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Orthogonal polynomial systems and weighted Cesàro means

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Zusammenfassung

Das Ziel dieser Arbeit ist es, sowohl einen Beitrag zum Fachgebiet der orthogonalen Polynome, als auch zur Operatortheorie zu leisten. Ein zentrales Thema bildet die Herleitung des konjugierten orthogonalen Polynomsystems. Dabei betrachten wir Polynomsysteme $(R_n(x))_{n \in \mathbb{N}_0}$, die derart normalisiert sind, dass $R_n(1) = 1$, für alle $n \in \mathbb{N}_0$, und die orthogonal bezüglich eines (Borel-)Wahrscheinlichkeitsmaßes μ sind, dessen Träger im Intervall $[-1, 1]$ enthalten ist. In diesem Zusammenhang definieren wir das „konjugierte orthogonale Polynomsystem" $(R_n(x))_{n \in \mathbb{N}_0}$ als diejenige Folge $(R_n^*(x))_{n \in \mathbb{N}_0}$ orthogonaler Polynome, die orthogonal bezüglich des Wahrscheinlichkeitsmaßes μ^* ist, wobei $\mathrm{d}\mu^*(x) = c(1 - x^2) \, \mathrm{d}\mu(x)$ für eine positive Konstante c. Im Zuge der Herleitung des konjugierten Systems, befassen wir uns mit zwei weiteren zugehörigen Polynomsystemen. Ausgehend vom ursprünglichen System werden wir das Haarmaß und die Koeffizienten in den Drei-Term-Rekursionen der neuen Polynomsysteme allgemein herleiten.

Das zweite Hauptthema dieser Arbeit ist der gewichtete Cesàro-Operator C_h auf gewichteten $\ell^p(h)$-Folgenräumen mit $1 < p < \infty$, wobei h ein positives diskretes Maß auf $\mathbb{N}_0$ bezeichnet. Wir untersuchen die Beschränktheit des Cesàro-Operators und bestimmen seine Operatornorm. Zusätzlich wird ein Ergebnis, das das Spektrum des Cesàro-Operators auf $\ell^2(h)$ betrifft, präsentiert. Schließlich untersuchen wir den gewichteten Cesàro-Operator auf $\ell^2(h)$ bezüglich verallgemeinerter Normalitätskonzepte. Genauer gesagt bestimmen wir eindeutig diejenigen Gewichte h, für die C_h hyponormal ist. Darüber hinaus zeigen wir, dass der Cesàro-Operator nicht immer die Eigenschaft normaloider Operatoren erfüllt und dass es keine Gewichtsfolge h gibt, für die C_h quasinormal ist.

Außerdem stellen wir einen Zusammenhang zwischen orthogonalen Polynomsystemen und dem gewichteten Cesàro-Operator her. Dabei spielen unter anderem die Christoffel-Darboux-Formel und der Tridiagonaloperator, welcher durch die Rekursionsrelation der entsprechenden Polynome definiert ist, eine Rolle. Schließlich diskutieren wir als Beispiele homogene Baumpolynome und Karlin-McGregor Polynome, für die einerseits die konjugierten Systeme explizit bestimmt werden und andererseits die zugehörigen Cesàro-Operatoren, insbesondere im Bezug auf Hyponormalität, untersucht werden.

Abstract

The purpose of this thesis is to make a contribution to both, the field of orthogonal polynomials and that of operator theory. One central topic is the definition of the conjugate orthogonal polynomial system. We consider polynomial systems $(R_n(x))_{n \in \mathbb{N}_0}$ which are normalized such that $R_n(1) = 1$, for all $n \in \mathbb{N}_0$, and which are orthogonal with respect to a probability (Borel) measure μ, whose support is contained in the interval $[-1, 1]$. Then, the so called "conjugate orthogonal polynomial system" $(R_n(x))_{n \in \mathbb{N}_0}$ will be defined as the sequence $(R_n^*(x))_{n \in \mathbb{N}_0}$ of polynomials which is orthogonal with respect to the probability measure μ^*, where $\mathrm{d}\mu^*(x) = c(1 - x^2)\,\mathrm{d}\mu(x)$ for some positive constant c. In the course of deducing the conjugate system, two further orthogonal polynomial systems are studied. Based on the initial system, we will in general determine the Haar measures and coefficients in the three-term recurrence relations of the new systems.

The second main issue of this thesis will be the weighted Cesàro operator C_h, acting on weighted $\ell^p(h)$-sequence spaces with $1 < p < \infty$, where h denotes a positive discrete measure on $\mathbb{N}_0$. The boundedness of the Cesàro operator will be investigated and the norm of C_h will be determined. We also present some results concerning the spectrum, when we consider the weighted Cesàro operator in the Hilbert space $\ell^2(h)$. Finally, the weighted Cesàro operator in $\ell^2(h)$ is investigated in terms of several concepts of normality. Moreover, we classify exactly those h for which C_h is hyponormal. Furthermore, we show that the Cesàro operator is not always normaloid and prove that quasinormality is not satisfied for any choice of h.

We will establish connections between the weighted Cesàro operator and orthogonal polynomial systems. In particular, the Chistoffel-Darboux Identity and the recurrence relation of the respective polynomials, are involved. Finally, we exhibit polynomials related to homogeneous trees and Karlin-McGregor polynomials as examples for which on the one hand, the conjugate systems will be determined, and on the other hand, the corresponding Cesàro operators will be investigated in view of hyponormality, in particular.

Contents

Introduction

Both, the field of orthogonal polynomials and that of functional analysis, in particular the theory of normal operators and spectral theory, are long established. Following [Chi78, p. vii], systems of orthogonal polynomials have already been taken into account two centuries ago. Functional analysis has its origin at the beginning of the 20th century, cf. [Wer07, p. vii]. Mathematicians like Hilbert, Schmidt or Riesz introduced important concepts in their work. Lateron, Banach and van Neumann, who coined the terms normed vector space and Hilbert space, respectively, unitized these concepts. Since then, the theory of orthogonal polynomials and functional analysis were continuously further developed and is a crucial tool in theoretical physics, in particular quantum mechanics, as well as probability theory, statistics and approximation theory.

Initially, the research of the author was concerned with Hardy's inequality in various forms, which can be traced back to [Har20], [Har25], [Cop27] or [Har28] and will be presented in Section 3.1.1. Hardy's inequality plays an important role in the theory of Fourier series, see for example [Bel44], [Kaw44] or [HLP88]. The topic is still current. An overview on the prehistory of Hardy's inequality can be found in [KMP06]. Furthermore, an interesting transfer of Hardy's inequality to non-commutative operator spaces was recently published in [Han09]. During those studies, our attention was drawn to the 1965 paper of Brown, Halmos and Shields, see [BHS65], which treated the Cesàro operator C in ℓ^2, where

$$C\alpha(n) = \frac{1}{n+1} \sum_{k=0}^{n} \alpha(k), \quad \text{for all } n \in \mathbb{N}_0, \text{ with } (\alpha(n))_{n \in \mathbb{N}_0} \in \ell^2.$$

Inspired by the weighted version of Hardy's inequality, see [Har25], we started to investigate a modified averaging operator C_h, acting in weighted sequence spaces. This "weighted Cesàro operator" will be introduced in the main part of the thesis, more precisely, in Chapter 3.
Nevertheless, we want to remark on the continuous topicality of Cesàro operators in both, sequence and function spaces. Over the years various authors studied the properties of Cesàro operators, for example Cesàro operators in Hardy spaces H^p, see [Sis87], [CS99] or [Miy04]. Important parts of these investigations were also a classification of the spectra of Cesàro operators, see [BHS65], [Mad89] and [CR13b]. Moreover, the Cesàro operator is related to the Cesàro summability method, which was for instance described in [Boo00, p. 100 ff.]. Classifying those complex sequences for which the image under C is contained in the classical sequence space ℓ^p for some $1 < p < \infty$, Curbera and Ricker recently made a contribution to characterizing the Cesàro operator further, see [CR13a].

Another part of [BHS65] aroused our interest, namely the investigation, whether C is a hyponormal operator. Subsequently, the author's studies were concerned with several concepts of normality, which form a significant chapter in the theory of bounded linear operators in Hilbert spaces. Weakening the conditions for the normality of an operator, different authors have introduced new classes of not necessarily normal operators which will be recalled in Section 3.3.1. So, for instance, in "A Hilbert Space Problem Book", [Hal82], Halmos studied the properties of normal, quasinormal, subnormal, hyponormal, normaloid, convexoid and spectraloid operators. Stampfli mainly focused on the property of hyponormality. One of his most important results is the answer to the question, under which conditions hyponormal operators are normal, see [Sta62]. Further assertions were for instance proved in [Sta65] and [Sta79]. A nice introduction to hyponormal operators was given by Martin and Putinar in [MP89]. In 1966 and 1967, respectively, a new class of operators, namely paranormal operators, was introduced by Furuta on the one hand, and on the other hand, by Istrăţescu, Saitô and Yoshino, see [Fur67] and [ISY66]. The term "paranormal" was established by Furuta. In the paper of Istrăţescu, Saitô and Yoshino paranormal operators were called "operators of class (N)". Furthermore, they extended Stampfli's result about hyponormal operators to the class of paranormal operators. Moreover, Ando also investigated, under which conditions paranormality leads to normality, see [And72]. In [Fur71], [FHN67] and [FN71], Furuta, Horie, Nakamoto and Takeda proved further properties of paranormal operators, convexoid operators and the numerical range of operators. Interesting results are obtained, when the Cesàro operator is investigated in terms of generalized concepts of normality. In their 1965 paper, see [BHS65], Brown, Halmos and Shields proved the hyponormality of the Cesàro operator C in ℓ^2. Later on, Kriete, Trutt and Cowen showed that C is even subnormal, see [Cow84] and [KT71]. Subnormal operators in general were for instance discussed in [Bra55], not forgetting Conway's recommendable overview on the theory of subnormal operators in [Con91]. Furthermore, subnormality is sometimes connected with weighted shift operators, see [Sta66] or [Lam76], an operator class which is related to the Cesàro operator. In his 1973 paper, Putnam made an important contribution to the classification of the different operator classes named above, including considerations on the spectra, see [Put73]. Beforehand, Putnam dealt with the property of hyponormality, in particular, see [Put70] or [Put72]. Of course there are further terms, like "essentially normal" or "essentially normaloid", which will not be discussed in this thesis. However, many authors involved the essential and the approximative spectrum, respectively, in their treatises on generalized concepts of normality, see [Pat78], [Wil94] or [Fel99].

Besides considering the weighted Cesàro operator in view of those weak normality conditions, we established a connection between the Cesàro operator and orthogonal polynomials which form the second main topic of our thesis.

In the context of orthogonal polynomials, well-known treatises are for instance the books of Szegő ([Sze75]) which was first published in 1939, and Chihara ([Chi78]). We also want to mention people like Askey and Ismail here, who enriched the theory of orthogonal polynomials during the last century. The "classical orthogonal polymials" are Jacobi polynomials (including Legendre, Tchebichef and ultraspherical or Gegenbauer polynomials), Laguerre polynomials and Hermite polynomials, see [Chi78, p. 142 ff.], which are

oft-quoted in literature. The appearance of Cesàro means in relation to Fourier series, as well as orthogonal polynomials (when we consider for instance the Christoffel-Darboux Identity or tridiagonal operators related to the recurrence relation), draw our attention to the classical concept of conjugacy. In [MS65], Muckenhoupt and Stein investigated classical polynomial expansions in analogy to ordinary Fourier series. In particular, they defined conjugacy for ultraspherical expansions. Concerning ultraspherical coefficients, there are also important results due to Askey and Wainger, see for instance [AW66a] and [AW66b]. In [Ste70], Stein introduced a general principle for the definition of conjugacy with respect to polynomial expansions which was applied by Muckenhoupt to define conjugacy for Hermite ([Muc69]) and Laguerre ([Muc70]) expansions. Gosselin and Stempak seized this idea of conjugacy and developed a theory, not only for polynomial expansions, but also for Hermite and ultraspherical functions, see [Ste93]. Finally, we shall also mention Li, who extended the theory of Muckenhoupt and Stein, to the class of Jacobi polynomials, in his two papers [Li96] and [Li97].

In this thesis, the approach to conjugate systems is a different one. We will consider orthogonal polynomial systems, satisfying a recurrence relation of the type

$$R_0(x) = 1, \quad R_1 = (x - b_0)/a_0,$$

$$xR_n(x) = a_n R_{n+1}(x) + b_n R_n(x) + c_n R_{n-1}(x), \quad n \in \mathbb{N}.$$

where the coefficients a_n, b_n and c_n are real numbers, satisfying some additional properties, such that we can assume that $R_n(1) = 1$ and that the support of the corresponding orthogonalization measure μ is contained in the real interval $[-1, 1]$. A more detailed definition will be part of Chapter 2. The conjugate orthogonal polynomial system will be defined as $(R_n^*(x))_{n \in \mathbb{N}_0}$, where $R_n^*(1) = 1$, which is orthogonal with respect to a measure μ^*, satisfying $\mathrm{d}\mu^*(x) = c(1 - x^2)\, \mathrm{d}\mu(x)$ for some positive constant c. The assumptions regarding the initial system ensure the well-definedness of the conjugate system. It is our aim to deduce the conjugate systems for arbitrary polynomial systems which satisfy the conditions above. Before getting around to a summary of the main part of the thesis, we want to mention the connection between orthogonal polynomials and the Cesàro operator. For example, it occurred to us that the definition of a related orthogonal polynomial system, namely the system orthogonal with respect to μ^-, where $\mathrm{d}\mu^-(x) = c'(1-x)\, \mathrm{d}\mu(x)$ for some positive constant c', can be associated with the definition of the respective weighted Cesàro operator. For the author, this was a decisive factor to bring together the field of operator theory and that of orthogonal polynomials.

The main part of the thesis is organized as follows:

Chapter 1 contains basics from the theory of orthogonal polynomials and operator theory. Orthogonal polynomial systems with respect to a moment functional will be introduced in general. We also want to recall important assertions regarding the existence and uniqueness of orthogonal polynomial systems. Furthermore, we outline important properties of classes of bounded linear operators and their spectra.

In Chapter 2 we are concerned with the introduction of the conjugate orthogonal polyno-

mial system. If $(R_n(x))_{n \in \mathbb{N}_0}$ is an orthogonal polynomial system with respect to a probability measure μ, then the so called "conjugate orthogonal polynomial system" $(R_n(x))_{n \in \mathbb{N}_0}$ is be defined as the sequence $(R_n^*(x))_{n \in \mathbb{N}_0}$ of polynomials which is orthogonal with respect to the probability measure μ^*, where $\mathrm{d}\mu^*(x) = c(1 - x^2)\,\mathrm{d}\mu(x)$ for some positive constant c. In the first section we will give a review on the classical definition of conjugacy for polynomial expansions which is due to Muckenhoupt and Stein. The remaining part of the chapter deals with the explicit deduction of two polynomial systems which are also related to the initial system, and the conjugate system. In particular, we determine the shape of each polynomial and the coefficients in the recurrence relations. Furthermore, we discuss some properties of the function spaces $L^p(\mu)$ and $L^p(\mu^*)$, where $1 \leq p < \infty$. In Chapter 3, the Cesàro operator in weighted ℓ^p-sequence spaces will be investigated. Involving Hardy's inequality, the boundedness of the Cesàro operator will be deduced. Dependent on the choice of the weights, we can determine the operator norm. In the second section of the chapter, we will deal with the spectrum of the Cesàro operator. Some results of [BHS65], concerning the ℓ^2-case, can be extended to the more general case. Finally, we want to investigate the Cesàro operator in terms of generalized normality concepts which have already been mentioned above. Those weights shall be classified for which the Cesàro operator satisfies certain weak normality conditions. In particular, the condition of hyponormality will be focal.

Chapter 4 establishes a relation between the Cesàro operator and orthogonal polynomial systems and the respective related systems which were introduced in Chapter 2. The definition of the Cesàro operator reminds of the definition of one of the related systems. The second section of the chapter deals with a tridiagonal operator T_1 which is related to the recurrence relations of certain orthogonal polynomial systems. It is shown that the Cesàro operator plays an important role to determine the inverse of $\mathsf{id} - T_1$. In the remaining part of the chapter, we will establish a connection between the Cesàro operator and polynomial hypergroups. Moreover, we show that if the weights satisfy a condition of Szwarc ([Szw92a] and [Szw92b]), we can infer that the Cesàro operator is hyponormal, choosing certain Haar measures of polynomials as weights.

In Chapter 5, two classes of orthogonal polynomials shall be discussed, namely polynomials related to homogeneous trees and Karlin-McGregor polynomials (both normalized as described above), which were for instance discussed in [Las83] and [FL00]. Contrary to Jacobi polynomials, those two classes are examples for polynomials, whose conjugate systems and related systems are not included in the respective class again. Hence, it is very interesting to study the shape and properties of the arising systems. In particular, we are interested in the question, whether the corresponding Cesàro operators (weighted with Haar measures) are hyponormal or at least normaloid. Regarding polynomials related to homogeneous trees, we will show that for certain parameters, the conjugate system and the related systems induce a polynomial hypergroup. A short outlook will close the thesis.

1 Basics from orthogonal polynomials and operator theory

In this first chapter an overview of operator theory, spectral theory and the theory of orthogonal polynomials is given.

1.1 Elementary theory of orthogonal polynomials

This section mainly follows [Chi78, Chapter I]. Another highly recommended treatise on the topic of orthogonal polynomials can be found in [Sze75].

1.1.1 Moment functional and orthogonality

In [Chi78, p. 6 ff.] the linear functional $\mathcal{L}$ and its corresponding orthogonal polynomials are introduced in a general way. Chihara considered polynomials with complex coefficients in one real variable. During this first chapter, we adhere to this general assumption and "polynomial" will denote a polynomials with complex coefficients.

Definition 1.1 (Moment functional, [Chi78, p. 6 f., Definition 2.1])**.** *Let $(\mu_n)_{n \in \mathbb{N}_0}$ be a sequence of complex numbers and let $\mathcal{L}$ be a complex valued function defined on the vector space of all polynomials by*

$$\mathcal{L}[x^n] = \mu_n, \quad \text{for all } n \in \mathbb{N}_0,$$

$$\mathcal{L}[\alpha_1 \pi_1(x) + \alpha_2 \pi_2(x)] = \alpha_1 \mathcal{L}[\pi_1(x)] + \alpha_2 \mathcal{L}[\pi_2(x)],$$

*for all complex numbers α_1, α_2 and all polynomials $\pi_1(x)$, $\pi_2(x)$. Then, $\mathcal{L}$ is called **moment functional** determined by the formal **moment sequence** $(\mu_n)_{n \in \mathbb{N}_0}$. The number μ_n is called the **moment** of order n.*

Definition 1.2 (OPS wrt a moment functional, [Chi78, p. 7, Definition 2.2])**.** *A sequence $(P_n(x))_{n \in \mathbb{N}_0}$ is called **orthogonal polynomial system with respect to a moment functional** $\mathcal{L}$, provided for all nonnegative integers m and n,*

   (i) $P_n(x)$ is a polynomial of degree n, symbolically, $\deg P_n = n$.

   (ii) $\mathcal{L}[P_m(x)P_n(x)] = 0$, for $m \neq n$,

   (iii) $\mathcal{L}[P_n^2(x)] \neq 0$.

Different from Chihara, the term "system" is used instead of "sequence" during the thesis. Hereinafter, "orthogonal polynomial system (with respect to the moment functional $\mathcal{L}$)" will be abbreviated to OPS (wrt $\mathcal{L}$). If $(P_n(x))_{n \in \mathbb{N}_0}$ is and OPS wrt $\mathcal{L}$ and in addition $\mathcal{L}[P_n^2(x)] = 1$, for all $n \in \mathbb{N}_0$, then it will be called orthonormal polynomial system (abbreviated to ONPS). Orthogonal polynomials are mostly considered wrt a certain weight function or wrt finite Borel measures on $\mathbb{R}$. The concept of orthogonality with respect to a moment functional is more general. The following remarks can also be found on [Chi78, p. 8]:

- Definition 1.2(*i*) and (*iii*) imply that if there exits an OPS wrt $\mathcal{L}$, we have

$$\mu_0 \neq 0 \quad \text{and} \quad P_0(x) \neq 0$$

  Thus, no OPS can exist if $\mathcal{L}[1] = 0$.

- If, for example, $\mu_0 = \mu_1 = \mu_2$, there exist no OPS

Next, we want to quote some equivalents of Definition 1.2, which were also stated in [Chi78].

Theorem 1.3 ([Chi78, p. 8, Theorem 2.1]). *Let $\mathcal{L}$ be a moment functional and let $(P_n(x))_{n \in \mathbb{N}_0}$ be a sequence of polynomials. Then, the following are equivalent:*

(i) $(P_n(x))_{n \in \mathbb{N}_0}$ is an OPS wrt $\mathcal{L}$.

(ii) $\mathcal{L}[\pi(x)P_n(x)] = 0$ for every polynomial $\pi(x)$ of degree $m < n$, while $\mathcal{L}[\pi(x)P_n(x)] \neq 0$, if $m = n$.

(iii) $\mathcal{L}[x^m P_n(x)] = K_n \delta_{mn}$, where $K_n \neq 0$, $m = 0, 1, 2, ..., n$ and δ_{mn} denotes Kronecker's delta which is

$$\delta_{mn} = \begin{cases} 1, & \text{for } m = n, \\ 0, & \text{for } m \neq n. \end{cases} \tag{1.1}$$

Theorem 1.4 ([Chi78, p. 9, Theorem 2.2]). *Let $(P_n(x))_{n \in \mathbb{N}_0}$ be an OPS wrt $\mathcal{L}$. Then, for every polynomial $\pi(x)$ of degree n, we obtain*

$$\pi(x) = \sum_{k=0}^{n} c_k P_k(x),$$

where

$$c_k = \frac{\mathcal{L}[\pi(x)P_k(x)]}{\mathcal{L}[P_k^2(x)]}, \quad \text{for } k = 0, 1, 2..., n.$$

One important consequence of the equivalent definition is the following fact, cf. [Chi78, p. 9. f.]: If $(P_n(x))_{n \in \mathbb{N}_0}$ is an OPS wrt $\mathcal{L}$, then each $P_n(x)$ is uniquely determined up to an arbitrary non-zero factor, i.e. if $(Q_n(x))_{n \in \mathbb{N}_0}$ is also an OPS wrt $\mathcal{L}$, then there exist constants c_n, $n \in \mathbb{N}_0$, such that

$$Q_n(x) = c_n P_n(x), \quad \text{for all } n \in \mathbb{N}_0.$$

Hence, the corresponding OPS can uniquely be determined by assuming additional conditions. Let $\mathcal{L}$ be a moment functional such that an OPS exists. Then, for instance, one of the following additional assumptions uniquely determines an OPS:

- The system $(P_n(x))_{n \in \mathbb{N}_0}$ is normalized such that the leading coefficient of $P_n(x)$ equals 1 for each $n \in \mathbb{N}_0$. In this case, $(P_n(x))_{n \in \mathbb{N}_0}$ is called **monic**.

- Let $(c_n)_{n \in \mathbb{N}_0}$ be a sequence of arbitrary non-zero numbers. Assume that $P_n(x_0) = c_n$, $n \in \mathbb{N}_0$, where $x_0 \in \mathbb{C}$ is chosen such that $P_n(x_0) \neq 0$ for each $n \in \mathbb{N}_0$.

In this thesis we will mainly deal with real orthogonal polynomial systems which are orthogonal wrt a certain Borel measure μ. Moreover, the support of μ is assumed to be contained in $[-1, 1]$ and the corresponding OPS $(R_n(x))_{n \in \mathbb{N}_0}$ is uniquely determined by $R_n(1) = 1$. However, before considering the special case, when $\mathcal{L}$ is determined by a Borel measure, the question of the existence of an OPS will be treated. Therefore, we introduce the determinants

$$\Delta_n = \det(\mu_{i+j})_{i,j=0}^n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix},$$

cf. [Chi78, p. 11]. A moment functional $\mathcal{L}$ can be classified based on the properties of the so called moment determinants Δ_n.

Definition 1.5. *Let $\mathcal{L}$ be a moment functional.*

*(i) $\mathcal{L}$ is called **quasi-definite**, if $\Delta_n \neq 0$, for all $n \in \mathbb{N}_0$.*

*(ii) $\mathcal{L}$ is called **positive-definite**, if the moments μ_n are all real and $\Delta_n > 0$, for all $n \in \mathbb{N}_0$.*

*(iii) $\mathcal{L}$ is called **symmetric**, if $\mu_{2k+1} = 0$, for all $k \in \mathbb{N}_0$.*

Important results that involve these properties and the question of the existence of an OPS are the following:

Theorem 1.6 ([Chi78, p. 11, Theorem 3.1]). *Let $\mathcal{L}$ be a moment functional with moment sequence $(\mu_n)_{n \in \mathbb{N}_0}$. Then there exists an OPS wrt $\mathcal{L}$ if and only if, $\mathcal{L}$ is quasi-definite, i.e.*

$$\Delta_n \neq 0, \quad \text{for all } n \in \mathbb{N}_0.$$

Theorem 1.7 ([Chi78, p. 15 f., Theorem 3.3]). *Let $\mathcal{L}$ be positive-definite. Then, the moments of $\mathcal{L}$ are all real and a corresponding real OPS exists.*

An inverse assertion is also true. Let $(P_n(x))_{n\in\mathbb{N}_0}$ be an OPS wrt $\mathcal{L}$. If $(P_n(x))_{n\in\mathbb{N}_0}$ is real and $\mathcal{L}[P_n^2(x)] > 0$, for all $n \in \mathbb{N}_0$, then $\mathcal{L}$ is positive definite, see [Chi78, p. 16, Corollary]. The case, when $\mathcal{L}$ is positive definite is exactly the case, when $\mathcal{L}$ is defined by a Borel measure on $\mathbb{R}$.

1.1.2 Recurrence relation and Favard's Theorem

The first theorem in this subsection is a slightly modified version of the tree-term recurrence relation in [Chi78, p. 18, Theorem 4.1]. Different from Chihara, we use the term "relation" instead of "formula".

Theorem 1.8 (Recurrence relation, monic version). *Let $\mathcal{L}$ be a quasi-definite moment functional and let $(P_n(x))_{n\in\mathbb{N}_0}$ be the monic OPS wrt $\mathcal{L}$. Then, there exist sequences of complex constants $(\gamma_n)_{n\in\mathbb{N}_0}$ and $(\lambda_n)_{n\in\mathbb{N}_0}$, with $\lambda_n \neq 0$, such that $\mathcal{L}[1] = \lambda_0$ and*

$$P_0(x) = 1, \quad P_1(x) = x - \gamma_0,$$

$$P_{n+1}(x) = (x - \gamma_n)P_n(x) - \lambda_n P_{n-1}(x), \quad n \in \mathbb{N}. \tag{1.2}$$

Moreover, if $\mathcal{L}$ is positive definite, then γ_n is real and $\lambda_n > 0$, for all $n \in \mathbb{N}_0$.

Note, that the analogue recurrence relations for non-monic polynomials can easily be derived. Let $\mathcal{L}$ be a quasi-definite moment functional and let $(Q_n(x))_{n\in\mathbb{N}_0}$ be an arbitrary OPS wrt $\mathcal{L}$, where the leading coefficient of $Q_n(x)$ shall be denoted by $k_n \in \mathbb{C}\backslash\{0\}$, for all $n \in \mathbb{N}_0$. If we define $P_n(x)$ by $P_n(x) = \frac{Q_n(x)}{k_n}$, for all $n \in \mathbb{N}_0$, then $(P_n(x))_{n\in\mathbb{N}_0}$ is the corresponding monic OPS which satisfies a recurrence relation as in (1.2). Straightforward computation yields a recurrence relation for $(Q_n(x))_{n\in\mathbb{N}_0}$. The next theorem gives criteria, when the recurrence relation can be simplified.

Theorem 1.9 ([Chi78, p. 21, Theorem 4.3]). *Let $(P_n(x))_{n\in\mathbb{N}_0}$ be the monic OPS wrt a quasi-definite moment functional $\mathcal{L}$. Then, the following are equivalent:*

(i) $\mathcal{L}$ is symmetric.

(ii) $P_n(-x) = (-1)^n P_n(x)$, for all $n \in \mathbb{N}_0$.

(iii) In the corresponding recurrence relation (1.2), $\gamma_n = 0$, for all $n \in \mathbb{N}_0$.

The next well-known theorem deals with the problem of characterizing those sequences of polynomials which define an OPS. In [Chi78] this very important assertion was presented as the converse of Theorem 1.8.

Theorem 1.10 (Favard's Theorem)**.** *Let $(\gamma_n)_{n\in\mathbb{N}_0}$ and $(\lambda_n)_{n\in\mathbb{N}_0}$ be arbitrary sequences of complex numbers and let $(P_n(x))_{n\in\mathbb{N}_0}$ be defined by the recurrence relation (1.2). Then, there exists a moment functional such that*

$$\mathcal{L}[1] = \lambda_0 \quad and \quad \mathcal{L}[P_m(x)P_n(x)] = 0, \quad for\ all\ m \neq n,\ where\ m, n \in \mathbb{N}_0.$$

The following statements are true:

(1) *$\mathcal{L}$ is quasi-definite and $(P_n(x))_{n\in\mathbb{N}_0}$ is the corresponding monic OPS, if and only if $\lambda_n \neq 0$, for all $n \in \mathbb{N}_0$.*

(2) *$\mathcal{L}$ is positive-definite and $(P_n(x))_{n\in\mathbb{N}_0}$ is the corresponding real monic OPS, if and only if $\gamma_n \in \mathbb{R}$ and $\lambda_n > 0$, for all $n \in \mathbb{N}_0$.*

Finally, we want to recall the **Christoffel-Darboux Identity** which can be found in two different versions for monic polynomials in [Chi78, p. 23 f.].

Theorem 1.11 (Christoffel-Darboux Identity)**.** *Let $(P_n(x))_{n\in\mathbb{N}_0}$ satisfy (1.2) with $\lambda_n \neq 0$, for all $n \in \mathbb{N}_0$. Then,*

$$\sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{\lambda_0\lambda_1\cdots\lambda_k} = (\lambda_0\lambda_1\cdots\lambda_n)^{-1}\frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{x - y}. \tag{1.3}$$

Theorem 1.12 (Confluent form)**.** *Let $(P_n(x))_{n\in\mathbb{N}_0}$ satisfy (1.2) with $\lambda_n \neq 0$, for all $n \in \mathbb{N}_0$. Then,*

$$\sum_{k=0}^{n} \frac{P_k^2(x)}{\lambda_0\lambda_1\cdots\lambda_k} = \frac{P_{n+1}'(x)P_n(x) - P_n'(x)P_{n+1}(x)}{\lambda_0\lambda_1\cdots\lambda_n}. \tag{1.4}$$

The corresponding assertions for an OPS which is not monic can be deduced from the respective normalization.

1.1.3 Zeros

In this subsection, important results concerning the zeros of an OPS wrt a positive-definite moment functional will be recalled.

Definition 1.13 (Supporting set, [Chi78, p. 26, Definition 5.1])**.** *Let $E \subset (-\infty, \infty)$. A moment functional $\mathcal{L}$ is said to be positive-definite on E, if and only if $\mathcal{L}[\pi(x)] > 0$ for every real polynomial $\pi(x)$ which is non-negative on E and does not vanish identically on E. The set E is called **supporting set** for $\mathcal{L}$.*

In general, there is no smallest infinite supporting set. Moreover, positive-definiteness on any infinite supporting set implies positive definiteness, but the converse is not true

in general, cf. [Chi78, p. 26 f.]. The question, whether there exists a smallest closed supporting set, can be answered positively in the case, when $\mathcal{L}$ has a bounded supporting set. In general, the question of existence is coherent with the Hamburger moment problem, see [Chi78, p. 71 ff.]. The following theorem sums up two crucial statements for the zeros of orthogonal polynomial systems.

Theorem 1.14. *Let $\mathcal{L}$ be a positive-definite moment functional and $(P_n(x))_{n \in \mathbb{N}_0}$ the monic OPS wrt $\mathcal{L}$.*

(i) *Let I be an interval which is a supporting set for $\mathcal{L}$. Then the zeros of $P_n(x)$ are all real, simple and are contained in the interior of I.*

(ii) *Denote the zeros of $P_n(x)$ by x_{ni}, $i = 1, 2, ..., n$, where*

$$x_{n,1} < x_{n,2} < ... < x_{n,n}, \quad \text{for all } n \in \mathbb{N}_0.$$

Then, the zeros of $P_n(x)$ and $P_{n+1}(x)$ mutually separate each other, i.e.

$$x_{n+1,i} < x_{n,i} < x_{n+1,i+1}, \quad \text{for all } i = 1, 2, ..., n \text{ and } n \in \mathbb{N}.$$

In particular, the limits

$$\xi_i := \lim_{n \to \infty} x_{n,i} \quad \text{and} \quad \eta_j := \lim_{n \to \infty} x_{n,n-j+1}$$

exist for all $i, j \in \mathbb{N}$ in the extended real number line $\mathbb{R} \cup \{-\infty, \infty\}$.

The assertions were proved in [Chi78, p. 27 ff.] and give rise to the following definition:

Definition 1.15 (True interval of orthogonality, [Chi78, p. 27, Definition 5.2])**.** *The closed interval $[\xi_1, \eta_1]$ is called the **true interval of orthogonality**.*

Following [Chi78, p. 35, Exercise 6.4], the set of all zeros is a supporting set for $\mathcal{L}$. Since all zeros are contained in any interval which is a supporting set for $\mathcal{L}$, the true interval of orthogonality is the smallest interval which is a supporting set for $\mathcal{L}$. In the main part of the thesis, we will basically deal with orthogonal polynomial systems, whose zeros are contained in the interval $[-1, 1]$.

1.1.4 Orthogonality wrt Borel measures and representation theorem

During this subsection, it will primarily be referred to [Chi78, Chapter II]. As remarked before, in the principle part of the thesis, we shall exclusively deal with sequences of real polynomials which are orthogonal wrt a Borel measure with compact support, i.e. wrt a positive-definite moment functional.

Definition 1.16. *Let μ be a Borel measure on $\mathbb{R}$, such that $\int_{\mathbb{R}} x^n \, \mathrm{d}\mu(x)$ exists and is finite for each $n \in \mathbb{N}_0$, then we can define a moment functional $\mathcal{L} = \mathcal{L}_\mu$ by*

$$\mathcal{L}_\mu[x^n] = \int_{\mathbb{R}} x^n \, \mathrm{d}\mu(x), \quad \text{for all } n \in \mathbb{N}_0.$$

*We call μ **representation measure** for $\mathcal{L}_\mu$. If there exists an OPS $(P_n(x))_{n \in \mathbb{N}_0}$ wrt $\mathcal{L}_\mu$, then μ is called an **orthogonalization measure** and $(P_n(x))_{n \in \mathbb{N}_0}$ is referred to as **OPS wrt** μ.*

Define the function ψ by

$$\psi(x) = \mu((-\infty, x]), \quad \text{for all } x \in \mathbb{R},$$

then ψ is non-decreasing and bounded, i.e. a distribution function in the sense of [Chi78, p. 51, Definition 1.1]. ψ is called representative of $\mathcal{L}$. Moreover, the function ψ in our definition is uniquely determined among substantially equal representatives in the sense of Chihara (see [Chi78, p. 52, Definition 1.2]), since ψ satisfies $\lim_{x \to \infty} \psi(x) = 0$. By $\mathcal{S}$, we denote the support of μ which is given by

$$\mathcal{S} = \mathsf{supp}\,\mu = \{x \in \mathbb{R} : \psi(x + \delta) - \psi(x - \delta) > 0, \text{ for all } \delta > 0\} \tag{1.5}$$

In [Chi78, p. 51] the set in (1.5) was called spectrum of ψ. Note that beforehand, Chihara introduced the concept of bounded variations, which is more general than that of finite Borel measures. In this general sense, the corresponding moment functional could be non-quasi-definite or quasi-definite and non-positive-definite, cf. [Chi78, p. 51]. However, it was stated that for a bounded Borel measure μ, the corresponding moment functional $\mathcal{L}_\mu$ is positive-definite with supporting set $\mathcal{S}$, if the set $\mathcal{S}$ is infinite, we write $\mathsf{card}\,\mathcal{S} = \infty$. In particular, an OPS wrt $\mathcal{L}_\mu$ exists if $\mathsf{card}\,\mathcal{S} = \infty$. In order to verify the converse, Chihara utilized Helly's Selection Principle and Helly's second theorem. The assertion was finally proved in [Chi78, p. 57 f., Theorem 3.1].

Theorem 1.17 (representation theorem)**.** *A moment functional $\mathcal{L}$ is positive-definite, if and only if there exists some finite Borel measure μ on $\mathbb{R}$ with $\mathsf{card}\,\mathcal{S} = \infty$ and $\mathcal{L} = \mathcal{L}_\mu$.*

The question arises, when the orthogonalization measure μ is uniquely determined.

Definition 1.18 ([Chi78, p. 58, Definition 3.1])**.** *A positive-definite moment functional $\mathcal{L}$ is **determinate**, if any two representatives of $\mathcal{L}$ are substantially equal (in the sense of Chihara). Otherwise, $\mathcal{L}$ is called indeterminate.*

As remarked before, we consider among substantially equal representatives the one which vanishes at $-\infty$. Hence, determinate means that there exists exactly one Borel (probability) measure μ on $\mathbb{R}$ with $\mathsf{card}\,\mathcal{S} = \infty$ and $\mathcal{L} = \mathcal{L}_\mu$. Finally, we obtain the following important result:

Theorem 1.19 ([Chi78, p. 67, Theorem 5.6]). *Let $\mathcal{L}$ be a positive-definite moment functional and denote by $[\xi_1, \eta_1]$ the corresponding true interval of orthogonality. Then, $\mathcal{L}$ is determinate, if $[\xi_1, \eta_1]$ is bounded.*

The proof to this theorem involves the fact that $\mathcal{S} \subset [\xi_1, \eta_1]$ for each representation measure μ for $\mathcal{L}$. The converse however is not true. Consider the Hermite polynomials as an example.

1.2 Operator and spectral theory

In this section T always denotes a bounded linear operator in a complex Hilbert space H. Notations and basic theorems in operator theory and, in particular, spectral theory shall be recalled. Moreover, we will focus on the property of normality and also more general normality concepts. This section will mainly follow [Wer07]. Readers not familiar with the basic properties of a bounded linear operator and its dual operator in Hilbert spaces find a nice introduction in [Wer07, Chapter I-V].

1.2.1 Normal operators

Let H be a complex Hilbert space with scalar product $\langle \, , \, \rangle_H$, which induces the norm $\| \, . \, \|_H$. The algebra of bounded linear operators in H is referred to as $B(H)$. For an operator $T \in B(H)$ the dual, or adjoint, operator T^* is defined by

$$\langle Tx, y \rangle_H = \langle x, T^*y \rangle_H, \quad \text{for all } x, y \in H.$$

By $\mathsf{ran}\, T$ and $\mathsf{ker}\, T$, the kernel and range of the operator T, respectively, are denoted, where

$$\mathsf{ran}\, T = \{Tx : \ x \in H\} \quad \text{and} \quad \mathsf{ker}\, T = \{x \in H : \ Tx = 0\}.$$

As usual,

$$\|T\| = \sup_{x \in H, \, \|x\|_H = 1} \|Tx\|_H$$

is the operator norm of T. We want to recall the definition of normality, see also [Wer07, p. 237, Definition V.5.3].

Definition 1.20. *Let H be a Hilbert space and let $T \in B(H)$. The operator T is called* **normal**, *if T commutes with its dual T^*, symbolically,*

$$TT^* = T^*T.$$

The normality condition in Definition 1.1 is equivalent to

$$\|Tx\|_H = \|T^*x\|_H, \quad \text{for all } x \in H,$$

see [Wer07, p. 241]. Obviously, if T is a normal operator, polynomials in T are normal operators.

In addition to the definition above, we want to give the definition of some well-known classes of normal operators, which were also classified in [Wer07, p. 237 ff.].

Definition 1.21. *Let H be a Hilbert space and let $T \in B(H)$. T is called*

(i) **self-adjoint**, *if $T = T^*$,*

(ii) **unitary**, *if $TT^* = \mathsf{id}_H = T^*T$, where id_H denotes the identity in H,*

(iii) **orthogonal projection**, *if $T^2 = T$ and $T = T^*$,*

(iv) **positive**, *if $\langle Tx, x \rangle_H \geq 0$, for all $x \in B(H)$.*

1.2.2 Spectrum and numerical range

In this subsection some important results in spectral theory will be cited. We follow the definitions in [Wer07, Chapter VI], but focus on bounded operators in Hilbert spaces.

Definition 1.22. *Let H be a Hilbert space and $T \in B(H)$.*

(i) *The* **resolvent set** *of T is*

$$\rho(T) = \{\lambda \in \mathbb{C} : \text{ the operator } (\lambda\mathsf{id}_H - T)^{-1} \text{ exists in } B(H)\}.$$

(ii) *The function*

$$R : \rho(T) \to B(H), \ R_\lambda := R_\lambda(T) := (\lambda\mathsf{id}_H - T)^{-1}$$

is called **resolvent function**.

(iii) *The* **spectrum of** *T is*

$$\sigma(T) = \mathbb{C} \backslash \rho(T).$$

Moreover, the **point spectrum** $\sigma_p(T)$, the **continuous spectrum** $\sigma_c(T)$ and the **residual spectrum** $\sigma_r(T)$ are defined as follows:

- $\sigma_p(T) = \{\lambda \in \mathbb{C} : (\lambda \mathrm{id}_H - T) \text{ is not injective}\}$,

- $\sigma_c(T) = \{\lambda \in \mathbb{C} : (\lambda \mathrm{id}_H - T) \text{ is injective, not surjective, and has dense range}\}$,

- $\sigma_r(T) = \{\lambda \in \mathbb{C} : (\lambda \mathrm{id}_H - T) \text{ is injective, not surjective, without dense range}\}$.

By the theorem of the continuity of the inverse, we obtain

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T),$$

see [Wer07, p. 256]. The elements of $\sigma_p(T)$ are called **eigenvalues** or proper values of T. A non-zero vector $x \in H$, satisfying $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$, is called **eigenvector** or proper vector of T. We also want to introduce the **approximate spectrum**

$$\pi(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is an approximate eigenvalue of } T\},$$

where $\lambda \in \mathbb{C}$ is called **approximate eigenvalue**, if there exists a sequence $(x_n)_{n \in \mathbb{N}_0} \subset H$, such that

$$\|x_n\|_H = 1, \ n \in \mathbb{N}_0, \quad \text{and} \quad Tx_n - \lambda x_n \to 0, \quad \text{for} \quad n \to \infty.$$

The following properties are well-known, see [Wer07, p. 257 f.].

Theorem 1.23. *Let H be a complex Hilbert space and $T \in B(H)$.*

(i) The spectrum $\sigma(T)$ is compact, in particular,

$$|\lambda| \leq \|T\|, \quad \text{for all } \lambda \in \sigma(T).$$

(ii) The spectrum of T satisfies $\sigma(T) \neq \emptyset$.

Theorem 1.24 ([Wer07, p. 256, Theorem VI.1.2])**.** *Let $T \in B(H)$ and T^* its dual operator, then*

$$\sigma(T^*) = \{\overline{\lambda} : \lambda \in \sigma(T)\}.$$

The spectral radius and its basic properties shall also be introduced here.

Definition 1.25 ([Wer07, p. 259, Definition VI.1.5])**.** *Let $T \in B(H)$. We define the* ***spectral radius*** *of T by*

$$r(T) := \inf_{n \in \mathbb{N}_0} \|T^n\|^{\frac{1}{n}} = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}}.$$

Theorem 1.26. *Let H be a complex Hilbert space and $T \in B(H)$, then the following assertions are true:*

(i) *The spectral radius satisfies*

$$|\lambda| \leq r(T), \quad \text{for all } \lambda \in \sigma(T).$$

(ii) *There exists $\lambda \in \sigma(T)$, satisfying $|\lambda| = r(T)$, which implies that*

$$r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}.$$

The following result is a well-known property for normal operators:

Theorem 1.27 ([Wer07, p. 270, Theorem VI.1.7]). *Let H be a Hilbert space and $T \in B(H)$. If T is normal, then*

$$r(T) = \|T\|.$$

Another related term is the numerical range of an operator.

Definition 1.28. *Let H be a Hilbert space and $T \in B(H)$. The **numerical range** of T is defined by*

$$W(T) = \{\langle Tx, x \rangle_H : \|x\|_H = 1\}.$$

Furthermore, the numerical radius is defined by

$$w(T) = \sup\{|z| : z \in W(T)\}.$$

The Cauchy-Schwarz inequality shows that $W(T)$ is bounded and therefore, we can infer that $\overline{W(T)}$ is compact. The following Lemma relates the spectrum and numerical range of an operator.

Lemma 1.29. *Let $T \in B(H)$. $W(T)$ is convex and we have $\sigma(T) \subset \overline{W(T)}$, or, equivalently, co $\sigma(T) \subset \overline{W(T)}$, where co denotes the convex hull.*

The following result is well-known, for instance in [Hal82, p. 161 f., 218].

Lemma 1.30. *Let $T \in B(H)$. Then, the following inequality holds:*

$$r(T) \leq w(T) \leq \|T\|.$$

In [Wer07, Chapter VI and VII], detailed deductions for important spectral theorems can be found. Initially, the spectral theory of compact operators was studied. In [Wer07, p. 269 ff., Theorem VI.3.2 and Corollary VI.3.3] we obtain the spectral theorem for compact normal operator in two different versions. In the general case of bounded linear operators the spectral measure was introduced in [Wer07, p. 327, Definition VII.1.9]. Finally, the spectral theorem for self-adjoint bounded linear operators was proved. The theorem of the polar decomposition of bounded linear operators exists in a compact ([Wer07, p. 273, Theorem VI.3.5]) and a more general, bounded ([Wer07, p. 334, Corollary VII.1.17]) version. This theorem is a crucial tool in spectral theory. An arbitrary bounded linear operator is related to a self-adjoint operator and properties of the spectrum can be derived from the spectrum of the respective self-adjoint operator.

1.2.3 Compact operators

In this subsection, the concept of compact operators will briefly be introduced. The property of compactness yields some interesting results for the spectrum of an operator.

Definition 1.31 (Compact operator)**.** *A bounded linear operator T in a Hilbert space H is called **compact**, symbolically $T \in K(H)$, if T maps each bounded subset of H to a relatively compact subset of H.*

An important property of this class is that if $T \in K(H)$, then $T^* \in K(H)$ and vice versa. A detailed introduction to the theory of compact operators can be found in [Wer07, p. 65 ff.]. Concerning the spectrum of a compact operator, the following, which was proved in [Wer07, p. 267 f., Theorem VI.2.5] for Banach spaces in general, can be inferred:

Theorem 1.32. *Let $T \in K(H)$.*

(i) If the dimension of H is infinite, dim $H = \infty$, it follows that $0 \in \sigma(T)$.

(ii) The set $\sigma(T)\backslash\{0\}$ is either empty, or finite, or countably infinite.

(iii) Each value $\lambda \in \sigma(T)\backslash\{0\}$ is an eigenvalue of T and the corresponding eigenspace is finite-dimensional.

(iv) If $\sigma(T)$ is infinite, then 0 is the only limit point of $\sigma(T)$.

Finally, two subclasses of compact operators, which were described in [Wer07, p. 284 ff.], shall be introduced.

Definition 1.33 (Nuclear operator)**.** *Let $T \in B(H)$. The operator T is called **nuclear**, symbolically $T \in N(H)$, if ther exist sequences $(x_n)_{n\in\mathbb{N}_0}, (y_n)_{n\in\mathbb{N}_0} \subset H$ with*

$$\sum_{n=0}^{\infty} \|x_n\|_H \|y_n\|_H < \infty,$$

such that

$$Tx = \sum_{n=0}^{\infty} \langle x, x_n \rangle_H y_n, \quad \text{for all } n \in \mathbb{N}_0. \tag{1.6}$$

The representation of T in (1.6) is called a nuclear representation.

Definition 1.34 (Hilbert-Schmidt operator). *Let $T \in B(H)$. The operator T is called* **Hilbert-Schmidt operator**, *symbolically $T \in HS(H)$, if there exists an orthonormal basis $(g_n)_{n \in \mathbb{N}_0} \subset H$ of H for which*

$$\sum_{n=0}^{\infty} \|Tg_n\|_H^2 < \infty.$$

The equivalence of our definition to [Wer07, p. 296, Definition VI.6.1] was proved in [Wer07, p. 296, Theorem VI.6.2]. It is a well-known fact that the operator classes satisfy the inclusion

$$N(H) \subset HS(H) \subset K(H).$$

Furthermore, if $T \in N(H)$ and $T \in HS(H)$, respectively, it follows that $T^* \in N(H)$ and $T^* \in HS(H)$, respectively.

2 The conjugate orthogonal polynomial system

In this chapter orthogonal polynomial systems and their conjugate system will be considered. First, we want to recall some further facts about orthogonal polynomials, see Theorem 1.8, [Chi78, pp. 18 ff.] and [LOR07].

Definition 2.1. *Let μ be a probability measure on the real line. Denote the support of μ,* supp *μ, by $\mathcal{S}$ and assume* card *$\mathcal{S} = \infty$. Let the sequence $(R_n(x))_{n \in \mathbb{N}_0}$ denote the **orthogonal polynomial system (OPS)** with respect to μ which is normalized such that $R_n(1) = 1$, for all $n \in \mathbb{N}_0$. Then, $(R_n(x))_{n \in \mathbb{N}_0}$ satisfies a three-term recurrence relation*

$$R_0(x) = 1, \quad R_1 = (x - b_0)/a_0,$$

$$xR_n(x) = a_n R_{n+1}(x) + b_n R_n(x) + c_n R_{n-1}(x), \quad n \in \mathbb{N}. \tag{2.1}$$

The coefficients are real numbers with $c_n a_{n-1} > 0$, $n \in \mathbb{N}$. Conversely, if we define $(R_n(x))_{n \in \mathbb{N}_0}$ by (2.1), there is a measure μ with the assumed properties.

The focus is on those orthogonal polynomial systems which additionally have the property that $a_0 + b_0 = 1$ and $a_n + b_n + c_n = 1$, $n \in \mathbb{N}$. Then, the Haar measure h satisfies

$$h(n)^{-1} = \int_{\mathcal{S}} R_n^2(x) \, \mathrm{d}\mu(x), \quad n \in \mathbb{N}_0, \tag{2.2}$$

and

$$h(n+1) = \frac{a_n}{c_{n+1}} h(n), \quad \text{for} \quad n \in \mathbb{N}_0,$$

see [FLS04] or [LOR07], where this type of recurrence relation was also used. Furthermore, we focus on those sequences of orthogonal polynomials, where the support of the orthogonalization measure is contained in the interval $[-1, 1]$. Following this assumption, it is obvious that the defined class of orthogonal polynomial systems includes the class of random walk polynomials, which were detailedly discussed in [vDS93] and [CSvD98], respectively. If x_0 is a zero of $R_n(x)$ for some $n \in \mathbb{N}_0$, it can be inferred that $x_0 \in (-1, 1)$, see Theorem 1.14 and [Chi78, p. 27]. For an OPS as defined above we define the conjugate orthogonal polynomial system which is related to the original system as follows:

Definition 2.2 (Conjugate orthogonal polynomial system)**.** *Let μ be a probability measure on $[-1, 1]$. Denote the support of μ by $\mathcal{S}$ and assume* card $\mathcal{S} = \infty$. *Let $(R_n(x))_{n \in \mathbb{N}_0}$ denote the orthogonal polynomial system with respect to μ which is normalized such that $R_n(1) = 1$, for all $n \in \mathbb{N}_0$. Then the **conjugate orthogonal polynomial system (COPS)** $(R_n^*(x))_{n \in \mathbb{N}_0}$ is defined as the sequence which is orthogonal with respect to the probability measure μ^* with $\mathrm{d}\mu^* = c(1 - x^2)\,\mathrm{d}\mu$, $c \in \mathbb{R}$.*

Now the aim is to deduce a representation of the COPS with respect to the original OPS. Moreover, we will determine the corresponding linearization coefficients (the coefficients in the three-term recurrence relation) and the Haar measure. In a short introduction, we will take up the topic of conjugate functions that goes back to Muckenhoupt and Stein. After that, we will introduce two related OPS that will be useful for the calculation of the COPS in the last section.

2.1 The conjugate series for Jacobi polynomials

In [MS65], Muckenhoupt and Stein investigated classical polynomial expansions in analogy to ordinary Fourier series. In particular, they defined conjugacy for ultraspherical expansions. The definition of conjugacy relies on the relation of the Poisson integral and the conjugate Poisson integral by suitable Cauchy Riemann equations. For $\lambda > -\frac{1}{2}$ let $(P_n^\lambda(x))_{n \in \mathbb{N}_0}$ denote the ultraspherical (or Gegenbauer) polynomials, normalized as in [Chi78, p. 144]. Muckenhoupt and Stein in particular considered the system $(P_n^\lambda(\cos \theta))_{n \in \mathbb{N}_0}$ which is orthogonal and complete over $(0, \pi)$ with respect to the measure m_λ with $\mathrm{d}m_\lambda(\theta) = (\sin \theta)^{2\lambda}\,\mathrm{d}\theta$, see [MS65]. Precisely, to a series

$$f(\theta) \sim \sum_{k=0}^{\infty} a_k P_k^\lambda(\cos \theta),$$

they associated the conjugate series

$$\widetilde{f}(\theta) \sim 2\lambda \sum_{k=1}^{\infty} \frac{a_k}{k + 2\lambda} \sin \theta P_{k-1}^{\lambda+1}(\cos \theta).$$

The mapping $f(\theta) \to \widetilde{f}(\theta)$ is a generalized Hilbert transform and a bounded operator in L^p. Muckenhoupt and Stein defined a similar notation of conjugacy for Hankel transforms and for Fourier-Bessel series, see [MS65]. Later on, Muckenhoupt developed a conjugate function theory for Hermite ([Muc69]) and Laguerre ([Muc70]) expansions which involved a general principle for the definition of conjugacy, given by Stein in [Ste70]. Based on this theory, Gosselin and Stempak developed a conjugacy theory for expansions with respect to the system of corresponding orthonormal functions, in particular, Hermite functions and ultrashperical functions, see [Ste93]. In his two papers, [Li96] and [Li97], Li generalized the

concept of conjugacy for ultraspherical (polynomial) expansions and introduced conjugate Jacobi series. For $\alpha, \beta > 1$ let $(P_n^{(\alpha,\beta)}(x))_{n \in \mathbb{N}_0}$ be the Jacobi Polynomials, normalized, such that

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n},$$

for all $n \in \mathbb{N}_0$, see [Chi78, p. 144]. The relation between the ultraspherical polynomials and the Jacobi polynomials is

$$P_n^\lambda(x) = \binom{2\alpha}{\alpha}^{-1} \binom{n+2\alpha}{\alpha} P_n^{(\alpha,\alpha)}(x),$$

where $\alpha = \lambda - \frac{1}{2} \neq -\frac{1}{2}$, see [Chi78, p. 144]. For $\alpha = -\frac{1}{2}$, one obtains the Tschebichef polynomials of the first kind, see [Chi78, p. 143]. Precisely, Li considered the system $(P_n^{(\alpha,\beta)}(\cos\theta))_{n \in \mathbb{N}_0}$. These polynomials are orthogonal and complete over $(0, \pi)$ with respect to the measure $m^{(\alpha,\beta)}$ with $\mathrm{d}m^{(\alpha,\beta)}(\theta) = 2^{\alpha+\beta+1} \sin^{2\alpha+1}(\theta/2) \cos^{2\beta+1}(\theta/2) \,\mathrm{d}\theta$, see [Li96]. Li defined

$$(R_n^{(\alpha,\beta)}(\cos\theta))_{n \in \mathbb{N}_0} := \left(\frac{P_n^{(\alpha,\beta)}(\cos\theta)}{P_n^{(\alpha,\beta)}(1)} \right)_{n \in \mathbb{N}_0}.$$

For $1 \leq p < \infty$, let $L^p(m^{(\alpha,\beta)})$ be the space of all functions f for which

$$\int_0^\pi |f(\theta)|^p \,\mathrm{d}m^{(\alpha,\beta)}(\theta) > \infty.$$

For $f \in L^1(m^{(\alpha,\beta)})$, the Jacobi series was defined as

$$f(\theta) \sim \sum_{k=0}^\infty \hat{f}(k)\omega_k^{(\alpha,\beta)} R_k^{(\alpha,\beta)}(\cos\theta),$$

where

$$\hat{f}(k) = \int_0^\pi f(\varphi)R_k^{(\alpha,\beta)}(\cos\varphi) \,\mathrm{d}m^{(\alpha,\beta)}(\varphi),$$

$$\omega_k^{(\alpha,\beta)} = \left[\int_0^\pi (R_k^{(\alpha,\beta)}(\cos\varphi))^2 \,\mathrm{d}m^{(\alpha,\beta)}(\varphi) \right]^{-1}.$$

Considering the associated harmonic function (Poisson Integral) and defining the conjugate harmonic function (conjugate Poisson Integral), leads to a generalized Hilbert transform $f \to \tilde{f}$ with

$$\tilde{f}(\theta) \sim \sum_{k=1}^{\infty} \frac{k}{2\alpha+2} \hat{f}(k) \omega_k^{(\alpha,\beta)} R_{k-1}^{(\alpha+1,\beta+1)}(\cos\theta) \sin\theta, \qquad (2.3)$$

or, equivalently,

$$\tilde{f}(\theta) \sim \sum_{k=1}^{\infty} \frac{2\alpha+2}{k+\alpha+\beta+1} \hat{f}(k) \omega_{k-1}^{(\alpha+1,\beta+1)} R_{k-1}^{(\alpha+1,\beta+1)}(\cos\theta) \sin\theta,$$

see [Li96] and [Li97]. Moreover, Li proved the L^1 weak-boundedness and the L^p boundedness, for $1 < p < \infty$, of the conjugacy mapping and considered Abel and Cesàro means of the conjugate Jacobi series.

According to Definition 2.1, we rewrite (2.3) for normalized Jacobi polynomials $(R_n^{(\alpha,\beta)}(x))_{n\in\mathbb{N}_0}$ which are orthonormal and complete over $(-1,1)$ with respect to the corresponding probability measure $\mu^{(\alpha,\beta)}$. Definition 2.2 tells us that the corresponding COPS is given by $(R_n^{(\alpha+1,\beta+1)}(x))_{n\in\mathbb{N}_0}$. In particular, for $\mu^{(\alpha,\beta)}$ one obtains

$$\mathrm{d}\mu^{(\alpha,\beta)}(x) = 2^{-\alpha-\beta-1} \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\beta+1)\Gamma(\alpha+1)} (1-x)^\alpha (1+x)^\beta \,\mathrm{d}x \qquad (2.4)$$

which can be inferred, for example, from [Chi78, p. 148]. Furthermore, the Haar measure $h^{(\alpha,\beta)} = (h^{(\alpha,\beta)}(n))_{n\in\mathbb{N}_0}$ of the Jacobi Polynomials is given by

$$h^{(\alpha,\beta)}(k) = \left[\int_{-1}^{1} (R_k^{(\alpha,\beta)}(x))^2 \,\mathrm{d}\mu^{(\alpha,\beta)}(x) \right]^{-1}$$

$$= \frac{(\alpha+1)_k (\alpha+\beta+1)_k (2k+\alpha+\beta+1)}{(\beta+1)_k k! (\alpha+\beta+1)}, \qquad (2.5)$$

for $k \in \mathbb{N}_0$, where we denote by $(a)_n$ the Pochhammer symbol for $a \in \mathbb{R}$ and $n \in \mathbb{N}_0$ which is

$$(a)_n = \begin{cases} 1, & \text{for} \quad n = 0, \\ a(a+1)\dots(a+n-1), & \text{for} \quad n \in \mathbb{N}. \end{cases}$$

In the following, the relation

$$\frac{h^{(\alpha,\beta)}(k)}{h^{(\alpha+1,\beta+1)}(k-1)} = \frac{(\alpha+\beta+3)(\alpha+\beta+2)(\alpha+1)}{(k+\alpha+\beta+1)(\beta+1)k}, \qquad (2.6)$$

is utilized for $k \in \mathbb{N}_0$. Let $f \in L^1(\mu^{(\alpha,\beta)})$ have the representation

$$f(x) \sim \sum_{k=0}^{\infty} h^{(\alpha,\beta)}(k)a(k)R_k^{(\alpha,\beta)}(x),$$

where

$$a(k) = \int_{-1}^{1} f(x)R_k^{(\alpha,\beta)}(x) \, d\mu^{(\alpha,\beta)}(x).$$

By using (2.3), (2.5) and (2.6) the conjugate Jacobi series $\tilde{f}$ reads

$$\tilde{f}(x) \sim \sum_{k=1}^{\infty} h^{(\alpha,\beta)}(k)\frac{ka(k)}{2\alpha+2}\sqrt{1-x^2}R_{k-1}^{(\alpha+1,\beta+1)}(x),$$

or, equivalently,

$$\tilde{f}(x) \sim \sum_{k=1}^{\infty} h^{(\alpha+1,\beta+1)}(k-1)\frac{(\alpha+\beta+3)(\alpha+\beta+2)a(k)}{2(k+\alpha+\beta+1)(\beta+1)}\sqrt{1-x^2}R_{k-1}^{(\alpha+1,\beta+1)}(x). \qquad (2.7)$$

The system $(\sqrt{1-x^2}R_n^{(\alpha+1,\beta+1)}(x))_{n\in\mathbb{N}_0}$ is orthogonal wrt the measure $\mu^{(\alpha,\beta)}$ and, taking (2.4) into account, satisfies the relation

$$\int_{-1}^{1}(\sqrt{1-x^2}R_n^{(\alpha+1,\beta+1)}(x))^2 \, d\mu^{(\alpha,\beta)}(x)$$

$$= \frac{4(\alpha+1)(\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)}\int_{-1}^{1}(R_n^{(\alpha+1,\beta+1)}(x))^2 \, d\mu^{(\alpha+1,\beta+1)}(x).$$

The definition of the conjugate series in the sense of Muckenhoupt and Stein relates the OPS $(R_n^{(\alpha,\beta)}(x))_{n\in\mathbb{N}_0}$ to its COPS $(R_n^{(\alpha+1,\beta+1)}(x))_{n\in\mathbb{N}_0}$. In particular, let $f \in L^2(\mu^{(\alpha,\beta)})$. Then, the $L^2(\mu^{(\alpha,\beta)})$ norm of the conjugate series in (2.8) coincides up to the constant

$\frac{4(\alpha+1)(\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)}$ with the $L^2(\mu^{(\alpha+1,\beta+1)})$ norm of the series

$$f^*(x) \sim \sum_{k=1}^{\infty} h^{(\alpha+1,\beta+1)}(k-1)\frac{(\alpha+\beta+3)(\alpha+\beta+2)a(k)}{2(k+\alpha+\beta+1)(\beta+1)}R_{k-1}^{(\alpha+1,\beta+1)}(x). \qquad (2.8)$$

From the boundedness of the conjugacy mapping for $1 \leq p < \infty$, proved in [Li96] by considering the definition of the conjugate function in integral form, it can be inferred that the series f^* in (2.8) is an element of $L^2(\mu^{(\alpha,\beta)})$. In view of the conjugate series, Stein introduced a general principle for the definition of conjugacy with respect to polynomial expansions in [Ste70] which was applied by Muckenhoupt to define conjugacy for Hermite ([Muc69]) and Laguerre ([Muc70]) expansions. We are interested in the COPS instead which can be determined for each OPS that is orthogonal wrt a measure in (-1,1). In the following, we will in general deduce the corresponding COPS and discuss its properties.

2.2 Two related orthogonal systems

In the following, the COPS and two other related OPS will be determined by direct calculation. Since $d\mu^*(x)$ is up to a constant equal to

$$(1 - x^2)\,d\mu(x) = (1 - x)(1 + x)\,d\mu(x),$$

an approach of the calculation of the conjugate system is to determine the sequences orthogonal with respect to μ^- and μ^+ which are up to a constant given by

$$d\mu^-(x) \sim (1 - x)\,d\mu(x) \quad \text{and}$$
$$d\mu^+(x) \sim (1 + x)\,d\mu(x),$$

respectively. We will denote them by $(R_n^-(x))_{n\in\mathbb{N}_0}$ and $(R_n^+(x))_{n\in\mathbb{N}_0}$, respectively. A useful tool to deduce the related sequences of orthogonal polynomials is the Christoffel-Darboux Identity, see Theorem 1.11 and [Chi78, p. 23]. For an OPS as defined in (2.1), we obtain

$$\sum_{k=0}^{n} h(k)R_n(x)R_n(y) = a_n h(n)\frac{R_n(x)R_{n+1}(y) - R_{n+1}(x)R_n(y)}{y - x}, \qquad (2.9)$$

where $n \in \mathbb{N}_0, x, y \in \mathbb{R}$ and $x \neq y$. Substituting y in (2.9) by 1 and -1, respectively, yields the corresponding terms in the denominator.

2.2.1 Orthogonal polynomial system with respect to μ^-

Let $(R_n(x))_{n\in\mathbb{N}_0}$ be an OPS as described in Definition 2.1. Define $(\widetilde{R}_n^-(x))_{n\in\mathbb{N}_0}$ by

$$\widetilde{R}_n^-(x) = \sum_{k=0}^n h(k) R_k(x) = a_n h(n) \frac{R_n(x) - R_{n+1}(x)}{1 - x},$$

for all $n \in \mathbb{N}_0$, where we used (2.9) for the second equality. Denote by $(R_n^-(x))_{n\in\mathbb{N}_0}$ the sequence $\left(\frac{\widetilde{R}_n^-(x)}{\widetilde{R}_n^-(1)}\right)_{n\in\mathbb{N}_0}$ which is in particular given by

$$R_n^-(x) = \frac{1}{H(n)} \sum_{k=0}^n h(k) R_k(x) = \frac{a_n h(n)}{H(n)} \frac{R_n(x) - R_{n+1}(x)}{1 - x}, \tag{2.10}$$

for all $n \in \mathbb{N}_0$, where $H(n) := \sum_{k=0}^n h(k)$, $H(-1) := 0$.

Proposition 2.3. *The sequence $(R_n^-(x))_{n\in\mathbb{N}_0}$ of polynomials is orthogonal and complete over $(-1, 1)$ with respect to the probability measure μ^- with*

$$\mathrm{d}\mu^- = \frac{1-x}{1-b_0}\, \mathrm{d}\mu.$$

The Haar measure h^- is given by

$$h^-(n) = \frac{(1-b_0)H(n)^2}{a_n h(n)},$$

for all $n \in \mathbb{N}_0$.

    Proof: Observe first that $R_n^-(x)$ is a polynomial of degree n. Furthermore, if $\mathsf{supp}\,\mu \subset [-1, 1]$, then $\mathsf{supp}\,\mu^- \subset [-1, 1]$. In particular, μ^- is well-defined, since the fact that the zero of $R_1(x)$ is contained in (-1,1) implies $b_0 \in (-1, 1)$. The measure μ^- is a probability measure, since

$$\int_{\mathbb{R}} \mathrm{d}\mu^-(x) = \frac{1}{1-b_0} \int_{-1}^1 R_0(x) R_0(x)(1-x)\, \mathrm{d}\mu(x)$$

$$= \frac{1}{1-b_0} \int_{-1}^1 R_0(x)((1-b_0)R_0(x) - a_0 R_1(x))\, \mathrm{d}\mu(x) = 1.$$

Now, let $n \in \mathbb{N}_0$ and $q(x)$ be a polynomial with $\deg q = m < n$. Then, we have

$$\int_{-1}^{1} R_n^-(x)q(x) \, \mathrm{d}\mu^-(x) = \frac{a_n h(n)}{(1-b_0)H(n)} \int_{-1}^{1} (R_n(x) - R_{n+1}(x))q(x) \, \mathrm{d}\mu(x) = 0, \quad (2.11)$$

since $(R_n(x))_{n \in \mathbb{N}_0}$ is an OPS wrt μ. From (2.11), it follows that

$$\int_{-1}^{1} R_n^-(x)R_m^-(x) \, \mathrm{d}\mu^-(x) = 0, \qquad \text{for all } m, n \in \mathbb{N}_0, \, m \neq n.$$

Finally, the Haar measure h^- of the sequence $(R_n^-(x))_{n \in \mathbb{N}_0}$ is given by

$$[h^-(n)]^{-1} = \int_{-1}^{1} (R_n^-(x))^2 \, \mathrm{d}\mu^-(x)$$

$$= \frac{a_n h(n)}{(1-b_0)H(n)} \int_{-1}^{1} R_n^-(x)(R_n(x) - R_{n+1}(x)) \, \mathrm{d}\mu(x)$$

$$= \frac{a_n h(n)}{(1-b_0)H(n)^2} \int_{-1}^{1} h(n)(R_n(x))^2 \, \mathrm{d}\mu(x) = \frac{a_n h(n)}{(1-b_0)H(n)^2},$$

for all $n \in \mathbb{N}_0$.

$\square$

Additionally, we want to determine the linearization coefficients of the OPS $(R_n^-(x))_{n \in \mathbb{N}_0}$. From (2.1) and (2.10), we can infer that

$$xR_n^-(x) = \frac{a_n h(n)}{H(n)} \frac{xR_n(x) - xR_{n+1}(x)}{1-x}$$

$$= \frac{a_n h(n)}{H(n)} \left[\frac{a_n R_{n+1}(x) + b_n R_n(x) + c_n R_{n-1}(x)}{1-x} \right.$$

$$\left. - \frac{a_{n+1}R_{n+2}(x) + b_{n+1}R_{n+1}(x) + c_{n+1}R_n(x)}{1-x} \right].$$

Moreover, substituting b_n by $1 - a_n - c_n$ and b_{n+1} by $1 - a_{n+1} - b_{n+1}$, respectively, and using the property of the Haar measure h, results in

$$xR_n^-(x) = \frac{a_n h(n)}{H(n)} \frac{a_{n+1}(R_{n+1}(x) - R_{n+2}(x))}{1 - x}$$

$$+ \frac{a_n h(n)}{H(n)} \frac{(1 - c_{n+1} - a_n)(R_n(x) - R_{n+1}(x))}{1 - x}$$

$$+ \frac{a_n h(n)}{H(n)} \frac{c_n(R_{n-1}(x) - R_n(x))}{1 - x}$$

$$= \frac{c_{n+1}H(n+1)}{H(n)} R_{n+1}^-(x) + (1 - c_{n+1} - a_n)R_n^-(x) + \frac{a_n H(n-1)}{H(n)} R_{n-1}^-(x).$$

The following theorem sums up the results.

Theorem 2.4. *Let $(R_n(x))_{n \in \mathbb{N}_0}$ be an OPS with respect to a probability measure μ on $[-1, 1]$, satisfying (2.1). Let the sequence $(R_n^-(x))_{n \in \mathbb{N}_0}$ be defined as in (2.10). Then, $(R_n^-(x))_{n \in \mathbb{N}_0}$ is an OPS with respect to the probability measure μ^- on $[-1, 1]$, $\mathrm{d}\mu^-(x) = \frac{1-x}{1-b_0} \, \mathrm{d}\mu(x)$. Furthermore, the system $(R_n^-(x))_{n \in \mathbb{N}_0}$ satisfies the recurrence relation*

$$R_0^-(x) = 1, \quad R_1^-(x) = (x - b_0^-)/a_0^-,$$

$$xR_n^-(x) = a_n^- R_{n+1}^-(x) + b_n^- R_n^-(x) + c_n^- R_{n-1}^-(x), \qquad n \in \mathbb{N}.$$

Moreover, for $n \in \mathbb{N}_0$, the coefficients are given by

$$a_n^- = \frac{c_{n+1}H(n+1)}{H(n)},$$

$$b_n^- = 1 - c_{n+1} - a_n,$$

$$c_n^- = \frac{a_n H(n-1)}{H(n)},$$

and satisfy $a_n^- + b_n^- + c_n^- = 1$. For the Haar measure $h^- = (h^-(n))_{n \in \mathbb{N}_0}$ of $(R_n^-(x))_{n \in \mathbb{N}_0}$, we obtain

$$h^-(n) = \frac{(1 - b_0)H(n)^2}{a_n h(n)}.$$

2.2.2 Orthogonal polynomial system with respect to μ^+

In the same way as in the subsection before, the OPS wrt μ^+, its linearization coefficients and its Haar measure will be determined. Let $(R_n(x))_{n\in\mathbb{N}_0}$ be an OPS as described in Definition 2.1. Define $(\widetilde{R}_n^+(x))_{n\in\mathbb{N}_0}$ by

$$\widetilde{R}_n^+(x) = \sum_{k=0}^{n} h(k)R_k(x)R_k(-1) = a_n h(n)\frac{R_n(-1)R_{n+1}(x) - R_{n+1}(-1)R_n(x)}{1+x},$$

for all $n \in \mathbb{N}_0$, where (2.9) was used for the second equality. Denote by $(R_n^+(x))_{n\in\mathbb{N}_0}$ the sequence $\left(\frac{\widetilde{R}_n^+(x)}{\widetilde{R}_n^+(1)}\right)_{n\in\mathbb{N}_0}$ which is in particular given by

$$R_n^+(x) = \frac{1}{K(n)} \sum_{k=0}^{n} h(k)R_k(x)R_k(-1)$$

$$= \frac{a_n h(n)}{K(n)} \frac{R_n(-1)R_{n+1}(x) - R_{n+1}(-1)R_n(x)}{1+x}, \qquad (2.12)$$

for all $n \in \mathbb{N}_0$, where

$$K(n) := \sum_{k=0}^{n} h(k)R_k(-1) = a_n h(n)\frac{R_n(-1) - R_{n+1}(-1)}{2}, \qquad K(-1) := 0.$$

The second equation again follows from the Christoffel-Darboux Identity. Observe that $K(n) = \widetilde{R}_n^+(1) \neq 0$, for all $n \in \mathbb{N}_0$, since $\mathsf{supp}\ \mu \subset [-1,1]$ implies $\mathsf{supp}\ \mu^+ \subset [-1,1]$, by definition. Hence, utilizing Theorem 1.14, all zeros of $(R_n(x))_{n\in\mathbb{N}_0}$ and $(\widetilde{R}_n^+(x))_{n\in\mathbb{N}_0}$, respectively, are contained in the interval $(-1,1)$. The normalization $R_n(1) = 1$, for all $n \in \mathbb{N}_0$, implies

$$\mathsf{sgn}\ R_n(-1) = (-1)^n, \quad \text{for all } n \in \mathbb{N}_0,$$

where sgn denotes the sign function. Considering the definition of $(K(n))_{n\in\mathbb{N}_0}$, an immediate consequence is

$$\mathsf{sgn}\ K(n) = (-1)^n, \quad \text{for all } n \in \mathbb{N}_0.$$

In the symmetric case, i.e. if $b_n = 0$, for all $n \in \mathbb{N}_0$, it follows directly from Theorem 1.9, as well as by straightforward computation that $R_n(-1) = (-1)^n$. Hence, the expressions above can be simplified to

$$K(n) = (-1)^n a_n h(n)$$

and

$$R_n^+(x) = \frac{(-1)^n}{a_n h(n)} \sum_{k=0}^{n} h(k) R_k(x)(-1)^k = \frac{R_{n+1}(x) + R_n(x)}{1 + x},$$

respectively, where $R_n^+(-1) = (-1)^n \frac{H(n)}{a_n h(n)}$, for all $n \in \mathbb{N}_0$.

Proposition 2.5. *The sequence $(R_n^+(x))_{n \in \mathbb{N}_0}$ of polynomials is orthogonal and complete over $(-1, 1)$ with respect to the probability measure μ^+ with*

$$\mathrm{d}\mu^+ = \frac{1 + x}{1 + b_0} \, \mathrm{d}\mu.$$

The Haar measure h^+ is given by

$$h^+(n) = \frac{(1 + b_0) K(n)^2}{-a_n h(n) R_n(-1) R_{n+1}(-1)},$$

for all $n \in \mathbb{N}_0$.

   Proof: Observe first that $R_n^+(x)$ is a polynomial of degree n. Furthermore, if $\mathsf{supp}\ \mu \subset [-1, 1]$, then $\mathsf{supp}\ \mu^+ \subset [-1, 1]$. In particular, μ^+ is well-defined, since the fact that the zero of $R_1(x)$ is contained in (-1,1), implies $b_0 \in (-1, 1)$. The measure μ^+ is a probability measure, since

$$\int_{\mathbb{R}} \mathrm{d}\mu^+(x) = \frac{1}{1 + b_0} \int_{-1}^{1} R_0(x) R_0(x)(1 + x) \, \mathrm{d}\mu(x)$$

$$= \frac{1}{1 + b_0} \int_{-1}^{1} R_0(x)((1 + b_0) R_0(x) + a_0 R_1(x)) \, \mathrm{d}\mu(x) = 1.$$

Now, let $n \in \mathbb{N}_0$ and $q(x)$ be a polynomial with $\deg q = m < n$. Then,

$$\int_{-1}^{1} R_n^+(x) q(x) \, \mathrm{d}\mu^+(x)$$

$$= \frac{a_n h(n)}{(1 + b_0) K(n)} \int_{-1}^{1} (R_n(-1) R_{n+1}(x) - R_{n+1}(-1) R_n(x)) q(x) \, \mathrm{d}\mu(x) = 0, \qquad (2.13)$$

since $(R_n(x))_{n \in \mathbb{N}_0}$ is an OPS wrt μ. From (2.13), it follows that

$$\int_{-1}^{1} R_n^+(x) R_m^+(x) \, \mathrm{d}\mu^+(x) = 0, \qquad \text{for all } m, n \in \mathbb{N}_0, \, m \neq n.$$

Finally, the Haar measure h^+ of the sequence $(R_n^+(x))_{n \in \mathbb{N}_0}$ is, for all $n \in \mathbb{N}_0$, given by

$$[h^+(n)]^{-1} = \int_{-1}^{1} (R_n^+(x))^2 \, \mathrm{d}\mu^+(x)$$

$$= \frac{a_n h(n)}{(1 + b_0) K(n)} \int_{-1}^{1} R_n^+(x) (R_n(-1) R_{n+1}(x) - R_{n+1}(-1) R_n(x)) \, \mathrm{d}\mu(x)$$

$$= \frac{-a_n h(n) R_n(-1) R_{n+1}(-1)}{(1 + b_0) K(n)^2} \int_{-1}^{1} h(n) (R_n(x))^2 \, \mathrm{d}\mu(x)$$

$$= \frac{-a_n h(n) R_n(-1) R_{n+1}(-1)}{(1 + b_0) K(n)^2}.$$

$\square$

Additionally, like in the subsection before, we want to determine the linearization co-efficients of the OPS $(R_n^+(x))_{n \in \mathbb{N}_0}$. From (2.1) and (2.12), it can be inferred that

$$x R_n^+(x) = \frac{a_n h(n)}{K(n)} \frac{x R_n(-1) R_{n+1}(x) - x R_{n+1}(-1) R_n(x)}{1 + x}$$

$$= \frac{a_n h(n)}{K(n)} \left[\frac{R_n(-1)(a_{n+1} R_{n+2}(x) + b_{n+1} R_{n+1}(x) + c_{n+1} R_n(x))}{1 + x} \right.$$

$$\left. - \frac{R_{n+1}(-1)(a_n R_{n+1}(x) + b_n R_n(x) + c_n R_{n-1}(x))}{1 + x} \right].$$

Moreover, using the property of the Haar measure h yields

$$xR_n^+(x) = \frac{a_n h(n)}{K(n)} \frac{a_{n+1} R_n(-1)(R_{n+1}(-1)R_{n+2}(x) - R_{n+2}(-1)R_{n+1}(x))}{R_{n+1}(-1)(1+x)}$$

$$- \frac{a_n h(n)}{K(n)} \frac{(1 + \frac{c_{n+1} R_n(-1)}{R_{n+1}(-1)} + \frac{a_n R_{n+1}(-1)}{R_n(-1)})(R_n(-1)R_{n+1}(x) - R_{n+1}(-1)R_n(x))}{1+x}$$

$$+ \frac{a_n h(n)}{K(n)} \frac{c_n R_{n+1}(-1)(R_{n-1}(-1)R_n(x) - R_n(-1)R_{n-1}(x))}{R_n(-1)(1+x)}.$$

Altogether,

$$xR_n^+(x) = \frac{c_{n+1} R_n(-1)K(n+1)}{R_{n+1}(-1)K(n)} R_{n+1}^+(x)$$

$$+ \left(-1 - \frac{c_{n+1} R_n(-1)}{R_{n+1}(-1)} - \frac{a_n R_{n+1}(-1)}{R_n(-1)} \right) R_n^+(x)$$

$$+ \frac{a_n R_{n+1}(-1)K(n-1)}{R_n(-1)K(n)} R_{n-1}^+(x).$$

The following theorem sums up the results.

Theorem 2.6. *Let $(R_n(x))_{n \in \mathbb{N}_0}$ be an OPS with respect to a probability measure μ on $[-1, 1]$, satisfying (2.1). Let the sequence $(R_n^+(x))_{n \in \mathbb{N}_0}$ be defined as in (2.12). Then, $(R_n^+(x))_{n \in \mathbb{N}_0}$ is an OPS with respect to the probability measure μ^+ on $[-1, 1]$, $\mathrm{d}\mu^+(x) = \frac{1+x}{1+b_0} \mathrm{d}\mu(x)$. Furthermore, the system $(R_n^+(x))_{n \in \mathbb{N}_0}$ satisfies the recurrence relation*

$$R_0^+(x) = 1, \quad R_1^+(x) = (x - b_0^+)/a_0^+,$$

$$xR_n^+(x) = a_n^+ R_{n+1}^+(x) + b_n^+ R_n^+(x) + c_n^+ R_{n-1}^+(x), \qquad n \in \mathbb{N}.$$

Moreover, for $n \in \mathbb{N}_0$, the coefficients are given by

$$a_n^+ = \frac{c_{n+1} R_n(-1)K(n+1)}{R_{n+1}(-1)K(n)},$$

$$b_n^+ = -1 - \frac{c_{n+1} R_n(-1)}{R_{n+1}(-1)} - \frac{a_n R_{n+1}(-1)}{R_n(-1)},$$

$$c_n^+ = \frac{a_n R_{n+1}(-1) K(n-1)}{R_n(-1) K(n)},$$

and satisfy $a_n^+ + b_n^+ + c_n^+ = 1$. For the Haar measure $h^+ = (h^+(n))_{n \in \mathbb{N}_0}$ of $(R_n^+(x))_{n \in \mathbb{N}_0}$, we obtain

$$h^+(n) = \frac{(1 + b_0) K(n)^2}{-a_n h(n) R_n(-1) R_{n+1}(-1)}.$$

In the case, when μ is a symmetric measure, the expressions in Theorem 2.6 can be simplified.

Corollary 2.7. *Let $(R_n(x))_{n \in \mathbb{N}_0}$ be an OPS with respect to a symmetric probability measure μ on $[-1, 1]$, satisfying (2.1) with $b_n = 0$, for all $n \in \mathbb{N}_0$. Let the sequence $(R_n^+(x))_{n \in \mathbb{N}_0}$ be defined as in (2.12). Then, $(R_n^+(x))_{n \in \mathbb{N}_0}$ is an OPS with respect to the probability measure μ^+ on $[-1, 1]$, $\mathrm{d}\mu^+(x) = (1 + x)\,\mathrm{d}\mu(x)$. Furthermore, the system $(R_n^+(x))_{n \in \mathbb{N}_0}$ satisfies the recurrence relation*

$$R_0^+(x) = 1, \quad R_1^+(x) = (x - b_0^+)/a_0^+,$$

$$x R_n^+(x) = a_n^+ R_{n+1}^+(x) + b_n^+ R_n^+(x) + c_n^+ R_{n-1}^+(x), \qquad n \in \mathbb{N}.$$

Moreover, for $n \in \mathbb{N}_0$, the coefficients are given by

$$a_n^+ = a_{n+1},$$

$$b_n^+ = -1 + c_{n+1} + a_n,$$

$$c_n^+ = c_n,$$

and satisfy $a_n^+ + b_n^+ + c_n^+ = 1$. For the Haar measure $h^+ = (h^+(n))_{n \in \mathbb{N}_0}$ of $(R_n^+(x))_{n \in \mathbb{N}_0}$, we obtain $h^+(n) = a_n h(n)$. Moreover, the sequence H^+ can be directly computed. In particular, we have

$$H^+(n) = \frac{a_n h(n) + H(n)}{2}.$$

The last assertion in Corollary 2.7 follows from the equation

$$2 \sum_{k=0}^{n} a_k h(k) = \sum_{k=0}^{n} (a_k h(k) + c_{k+1} h(k+1)) = a_n h(n) + \sum_{k=0}^{n} (a_k + c_k) h(k).$$

2.3 Computation of the conjugate orthogonal polynomial system

In this section we attend to the deduction of the conjugate polynomial system. First, a general relation will be derived from the results in Section 2.2. Afterwards, the special case, when the measure of the initial OPS is symmetric, shall be treated.

2.3.1 A general formula

It can be inferred from Section 2.2 that the COPS $(R_n^*(x))_{n\in\mathbb{N}_0}$ of $(R_n(x))_{n\in\mathbb{N}_0}$ is given by $(R_n^{+-}(x))_{n\in\mathbb{N}_0}$ and $(R_n^{-+}(x))_{n\in\mathbb{N}_0}$, respectively. Let the systems $(R_n(x))_{n\in\mathbb{N}_0}$, $(R_n^-(x))_{n\in\mathbb{N}_0}$ and $(R_n^+(x))_{n\in\mathbb{N}_0}$ be defined as in (2.1), (2.10) and (2.12), respectively. By $(R_n^*(x))_{n\in\mathbb{N}_0}$, denote the sequence of polynomials which is given by

$$R_n^*(x) = \frac{1}{K^-(n)} \sum_{k=0}^{n} h^-(k) R_k^-(x) R_k^-(-1)$$

$$= \frac{a_n^- h^-(n)}{K^-(n)} \frac{R_n^-(-1)R_{n+1}^-(x) - R_{n+1}^-(-1)R_n^-(x)}{1+x} \tag{2.14}$$

for all $n \in \mathbb{N}_0$. The aim is to rewrite $R_n^*(x)$ by using the coefficients and weights of the system $(R_n(x))_{n\in\mathbb{N}}$. Observe that

$$R_n^-(-1) = \frac{1}{H(n)} \sum_{k=0}^{n} h(k) R_k(-1) = \frac{K(n)}{H(n)},$$

$$K^-(n) = a_n^- h^-(n) \frac{R_n^-(-1) - R_{n+1}^-(-1)}{2} \tag{2.15}$$

$$= a_n^- h^-(n) \frac{K(n)H(n+1) - K(n+1)H(n)}{2H(n)H(n+1)}$$

$$= \frac{c_{n+1}(1-b_0)H(n)H(n+1)}{a_n h(n)} \frac{h(n+1)K(n) - h(n+1)R_{n+1}(-1)H(n)}{2H(n)H(n+1)}$$

$$= \frac{(1-b_0)(K(n) - R_{n+1}(-1)H(n))}{2}.$$

Applying (2.15) and Theorem 2.4 to equation (2.14) yields

$$
R_n^*(x) = \frac{2H(n)H(n+1)}{K(n)H(n+1) - K(n+1)H(n)} \times
$$

$$
\times \left[\frac{\frac{a_{n+1}h(n+1)K(n)}{H(n)H(n+1)} \frac{R_{n+1}(x) - R_{n+2}(x)}{1-x}}{1+x} - \frac{\frac{a_n h(n)K(n+1)}{H(n)H(n+1)} \frac{R_n(x) - R_{n+1}(x)}{1-x}}{1+x} \right]
$$

$$
= \frac{2}{K(n) - R_{n+1}(-1)H(n)} \times \tag{2.16}
$$

$$
\times \frac{-c_{n+1}K(n+1)R_n(x) + (a_{n+1}K(n) + c_{n+1}K(n+1))R_{n+1}(x) - a_{n+1}K(n)R_{n+2}(x)}{1 - x^2}.
$$

Proposition 2.8. *The sequence* $(R_n^*(x))_{n \in \mathbb{N}_0}$ *of polynomials is orthogonal and complete over* $(-1, 1)$ *with respect to the probability measure* μ^* *with*

$$
\mathrm{d}\mu^* = \frac{1 - x^2}{(1 - c_1 + b_0)(1 - b_0)} \, \mathrm{d}\mu.
$$

The Haar measure h^* *is given by*

$$
h^*(n) = \frac{(1 - c_1 + b_0)(1 - b_0)h(n+1)(K(n) - R_{n+1}(-1)H(n))^2}{-4K(n)K(n+1)},
$$

for all $n \in \mathbb{N}_0$.

   Proof: Since $(R_n^*(x))_{n \in \mathbb{N}_0} = (R_n^{-+}(x))_{n \in \mathbb{N}_0}$, we can infer from Proposition 2.3 and Proposition 2.5 that $(R_n^*(x))_{n \in \mathbb{N}_0}$ is orthogonal and complete over $(-1, 1)$ with respect to the probability measure $\mu^* := \mu^{-+}$ with

$$
\mathrm{d}\mu^* = \frac{1 + x}{(1 + b_0^-)} \, \mathrm{d}\mu^- = \frac{(1 + x)(1 - x)}{(1 - c_1 + b_0)(1 - b_0)} \, \mathrm{d}\mu.
$$

Furthermore, $h^* = h^{-+}$ can be computed directly by considering the results in Section 2.2. For all $n \in \mathbb{N}_0$, one obtains

$$h^*(n) = \frac{(1 + b_0^-)K^-(n)^2}{-a_n^- h^-(n)R_n^-(-1)R_{n+1}^-(-1)}$$

$$= \frac{(1 - c_1 + b_0)K^-(n)(K(n)H(n+1) - H(n)K(n+1))}{-2H(n)H(n+1)\frac{K(n)}{H(n)}\frac{K(n+1)}{H(n+1)}}$$

$$= \frac{(1 - c_1 + b_0)h(n+1)K^-(n)(K(n) - R_{n+1}(-1)H(n))}{-2K(n)K(n+1)}$$

$$= \frac{(1 - c_1 + b_0)(1 - b_0)h(n+1)(K(n) - R_{n+1}(-1)H(n))^2}{-4K(n)K(n+1)}.$$

$\square$

Following the results in Section 2.2, we can moreover determine the linearization coefficients $(a_n^*)_{n \in \mathbb{N}_0}$, $(b_n^*)_{n \in \mathbb{N}_0}$ and $(c_n^*)_{n \in \mathbb{N}_0}$ for the COPS $(R_n^*(x))_{n \in \mathbb{N}_0}$. For all $n \in \mathbb{N}_0$, it follows that

$$a_n^* = a_n^{-+} = \frac{c_{n+1}^- R_n^-(-1)K^-(n+1)}{R_{n+1}^-(-1)K^-(n)}$$

$$= \frac{a_{n+1}H(n)}{H(n+1)}\frac{K(n)}{H(n)}\frac{H(n+1)}{K(n+1)}\frac{K(n+1) - R_{n+2}(-1)H(n+1)}{K(n) - R_{n+1}(-1)H(n)}$$

$$= \frac{a_{n+1}K(n)(K(n+1) - R_{n+2}(-1)H(n+1))}{K(n+1)(K(n) - R_{n+1}(-1)H(n))},$$

by using Theorem 2.4, Theorem 2.6 and (2.15). Moreover, we obtain

$$b_n^* = b_n^{-+} = -1 - \frac{c_{n+1}^- R_n^-(-1)}{R_{n+1}^-(-1)} - \frac{a_n^- R_{n+1}^-(-1)}{R_n^-(-1)}$$

$$= -1 - \frac{a_{n+1}K(n)}{K(n+1)} - \frac{c_{n+1}K(n+1)}{K(n)}$$

and the computation of $(c_n^*)_{n \in \mathbb{N}_0}$ results in

$$c_n^* = c_n^{-+} = \frac{a_n^- R_{n+1}^-(-1)K^-(n-1)}{R_n^-(-1)K^-(n)}$$

$$= \frac{c_{n+1}H(n+1)}{H(n)} \frac{K(n+1)}{H(n+1)} \frac{H(n)}{K(n)} \frac{K(n-1) - R_n(-1)H(n-1)}{K(n) - R_{n+1}(-1)H(n)}$$

$$= \frac{c_{n+1}K(n+1)(K(n-1) - R_n(-1)H(n-1))}{K(n)(K(n) - R_{n+1}(-1)H(n))}.$$

The following theorem sums up the results.

Theorem 2.9. *Let* $(R_n(x))_{n \in \mathbb{N}_0}$ *be an OPS with respect to a probability measure* μ *on* $[-1, 1]$*, satisfying (2.1). Let the sequence* $(R_n^*(x))_{n \in \mathbb{N}_0}$ *be defined as in (2.14) and (2.16), respectively. Then,* $(R_n^*(x))_{n \in \mathbb{N}_0}$ *is an OPS with respect to the probability measure* μ^* *on* $[-1, 1]$*,* $\mathrm{d}\mu^*(x) = \frac{1-x^2}{(1-b_0)(1-c_1+b_0)} \, \mathrm{d}\mu(x)$*. Furthermore, the system* $(R_n^*(x))_{n \in \mathbb{N}_0}$ *satisfies the recurrence relation*

$$R_0^*(x) = 1, \quad R_1^*(x) = (x - b_0^*)/a_0^*,$$

$$xR_n^*(x) = a_n^* R_{n+1}^*(x) + b_n^* R_n^*(x) + c_n^* R_{n-1}^*(x), \qquad n \in \mathbb{N}.$$

Moreover, for $n \in \mathbb{N}_0$*, the coefficients are given by*

$$a_n^* = \frac{a_{n+1}K(n)(K(n+1) - R_{n+2}(-1)H(n+1))}{K(n+1)(K(n) - R_{n+1}(-1)H(n))},$$

$$b_n^* = -1 - \frac{a_{n+1}K(n)}{K(n+1)} - \frac{c_{n+1}K(n+1)}{K(n)},$$

$$c_n^* = \frac{c_{n+1}K(n+1)(K(n-1) - R_n(-1)H(n-1))}{K(n)(K(n) - R_{n+1}(-1)H(n))},$$

and satisfy $a_n^* + b_n^* + c_n^* = 1$*. For the Haar measure* $h^* = (h^*(n))_{n \in \mathbb{N}_0}$ *of* $(R_n^*(x))_{n \in \mathbb{N}_0}$*, we obtain*

$$h^*(n) = \frac{(1 - c_1 + b_0)(1 - b_0)h(n+1)(K(n) - R_{n+1}(-1)H(n))^2}{-4K(n)K(n+1)}.$$

2.3.2 The symmetric case

In the section before, a general formula for the COPS $(R_n^*(x))_{n\in\mathbb{N}_0}$ has been deduced. Now the case, when μ is a symmetric measure, will be treated. Then, different from the measures μ^- and μ^+, the measure μ^* is also symmetric. The initial system $(R_n(x))_{n\in\mathbb{N}_0}$ satisfies a three-term recurrence relation

$$R_0(x) = 1, \quad R_1 = x,$$

$$xR_n(x) = a_n R_{n+1}(x) + c_n R_{n-1}(x), \quad n \in \mathbb{N}, \tag{2.17}$$

where $a_n + c_n = 1$ and $a_n c_{n+1} > 0$. From Section 2.2, we know that

$$K(n) = (-1)^n a_n h(n) \qquad \text{and} \qquad R_n(-1) = (-1)^n,$$

for all $n \in \mathbb{N}_0$. Hence, (2.15) can be simplified to

$$R_n^-(-1) = \frac{(-1)^n a_n h(n)}{H(n)}$$

and

$$K^-(n) = \frac{(-1)^n (a_n h(n) + H(n))}{2}.$$

Therefore, we obtain in (2.14)

$$R_n^*(x) = \frac{1}{K^-(n)} \sum_{k=0}^{n} h^-(k) R_k^-(x) R_k^-(-1)$$

$$= \frac{2(-1)^n}{a_n h(n) + H(n)} \sum_{k=0}^{n} H(k)(-1)^k R_k^-(x)$$

$$= \frac{2(-1)^n}{a_n h(n) + H(n)} \sum_{k=0}^{n} h(k) R_k(x) \sum_{m=k}^{n} (-1)^m. \tag{2.18}$$

Equation (2.16) can be simplified to

$$R_n^*(x) = \frac{2(-1)^n}{a_n h(n) + H(n)} \times$$

$$\times \frac{-c_{n+1}(-1)^{n+1} a_{n+1} h(n+1) R_n(x) - a_{n+1}(-1)^n a_n h(n) R_{n+2}(x)}{1 - x^2}$$

$$= \frac{2a_n h(n)}{a_n h(n) + H(n)} \frac{a_{n+1} R_n(x) - a_{n+1} R_{n+2}(x)}{1 - x^2}$$

$$= \frac{2a_n h(n)}{a_n h(n) + H(n)} \frac{R_n(x) - x R_{n+1}(x)}{1 - x^2}. \tag{2.19}$$

The following theorem sums up further results.

Theorem 2.10. *Let $(R_n(x))_{n \in \mathbb{N}_0}$ be an OPS with respect to a probability measure μ on $[-1, 1]$, satisfying (2.17). Let the sequence $(R_n^*(x))_{n \in \mathbb{N}_0}$ be defined as in (2.18) and (2.19), respectively. Then, $(R_n^*(x))_{n \in \mathbb{N}_0}$ is an OPS with respect to the probability measure μ^* on $[-1, 1]$, $\mathrm{d}\mu^*(x) = \frac{1-x^2}{a_1} \, \mathrm{d}\mu(x)$. Furthermore, the system $(R_n^*(x))_{n \in \mathbb{N}_0}$ satisfies the recurrence relation*

$$R_0^*(x) = 1, \quad R_1^*(x) = x,$$

$$x R_n^*(x) = a_n^* R_{n+1}^*(x) + c_n^* R_{n-1}^*(x), \qquad n \in \mathbb{N}.$$

Moreover, for $n \in \mathbb{N}_0$, the coefficients are given by

$$a_n^* = \frac{c_{n+1}(a_{n+1} h(n+1) + H(n+1))}{a_n h(n) + H(n)},$$

$$c_n^* = \frac{a_{n+1}(a_{n-1} h(n-1) + H(n-1))}{a_n h(n) + H(n)},$$

and satisfy $a_n^* + c_n^* = 1$. For the Haar measure $h^* = (h^*(n))_{n \in \mathbb{N}_0}$ of $(R_n^*(x))_{n \in \mathbb{N}_0}$, we obtain

$$h^*(n) = \frac{a_1(a_n h(n) + H(n))^2}{4 a_{n+1} a_n h(n)}.$$

Example 2.11. *As an example we want to consider the normalized Karlin-McGregor polynomials* $(R_n^{(a,b)}(x))_{n \in \mathbb{N}_0}$, $a, b > 1, a \neq b$ *which are defined by the recurrence relation*

$$R_0^{(a,b)}(x) = 1, \quad R_1^{(a,b)}(x) = x,$$

$$x R_n^{(a,b)}(x) = a_n R_{n+1}^{(a,b)}(x) + c_n R_{n-1}^{(a,b)}(x), \quad n \in \mathbb{N},$$

where

$$a_n = \begin{cases} 1, & n = 0, \\ \dfrac{a-1}{a}, & n \in \{1, 3, 5, ...\}, \\ \dfrac{b-1}{b}, & n \in \{2, 4, 6, ...\}, \end{cases} \quad and \quad c_n = \begin{cases} 0, & n = 0, \\ \dfrac{1}{a}, & n \in \{1, 3, 5, ...\}, \\ \dfrac{1}{b}, & n \in \{2, 4, 6, ...\}. \end{cases}$$

The Karlin-Mc Gregor polynomials were for example discussed in [FL00] and are orthogonal and complete over $I_{a,b} \subset (-1, 1)$ wrt the orthogonalization measure $\mu^{(a,b)}$ which satisfies

$$\mathrm{d}\mu^{(a,b)}(x) = \frac{b\sqrt{4\frac{b-1}{ab}x^2 - \left(x^2 + \frac{b-a}{ab}\right)^2}}{2\pi|x|(1-x^2)} \, \mathrm{d}x.$$

The measure shows that using our definition, the COPS of Karlin-McGregor polynomials is not included in the class of Karlin-McGregor polynomials again. This would be the case, if we consider for example Jacobi Polynomials. In general, the corresponding Haar measure is given by

$$h(n) = \begin{cases} 1, & n = 0, \\ a(a-1)^{\frac{n-1}{2}}(b-1)^{\frac{n-1}{2}}, & n \in \{1, 3, 5, ...\}, \\ b(a-1)^{\frac{n}{2}}(b-1)^{\frac{n-2}{2}}, & n \in \{2, 4, 6, ...\} \end{cases}$$

and computing H for $(a-1)(b-1) \neq 1$ yields

$$H(n) = \begin{cases} 1, & n = 0, \\ \dfrac{2a - (a-1)^{\frac{n+1}{2}}(b-1)^{\frac{n-1}{2}}(b + a(b-1))}{1 - (a-1)(b-1)}, & n \in \{1, 3, 5, ...\}, \\ \dfrac{2a - (a-1)^{\frac{n}{2}}(b-1)^{\frac{n}{2}}(b(a-1) + a)}{1 - (a-1)(b-1)}, & n \in \{2, 4, 6, ...\}. \end{cases}$$

In the special case $(a-1)(b-1) = 1$, which is satisfied if $a = \frac{b}{b-1}$, $a \neq 2$, the expression can be simplified to

$$h(n) = \begin{cases} 1, & n = 0, \\ a, & n \in \mathbb{N} \end{cases}$$

and as a consequence,

$$H(n) = 1 + an, \quad \text{for all } n \in \mathbb{N}_0.$$

First observe that the COPS of $(R_n^{(a,b)}(x))_{n \in \mathbb{N}_0}$ is orthogonal and complete over $I_{a,b} \subset (-1,1)$ with respect to the measure $\mu^{(a,b)^*}$, where

$$d\mu^{(a,b)^*}(x) = \frac{ab\sqrt{4\frac{b-1}{ab}x^2 - \left(x^2 + \frac{b-a}{ab}\right)^2}}{2(a-1)\pi|x|} \, dx.$$

The coefficients of the COPS can be computed due to the formulas in Theorem 2.10. First, consider the case, when $(a-1)(b-1) \neq 1$. One obtains

$$a_n h(n) + H(n) = \begin{cases} 2, & n = 0, \\[2mm] \dfrac{2(a - a(a-1)^{\frac{n+1}{2}}(b-1)^{\frac{n+1}{2}})}{1 - (a-1)(b-1)}, & n \in \{1, 3, 5, ...\}, \\[3mm] \dfrac{2(a - b(a-1)^{\frac{n+2}{2}}(b-1)^{\frac{n}{2}})}{1 - (a-1)(b-1)}, & n \in \{2, 4, 6, ...\}. \end{cases} \tag{2.20}$$

Using (2.20), we can infer that

$$a_n^* = \begin{cases} 1, & n = 0, \\[2mm] \dfrac{a - b(a-1)^{\frac{n+3}{2}}(b-1)^{\frac{n+1}{2}}}{b(a - a(a-1)^{\frac{n+1}{2}}(b-1)^{\frac{n+1}{2}})}, & n \in \{1, 3, 5, ...\}, \\[3mm] \dfrac{a - a(a-1)^{\frac{n+2}{2}}(b-1)^{\frac{n+2}{2}}}{a(a - b(a-1)^{\frac{n+2}{2}}(b-1)^{\frac{n}{2}})}, & n \in \{2, 4, 6, ...\} \end{cases}$$

and

$$
c_n^* = \begin{cases} 0, & n = 0, \\[2mm] \dfrac{(b-1)(a-b(a-1)^{\frac{n+1}{2}}(b-1)^{\frac{n-1}{2}})}{b(a-a(a-1)^{\frac{n+1}{2}}(b-1)^{\frac{n+1}{2}})}, & n \in \{1,3,5,...\}, \\[3mm] \dfrac{(a-1)(a-a(a-1)^{\frac{n}{2}}(b-1)^{\frac{n}{2}})}{a(a-b(a-1)^{\frac{n+2}{2}}(b-1)^{\frac{n}{2}})}, & n \in \{2,4,6,...\}. \end{cases}
$$

Finally, computing the Haar measure of the COPS, one obtains

$$
h(n)^* = \begin{cases} 1, & n = 0, \\[2mm] \dfrac{b(a-a(a-1)^{\frac{n+1}{2}}(b-1)^{\frac{n+1}{2}})^2}{a(a-1)^{\frac{n-1}{2}}(b-1)^{\frac{n+1}{2}}(1-(a-1)(b-1))^2}, & n \in \{1,3,5,...\}, \\[3mm] \dfrac{(a-b(a-1)^{\frac{n+2}{2}}(b-1)^{\frac{n}{2}})^2}{(a-1)^{\frac{n}{2}}(b-1)^{\frac{n}{2}}(1-(a-1)(b-1))^2}, & n \in \{2,4,6,...\}. \end{cases}
$$

The expressions can be simplified in the case, when $(a-1)(b-1) = 1$. Observe that

$$
a_n h(n) + H(n) = \begin{cases} 2 + na, & n \in \{0,2,4,...\}, \\[2mm] (n+1)a, & n \in \{1,3,5,...\}. \end{cases}
$$

Moreover, it follows that

$$
a_n^* = \begin{cases} \dfrac{n+2}{2+na}, & n \in \{0,2,4,...\}, \\[3mm] \dfrac{(a-1)(2+(n+1)a)}{(n+1)a^2}, & n \in \{1,3,5,...\} \end{cases}
$$

and

$$
c_n^* = \begin{cases} \dfrac{n(a-1)}{2+na}, & n \in \{0,2,4,...\}, \\[3mm] \dfrac{2+(n-1)a}{(n+1)a^2}, & n \in \{1,3,5,...\}. \end{cases}
$$

Computing the Haar measure results in

$$h(n)^* = \begin{cases} \dfrac{(2 + na)^2}{4}, & n \in \{0, 2, 4, ...\}, \\[3mm] \dfrac{(n + 1)^2 a^2}{4}, & n \in \{1, 3, 5, ...\}. \end{cases}$$

2.4 Properties of the function spaces $L^p(\mu)$ and $L^p(\mu^*)$

In this section, we draw attention to the function spaces $L^p(\mu)$ and $L^p(\mu^*)$, respectively, for $1 \leq p < \infty$. Assume that $(R_n(x))_{n \in \mathbb{N}_0}$ is an OPS as in Definition 2.1, orthogonal wrt the probability measure μ, and denote by $(R_n^*(x))_{n \in \mathbb{N}_0}$ its COPS. For $1 \leq p < \infty$, let $L^p(\mu)$ be the space of functions f for which

$$\|f\|_{p,\mu} := \left[\int_{-1}^{1} |f(x)|^p \, \mathrm{d}\mu(x) \right]^{\frac{1}{p}} < \infty.$$

Obviously, one obtains

$$L^p(\mu) \subset L^p(\mu^*),$$

since for all $f \in L^p(\mu)$, it can be inferred that

$$\int_{-1}^{1} |f(x)|^p \, \mathrm{d}\mu^*(x) \leq \frac{1}{(1 - c_1 + b_0)(1 - b_0)} \int_{-1}^{1} |f(x)|^p \, \mathrm{d}\mu(x).$$

For $f \in L^1(\mu)$, the corresponding polynomial series is given by

$$f(x) \sim \sum_{n=0}^{\infty} h(n)a(n)R_n(x),$$

where

$$a(n) = \int_{-1}^{1} f(x)R_n(x) \, \mathrm{d}\mu(x), \qquad n \in \mathbb{N}_0.$$

Consider the spaces $\ell^p(h)$, for $1 \le p < \infty$, where h is the Haar measure of the respective OPS. $\ell^p(h)$ is the space of all complex sequences a for which

$$\|a\|_{p,h} := \left[\sum_{n=0}^{\infty} h(n)|a(n)|^p \right]^{\frac{1}{p}} < \infty.$$

For $p = \infty$, it follows that $\ell^\infty(h) = \ell^\infty$. These spaces will be further discussed in Chapter 3, where we introduce the weighted Cesàro operator. Let $\sup_{n \in \mathbb{N}_0} \sup_{x \in [-1,1]} |R_n(x)|$ be bounded by a constant $M > 0$ and consider the transform

$$\mathcal{P}_h: \quad L^1(\mu) \to \ell^\infty,$$

$$f \sim \sum_{n=0}^{\infty} h(n)a(n)R_n \mapsto a = (a(n))_{n \in \mathbb{N}_0}. \tag{2.21}$$

$\mathcal{P}_h$ is a bounded linear operator from $L^1(\mu)$ to ℓ^∞, since

$$|a(n)| \le \int_{-1}^{1} |f(x)|\, |R_n(x)|\, \mathrm{d}\mu(x)$$

$$\le M\|f\|_{1,\mu} < \infty, \tag{2.22}$$

for all $n \in \mathbb{N}_0$. In particular, we obtain $\|\mathcal{P}_h\|_{L^1(h) \to \ell^\infty} \le M$. Furthermore, $\mathcal{P}_h$ is a bounded linear operator from $L^2(\mu)$ to $\ell^2(h)$ and the correspondence between these two spaces is one-to-one, since we have

$$\|f\|_{2,\mu}^2 = \int_{-1}^{1} \left| \sum_{n=0}^{\infty} h(n)a(n)R_n(x) \right|^2 \mathrm{d}\mu(x)$$

$$= \sum_{n=0}^{\infty} h(n)|a(n)|^2 = \|a\|_{2,h}^2,$$

for all $f \in L^2(\mu)$ and hence, $\|\mathcal{P}_h\|_{L^2(h) \to \ell^2(h)} = 1$. Moreover,

$$\mathcal{P}_h(R_n) = \frac{e_n}{h(n)},$$

for all $n \in \mathbb{N}_0$, where e_n denotes the nth unit sequence. The Riesz-Thorin theorem, see [Wer07, p. 73], implies that $\mathcal{P}_h$ is a bounded linear operator from $L^p(\mu)$ to $\ell^q(h)$, for $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

3 The Cesàro operator in weighted ℓ^p-sequence spaces and the generalized concepts of normality

In this chapter, the Cesàro operator in weighted ℓ^p-spaces is discussed. In particular, we are interested in the properties of the Cesàro operator in weighted ℓ^2-Hilbert spaces.

Definition 3.1. *For a sequence $h = (h(n))_{n \in \mathbb{N}}$ of positive numbers, called weights and a sequence $a = (a(n))_{n \in \mathbb{N}}$ of complex numbers the **discrete weighted Cesàro operator** C_h is defined by*

$$C_h a(n) = \frac{1}{H(n)} \sum_{k=0}^{n} h(k)a(k), \quad with \quad H(n) = \sum_{k=0}^{n} h(k) \quad and \quad n \in \mathbb{N}_0. \qquad (3.1)$$

W.l.o.g. we assume $h(0) = 1$. Let $1 < p < \infty$ and

$$\ell^p(h) = \{a = (a(n))_{n \in \mathbb{N}} : a(n) \in \mathbb{C}, \|a\|_{p,h}^p := \sum_{n=0}^{\infty} h(n) |a(n)|^p < \infty\}. \qquad (3.2)$$

Computation shows that the dual operator C_h^* of C_h in $\ell^q(h)$, $\frac{1}{p} + \frac{1}{q} = 1$, is given by

$$C_h^* a(n) = \sum_{k=n}^{\infty} \frac{h(k)a(k)}{H(k)}, \quad \text{for all } n \in \mathbb{N}_0, \qquad (3.3)$$

where we used the duality relation between $\ell^q(h)$ and $\ell^p(h)$. In the Hilbert space $\ell^2(h)$, the inner product is defined by

$$\langle a, b \rangle_h = \sum_{n=0}^{\infty} h(n)a(n)\overline{b(n)}, \quad a, b \in \ell^2(h). \qquad (3.4)$$

One obtains the classical sequence space ℓ^2, when choosing $h = (1, 1, 1, ...)$. Let e_j be the jth unit sequence, $e_j(i) = \delta_{ij}$, $i, j \in \mathbb{N}_0$. Since C_h and C_h^* are operators in a sequence space, they have matrix representations with respect to the basis $(e_j)_{j \in \mathbb{N}}$ of $l^2(h)$ (in the

following also denoted by C_h and C_h^*, respectively). From (3.1) and (3.3) it can be inferred that

$$
C_h = \begin{pmatrix} \frac{h(0)}{H(0)} & & & 0 \\ \frac{h(0)}{H(1)} & \frac{h(1)}{H(1)} & & \\ \frac{h(0)}{H(2)} & \frac{h(1)}{H(2)} & \frac{h(2)}{H(2)} & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},
$$

$$
C_h^* = \begin{pmatrix} \frac{h(0)}{H(0)} & \frac{h(1)}{H(1)} & \frac{h(2)}{H(2)} & \cdots \\ & \frac{h(1)}{H(1)} & \frac{h(2)}{H(2)} & \cdots \\ 0 & & \frac{h(2)}{H(2)} & \cdots \\ & & & \ddots \end{pmatrix}.
$$
(3.5)

The (unweighted) Cesàro operator C in ℓ^2 has been particularized discussed in [BHS65]. In the first section of this chapter, we will be concerned with the boundedness and the norm of the Cesàro operator and its dual operator C_h^* in $\ell^p(h)$. Afterwards, we focus our attention on the spectrum of C_h and C_h^*, respectively. Finally, the weighted Cesàro operator in $\ell^2(h)$ is investigated in terms of several concepts of normality.

3.1 Boundedness of the Cesàro operator in $\ell^p(h)$

3.1.1 Hardy's inequality

In the 1920s a crucial inequality was proved by Hardy et al. Let $p > 1$, $a_n \geq 0$, $\lambda_n > 0$, for all $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} \lambda_n a_n^p < \infty$; let $\Lambda_n = \sum_{k=1}^{n} \lambda_k$, $A_n = \sum_{k=1}^{n} \lambda_k a_k$, then

$$
\sum_{n=1}^{\infty} \lambda_n \left(\frac{A_n}{\Lambda_n} \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} \lambda_n a_n^p.
$$
(3.6)

Various proofs can be found in [Har20],[Har25], [Har28], [HLP88, 9.8 and 9.10] or [Lan26]. In [Cop27], Copson proved the respective dual inequality. Let $p > 1$, $b_n \geq 0$, $\lambda_n > 0$, for all $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} \lambda_n b_n^p < \infty$; let $\Lambda_n = \sum_{k=1}^{n} \lambda_k$, $B_n = \sum_{k=n}^{\infty} \frac{\lambda_k b_k}{\Lambda_k}$, then

$$
\sum_{n=1}^{\infty} \lambda_n B_n^p \leq p^p \sum_{n=1}^{\infty} \lambda_n b_n^p.
$$
(3.7)

These two inequalities are very helpful for our purpose to prove the boundedness of the Cesàro operator. Inequality (3.6) says that if $a = (a(n))_{n \in \mathbb{N}_0} \in \ell^p(h)$, for $1 < p < \infty$, and an arbitrary weight sequence h, we obtain

$$\|C_h a\|_{p,h} = \left(\sum_{n=0}^{\infty} h(k) \left| \frac{1}{H(n)} \sum_{k=0}^{n} h(k) a(k) \right|^p \right)^{\frac{1}{p}}$$

$$\leq \left(\sum_{n=0}^{\infty} h(k) \left(\frac{1}{H(n)} \sum_{k=0}^{n} h(k)|a(k)| \right)^p \right)^{\frac{1}{p}}$$

$$\leq \frac{p}{p-1} \left(\sum_{n=0}^{\infty} h(n)|a(n)|^p \right)^{\frac{1}{p}} = \frac{p}{p-1} \|a\|_{p,h}.$$

Analogously, applying inequality (3.7) to a sequence $b = (b(n))_{n \in \mathbb{N}_0} \in \ell^q(h)$, where $q = \frac{p}{p-1}$, yields

$$\|C_h^* b\|_{q,h} \leq q \|b\|_{q,h}.$$

Let us register the boundedness of the Cesàro operator in the following proposition.

Proposition 3.2. *The Cesàro operator in $\ell^p(h)$ is a bounded linear operator, we write $C_h \in B(\ell^p(h))$. In particular, it is bounded by $\|C_h\| \leq \frac{p}{p-1}$.*

3.1.2 The norm of the Cesàro operator

In [BHS65], the authors found a relation between the Cesàro operator in ℓ^2 and a diagonal operator, in particular they showed that

$$(\mathsf{id} - C) \circ (\mathsf{id} - C^*) = (\mathsf{id} - D),$$

where id denotes the identity matrix and

$$D = \begin{pmatrix} 1 & & 0 & \\ & \frac{1}{2} & & \\ 0 & & \frac{1}{3} & \\ & & & \ddots \end{pmatrix}.$$

A similar relation occurs, when considering the weighted Cesàro operator in $\ell^2(h)$. First, observe that for an arbitrary sequence $a \in \ell^2(h)$ and, for all $n \in \mathbb{N}_0$, one obtains

$$(C_h \circ C_h^*)(a)(n) = \frac{1}{H(n)} \sum_{k=0}^{n} h(k) \sum_{m=k}^{\infty} \frac{h(m)a(m)}{H(m)}$$

$$= \frac{1}{H(n)} \sum_{m=0}^{n} \frac{h(m)a(m)}{H(m)} \sum_{k=0}^{m} h(k) + \sum_{m=n+1}^{\infty} \frac{h(m)a(m)}{H(m)}$$

$$= C_h a(n) + C_h^* a(n) - \frac{h(n)}{H(n)} a(n)$$

from which we can infer that

$$(\mathsf{id} - C_h) \circ (\mathsf{id} - C_h^*) = (\mathsf{id} - D_h),$$

where D_h denotes the matrix

$$D_h = \begin{pmatrix} 1 & & & 0 \\ & \frac{h(1)}{H(1)} & & \\ 0 & & \frac{h(2)}{H(2)} & \\ & & & \ddots \end{pmatrix}.$$

The matrix representations of C_h and C_h^*, respectively, can be found in (3.5) and $C_h \circ C_h^*$ has the matrix representation

$$C_h C_h^* = \begin{pmatrix} 1 & \frac{h(1)}{H(1)} & \frac{h(2)}{H(2)} & \\ \frac{h(0)}{H(1)} & \frac{h(1)}{H(1)} & \frac{h(2)}{H(2)} & \\ \frac{h(0)}{H(2)} & \frac{h(1)}{H(2)} & \frac{h(2)}{H(2)} & \\ & & & \ddots \end{pmatrix}. \tag{3.8}$$

The same arguments as in [BHS65] imply

$$\|C_h\| \leq 1 + \|\mathsf{id} - C_h\|$$

$$\leq 1 + \|\mathsf{id} - D_h\|^{-\frac{1}{2}} = 1 + \left(\sup_{n \in \mathbb{N}_0} \frac{H(n)}{H(n+1)} \right)^{\frac{1}{2}}. \tag{3.9}$$

This shows that for some weights, the constant $\frac{p}{p-1}$ in (3.6) cannot be sharp, i.e the constant is not the best possible. Otherwise, one obtains the sharpness of $\frac{p}{p-1}$, by making some additional assumption: Let $1 < q < \infty$ and furthermore, choose h such that $\lim_{n \to \infty} \frac{H(n)}{H(n+1)} = 1$ and $\lim_{n \to \infty} H(n) = \infty$. Then, for all $\delta > 0$, there exists $N = N(\delta) \in \mathbb{N}_0$ such that

$$\frac{H(n)}{H(n+1)} > 1 - \delta.$$

Additionally, let $\alpha = \frac{1}{q-\epsilon}$ for some $\epsilon > 0$. Define the sequence b^N by

$$b^N(n) = \begin{cases} 0 & \text{for} \quad n = 0, 1, ..., N, \\ \dfrac{1}{H(n)^\alpha} & \text{for} \quad n = N+1, N+2, ..., \end{cases}$$

for all $n \in \mathbb{N}_0$. Observe that

$$\|b^N\|_{q,h}^q = \sum_{n=N+1}^\infty \frac{h(n)}{H(n)^{\alpha q}} \leq \int_{H(N)}^\infty \frac{1}{x^{\alpha q}} \, \mathrm{d}x < \infty.$$

Considering the norm of $C_h^* b^N$, we obtain

$$\left\| C_h^* b^N \right\|_{q,h}^q = \sum_{n=N+1}^\infty h(n) \left(\sum_{k=n}^\infty \frac{h(k)}{H(k)^{\alpha+1}} \right)^q$$

$$> \sum_{n=N+1}^\infty h(n) \left(\sum_{k=n}^\infty \frac{(1-\delta)^{\alpha+1} h(k)}{H(k-1)^{\alpha+1}} \right)^q$$

$$> (1-\delta)^{q(\alpha+1)} \sum_{n=N+1}^\infty h(n) \left(\int_{H(n)}^\infty \frac{1}{x^{\alpha+1}} \right)^q$$

$$= (1-\delta)^{\alpha+1} (q-\epsilon)^q \sum_{n=N+1}^\infty \frac{h(n)}{H(n)^{\alpha q}}$$

$$= (1-\delta)^{\alpha+1} (q-\epsilon)^q \|b^N\|_{q,h}^q.$$

Letting $\delta, \epsilon \to 0$, results in the sharpness of the constant q which means that $\|C_h^*\| = q$ in $\ell^q(h)$. The following proposition is a direct consequence:

Proposition 3.3. *Let $1 < p < \infty$ and let the sequence of weights satisfy $\lim_{n \to \infty} \frac{H(n)}{H(n+1)} = 1$ and $\lim_{n \to \infty} H(n) = \infty$. Then, the weighted Cesàro operator in $\ell^p(h)$ has norm*

$$\|C_h\| = \frac{p}{p-1}.$$

3.2 The spectrum of the weighted Cesàro operator

In [BHS65] the spectrum of the Cesàro operator in ℓ^2 was investigated. Brown, Halmos and Shields proved the following result:

Theorem 3.4 (Brown, Halmos and Shields, 1965)**.** *Let C be the Cesàro operator in ℓ^2 and C^* its dual. Then the following statements hold:*

(i) The point spectrum of C is empty.

(ii) If $|1 - \lambda| < 1$, then λ is a simple eigenvalue of C^*.

(iii) The point spectrum of C^* is the open disc $\{\lambda : |1 - \lambda| < 1\}$.

(iv) The spectrum of C is the closed disc $\{\lambda : |1 - \lambda| \leq 1\}$.

For the proof, they used the properties of the operators $\mathsf{id} - C$ and $\mathsf{id} - C^*$, respectively, and deduced the point spectrum of C^* by direct computation. The respective results, one obtains for the weighted Cesàro operator in general, are the following:

Theorem 3.5. *Let h be a sequence of positive weights and let C_h be the Cesàro operator in $\ell^2(h)$ and C_h^* its dual. Then the following statements hold:*

(i) The point spectrum $\sigma_p(C_h)$ of C_h is included in the set $\left\{ \frac{h(n)}{H(n)} : n \in \mathbb{N}_0 \right\}$.

(ii) The point spectrum $\sigma_p(C_h^*)$ of C_h^* contains the set $\left\{ \frac{h(n)}{H(n)} : n \in \mathbb{N}_0 \right\}$ and is included in the closed disc $\left\{ \lambda : |1 - \lambda| \leq \left(\sup_{n \in \mathbb{N}_0} \frac{H(n)}{H(n+1)} \right)^{\frac{1}{2}} \right\}$.

(iii) The spectrum $\sigma(C_h)$ of C_h is contained in the closed disc $\left\{ \lambda : |1 - \lambda| \leq \left(\sup_{n \in \mathbb{N}_0} \frac{H(n)}{H(n+1)} \right)^{\frac{1}{2}} \right\}$.

Proof: Ad (i): Let $C_h a = \lambda a$ for some complex sequence a and some complex value λ. By definition we obtain for $n \in \mathbb{N}$

$$a(n) = \frac{H(n)}{h(n)} C_h a(n) - \frac{H(n-1)}{h(n)} C_h a(n-1)$$

$$\Leftrightarrow \quad a(n) = \lambda \left(\frac{H(n)}{h(n)} a(n) - \frac{H(n-1)}{h(n)} a(n-1) \right)$$

$$\Leftrightarrow \quad a(n) \left(\lambda \frac{H(n)}{h(n)} - 1 \right) = \lambda \frac{H(n-1)}{h(n)} a(n-1).$$

Choosing the smallest $m \in \mathbb{N}_0$ such that $a(m) \neq 0$, it follows that $\lambda_m = \frac{h(m)}{H(m)}$. Thus, the set $\{\lambda_m : m \in \mathbb{N}_0\}$ of possible eigenvalues of C_h coincides with the set $\left\{ \frac{h(n)}{H(n)} : n \in \mathbb{N}_0 \right\}$. The eigensequence a^m of C_h with respect to the eigenvalue λ_m is up to a constant given by

$$a^m(n) = \begin{cases} 0, & \text{for } n = 0, 1, ..., m-1, \\[2mm] 1, & \text{for } n = m, \\[2mm] \displaystyle\prod_{k=m+1}^{n} \frac{\frac{H(k-1)}{h(k)}}{\frac{H(k)}{h(k)} - \frac{H(m)}{h(m)}}, & \text{for } n = m+1, m+2, ... \,. \end{cases}$$

A first criterion for the well-definedness of a^m in $\ell^2(h)$ is

$$\frac{h(n)}{H(n)} \neq \frac{h(m)}{H(m)},$$

for all $n > m$. The second criterion is that the norm of a^m is finite, i.e.

$$\|a^m\|_{2,h}^2 = \sum_{n=m}^{\infty} h(n) a^m(n)^2 < \infty.$$

In [BHS65], the authors showed that in the unweighted case, the sequences a^m are not contained in ℓ^2, for all $m \in \mathbb{N}_0$. Nevertheless, we can find examples of weights, where a^m

is contained in $\ell^2(h)$, for some $m \in \mathbb{N}_0$. In particular, we will determine a class of examples, where the set $\left\{ \frac{h(n)}{H(n)} : n \in \mathbb{N}_0 \right\}$ is included in $\sigma_p(C_h)$. Therefore, let

$$h(n) = z^n, \qquad \text{for all } n \in \mathbb{N}_0,$$

where $0 < z < 1$. Hence, it follows that

$$H(n) = \frac{1 - z^{n+1}}{1 - z}.$$

We show that $\|a^m\|_{2,h}^2 < \infty$, for all $m \in \mathbb{N}_0$. Let $m \in \mathbb{N}_0$ be arbitrary and let $n > m$, then one obtains

$$\frac{h(n)(a^m(n))^2}{h(n-1)(a^m(n-1))^2} = z \left(\frac{H(n-1)}{H(n) - \frac{H(m)}{h(m)} h(n)} \right)^2$$

$$= z \left(\frac{1 - z^n}{1 - z^{n-m}} \right)^2$$

$$\leq z \left(1 + z^n \frac{z^{-m} - 1}{1 - z} \right)^2.$$

Since $z < 1$, there exist $N_m \in \mathbb{N}_0$, $N_m \geq m$, such that $z^n < \frac{1-z}{z^{-m}-1}(z^{-\frac{1}{2}} - 1)$, for all $n \geq N_m$. Therefore, it can be inferred that

$$\frac{h(n)(a^m(n))^2}{h(n-1)(a^m(n-1))^2} < z \left(1 + z^{N_m} \frac{z^{-m} - 1}{1 - z} \right)^2 < 1,$$

for all $n > N_m$, which implies that the summands in $\|a^m\|_{2,h}^2$ decay exponentially from $h(N_m)|a^m(N_m)|^2$ on.

Ad (ii): Because of (3.9), we obtain

$$\sigma_p(C_h^*) \subset \left\{ \lambda : |1 - \lambda| \leq \left(\sup_{n \in \mathbb{N}_0} \frac{H(n)}{H(n+1)} \right)^{\frac{1}{2}} \right\}.$$

As in [BHS65], we can compute candidates for eigensequences of C_h^*. Let $C_h^* b = \lambda b$ for some complex sequence b and some complex value λ. By definition, we obtain

$$\frac{h(n)}{H(n)} b(n) = C_h^*(b)(n) - C_h^*(b)(n+1)$$

$$\Leftrightarrow \quad b(n+1) = \left(1 - \frac{h(n)}{\lambda H(n)} \right) b(n),$$

for all $n \in \mathbb{N}_0$. A possible eigensequence of C_h^* with respect to the eigenvalue λ is the sequence b^λ, where

$$b^\lambda(n) = \prod_{k=0}^{n-1} \left(1 - \frac{h(k)}{\lambda H(k)} \right),$$

for all $n \in \mathbb{N}_0$. If $m \in \mathbb{N}_0$ is minimal such that $\lambda = \frac{h(m)}{H(m)}$, then b^λ is a finite sequence and thus, an element of $\ell^2(h)$. Moreover, one obtains $C_h^* b^\lambda = \lambda b^\lambda$. We can conclude that independently from the choice of h, we have

$$\left\{ \frac{h(n)}{H(n)} \; : \; n \in \mathbb{N}_0 \right\} \subset \sigma_p(C_h^*).$$

If $\lambda \neq \frac{h(n)}{H(n)}$, for all $n \in \mathbb{N}_0$, the question, whether b^λ is well-defined, cannot be answered positively. In fact, the proof in [BHS65] relies on the analytic result that

$$\lambda = \sum_{n=0}^{\infty} \frac{1}{n+1} \prod_{k=1}^{n} \left(1 - \frac{1}{\lambda k} \right).$$

Assume that in the general case, $C_h^* b^\lambda = \lambda b^\lambda$. In particular, $C_h^* b^\lambda(0) = \lambda b^\lambda(0) = \lambda$ which implies that

$$\lambda = \sum_{n=0}^{\infty} \frac{h(n)}{H(n)} \prod_{k=0}^{n-1} \left(1 - \frac{h(k)}{\lambda H(k)} \right)$$

has to be satisfied, independent of the choice of the weights. This is not true for arbitrary weights, in general.

Ad (*iii*): This statement is a consequence of (3.9) and (*ii*).

$\square$

Note that there are examples of weights for which

$$\sigma(C_h) = \{\lambda : |1 - \lambda| \leq 1\},$$

see [BHS65]. But in general, the spectrum is a subset of the set above and has to be investigated separately, for each class of weight sequence.

3.3 Cesàro operator and the generalized concepts of normality

In this section, we will introduce several operator classes that are related to normal operators. We investigate, to which class of operators the weighted Cesàro operator belongs to. Some of the key results have already been published in [Wag13].

3.3.1 Classes of operators with a weak normality condition

We will now focus our attention on the weighted Cesàro operator in $\ell^2(h)$ and the property of normality in Hilbert spaces. By weakening the conditions of normality in various ways, one obtains the following classes of not necessarily normal operators, see [Fur67] and [Hal82, problems 137, 195, 203 and 216]:

Definition 3.6 (generalized concepts of normality)**.** *Let H be a Hilbert space and T be a bounded linear operator in H, symbolically $T \in B(H)$. Then, T is called*

1. ***normal***, *if and only if $T^*T = TT^*$.*

2. ***quasinormal***, *if and only if $T^*TT = TT^*T$.*

3. ***subnormal***, *if and only if T has a normal extension, i.e. there exists a Hilbert space K, H can be embedded in K, and a normal operator $N \in B(K)$ which has the shape $N = \begin{pmatrix} T & B \\ 0 & A \end{pmatrix}$, where A, B are bounded operators.*

4. ***hyponormal***, *if and only if $T^*T \geq TT^*$, i.e. $T^*T - TT^*$ is positive.*

5. ***paranormal***, *if and only if $\|T^2x\| \geq \|Tx\|^2$, for all $x \in H$ with $\|x\| = 1$.*

6. ***normaloid***, *if and only if $r(T) = \|T\|$.*

As shown in [Fur67], one obtains the following inclusion relations for the operator classes and all inclusions are proper, if appropriate spaces are chosen:

normal operators $\subset$ quasinormal operators $\subset$ subnormal operators

$\subset$ hyponormal operators $\subset$ paranormal operators $\subset$ normaloid operators.

In their 1965 paper, Brown, Halmos and Shields showed that the Cesàro operator in ℓ^2 is hyponormal, see [BHS65]. Later on, Kriete, Trutt ([KT71]) and Cowen ([Cow84]) proved the subnormality of the Cesàro operator in ℓ^2. Here, the properties of the weighted Cesàro operator C_h in $\ell^2(h)$ shall be investigated. To which class of operators from Definition 3.6 the operator C_h belongs, depends on the sequence h.

The remaining part of the section is organized as follows: First, necessary and sufficient conditions for the hyponormality of the Cesàro operator are studied. Then, the Haar measures of Jacobi polynomials and polynomials related to homogeneous trees are discussed as examples of weights for which C_h becomes hyponormal. Afterwards, we exhibit sequences of weights for which C_h is not paranormal and not normaloid, respectively. Last but not least, we show that C_h never satisfies the conditions of quasinormality, independently from the choice of h.

3.3.2 Hyponormality of the Cesàro operator in $\ell^2(h)$

Let e_j be the jth unit sequence, $e_j(i) = \delta_{ij}$, $i, j \in \mathbb{N}_0$. Due to the fact that C_h and C_h^* are operators in a sequence space, they have matrix representations with respect to the basis $(e_j)_{j \in \mathbb{N}_0}$ of $\ell^2(h)$, see (3.5) Direct computation yields the matrix representations of $C_h \circ C_h^*$ and $C_h^* \circ C_h$ with respect to $(e_j)_{j \in \mathbb{N}_0}$:

$$
C_h C_h^* = \begin{pmatrix}
h(0)\alpha_0 & h(1)\alpha_1 & h(2)\alpha_2 & \cdots \\
h(0)\alpha_1 & h(1)\alpha_1 & h(2)\alpha_2 & \cdots \\
h(0)\alpha_2 & h(1)\alpha_2 & h(2)\alpha_2 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad \text{with } \alpha_n = \frac{1}{H(n)} \tag{3.10}
$$

and

$$
C_h^* C_h = \begin{pmatrix}
h(0)\beta_0 & h(1)\beta_1 & h(2)\beta_2 & \cdots \\
h(0)\beta_1 & h(1)\beta_1 & h(2)\beta_2 & \cdots \\
h(0)\beta_2 & h(1)\beta_2 & h(2)\beta_2 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad \text{with } \beta_n = \sum_{k=n}^{\infty} \frac{h(k)}{H(k)^2}. \tag{3.11}
$$

The associated matrices of $C_h \circ C_h^*$ and $C_h^* \circ C_h$ in (3.10) and (3.11), respectively, have the same shape. Despite the prefactor $h(j)$ in the jth column, the above matrices are 'L-shaped' as the ones analysed in [BHS65].

Theorem 3.7. *The weighted Cesàro operator C_h in $\ell^2(h)$ is hyponormal (i.e. $C_h^* C_h - C_h C_h^*$ is positive), if and only if*

$$(1) \qquad \sum_{k=n}^{\infty} \frac{h(k)}{H(k)^2} - \frac{1}{H(n)} \geq 0, \qquad \text{for all } n \in \mathbb{N}_0, \quad \text{and}$$

$$(2) \qquad H(n)^2 \geq H(n-1)H(n+1), \qquad \text{for all } n \in \mathbb{N}_0, \quad \text{where } H(-1) := 0.$$

Proof: Let

$$T := C_h^* C_h - C_h C_h^* \quad \text{and} \quad \gamma_n := \beta_n - \alpha_n = \sum_{k=n}^{\infty} \frac{h(k)}{H(k)^2} - \frac{1}{H(n)}.$$

Utilizing (3.10) and (3.11), it follows that

$$\begin{pmatrix} h(0)\gamma_0 & h(1)\gamma_1 & h(2)\gamma_2 & \cdots \\ h(0)\gamma_1 & h(1)\gamma_1 & h(2)\gamma_2 & \cdots \\ h(0)\gamma_2 & h(1)\gamma_2 & h(2)\gamma_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad (3.12)$$

is the associated matrix of T.

In [BHS65] the positivity of the matrix T acting in ℓ^2 was proved by considering the determinants of its finite sections. In order to include the case when the matrix T is positive semidefinite, we give a more detailed proof for the positivity of the operator T here.

First observe that $\|T\| \leq 2\|C_h\|^2 \leq 8$ and that $\langle Ta, b \rangle_h = \langle a, Tb \rangle_h$ for all $a, b \in \ell^2(h)$. Moreover, every $a \in \ell^2(h)$ has the unique representation

$$a = \sum_{n=0}^{\infty} a(n)e_n,$$

where e_n shall denote the nth unit sequence. For $n \in \mathbb{N}_0$, define a_n by

$$a_n = \sum_{k=0}^{n} a(k)e_k.$$

For all $0 \neq a \in \ell^2(h)$ and all $\epsilon > 0$ there exists $N = N_{a,\epsilon} \in \mathbb{N}_0$ such that

$$\|a - a_n\|_{2,h} < \frac{\epsilon}{16\|a\|_{2,h}}, \tag{3.13}$$

for all $n \geq N_{a,\epsilon}$.

Finally, define by c_k the sequence

$$c_k := \frac{e_k}{h(k)} - \frac{e_{k+1}}{h(k+1)}, \quad \text{for all } k \in \mathbb{N}_0$$

and let

$$\mathcal{B}_n = \{c_0, c_1, ..., c_{n-1}, e_n\}, \quad \text{for all } n \in \mathbb{N}_0.$$

It follows that $\mathsf{span}(\mathcal{B}_n) = \mathbb{C}^n \bigoplus \{0\}$ and that for each $n \in \mathbb{N}_0$, a_n has the representation

$$a_n = \sum_{k=0}^{n-1} a^{(n)}(k) c_k + a^{(n)}(n) e_n \tag{3.14}$$

with unique coefficients $a^{(n)}(k)$, $0 \leq k \leq n$. Using the definition of T and particularly (3.12), one obtains

$$Tc_k = \gamma_k \sum_{m=0}^{k} e_m + \sum_{m=k+1}^{\infty} \gamma_m e_m - \gamma_{k+1} \sum_{m=0}^{k+1} e_m - \sum_{m=k+2}^{\infty} \gamma_m e_m = (\gamma_k - \gamma_{k+1}) \sum_{m=0}^{k} e_m,$$

for all $k \in \mathbb{N}_0$, and hence,

$$\langle Tc_k, e_n \rangle_h = 0, \qquad\qquad \text{for} \quad 0 \leq k < n,$$

$$\langle Tc_k, c_j \rangle_h = \delta_{kj}(\gamma_k - \gamma_{k+1}), \qquad\qquad \text{for} \quad k, j \in \mathbb{N}_0, \tag{3.15}$$

$$\langle Te_n, e_m \rangle_h = h(n)h(m)\gamma_{\mathsf{max}(n,m)}, \qquad\qquad \text{for} \quad n, m \in \mathbb{N}_0.$$

Now, let us assume that T is positive. Then $\langle a, Ta \rangle_h \geq 0$, for all $a \in \ell^2(h)$.
In particular, by (3.15) we have

$$0 \leq \langle e_n, Te_n \rangle_h = h(n)^2 \gamma_n \Leftrightarrow \sum_{k=n}^{\infty} \frac{h(k)}{H(k)^2} - \frac{1}{H(n)} \geq 0$$

and

$$0 \leq \langle c_n, Tc_n \rangle_h = \gamma_n - \gamma_{n+1} \quad \Leftrightarrow \sum_{k=n}^{\infty} \frac{h(k)}{H(k)^2} - \frac{1}{H(n)} \geq \sum_{k=n+1}^{\infty} \frac{h(k)}{H(k)^2} - \frac{1}{H(n+1)}$$

$$\Leftrightarrow \frac{h(n)}{H(n)^2} - \frac{1}{H(n)} \geq -\frac{1}{H(n+1)}$$

$$\Leftrightarrow \frac{H(n-1)}{H(n)^2} \leq \frac{1}{H(n+1)}$$

$$\Leftrightarrow H(n)^2 \geq H(n+1)H(n-1),$$

for all $n \in \mathbb{N}_0$, which shows that the conditions (1) and (2) hold for hyponormal T. Conversely, let us assume that the conditions (1) and (2) are satisfied, i.e. $\gamma_n \geq \gamma_{n+1} \geq 0$, for all $n \in \mathbb{N}_0$. Let $0 \neq a \in \ell^2(h)$ be arbitrary. According to (3.13), for all $\epsilon > 0$ we obtain

$$\langle Ta, a \rangle_h = \langle Ta_n, a_n \rangle_h + \langle Ta_n, a - a_n \rangle_h + \langle T(a - a_n), a \rangle_h$$

$$\geq \langle Ta_n, a_n \rangle_h - \|a - a_n\|_{2,h} \|T\|(\|a\| + \|a_n\|)$$

$$> \langle Ta_n, a_n \rangle_h - \epsilon$$

by choosing some $n \geq N_{a,\epsilon}$. By assumption and from (3.14) and (3.15), we can infer that

$$\langle Ta_n, a_n \rangle_h = \sum_{m,k=0}^{n-1} a^{(n)}(m)\overline{a^{(n)}(k)}\langle c_m, Tc_k \rangle_h + \sum_{k=0}^{n-1} a^{(n)}(k)\overline{a^{(n)}(n)}\langle Tc_k, a_n \rangle_h$$

$$+ \sum_{k=0}^{n-1} a^{(n)}(n)\overline{a^{(n)}(k)}\langle Te_n, c_k \rangle_h + a^{(n)}(n)\overline{a^{(n)}(n)}\langle e_n, Te_n \rangle_h$$

$$= \sum_{k=0}^{n-1} |a^{(n)}(k)|^2(\gamma_k - \gamma_{k+1}) + h(n)^2 |a^{(n)}(n)|^2 \gamma_n \geq 0.$$

Hence, for all $a \in \ell^2(h)$ and all $\epsilon > 0$, it follows that $\langle Ta, a \rangle_h > -\epsilon$ and therefore, $\langle Ta, a \rangle_h \geq 0$.

$\square$

Before discussing several examples, the next theorem will give equivalent conditions for the hyponormality of the Cesàro operator.

Theorem 3.8. *The weighted Cesàro operator C_h in $\ell^2(h)$ is hyponormal, if and only if*

$$(1)' \qquad H := \lim_{n\to\infty} H(n) = \infty \qquad\qquad and$$

$$(2)' \qquad \frac{h(n)}{H(n)} \geq \frac{h(n+1)}{H(n+1)}, \qquad\qquad for\ all\ n \in \mathbb{N}_0.$$

Proof: First note that the conditions (2)' and (2) are equivalent, because for all $n \in \mathbb{N}_0$ and $H(-1) := 0$, we have

$$\frac{h(n)}{H(n)} \geq \frac{h(n+1)}{H(n+1)} \quad \Leftrightarrow \quad \frac{H(n) - H(n-1)}{H(n)} \geq \frac{H(n+1) - H(n)}{H(n+1)}$$

$$\Leftrightarrow \quad \frac{H(n)}{H(n+1)} \geq \frac{H(n-1)}{H(n)}$$

$$\Leftrightarrow \quad H(n)^2 \geq H(n-1)H(n+1).$$

If additionally condition (1)' is satisfied, we obtain

$$\sum_{k=n}^{\infty} \frac{h(k)}{H(k)^2} - \frac{1}{H(n)} = \sum_{k=n}^{\infty} \frac{h(k)}{H(k)^2} - \sum_{k=n}^{\infty} \left(\frac{1}{H(k)} - \frac{1}{H(k+1)} \right)$$

$$= \sum_{k=n}^{\infty} \frac{1}{H(k)} \left(\frac{h(k)}{H(k)} - \frac{h(k+1)}{H(k+1)} \right) \overset{(2)'}{\geq} 0$$

which is (1). On the other hand, if $H < \infty$, we have

$$\sum_{k=n}^{\infty} \frac{h(k)}{H(k)^2} \leq \int_{H(n)}^{H} \frac{1}{x^2}\,\mathrm{d}x + \frac{h(n)}{H(n)^2}$$

$$= -\frac{1}{H} + \frac{1}{H(n)} + \frac{h(n)}{H(n)^2} \overset{n\to\infty}{\to} 0.$$

Thus,

$$\lim_{n\to\infty} \left(\sum_{k=n}^{\infty} \frac{h(k)}{H(k)^2} - \frac{1}{H(n)} \right) = -\frac{1}{H} < 0,$$

and (1) is not satisfied.

$\square$

In the following two examples, h is the Haar measure of certain orthogonal polynomial sequences which are defined as in Chapter 2.

Example 3.9 (Haar weights of the normalized Jacobi polynomials)**.** *Let $\alpha, \beta > -1$ and $(R_n^{(\alpha,\beta)}(x))_{n \in \mathbb{N}_0}$ be defined by (2.1), where*

$$a_n \ = \ \frac{2(n + \alpha + 1)(n + \alpha + \beta + 1)}{(2n + \alpha + \beta + 1)(2n + \alpha + \beta + 2)},$$

$$b_n \ = \ \frac{\beta^2 - \alpha^2}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)},$$

$$c_n \ = \ \frac{2n(n + \beta)}{(2n + \alpha + \beta + 1)(2n + \alpha + \beta)},$$

see [Las83] or [Las94]. Then, we obtain for the Haar weights

$$h(n) = \frac{(\alpha + 1)_n (\alpha + \beta + 1)_n (2n + \alpha + \beta + 1)}{(\beta + 1)_n n! (\alpha + \beta + 1)},$$

where we denote by $(a)_n$ the Pochhammer symbol for $a \in \mathbb{R}$ and $n \in \mathbb{N}_0$ which is

$$(a)_n = \begin{cases} 1, & \text{for} \quad n = 0, \\ a(a + 1) \ldots (a + n - 1), & \text{for} \quad n \in \mathbb{N}. \end{cases}$$

Inductively, one has

$$H(n) = \frac{(\alpha + \beta + 1)_{n+1} (\alpha + 2)_n}{(\beta + 1)_n n! (\alpha + \beta + 1)}.$$

We want to check, whether the conditions in Theorem 3.8 are satisfied. Since

$$\alpha + \beta + 2 > \beta + 1 > 0 \quad \text{and} \quad \alpha + 2 > 1,$$

it follows that

$$H(n) = \frac{(\alpha + \beta + 2)_n}{(\beta + 1)_n} \cdot \frac{(\alpha + 2)_n}{(1)_n} \overset{n \to \infty}{\to} \infty$$

which is condition (1)'. To verify condition (2)', observe that for $n = 1, 2, 3, ...$

$$\frac{h(n)}{H(n)} - \frac{h(n+1)}{H(n+1)} = \frac{(\alpha+1)(2n+\alpha+\beta+1)}{(n+\alpha+\beta+1)(n+\alpha+1)} - \frac{(\alpha+1)(2n+\alpha+\beta+3)}{(n+\alpha+\beta+2)(n+\alpha+2)}.$$

First note that $1 = \frac{h(0)}{H(0)} > \frac{h(1)}{H(1)}$ by definition. Thus, we have to check, whether

$$(2n+\alpha+\beta+1)(n+\alpha+\beta+2)(n+\alpha+2)$$
$$\geq (2n+\alpha+\beta+3)(n+\alpha+\beta+1)(n+\alpha+1) \qquad (3.16)$$

holds for all $n \in \mathbb{N}$.

$$
\begin{aligned}
(3.16) \quad &\Leftrightarrow \quad (2n+\alpha+\beta+1) \\
&\qquad \times ((n+\alpha+\beta+1)(n+\alpha+1) + n+\alpha+\beta+1 + n+\alpha+2) \\
&\qquad \geq ((2n+\alpha+\beta+1)+2)(n+\alpha+\beta+1)(n+\alpha+1) \\[6pt]
&\Leftrightarrow \quad (2n+\alpha+\beta+1)(2n+2\alpha+\beta+3) \\
&\qquad \geq (2n+2\alpha+2\beta+2)(n+\alpha+1) \\[6pt]
&\Leftrightarrow \quad ((n+\beta)+(n+\alpha+1))((2n+2\alpha+\beta+2)+1) \\
&\qquad \geq ((2n+2\alpha+\beta+2)+\beta)(n+\alpha+1) \\[6pt]
&\Leftrightarrow \quad (n+\beta)(2n+2\alpha+\beta+2) + (n+\beta) + (n+\alpha+1) \geq \beta(n+\alpha+1) \\[6pt]
&\Leftrightarrow \quad n(2n+2\alpha+\beta+2) + \beta(n+\alpha+\beta+1) \\
&\qquad + (n+\beta) + (n+\alpha+1) \geq 0 \\[6pt]
&\Leftrightarrow \quad n(2n+2\alpha+\beta+3) + (\beta+1)(n+\alpha+\beta+1) \geq 0
\end{aligned}
$$

which is satisfied, since for $n \in \mathbb{N}$ and $\alpha, \beta > -1$, both summands are positive. Therefore, the weights of the normalized Jacobi polynomials define a hyponormal Cesàro operator.

Example 3.10 (Haar weights of polynomials connected with homogeneous trees)**.** *Let $a \geq 2$ and $(R_n^a(x))_{n \in \mathbb{N}_0}$ be defined by (2.1), where $a_0 = 1$ and $a_n = \frac{a-1}{a}$, $c_n = \frac{1}{a}$, $n \in \mathbb{N}$. We obtain $h(0) = 1$ and, by using $h(n+1) = \frac{a_n}{c_{n+1}}h(n)$,*

$$h(n) = a(a-1)^{n-1}, \quad \text{for} \quad n \in \mathbb{N},$$

see [Las83]. For $a = 2$, these are the weights for the Tchebichef polynomials of first kind, which are in the class of the Jacobi polynomials. Now, let $a \neq 2$ and observe that

$$H(n) = 1 + a\sum_{k=0}^{n-1}(a-1)^k = 1 + a\frac{(a-1)^n - 1}{(a-1) - 1} = \frac{h(n+1) - 2}{a-2}, \quad n \in \mathbb{N}_0.$$

Thus, a necessary condition for the hyponormality of the corresponding Cesàro operator is

$$\lim_{n \to \infty} H(n) = \infty \Leftrightarrow a - 1 > 1 \Leftrightarrow a > 2.$$

We show that in this case condition (2)' is satisfied either. By definition it follows that

$$\frac{h(0)}{H(0)} = 1 > \frac{h(n)}{H(n)} = 1 - \frac{H(n-1)}{H(n)}, \quad \text{for all} \quad n \in \mathbb{N}.$$

Furthermore, for $n \in \mathbb{N}$, one obtains

$$\frac{h(n)}{H(n)} - \frac{h(n+1)}{H(n+1)} = \frac{1}{H(n)H(n+1)}(h(n)H(n+1) - h(n+1)H(n))$$

$$= \frac{h(n)}{H(n)H(n+1)}(H(n+1) - (a-1)H(n))$$

$$= \frac{h(n)}{H(n)H(n+1)}\left(\frac{h(n+2) - 2}{a-2} - (a-1)\frac{h(n+1) - 2}{a-2}\right)$$

$$= \frac{h(n)}{H(n)H(n+1)}\left(\frac{h(n+1)(a-1) - 2 - (a-1)h(n+1) + 2(a-1)}{a-2}\right)$$

$$= \frac{2h(n)}{H(n)H(n+1)}.$$

Therefore, for all $n \in \mathbb{N}_0$, it can be inferred that

$$\frac{h(n)}{H(n)} - \frac{h(n+1)}{H(n+1)} \geq 0.$$

3.3.3 Necessary conditions for paranormal and normaloid Cesàro operators in $\ell^2(h)$

As the conditions of Theorem 3.7 and Theorem 3.8, respectively, are not always satisfied, there must be some h for which C_h is not hyponormal. Now the question arises, whether the Cesàro satisfies the weaker conditions for paranormal or normaloid operators. The following theorem exhibits some weights for which the weaker condition for paranormality is not satisfied.

Theorem 3.11. *Let $\lim_{n\to\infty} H(n) =: H < \infty$. Then, the Cesàro operator in $\ell^2(h)$ is not paranormal.*

Proof: We will give an example for a sequence in $\ell^2(h)$ for which the inequality in the condition for the paranormality of the Cesàro operator is not satisfied. For $n \in \mathbb{N}_0$, define the sequence χ_n by

$$\chi_n(k) = \begin{cases} 1, & \text{for } k = 0, 1, 2, ..., n, \\ 0, & \text{for } k = n+1, n+2, n+3, ..., \end{cases} \tag{3.17}$$

i.e. $\chi_n = (1, 1, ..., 1, \mathbf{1}, 0, 0, ...)$, where the bold element marks position n. Since $H(n) \to H$ there exists $N \in \mathbb{N}_0$ such that $\frac{H(N)^2}{H^2} > \frac{1}{2}$. Then,

$$C_h\chi_N = \left(1, 1, ..., 1, \mathbf{1}, \frac{H(N)}{H(N+1)}, \frac{H(N)}{H(N+2)}, ...\right) \quad \text{and}$$

$$C_h^2\chi_N = \left(1, 1, ..., 1, \mathbf{1}, \frac{H(N) + \frac{h(N+1)H(N)}{H(N+1)}}{H(N+1)}, \frac{H(N) + \frac{h(N+1)H(N)}{H(N+1)} + \frac{h(N+2)H(N)}{H(N+2)}}{H(N+2)}, ...\right).$$

Observe that $\|\chi_N\|^2 = H(N)$ and

$$\|C_h\chi_N\|^2 = H(N) + \sum_{n=N+1}^{\infty} h(n)\frac{H(N)^2}{H(n)^2} > H(N) + \frac{H(N)^2}{H^2}(H - H(N)),$$

$$\|C_h^2\chi_N\|^2 = H(N) + \sum_{n=N+1}^{\infty} h(n)\left(\frac{H(N) + \sum_{k=N+1}^{n} h(k)\frac{H(N)}{H(k)}}{H(n)}\right)^2 < H.$$

Considering the choice of N, we can conclude that

$$\|C_h\chi_N\|^4 > H(N)^2 + H(N)(H - H(N)) = H(N)H > \|\chi_N\|^2\|C_h^2\chi_N\|^2.$$

$\square$

Theorem 3.11 shows that a necessary condition for the paranormality of the weighted Cesàro operator is the divergence of the sequence H. An even stronger assertion proves to be true. In fact, the next theorem shows that the convergence of the series H is a sufficient criteria for the Cesàro operator to be not normaloid.

Theorem 3.12. *Let* $\lim_{n\to\infty} H(n) =: H < \infty$ *. Then, the Cesàro operator is not normaloid.*

Proof: We will show, that for weights with the above property, the respective Cesàro operator is a Hilbert-Schmidt operator, symbolically $C_h \in HS(\ell^2(h))$, and hence, is compact. Following Definition 1.34, we show that there exists an orthonormal bases $(g_n)_{n\in\mathbb{N}_0} \subset \ell^2(h)$, such that $\sum_{n=0}^{\infty} \|C_h^* g_n\|_{2,h}^2 < \infty$. Let e_n denote the nth unit sequence and define g_n by $g_n := \frac{e_n}{\sqrt{h(n)}}$ for all $n \in \mathbb{N}_0$. It follows that

$$C_h^* g_n = \frac{\sqrt{h(n)}}{H(n)} \sum_{k=0}^{n} e_k, \quad \text{for all } n \in \mathbb{N}_0,$$

which implies

$$\|C_h^* g_n\|_{2,h}^2 = \frac{h(n)}{H(n)^2} \sum_{k=0}^{n} h(k) = \frac{h(n)}{H(n)}, \quad \text{for all } n \in \mathbb{N}_0.$$

Altogether, we have

$$\sum_{n=0}^{\infty} \|C_h^* g_n\|_{2,h}^2 = \sum_{n=0}^{\infty} \frac{h(n)}{H(n)} < H < \infty,$$

by the choice of h. It follows that C_h^* and C_h are contained in $HS(\ell^2(h))$ and thus compact. Since C_h is compact, we can infer from spectral theory, see Theorem 1.32 and Theorem 3.5 that the spectrum of C_h is given by

$$\sigma(C_h) = \{0\} \cup \left\{ \frac{h(n)}{H(n)} : n \in \mathbb{N}_0 \right\}.$$

In particular, every value besides zero is an eigenvalue of C_h, i.e.

$$\sigma_p(C_h) = \sigma(C_h) \backslash \{0\}.$$

Hence, for the spectral radius of C_h, we obtain $r(C_h) = 1$.
Otherwise, C_h satisfies $\|C_h\| > 1$, since for example, $\|e_0\|_{2,h}^2 = 1$ and

$$\|C_h e_0\|_{2,h}^2 = 1 + \sum_{n=1}^{\infty} \frac{h(n)}{H(n)^2} > 1.$$

Thus, we have $r(C_h) < \|C_h\|$ and the assertion follows.

$\square$

Note that if additionally $\sum_{k=0}^{\infty} \sqrt{h(k)} < \infty$, one can easily show that C_h is nuclear. A nuclear representation in the sense of Definition 1.33 is given by

$$C_h a = \sum_{n=0}^{\infty} \langle a, y_n \rangle_h e_n, \qquad (3.18)$$

where $a \in \ell^2(h)$, $y_n = \frac{1}{H(n)} \sum_{k=0}^{n} e_k$ and e_n denotes the nth unit sequence for all $n \in \mathbb{N}_0$. This can be verified by the following computation:

$$\sum_{n=0}^{\infty} \|y_n\|_{2,h} \|e_n\|_{2,h} = \sum_{n=0}^{\infty} \sqrt{h(n)} \left(\frac{1}{H(n)^2} \sum_{k=0}^{n} h(k) \right)^{\frac{1}{2}}$$

$$< \sum_{n=0}^{\infty} \sqrt{h(n)} < \infty$$

which shows that the right hand side in equation (3.18) is a nuclear representation of C_h by definition. Theorem 3.12 shows that a necessary condition for the fact that C_h is normaloid, is

$$\sum_{n=0}^{\infty} h(n) = \infty.$$

3.3.4 The weighted Cesàro operator and quasinormality

On the one hand, the weights of Theorem 3.12 do not satisfy any weak normality condition stronger than the condition of normaloid operators. But on the other hand, the Cesàro operator is subnormal in the unweighted case, see [KT71] and [Cow84]. Here, the question arises, whether the even stronger condition, quasinormality, can be satisfied. The next theorem answers this question in the negative and also implies, that subnormality is the strongest property (in terms of the generalized concepts of normality) C_h can have.

Theorem 3.13. *The weighted Cesàro operator C_h in $\ell^2(h)$ is not quasinormal independently from the choice of weights.*

Proof: By definition, C_h is quasinormal if and only if $(C_h^* C_h - C_h C_h^*)C_h = 0$. Let us define T and γ_n as in the proof of Theorem 3.7. Considering the matrix representations for the operators yields

$$
TC_h(i,j) = (h(0)\gamma_i, ..., h(i)\gamma_i, h(i+1)\gamma_{i+1}, ...)
$$

$$
\times (\underbrace{0, ..., 0}_{j\ times}, \frac{h(j)}{H(j)}, \frac{h(j)}{H(j+1)}, \frac{h(j)}{H(j+2)}, ...)^T
$$

$$
= \sum_{k=j}^{\infty} h(k)\gamma_{\max(i,k)} \frac{h(j)}{H(k)}
$$

$$
= h(j)
\begin{cases}
\gamma_i \displaystyle\sum_{k=j}^{i-1} \frac{h(k)}{H(k)} + \sum_{k=i}^{\infty} \frac{h(k)}{H(k)}\gamma_k, & \text{for } i > j, \\[2ex]
\displaystyle\sum_{k=j}^{\infty} \frac{h(k)}{H(k)}\gamma_k, & \text{for } i \leq j,
\end{cases}
\tag{3.19}
$$

for all $i, j \in \mathbb{N}_0$. Assume that C_h is quasinormal for some sequence h. Then, by definition, $\sum_{k=n}^{\infty} \frac{h(k)}{H(k)}\gamma_k = 0$ for all $n \in \mathbb{N}_0$. Hence,

$$
\gamma_n = \frac{H(n)}{h(n)} \left(\sum_{k=n}^{\infty} \frac{h(k)}{H(k)}\gamma_k - \sum_{k=n+1}^{\infty} \frac{h(k)}{H(k)}\gamma_k \right) = 0, \quad \text{for all } n \in \mathbb{N}_0,
\tag{3.20}
$$

which is the condition for normality, i.e. $T = 0$. The definition of γ_n and (3.20) imply

$$
\frac{1}{H(n)} = \sum_{k=n}^{\infty} \frac{h(k)}{H(k)^2} = \frac{h(n)}{H(n)^2} + \sum_{k=n+1}^{\infty} \frac{h(k)}{H(k)^2} = \frac{h(n)}{H(n)^2} - \frac{1}{H(n+1)},
\tag{3.21}
$$

or, equivalently,

$$\frac{1}{H(n+1)} = \frac{1}{H(n)}\left(\frac{h(n)}{H(n)} - 1\right), \quad \text{for all } n \in \mathbb{N}_0.$$

Then, in particular for $n = 0$, we have

$$0 \neq \frac{1}{H(1)} = \frac{1}{H(0)}\left(\frac{h(0)}{H(0)} - 1\right) = 0$$

which is a contradiction. Thus, there exists no sequence h of weights for which C_h is quasinormal or normal.

$\square$

3.4 Remarks on the Cesàro operator in ℓ^∞ and c

For the sake of completeness, this section contains some remarks on the behaviour of the weighted Cesàro operators in two further well-known sequence spaces. As usual, we denote by ℓ^∞ and c, respectively, the space of all bounded and that of all convergent sequences, respectively, which are defined by

$$\ell^\infty = \{a = (a(n))_{n\in\mathbb{N}_0} : a(n) \in \mathbb{C}, \|a\|_\infty := \sup_{n\in\mathbb{N}_0} |a(n)| < \infty\},$$

$$c = \{a = (a(n))_{n\in\mathbb{N}_0} \in \ell^\infty : a \text{ converges, i.e. } \lim_{n\to\infty} a(n) \text{ exists}\}.$$

Obviously, $C_h : \ell^\infty \to \ell^\infty$ is bounded for an arbitrary sequence h of weights with $\|C_h\| = 1$. Correspondingly, the question arises, whether C_h maps c into itself, symbolically, $C_h c \subset c$. Using results from the theory of summability, particularly applying [Boo00, p. 46 ff., Theorem 2.3.7 (of Toeplitz, Silverman, Kojima and Schur) I], the question can be answered positively, independently from the choice of weights, i.e. $C_h c \subset c$, for all h. Note that, according to [Boo00, p. 21], C_h is called a **conservative** matrix method. Moreover, we obtain the following:

- By [Boo00, p. 46 f., Theorem 2.3.7 II] it can be easily verified that C_h even preserves the limit for certain weights, more precisely,

$$\lim_{n\to\infty} C_h a(n) = \lim_{n\to\infty} a(n), \qquad \text{for all } a \in c,$$

if and only if $\lim_{n\to\infty} H(n) = \infty$. In this case, the matrix (summability) method C_h is called **regular**, cf. [Boo00, p. 23]. When $\lim_{n\to\infty} H(n) = H < \infty$, a suitable counter-example is given by e_0, since

$$\lim_{n\to\infty} C_h e_0(n) = \frac{1}{H} \neq 0 = \lim_{n\to\infty} e_0(n).$$

- Following [Boo00, p. 51, Theorem 2.4.1 (of Schur)], one obtains $C_h \ell^\infty \subset c$, if and only if $\lim_{n\to\infty} H(n) = H < \infty$ and, in keeping with [Boo00, p. 21], C_h is **coercive** in this case. If $(H(n))_{n\in\mathbb{N}_0}$ converges, another interesting circumstance is that $\ell^\infty \subset \ell^p(h) \subset \ell^1(h)$, for $p > 1$.

4 Relations between the Cesàro operator, orthogonal polynomial systems and the conjugate orthogonal polynomial system

4.1 Cesàro operator and the system $(R_n^-(x))_{n \in \mathbb{N}_0}$

The first section in this chapter concerns the relations between the Cesàro operator and the Christoffel-Darboux Identity. Let $(R_n(x))_{n \in \mathbb{N}_0}$ be as in Definition 2.1 and furthermore, fix $x_0 \in \mathcal{S}$ and $N \in \mathbb{N}_0$. Then, $r_{x_0,N}$, defined by

$$r_{x_0,N}(n) = \begin{cases} R_n(x_0), & \text{for } n = 0, 1, ..., N, \\ 0, & \text{for } n = N+1, N+2, ..., \end{cases}$$

is a sequence in $\ell^p(h)$, $1 < p < \infty$. Calculating $C_h r_{x_0,n}$, where h is the Haar measure of $(R_n(x))_{n \in \mathbb{N}_0}$, we obtain

$$C_h r_{x_0,N}(n) = \frac{1}{H(n)} \sum_{k=0}^{n} h(k) R_k(x_0) = R_n^-(x_0) = r_{x_0,N}^-(n), \tag{4.1}$$

for all $n \leq N$. The system $(R_n^-(x))_{n \in \mathbb{N}_0}$ was, up to the the factors $H(n)$, defined by the Christoffel-Darboux Identity in (2.9) for $(R_n(x))_{n \in \mathbb{N}_0}$ with $y = 1$. On the one hand, equation (4.1) describes a relation between the system $(R_n^-(x))_{n \in \mathbb{N}_0}$ on the other hand, by using the transform in (2.21), we obtain another formula. Let h be the Haar measure of $(R_n(x))_{n \in \mathbb{N}_0}$. The operator $\mathcal{P}_h$ maps $L^2(\mu)$ to $\ell^2(h)$ and shows the one-to-one correspondence between those two spaces, see Chapter 2. For $n \in \mathbb{N}_0$, the polynomial R_n is mapped to the sequence $\frac{e_n}{h(n)}$ and the inverse $\mathcal{P}_h^{-1}$ of $\mathcal{P}_h$ is well-defined. Since for all $n \in \mathbb{N}_0$,

$$C_h^* \frac{e_n}{h(n)} = \frac{1}{H(n)} \sum_{k=0}^{n} e_k$$

and

$$\mathcal{P}_h^{-1} e_n = h(n) R_n,$$

we obtain

$$R_n^- = \frac{1}{H(n)} \sum_{k=0}^{n} h(k) R_k = (\mathcal{P}_h^{-1} \circ C_h^*) \frac{e_n}{h(n)}.$$

4.2 On the spectrum of tridiagonal operators and the support of orthogonalization measures

In [LOW13], a certain class of orthogonal polynomials was studied. The notations here will slightly differ from those in the paper in order to keep a consistent style in the whole thesis. Let $(R_n(x))_{n\in\mathbb{N}_0}$ satisfy (2.1), namely,

$$R_0(x) = 1, \quad R_1(x) = (x - b_0)/a_0,$$

$$x R_n(x) = a_n R_{n+1}(x) + b_n R_n(x) + c_n R_{n-1}(x), \quad n \in \mathbb{N}, \tag{4.2}$$

where $a_0 + b_0 = 1$ and additionally $a_n + b_n + c_n = 1$, $a_n, c_n > 0$ and $|b_n| \leq M$ for some $M > 0$, for all $n \in \mathbb{N}_0$. Let μ be the corresponding orthogonalization (probability) measure and denote by $\mathcal{S}$ the support of μ.

4.2.1 The tridiagonal operator T_1

Let us consider the operator T_1 on the space $\ell^2(h)$, where h is the Haar measure of $(R_n(x))_{n\in\mathbb{N}_0}$. The tridiagonal operator T_1 is determined by the recurrence relation (4.2). For $a \in \ell^2(h)$, we have

$$T_1 a(n) = a_n a(n+1) + b_n a(n) + c_n a(n-1),$$

for all $n \in \mathbb{N}$, where $T_1 a(0) = a_0 a(1) + b_0 a(0)$. The corresponding matrix representation with respect to the basis $(e_j)_{j \in \mathbb{N}_0}$ of $l^2(h)$ is given by

$$
T_1 = \begin{pmatrix}
b_0 & a_0 & 0 & 0 & 0 & \cdots \\
c_1 & b_1 & a_1 & 0 & 0 & \cdots \\
0 & c_2 & b_2 & a_2 & 0 & \cdots \\
0 & 0 & c_3 & b_3 & a_3 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots
\end{pmatrix}.
$$

The following two propositions and two corollaries sum up the results concerning the operator T_1. Detailed proofs can be looked up in [LOW13, Section 2].

Proposition 4.1. *Let T_1 be defined as above.*

1. *T_1 is a bounded linear operator on $l^1(h)$ with operator norm $\|T_1\|_1 \leq 2M + 1$.*

2. *T_1 is a self-adjoint bounded operator on $l^2(h)$ with operator norm $\|T_1\|_2 \leq 2M + 1$.*

Both statements of this proposition can be proved by direct computation, involving the inner product in $\ell^2(h)$ and the assumed boundedness of the linearization coefficients in (4.2).

The following results concern the spectrum of T_1 in $\ell^2(h)$. Therefore, we introduce the numerical range of T_1 which is

$$
W(T_1) = \{\langle T_1 b, b \rangle_h : b \in \ell^2(h), \|b\|_{2,h} = 1\}.
$$

Since T_1 is self-adjoint, the following inclusion relation is satisfied:

$$
\{m(T_1), M(T_1)\} \subseteq \sigma(T_1) \subseteq \mathrm{co}(\sigma(T_1)) = \overline{W(T_1)} = [m(T_1), M(T_1)]
$$

where $\mathrm{co}(\sigma(T_1))$ denotes the convex hull of $\sigma(T_1)$, $m(T_1) = \inf W(T_1)$ and $M(T_1) = \sup W(T_1)$, see [Hal82, Intro]. Moreover, $\|T_1\| = \max\{|m(T_1)|, |M(T_1)|\}$. $\mathrm{co}(\sigma(T_1)) = \overline{W(T_1)}$ follows from the fact that self-adjoint operators are convexoid, see for example [Sta65].

Proposition 4.2. *Let $b \in l^2(h)$ be arbitrary. Then,*

$$
\langle (\mathrm{id} - T_1) b, b \rangle_h = \sum_{n=0}^{\infty} a_n |b(n) - b(n+1)|^2 h(n).
$$

The equation can be verified by straightforward computation. An immediate consequence is the following corollary.

Corollary 4.3. *The closure* $\overline{W(T_1)}$ *of the numerical range of* T_1 *is a subset of* $[-(2M + 1), 1]$. *In particular,* $\sigma(T_1) \subseteq [-(2M + 1), 1]$.

For the last corollary the transform $\mathcal{P}_h$ in (2.21) is utilized again. Let M_1 be the multiplication operator in $L^2(\mu)$, defined by

$$M_1(f) = \mathcal{P}_h^{-1} \circ T_1 \circ \mathcal{P}_h(f).$$

The properties of the spectrum of multiplication operators and the relation between M_1 and T_1, see [LOW13], imply the following:

Corollary 4.4. *Let* $(R_n(x))_{n \in \mathbb{N}_0}$ *be defined as in (4.2). Then, the following is true:*

$$\mathcal{S} = \operatorname{supp} \mu = \sigma(T_1) \subseteq [-(2M + 1), 1].$$

4.2.2 A characterization of $1 \in \mathcal{S}$

The main results in [LOW13] were the characterization of $1 \in \mathcal{S}$ and, in case of $1 \notin \mathcal{S}$, the presentation of an explicit form of the inverse. We will particularize these results in the following and will see that there is a relation between the inverse and the Cesàro operator.

Theorem 4.5. *If* $1 \notin \mathcal{S} = \sigma(T_1)$, *then* $\left\{ \frac{H(n)}{a_n h(n)} : n \in \mathbb{N}_0 \right\}$ *is bounded.*

Proof: For $n \in \mathbb{N}_0$, denote by χ_n the sequence in (3.17) with $\chi_n(k) = 1$, for $k = 0, 1, ..., n$, and $\chi_n(k) = 0$, for $k = n + 1, n + 2, ...$. An easy computation shows that

$$(\operatorname{id} - T_1)(\chi_n)(k) = \begin{cases} a_n, & \text{for } k = n, \\ -c_{n+1}, & \text{for } k = n + 1, \\ 0, & \text{else.} \end{cases}$$

Hence, we obtain

$$\|(\operatorname{id} - T_1)(\chi_n)\|_{2,h}^2 = a_n^2 h(n) + c_{n+1}^2 h(n + 1) = a_n(a_n + c_{n+1})h(n).$$

Since $1 \notin \sigma(T_1)$, there exists $A = (\operatorname{id} - T_1)^{-1} \in B(\ell^2(h))$. It follows that

$$\|A \circ (\operatorname{id} - T_1)(\chi_n)\|_{2,h}^2 = \|\chi_n\|_{2,h}^2 = \sum_{k=0}^{n} h(k) = H(n)$$

and

$$\|A \circ (\mathsf{id} - T_1)(\chi_n)\|_{2,h}^2 \leq \|A\|^2 \|(\mathsf{id} - T_1)(\chi_n)\|_{2,h}^2 = \|A\|^2 a_n h(n)(a_n + c_{n+1})$$

$$\leq (2M + 2)\|A\|^2 a_n h(n)$$

so that

$$H(n) \leq (2M + 2)\|A\|^2 a_n h(n), \qquad \text{for all } n \in \mathbb{N}_0.$$

Therefore, the set $\left\{ \frac{H(n)}{a_n h(n)} : n \in \mathbb{N}_0 \right\}$ is bounded. In order to prove the converse implication, we begin with determining a sequence $\alpha = (\alpha(n))_{n \in \mathbb{N}_0}$, such that $(\mathsf{id} - T_1)(\alpha) = e_0$.

Lemma 4.6. *A sequence $\alpha = (\alpha(n))_{n \in \mathbb{N}_0}$ satisfies $(\mathsf{id} - T_1)(\alpha) = e_0$, if and only if $\alpha(n+1) = \alpha(0) - \sum_{k=0}^{n} \frac{1}{a_k h(k)}$, for $n \in \mathbb{N}_0$.*

Proof: Computation shows that $(\mathsf{id} - T_1)(\alpha)(0) = 1$, if and only if $\alpha(0) - \alpha(1) = \frac{1}{a_0}$. For $n \in \mathbb{N}$, we can infer that

$$(\mathsf{id} - T_1)(\alpha)(n) = \alpha(n) - (a_n \alpha(n+1) + b_n \alpha(n) + c_n \alpha(n-1)) = 0$$

is equivalent to $a_n(\alpha(n+1) - \alpha(n)) = c_n(\alpha(n) - \alpha(n-1))$. By iteration, we obtain

$$\alpha(n+1) - \alpha(n) = \frac{c_n}{a_n}(\alpha(n) - \alpha(n-1)) = \frac{c_n c_{n-1} \cdots c_1}{a_n a_{n-1} \cdots a_1} \frac{-1}{a_0} = \frac{-1}{a_n h(n)}.$$

$\square$

Next we have to study, whether the sequence $\alpha = (\alpha(n))_{n \in \mathbb{N}_0}$ of Lemma 4.6 is contained in $\ell^2(h)$.

Definition 4.7. *We say that $(R_n(x))_{n \in \mathbb{N}_0}$ **satisfies condition (B)**, if and only if $\left\{ \frac{H(n)}{a_n h(n)} : n \in \mathbb{N}_0 \right\}$ is bounded.*

Lemma 4.8. *Assume that the OPS $(R_n(x))_{n \in \mathbb{N}_0}$ satisfies condition (B). Then,*

$$\sum_{k=0}^{\infty} \frac{1}{a_k h(k)} < \infty.$$

Proof: By property (B) we know that there is some $K > 0$ such that

$$\sum_{k=0}^{n} h(k) \left(\frac{1}{a_k h(k)} \right)^2 \leq K \sum_{k=0}^{n} h(k) \left(\frac{1}{H(k)} \right)^2,$$

for all $n \in \mathbb{N}_0$. Since $C_h e_0 = \left(\frac{1}{H(n)} \right)_{n \in \mathbb{N}_0}$, Hardy's result on weighted Cesàro operators implies that $\left(\frac{1}{H(n)} \right)_{n \in \mathbb{N}_0} \in \ell^2(h)$. Hence, $\left(\frac{1}{a_n h(n)} \right)_{n \in \mathbb{N}_0} \in \ell^2(h)$ which is equivalent to

$$\sum_{k=0}^{\infty} \frac{1}{a_k^2 h(k)} < \infty.$$

Since $a_k^2 \leq (M+1)a_k$, it follows that $\sum_{k=0}^{\infty} \frac{1}{a_k h(k)} < \infty.$

$\square$

Following Lemma 4.6, the sequence $\alpha = (\alpha(n))_{n \in \mathbb{N}_0}$ is further on defined by

$$\alpha(n) = \sum_{k=n}^{\infty} \frac{1}{a_k h(k)}, \qquad \text{for } n \in \mathbb{N}_0, \tag{4.3}$$

(provided the series converges). In order to prove that $\alpha \in \ell^2(h)$ whenever (B) holds, we apply the dual of the weighted Cesàro operator $C_h^* \in B(\ell^2(h))$ which has already been discussed in Chapter 3. Define the sequence $\beta = (\beta(n))_{n \in \mathbb{N}_0}$ by

$$\beta(n) = \frac{H(n)}{a_n (h(n))^2}, \qquad \text{for all } n \in \mathbb{N}_0. \tag{4.4}$$

From (3.3), we can infer that $C_h^* \beta(n) = \alpha(n)$.

Lemma 4.9. *Assume that $(R_n(x))_{n \in \mathbb{N}_0}$ satisfies condition (B). Then, $(\beta(n))_{n \in \mathbb{N}_0}$ is an element of $\ell^2(h)$.*

Proof: We have $\beta(n) \leq K \frac{1}{h(n)}$, for all $n \in \mathbb{N}_0$. Hence, it remains to show that $\left(\frac{1}{h(n)} \right)_{n \in \mathbb{N}_0}$ is an element of $\ell^2(h)$. Applying Lemma 4.8 yields

$$\sum_{k=0}^{\infty} h(k) \frac{1}{(h(k))^2} \leq (M+1) \sum_{k=0}^{\infty} \frac{1}{a_k h(k)} < \infty.$$

$\square$

Since $C_h^* \beta = \alpha$, an immediate consequence of Lemma 4.9 is the subsequent proposition.

Proposition 4.10. *Assume that $(R_n(x))_{n \in \mathbb{N}_0}$ satisfies condition (B). Then, the sequence $(\alpha(n))_{n \in \mathbb{N}_0}$ which is given by (4.3), is the unique solution in $\ell^2(h)$ of the equation $(\mathrm{id} - T_1)(\alpha) = e_0$.*

Assuming (B), our next aim is to find a sequence $\alpha^m \in l^2(h)$, such that $(\mathrm{id} - T_1)(\alpha^m) = e_m$, for every $m \in \mathbb{N}$. For this purpose, it is useful to slightly modify the recurrence relation in (4.2). The OPS $(R_n(x))_{n \in \mathbb{N}_0}$ satisfies

$$R_1(x)R_n(x) = \tilde{a}_n R_{n+1} + \tilde{b}_n R_n(x) + \tilde{c}_n R_{n-1}, \qquad n \in \mathbb{N}_0, \tag{4.5}$$

where $\tilde{a}_n = \frac{1}{a_0} a_n$, $\tilde{b}_n = \frac{1}{a_0}(b_n - b_0)$ and $\tilde{c}_n = \frac{1}{a_0} c_n$. Note that $\tilde{a}_n + \tilde{b}_n + \tilde{c}_n = 1$ and $\tilde{a}_n, \tilde{c}_n > 0$. Based on the equations (4.5), we introduce the modification S_1 of T_1 by

$$S_1 = \frac{1}{a_0} T_1 - \frac{b_0}{a_0} \mathrm{id} \in B(\ell^2(h)). \tag{4.6}$$

For $n \in \mathbb{N}_0$, denote by d_n the sequence $\frac{e_n}{h(n)}$. Obviously, $d_0 = e_0$. Hence we have $S_1 d_0 = d_1$ and $S_1 d_n = \tilde{c}_n d_{n-1} + \tilde{b}_n d_n + \tilde{a}_n d_{n+1}$, $n \in \mathbb{N}_0$. Moreover, we introduce a sequence of operators $(S_n)_{n \in \mathbb{N}_0}$, by setting

$$S_{n+1} = \frac{1}{\tilde{a}_n}\left(S_1 \circ S_n - \tilde{b}_n S_n - \tilde{c}_n S_{n-1}\right), \qquad n \in \mathbb{N}. \tag{4.7}$$

(with $S_0 = \mathrm{id}$). Obviously, $S_n \in B(\ell^2(h))$.

Proposition 4.11. *The operators S_m act as follows:*

(1) $\qquad\qquad S_m d_0 = d_m, \qquad\qquad$ *for $m \in \mathbb{N}_0$.*

(2) $\qquad\qquad S_m \alpha(k) = \alpha(m), \qquad\qquad$ *for $k = 0, 1, ..., m$, and*

$\qquad\qquad S_m \alpha(k) = \alpha(k), \qquad\qquad$ *for $k = m+1, m+2, ...$.*

Proof: Ad (1): Assume that $S_m d_0 = d_m$ and $S_{m-1} d_0 = d_{m-1}$ has already been shown. Then, by (4.7), it follows

$$\tilde{a}_m S_{m+1} d_0 = S_1 d_m - \tilde{b}_m d_m - \tilde{c}_m d_{m-1} = \tilde{a}_m d_{m+1}.$$

Ad (2): For $m = 1$, we obtain

$$S_1\alpha = \frac{1}{a_0}T_1\alpha - \frac{b_0}{a_0}\alpha = \frac{1}{a_0}(\alpha - d_0) - \frac{b_0}{a_0}\alpha = \alpha - \frac{d_0}{a_0}$$

$$= (\alpha(1), \alpha(1), \alpha(2), \alpha(3), ...).$$

Again, we use induction and assume that the statement holds for m and $m - 1$. Then, for $k = 0, ..., m - 1$, we have

$$S_{m+1}\alpha(k) = \frac{1}{\tilde{a}_m}(\alpha(m) - \tilde{b}_m\alpha(m) - \tilde{c}_m\alpha(m - 1))$$

$$= \alpha(m) + \frac{\tilde{c}_m}{\tilde{a}_m}(\alpha(m) - \alpha(m - 1)) = \alpha(m) - \frac{c_m}{a_m}\frac{1}{a_{m-1}h(m - 1)}$$

$$= \alpha(m) - \frac{1}{a_m h(m)} = \alpha(m + 1).$$

For $k = m$, it follows that

$$S_{m+1}\alpha(m) = \frac{1}{\tilde{a}_m}(\tilde{a}_m\alpha(m + 1) + \tilde{b}_m\alpha(m) + \tilde{c}_m\alpha(m))$$

$$- \frac{\tilde{b}_m}{\tilde{a}_m}\alpha(m) - \frac{\tilde{c}_m}{\tilde{a}_m}\alpha(m) = \alpha(m + 1).$$

For $k = m + 1, m + 2, ...$, we have

$$S_{m+1}\alpha(k) = \frac{1}{\tilde{a}_m}\left(\tilde{a}_k\alpha(k + 1) + \tilde{b}_k\alpha(k) + \tilde{c}_k\alpha(k - 1)\right) - \frac{\tilde{b}_m}{\tilde{a}_m}\alpha(k) - \frac{\tilde{c}_m}{\tilde{a}_m}\alpha(k)$$

$$= \frac{1}{\tilde{a}_m}\left[\tilde{a}_k\left(\alpha(k) - \frac{1}{a_k h(k)}\right) + \tilde{b}_k\alpha(k)\right.$$

$$\left. + \tilde{c}_k\left(\alpha(k) + \frac{1}{a_{k-1}h(k - 1)}\right)\right] - \frac{\tilde{b}_m}{\tilde{a}_m}\alpha(k) - \frac{\tilde{c}_m}{\tilde{a}_m}\alpha(k)$$

$$= \frac{1}{\tilde{a}_m}\left[\alpha(k) - \frac{1}{a_0 h(k)} + \frac{1}{a_0 h(k)}\right] - \frac{\tilde{b}_m}{\tilde{a}_m}\alpha(k) - \frac{\tilde{c}_m}{\tilde{a}_m}\alpha(k) = \alpha(k).$$

$\square$

Proposition 4.12. *Assume that $(R_n(x))_{n \in \mathbb{N}_0}$ satisfies condition (B). Then, the elements $S_m \alpha \in \ell^2(h)$ satisfy $(\mathrm{id} - T_1)(S_m \alpha) = d_m$ for each $m \in \mathbb{N}$.*

Proof: Obviously the operator S_m commutes with $\mathrm{id} - T_1$. Hence, we obtain

$$(\mathrm{id} - T_1)(S_m \alpha) = S_m(\mathrm{id} - T_1)(\alpha) = S_m d_0 = d_m.$$

$\square$

For fixed $m \in \mathbb{N}_0$, define the sequence β^m by

$$\beta^m(k) = \begin{cases} 0, & \text{for } k = 0, 1, ..., m-1, \\ \dfrac{H(k)}{a_k(h(k))^2}, & \text{for } k = m, m+1, ... \end{cases}$$

Following the proof of Lemma 4.9, $\beta^m \in \ell^2(h)$ whenever (B) is satisfied. Moreover,

$$C_h^*(\beta^m)(n) = \sum_{k=m}^{\infty} \frac{\beta^m(k)h(k)}{H(k)} = \sum_{k=m}^{\infty} \frac{1}{a_k h(k)} = \alpha(m), \quad \text{for } n = 0, 1, ..., m, \text{ and}$$

$$C_h^*(\beta^m)(n) = \sum_{k=n}^{\infty} \frac{1}{a_k h(k)} = \alpha(n), \quad \text{for } n = m+1, m+2, ...$$

Proposition 4.11(2) says that $C_h^*(\beta^m) = S_m(\alpha)$. We put $\beta^0 = \beta$. Now we can combine the results above to determine the inverse operator of $\mathrm{id} - T_1$, provided condition (B) is satisfied. Let

$$\varphi = (\varphi(k))_{k \in \mathbb{N}_0}, \quad \text{with} \quad \varphi(k) = \frac{H(k)^2}{a_k(h(k))^2}. \tag{4.8}$$

If $\frac{H(k)}{a_k h(k)} \leq K$, then $\varphi(k) \leq K^2(M+1)$. Hence, φ is a bounded sequence if (B) holds. The multiplication with $\varphi \in \ell^\infty$ defines a bounded operator M_φ in $\ell^2(h)$, where $M_\varphi(\gamma)(n) = \varphi(n)\gamma(n)$, $\gamma \in \ell^2(h)$. The following theorem is one of the key results in the paper. It shows the relation between the Cesàro operator and its dual and the inverse of $\mathrm{id} - T_1$, if this inverse exists.

Theorem 4.13. *Assume that $(R_n(x))_{n \in \mathbb{N}_0}$ satisfies (B). Then $C_h^* \circ M_\varphi \circ C_h \in B(\ell^2(h))$ is the inverse of $\mathrm{id} - T_1$, where φ is the sequence in (4.8).*

Proof: Observe that $C_h(d_m)(k) = 0$, for $k = 0, ..., m-1$, and $C_h(d_m)(k) = \frac{1}{H(k)}$, for $k = m, m+1, ...$. It follows that $M_\varphi \circ C_h(d_m) = \beta^m$. Hence, $C_h^* \circ M_\varphi \circ C_h(d_m) = S_m(\alpha)$.

In particular,

$$(\mathsf{id} - T_1) \circ (C_h^* \circ M_\varphi \circ C_h)(d_m) = d_m.$$

Furthermore, $a_0(\mathsf{id} - S_1) = \mathsf{id} - T_1$ and it can be inferred that

$$
\begin{aligned}
C_h^* \circ M_\varphi \circ C_h \circ (\mathsf{id} - T_1)(d_0) &= a_0(C_h^* \circ M_\varphi \circ C_h) \circ (\mathsf{id} - S_1)(d_0) \\
&= a_0(C_h^* \circ M_\varphi \circ C_h)(d_0 - d_1) \\
&= a_0(\alpha - S_1(\alpha)) = (\mathsf{id} - T_1)(\alpha) = d_0
\end{aligned}
$$

and for $m \in \mathbb{N}$

$$
\begin{aligned}
C_h^* \circ M_\varphi \circ C_h \circ (\mathsf{id} - T_1)(d_m) &= a_0(C_h^* \circ M_\varphi \circ C_h) \circ (\mathsf{id} - S_1)(d_m) \\
&= a_0(C_h^* \circ M \circ C_h)\left(d_m - (\tilde{a}_m d_{m+1} + \tilde{b}_m d_m + \tilde{c}_m d_{m-1})\right) \\
&= a_0\left(S_m(\alpha) - (\tilde{a}_m S_{m+1}(\alpha) + \tilde{b}_m S_m(\alpha) + \tilde{c}_m S_{m-1}(\alpha))\right) \\
&= a_0(S_m(\alpha) - S_1 \circ S_m(\alpha)) = a_0(\mathsf{id} - S_1)(S_m(\alpha)) \\
&= (\mathsf{id} - T_1)(S_m(\alpha)) = d_m.
\end{aligned}
$$

Since $C_h^* \circ M_\varphi \circ C_h$ is a bounded linear operator in $\ell^2(h)$, we obtain that

$$C_h^* \circ M_\varphi \circ C_h \circ (\mathsf{id} - T_1) = \mathsf{id} = (\mathsf{id} - T_1) \circ C_h^* \circ M_\varphi \circ C_h,$$

i.e. $(\mathsf{id} - T_1)^{-1} = C_h^* \circ M_\varphi \circ C_h$.

$\qquad \square$

Collecting the results yields the following characterization.

Corollary 4.14. *Assume that $(R_n(x))_{n \in \mathbb{N}_0}$ is generated by (4.2). Then, $1 \notin \mathsf{supp}\ \mu = \sigma(T_1)$, if and only if (B) holds.*

After submission of the paper, Ryszard Szwarc called our attention to a result in [Bec00, Theorem 2.3] which is similar to Corollary 4.14. He pointed out that in [Bec00] there is no formula of the inverse as in Theorem 4.13. We also want to add the following corollary which concerns the Cesàro operator.

Corollary 4.15. *Assume that $(R_n(x))_{n \in \mathbb{N}_0}$ is generated by (4.2). Then, $0 \notin \sigma(C_h)$ and $0 \notin \sigma(C_h^*)$, respectively, if and only if (B) holds.*

Proof: Condition (B) implies that $\frac{H(n)}{a_n h(n)} \leq K$ for some $K > 0$ and all $n \in \mathbb{N}_0$. Moreover, the choice of the linearization coefficients implies $0 < a_n < M + 1$ for some $M > 0$ and all $n \in \mathbb{N}_0$. Thus,

$$\frac{H(n)}{H(n+1)} = 1 - \frac{h(n+1)}{H(n+1)} \leq 1 - \frac{1}{K(M+1)} < 1, \quad \text{for all } n \in \mathbb{N}_0.$$

The assertion follows by Theorem 3.5(*ii*) and (*iii*).

$\square$

In case of $0 \notin \sigma(C_h)$ and $0 \notin \sigma(C_h^*)$, respectively, we can explicitly determine the inverse operators C_h^{-1} and C_h^{*-1}. Computing the matrix representations with respect to the basis $(e_j)_{j \in \mathbb{N}_0}$ of $\ell^2(h)$, results in

$$C_h^{-1} = \begin{pmatrix} \frac{H(0)}{h(0)} & & & 0 \\ -\frac{H(0)}{h(1)} & \frac{H(1)}{h(1)} & & \\ 0 & -\frac{H(1)}{h(2)} & \frac{H(2)}{h(2)} & \\ & & \ddots & \ddots \end{pmatrix},$$

$$C_h^{*-1} = \begin{pmatrix} \frac{H(0)}{h(0)} & -\frac{H(0)}{h(0)} & & 0 \\ & \frac{H(1)}{h(1)} & -\frac{H(1)}{h(1)} & \\ 0 & & \frac{H(2)}{h(2)} & \ddots \\ & & & \ddots \end{pmatrix}.$$

4.2.3 Inferences on the spectrum of T_2

Since the operator T_1 in $\ell^2(h)$ is associated with the multiplication by the moment x in $L^2(\mu)$, it is natural to define operators $T_n := T_1^n$, acting in $\ell^2(h)$, for $n \in \mathbb{N}_0$, which are directly related to the multiplication by the respective moments x^n in $L^2(\mu)$. This is also consistent in view of the definition of the operators S_n in (4.7) which are associated to the multiplication by $R_n(x)$ for $n \in \mathbb{N}_0$.

Considering the operator T_2 particularly yields a nice result, also with respect to the deduction of the related system $(R_n^+(x))_{n \in \mathbb{N}_0}$ in Chapter 2. We are interested in the question, whether $1 \in \sigma(T_2)$ and, in case of $1 \notin \sigma(T_2)$, we want to determine the inverse of $(\text{id} - T_2)$.

In the following, let $(R_n(x))_{n\in\mathbb{N}_0}$ be an OPS as defined in (4.2) wrt the measure μ, where we additionally assume that $\mathcal{S} = \mathsf{supp}\,\mu \subset [-1, 1]$.

Lemma 4.16. *Assume that $(R_n(x))_{n\in\mathbb{N}_0}$ is generated by (4.2). Then, $1 \notin \sigma(T_2)$, if and only if $1 \notin \sigma(T_1)$ and $-1 \notin \sigma(T_1)$.*

    Proof: Spectral theory implies that $\sigma(T_2) = \sigma(T_1^2) = \sigma(T_1)^2 = \{\lambda^2 : \lambda \in \sigma(T_1)\}$. Observe also that $(\mathsf{id} - T_2) = (\mathsf{id} - T_1) \circ (\mathsf{id} + T_1) = (\mathsf{id} + T_1) \circ (\mathsf{id} - T_1)$.

<div align="right">□</div>

    Utilizing Theorem 4.13 and Corollary 4.14, the following can be stated:

Corollary 4.17. *Assume that $(R_n(x))_{n\in\mathbb{N}_0}$ is generated by (4.2). Then, a necessary condition for $1 \notin \sigma(T_2)$ is that $(R_n(x))_{n\in\mathbb{N}_0}$ satisfies condition (B) in Definition 4.7.*

    In the subsection before, it was proved that $(\mathsf{id} - T_1)^{-1} = C_h^* \circ M_\varphi \circ C_h$, where C_h denoted the Cesàro operator with respect to the Haar measure of the OPS $(R_n(x))_{n\in\mathbb{N}_0}$ and M_φ was defined by $M_\varphi a(n) = \frac{H(n)^2}{a_n h(n)^2} a(n)$, for all $n \in \mathbb{N}_0$ and $a \in \ell^2(h)$. We want to introduce an operator C_k which shall be defined by

$$C_k a(n) = \frac{1}{K(n)} \sum_{m=0}^{n} h(m) R_m(-1) a(m), \quad \text{for all } n \in \mathbb{N}_0, \tag{4.9}$$

where $K(n)$ is defined as in Chapter 2. Observe that there is an obvious connection between (4.9) and the definition of the system $(R_n^+(x))_{n\in\mathbb{N}_0}$ in (2.12). The operator C_k is well-defined, since $\mathsf{supp}\,\mu \subset [-1, 1]$ implies $\mathsf{supp}\,\mu^- \subset [-1, 1]$ and hence, by using (2.15) and Theorem 1.14, we have

$$K(n) = H(n) R_n^-(-1) \neq 0, \quad \text{for all } n \in \mathbb{N}_0.$$

Since $R_n(1) = 1 > 0$ and since all zeros of $(R_n(x))_{n\in\mathbb{N}_0}$ are contained in the interval $(-1, 1)$, one obtains

$$\mathsf{sgn}\, R_n(-1) = (-1)^n, \quad \text{for all } n \in \mathbb{N}_0,$$

where sgn denotes the sign function. Considering the definition of $(K(n))_{n\in\mathbb{N}_0}$, an immediate consequence is

$$\mathsf{sgn}\, K(n) = (-1)^n, \quad \text{for all } n \in \mathbb{N}_0.$$

Moreover, we can conclude for all $n \in \mathbb{N}_0$ that

$$|K(n)| + |K(n+1)| = h(n+1)|R_{n+1}(-1)|,$$

$$|R_n(-1)| + |R_{n+1}(-1)| = \frac{2}{a_n h(n)}|K(n)|, \tag{4.10}$$

$$a_n|R_{n+1}(-1)| + c_n|R_{n-1}(-1)| = (1+b_n)|R_n(-1)|.$$

However, it is not clear, whether C_k is a bounded operator in $\ell^2(h)$. The dual operator C_k^* of C_k is given by

$$C_k^* a(n) = R_n(-1) \sum_{m=n}^{\infty} \frac{h(m)a(m)}{K(m)}, \quad \text{for all } n \in \mathbb{N}_0. \tag{4.11}$$

Since C_k and C_k^* in (4.9) and (4.11), respectively, are operators in a sequence space, they have matrix representations with respect to the basis $(e_j)_{j\in\mathbb{N}}$ of $\ell^2(h)$ (in the following also denoted by C_k and C_k^*, respectively).

$$C_k = \begin{pmatrix} \frac{h(0)R_0(-1)}{K(0)} & & & & 0 \\ \frac{h(0)R_0(-1)}{K(1)} & \frac{h(1)R_1(-1)}{K(1)} & & \\ \frac{h(0)R_0(-1)}{K(2)} & \frac{h(1)R_1(-1)}{K(2)} & \frac{h(2)R_2(-1)}{K(2)} & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$C_k^* = \begin{pmatrix} \frac{h(0)R_0(-1)}{K(0)} & \frac{h(1)R_0(-1)}{K(1)} & \frac{h(2)R_0(-1)}{K(2)} & \cdots \\ & \frac{h(1)R_1(-1)}{K(1)} & \frac{h(2)R_1(-1)}{K(2)} & \cdots \\ 0 & & \frac{h(2)R_2(-1)}{K(2)} & \cdots \\ & & & \ddots \end{pmatrix}. \tag{4.12}$$

The boundedness of the operators in (4.12) depends on the choice of the OPS. If C_k is bounded, matrix computations show that

$$C_k \circ (\text{id} + T_1) \circ C_k^* =: M_\psi^{-1}, \tag{4.13}$$

where M_ψ^{-1} is a multiplication operator, satisfying

$$M_\psi^{-1}a(n) = \psi(n)^{-1}a(n), \quad \text{for all } a \in \ell^2(h) \text{ and } n \in \mathbb{N}_0,$$

with $\psi = (\psi(n))_{n\in\mathbb{N}_0} = -\frac{K(n)^2}{a_n h(n)^2 R_n(-1)R_{n+1}(-1)}$. In the following, the question, when C_k and M_ψ^{-1} are bounded, shall be treated.

Definition 4.18. *Let $(R_n(x))_{n\in\mathbb{N}_0}$ be generated by (4.2). We say that $(R_n(x))_{n\in\mathbb{N}_0}$ **satisfies condition (B)'**, if and only if the set*

$$
\left\{ \frac{\sum_{k=0}^n h(k)R_k(-1)^2}{a_n h(n)(a_n R_{n+1}(-1)^2 + c_{n+1}R_n(-1)^2)} : n \in \mathbb{N}_0 \right\}
$$

is bounded.

Lemma 4.19. *Assume that $(R_n(x))_{n\in\mathbb{N}_0}$ is generated by (4.2) and furthermore, let $(R_n(x))_{n\in\mathbb{N}_0}$ satisfy condition (B)'. Then, one obtains the following:*

(i) The operator C_k is bounded.

(ii) The inverse of C_k exists in $B(\ell^2(h))$.

(iii) The operator M_ψ^{-1} and $M_\psi := (M_\psi^{-1})^{-1}$ are bounded.

Proof: Condition (B)' implies that

$$
\sum_{k=0}^n h(k)R_k(-1)^2 \leq K_1 a_n h(n)(a_n R_{n+1}(-1)^2 + c_{n+1}R_n(-1)^2),
$$

for all $n \in \mathbb{N}_0$ and some positive number $K_1 \geq 1$.

Ad (i): By assumption, we have $|b_n| < 1$ and $c_n < 2$, for all $n \in \mathbb{N}_0$, and hence, using (4.10), results in

$$
a_n h(n)(a_n R_{n+1}(-1)^2 + c_{n+1}R_n(-1)^2)
$$

$$
< a_n h(n)|R_n(-1)|\left((1+b_n)|R_{n+1}(-1)| + c_{n+1}|R_n(-1)|\right) < 4|R_n(-1)K(n)|. \quad (4.14)
$$

Let $a \in \ell^2(h)$ be arbitrary. Utilizing (4.14), we obtain

$$
\|C_k a\|_{2,h}^2 = \sum_{n=0}^\infty h(n)\left| \frac{1}{K(n)}\sum_{m=0}^n h(m)R_m(-1)a(m) \right|^2
$$

$$
\leq \sum_{n=0}^\infty h(n)R_n(-1)^2 \frac{\left(\sum_{m=0}^n h(m)R_m(-1)^2\right)^2}{R_n(-1)^2 K(n)^2}
$$

$$
\times \left(\frac{\sum_{m=0}^n h(m)R_m(-1)^2|R_m(-1)^{-1}a(m)|}{\sum_{m=0}^n h(m)R_m(-1)^2} \right)^2 \leq
$$

$$\leq 16K_1^2 \sum_{n=0}^{\infty} h(n) R_n(-1)^2 \left(\frac{\sum_{m=0}^{n} h(m) R_m(-1)^2 |R_m(-1)^{-1} a(m)|}{\sum_{m=0}^{n} h(m) R_m(-1)^2} \right)^2.$$

Applying Hardy's inequality (3.6) for $p = 2$, $\lambda_n = h(n) R_n(-1)^2$ and $(R_n(-1)^{-1} a(n))_{n \in \mathbb{N}_0} \in \ell^2(\lambda)$ yields

$$\|C_k a\|_{2,h}^2 \leq 64 K_1^2 \sum_{n=0}^{\infty} h(n) R_n(-1)^2 |R_n(-1)^{-1} a(n)| = 64 K_1^2 \|a\|_{2,h}^2$$

which shows the boundedness of C_k and C_k^*, respectively.

Ad (ii): We start with determining the inverse image α_j of the jth unit sequence e_j, for $j \in \mathbb{N}_0$, under C_k:

$$\alpha_j(n) = \begin{cases} \dfrac{K(j)}{h(j) R_j(-1)}, & \text{for } n = j, \\[3mm] -\dfrac{K(j+1)}{h(j) R_j(-1)}, & \text{for } n = j+1, \\[3mm] 0, & \text{else,} \end{cases}$$

in particular, the inverse image is well-defined for all $j \in \mathbb{N}_0$. Therefore, C_k^{-1} exists as an operator with matrix representation

$$C_k^{-1} = \begin{pmatrix} \frac{K(0)}{h(0) R_0(-1)} & & & 0 \\ -\frac{K(0)}{h(1) R_1(-1)} & \frac{K(1)}{h(1) R_1(-1)} & & \\ 0 & -\frac{K(1)}{h(2) R_2(-1)} & \frac{K(2)}{h(2) R_2(-1)} & \\ & & \ddots & \ddots \end{pmatrix}, \tag{4.15}$$

with respect to $(e_j)_{j \in \mathbb{N}_0}$ and it remains to show that $C_k^{-1} \in B(\ell^2(h))$. Let $a \in \ell^2(h)$ be arbitrary and put $a(-1) = 0$. Calculation shows that

$$\|C_k a\|_{2,h}^2 = \sum_{n=0}^{\infty} h(n) \left| \frac{K(n)}{h(n) R_n(-1)} a(n) - \frac{K(n-1)}{h(n) R_n(-1)} a(n-1) \right|^2$$

$$\leq 2 \sum_{n=0}^{\infty} h(n) |a(n)|^2 \left(\frac{K(n)^2}{h(n)^2 R_n(-1)^2} + \frac{K(n)^2}{h(n) h(n+1) R_{n+1}(-1)^2} \right).$$

From (4.10), it follows that $\frac{K(n)^2}{h(n)^2 R_n(-1)^2} < 1$, for all $n \in \mathbb{N}_0$. Moreover, we have

$$\frac{K(n)^2}{h(n)h(n+1)R_{n+1}(-1)^2} < \frac{2|K(n)|}{c_{n+1}h(n+1)|R_{n+1}(-1)|}.$$

If $c_{n+1} \geq \frac{1}{4K_1}$, we can infer that $\frac{2|K(n)|}{c_{n+1}h(n+1)|R_{n+1}(-1)|} < 8K_1$. In the case, when $c_{n+1} < \frac{1}{4K_1}$, condition $(B)'$ implies

$$h(n)R_n(-1)^2 < K_1 a_n^2 h(n) R_{n+1}(-1)^2 + \frac{1}{2}h(n)R_n(-1)^2$$

which is equivalent to $|R_{n+1}(-1)| > \frac{1}{\sqrt{8K_1}}|R_n(-1)|$. Hence, we obtain

$$\frac{K(n)^2}{h(n)h(n+1)R_{n+1}(-1)^2} < \sqrt{8K_1} < 8K_1$$

which shows the boundedness of C_k^{-1}.

Ad (iii): Let $a \in \ell^2(h)$. The definition of M_ψ^{-1} implies that

$$M_\psi a(n) = -\frac{K(n)^2}{a_n h(n)^2 R_n(-1)R_{n+1}(-1)}a(n),$$

for all $n \in \mathbb{N}_0$. Considering (4.10), (4.14) and condition (B)', we can infer that

$$h(n)|R_n(-1)| < 4|K(n)| \quad \text{and} \quad a_n h(n)|R_{n+1}(-1)| < 2|K(n)|,$$

as well as

$$K(n)^2 < h(n)^2 R_n(-1)^2 < K_1 a_n h(n)^2 (a_n R_{n+1}(-1)^2 + c_{n+1} R_n(-1)^2)$$

$$< K_1 a_n (2 + b_n + b_{n+1})h(n)^2 |R_n(-1)R_{n+1}(-1)|,$$

for all $n \in \mathbb{N}_0$, which shows that M_ψ^{-1} and M_ψ, respectively, are bounded by $\|M_\psi^{-1}\| < 8$ and $\|M_\psi\| < 4K_1$.

$\square$

Corollary 4.20. *Assume that $(R_n(x))_{n\in\mathbb{N}_0}$ is generated by (4.2) and furthermore, let $(R_n(x))_{n\in\mathbb{N}_0}$ satisfy condition (B)'. Then, $0 \notin \sigma(C_k)$.*

Theorem 4.21. *Let $(R_n(x))_{n \in \mathbb{N}_0}$ be generated by (4.2). Then, $-1 \notin \sigma(T_1)$, if and only if $(R_n(x))_{n \in \mathbb{N}_0}$ satisfies condition (B)'. Moreover, the inverse of $(\mathsf{id} + T_1)$ is given by*

$$(\mathsf{id} + T_1)^{-1} = C_k^* \circ M_\psi \circ C_k.$$

Proof: Assume that $-1 \notin \sigma(T_1)$. In the following, similar arguments as in the proof of Theorem 4.5 are used. For $n \in \mathbb{N}_0$, denote by ζ_n the sequence defined by $\zeta_n(k) = R_k(-1)$, for $k = 0, ..., n$ and $\zeta_n(k) = 0$, for $k = n + 1, n + 2, ...$. An easy computation shows that

$$(\mathsf{id} + T_1)(\zeta_n)(k) = \begin{cases} 0, & \text{for } k \in \mathbb{N}_0 \backslash \{n, n+1\}, \\ -a_n R_{n+1}(-1), & \text{for } k = n \quad \text{and} \\ c_{n+1} R_n(-1), & \text{for } k = n + 1. \end{cases}$$

Hence, one obtains

$$\|(\mathsf{id} + T_1)(\zeta_n)\|_{2,h}^2 = h(n)|a_n R_n(-1)|^2 + h(n+1)|c_{n+1} R_n(-1)|^2$$

$$= a_n h(n)(a_n R_{n+1}(-1)^2 + c_{n+1} R_n(-1)^2).$$

Since $-1 \notin \sigma(T_1)$, there exists $B = (\mathsf{id} + T_1)^{-1} \in B(\ell^2(h))$. It follows that

$$\|B \circ (\mathsf{id} + T_1)(\zeta_n)\|_{2,h}^2 = \|\zeta_n\|_{2,h}^2 = \sum_{k=0}^{n} h(k) R_k(-1)^2 \qquad \text{and}$$

$$\|B \circ (\mathsf{id} + T_1)(\zeta_n)\|_{2,h}^2 \leq \|B\|^2 \|(\mathsf{id} + T_1)(\zeta_n)\|_{2,h}^2$$

$$= \|B\|^2 a_n h(n)(a_n R_{n+1}(-1)^2 + c_{n+1} R_n(-1)^2)$$

so that

$$\sum_{k=0}^{n} h(k) R_k(-1)^2 \leq \|B\|^2 a_n h(n)(a_n R_{n+1}(-1)^2 + c_{n+1} R_n(-1)^2),$$

for all $n \in \mathbb{N}_0$, which is condition (B)'. Inferring from Lemma 4.19, C_k, C_k^{-1}, C_k^*, C_k^{*-1} and M_ψ are well-defined bounded operators. Rearranging and inverting (4.13) yields

$$(\mathsf{id} + T_1)^{-1} = C_k^* \circ M_\psi \circ C_k.$$

Conversely, let condition (B)' be satisfied. Then, Lemma 4.19 implies that $C_k^* \circ M_\psi \circ C_k$ is a well-defined operator in $B(\ell^2(h))$ with

$$(\text{id} + T_1) \circ C_k^* \circ M_\psi \circ C_k = C_k^* \circ M_\psi \circ C_k \circ (\text{id} + T_1) = \text{id}.$$

This shows $C_k^* \circ M_\psi \circ C_k = (\text{id} + T_1)^{-1}$ and, in particular, the existence of $(\text{id} + T_1)^{-1}$ in $B(\ell^2(h))$. Hence, we can conclude that $-1 \notin \sigma(T_1)$.

$\square$

Theorem 4.22. *Let $(R_n(x))_{n \in \mathbb{N}_0}$ be generated by (4.2). Then, $1 \notin \sigma(T_2)$, if and only if $(R_n(x))_{n \in \mathbb{N}_0}$ satisfies condition (B) and (B)'. Moreover, the inverse of $(\text{id} - T_2)$ is given by*

$$(\text{id} - T_2)^{-1} = C_h^* \circ M_\varphi \circ C_h \circ C_k^* \circ M_\psi \circ C_k = C_k^* \circ M_\psi \circ C_k \circ C_h^* \circ M_\varphi \circ C_h.$$

Proof: First recall that $(\text{id} - T_2) = (\text{id} - T_1) \circ (\text{id} + T_1) = (\text{id} + T_1) \circ (\text{id} - T_1)$ and that $1 \notin \sigma(T_2)$ is equivalent to $1 \notin \sigma(T_1)$ and $-1 \notin \sigma(T_1)$. Following Theorem 4.13 and Corollary 4.14, the inverse of $\text{id} - T_1$ exists in $B(\ell^2(h))$, if and only if condition (B) is satisfied, where $(\text{id} - T_1)^{-1} = C_h^* \circ M_\varphi \circ C_h$. On the other hand, Theorem 4.21 implies that the inverse of $\text{id} + T_1$ exists in $B(\ell^2(h))$, if and only if condition (B)' is satisfied, where $(\text{id} + T_1)^{-1} = C_k^* \circ M_\psi \circ C_k$. Altogether, the assertion follows.

$\square$

Corollary 4.23. *Assume that $(R_n(x))_{n \in \mathbb{N}_0}$ is generated by (4.2) and additionally assume that $(R_n(x))_{n \in \mathbb{N}_0}$ is symmetric, i.e. $b_n = 0$, for all $n \in \mathbb{N}_0$. Furthermore, let condition (B) be satisfied. Then, $1 \notin \sigma(T_2)$.*

Proof: The assertion follows by verifying that (B) and (B)' coincide in the symmetric case.

$\square$

For an OPS as in Corollary 4.23, the expressions for the respective operators can be simplified. Computing the operator C_k and its dual, results in

$$C_k = \begin{pmatrix} \frac{h(0)}{a_0 h(0)} & & & 0 \\ -\frac{h(0)}{a_1 h(1)} & \frac{h(1)}{a_1 h(1)} & & \\ \frac{h(0)}{a_2 h(2)} & -\frac{h(1)}{a_2 h(2)} & \frac{h(2)}{a_2 h(2)} & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad C_k^* = \begin{pmatrix} \frac{1}{a_0} & -\frac{1}{a_1} & \frac{1}{a_2} & \cdots \\ & \frac{1}{a_1} & -\frac{1}{a_2} & \cdots \\ 0 & & \frac{1}{a_2} & \cdots \\ & & & \ddots \end{pmatrix}.$$

Moreover, we obtain

$$
C_k^{-1} = \begin{pmatrix} a_0 & & 0 \\ c_1 & a_1 & \\ 0 & c_2 & a_2 \\ & & & \ddots \end{pmatrix}, \qquad C_k^{*-1} = \begin{pmatrix} a_0 & a_0 & 0 \\ & a_1 & a_1 \\ 0 & & a_2 & a_2 \\ & & & \ddots \end{pmatrix}
$$

and $\psi(n) = a_n$, for all $n \in \mathbb{N}_0$.

4.3 Hyponormality and the orthogonal polynomial system

In this section we want to investigate, whether there is a connection between the hyponormality of the Cesàro operator and the fact that the relative OPS forms a polynomial hypergroup. Consider an OPS $(R_n(x))_{n \in \mathbb{N}_0}$ as defined in (2.1). For $m, n \in \mathbb{N}_0$, the polynomial $R_m R_n$ is a polynomial of degree $m + n$ and can be uniquely represented by

$$
R_m R_n(x) = \sum_{k=|m-n|}^{m+n} g(m, n, k) R_k(x), \tag{4.16}
$$

where the coefficients $g(m, n, k)$ are real numbers which sum up to 1, see [Las83], [Las94], [Szw92b] and [Szw92a]. In particular, choosing $m = 1$ in (4.16) yields the modified three-term recurrence relation

$$
R_1 R_n(x) = R_n R_1(x) = \widetilde{a}_n R_{n+1}(x) + \widetilde{b}_n R_n(x) + \widetilde{c}_n R_{n-1}(x), \tag{4.17}
$$

where

$$
\widetilde{a}_n = g(1, n, n+1) = g(n, 1, n+1) = \frac{1}{a_0} a_n,
$$

$$
\widetilde{b}_n = g(1, n, n) = g(n, 1, n) = \frac{1}{a_0}(b_n - b_0),
$$

$$
\widetilde{c}_n = g(1, n, n-1) = g(n, 1, n-1) = \frac{1}{a_0} c_n,
$$

for $n \in \mathbb{N}$, see also (4.5). If the so called linearization coefficients $g(m, n, k)$ are assumed to be non-negative, $\mathbb{N}_0$ together with a convolution defined by the linearization coeffi-

cients becomes a commutative, discrete hypergroup, a polynomial hypergroup, see [Las83], [Las94], [Szw92b] and [Szw92a]. Example 3.9 showed that for all parameters $\alpha, \beta > -1$, the Haar measures of the Jacobi polynomials define hyponormal Cesàro operators. On the other hand, if $\alpha < \beta$, we know that the respective Jacobi polynomials do not induce a polynomial hypergroup, see [Las94] or [Szw92a]. Moreover, in Chapter 5 it is shown that there are polynomials which define polynomial hypergroups but non-hypononormal Cesàro operators. However, we are interested in the behaviour of the Cesàro operator, when we assume a special property for the weights that was established by Szwarc in [Szw92a, Theorem 1].

Theorem 4.24 (Szwarc, 1992)**.** *If polynomials $(P_n(x))_{n \in \mathbb{N}_0}$ satisfy*

$$xP_n(x) = a'_n P_{n+1}(x) + b'_n P_n(x) + c'_n P_{n-1}(x)$$

and

(i) *$(c'_n)_{n \in \mathbb{N}_0}$, $(b'_n)_{n \in \mathbb{N}_0}$, and $(a'_n + c'_n)_{n \in \mathbb{N}_0}$, are increasing sequences with $a'_n, c'_n \geq 0$,*

(ii) *$c'_n \leq a'_n$, for all $n \in \mathbb{N}_0$,*

then $g'(m, n, k) \geq 0$, where $g'(m, n, k)$ are the linearization coefficients of the system $(P_n)_{n \in \mathbb{N}_0}$.

As we consider polynomials $(R_n(x))_{n \in \mathbb{N}_0}$, normalized such that $R_n(1) = 1$, the coefficients a_n, b_n and c_n sum up to 1. An immediate consequence is the following corollary.

Corollary 4.25. *If polynomials $(R_n(x))_{n \in \mathbb{N}_0}$ satisfy (2.1) and*

(i) *$(c_n)_{n \in \mathbb{N}_0}$ is an increasing sequence and $(b_n)_{n \in \mathbb{N}_0}$ and $(a_n + c_n)_{n \in \mathbb{N}_0}$ are constant sequences with $a_n, c_n \geq 0$,*

(ii) *$c_n \leq a_n$, for all $n \in \mathbb{N}_0$,*

then $g(m, n, k) \geq 0$, see (4.16).

The conditions in Corollary 4.25 are very strong. In the remaining part of this subsection, we want to prove assertions concerning the related orthogonal polynomial systems and the related Cesàro operator, whenever an OPS $(R_n(x))_{n \in \mathbb{N}_0}$ satisfies those conditions.

Proposition 4.26. *Let $(R_n(x))_{n \in \mathbb{N}_0}$ be defined as in Corollary 4.25. Denote by h and C_h the Haar measure and the corresponding Cesàro operator, respectively. Then, C_h is hyponormal.*

Proof: Since $c_n \leq a_n \leq a_{n-1}$, for all $n \in \mathbb{N}$, we have

$$h(n) = \frac{a_{n-1}}{c_n} h(n-1) \geq h(n-1)$$

and that implies $\lim_{n \to \infty} H(n) = \infty$ which is condition (1)' in Theorem 3.8. Furthermore, we obtain

$$\frac{h(n)}{h(n+1)} = \frac{c_{n+1}}{a_n} \geq \frac{c_n}{a_{n-1}} = \frac{h(n-1)}{h(n)}, \tag{4.18}$$

for all $n \in \mathbb{N}$, and the second condition can be inferred from induction. Obviously,

$$\frac{h(0)}{h(1)} \geq \frac{H(0)}{H(1)}$$

and if $\frac{h(n)}{h(n+1)} \geq \frac{H(n)}{H(n+1)}$ for some $n \in \mathbb{N}_0$, by (4.18) it follows that

$$\frac{h(n+1)}{h(n+2)} \geq \frac{H(n)}{H(n+1)}$$

which is equivalent to

$$\frac{h(n+1)}{h(n+2)} \geq \frac{H(n+1)}{H(n+2)}.$$

$$\square$$

Reducing the investigations to the symmetric case yields another interesting result.

Proposition 4.27. *Let $(R_n(x))_{n \in \mathbb{N}_0}$ be defined as in Corollary 4.25 and let the respective orthogonalization measure μ be symmetric which implies $b_n = 0$, for all $n \in \mathbb{N}_0$. Denote by h and C_h the Haar measure and the corresponding Cesàro operator, respectively. Let furthermore $(R_n^+(x))_{n \in \mathbb{N}_0}$ be defined as in Corollary 2.7. Then, the Cesàro operator C_{h^+} in $\ell^2(h^+)$ is hyponormal.*

Proof: From Corollary 2.7, we know that

$$a_n^+ = a_{n+1}, \quad b_n^+ = -1 + c_{n+1} + a_n \quad \text{and} \quad c_n^+ = c_n,$$

for all $n \in \mathbb{N}_0$. Since $(a_n)_{n \in \mathbb{N}_0}$ and $(c_n)_{n \in \mathbb{N}_0}$ were chosen such that $a_i \geq c_j$, for all $i, j \in \mathbb{N}_0$, it follows that

$$h^+(n+1) = \frac{a_{n+1}}{c_{n+1}} h^+(n) \geq h^+(n),$$

for all $n \in \mathbb{N}_0$, implying $\lim_{n \to \infty} H(n) = \infty$ which is condition (1)' in Theorem 3.8. Furthermore, by the choice of the linearization coefficients, we have

$$\frac{h^+(n)}{h^+(n+1)} = \frac{c_{n+1}}{a_{n+1}} \geq \frac{c_n}{a_n} = \frac{h^+(n)}{h^+(n-1)},$$

for all $n \in \mathbb{N}$. The same argument as in the proof of Proposition 4.26 implies condition (2)' for the sequence of weights h^+ which shows the hyponormality of C_{h^+}.

$\square$

5 Examples: Polynomials related to homogeneous trees and Karlin-McGregor polynomials

In this chapter, the results from the previous chapters shall be illuminated by considering various examples. More precisely, we exhibit polynomials related to homogeneous trees and Karlin-McGregor polynomials as examples. Both, polynomials related to homogeneous trees and Karlin-McGregor polynomials have already made an appearance in Example 3.10 and Example 2.11, respectively, and will be investigated more detailed in this chapter. We will also focus our attention on the COPS and the related polynomial systems of these two classes of orthogonal polynomials. In particular, we are interested in the behaviour of the respective Cesàro operators, where hyponormality will be of peculiar interest. Furthermore, we want to add some results, concerning the question, when related systems and COPS of polynomials related to homogeneous trees induce polynomial hypergoups.

5.1 Polynomials related to homogeneous trees

In this section, we consider the normalized polynomials related to homogeneous trees, $(R_n^a(x))_{n \in \mathbb{N}_0}$, where $a \in \mathbb{R}$, $a > 1$, which have already been introduced in Example 3.10 and which are defined by the following recurrence relation:

$$R_0^a(x) = 1, \quad R_1^a(x) = x,$$

$$xR_n^a(x) = a_n R_{n+1}^a(x) + c_n R_{n-1}^a(x), \quad n \in \mathbb{N}, \tag{5.1}$$

where

$$a_n = \begin{cases} 1, & n = 0, \\ \dfrac{a-1}{a}, & n \in \mathbb{N}, \end{cases} \quad \text{and} \quad c_n = \begin{cases} 0, & n = 0, \\ \dfrac{1}{a}, & n \in \mathbb{N}. \end{cases}$$

Polynomials related to homogeneous trees are orthogonal and complete over $I_a \subset (-1, 1)$ with respect to the orthogonalization measure μ^a which satisfies

$$\mathrm{d}\mu^a(x) = \frac{a\sqrt{4\frac{a-1}{a^2} - x^2}}{2\pi(1 - x^2)}\,\mathrm{d}x.$$

The shape of the measure shows that neither the COPS of polynomials related to homogeneous trees, nor the systems $(R_n^{a-}(x))_{n\in\mathbb{N}_0}$ and $(R_n^{a+}(x))_{n\in\mathbb{N}_0}$ are included in the class of these polynomials again. In Example 2.11, we made the same observations for Karlin-McGregor polynomials. For $a = 2$ the polynomials in (5.1) coincide up to constants with the Tchebichef polynomials of first kind, see [Chi78, p. 1]. In this case, it is known that $(R_n^a(x))_{n\in\mathbb{N}_0}$, $(R_n^{a-}(x))_{n\in\mathbb{N}_0}$ and $(R_n^{a*}(x))_{n\in\mathbb{N}_0}$ induce polynomial hypergroups and that the respective weights define hyponormal Cesàro operators, see [Las83] and Example 3.9. The weights of $(R_n^{a+}(x))_{n\in\mathbb{N}_0}$ also define a hyponormal Cesàro operator. In the following, we consider the case, when $a \neq 2$. In Example 3.10, we recalled that the Haar measure of $(R_n^a(x))_{n\in\mathbb{N}_0}$ and the sequence H are given by

$$h(n) = \begin{cases} 1, & n = 0, \\ a(a-1)^{n-1}, & n \in \mathbb{N}, \end{cases}$$

and

$$H(n) = \frac{a(a-1)^n - 2}{a - 2},$$

respectively. Moreover, it has been proved that the corresponding Cesàro operator is hyponormal if and only if $a > 2$. Because of

$$\sum_{n=0}^{\infty} h(n) = \frac{2}{2 - a} < \infty,$$

if $a < 2$, it can be inferred from Theorem 3.12 that the corresponding Cesàro operator is not even normaloid.

5.1.1 The conjugate system and the related systems of polynomials related to homogeneous trees

In this subsection, we determine the coefficients and weights of the respective orthogonal polynomial systems by utilizing Theorem 2.4, Corollary 2.7 and Theorem 2.10. Furthermore, we assume $a \neq 2$.

- The system $(R_n^{a-}(x))_{n \in \mathbb{N}_0}$ is orthogonal with respect to the measure μ^{a-}, where

$$\mathrm{d}\mu^{a-}(x) = \frac{a\sqrt{4\frac{a-1}{a^2} - x^2}}{2\pi(1+x)} \, \mathrm{d}x.$$

Using Theorem 2.4, one obtains for the linearization coefficients $a_0^- = \frac{a+1}{a}$, $b_0^- = -\frac{1}{a}$ and for $n \in \mathbb{N}$,

$$a_n^- = \frac{1}{a} \frac{a(a-1)^{n+1} - 2}{a(a-1)^n - 2},$$

$$b_n^- = 0,$$

$$c_n^- = \frac{a-1}{a} \frac{a(a-1)^{n-1} - 2}{a(a-1)^n - 2}.$$

The Haar measure h^- satisfies

$$h^-(n) = \frac{(a(a-1)^n - 2)^2}{(a-2)^2(a-1)^n},$$

for all $n \in \mathbb{N}_0$. Computing the sequence of sums H^-, results in the following, for $n \in \mathbb{N}_0$:

$$H^-(n) = \frac{(a^2 + 4(a-1)^{-n})((a-1)^{n+1} - 1)}{(a-2)^3} - \frac{4(n+1)a}{(a-2)^2}.$$

- The system $(R_n^{a+}(x))_{n \in \mathbb{N}_0}$ is orthogonal with respect to the measure μ^{a+}, where

$$\mathrm{d}\mu^{a+}(x) = \frac{a\sqrt{4\frac{a-1}{a^2} - x^2}}{2\pi(1-x)} \, \mathrm{d}x.$$

Utilizing Corollary 2.7, we can conclude that $a_0^+ = \frac{a-1}{a}$, $b_0^+ = \frac{1}{a}$ and for $n \in \mathbb{N}$,

$$a_n^+ = \frac{a-1}{a},$$

$$b_n^+ = 0,$$

$$c_n^+ = \frac{1}{a}.$$

The Haar measure h^+ satisfies

$$h^+(n) = (a-1)^n,$$

for all $n \in \mathbb{N}_0$. Additionally, determining the sequence H^+ yields

$$H^+(n) = \frac{(a-1)^{n+1} - 1}{a-2}.$$

- The system $(R_n^{a*}(x))_{n \in \mathbb{N}_0}$ is orthogonal with respect to the measure μ^{a*}, where

$$d\mu^{a*}(x) = \frac{a^2 \sqrt{4\frac{a-1}{a^2} - x^2}}{2(a-1)\pi} \, dx.$$

It can be easily verified that for all $n \in \mathbb{N}_0$, we have

$$a_n h(n) + H(n) = \frac{2((a-1)^{n+1} - 1)}{a-2}.$$

From this and from Theorem 2.10, we can infer that

$$a_n^* = \frac{1}{a} \frac{(a-1)^{n+2} - 1}{(a-1)^{n+1} - 1},$$

$$b_n^* = 0,$$

$$c_n^* = \frac{a-1}{a} \frac{(a-1)^n - 1}{(a-1)^{n+1} - 1},$$

for all $n \in \mathbb{N}_0$. Furthermore, for the Haar measure, one obtains

$$h^*(n) = \frac{((a-1)^{n+1} - 1)^2}{(a-2)^2(a-1)^n},$$

for all $n \in \mathbb{N}_0$. Calculating $H^*(n)$ for all $n \in \mathbb{N}_0$ yields

$$H^*(n) = \frac{((a-1)^2 + (a-1)^{-n})((a-1)^{n+1} - 1)}{(a-2)^3} - \frac{2(n+1)(a-1)}{(a-2)^2}.$$

5.1.2 Inferences on the properties of the polynomial systems and the corresponding Cesàro operators

From [Las83], we know that the OPS $(R_n^a(x))_{n\in\mathbb{N}_0}$ induces a polynomial hypergroup for $a \geq 2$. In particular, the linearization coefficients $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ and $(c_n)_{n\in\mathbb{N}_0}$ satisfy the conditions of Szwarc in Corollary 4.25, where $b_n = 0$ for all $n \in \mathbb{N}_0$. In the following, the results, whether the OPS, the COPS or the systems $(R_n^{a-}(x))_{n\in\mathbb{N}_0}$ or $(R_n^{a+}(x))_{n\in\mathbb{N}_0}$ induce polynomial hypergroups, will be summed up.

Theorem 5.1. *Let $(R_n^a(x))_{n\in\mathbb{N}_0}$ be the sequence of polynomials related to homogeneous trees, where $a > 1$. Then,*

   (i) *for $a \geq 2$, the system $(R_n^a(x))_{n\in\mathbb{N}_0}$ induces a polynomial hypergroup,*

   (ii) *for $a > 1$, the system $(R_n^{a-}(x))_{n\in\mathbb{N}_0}$ induces a polynomial hypergroup,*

   (iii) *the system $(R_n^{a+}(x))_{n\in\mathbb{N}_0}$ does not induce a polynomial hypergroup for any choice of a and*

   (iv) *for $a > 1$, the system $(R_n^{a*}(x))_{n\in\mathbb{N}_0}$ induces a polynomial hypergroup.*

   Proof: Ad (i): This was proved in [Las83], for example.
Ad (ii): Let $a \neq 2$. We go back to the notation in Chapter 4.

$$R_1^{a-} R_n^{a-}(x) = \widetilde{a}_n^- R_{n+1}^{a-}(x) + \widetilde{b}_n^- R_n^{a-}(x) + \widetilde{c}_n^- R_{n-1}^{a-}(x),$$

where

$$\widetilde{a}_n^- = \frac{1}{a_0^-}a_n^-, \quad \widetilde{b}_n^- = \frac{1}{a_0^-}(b_n^- - b_0^-) \quad \text{and} \quad \widetilde{c}_n^- = \frac{1}{a_0^-}c_n^-,$$

for $n \in \mathbb{N}$, see also (4.5). A result of Askey, which was cited and proved in [Las83], says that $(R_n^{a-}(x))_{n\in\mathbb{N}_0}$ induces a polynomial hypergroup, if $\widetilde{b}_{n+1}^- \geq \widetilde{b}_n^-$, for all $n \in \mathbb{N}$, and $\widetilde{c}_2^- \widetilde{a}_1^- \geq \widetilde{c}_1^-$, $\widetilde{c}_{n+1}^- \widetilde{a}_n^- \geq \widetilde{c}_n^- \widetilde{a}_{n-1}^-$, for $n \in \mathbb{N}$, $n \geq 2$. We have $\widetilde{a}_0^- = 1$, $\widetilde{b}_0^- = \widetilde{c}_0^- = 0$ and

$$\widetilde{a}_n^- = \frac{1}{a+1}\frac{H(n+1)}{H(n)},$$

$$\widetilde{b}_n^- = \frac{1}{a+1},$$

$$\widetilde{c}_n^- = \frac{a-1}{a+1}\frac{H(n-1)}{H(n)},$$

for $n \in \mathbb{N}$, which yields

$$\widetilde{b}_{n+1}^- \geq \widetilde{b}_n^-,$$

for all $n \in \mathbb{N}_0$ and furthermore,

$$\widetilde{c}_1^- = \frac{a-1}{(a+1)^2} = \widetilde{c}_{n+1}^- \widetilde{a}_n^-,$$

for all $n \in \mathbb{N}$. This shows that $(R_n^{a-}(x))_{n \in \mathbb{N}_0}$ induces a polynomial hypergroup for arbitrary $a > 1$.

Ad (iii): Using the definitions and the calculations in the subsection before, we obtain for all $n \in \mathbb{N}$ and all $a > 1$ that

$$\widetilde{b}_n^+ = -\frac{1}{a+1} < 0$$

which implies that $(R_n^{a+}(x))_{n \in \mathbb{N}_0}$ cannot induce a polynomial hypergroup.

Ad (iv): Let $a \neq 2$. Since the COPS $(R_n^{a*}(x))_{n \in \mathbb{N}_0}$ is symmetric, the linearization coefficients coincide with the coefficients of the modified recurrence relation. We show that $(a_n^*)_{n \in \mathbb{N}_0}$, $(b_n^*)_{n \in \mathbb{N}_0}$ and $(c_n^*)_{n \in \mathbb{N}_0}$ satisfy the conditions in Corollary 4.25. Since $b_n^* = 0$, for all $n \in \mathbb{N}_0$, it remains to show that $a_n^* \geq c_n^*$ and $a_n^* \geq a_{n+1}^*$, for all $n \in \mathbb{N}_0$. Obviously, $a_0^* > c_0^*$ and for $n \in \mathbb{N}$ calculation yields

$$a_n^* - c_n^* = \frac{(a-2)((a-1)^{n+1}+1)}{a((a-1)^{n+1}-1)}$$

which is positive for each $a > 1$. Moreover, we obtain

$$\frac{a_n^*}{a_{n+1}^*} = \frac{((a-1)^{n+2}-1)^2}{((a-1)^{n+3}-1)((a-1)^{n+1}-1)}$$

$$= \frac{(a-1)^{2n+4}+1-2(a-1)(a-1)^{n+1}}{(a-1)^{2n+4}+1-((a-1)^2+1)(a-1)^{n+1}} \geq 1,$$

for all $n \in \mathbb{N}_0$ and the assertion follows.

$\square$

The next theorem deals with the question, what we can infer for the respective Cesàro operators C_{h^-}, C_{h^+} and C_{h^*}, when h is the Haar measure of the OPS $(R_n^a(x))_{n \in \mathbb{N}_0}$ for some $a > 1$. In Example 3.10, we have already seen that the Cesàro operator in $\ell^2(h)$ is hyponormal, if and only if $a \geq 2$.

Theorem 5.2. *Let $(R_n^a(x))_{n \in \mathbb{N}_0}$ be the sequence of polynomials related to homogeneous trees, where $a > 1$. Then,*

(i) *for $a \geq 2$, the Cesàro operator in $\ell^2(h)$ is hyponormal and for $a < 2$, C_h is not normaloid,*

(ii) *for $a > 1$, the Cesàro operator in $\ell^2(h^-)$ is hyponormal,*

(iii) *for $a \geq 2$, the Cesàro operator in $\ell^2(h^+)$ is hyponormal and for $a < 2$, C_{h^+} is not normaloid and*

(iv) *for $a > 1$, the Cesàro operator in $\ell^2(h^*)$ is hyponormal.*

Proof: Ad (i): The assertion follows from Example 3.10, Theorem 3.12 and from the remark before.
Ad (ii): Let $a > 1$ be arbitrary. We utilize Theorem 3.8 to show the hyponormality of the corresponding Cesàro operator C_{h^-}. Since for all $n \in \mathbb{N}$, we have

$$a_n^- - c_n^- = \frac{(a-2)(a(a-1)^n + 2)}{a(a(a-1)^n - 2)} > 0,$$

it follows that $h^-(n+1) \geq h^-(n)$ and thus, $\lim_{n \to \infty} H^-(n) = \infty$ which is condition (1)'. Moreover, an easy computation shows

$$\frac{h^-(n)}{h^-(n+1)} = \frac{((a-1)^n - 2)^2(a-1)}{((a-1)^{n+1} - 2)^2} \geq \frac{((a-1)^{n-1} - 2)^2(a-1)}{((a-1)^n - 2)^2} = \frac{h^-(n-1)}{h^-(n)},$$

for $n \in \mathbb{N}$. The same argument as in the proof of Proposition 4.26 implies condition (2)' for the sequence of weights h^- which shows the hyponormality of C_{h^-}.
Ad (iii): Let $1 < a < 2$. Then,

$$\lim_{n \to \infty} H^+(n) = \lim_{n \to \infty} \frac{(a-1)^{n+1} - 1}{a - 2} = \frac{1}{2 - a} < \infty$$

which contradicts condition (1)' in Theorem 3.8. Moreover, we can infer from Theorem 3.12 that C_{h^+} is not normaloid.
If $a = 2$, the OPS $(R_n^{a+}(x))_{n \in \mathbb{N}_0}$ coincides with the system of normalized Jacobi polynomials for $\alpha = -\frac{1}{2}$ and $\beta = \frac{1}{2}$. The hyponormality of the corresponding Cesàro operator follows from Example 3.9.

Let $a > 2$. Obviously, we obtain $\lim_{n\to\infty} H^+(n) = \infty$ which is condition (1)' in Theorem 3.8. Furthermore,

$$\frac{h(n)}{H(n)} = \frac{(a-2)}{(a-1)-(a-1)^{-n}} > \frac{(a-2)}{(a-1)-(a-1)^{-(n+1)}} = \frac{h(n+1)}{H(n+1)},$$

is satisfied for all $n \in \mathbb{N}_0$ which is condition (2)'. Hence, the hyponormality of the corresponding Cesàro operator C_{h^+} can be inferred.

Ad (iv): In the proof of Theorem 5.1, it was shown that $(a_n^*)_{n\in\mathbb{N}_0}$, $(b_n^*)_{n\in\mathbb{N}_0}$ and $(c_n^*)_{n\in\mathbb{N}_0}$ satisfy the conditions in Corollary 4.25. By Proposition 4.26, the corresponding Cesàro operator C_{h^*} is hyponormal.

$\square$

5.2 Karlin-McGregor polynomials

In this section, we consider the normalized Karlin-McGregor polynomials, $(R_n^{(a,b)}(x))_{n\in\mathbb{N}_0}$, where $a, b \in \mathbb{R}$, $a, b > 1$, which have already been introduced in Chapter 2 and are defined by the following recurrence relation, see Example 2.11.

$$R_0^{(a,b)}(x) = 1, \quad R_1^{(a,b)}(x) = x,$$

$$xR_n^{(a,b)}(x) = a_n R_{n+1}^{(a,b)}(x) + c_n R_{n-1}^{(a,b)}(x), \quad n \in \mathbb{N}, \tag{5.2}$$

where

$$a_n = \begin{cases} 1, & n = 0, \\ \dfrac{a-1}{a}, & n \in \{1, 3, 5, ...\}, \\ \dfrac{b-1}{b}, & n \in \{2, 4, 6, ...\} \end{cases} \quad \text{and} \quad c_n = \begin{cases} 1, & n = 0, \\ \dfrac{1}{a}, & n \in \{1, 3, 5, ...\}, \\ \dfrac{1}{b}, & n \in \{2, 4, 6, ...\}. \end{cases}$$

The Karlin-McGregor polynomials are orthogonal and complete over $I_{a,b} \subset (-1, 1)$ with respect to the orthogonalization measure $\mu^{(a,b)}$ which satisfies

$$\mathrm{d}\mu^{(a,b)}(x) = \frac{b\sqrt{4\frac{b-1}{ab}x^2 - \left(x^2 + \frac{b-a}{ab}\right)^2}}{2\pi|x|(1-x^2)} \, \mathrm{d}x.$$

We want to recap the formulas for the Haar measure h and the sequence H which are given by

$$h(n) = \begin{cases} 1, & n = 0, \\ a(a-1)^{\frac{n-1}{2}}(b-1)^{\frac{n-1}{2}}, & n \in \{1, 3, 5, ...\}, \\ b(a-1)^{\frac{n}{2}}(b-1)^{\frac{n-2}{2}}, & n \in \{2, 4, 6, ...\} \end{cases}$$

and

$$H(n) = \begin{cases} 1, & n = 0, \\ \dfrac{2a - (b + a(b-1))(a-1)^{\frac{n+1}{2}}(b-1)^{\frac{n-1}{2}}}{1 - (a-1)(b-1)}, & n \in \{1, 3, 5, ...\}, \\ \dfrac{2a - (a + b(a-1))(a-1)^{\frac{n}{2}}(b-1)^{\frac{n}{2}}}{1 - (a-1)(b-1)}, & n \in \{2, 4, 6, ...\}, \end{cases} \qquad (5.3)$$

if $(a-1)(b-1) \neq 1$. Choosing $a = b$, leads to polynomials related to homogeneous trees which have already been discussed in the section before. In the following, we will assume $a \neq b$. If the parameters a and b satisfy $(a-1)(b-1) = 1$, one obtains

$$h(n) = \begin{cases} 1, & n = 0, \\ a, & n \in \mathbb{N}, \end{cases} \qquad (5.4)$$

and

$$H(n) = 1 + an, \quad \text{for all } n \in \mathbb{N}_0,$$

see Example 2.11. There, we also determined the coefficients and Haar measure of the respective COPS $(R_n^{(a,b)*}(x))_{n \in \mathbb{N}_0}$. Considering the measures $\mu^{(a,b)-}$, $\mu^{(a,b)+}$ and $\mu^{(a,b)*}$ shows that the respective orthogonal polynomial systems are not contained in the class of Karlin-McGregor polynomials again. It is of great interest to us which properties in terms of the generalized concepts of normality, in particular hyponormality, the corresponding Cesàro operators will have.

5.2.1 The conjugate system and the related systems of Karlin-McGregor polynomials

First of all, we want to determine the Haar measure of the orthogonal polynomial systems $(R_n^{(a,b)-}(x))_{n\in\mathbb{N}_0}$, $(R_n^{(a,b)+}(x))_{n\in\mathbb{N}_0}$, and $(R_n^{(a,b)*}(x))_{n\in\mathbb{N}_0}$, by utilizing Theorem 2.4, Corollary 2.7 and Theorem 2.10. Furthermore, we assume $a \neq b$.

- The system $(R_n^{(a,b)-})_{n\in\mathbb{N}_0}$ is orthogonal with respect to the measure $\mu^{(a,b)-}$, where

$$\mathrm{d}\mu^{(a,b)-}(x) = \frac{b\sqrt{4\frac{b-1}{ab}x^2 - \left(x^2 + \frac{b-a}{ab}\right)^2}}{2\pi|x|(1+x)}\ \mathrm{d}x.$$

For $(a-1)(b-1) \neq 1$, the Haar measure h^- satisfies

$$h^-(n) = \begin{cases} 1, & n = 0, \\[2mm] \dfrac{(2a - (b + a(b-1))(a-1)^{\frac{n+1}{2}}(b-1)^{\frac{n-1}{2}})^2}{(1 - (a-1)(b-1))^2(a-1)^{\frac{n+1}{2}}(b-1)^{\frac{n-1}{2}}}, & n \in \{1,3,5,...\}, \\[4mm] \dfrac{(2a - (a + b(a-1))(a-1)^{\frac{n}{2}}(b-1)^{\frac{n}{2}})^2}{(1 - (a-1)(b-1))^2(a-1)^{\frac{n}{2}}(b-1)^{\frac{n}{2}}}, & n \in \{2,4,6,...\}. \end{cases} \tag{5.5}$$

In the case, when $(a-1)(b-1) = 1$, we obtain

$$h^-(n) = \begin{cases} (1 + an)^2, & n \in \{0,2,4,...\}, \\[3mm] \dfrac{(1 + an)^2}{a - 1}, & n \in \{1,3,5,...\}. \end{cases} \tag{5.6}$$

- The system $(R_n^{(a,b)+}(x))_{n\in\mathbb{N}_0}$ is orthogonal with respect to the measure $\mu^{(a,b)+}$, where

$$\mathrm{d}\mu^{(a,b)+}(x) = \frac{b\sqrt{4\frac{b-1}{ab}x^2 - \left(x^2 + \frac{b-a}{ab}\right)^2}}{2\pi|x|(1-x)}\ \mathrm{d}x.$$

For $(a-1)(b-1) \neq 1$, the Haar measure h^+ satisfies

$$h^+(n) = \begin{cases} (a-1)^{\frac{n}{2}}(b-1)^{\frac{n}{2}}, & n \in \{0,2,4,...\}, \\[3mm] (a-1)^{\frac{n+1}{2}}(b-1)^{\frac{n-1}{2}}, & n \in \{1,3,5,...\} \end{cases} \tag{5.7}$$

and choosing $(a-1)(b-1) = 1$, results in

$$h^+(n) = \begin{cases} 1, & n \in \{0, 2, 4, ...\}, \\ (a-1), & n \in \{1, 3, 5, ...\}. \end{cases} \tag{5.8}$$

- Finally, as we have already seen in Example 2.11, the system $(R_n^{a*}(x))_{n \in \mathbb{N}_0}$ is orthogonal with respect to the measure $\mu^{(a,b)*}$, where

$$\mathrm{d}\mu^{(a,b)*}(x) = \frac{ab\sqrt{4\frac{b-1}{ab}x^2 - \left(x^2 + \frac{b-a}{ab}\right)^2}}{2(a-1)\pi|x|} \, \mathrm{d}x.$$

The coefficients of the COPS can be computed due to the formulas in Theorem 2.10. First, consider the case, when $(a-1)(b-1) \neq 1$. One obtains

$$h(n)^* = \begin{cases} 1, & n = 0, \\[2mm] \dfrac{b(a - a(a-1)^{\frac{n+1}{2}}(b-1)^{\frac{n+1}{2}})^2}{a(a-1)^{\frac{n-1}{2}}(b-1)^{\frac{n+1}{2}}(1-(a-1)(b-1))^2}, & n \in \{1, 3, 5, ...\}, \\[4mm] \dfrac{(a - b(a-1)^{\frac{n+2}{2}}(b-1)^{\frac{n}{2}})^2}{(a-1)^{\frac{n}{2}}(b-1)^{\frac{n}{2}}(1-(a-1)(b-1))^2}, & n \in \{2, 4, 6, ...\}. \end{cases} \tag{5.9}$$

In the case, when $(a-1)(b-1) = 1$, the expression can be simplified to

$$h(n)^* = \begin{cases} \dfrac{(2+na)^2}{4}, & n \in \{0, 2, 4, ...\}, \\[4mm] \dfrac{(n+1)^2 a^2}{4}, & n \in \{1, 3, 5, ...\}. \end{cases} \tag{5.10}$$

5.2.2 Inferences on the properties of the corresponding Cesàro operators

Denote by $(R_n^{(a,b)}(x))_{n \in \mathbb{N}_0}$ the Karlin-McGregor polynomials with parameters $a, b > 1$, normalized as the polynomials in Chapter 2, and furthermore, assume $a \neq b$. To start with, the weights of Karlin-McGregor polynomials and the corresponding Cesàro operators are investigated.

Proposition 5.3. *Let* $(R_n^{(a,b)}(x))_{n \in \mathbb{N}_0}$ *be the sequence of Karlin-McGregor polynomials with parameters* $a, b > 1$, $a \neq b$*. Then,*

(i) *for* $(a-1)(b-1) = 1$*, the Cesàro operator in* $\ell^2(h)$ *is hyponormal,*

(ii) *for* $(a-1)(b-1) \neq 1$*, the Cesàro operator is not hyponormal and in particular, for* $(a-1)(b-1) < 1$*,* C_h *is not normaloid.*

Proof: Ad (i): For $(a-1)(b-1) = 1$, the Haar measure is defined by $h(0) = 1$, $h(n) = a$, for $n \in \mathbb{N}$, see Example 2.11 and (5.4). Thus, we have $\lim_{n \to \infty} H(n) = \infty$ and

$$1 = \frac{h(0)}{H(0)} > \frac{h(n)}{H(n)} = \frac{a}{1+an} > \frac{1}{1+a(n+1)} = \frac{h(n+1)}{H(n)},$$

for all $n \in \mathbb{N}$, which are conditions (1)' and (2)' in Theorem 3.8.
Ad (ii): Let $(a-1)(b-1) < 1$. From (5.3), it can be inferred that

$$\lim_{n \to \infty} H(n) = \frac{2a}{1-(a-1)(b-1)} < \infty.$$

Applying Theorem 3.12 yields that C_h cannot be normaloid.
Let $(a-1)(b-1) > 1$. Utilizing the definition of the sequence H in (5.3), we obtain

$$\lim_{k \to \infty} \frac{h(2k)}{H(2k)} = \frac{a((a-1)(b-1)-1)}{(a-1)(b+a(b-1))}$$

and, on the other hand, one has

$$\lim_{k \to \infty} \frac{h(2k+1)}{H(2k+1)} = \frac{b((a-1)(b-1)-1)}{(b-1)(a+b(a-1))}.$$

Hyponormality can only be satisfied, if the two limits coincide. Straightforward computation shows that this would imply $a = b$ or $a = \frac{b}{b-1}$ which contradicts the assumptions. Hence, C_h is not hyponormal.

$\square$

Now, C_h is hyponormal, if and only if $a = \frac{b}{b-1}$. Otherwise, following [FL00], $(R_n^{(a,b)}(x))_{n \in \mathbb{N}_0}$ induces a polynomial hypergroup if $a, b \geq 2$. Hence, inducing a polynomial hypergroup does not imply the hypernormality of the corresponding Cesàro operator, in general.

The following proposition deals with the system $(R_n^{(a,b)-}(x))_{n \in \mathbb{N}_0}$.

Proposition 5.4. *Let* $(R_n^{(a,b)}(x))_{n \in \mathbb{N}_0}$ *be the sequence of Karlin-McGregor polynomials with parameters* $a, b > 1$, $a \neq b$. *Then,*

  (i) *for* $(a-1)(b-1) \leq 1$, *the Cesàro operator in* $\ell^2(h^-)$ *is not hyponormal,*

  (ii) *for* $(a-1)(b-1) > 1$, *a necessary condition for the hyponormality of* C_{h^-} *is*
      $(a-1)(b+a(b-1))^4 = (b-1)(a+b(a-1))^4$.

  Proof: Let $(a-1)(b-1) = 1$. From (5.6), we can infer that

$$\lim_{k \to \infty} \frac{h^-(2k)}{h^-(2k+1)} = a - 1$$

and, on the other hand,

$$\lim_{k \to \infty} \frac{h^-(2k-1)}{h^-(2k)} = \frac{1}{a-1}.$$

Furthermore, we obtain that $H^-(n)$ is a polynomial of degree 3. Calculation shows that the leading coefficient is given by $\frac{a}{6(a-1)}$, for all $n \in \mathbb{N}_0$. This implies

$$\lim_{k \to \infty} \frac{H^-(2k)}{H^-(2k+1)} = \lim_{k \to \infty} \frac{H^-(2k-1)}{H^-(2k)} = 1.$$

Since a was chosen such that $a \neq 2$, it follows that condition (2)' of Theorem 3.8 cannot be satisfied.

Let $(a-1)(b-1) < 1$. Then, the crucial summand of $h^-(n)$ is

$$\frac{4a^2}{(1-(a-1)(b-1))^2}(a-1)^{-\frac{n+1}{2}}(b-1)^{-\frac{n-1}{2}}$$

for $n \in \{1, 3, 5, ...\}$ and

$$\frac{4a^2}{(1-(a-1)(b-1))^2}(a-1)^{-\frac{n}{2}}(b-1)^{-\frac{n}{2}}$$

for $n \in \{2, 4, 6, ...\}$, respectively, see (5.5). Hence, computing the sequence H^- approximately yields

$$\lim_{k \to \infty} \frac{h^-(2k)}{H^-(2k)} = \frac{1-(a-1)(b-1)}{b}$$

and

$$\lim_{k \to \infty} \frac{h^-(2k+1)}{H^-(2k+1)} = \frac{1-(a-1)(b-1)}{a}.$$

Since by assumption $a \neq b$, the two limits differ from each other which contradicts condition (2)' in Theorem 3.8.

Let $(a-1)(b-1) > 1$. Then, for large $n \in \mathbb{N}_0$, $h^-(n)$ is approximately given by

$$\frac{(b+a(b-1))^2(a-1)^{\frac{n+1}{2}}(b-1)^{\frac{n-1}{2}}}{((a-1)(b-1)-1)^2}$$

for $n \in \{1, 3, 5, ...\}$ and

$$\frac{(a+b(a-1))^2(a-1)^{\frac{n}{2}}(b-1)^{\frac{n}{2}}}{((a-1)(b-1)-1)^2}$$

for $n \in \{2, 4, 6, ...\}$, respectively, see (5.5). Computing the sequence H^- approximately, results in

$$\lim_{k \to \infty} \frac{h^-(2k)}{H^-(2k)} = (1-(a-1)(b-1))\left((a-1)(b-1)+(a-1)\left(\frac{b+a(b-1)}{a+b(a-1)}\right)^2\right)^{-1}$$

and

$$\lim_{k \to \infty} \frac{h^-(2k+1)}{H^-(2k+1)} = (1-(a-1)(b-1))\left((a-1)(b-1)+(b-1)\left(\frac{a+b(a-1)}{b+a(b-1)}\right)^2\right)^{-1}.$$

The limits coincide, if

$$(a-1)(b+a(b-1))^4 = (b-1)(a+b(a-1))^4.$$

$\square$

Note that the equality above is satisfied for $a = b$. But for certain parameters $a \neq b$, it can also be satisfied and one is not able to avoid examining the equality more detailed. However, this shall not be part of this thesis and we will now attend to the system $(R_n^{(a,b)+}(x))_{n \in \mathbb{N}_0}$.

Proposition 5.5. *Let* $(R_n^{(a,b)}(x))_{n \in \mathbb{N}_0}$ *be the sequence of Karlin-McGregor polynomials with parameters* $a, b > 1$, $a \neq b$. *Then, the Cesàro operator in* $\ell^2(h^+)$ *is not hyponormal, independent of the choice of the parameters* a *and* b.

Proof: Let $(a-1)(b-1) = 1$. Following (5.8), we can infer that

$$\lim_{n \to \infty} \frac{H^+(n)}{H^+(n+1)} = 1.$$

On the other hand, one obtains for all $k \in \mathbb{N}$ that

$$\frac{h^+(2k)}{h^+(2k+1)} = \frac{1}{a-1} \neq a-1 = \frac{h^+(2k-1)}{h^+(2k)}.$$

Hence, condition (2)' in Theorem 3.8 cannot be satisfied.
Next, we consider the case, when $(a-1)(b-1) < 1$. Using (5.7), we can infer that

$$\sum_{n=0}^{\infty} h^+(n) < \infty.$$

Applying Theorem 3.12 yields that C_{h^+} is not normaloid and thus, not hyponormal.
Let $(a-1)(b-1) > 1$. An easy computation shows that

$$\lim_{k \to \infty} \frac{h^+(2k)}{H^+(2k)} = \frac{(a-1)(b-1)-1}{b(a-1)}$$

but otherwise,

$$\lim_{k \to \infty} \frac{h^+(2k)}{H^+(2k)} = \frac{(a-1)(b-1)-1}{a(b-1)}.$$

The limits coincide, if and only if $a = b$ which we excluded. Hence, C_{h^+} cannot be hyponormal.

$\square$

Finally, we focus our attention on the COPS of Karlin-McGregor polynomials.

Proposition 5.6. *Let* $(R_n^{(a,b)}(x))_{n \in \mathbb{N}_0}$ *be the sequence of Karlin-McGregor polynomials with parameters* $a, b > 1$, $a \neq b$ *and denote by* $(R_n^{(a,b)*}(x))_{n \in \mathbb{N}_0}$ *the respective COPS. Then,*

(i) *for $(a-1)(b-1) = 1$, the Cesàro operator in $\ell^2(h^*)$ is hyponormal, if and only if $a \in [a_r, a_R]$, where $a_r \in (1.39, 1.40)$ and $a_R \in (3.76, 3.77)$.*

(ii) *for $(a-1)(b-1) \neq 1$, the Cesàro operator is not hyponormal and in particular, for $(a-1)(b-1) < 1$, C_h is not normaloid.*

Proof: Let $(a-1)(b-1) = 1$. Utilizing (5.10), one obtains

$$h^*(2k) = (1+ka)^2 \qquad \text{and} \qquad h^*(2k+1) = (1+k)^2 a^2, \qquad \text{for all } k \in \mathbb{N}_0.$$

Straightforward computation of the sequence H^* yields

$$H^*(2k) = \frac{2}{3}a^2 k^3 + (a^2 + a)k^2 + \left(\frac{a^2}{3} + a + 1\right)k + 1,$$

$$H^*(2k+1) = \frac{2}{3}a^2 k^3 + (2a^2 + a)k^2 + \left(\frac{7}{3}a^2 + a + 1\right)k + a^2 + 1.$$

Obviously, for all $a > 1$, condition (1)' in Theorem 3.8 is satisfied. Subsequently, we want to investigate for which parameters a condition (2)' is also true. We have

$$\frac{H^*(2k)}{h^*(2k)} = \frac{2}{3}k + 1 - \frac{1}{3a} + \frac{(a^2 - 3a + 3)k + \frac{1}{a}}{3(ak+1)^2}$$

and

$$\frac{H^*(2k+1)}{h^*(2k+1)} = \frac{2}{3}k + \frac{2}{3} + \frac{1}{a} + \frac{a^2 - 3a + 3}{3a^2(k+1)},$$

for all $k \in \mathbb{N}_0$. The operator C_h^* is hyponormal, if $\left(\frac{H^*(n)}{h^*(n)}\right)_{n \in \mathbb{N}_0}$ is an increasing sequence. Therefore, we continue with determining

$$\frac{H(2k+1)}{h(2k+1)} - \frac{H(2k)}{h(2k)} \qquad \text{and} \qquad \frac{H^*(2k+2)}{h^*(2k+2)} - \frac{H^*(2k+1)}{h^*(2k+1)}, \qquad (5.11)$$

respectively, for all $k \in \mathbb{N}_0$. For the first expression in (5.11), we obtain

$$\frac{H^*(2k+1)}{h^*(2k+1)} - \frac{H^*(2k)}{h^*(2k)} = \frac{4}{3a} - \frac{1}{3} + \frac{(a^2 - 3a + 3)((2a - a^2)k + 1) - a(k+1)}{3a^2(k+1)(ak+1)^2}. \qquad (5.12)$$

For $k = 0$, the right hand side of (5.12) is positive, independent of the choice of a. Denote

the right summand on the right hand side of (5.12) by $c_1^a(k)$.
If $1 < a < 2$, we can infer that $c_1^a(k)$ is bounded below by

$$c_1^a(k) > -\frac{1}{3a(ak+1)^2} > -\frac{1}{3},$$

for all $k \in \mathbb{N}$. Since $\frac{4}{3a} > \frac{2}{3}$, positivity in (5.12) is satisfied.
If $2 < a < 3$, it can easily be verified that

$$c_1^a(k) > -\frac{4}{3a(ak+1)^2} > -\frac{1}{9},$$

for all $k \in \mathbb{N}$ and $\frac{4}{3a} - \frac{1}{3} > \frac{1}{9}$, respectively, which implies positivity in (5.12). Let $a \geq 3$.
We show that $(c_1^a(k))_{k \in \mathbb{N}}$ is an increasing sequence. Therefore, denote by f_a the continuous
function

$$f_a : \mathbb{R}^+ \to \mathbb{R}, \quad x \mapsto \frac{(a^2 - 3a + 3)((-a^2 + 2a)x + 1) - a(x+1)}{3a^2(x+1)(ax+1)^2}.$$

Then, the derivation f_a' of f_a is a continuous function in $\mathbb{R}^+$ which satisfies

$$f_a'(x) = 3^{-1}a^{-2}((x+1)(ax+1)^2)^{-2}$$
$$\times \left[(x+1)(ax+1)^2((a^2 - 3a + 3)(-a^2 + 2a) - a) \right.$$
$$\left. -((ax+1)^2 + 2a(x+1)(ax+1))((a^2 - 3a + 3)((-a^2 + 2a)x + 1) - a(x+1)) \right]$$

$$= 3^{-1}a^{-2}((x+1)(ax+1)^2)^{-2}$$
$$\times \left[-(ax+1)^2(a^2 - 3a + 3)(a^2 - 2a + 1) \right.$$
$$\left. +2a(x+1)(ax+1)(a^2 - 3a + 3)(ax(a-2) - 1) + 2a^2(x+1)^2(ax+1) \right]$$

$$= 3^{-1}a^{-2}((x+1)(ax+1)^2)^{-2}$$
$$\times \left[(ax+1)(a^2 - 3a + 3)(2a^3x^2 + a^3x - 4a^2x^2 - 2a^2x - 3ax - a^2 - 1) \right.$$
$$\left. +2a^2(x+1)^2(ax+1) \right]$$

$$\geq 3^{-1}a^{-2}((x+1)(ax+1)^2)^{-2}$$
$$\times \left[2a^2(x+1)^2(ax+1) + (ax+1)(a^2 - 3a + 1)(2a^2x^2 - a^2 - 1) \right]$$

which is nonnegative for all $x \geq 1$ and $a \geq 3$. Since $f_a(k) = c_1^a(k)$ for all $k \in \mathbb{N}$, we can infer that $c_1^a(k)$ is bounded below by $c_1^a(1)$ for all $k \in \mathbb{N}$. Hence, we have to investigate for which $a \geq 3$ the inequality

$$\frac{H^*(3)}{h^*(3)} - \frac{H^*(2)}{h^*(2)} = \frac{4}{3a} - \frac{1}{3} + \frac{(a^2 - 3a + 3)(-a^2 + 2a + 1) - 2a}{6a^2(a+1)^2} \geq 0, \tag{5.13}$$

is true. The inequality in (5.13) is equivalent to

$$-a^4 + 3a^3 + 2a^2 + 3a + 1 \geq 0. \tag{5.14}$$

The polynomial in (5.14) has one positive, real root a_R which is located in the interval $(3.76, 3.77)$ and the inequality is satisfied for $1 < a \leq a_R$. The second difference in (5.11) reduces to

$$\frac{H^*(2k+2)}{h^*(2k+2)} - \frac{H^*(2k+1)}{h^*(2k+1)} = 1 - \frac{4}{3a} + \frac{(a^2 - 3a + 3)(-2a(k+1) - 1) + a(k+1)}{3a^2(k+1)(ak+a+1)^2}, \tag{5.15}$$

for all $k \in \mathbb{N}_0$. Denote by $c_2^a(k)$ the right summand on the right hand side of equation (5.15). We show that $c_2^a(k)$ is bounded below by $c_2^a(0)$, for all $n \in \mathbb{N}_0$. Obviously, since $(a^2 - 3a + 3) \geq \frac{3}{4}$, it follows that $c_2^a(k) < 0$, for all $k \in \mathbb{N}_0$. Moreover, we can infer that

$$c_2^a(k) = \frac{(a^2 - 3a + 3)(-2a(k+1) - 1) + a(k+1)}{3a^2(k+1)(ak+a+1)^2}$$

$$\geq \frac{(a^2 - 3a + 3)(-2a(k+1) - (k+1)) + a(k+1)}{3a^2(k+1)(a+1)^2} = c_2^a(0),$$

for all $k \in \mathbb{N}_0$. Hence, a necessary condition for the hyponormality of C_{h^*} is

$$1 - \frac{4}{3a} + \frac{(a^2 - 3a + 3)(-2a - 1) + a}{3a^2(a+1)^2} \geq 0.$$

Simplifying this expression, results in

$$a^4 - 2a - 1 \geq 0. \tag{5.16}$$

The polynomial in (5.16) has one positive, real root a_r with $a_r \in (1.39, 1.40)$ and the inequality is satisfied, if $a \geq a_r$. In summary, it can be stated that C_{h^*} is hyponormal, if and only if

$$a_r \leq a \leq a_R,$$

where a_r and a_R, respectively, are the only positive, real roots of the polynomials in (5.16) and (5.14), respectively.

Let $(a - 1)(b - 1) < 1$. We use the same argument as in the proof of Proposition 5.4. Then, the crucial summand of $h^*(n)$ is

$$\frac{ab}{(1 - (a - 1)(b - 1))^2}(a - 1)^{-\frac{n-1}{2}}(b - 1)^{-\frac{n+1}{2}}$$

for $n \in \{1, 3, 5, ...\}$ and

$$\frac{a^2}{(1 - (a - 1)(b - 1))^2}(a - 1)^{-\frac{n}{2}}(b - 1)^{-\frac{n}{2}}$$

for $n \in \{2, 4, 6, ...\}$, respectively, see (5.9). Hence, computing the sequence H^* approximately yields

$$\lim_{k \to \infty} \frac{h^*(2k)}{H^*(2k)} = (1 - (a - 1)(b - 1))\frac{a}{a + b(a - 1)}$$

and

$$\lim_{k \to \infty} \frac{h^*(2k + 1)}{H^*(2k + 1)} = (1 - (a - 1)(b - 1))\frac{b}{b + a(b - 1)}.$$

Since we chose $a \neq b$ and $a \neq \frac{b}{b-1}$, the two limits differ from each other which contradicts condition (2)' in Theorem 3.8.

Let $(a - 1)(b - 1) > 1$. In this case, the crucial summand of $h^*(n)$ is

$$\frac{b(a - 1)^2}{(1 - (a - 1)(b - 1))^2}\, a(a - 1)^{\frac{n-1}{2}}(b - 1)^{\frac{n+1}{2}}$$

for $n \in \{1, 3, 5, ...\}$ and

$$\frac{b(a - 1)^2}{(1 - (a - 1)(b - 1))^2}\, b(a - 1)^{\frac{n}{2}}(b - 1)^{\frac{n}{2}}$$

for $n \in \{2, 4, 6, ...\}$, respectively, see (5.9). Thus, computing the sequence H^* approximately, results in

$$\lim_{k \to \infty} \frac{h^*(2k)}{H^*(2k)} = (1 - (a-1)(b-1)) \frac{b}{(b-1)(a + b(a-1))}$$

and

$$\lim_{k \to \infty} \frac{h^*(2k+1)}{H^*(2k+1)} = (1 - (a-1)(b-1)) \frac{a}{(a-1)(b + a(b-1))}.$$

The two limits coincide, if and only if $a \neq b$ or $a \neq \frac{b}{b-1}$, what we excluded. Hence, following Theorem 3.8, C_{h^*} is not hyponormal.

$\square$

Outlook

In the main part of the dissertation, we primarily treated polynomial systems $(R_n(x))_{n \in \mathbb{N}_0}$ which were orthogonal with respect to a probability measure μ with $\mathsf{supp}\, \mu \subset [-1, 1]$. Due to this assumption, the conjugate orthogonal polynomial system $(R_n^*(x))_{n \in \mathbb{N}_0}$ and the related systems $(R_n^-(x))_{n \in \mathbb{N}_0}$ and $(R_n^+(x))_{n \in \mathbb{N}_0}$ are well-defined. Moreover, the normalization was chosen such that $R_n(1) = R_n^-(1) = R_n^+(1) = R_n^*(1) = 1$, for all $n \in \mathbb{N}_0$, a normalization which is, by the way, principally used in the context of polynomial hypergroups.

For one thing, the COPS and the related systems might be useful to characterize certain classes of orthogonal polynomials, like ultraspherical polynomials as in [LO08].

Besides playing a decisive role in the deduction of the conjugate orthogonal polynomial system, the formulas of $(R_n^-(x))_{n \in \mathbb{N}_0}$ and $(R_n^+(x))_{n \in \mathbb{N}_0}$ which were interrelated with the definitions of the operators C_h and C_k in Section 4.2, turned out to be important for the investigations of the operator T_1. We were able to find necessary and sufficient conditions for $1 \in \mathcal{S} = \mathsf{supp}\, \mu = \sigma(T_1)$ and $-1 \in \mathcal{S}$, respectively.

Naturally, the question arises, whether one obtains necessary and sufficient criteria for $\lambda \in \mathcal{S}$ with $\lambda \in \mathbb{R}$. This would be a very interesting topic, since in many cases the true interval of orthogonality is unknown. Probably, if conditions (B) and (B)' are satisfied, it would be possible to approach the upper bound η_1 and the lower bound ξ_1 of $\mathcal{S}$, where we denote by $[\xi_1, \eta_1]$ the true interval of orthogonality of $(R_n(x))_{n \in \mathbb{N}_0}$, by deciding for which $\lambda \in \mathbb{R}$, starting from $\lambda = 1$ and $\lambda = -1$, respectively, the operator $\lambda \mathsf{id} - T_1$ is invertible. Correspondingly an interesting strategy would be a normalization of the considered polynomial systems at the upper bound η_1 or at the lower bound ξ_1 of $\mathsf{supp}\, \mu$. Obviously, (also mentioned in Chapter 1,) assuming the boundedness of $[\xi_1, \eta_1]$ yields the uniqueness of the corresponding orthogonalization measure and moreover, all the zeros of $(R_n(x))_{n \in \mathbb{N}_0}$ are contained in (ξ_1, η_1). Hence, it is possible to consider the OPS $(R_n(x; \eta_1))_{n \in \mathbb{N}_0}$ and the OPS $(R_n(x; \xi_1))_{n \in \mathbb{N}_0}$, respectively, wrt μ, where $R_n(\eta_1; \eta_1) = 1$ and $R_n(\xi_1; \xi_1) = 1$, respectively, for all $n \in \mathbb{N}_0$. Furthermore, one could determine the polynomial systems orthogonal wrt measures $\mu^{(\eta_1)}$ and $\mu^{(\xi_1)}$, satisfying

$$\mathrm{d}\mu^{(\eta_1)} = c(\eta_1 - x)\, \mathrm{d}\mu,$$

$$\mathrm{d}\mu^{(\xi_1)} = c'(x - \xi_1)\, \mathrm{d}\mu,$$

for some constants c and c', chosen such that $\mu^{(\eta_1)}$ and $\mu^{(\xi_1)}$ become probability measures. Alternatively, it is also possible to choose an arbitrary real value ξ which is no zero of $R_n(x)$, for all $n \in \mathbb{N}_0$, and consider the correspondingly normalized system.

The second annotation concerns the properties of the Cesàro operator, more detailed the studies on the generalized concepts of normality. The question, whether the Cesàro operator satisfies subnormality, was only marginally treated. The author consciously decided to exclude the investigations, since they would go far beyond the constraints of this thesis. In the fore should primarily be other properties of C_h, as well as the relation between the Cesáro operator and the OPS, the related systems, the COPS and the tridiagonal operator T_1.

Classifying those weights for which the Cesàro operator becomes subnormal would be, seen individually, an interesting topic. Presumably, methods in the papers of Kriete and Trutt ([KT71]) and Cowen ([Cow84]), which have already been mentioned in Section 3.3, and Conway's treatise on subnormal operator ([Con91]) will be very auxiliary to derive subnormality for the Cesàro operator in the weighted case. Otherwise, it is indicated that for each class of weights similar procedures as in the papers referred to above are necessary and moreover, that the chosen transforms and measures will also depend on the weights. For the author it is of peculiar interest, whether there are further sequences of weights for which C_h becomes subnormal. Additionally, the questions arises, whether there are examples for weights such that the corresponding Cesàro operator is hyponormal but not subnormal and whether, in case of subnormality, we can infer stronger properties (for instance in terms of the classification of polynomial hypergroups) for the respective OPS, the related systems and the COPS.

Certainly there are further interesting questions and approaches arising from the investigations in this thesis. The proved results will be a firm basis.

Bibliography

[And72]   T. Ando. Operators with a norm condition. *Acta Sci. Math. (Szeged)*, 33:169–178, 1972.

[AW66a]   R. Askey and S. Wainger. A transplantation theorem between ultraspherical series. *Illinois J. Math.*, 10:322–344, 1966.

[AW66b]   R. Askey and S. Wainger. A transplantation theorem for ultraspherical coefficients. *Pacific J. Math.*, 16:393–405, 1966.

[Bec00]   B. Beckermann. On the classification of the spectrum of second order difference operators. *Math. Nachr.*, 216:45–59, 2000.

[Bel44]   R. Bellman. A note on a theorem of Hardy on Fourier constants. *Bull. Amer. Math. Soc.*, 50:741–744, 1944.

[BHS65]   A. Brown, P. R. Halmos, and A. L. Shields. Cesàro operators. *Acta Sci. Math. (Szeged)*, 26:125–137, 1965.

[Boo00]   J. Boos. *Classical and modern methods in summability.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000. Assisted by Peter Cass, Oxford Science Publications.

[Bra55]   J. Bram. Subnormal operators. *Duke Math. J.*, 22:75–94, 1955.

[Chi78]   T. S. Chihara. *An introduction to orthogonal polynomials.* Gordon and Breach Science Publishers, New York, 1978. Mathematics and its Applications, Vol. 13.

[Con91]   J. B. Conway. *The theory of subnormal operators*, volume 36 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1991.

[Cop27]   E. T. Copson. Note on Series of Positive Terms. *J. London Math. Soc.*, S2(1):9–12, 1927.

[Cow84]   C. C. Cowen. Subnormality of the Cesàro operator and a semigroup of composition operators. *Indiana Univ. Math. J.*, 33(2):305–318, 1984.

[CR13a]   G. P. Curbera and W. J. Ricker. A feature of averaging. *Integral Equations Operator Theory*, 76(3):447–449, 2013.

[CR13b]   G. P. Curbera and W. J. Ricker. Spectrum of the Cesàro operator in ℓ^p. *Arch. Math. (Basel)*, 100(3):267–271, 2013.

[CS99]   J. A. Cima and A. G. Siskakis. Cauchy transforms and Cesàro averaging operators. *Acta Sci. Math. (Szeged)*, 65(3-4):505–513, 1999.

[CSvD98]   P. Coolen-Schrijner and E. A. van Doorn. Analysis of random walks using orthogonal polynomials. In *Proceedings of the VIIIth Symposium on Orthogonal Polynomials and Their Applications (Seville, 1997)*, volume 99, pages 387–399, 1998.

[Fel99]   N. S. Feldman. Essentially subnormal operators. *Proc. Amer. Math. Soc.*, 127(4):1171–1181, 1999.

[FHN67]   T. Furuta, M. Horie, and R. Nakamoto. A remark on a class of operators. *Proc. Japan Acad.*, 43:607–609, 1967.

[FL00]   F. Filbir and R. Lasser. Reiter's condition P_2 and the Plancherel measure for hypergroups. *Illinois J. Math.*, 44(1):20–32, 2000.

[FLS04]   F. Filbir, R. Lasser, and R. Szwarc. Reiter's condition P_1 and approximate identities for polynomial hypergroups. *Monatsh. Math.*, 143(3):189–203, 2004.

[FN71]   T. Furuta and R. Nakamoto. On the numerical range of an operator. *Proc. Japan Acad.*, 47:279–284, 1971.

[Fur67]   T. Furuta. On the class of paranormal operators. *Proc. Japan Acad.*, 43:594–598, 1967.

[Fur71]   T. Furuta. Certain convexoid operators. *Proc. Japan Acad.*, 47(suppl. I):888–893, 1971.

[Hal82]   P. R. Halmos. *A Hilbert space problem book*, volume 19 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1982. Encyclopedia of Mathematics and its Applications, 17.

[Han09]   F. Hansen. Non-commutative Hardy inequalities. *Bull. Lond. Math. Soc.*, 41(6):1009–1016, 2009.

[Har20]   G. H. Hardy. Note on a theorem of Hilbert. *Math. Z.*, 6(3-4):314–317, 1920.

[Har25]   G. H. Hardy. Notes on some points in the integral calculus. LX. *Messenger of Math.*, 54:150–156, 1925.

[Har28]   G. H. Hardy. Notes on some points in the integral calculus. LXVI. *Messenger of Math.*, 58:50–52, 1928.

[HLP88]   G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988. Reprint of the 1952 edition.

[ISY66]   V. Istrǎţescu, T. Saitô, and T. Yoshino. On a class of operators. *Tôhoku Math. J. (2)*, 18:410–413, 1966.

[Kaw44] T. Kawata. Notes on Fourier series. XII. On Fourier constants. *Proc. Imp. Acad. Tokyo*, 20:218–222, 1944.

[KMP06] A. Kufner, L. Maligranda, and L.-E. Persson. The prehistory of the Hardy inequality. *Amer. Math. Monthly*, 113(8):715–732, 2006.

[KT71] T. L. Kriete, III and D. Trutt. The Cesàro operator in l^2 is subnormal. *Amer. J. Math.*, 93:215–225, 1971.

[Lam76] A. Lambert. Subnormality and weighted shifts. *J. London Math. Soc. (2)*, 14(3):476–480, 1976.

[Lan26] E. Landau. A note on a theorem concerning series of positive terms. *J. London Math. Soc.*, 1:38–39, 1926.

[Las83] R. Lasser. Orthogonal polynomials and hypergroups. *Rend. Mat. (7)*, 3(2):185–209, 1983.

[Las94] R. Lasser. Orthogonal polynomials and hypergroups. II. The symmetric case. *Trans. Amer. Math. Soc.*, 341(2):749–770, 1994.

[Li96] Z. Li. Conjugate Jacobi series and conjugate functions. *J. Approx. Theory*, 86(2):179–196, 1996.

[Li97] Z. Li. On the Cesàro means of conjugate Jacobi series. *J. Approx. Theory*, 91(1):103–116, 1997.

[LO08] R. Lasser and J. Obermaier. A new characterization of ultraspherical polynomials. *Proc. Amer. Math. Soc.*, 136(7):2493–2498, 2008.

[LOR07] R. Lasser, J. Obermaier, and H. Rauhut. Generalized hypergroups and orthogonal polynomials. *J. Aust. Math. Soc.*, 82(3):369–393, 2007.

[LOW13] R. Lasser, J. Obermaier, and J. Wagner. On the spectrum of tridiagonal operators and the support of orthogonalization measures. *Arch. Math. (Basel)*, 100(3):289–299, 2013.

[Mad89] I. J. Maddox. Point spectra of Cesàro matrices. *Nederl. Akad. Wetensch. Indag. Math.*, 51(4):465–470, 1989.

[Miy04] A. Miyachi. Boundedness of the Cesàro operator in Hardy spaces. *J. Fourier Anal. Appl.*, 10(1):83–92, 2004.

[MP89] M. Martin and M. Putinar. *Lectures on hyponormal operators*, volume 39 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1989.

[MS65] B. Muckenhoupt and E. M. Stein. Classical expansions and their relation to conjugate harmonic functions. *Trans. Amer. Math. Soc.*, 118:17–92, 1965.

[Muc69] B. Muckenhoupt. Hermite conjugate expansions. *Trans. Amer. Math. Soc.*, 139:243–260, 1969.

[Muc70] B. Muckenhoupt. Conjugate functions for Laguerre expansions. *Trans. Amer. Math. Soc.*, 147:403–418, 1970.

[Pat78] S. M. Patel. Essentially normaloid and essentially convexoid operators. *Yokohama Math. J.*, 26(1):55–59, 1978.

[Put70] C. R. Putnam. Unbounded inverses of hyponormal operators. *Pacific J. Math.*, 35:755–762, 1970.

[Put72] C. R. Putnam. A similarity between hyponormal and normal spectra. *Illinois J. Math.*, 16:695–702, 1972.

[Put73] C. R. Putnam. Almost normal operators, their spectra and invariant subspaces. *Bull. Amer. Math. Soc.*, 79:615–624, 1973.

[Sis87] A. G. Siskakis. Composition semigroups and the Cesàro operator on H^p. *J. London Math. Soc. (2)*, 36(1):153–164, 1987.

[Sta62] J. G. Stampfli. Hyponormal operators. *Pacific J. Math.*, 12:1453–1458, 1962.

[Sta65] J. G. Stampfli. Hyponormal operators and spectral density. *Trans. Amer. Math. Soc.*, 117:469–476, 1965.

[Sta66] J. G. Stampfli. Which weighted shifts are subnormal? *Pacific J. Math.*, 17:367–379, 1966.

[Sta79] J. G. Stampfli. On selfadjoint derivation ranges. *Pacific J. Math.*, 82(1):257–277, 1979.

[Ste70] E. M. Stein. *Topics in harmonic analysis related to the Littlewood-Paley theory.* Annals of Mathematics Studies, No. 63. Princeton University Press, Princeton, N.J., 1970.

[Ste93] K. Stempak. Conjugate expansions for ultraspherical functions. *Tohoku Math. J. (2)*, 45(4):461–469, 1993.

[Sze75] G. Szegő. *Orthogonal polynomials.* American Mathematical Society, Providence, R.I., fourth edition, 1975. American Mathematical Society, Colloquium Publications, Vol. XXIII.

[Szw92a] R. Szwarc. Orthogonal polynomials and a discrete boundary value problem. I. *SIAM J. Math. Anal.*, 23(4):959–964, 1992.

[Szw92b] R. Szwarc. Orthogonal polynomials and a discrete boundary value problem. II. *SIAM J. Math. Anal.*, 23(4):965–969, 1992.

[vDS93] E. A. van Doorn and P. Schrijner. Random walk polynomials and random walk measures. In *Proceedings of the Seventh Spanish Symposium on Orthogonal Polynomials and Applications (VII SPOA) (Granada, 1991)*, volume 49, pages 289–296, 1993.

[Wag13] J. Wagner. On the Cesáro operator in weighted ℓ^2-sequence spaces and the generalized concept of normality. *Ann. Funct. Anal.*, 4(2):1–11, 2013.

[Wer07] D. Werner. *Funktionalanalysis*. Springer-Verlag, Berlin, 6. korrigierte Auflage, 2007.

[Wil94] L. R. Williams. The local spectra of pure quasinormal operators. *J. Math. Anal. Appl.*, 187(3):842–850, 1994.