed subset of $L^\infty_{w-*}(\Omega)$, thus closed in the stronger topology of $L^2(\Omega)$, so that

$$\bar{C}_* \subset B_{**},$$

and the process $\{U(t,\tau)\}_{t\geqslant\tau}$ is well defined in $\bar{C}_*$.

Let us assume for a moment that $\{U(t,\tau)\}_{t\geqslant\tau}$ is a closed process in $\bar{C}_*$. As a closure of a positively invariant set, $\bar{C}_*$ is then also positively invariant under $\{U(t,\tau)\}_{t\geqslant\tau}$. Let us assume further that the set

$$K_* := cl_{L^2(\Omega)\times\left(1+W_0^{1,\infty}(\Omega)\right)}\left(\bigcup_{\tau\in\mathbb{R}} U(\tau+1,\tau)\bar{C}_*\right) \tag{3.85}$$

is compact in $L^2(\Omega)\times\left(1+W_0^{1,\infty}(\Omega)\right)$. As a closure of a uniform time shift of $\bar{C}_*$, K_* is also uniformly absorbing. All assumptions of *Theorem 1.14* are then fulfilled since we are dealing with a closed process (the process $\{U(t,\tau)\}_{t\geqslant\tau}$) in a complete metric space (the set $\bar{C}_*$ equipped with a complete metric), and this process possesses a compact uniformly absorbing set (the set K_*). *Theorem 1.14* yields the existence of the pullback attractor, its sets are contained in the compact K_*, and, as we showed above, it is also the pullback attractor for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in each of the spaces $\mathcal{H}^p(\Omega)\times\left(1+W_0^{1,\infty}(\Omega)\right)$ for $p\in[1,\infty]$.

In order to finish the proof, it remains to check the closedness of $\{U(t,\tau)\}_{t\geqslant\tau}$ and the compactness of K_*.

Let us first prove the closedness of the process operators. In the proof of *Theorem 3.1* we encountered the local Lipschitz-type continuity property (3.39) for the solutions of (3.1)-(3.4). It can be translated into the following form:

$$\begin{aligned}&\left\|U(t,\tau)\left(M_\tau^{(1)},\rho_\tau^{(1)}\right)-U(t,\tau)\left(M_\tau^{(2)},\rho_\tau^{(2)}\right)\right\|_{H^{-1}(\Omega)\times L^2(\Omega)}\\ \leqslant& L(t-\tau,R_{**})\left\|\left(M_\tau^{(1)},\rho_\tau^{(1)}\right)-\left(M_\tau^{(2)},\rho_\tau^{(2)}\right)\right\|_{H^{-1}(\Omega)\times L^2(\Omega)}\end{aligned} \tag{3.86}$$

for all $\left(M_\tau^{(1)},\rho_\tau^{(1)}\right),\left(M_\tau^{(2)},\rho_\tau^{(2)}\right)\in\bar{C}_*$. The constant $L(t-\tau,R_{**})$ depends only on $t-\tau$, R_{**} and the parameters of the problem.

Recall that due to the embedding theorems for Sobolev spaces, we have

$$L^2(\Omega)\times W_0^{1,\infty}(\Omega)\hookrightarrow H^{-1}(\Omega)\times L^2(\Omega). \tag{3.87}$$

Let $\left\{\left(M_\tau^{(n)},\rho_\tau^{(n)}\right)\right\}_{n\in\mathbb{N}}\subset\bar{C}_*$ be a sequence of initial data convergent in $L^2(\Omega)\times\left(1+W_0^{1,\infty}(\Omega)\right)$ to some (M_τ,ρ_τ). Due to the continuous embedding (3.87), this sequence converges in $H^{-1}(\Omega)\times L^2(\Omega)$ to the same limit. From the property (3.86) we deduce that the sequence $\left\{U(t,\tau)\left(M_\tau^{(n)},\rho_\tau^{(n)}\right)\right\}_{n\in\mathbb{N}}$ converges to

$U(t,\tau)\,(M_\tau,\rho_\tau)$ in $H^{-1}(\Omega)\times L^2(\Omega)$ for all $t\geqslant\tau$.
Let us further assume that for some $t\geqslant\tau$ the sequence $\left\{U(t,\tau)\left(M_\tau^{(n)},\rho_\tau^{(n)}\right)\right\}_{n\in\mathbb{N}}$ is convergent in $L^2(\Omega)\times\left(1+W_0^{1,\infty}(\Omega)\right)$. Again, due to the continuity of the embedding (3.87), the limit is the same. This proves closedness.

To prove compactness of K_* in $L^2(\Omega)\times\left(1+W_0^{1,\infty}(\Omega)\right)$, we multiply the equation (3.1) by $(\alpha+1)\partial_t|M|^\alpha M$ and integrate (formally) over Ω:

$$(\alpha+1)\left(\partial_t M,\partial_t|M|^\alpha M\right)=\left(\Delta|M|^\alpha M+(\alpha+1)\hat f(t,M,\rho),\partial_t|M|^\alpha M\right).$$

Here:

$$\hat f(t,M,\rho)=-\nabla\cdot\left(|M|^\gamma\nabla\rho\right)+f(t,M,\rho).$$

After integrating by parts, we obtain that

$$\left(\frac{\alpha+1}{\frac{\alpha}{2}+1}\right)^2\left\|\partial_t|M|^{\frac{\alpha}{2}+1}\right\|^2=-\frac{1}{2}\partial_t\left\|\nabla|M|^{\alpha+1}\right\|^2+\frac{(\alpha+1)^2}{\frac{\alpha}{2}+1}\left(|M|^{\frac{\alpha}{2}}\hat f(t,M,\rho),\partial_t|M|^{\frac{\alpha}{2}+1}\right).$$

With the Cauchy-Schwarz inequality we have

$$\left(\frac{\alpha+1}{\frac{\alpha}{2}+1}\right)^2\left\|\partial_t|M|^{\frac{\alpha}{2}+1}\right\|^2\leqslant-\partial_t\left\|\nabla|M|^{\alpha+1}\right\|^2+(\alpha+1)^2\left\||M|^{\frac{\alpha}{2}}\hat f(t,M,\rho)\partial_t|M|^{\frac{\alpha}{2}+1}\right\|^2,$$

so that

$$\partial_t\left\|\nabla|M|^{\alpha+1}\right\|^2\leqslant(\alpha+1)^2\left\||M|^{\frac{\alpha}{2}}\hat f(t,M,\rho)\right\|^2.$$

It follows with multiplying by $t-\tau$, $t>\tau$ that

$$\partial_t\left((t-\tau)\left\|\nabla|M|^{\alpha+1}\right\|^2\right)\leqslant\left\|\nabla|M|^{\alpha+1}\right\|^2+(t-\tau)(\alpha+1)^2\left\||M|^{\frac{\alpha}{2}}\hat f(t,M,\rho)\right\|^2.$$

Integrating over $(\tau,\tau+1)$, we obtain that

$$\left\|\nabla|M(\tau+1)|^{\alpha+1}\right\|^2\leqslant\int_\tau^{\tau+1}\left\|\nabla|M(s)|^{\alpha+1}\right\|^2+(s-\tau)(\alpha+1)^2\left\||M(s)|^{\frac{\alpha}{2}}\hat f(s,M(s),\rho(s))\right\|^2\,ds.\quad(3.88)$$

It remains, therefore, to estimate the integral on the right side of (3.88). We have

$$\begin{aligned}\left\||M|^{\frac{\alpha}{2}}\hat f(s,M,\rho)\right\|&=\left\|\frac{\gamma}{\frac{\alpha}{2}+\gamma}\nabla|M|^{\frac{\alpha}{2}+\gamma}\cdot\nabla\rho+|M|^{\frac{\alpha}{2}+\gamma}\Delta\rho+|M|^{\frac{\alpha}{2}}f(s,M,\rho)\right\|\\&\leqslant\frac{\gamma}{\frac{\alpha}{2}+\gamma}\left\|\nabla\rho\right\|_\infty\left\|\nabla|M|^{\frac{\alpha}{2}+\gamma}\right\|+\left\|\Delta\rho\right\|_2\left\||M|^{\frac{\alpha}{2}+\gamma}\right\|_\infty\\&\quad+\left\||M|^{\frac{\alpha}{2}}f(s,M,\rho)\right\|.\end{aligned}$$

From the derivation of dissipative estimates for $||M||_\delta^\delta$ in *Section 3.4* we conclude that for all $(M_\tau, \rho_\tau) \in \bar{C}_*$ It then holds:

$$\int_\tau^{\tau+1} \left\| \nabla |M(s)|^{\alpha+1} \right\|^2 ds \leqslant B_1 \left(R_{**}\right), \tag{3.89}$$

$$\int_\tau^{\tau+1} \left\| \nabla |M(s)|^{\frac{\alpha}{2}+\gamma} \right\|^2 ds \leqslant B_2 \left(R_{**}\right), \tag{3.90}$$

where the constants $B_1 \left(R_{**}\right)$ and $B_2 \left(R_{**}\right)$ depend only on R_{**} and the parameters of the problem.

Further, due to the classical energy estimate and assumptions on g, we have for all $(M_\tau, \rho_\tau) \in \bar{C}_*$

$$\begin{aligned}\int_\tau^{\tau+1} \|\Delta\rho(s)\|^2 ds &\leqslant \|\nabla\rho(\tau)\|^2 + \int_\tau^{\tau+1} \|g(s, M(s), \rho(s))\, ds\|^2 \\ &\leqslant B_3 \left(R_{**}\right),\end{aligned} \tag{3.91}$$

the constant $B_3 \left(R_{**}\right)$ depends only on R_{**} and the parameters of the problem.

By combining (3.89)-(3.91) with (3.88), we arrive at the following smoothing estimate for M:

$$\left\| |M(\tau+1)|^{\alpha+1} \right\|_{H_0^1(\Omega)} \leqslant B_4 \left(R_{**}\right), \tag{3.92}$$

the constant $B_4 \left(R_{**}\right)$ depends only on R_{**} and the parameters of the problem. Finally, using *Lemma 1.1* for $q = \alpha + 1$, we obtain from (3.92) that

$$\|M(\tau+1)\|_{W^{\frac{1}{\alpha+1}-\theta, 2(\alpha+1)}(\Omega)} \leqslant B_M \left(R_{**}\right)$$

for an arbitrary $\theta \in (0, 1)$, the constant $B_M \left(R_{**}\right)$ depends only on R_{**}, θ and the parameters of the problem. We choose $\theta := \frac{1}{2}$, so that

$$\|M(\tau+1)\|_{W^{\frac{1}{2(\alpha+1)}, 2(\alpha+1)}(\Omega)} \leqslant B_M \left(R_{**}\right). \tag{3.93}$$

Next, we deal with equation (3.2). Since

$$\begin{aligned}(-\Delta)^{\frac{11}{12}} \rho(\tau+1) = &(-\Delta)^{\frac{11}{12}} e^{\Delta} \rho_\tau \\ &- \int_\tau^{\tau+1} (-\Delta)^{\frac{11}{12}} e^{(\tau+1-\omega)\Delta} g(\omega, M(\omega), \rho(\omega))\, d\omega\end{aligned}$$

we conclude with the properties (1.9)-(1.10) that, due to assumptions on g,

$$\begin{aligned}&\left\| (-\Delta)^{\frac{11}{12}} \rho(\tau+1) \right\|_6 \\ \leqslant &\left\| (-\Delta)^{\frac{11}{12}} e^{\Delta}\rho_\tau - \int_\tau^{\tau+1} (-\Delta)^{\frac{11}{12}} e^{(\tau+1-\omega)\Delta} g(\omega, M(\omega), \rho(\omega))\, d\omega \right\|_6 \\ \leqslant &A\left(\frac{5}{12}, 6\right) \|\nabla\rho_\tau\|_6 + A\left(\frac{11}{12}, 6\right) \int_\tau^{\tau+1} (\tau+1-\omega)^{-\frac{11}{12}} \|g(\omega, M(\omega), \rho(\omega))\|_6\, d\omega \\ \leqslant &B_\rho(R_{**}),\end{aligned} \tag{3.94}$$

and the constant $B_\rho(R_{**})$ depends only on R_{**} and the parameters of the problem.

With the smoothing properties (3.93)-(3.94) and the compact embeddings (see *Theorem 1.10*)

$$W^{\frac{1}{2(\alpha+1)},2(\alpha+1)}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega),$$
$$W^{\frac{11}{6},6}(\Omega) \hookrightarrow\hookrightarrow W^{1,\infty}(\Omega),$$

we obtain that $U(\tau+1,\tau)$ maps the set $\bar{C}_*$ into a compact subset of $L^2(\Omega) \times \Big(1 + W_0^{1,\infty}(\Omega)\Big)$, and that this set can be assumed to be one and the same for all $\tau \in \mathbb{R}$. Since K_* is contained in it (see the definition of K_* in (3.85)), it is also compact. *Theorem 3.3* is thus proven.

□

Remark 3.5 (Pullback attractor in 1D-case)**.** In case of one spatial dimension, there is no need to pass to a weaker topology. For $\Omega = (a,b)$ for some $-\infty < a < b < \infty$ the process $\{U(t,\tau)\}_{t\geqslant\tau}$ possesses the pullback attractor in the phase space $L^\infty(a,b) \times \Big(1 + W_0^{1,\infty}(a,b)\Big)$. The proof of this statement (can be found in [15]) is very similar to the proof of *Theorem 3.3*. It makes use of the compact embedding

$$H_0^1(a,b) \hookrightarrow\hookrightarrow C([a,b]),$$

see *Theorem 1.10*.

Chapter 4

A biofilm model with chemotaxis and volume-filling effects

4.1 The Model

In this chapter, we consider the following model:

$$\partial_t M = d_M \nabla \cdot \left(\frac{M^{\alpha_1}}{(1-M)^{\alpha_2}} \nabla M \right) - d_c \nabla \cdot (M^{\gamma_1}(1-M)^{\gamma_2} \nabla \rho) + f(M, \rho), \tag{4.1}$$

$$\partial_t \rho = d_\rho \Delta \rho - g(M, \rho) \tag{4.2}$$

satisfied in $(0, \infty) \times \Omega$, with the boundary conditions

$$M = 0, \ \rho = 1 \text{ in } (0, \infty) \times \partial\Omega \tag{4.3}$$

and the initial conditions

$$M(0, \cdot) = M_0, \ \rho(0, \cdot) = \rho_0 \text{ in } \Omega, \tag{4.4}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ is a nonempty smooth bounded domain, and the given constants d_M, d_c, d_ρ, α_1, α_2, γ_1, γ_2 satisfy

$$d_M, d_c, d_\rho > 0, \tag{4.5}$$

$$\alpha_1 > 0, \ \alpha_2 \in (0, 1), \tag{4.6}$$

and a 'balance' condition

$$\gamma_1 \geqslant \frac{\alpha_1}{2} + 1, \gamma_2 \geqslant -\frac{\alpha_2}{2} + 1. \tag{4.7}$$

We assume that the given functions $f, g : [0,1]^2 \to \mathbb{R}$ for all $M, \rho \in [0,1]$ satisfy the conditions

$$\partial_M f(M,\rho) = F_1 + \frac{M^{\frac{\alpha_1}{2}}}{(1-M)^{\frac{\alpha_2}{2}}} \partial_M f_2(M,\rho),\ \partial_\rho f \in L^\infty\left((0,1)^2\right), \tag{4.8}$$

$$g(M,\rho) = G_0 + G_1\rho + g_2(\rho)M \tag{4.9}$$

for such constants

$$F_1, G_0, G_1 \in \mathbb{R} \tag{4.10}$$

and functions

$$f_2 \in W^{1,\infty}\left((0,1)^2\right), g_2 \in W^{1,\infty}(0,1) \tag{4.11}$$

that the conditions

$$f(0,\rho) \geqslant 0,\ f(1,\rho) \leqslant 0 \text{ for all } \rho \in [0,1], \tag{4.12}$$

$$g(M,0) \leqslant 0,\ g(M,1) \geqslant 0 \text{ for all } M \in [0,1] \tag{4.13}$$

are fulfilled. For the initial data we assume that

$$M_0 \in L^\infty(\Omega),\ \rho_0 \in W^{1,\infty}(\Omega),\ 0 \leqslant M_0, \rho_0 \leqslant 1 \text{ almost everywhere in } \Omega. \tag{4.14}$$

Remark 4.1 (On assumptions)**.**

(1) It is clear from (4.6)-(4.7) that $\gamma_1 > 1, \gamma_2 > \frac{1}{2}$ should necessarily hold.

(2) The function f has, due to the assumption (4.8), the form

$$f(M,\rho) = f(0,\rho) + F_1 M + \int_0^M \frac{M^{\frac{\alpha_1}{2}}}{(1-M)^{\frac{\alpha_2}{2}}} \partial_M f_2(M,\rho)\, dM.$$

With $\alpha_2 \in (0,1)$ and $\partial_\rho f, \partial_M f_2 \in L^\infty\left((0,1)^2\right)$ it follows that $f \in C([0,1]^2)$.

(3) Since g is linear in M, (4.13) is equivalent to the following set of conditions:

$$G_0 \leqslant 0,\ G_0 + g_2(0) \leqslant 0,\ G_0 + G_1 \geqslant 0,\ G_0 + G_1 + g_2(1) \geqslant 0.$$

Example 4.1. The primary example of functions f and g satisfying the above assumptions is: a logistic-like growth function for the biomass growth

$$f(M,\rho) = k_3 M^{a-1}(1-M)^b \frac{M\rho}{\rho + k_2} \tag{4.15}$$

and a Monod's function for the nutrient uptake

$$g(M,\rho) = k_1 \frac{M\rho}{\rho + k_2}, \tag{4.16}$$

so that the growth rate is proportional to the consumption rate g. The given constants in (4.15)-(4.16) should satisfy

$$k_1, k_2, k_3 > 0,$$
$$a \geqslant \frac{\alpha_1}{2} + 1, b \geqslant -\frac{\alpha_2}{2} + 1. \tag{4.17}$$

The system (4.1)-(4.4) can be used to model biofilm formation with chemotaxis and volume filling effects present. A volume filling effect has been included into the biofilm prototype proposed in [32] and analyzed in [9, 14], as well as into several chemotaxis models for free-swimming populations, for which we refer the reader to [25] and to a survey of corresponding mathematical results [33]. It implies the existence of an upper threshold value for the biofilm density. This maximal density value corresponds to the tight packing state. In our modeling, we assume the threshold density to be normalized to 1.

As in the system (2.1)-(2.4), the unknown quantities in (4.1)-(4.4) are the biomass density M and the concentration ρ of the chemoattractant. Thus, $M = 1$ indicates the regions where the biomass is tightly packed. Note that, as in the case of free-swimming populations, the variable M can be also seen as volume fraction of the biomass in a two-phase modeling: a chemoattractant diffuses through a two-component mixture of biomass and fluid (water, for example) surrounding it (so that the fluid has the volume fraction $1 - M$).

The model (4.1)-(4.4) arises from an attempt to bring a volume filling effect into the model (2.1)-(2.4) for a biofilm with chemotaxis. The key difference between the new model and the original one lies in the possibility contained in equation (4.1) to control separately the asymptotics of the biomass motility coefficients at the both ends of density range. While in equation (4.1) one exponent α is responsible for the limiting behavior of the diffusion coefficient both as M tends to 0 and as it tends to infinity, in equation (4.1), we have two different independent exponents, α_1 and α_2, that regulate the limiting behavior at 0 and at 1, respectively. The same holds for the chemotaxis coefficient, where we now have two different independent exponents, γ_1 and γ_2, in place of γ, as in (4.1). This offers a wider parameter range. Still, for the same reasons as for the model (2.1)-(2.4), the exponents α_1 and γ_1 (α_2 and γ_2) cannot be chosen independently of each other. In place of the 'balance' condition eqrefbc1, a new 'balance' condition (4.7) is imposed to obtain a well-posed model. Note that in this work we deal only with the case where $\alpha_2 \in (0, 1)$, which, in Aronson's classification [3], corresponds to the so called fast diffusion, whereas the biofilm prototype from [9] includes the super-diffusion singularity instead.

The shape of the reaction term f is forced by analysis, especially, by the uniqueness proof. It differs considerably from the standard growth kinetics terms used in most biofilm studies, the sink/source density (net number of particles created/lost per unit time and per unit volume) being dependent on the biomass density M. In the important case $f(0, \cdot) \equiv 0$, $F_2 = 0$, there is, due to $\alpha_1 > 0$, a delay in the biomass growth wherever M is close to 0, especially during the early stages of biofilm formation. This corresponds to a 'lag-phase' coursed by the physiological adaptation which is needed during the onset of the biofilm growth. In the regions where M is close to 1, particularly in a mature biofilm, the slowing in the production of the new cells (observe that $\alpha_2 < 1$ holds) agrees well with the volume filling effect.

In this work, we consider weak solutions of the system (4.1)-(4.4). The definition is as follows:

Definition 4.1 (Weak solution)**.** *A pair of functions* $(M,\rho) : [0,\infty)\times\overline{\Omega} \to [0,1]^2$ *is said to be a weak solution of (4.1)-(4.4) for* $M_0 \in L^\infty(\Omega)$, $\rho_0 \in W^{1,\infty}(\Omega)$, $0 \leqslant M_0, \rho_0 \leqslant 1$ *almost everywhere in* Ω, *if for all* $T > 0$

(i) $\int_0^M \frac{M^{\alpha_1}}{(1-M)^{\alpha_2}}\, dM \in L^2\left((0,T); H_0^1(\Omega)\right)$, $\partial_t M \in L^2\left((0,T); H^{-1}(\Omega)\right)$;

(ii) $\rho - 1 \in C((0,T); W_0^{1,\infty}(\Omega))$;

(iii) (M,ρ) *satisfies the equation (4.1) in* $L^2\left((0,T); H^{-1}(\Omega)\right)$, $M(0) = M_0$ *in* $C((0,T); (L^2(\Omega), \sigma(L^2(\Omega), (L^2(\Omega))')))$*-sense and*

$$\rho(t) - 1 = e^{td_\rho\Delta}(\rho_0 - 1) - \int_0^t e^{(t-s)d_\rho\Delta} g(M(s),\rho(s))\, ds$$

in $W_0^{1,\infty}(\Omega)$.

Remark 4.2 (Initial condition)**.** From $M \in L^\infty((0,T); L^2(\Omega))$ and $\partial_t M \in L^2\left((0,T); H^{-1}(\Omega)\right)$, it follows with (1.5) for $p_0 = 2$, $E = L^2(\Omega)$ and $E_1 = H^{-1}(\Omega)$ and the compact embedding (see *Theorem 1.10*) $L^2(\Omega) \overset{d}{\hookrightarrow\hookrightarrow} H^{-1}(\Omega)$ that $M \in C((0,T); (L^2(\Omega), \sigma(L^2(\Omega), (L^2(\Omega))')))$. Therefore, the initial condition for M makes sense.

In the present work, we study the well-posedness and the long-time behavior of the system (4.1)-(4.4). We prove the following result on well-posedness:

Theorem 4.1 (Well-posedness)**.** *Let the given constants* $d_M, d_c, d_\rho, \alpha_1, \alpha_2, \gamma_1, \gamma_2$ *satisfy the assumptions (4.5)-(4.7) and the functions* f *and* g *satisfy the conditions (4.8)-(4.13). Then the initial boundary-value problem (4.1)-(4.4) is uniquely solvable (in the sense of Definition 4.1) for each pair of starting values* (M_0, ρ_0) *that satisfies the condition (4.14).*

The proof of *Theorem 4.1* is divided between *Sections 4.2* (global existence) and *4.3* (uniqueness).

In *Section 4.4*, we address the long-time behavior for our system and establish the existence of the global attractor. This chapter concludes with an illustration in *Section 4.5* of a possible model behavior in numerical simulations.

Remark 4.3 (Notation)**.**

(1) To shorten the notation, we introduce the functions

$$D(M) := \frac{M^{\alpha_1}}{(1-M)^{\alpha_2}},$$
$$\mathcal{E}(M) := \int_0^M D(M)\, dM.$$

(2) For the sake of convenience, we assume throughout this chapter that the constants B_i (appear below) for all indices i are only dependent upon the parameters of the problem, that is, upon the constants d_M, d_c, d_ρ, α_1, α_2, γ_1, γ_2, the functions f and g and the domain Ω, and **not** upon the initial data M_0, ρ_0 or the time variable t, or, unless stated otherwise, any other parameters.

4.2 Existence of solutions

Proof of *Theorem 4.1* **(Existence).** The main idea of the existence proof is to choose a suitable regularization sequence for the problem (4.1)-(4.4) and then apply the compactness method (see [21]).

Let us consider for arbitrary $T > 0$, $n \in \mathbb{N}$ a non-degenerate approximation of the problem (4.1)-(4.4), the system

$$\begin{aligned} \partial_t M_n =& d_M \Delta \left(\int_0^{M_n} \frac{\left(M + \frac{1}{n}\right)^{\alpha_1}}{(1-M)^{\alpha_2}} \, dM \right) - d_c \nabla \cdot \left(M_n^{\gamma_1} (1 - M_n)^{\gamma_2} \nabla \rho_n \right) \\ &+ f(M_n, \rho_n), \end{aligned} \tag{4.18}$$

$$\partial_t \rho_n = d_\rho \Delta \rho_n - g(M_n, \rho_n) \tag{4.19}$$

satisfied in $(0,T) \times \Omega$, with the same initial and boundary conditions as before:

$$M_n = 0, \qquad \rho_n = 1 \qquad \text{in } (0,T) \times \partial\Omega, \tag{4.20}$$

$$M_n(\cdot, 0) = M_0, \quad \rho_n(\cdot, 0) = \rho_0 \text{ in } \Omega. \tag{4.21}$$

To shorten the notation, we introduce for $\varepsilon \in [0,1]$ the functions

$$\begin{aligned} D_\varepsilon(M) &:= \frac{(M+\varepsilon)^{\alpha_1}}{(1-M)^{\alpha_2}}, \\ \mathcal{E}_\varepsilon(M) &:= \int_0^M D(M) \, dM, \end{aligned}$$

so that $D_0 = D$, $\mathcal{E}_0 = \mathcal{E}$. For any fixed $\varepsilon \in [0,1]$ the function D_ε is clearly monotonically increasing and

$$D_\varepsilon(M) \geqslant M^{\alpha_1} \text{ for all } M \in [0,1). \tag{4.22}$$

The function $\mathcal{E}_\varepsilon$ is due to assumptions on α_1 and α_2 continuous and bounded, and it holds that

$$\begin{aligned} \mathcal{E}_\varepsilon(M) \leqslant& (1+\varepsilon)^{\alpha_1} \int_0^M \frac{dM}{(1-M)^{\alpha_2}} \\ \leqslant& \frac{(1+\varepsilon)^{\alpha_1}}{1-\alpha_2} \left(1 - (1-M)^{1-\alpha_2}\right) \\ \leqslant& \frac{2^{\alpha_1}}{1-\alpha_2} =: B_1. \end{aligned} \tag{4.23}$$

We are now prepared to derive several a priori estimates that we then use to show the global existence of solutions of the problem (4.1)-(4.4). Observe first that for every $n \in \mathbb{N}$ the approximating equation (4.18) is clearly quasilinear and non-degenerate and that the equation (4.19) is even semi-linear. Therefore, we can conclude from the assumptions (4.7) on the constants, the assumptions (4.8)-(4.13) on the functions f and g, the assumptions (4.14) on the initial data and the boundary conditions (4.20) that

$$0 \leqslant M_n, \rho_n \leqslant 1 \text{ almost everywhere in } [0,T] \times \overline{\Omega} \text{ for all } n \in \mathbb{N}. \tag{4.24}$$

The proof can be done using the standard techniques, see, for example, [11].

As a consequence of the a priory uniform boundedness, the general theory from [20] may be extended to the non-degenerate problem (4.18)-(4.21) (for an alternative treatment via maximal regularity see [2]). It follows that this initial-boundary value problem possesses a unique classical solution.

Further, using the properties (1.9)-(1.10) we then get

$$\begin{aligned} ||\nabla \rho_n(t)||_\infty = & \left\| e^{t d_\rho \Delta} \nabla \rho_0 - \int_0^t \nabla e^{(t-s) d_\rho \Delta} (g(M_n(s), \rho_n(s)))\, ds \right\| \\ \leqslant & e^{-\beta d_\rho t} ||\nabla \rho_0||_\infty + A\left(\frac{3}{4}, 7\right) \int_0^t e^{-\beta d_\rho (t-s)} (t-s)^{-\frac{3}{4}} ||g||_\infty |\Omega|\, ds \\ \leqslant & e^{-\beta d_\rho t} ||\nabla \rho_0||_\infty + R_\infty \\ =: & B_2(||\nabla \rho_0||_\infty), \end{aligned} \tag{4.25}$$

where the constant R_∞ depends only on the parameters of the problem.

Next, we multiply the equation (4.18) by $\mathcal{E}_{\frac{1}{n}}(M_n)$ and integrate over $(0,T) \times \Omega$:

$$\begin{aligned} & \int_0^T \Big(\partial_t M_n(s), \mathcal{E}_{\frac{1}{n}}(M_n(s)) \Big)\, ds \\ = & d_M \int_0^T \Big(\nabla \cdot \Big(\nabla \mathcal{E}_{\frac{1}{n}}(M_n(s)) \Big), \mathcal{E}_{\frac{1}{n}}(M_n(s)) \Big)\, ds \\ & - d_c \int_0^T \Big(\nabla \cdot \left(M_n^{\gamma_1}(s) \left(1 - M_n(s)\right)^{\gamma_2} \nabla \rho_n(s) \right), \mathcal{E}_{\frac{1}{n}}(M_n(s)) \Big)\, ds \\ & + \int_0^T \Big(f\left(M_n(s), \rho_n(s)\right), \mathcal{E}_{\frac{1}{n}}(M_n(s)) \Big)\, ds. \end{aligned} \tag{4.26}$$

Integrating (4.26) by parts, we obtain that

$$\begin{aligned}
&\left(\int_{M_n(0)}^{M_n(T)} \mathcal{E}_{\frac{1}{n}}(M)\, dM, 1\right)\\
&= - d_M \int_0^T \left\|\nabla \mathcal{E}_{\frac{1}{n}}(M_n(s))\right\|^2 ds\\
&\quad + d_c \int_0^T \left(\nabla \mathcal{E}_{\frac{1}{n}}(M_n(s)), M_n^{\gamma_1}(s)\,(1 - M_n(s))^{\gamma_2}\, \nabla \rho_n(s)\right) ds\\
&\quad + \int_0^T \left(f\left(M_n(s), \rho_n(s)\right), \mathcal{E}_{\frac{1}{n}}(M_n(s))\right) ds
\end{aligned}$$

The Cauchy-Schwarz inequality, together with (4.23), (4.24) and (4.25), yields

$$\begin{aligned}
&\frac{d_M}{2} \int_0^T \left\|\nabla \mathcal{E}_{\frac{1}{n}}(M_n(s))\right\|^2 ds + \left(\int_0^{M_n(T)} \mathcal{E}_{\frac{1}{n}}(M)\, dM, 1\right)\\
&\leqslant \left(\int_0^{M_n(0)} \mathcal{E}_{\frac{1}{n}}(M)\, dM, 1\right) + \frac{d_c^2}{2 d_M} \int_0^T \left\|M_n^{\gamma_1}(s)\,(1 - M_n(s))^{\gamma_2}\, \nabla \rho_n\right\|^2 ds.\\
&\quad + \int_0^T \left(f\left(M_n, \rho_n\right), \mathcal{E}_{\frac{1}{n}}(M_n(s))\right) ds\\
&\leqslant \left(B_1 + \left(\frac{d_c^2}{2 d_M} + B_1 ||f||_\infty\right) T B_2(||\nabla \rho_0||_\infty)\right) |\Omega|\\
&=: B_3(||\nabla \rho_0||_\infty).
\end{aligned} \tag{4.27}$$

By using (4.27) as well, we can now estimate the right side of the equation (4.18) in $L^2((0,T), H^{-1}(\Omega))$ uniformly in $n \in \mathbb{N}$. We obtain

$$\int_0^T \|\partial_t M_n\|_{H^{-1}(\Omega)}^2\, ds \leqslant B_4(||\nabla \rho_0||_\infty), \tag{4.28}$$

where the constant $B_4(||\nabla \rho_0||_\infty)$ depends only on $||\nabla \rho_0||_\infty$ and the parameters of the problem. Further, (4.27) and (4.22) yield

$$\begin{aligned}
\int_0^T \left\|M_n^{\alpha_1+1}(s)\right\|_{H_0^1(\Omega)}^2 ds &\leqslant (\alpha_1 + 1)^2 \int_0^T \left\|\mathcal{D}_{\frac{1}{n}}(M_n(s)) \nabla M_n\right\|^2 ds\\
&= (\alpha_1 + 1)^2 \int_0^T \left\|\nabla \mathcal{E}_{\frac{1}{n}}(M_n(s))\right\|^2 ds\\
&\leqslant (\alpha_1 + 1)^2 B_4(||\nabla \rho_0||_\infty).
\end{aligned} \tag{4.29}$$

Moreover, applying *Lemma 1.1* for $q = \alpha_1 + 1$ and (for example) $s = \frac{1}{2}$ together with the Sobolev embedding theorem yields

$$\begin{aligned}
||M_n||_{W^{\frac{1}{2(\alpha_1+1)}, 2(\alpha_1+1)}(\Omega)}^{2(\alpha_1+1)} &\leqslant N^{2(\alpha_1+1)}(\alpha_1 + 1) \left\|M_n^{\alpha_1+1}\right\|_{W^{\frac{1}{2},2}(\Omega)}^2\\
&\leqslant B_5 \left\|M_n^{\alpha_1+1}\right\|_{H_0^1(\Omega)}^2.
\end{aligned} \tag{4.30}$$

Integrating (4.30) over $(0,T)$ and combining with (4.27)-(4.29) we conclude that

$$||M_n||_{W^{1,(2(\alpha_1+1),2)}\left((0,T);W^{\frac{1}{2(\alpha_1+1)},2(\alpha_1+1)}(\Omega),H^{-1}(\Omega)\right)} \leqslant B_6(||\nabla\rho_0||_\infty),$$

where the constant $B_6(||\nabla\rho_0||_\infty)$ depends only on $||\nabla\rho_0||_\infty$ and the parameters of the problem. The spaces $E_1 := W^{\frac{1}{2(\alpha_1+1)},2(\alpha_1+1)}(\Omega)$, $E := L^{2(\alpha_1+1)}(\Omega)$, $E_0 := H^{-1}(\Omega)$ satisfy the assumptions of *Theorem 1.12*, consequently, it holds (set $p_1 := 2(\alpha_1+1)$, $p_0 := 2$) that

$$W^{1,(2(\alpha_1+1),2)}\left((0,T);W^{\frac{1}{2(\alpha_1+1)},2(\alpha_1+1)}(\Omega),H^{-1}(\Omega)\right) \hookrightarrow\hookrightarrow L^{2(\alpha_1+1)}((0,T)\times\Omega),$$

and the set $\{M_n|\ n\in\mathbb{N}\}$ is thus compact in the space $L^{2(\alpha_1+1)}((0,T)\times\Omega)$.

For the second component, we now use the (4.25) to estimate the right side of the equation (4.19) in $L^\infty((0,T),W^{-1,\infty}(\Omega))$ and get

$$||\rho_n||_{W^{1,(\infty,\infty)}\left((0,T);W_0^{1,\infty}(\Omega),W^{-1,\infty}(\Omega)\right)} \leqslant B_7(||\nabla\rho_0||_\infty).$$

The spaces $E_1 := W_0^{1,\infty}(\Omega)$, $E := L^\infty(\Omega)$, $E_0 := W^{-1,\infty}(\Omega)$ satisfy the assumptions of *Theorem 1.12*, consequently, it holds (set $p_1 := p_0 := \infty$) that

$$W^{1,(\infty,\infty)}\left((0,T);W_0^{1,\infty}(\Omega),W^{-1,\infty}(\Omega)\right) \hookrightarrow\hookrightarrow C([0,T],L^\infty(\Omega)).$$

The set $\{\rho_n|\ n\in\mathbb{N}\}$ is thus compact in the space $C([0,T],L^\infty(\Omega))$, hence also in the larger space $L^\infty((0,T)\times\Omega)$.

By combining these results, we obtain there is a subsequence (n_m) such that

$$\begin{aligned}
&M_{n_m} \underset{m\to\infty}{\rightarrow} M \text{ in } L^{2(\alpha_1+1)}((0,T)\times\Omega),\\
&\rho_{n_m} \underset{m\to\infty}{\rightarrow} \rho \text{ in } L^\infty((0,T)\times\Omega),\\
&\nabla\rho_{n_m} \underset{m\to\infty}{\rightarrow} \nabla\rho \text{ in } L^\infty_{w-*}((0,T)\times\Omega).
\end{aligned} \tag{4.31}$$

for some $(M,\rho)\in L^{2(\alpha_1+1)}((0,T)\times\Omega)\times L^\infty((0,T)\times\Omega)$, and for a subsequence (not relabeled) the convergence is almost everywhere in the cylinder $(0,T)\times\Omega$.

It remains to check that (M,ρ) is indeed a solution of the original problem (4.1)-(4.4) in the sense of distributions. Recall first that $f,g\in W^{1,\infty}((0,1)^2)$, so that, with the second part of the Sobolev embedding theorem, we have that $f,g\in C([0,1]^2)$.

With the continuity argument and the dominant convergence theorem, we obtain that

$$\begin{aligned}
&f\left(M_{n_m},\rho_{n_m}\right) \underset{m\to\infty}{\rightarrow} f(M,\rho) \text{ in } L^2((0,T)\times\Omega),\\
&g(M_{n_m},\rho_{n_m}) \underset{m\to\infty}{\rightarrow} g(M,\rho) \text{ in } L^2((0,T)\times\Omega),\\
&\mathcal{E}_{\frac{1}{n_m}}(M_{n_m}) \underset{m\to\infty}{\rightarrow} \mathcal{E}(M) \text{ in } L^2((0,T)\times\Omega),\\
&M_n^{\gamma_1}\left(1-M_n\right)^{\gamma_2} \underset{m\to\infty}{\rightarrow} M^{\gamma_1}\left(1-M\right)^{\gamma_2} \text{ almost everywhere in } (\tau,T)\times\Omega.
\end{aligned} \tag{4.32}$$

Moreover, combining (4.32) with (4.31), we obtain with the dominant convergence theorem that

$$M_n^{\gamma_1}(1-M_n)^{\gamma_2}\nabla\rho_{n_m} \underset{m\to\infty}{\to} M^{\gamma_1}(1-M)^{\gamma_2}\nabla\rho \text{ in } L^\infty_{w-*}((0,T)\times\Omega).$$

Since the convergence in the distributional sense is weaker than the L^p convergence for any $p\in[1,\infty]$ or than the L^∞_{w-*} convergence and since differential operators are continuous in the space of distributions, it follows with the convergences we derived in this subsection that (M,ρ) solves the problem (4.1)-(4.4) in the sense of distributions. The existence part of *Theorem 4.1* is thus proven.

□

Remark 4.4. It follows from the proof that the solution (M,ρ) enjoys the estimates

$$\begin{aligned}&0\leqslant M,\rho\leqslant 1 \text{ almost everywhere in } [0,T]\times\overline{\Omega},\\ &\mathcal{E}(M,\rho)\leqslant B_1 \text{ almost everywhere in } [0,T]\times\overline{\Omega},\\ &\int_0^T \|\nabla\mathcal{E}(M(s))\|^2\,ds\leqslant B_8(\|\nabla\rho_0\|_\infty)\end{aligned}\tag{4.33}$$

and the dissipative estimate

$$\|\nabla\rho(t)\|_\infty \leqslant e^{-\beta d_\rho t}\|\nabla\rho_0\|_\infty + R_\infty.\tag{4.34}$$

4.3 Uniqueness of solutions

Proof of *Theorem 4.1* **(Uniqueness)**. Let us assume that the problem (4.1)-(4.4) has two different solutions (in the sense of *Definition 4.1*) (M_1,ρ_1), (M_2,ρ_2) with the same initial data:

$$M_1(0)=M_2(0),\quad \rho_1(0)=\rho_2(0).$$

Since both (M_1,ρ_1) and (M_2,ρ_2) are solutions of the equation (4.1), we get

$$\begin{aligned}\partial_t(M_1-M_2) =& d_M\Delta\int_{M_1}^{M_2}\frac{M^{\alpha_1}}{(1-M)^{\alpha_2}}\,dM\\ &- d_c\nabla\cdot\left(M_1^{\gamma_1}(1-M_1)^{\gamma_2}\nabla\rho_1 - M_2^{\gamma_1}(1-M_2)^{\gamma_2}\nabla\rho_2\right)\\ &+ \left(f(M_1,\rho_1)-f(M_2,\rho_2)\right).\end{aligned}\tag{4.35}$$

We want to estimate the difference M_1-M_2, and we choose to do so in the $\|\cdot\|_{H^{-1}(\Omega)}$ norm on an interval $[0,t]$ for arbitrary $t>0$. For this purpose, we multiply (4.35) by $(-\Delta)^{-1}(M_1 - M_2)$ and integrate over Ω:

$$\begin{aligned}
&\big(\partial_t(M_1 - M_2), (-\Delta)^{-1}(M_1 - M_2)\big)\\
=&d_M \left(\Delta \int_{M_1}^{M_2} \frac{M^{\alpha_1}}{(1-M)^{\alpha_2}}\, dM, (-\Delta)^{-1}(M_1 - M_2)\right)\\
&+ d_c \left(-\nabla \cdot \left(M_1^{\gamma_1} (1-M_1)^{\gamma_2} \nabla\rho_1 - M_2^{\gamma_1} (1-M_2)^{\gamma_2} \nabla\rho_2\right), (-\Delta)^{-1}(M_1 - M_2)\right)\\
&+ \big(f(M_1, \rho_1) - f(M_2, \rho_2), (-\Delta)^{-1}(M_1 - M_2)\big)\,. \qquad (4.36)
\end{aligned}$$

On the left side of the resulting equation, there appears:

$$\big(\partial_t(M_1 - M_2), (-\Delta)^{-1}(M_1 - M_2)\big) = \frac{1}{2}\frac{d}{dt}\left\|\nabla^{+*}(M_1 - M_2)\right\|^2. \qquad (4.37)$$

Suitable estimates for the terms on the right side of (4.36) are required now. The operator Δ is self-adjoint, therefore, using the Cauchy-Schwarz inequality, we obtain for the first summand that

$$\begin{aligned}
&\left(\Delta \int_{M_1}^{M_2} \frac{M^{\alpha_1}}{(1-M)^{\alpha_2}}\, dM, (-\Delta)^{-1}(M_1 - M_2)\right)\\
&= -\left(\int_{M_1}^{M_2} \frac{M^{\alpha_1}}{(1-M)^{\alpha_2}}\, dM, \int_{M_1}^{M_2} 1\, dM\right)\\
&\leqslant -\left\|\int_{M_1}^{M_2} \frac{M^{\frac{\alpha_1}{2}}}{(1-M)^{\frac{\alpha_2}{2}}}\, dM\right\|^2. \qquad (4.38)
\end{aligned}$$

The assumptions (4.7) and $M_1 \in [0,1]$ together with the property (1.6) and the Cauchy-Schwarz inequality lead for the second summand to

$$\begin{aligned}
&\left|\left(-\nabla \cdot \left(M_1^{\gamma_1} (1-M_1)^{\gamma_2} \nabla\rho_1 - M_2^{\gamma_1} (1-M_2)^{\gamma_2} \nabla\rho_2\right), (-\Delta)^{-1}(M_1 - M_2)\right)\right|\\
=&\left|\left(M_1^{\gamma_1} (1-M_1)^{\gamma_2} \nabla\rho_1 - M_2^{\gamma_1} (1-M_2)^{\gamma_2} \nabla\rho_2, \nabla^{+*}(M_1 - M_2)\right)\right|\\
\leqslant&\left|\left(\int_{M_1}^{M_2} \frac{d}{dM}\left(M^{\gamma_1} (1-M)^{\gamma_2}\right) dM \nabla\rho_2, \nabla^{+*}(M_1 - M_2)\right)\right|\\
&+ \left|\left(\left(M_1^{\gamma_1} (1-M_1)^{\gamma_2} \nabla(\rho_1 - \rho_2)\right), \nabla^{+*}(M_1 - M_2)\right)\right|\\
\leqslant&\left(\max\{\gamma_1, \gamma_2\} \|\nabla\rho_2\|_\infty \left\|\int_{M_1}^{M_2} \frac{M^{\frac{\alpha_1}{2}}}{(1-M)^{\frac{\alpha_2}{2}}}\, dM\right\| + \|\rho_1 - \rho_2\|_{H_0^1(\Omega)}\right)\cdot\\
&\cdot\left\|\nabla^{+*}(M_1 - M_2)\right\|. \qquad (4.39)
\end{aligned}$$

Further, we use assumptions on f and the Cauchy-Schwarz inequality and get

$$\begin{aligned} f(M_1,\rho_1)-f(M_2,\rho_2) =&(f(M_1,\rho_2)-f(M_2,\rho_2))+(f(M_1,\rho_1)-f(M_1,\rho_2))\\ =&\int_{M_1}^{M_2}(\partial_M f(M,\rho_2)-F_1)\ dM+F_1(M_1-M_2)\\ &+\int_{\rho_1}^{\rho_2}\partial_\rho f(M_1,\rho)\,d\rho\\ =&\int_{M_1}^{M_2}\frac{M^{\frac{\alpha_1}{2}}}{(1-M)^{\frac{\alpha_2}{2}}}\partial_M f_2(M,\rho)\,dM+F_1(M_1-M_2)\\ &+\int_{\rho_1}^{\rho_2}\partial_\rho f(M_1,\rho)\,d\rho, \end{aligned}$$

so that with the Poincaré, the Cauchy-Schwarz and the Young inequalities it follows that

$$\begin{aligned} &\left|\left(f(M_1,\rho_1)-f(M_2,\rho_2),(-\Delta)^{-1}(M_1-M_2)\right)\right|\\ \leqslant&\left|\left(\int_{M_1}^{M_2}\frac{M^{\frac{\alpha_1}{2}}}{(1-M)^{\frac{\alpha_2}{2}}}\partial_M f_2(M,\rho)\,dM,(-\Delta)^{-1}(M_1-M_2)\right)\right|+F_1\left\|\nabla^{+*}(M_1-M_2)\right\|^2\\ &+\left|\left(\int_{\rho_1}^{\rho_2}\partial_\rho f(M_1,\rho)\,d\rho,(-\Delta)^{-1}(M_1-M_2)\right)\right|\\ \leqslant&\left(||\partial_M f_2||_\infty\left\|\int_{M_1}^{M_2}\frac{M^{\frac{\alpha_1}{2}}}{(1-M)^{\frac{\alpha_2}{2}}}\,dM\right\|+||\partial_\rho f||_\infty||\rho_1-\rho_2||\right)\left\|(-\Delta)^{-1}(M_1-M_2)\right\|\\ &+F_1\left\|\nabla^{+*}(M_1-M_2)\right\|^2\\ \leqslant&\left(P(\Omega,2)||\partial_M f_2||_\infty\left\|\int_{M_1}^{M_2}\frac{M^{\frac{\alpha_1}{2}}}{(1-M)^{\frac{\alpha_2}{2}}}\,dM\right\|+P(\Omega,2)||\partial_\rho f||_\infty||\rho_1-\rho_2||_{H^1(\Omega)}+F_1\right)\cdot\\ &\cdot\left\|\nabla^{+*}(M_1-M_2)\right\|^2. \end{aligned} \tag{4.40}$$

Observe that it holds

$$\left\|\nabla^{+*}(M_1-M_2)\right\|=\|M_1-M_2\|_{H^{-1}(\Omega)} \tag{4.41}$$

due to the property (1.8). By combining (4.37)-(4.40), using (4.41), the Poincaré and the Cauchy-Schwarz inequalities, we can conclude from (4.36) that

$$\frac{1}{2}\frac{d}{dt}\|M_1-M_2\|^2_{H^{-1}(\Omega)}\leqslant\frac{d_\rho}{2}||\rho_1-\rho_2||^2_{H_0^1(\Omega)}+B_1(t,R)\|M_1-M_2\|^2_{H^{-1}(\Omega)}. \tag{4.42}$$

The constant $B_1(t,R)$ depends only on t,R and the parameters of the problem.

Now we turn to equation (4.2). Both (M_1,ρ_1) and (M_2,ρ_2) solve it, hence

$$\begin{aligned} \partial_t(\rho_1-\rho_2) =&d_\rho\Delta(\rho_1-\rho_2)-(g(M_1,\rho_1)-g(M_2,\rho_2))\\ =&d_\rho\Delta(\rho_1-\rho_2)-G_1(\rho_1-\rho_2)-(g_2(\rho_1)-g_2(\rho_2))M_1\\ &-g_2(\rho_2)(M_1-M_2). \end{aligned} \tag{4.43}$$

As usual, we multiply (4.43) by $\rho_1 - \rho_2$ and integrate over Ω

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\rho_1-\rho_2\|^2 =& d_\rho\left(\Delta(\rho_1-\rho_2),\rho_1-\rho_2\right) - G_1\|\rho_1-\rho_2\|^2\\ &-\left(M_1(g_2(\rho_1)-g_2(\rho_2)),\rho_1-\rho_2\right)\\ &-\left(M_1-M_2,g_2(\rho_2)(\rho_1-\rho_2)\right)\\ =&-d_\rho\|\nabla(\rho_1-\rho_2)\|^2 - G_1\|\rho_1-\rho_2\|^2\\ &-\left(M_1(g_2(\rho_1)-g_2(\rho_2)),\rho_1-\rho_2\right)\\ &-\left(\nabla^{+*}(M_1-M_2),g_2(\rho_2)\nabla(\rho_1-\rho_2)\right)\\ &-\left(\nabla^{+*}(M_1-M_2),(\rho_1-\rho_2)\frac{d}{d\rho}g_2(\rho_2)\nabla\rho_2\right)\\ \leqslant&-\frac{d_\rho}{2}\|\rho_1-\rho_2\|^2_{H^1_0(\Omega)} + B_2(t,R)||\rho_1-\rho_2||^2\\ &+B_2(t,R)\|M_1-M_2\|^2_{H^{-1}(\Omega)},\end{aligned} \tag{4.44}$$

while we again made use of the property (4.41), the Cauchy-Schwarz and the Young inequalities, and the constant $B_2(t,R)$ depends only on t,R and the parameters of the problem.

Finally, by adding (4.42) and (4.44) together, we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|M_1-M_2\|^2_{H^{-1}(\Omega)} + ||\rho_1-\rho_2||^2\right)\\ \leqslant& B_3(t,R)\left(\|M_1-M_2\|^2_{H^{-1}(\Omega)} + ||\rho_1-\rho_2||^2\right).\end{aligned} \tag{4.45}$$

Integrating (4.45), we conclude that

$$\begin{aligned}&\|M_1(t)-M_2(t)\|^2_{H^{-1}(\Omega)} + ||\rho_1(t)-\rho_2(t)||^2\\ \leqslant& B_4(t,R)\left(\|M_1(0)-M_2(0)\|^2_{H^{-1}(\Omega)} + ||\rho_1(0)-\rho_2(0)||^2\right)\end{aligned} \tag{4.46}$$

for some constant $B_4(t,R) > 0$ depending only on the parameters of the problem and on R and t. This proves uniqueness for the problem (4.1)-(4.4) since the solutions (M_1,ρ_1), (M_2,ρ_2) coincide at $t=0$. The uniqueness part of *Theorem 4.1* is thus proven.

□

4.4 Global attractor

The aim of this section is to apply the general theory from *Section 1.3* to the problem (4.1)-(4.4). We prove

Theorem 4.2. *Let the given constants $d_M, d_c, d_\rho, \alpha_1, \alpha_2, \gamma_1, \gamma_2$ satisfy the assumptions (4.5)-(4.7) and let the functions f and g satisfy the conditions (4.8)-(4.13). Then for all $p \in [1, \infty]$ the solutions of the problem (4.1)-(4.4) can be described by a semigroup $\{S(t)\}_{t \geqslant 0}$ that acts on the set*

$$B_1 := \left\{(M, \rho) \in L^\infty(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right) \mid 0 \leqslant M, \rho \leqslant 1 \text{ almost everywhere in } \Omega\right\}$$

equipped with the metric defined by

$$m_*^{(p)}((M_1, \rho_1), (M_2, \rho_2)) := \begin{cases} \max\left\{d_*^{(\infty)}(M_1, M_2), ||\rho_1 - \rho_2||_{W_0^{1,\infty}(\Omega)}\right\} & \text{for } p = \infty, \\ \left(||M_1 - M_2||_p^p + ||\rho_1 - \rho_2||^p_{W_0^{1,\infty}(\Omega)}\right)^{\frac{1}{p}} & \text{for } p \geqslant 1. \end{cases}$$

Here $d_*^{(\infty)}$ is the metric defined in (1.2) for $R := 1$.

The semigroup $\{S(t)\}_{t \geqslant 0}$ possesses the global attractor in $\left(B_1, m_*^{(p)}\right)$ that is independent of the concrete choice of p.

Remark 4.5 (Rate of convergence to the attractor)**.** The rate of convergence to the global attractor $\mathcal{A}$ may, of course, depend on p and can be arbitrarily slow.

Proof of *Theorem 4.2***.** We showed in *Theorem 4.1* that the problem (4.1)-(4.4), if considered in B_1, is well-posed: for each pair of initial values $(M_0, \rho_0) \in B_1$ there exists a unique solution (M, ρ) in terms of *Definition 4.1*.
We define the solving semigroup $\{S(t)\}_{t \geqslant 0}$ of the problem (4.1)-(4.4) on the phase space B_1 as follows: for all $t \geqslant 0$ let

$$S(t) : B_1 \to B_1,$$
$$S(t)(M_0, \rho_0) := (M(t), \rho(t)) \text{ for all } (M_0, \rho_0) \in B_1.$$

Let us now prove the existence of the global attractor for the semigroup $\{S(t)\}_{t \geqslant 0}$.

Observe first that the projection of the semigroup domain B_1 on the M component is the unit ball in $L^\infty(\Omega)$. Due to *Theorem 1.7(4)*, it is sufficient, therefore, to show the existence of a compact invariant set that attracts B_1 in the metric $m_*^{(2)}$. This set is then, necessarily, the global attractor in the metric $m_*^{(p)}$ for all $p \in [1, \infty]$.

As a closed subset of a complete metric space $\left(L^2(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right), m_*^{(2)}\right)$, the space $\left(B_1, m_*^{(2)}\right)$ is complete.

Let us assume for a moment that $\{S(t)\}_{t \geqslant 0}$ is a closed semigroup and that it possesses a compact absorbing set in $\left(B_1, m_*^{(2)}\right)$. All assumptions of *Theorem 1.13* are then fulfilled since we are dealing with a closed semigroup (the semigroup $\{S(t)\}_{t \geqslant 0}$) in a complete metric space (the space $\left(B_1, m_*^{(2)}\right)$), and this semigroup possesses a compact absorbing set. *Theorem 1.13* yields the existence of the global attractor, and, as we showed above, it is also the global attractor for the semigroup $\{S(t)\}_{t\geqslant 0}$ in each of the spaces $\left(B_1, m_*^{(p)}\right)$ for $p \in [1,\infty]$.

In order to finish the proof, it remains to check the closedness of $\{S(t)\}_{t\geqslant 0}$ and the existence of a compact absorbing set.

Let us first prove the closedness of the semigroup operators. In the proof of *Theorem 4.1* we encountered the local Lipschitz-type continuity property (4.46) for the solutions of (4.1)-(4.4). It can be translated into the following form:

$$\begin{aligned} &\left\| S(t)\left(M_0^{(1)},\rho_0^{(1)}\right) - S(t)\left(M_0^{(2)},\rho_0^{(2)}\right)\right\|_{H^{-1}(\Omega)\times L^2(\Omega)} \\ \leqslant &L(t,R)\left\|\left(M_0^{(1)},\rho_0^{(1)}\right) - \left(M_0^{(2)},\rho_0^{(2)}\right)\right\|_{H^{-1}(\Omega)\times L^2(\Omega)} \end{aligned} \tag{4.47}$$

for $\left(M_0^{(1)},\rho_0^{(1)}\right), \left(M_0^{(2)},\rho_0^{(2)}\right) \in B_1,\ R := \max\{||\nabla\rho_1||_\infty, ||\nabla\rho_2||_\infty\}$. The constant $L(t,R)$ depends only on t, R and the parameters of the problem.

Recall that due to the embedding theorems for Sobolev spaces, we have

$$L^2(\Omega)\times W^{1,\infty}(\Omega) \hookrightarrow H^{-1}(\Omega)\times L^2(\Omega). \tag{4.48}$$

Let $\left\{\left(M_0^{(n)},\rho_0^{(n)}\right)\right\}_{n\in\mathbb{N}} \subset B_1$ be a sequence of initial data convergent in $L^2(\Omega)\times \left(1+W_0^{1,\infty}(\Omega)\right)$ to some (M_0,ρ_0). Due to the continuous embedding (4.48), this sequence converges in $H^{-1}(\Omega)\times L^2(\Omega)$ to the same limit. From the property (4.47) we deduce that the sequence $\left\{S(t)\left(M_0^{(n)},\rho_0^{(n)}\right)\right\}_{n\in\mathbb{N}}$ converges to $S(t)\left(M_0,\rho_0\right)$ in $H^{-1}(\Omega)\times L^2(\Omega)$ for all $t\geqslant 0$.
Let us further assume that for some $t\geqslant 0$ the sequence $\left\{S(t)\left(M_0^{(n)},\rho_0^{(n)}\right)\right\}_{n\in\mathbb{N}}$ is convergent in $L^2(\Omega)\times\left(1+W_0^{1,\infty}(\Omega)\right)$. Again, due to the continuity of the embedding (4.48), the limit is the same. This proves closedness.

Next, the dissipative estimate (4.34) provides the existence of a ball B_* centered at $(0,1)$ of a radius $R_* := (1+4R_\infty^2)^{\frac{1}{2}}$ in the $m_*^{(2)}$ metric, which absorbs all bounded sets of B_1. If we prove that $S(1)B_*$ is a relatively compact set, then $cl_{\left(B_1,m_*^{(2)}\right)}(S(1)B_*)$ is a compact absorbing set for the semigroup.

To prove this, we multiply the equation (4.1) by $(\alpha_1+1)\partial_t M^{\alpha_1+1}$ and integrate (formally) over Ω:

$$\begin{aligned} (\alpha_1+1)\left(\partial_t M, \partial_t M^{\alpha_1+1}\right) = &\left(\nabla\cdot\left(\frac{1}{(1-M)^{\alpha_2}}\nabla M^{\alpha_1+1}\right), \partial_t M^{\alpha_1+1}\right) \\ &+ \frac{(\alpha_1+1)^2}{\frac{\alpha_1}{2}+1}\left(M^{\frac{\alpha_1}{2}}\hat{f}(M,\rho), \partial_t M^{\frac{\alpha_1}{2}+1}\right). \end{aligned}$$

Here:

$$\hat{f}(M,\rho) = -\,d_c\nabla\cdot\left(M^{\gamma_1}(1-M)^{\gamma_2}\nabla\rho\right) + f(M,\rho).$$

After integrating by parts, we obtain that

$$\left(\frac{\alpha_1+1}{\frac{\alpha_1}{2}+1}\right)^2 \left\|\partial_t M^{\frac{\alpha_1}{2}+1}\right\|^2 = -\frac{1}{2}\partial_t \left\|\frac{1}{(1-M)^{\frac{\alpha_2}{2}}}\nabla M^{\alpha_1+1}\right\|^2 + \frac{(\alpha_1+1)^2}{\frac{\alpha_1}{2}+1}\left(M^{\frac{\alpha_1}{2}}\hat{f}(M,\rho), \partial_t M^{\frac{\alpha_1}{2}+1}\right).$$

With the Cauchy-Schwarz inequality we have

$$\left(\frac{\alpha_1+1}{\frac{\alpha_1}{2}+1}\right)^2 \left\|\partial_t M^{\frac{\alpha_1}{2}+1}\right\|^2 \leqslant -\partial_t \left\|\frac{1}{(1-M)^{\frac{\alpha_2}{2}}}\nabla M^{\alpha_1+1}\right\|^2 + (\alpha_1+1)^2\left\|M^{\frac{\alpha_1}{2}}\hat{f}(M,\rho)\partial_t M^{\frac{\alpha_1}{2}+1}\right\|^2.$$

so that

$$\partial_t \left\|\frac{1}{(1-M)^{\frac{\alpha_2}{2}}}\nabla M^{\alpha_1+1}\right\|^2 \leqslant (\alpha_1+1)^2\left\|M^{\frac{\alpha_1}{2}}\hat{f}(M,\rho)\right\|^2.$$

It follows with multiplying by t that

$$\partial_t\left(t\left\|\frac{1}{(1-M)^{\frac{\alpha_2}{2}}}\nabla M^{\alpha_1+1}\right\|^2\right) \leqslant \left\|\frac{1}{(1-M)^{\frac{\alpha_2}{2}}}\nabla M^{\alpha_1+1}\right\|^2 + t(\alpha_1+1)^2\left\|M^{\frac{\alpha_1}{2}}\hat{f}(M,\rho)\right\|^2.$$

Integrating over $(0,1)$, we obtain that

$$\left\|\frac{1}{(1-M(1))^{\frac{\alpha_2}{2}}}\nabla M^{\alpha_1+1}(1)\right\|^2 \leqslant \int_0^1 \left\|\frac{1}{(1-M(s))^{\frac{\alpha_2}{2}}}\nabla M^{\alpha_1+1}(s)\right\|^2 + s(\alpha_1+1)^2\left\|M^{\frac{\alpha_1}{2}}(s)\hat{f}(M(s),\rho(s))\right\|^2 ds. \tag{4.49}$$

Let us estimate the integral on the right side of (4.49). Due to the assumption (4.7) on γ_1 and γ_2 we have

$$\begin{aligned}\left|M^{\frac{\alpha_1}{2}}\hat{f}(M,\rho)\right| \leqslant& |\gamma_1(1-M)-\gamma_2 M| M^{\gamma_1+\frac{\alpha_1}{2}-1}(1-M)^{\gamma_2-1}|\nabla M|\,\|\nabla\rho\|_\infty \\ &+ M^{\gamma_1+\frac{\alpha_1}{2}}(1-M)^{\gamma_2}|\Delta\rho| + M^{\frac{\alpha_1}{2}}|f(M,\rho)| \\ \leqslant& (\gamma_1+\gamma_2)\frac{1}{(1-M)^{\frac{\alpha_2}{2}}}\left|\nabla M^{\alpha_1+1}\right|\|\nabla\rho\|_\infty + |\Delta\rho|_2 + ||f||_\infty\end{aligned} \tag{4.50}$$

and

$$\frac{1}{(1-M)^{\frac{\alpha_2}{2}}}\left|\nabla M^{\alpha_1+1}\right|^2 \leqslant \frac{1}{(1-M)^{\alpha_2}}\left|\nabla M^{\alpha_1+1}\right|^2 = (\alpha+1)\left|\nabla\mathcal{E}(M)\right|^2. \tag{4.51}$$

Further, due to the classical energy estimate, we have for all $(M_0, \rho_0) \in B_*$

$$\int_0^1 \|\Delta\rho(s)\|^2 \, ds \leqslant \|\nabla\rho_0\|^2 + \int_0^1 \|g(M(s),\rho(s))\, ds\|^2$$
$$\leqslant \|\nabla\rho_0\|^2 + ||g||_\infty. \tag{4.52}$$

By combining (4.50)-(4.52) with (4.49) and (4.33), we get the following smoothing estimate for M:

$$\left\| |M(1)|^{\alpha_1+1} \right\|_{H_0^1(\Omega)} \leqslant B_1(R_*), \tag{4.53}$$

the constant $B_1(R_*)$ depends only on R_* and the parameters of the problem. Finally, using *Lemma 1.1* for $q = \alpha_1 + 1$, we obtain from (4.53) that

$$\|M(1)\|_{W^{\frac{1}{\alpha_1+1}-\theta,2}(\Omega)} \leqslant B_M(R_*) \tag{4.54}$$

for an arbitrary $\theta \in (0,1)$, the constant $B_M(R_*)$ depends only on R_*, θ and the parameters of the problem. We choose $\theta := \frac{1}{2}$ in (4.54), so that

$$\|M(1)\|_{W^{\frac{1}{2},2}(\Omega)} \leqslant B_M(R_*). \tag{4.55}$$

Next, we deal with equation (4.2). Since

$$(-\Delta)^{\frac{11}{12}} \rho(1) = (-\Delta)^{\frac{11}{12}} e^{\Delta}\rho_0 - \int_0^1 (-\Delta)^{\frac{11}{12}} e^{(1-\omega)\Delta} g(M(\omega),\rho(\omega))\, d\omega$$

we conclude with the properties (1.9)-(1.10) that, due to assumptions on g,

$$\left\|(-\Delta)^{\frac{11}{12}} \rho(1)\right\|_6 \leqslant \left\|(-\Delta)^{\frac{11}{12}} e^{\Delta}\rho_0 - \int_0^1 (-\Delta)^{\frac{11}{12}} e^{(1-s)\Delta} g(M(s),\rho(s))\, ds\right\|_6$$
$$\leqslant A\left(\frac{5}{12},6\right)\|\nabla\rho_0\|_6 + A\left(\frac{11}{12},6\right)\int_0^1 (1-s)^{-\frac{11}{12}}\, d\omega$$
$$\leqslant B_\rho(R_*), \tag{4.56}$$

and the constant $B_\rho(R_*)$ depends only on R_* and the parameters of the problem.

With the smoothing properties (4.55)-(4.56), the compact embeddings (see *Theorem 1.10*)

$$W^{\frac{1}{2(\alpha+1)},2(\alpha+1)}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega),$$
$$W^{\frac{11}{6},6}(\Omega) \hookrightarrow\hookrightarrow W^{1,\infty}(\Omega),$$

we obtain that $S(1)$ maps the set B_* into a relative compact subset of $L^2(\Omega) \times \left(1 + W_0^{1,\infty}(\Omega)\right)$. *Theorem 4.2* is thus proven.

□

Remark 4.6 (Global attractor in 1D-case)**.** In case of one spatial dimension, there is no need to pass to a weaker topology. For $\Omega = (a,b)$ for some $-\infty < a < b < \infty$ the semigroup $\{S(t)\}_{t\geqslant 0}$ possesses the global attractor in the phase space $L^\infty(a,b) \times \Big(1 + W_0^{1,\infty}(a,b)\Big)$. The proof of this statement is very similar to the proof of *Theorem 4.2*. It makes use of the compact embedding

$$H_0^1(a,b) \hookrightarrow\hookrightarrow C([a,b]),$$

see *Theorem 1.10*. We leave the details to the reader.

4.5 Numerical simulations

We conclude this chapter with a presentation of numerical simulation results that illustrate possible model behavior. The simulation was performed by Hermann Eberl. For computational convenience, we restrict ourselves to the case of one spatial dimension. Our goal is to investigate the potential effect of chemotaxis in early stages of biofilm colony formation, for a generic biofilm rather than a particular biological system. We will do this by comparing the simulations of the biofilm-chemotaxis model with the simulations of the corresponding biofilm model without chemotaxis.

Table 4.1: *Model parameters used in the simulations*

parameter	**symbol**	**value**	**unit**
system length	L	$5 \cdot 10^{-4}$	m
biomass motility coefficient (diffusion)	d_M	*varied*	$m^2 d^{-1}$
biomass motility coefficient (chemotaxis)	d_c	*varied*	$m^2 d^{-1}$
substrate diffusion coefficient	d_ρ	10^4	$m^2 d^{-1}$
maximum growth rate	k_1	6	d^{-1}
half saturation concentration	k_2	0.2	–
maximum substrate uptake rate	k_3	95238.1	d^{-1}
biomass diffusion exponent	α_1	4	–
biomass diffusion exponent	α_2	0.5	–
chemotaxis exponent	γ_1	3	–
chemotaxis exponent	γ_2	3	–
logistic growth exponent	a	3	–
logistic growth exponent	b	0.8	–

The simulations are done in a domain of length $L = 0.5mm$. For the reaction part, we use the functions from *Example 4.1*. The parameters $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ that describe the spatial movement of the biomass and the growth parameters a, b are chosen in accordance with (4.6) and (4.17), respectively, so that to ensure the existence of a unique solution of the problem (4.1)-(4.4). The remaining growth kinetics parameters and the chemotaxis diffusion coefficient are taken from Benchmark Problem 1 of the International Water Association's Taskgroup on Biofilm Modeling [32], where the maximum uptake rate k_3 in (4.16) is compounded from the maximum specific growth rate k_1, a yield coefficient and the maximum cell density. The half saturation concentration k_2 is chosen clearly smaller 1, i.e. we consider the case of biomass growth that is not initially limited by the chemoattractant. These parameters are kept constant for all simulations. The biomass motility parameters d_M and d_c are varied to investigate different scenarios. All model parameters are collected in *Table 4.1*.

The numerical method that we use in these simulations is a straightforward adaptation of the finite difference scheme [7] for the density-dependent diffusion-reaction biofilm model. This method is able to deal with both the degeneracy and the singularity in the biomass diffusion equation with sufficient accuracy, while requiring only moderate spatial refinement [7, 23]; in our simulations we use 200 grid points. In the numerical treatment, the additional chemotaxis terms in (4.2) are treated as convective terms with density dependent convective velocity.

The system which we simulate corresponds to a standard biofilm growth scenario. We use the initial data

$$M_0(x) = \begin{cases} m_0 & \text{for } \frac{L}{2} - r \leqslant x \leqslant \frac{L}{2} + r, \\ 0 & \text{else} \end{cases} \quad \text{for } r = 0.05L,\ m_0 = 0.1 \tag{4.57}$$

$$\rho_0 \equiv 1. \tag{4.58}$$

The region where $M = 0$ is the aqueous phase, the region with $M > 0$ is the actual biofilm. Due to growth, both regions change in time. It is easy to verify that these symmetric initial data will lead to a symmetric solution, which is unique due to *Theorem 4.1*. This solution will have $M_x = \rho_x = 0$ for $x = L/2$. Hence, the solution of the problem restricted to the interval $0 < x < L/2$ can be interpreted as the solution of the system with a biofilm originally in a small pocket on an impermeable substratum at $L/2$. As we present and discuss the solution, we, therefore, restrict ourselves to the interval $0 < x < L/2$.

The nutrients are added into the system at $x = 0$, due to (4.3), i.e. at the boundary on the opposite side of the substratum. Thus chemotaxis is expected to lead to a faster expansion of the biofilm toward the nutrient source. A particularity of the Dirichlet boundary conditions is that, by virtue of the maximum principle, a higher amount of biomass leads to steeper chemoattractant gradients at the boundary, i.e. to improved environmental conditions. In *Figure 4.1* we plot the solution (M, ρ) of (4.1)-(4.3), (4.57)-(4.58) for biomass motility coefficients $d_M = d_c = 10^{-12}$ as surface data over the x-t-plane. In the beginning, biomass growth is very slow and appears to be almost stationary. After some time, biomass density in the biofilm pocket starts increasing without the biofilm region expanding. Initially, the biomass density increases faster in the outer layer of the biofilm (close to the biofilm/aqueous phase interface) than in the inner layer (at the substratum), due to higher nutrient availability and no pressure to diffuse. Once the biomass density reaches values close to unity, the biofilm region starts expanding and the biomass density attains a value $M \approx 1$ in the interior of the biofilm. The chemoattractant concentration

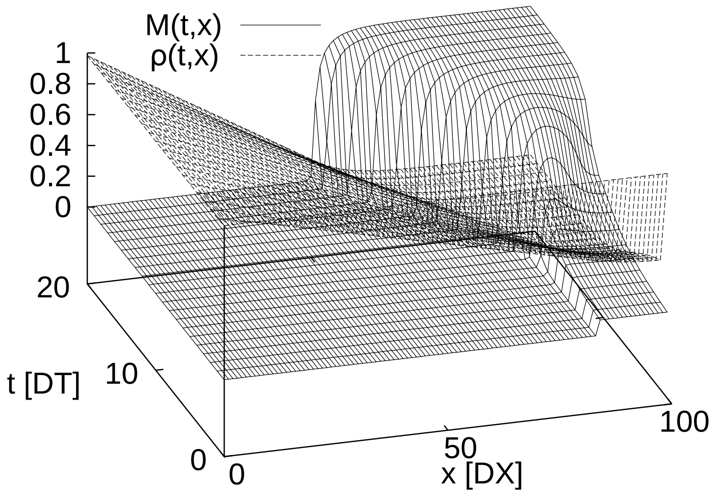


Figure 4.1: *Biomass density $M(t,x)$ and chemoattractant concentration ρ as a solution of (4.1)-(4.3), (4.57)-(4.58) with parameters according to Table 4.1 and biomass motility coefficients $d_M = d_c = 10^{-10}$.*

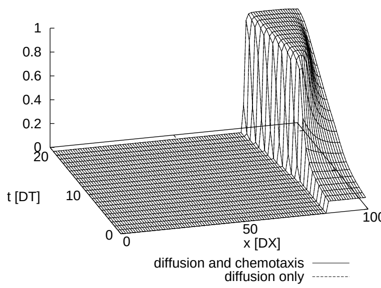
field coincides with the biomass density. It attains its minimum at the substratum. The more biomass there is in the system the lower is the chemoattractant concentration. The chemoattractant gradient at the boundary $x = 0$ increases as biomass grows.

In order to assess the contribution of chemotaxis to biomass movement, relative to diffusion, we plot in *Figure 4.2* the biomass densities for various choices of the biomass motility coefficients. Together with the solution of our original model (4.2) with $d_M > 0$, $d_c > 0$ we plot the solution of the corresponding biofilm model without chemotaxis-term ($d_M > 0$, $d_c = 0$), i.e. the solution of

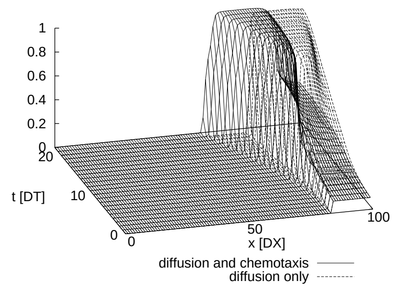
$$\partial_t M = d_M \nabla \cdot \left(\frac{M^{\alpha_1}}{(1-M)^{\alpha_2}} \nabla M \right) + f(M, \rho).$$

In the left column of *Figure 4.2* we show simulations of the model with the same biomass motility coefficient for both processes, diffusion and chemotaxis, i.e. $d_M = d_c$. These parameters range here from 10^{-12} to 10^{-8}. This coefficient controls how fast the biofilm region expands and to which maximum biomass density it grows. For $d_M = 10^{-12}$, expansion is very slow and the biomass density reaches values close to $M \approx 1$ inside the biofilm (*Figure 4.2*(a)). For $d_M = 10^{-10}$ the biofilm expands faster, still growing close to maximum cell density (*Figure 4.2(c)*). For the largest value, the biofilm expands quickly, but does not exceed values of $M \approx 0.6$ *Figure 4.2(e)*). This choice of parameters is therefore considered too big to be realistic. In all three cases with the same biomass motility coefficient for both spatial processes, the solution of the model with and without chemotaxis are essentially indistinguishable, indicating that chemotaxis does not contribute noteworthy to biofilm formation in such cases.

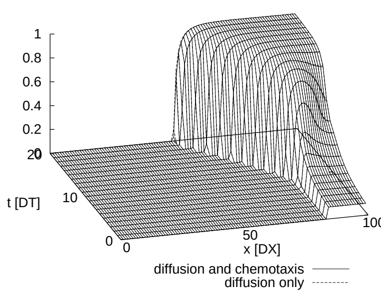
In the right column of *Figure 4.2*, we use different biomass motility coefficients d_M, d_c. In all cases we choose the chemotaxis coefficient to be higher than the diffusive one, $d_M \ll d_c$. We notice distinct differences in the biomass densities of the models with and without chemotaxis, in the cases of *Figure 4.2(b)* and *Figure 4.2(d)*, where $d_c = 10^{-8}$ but not so in *Figure 4.2(f)*, where $d_c = 10^{-10}$. In the case of *Figure 4.2(b)*, with $d_M = 10^{-12}$ and $d_c = 10^{-8}$ chemotaxis leads to a very different biofilm structure than obtained by the model without chemotaxis. The chemotaxis effect pulls biomass toward the nutrient source and leads to a biofilm that is much denser close to the biofilm/water interface than in the inner layers close to the substratum. This could be understood as the 1D analogy of mushroom type biofilm colonies in 2D/3D. In the case of *Figure 4.2(d)*, with $d_M = 10^{-10}$ and $d_c = 10^{-8}$, on the other hand the differences are not as pronounced. Interestingly, it appears that the biofilm without chemotaxis grows bigger and denser in this case than the one with chemotaxis. The chemoattractant concentration ρ in all cases is similar as shown in *Figure 4.2*. These simulations were repeated several times with different exponents of the chemotaxis model and different initial data (smaller initial inoculum or non-constant biomass distribution in the inoculum). In all cases the results were qualitatively the same (data not shown). This suggests that in early stages chemotaxis will only affect biofilm structure quantitatively if the biomass motility coefficient is substantially larger than the biomass motility coefficient due to diffusive biomass spreading. It is reasonable to assume that this parameter depends on the material properties of the particular biofilm (species and environment), in particular of the EPS, in which cells are embedded, but also parameters that describe the ability of the cells to move, e.g., by flagellar motion or twitching motility.



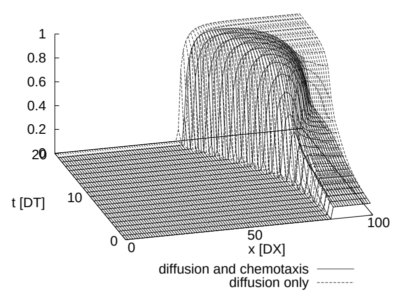
(a) $d_M = d_c = 10^{-12}$



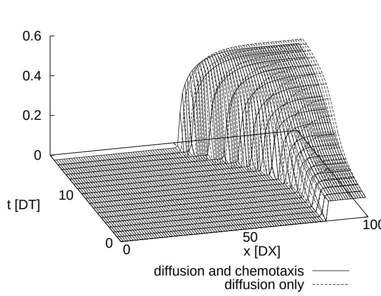
(b) $d_M = 10^{-12}, d_c = 10^{-8}$



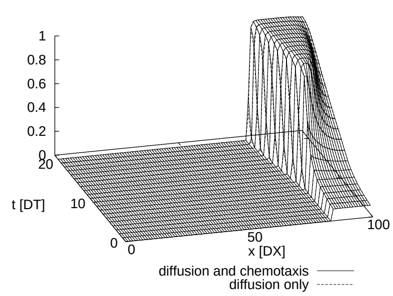
(c) $d_M = d_c = 10^{-10}$



(d) $d_M = 10^{-10}, d_c = 10^{-8}$



(e) $d_M = d_c = 10^{-8}$



(f) $d_M = 10^{-12}, d_c = 10^{-10}$

Figure 4.2: *Comparison of the chemotaxis-diffusion biofilm model with the diffusion-only biofilm model. Plotted are biomass densities M for various biomass motility coefficients. The units in x direction are grid point spacings $\Delta x = L/200$, the units in t direction are output time steps $\Delta T = 2d$.*

Appendix A

An auxiliary Lemma

Consider first the differential inequality

$$\frac{d}{dt}y \leqslant -\omega_y y + d_y y^{\zeta_y}$$

assuming that $y \geqslant 1$, $\zeta_y \in (0,1)$, $d_y \in L^1_b(\mathbb{R})$ so that with some computation the estimate

$$(y(t))^{1-\zeta_y} \leqslant (y(\tau))^{1-\zeta_y} e^{-\omega_y(1-\zeta_y)t} + (1-\zeta_y)\int_\tau^t e^{-\omega_y(1-\zeta_y)(t-s)} d_y(s)\, ds$$

follows.

Lemma A.1. *Let $z_1, z_2, z_3 : [\tau, +\infty) \to [0, +\infty)$ be such functions that*

$$\begin{aligned}
&z_1(t) \leqslant \psi_1(z_1(\tau))e^{-\omega_1 t} + D_1,\\
&z_2(t) \leqslant \psi_2(z_2(\tau))e^{-\omega_2 t} + D_2,\\
&z_3(t) \leqslant z_3(\tau)e^{-\omega_3 t} + \int_\tau^t e^{-\omega_3(t-s)} d_3(t,s) z_1(s)\, ds,\\
&z_1(\tau), z_2(\tau), z_3(\tau) \geqslant 1,
\end{aligned} \tag{A.1}$$

for some constants $\omega_1, \omega_2, \omega_3 > 0$ and $D_1, D_2 \geqslant 1$, some non-decreasing functions $\psi_1, \psi_2 : [1, +\infty) \to [1, +\infty)$ and some $d_3 \in L^\infty(\mathbb{R}^+_\tau, L^1_b(\mathbb{R}^+_\tau))$. Then it holds that

(1) $(z_1 + z_2)(t) \leqslant (\psi_1 + \psi_2)((z_1 + z_2)(\tau))e^{-\min\{\omega_1,\omega_2\}t} + D_1 + D_2$.

(2) $z_1 z_2(t) \leqslant 3D_1 D_2 \psi_1 \psi_2 (z_1 z_2(\tau))e^{-\min\{\omega_1,\omega_2\}t} + D_1 D_2$.

(3) $z_1^\sigma(t) \leqslant \max\left\{1, 2^{\sigma-1}\right\}\left(\psi_1^\sigma(z_1(\tau))e^{-\sigma\omega_1 t} + D_1^\sigma\right) \quad \forall \sigma > 0.$

(4) For $\omega_1 \neq \omega_3$

$$z_3(t) \leqslant \left(\psi_1(z_1(\tau))\frac{1}{1-e^{-|\omega_1-\omega_3|}}e^{-\min\{\omega_1,\omega_3\}t} + D_1\frac{1}{1-e^{-\omega_3}}\right) \cdot ||d_3||_{L^\infty(\mathbb{R}^+_\tau, L^1_b(\mathbb{R}^+_\tau))} + z_3(\tau)e^{-\omega_3 t} \quad \text{(A.2)}$$

and for $\omega_3 = \omega_1$

$$z_3(t) \leqslant \left(\psi_1(z_1(\tau))\lceil t\rceil e^{-\omega_1 t} + D_1\frac{1}{1-e^{-\omega_1}}\right)||d_3||_{L^\infty(\mathbb{R}^+_\tau, L^1_b(\mathbb{R}^+_\tau))} + z_3(\tau)e^{-\omega_1 t}.$$

For $\omega_1 < \omega_3$*, we also have*

$$z_3(t) \leqslant z_3(\tau)e^{-\omega_3 t} + z_1(t)\int_\tau^t e^{-(\omega_3-\omega_1)(t-s)}d_3(t,s)\,ds. \quad \text{(A.3)}$$

Proof. We only check the property (A.2). Since

$$\begin{aligned}
&\int_\tau^t e^{-\omega_3(t-s)}e^{-\omega_1 s}d_3(t,s)\,ds \\
=&e^{-\min\{\omega_1,\omega_3\}t}\begin{cases} \int_\tau^t e^{-|\omega_1-\omega_3|(t-s)}d_3(t,s)\,ds & \text{if } \omega_1<\omega_3 \\ \int_\tau^t e^{-|\omega_1-\omega_3|s}d_3(t,s)\,ds & \text{if } \omega_1>\omega_3 \end{cases} \\
\leqslant&\frac{1}{1-e^{-|\omega_1-\omega_3|}}e^{-\min\{\omega_1,\omega_3\}t}||d_3||_{L^\infty(\mathbb{R}^+_\tau, L^1_b(\mathbb{R}^+_\tau))},
\end{aligned}$$

we conclude from (A.1) that

$$\begin{aligned}
&\int_\tau^t e^{-\omega_3(t-s)}d_3(t,s)z_1(s)\,ds \\
\leqslant&\int_\tau^t e^{-\omega_3(t-s)}d_3(t,s)\left(\psi_1(z_1(s))e^{-\omega_1(t-s)} + D_1\right)ds \\
\leqslant&\left(\psi_1(z_1(\tau))\frac{1}{1-e^{-|\omega_1-\omega_3|}}e^{-\min\{\omega_1,\omega_3\}t} + D_1\frac{1}{1-e^{-\omega_3}}\right) \\
&\cdot ||d_3||_{L^\infty(\mathbb{R}^+_\tau, L^1_b(\mathbb{R}^+_\tau))},
\end{aligned}$$

and (A.2) follows.

□

Bibliography

[1] R.A. Adams, *Sobolev spaces*, Academic press, New York (1975).

[2] H. Amann, *Linear and Quasilinear Parabolic Problems*, Volume 1, Abstract Linear Theory, Birkhäuser, Basel (1995).

[3] D.G. Aronson, *The porous medium equation*, in Some Problems in Nonlinear Diffusion (A. Fasano, M. Primicerio, eds), Lecture Notes in Mathematics, 1224, Springer, Berlin (1986).

[4] A.V. Babin, M.I. Vishik, *Attractors of evolution equations*, North-Holland, Amsterdam (1992).

[5] V.V. Chepyzhov, M.I. Vishik, *Attractors for equations of mathematical physics*, AMS Colloquium Publications, 49, Providence (2002).

[6] V.V. Chepyzhov, M.I. Vishik, *Attractors of nonautonomous dynamical systems and their dimension*, Journal de Mathématiques Pures et Appliquées, 73, 279-333 (1994).

[7] H.J. Eberl, L. Demaret, *A finite difference scheme for a degenerated diffusion equation arising in microbial ecology*, Electronic Journal of Differential Equations, CS15, 77-95 (2007).

[8] M.A. Efendiev, *Attractors of Degenerate Parabolic Type Equations*, to appear in AMS.

[9] M.A. Efendiev, *Evolution Equations Arising in the Modelling of Life Sciences*, International Series of Numerical Mathematics, 163, Birkhäuser (2013).

[10] M.A. Efendiev, *Finite and Infinite Dimensional Attractors for Evolution Equations of Mathematical Physics*, Gakkotoscho International Series, Tokyo, 33 (2010).

[11] M.A. Efendiev, S. Sonner, *On verifying mathematical models with diffusion, transport and interaction*, Advances in Mathematical Sciences and Applications, 32, 41-67 (2010).

[12] M.A. Efendiev, T. Senba, *On the well-posedness of a class of PDEs including porous medium and chemotaxis effect*, Advances in Differential Equations, 16(9-10), 937-954 (2011).

[13] M.A. Efendiev, Y. Yamamoto, A. Yagi, *Exponential attractors for non-autonomous dissipative system*, Journal of the Mathematical Society of Japan, 63, 2, 647-673 (2011).

[14] M.A. Efendiev, H. Eberl, and S. Zelik, *Existence and long-time behaviour of biofilm model*, Communications on Pure and Applied Analysis, 8, 2, 509-531 (2009).

[15] M.A. Efendiev, A. Zhigun, *On a Global Uniform Pull-back Attractor of a Class of PDEs with Degenerate Diffusion and Chemotaxis in One Dimension*, Recent Trends in Dynamical Systems: Proceedings of a Conference in Honor of Jürgen Scheurle, Springer Proceedings in Mathematics & Statistics, Birkhäuser (2013).

[16] R. Hagen, S. Roch, B. Silbermann, *C^*-algebras and Numerical Analysis*, CRC Press (2001).

[17] A. Haraux, *Systèmes Dynamiques Dissipatifs et Applications*, Masson (1991).

[18] D. Horstmann, *From 1970 Until Present: The Keller-Segel Model in Chemotaxis and its Consequences*, Part I, Jahresbericht der DMV, 105(3), 103-165 (2003).

[19] P. Kloeden, *Pullback attractors of nonautonomous semidynamical systems*, Stochastics and Dynamics, 3, 101–112 (2003).

[20] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural'ceva, *Linear and Quasi-linear Equations of Parabolic Type*, Transl. Math. AMS. Providence, R.I. (1968).

[21] J.L. Lions, *Quelques mthodes de rsolution des problmes aux limites non linaires*, Dunod, Paris (1969).

[22] J.L. Lions, E. Magenes, *Non-homogeneous Boundary Value problems and its Applications*, Springer-Verlag (1972).

[23] N. Muhammad, H.J. Eberl, *Model parameter uncertainties in a dual-species biofilm competition model affect ecological output parameters much stronger than morphological ones*, Mathematical Biosciences, 233(1), 1-18 (2011).

[24] V. Pata, S. Zelik, *A result on the existence of global attractors for semi-groups of closed operators*, Communications on Pure and Applied Analysis, 6, 481-486 (2007).

[25] K. Painter, T. Hillen, *Volume-filling and quorum-sensing in models for chemosensitive movement*, Canadian Applied Mathematics Quarterly, 10, 501-543 (2002).

[26] W. Rudin, *Functional Analysis*, McGraw-Hill, New York (1991).

[27] Walter Rudin, *Real and Complex Analysis*, McGraw-Hill (1987).

[28] B. Schmalfuss, *Attractors for the non-autonomous dynamical systems* EQUADIFF 99, World Scientific, 1 (2000).

[29] S. Sonner, *Systems of Quasi-Linear PDEs Arising in the Modelling of Biofilms and Related Dynamical Questions*, Ph. D. thesis, Technical University of Munich, Germany (2012).

[30] H. Triebel, *Interpolation theory, function spaces, differential operators*2nd ed., Johann Ambrosius Barth, Heidelberg (1995)

[31] J.L. Vazquez, *The Porous Medium Equation: Mathematical Theory*, Oxford (2007).

[32] O. Wanner, H. Eberl, E. Morgenroth, D. Noguera, C. Picioreanu, B. Rittmann B, M. van Loosdrecht, *Mathematical Modeling of Biofilms*, IWA Publishing, London, (2006).

[33] D. Wrzosek, *Volume filling effect in modeling chemotaxis*, Mathematical Modelling of Natural Phenomena, 5, 123-147 (2010).

[34] A. Yagi, *Abstract Parabolic Evolution Equations and their Applications*, Springer Monographs in Mathematics, Springer-Verlag Berlin Heidelberg (2010).