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Kinetic limit for Wave Propagation in a Continuous, Weakly Random Medium

Self-averaging and Convergence to a Linear Boltzmann Equation

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Abstract

In this thesis, we investigate the dynamics of a scalar wave field in two or more space dimensions, traveling through a weakly disordered medium. The disorder is modeled by random spatial fluctuations of the wave speed, but frozen in time, and is of order $\sqrt{\varepsilon}$, $0 < \varepsilon \ll 1$. On the kinetic scale, with space and time of order ε^{-1}, we prove that the Wigner function almost surely converges to the solution of a linear Boltzmann equation as $\varepsilon \to 0$. Essentially, the only requirements for the initial data are natural ones — bounded energy and tightness on the kinetic scale.

The proof of the result consists of two steps. First, the limit of the *disorder-averaged* Wigner function is identified by a mathematically rigorous expansion into Feynman graphs. All diagrams except for the "ladder" diagrams are shown to vanish like ε^C, $C > 0$. Using a more involved graph expansion, the l-th moments of the random *fluctuations* of the Wigner function are then shown to scale like ε^{Cl}, from which we conclude almost sure convergence.

Kurzzusammenfassung

In dieser Arbeit untersuchen wir die Dynamik einer skalaren Welle in mindestens zwei Raumdimensionen, die durch ein leicht inhomogenes Medium propagiert, dessen Wellengeschwindigkeit zufällige, räumliche, zeitlich konstante Schwankungen der Größenordnung $\sqrt{\varepsilon}$, $0 < \varepsilon \ll 1$ aufweist. Wir zeigen, dass die Wignerfunktion auf der kinetischen Skala, also auf Raum- und Zeitskalen der Ordnung ε^{-1}, im Limes $\varepsilon \to 0$ fast sicher gegen die Lösung einer linearen Boltzmann-Gleichung konvergiert. Im Wesentlichen sind die einzigen Voraussetzungen an die Anfangswerte der Wellengleichungen die natürlichen Annahmen von beschränkter Energie und Straffheit auf der kinetischen Skala.

Der Beweis zerfällt in zwei Schritte. Zunächst wird der Limes der über alle Realisierungen des Mediums *gemittelten* Wignerfunktion mit Hilfe einer Entwicklung in Feynman-Graphen bewiesen. Dabei verschwinden alle Graphen, die keine „Leiter"-Form aufweisen wie ε^C, $C > 0$. Anschließend zeigen wir mithilfe komplizierterer Feynman-Graphen, dass die l-ten Momente der zufälligen *Schwankungen* der Wignerfunktion wie ε^{Cl} skalieren, und schließen daraus auf fast sichere Konvergenz.

Contents

1. Introduction

1.1. Physics background

Any realistic model for wave motion should allow for the possibility of a spatially heterogeneous medium. Among the classical examples of waves interacting with inhomogeneous media are cases in which the underlying environment features only one or a few, macroscopically observable transitions between different material coefficients, such as the refraction of light (an electromagnetic wave) at the interface of water and air, or the shoaling of water waves entering from deep into shallower water.

However, a heterogeneous medium may also appear homogenous on a large observation scale, while inhomogeneities determine the structure on a much smaller scale. This microscopic structure of the medium is typically unknown or hard to identify, and all the information available may be the average distance of neighboring inhomogeneities, the typical fluctuations of the material coefficients and the like. In this situation it is often appropriate to assume the medium to be random, i.e. to model the coefficients of the medium as random fields.

Examples from different branches of physics include seismic waves scattering off heterogeneities in the earth's crust [43, 44], the use of ultrasound to detect the position of targets in a medium with sound speed fluctuations, [5], or laser light entering a suspension of submicrometer polystyrene balls in water, [42].

The physical model this thesis will focus on is in several ways a special case of the above-mentioned examples. First, in what is a slight simplification compared to the vector-valued elastic or electromagnetic waves mentioned above, we shall only consider a scalar wave equation

$$\frac{\partial^2}{\partial t^2} u(x,t) = c(x)^2 \Delta u(x,t), \tag{1.1}$$

so that instead of several possibly random parameters (for example magnetic and electric susceptibility tensors), the only source of randomness will be the wave speed, i.e. the scalar random field $c(x)$, $x \in \mathbb{R}^d$.

As is already apparent from the notation, the random field $c(x)$ is assumed to depend only on position x, but not on the time variable t, (or, realistically, the medium configuration varies on a much larger time scale than the time it takes the waves to pass through the medium. For seismic waves, these two time scales are millennia and seconds, respectively; for the polystyrene suspension, Brownian motion only requires milliseconds to alter the configuration of scatterers by a full wavelength, but light crosses the sample ($\sim$ 1cm) ten orders of magnitude faster, [42]). A random wave speed $c(x,t)$ quickly decorrelating in time would substantially simplify the analysis of the long-time behavior of (1.1). In that case long-term dependencies in the scattering history of the wave are a priori impossible, and a markovian limit is rather easily obtained. With disorder $c(x)$ constant in time, however, one has to exclude such correlations by carefully tracking the spread of the wave packet.

The random fluctuations of c will be assumed to be small compared to the average wave speed. Normalizing the average wave speed to 1, one can thus write

$$c(x) = 1 + \sqrt{\varepsilon}\xi(x), \tag{1.2}$$

with a random disorder term ξ scaled by a prefactor $0 < \sqrt{\varepsilon} \ll 1$.

The initial data $\left(u(\cdot,0), \frac{\partial}{\partial t}u(\cdot,0)\right)$ for (1.1) is chosen to be deterministic, in particular independent of the randomness of the medium, and to have bounded energy $E(0) = \int E(x,0)\mathrm{d}x$, with the energy density

$$E(x,t) = \frac{1}{2}\left|\nabla u(x,t)\right|^2 + \frac{1}{2c(x)^2}\left|\frac{\partial}{\partial t}u(x,t)\right|^2. \tag{1.3}$$

Finally, the typical wavelength of the initial conditions $\left(u(\cdot,0), \frac{\partial}{\partial t}u(\cdot,0)\right)$ is taken to be of the same order of magnitude as the correlation length of the wave speed fluctuations ξ. This should prove to be the regime with the most interesting interaction between medium and wave (for large wave lengths, one expects little scattering to occur due to the λ^{-d-1} wavelength dependence of the Rayleigh scattering cross section; for wavelengths much shorter than the scale on which the medium varies, one essentially is in the setting of a semiclassical limit, and the wave nature of the scattering process is lost).

For a wave scattering off a single, compactly supported inhomogeneity of strength $\sqrt{\varepsilon}$, the scattering amplitude scales like $\sqrt{\varepsilon}$ as $\varepsilon \to 0$, so the portion of the wave energy that is scattered (in the quantum case, this would rather be the probability of a scattering occuring) is of order $\sqrt{\varepsilon}^2 = \varepsilon$, [17]. As ξ has a correlation length of order 1, a wave packet encounters $\mathcal{O}(t)$ such individual inhomogeneities while traveling for a duration of microscopic time t, and one consequently expects a fraction εt of the energy to be scattered. The shortest time scale on which the weak disorder is expected to have a noticeable effect are thus microscopic times of order $t \sim \varepsilon^{-1}$, motivating the use of a macroscopic time coordinate $T = \varepsilon t$. To keep the average speed of wave propagation unscaled, we also set the macroscopic space variable to be $X = \varepsilon x$. This constitutes the *kinetic scaling*. The energy density $E^\varepsilon(X,T) = E(X/\varepsilon, T/\varepsilon)$ of the wave does not obey an autonomous evolution equation, but the *Wigner transform*, to be defined in Section 1.2, equation (1.9),

$$W^\varepsilon(X,K,T) = \varepsilon^{-d}W(X/\varepsilon, K, T/\varepsilon) \tag{1.4}$$

typically does, [2]. Here, K is the wave-number, which remains unscaled. For each time, W^ε is a function on phase space that can be thought of as a “wavenumber-resolved” energy density, [23], from which the spatial energy density can be retrieved by

$$E^\varepsilon(X,T) = \int_{\mathbb{R}^d} \mathrm{d}K W^\varepsilon(X,K,T). \tag{1.5}$$

In applications, direct simulation of the complicated random dynamics of W^ε is avoided by approximating W^ε by a solution $\overline{W}$ of the linear Boltzmann equation

$$\begin{aligned} \frac{\partial}{\partial T}\overline{W}(X,K,T)(x,k) &= \mp \frac{K}{|K|}\cdot \nabla_X \overline{W}(X,K,T) \\ &\quad + |2\pi K|^2 \int_{\mathbb{R}^d} \mathrm{d}K' \delta(|K|-|K'|)\widehat{g}_2(K-K')\left(\overline{W}(X,K',T)-\overline{W}(X,K,T)\right). \end{aligned} \tag{1.6}$$

Here, $\widehat{g}_2$ is the power spectrum of ξ, which is the Fourier transform of the correlation function $g_2(x) = \mathbb{E}[\xi(0)\xi(x)]$. The sign $\mp$ arises from the fact that the unperturbed wave motion exhibits both wave modes that travel according to a dispersion relation $\omega(k) = +2\pi|k|$, and modes with $\omega(k) = -2\pi|k|$.

This approximation is widely employed (the review by Ryzhik, Papanicolaou, Keller, [35] provides details for several physical settings); numerical simulations of the energy transport via wave equation and the linear Boltzmann equation coincide remarkably well, [3], and the use of equation (1.6) for imaging in random media yields satisfactory experimental results, [4].

Mathematically, the use of this approximation has so far not been fully justfied. In particular, as W^ε is a random quantity, while $\overline{W}$ is deterministic, the kind of convergence needs to be clarified — does $\overline{W}$ only approximate the disorder-averaged Wigner function $\mathbb{E}W^\varepsilon$, or does the convergence $W^\varepsilon \to \overline{W}$ hold in probability or almost surely, i.e. regardless of the microscopic details of the medium?

A convergence in probability result for the kinetic limit of waves in a weakly random medium was obtained by Bal, Komorowski and Ryzhik, [2], but only for initial states with wavelengths much shorter than the correlation length of the medium (i.e. parallel to the weak coupling limit and the rescaling of time and space variables, the authors take a high frequency limit and rescale the momentum). The limit dynamics resembles (1.6), with the Boltzmann collision kernel replaced by the small angle approximation of the jump process, leading to a diffusion of the momentum on spheres of constant $|K|$. Their methods cannot be extended to our case of wavelengths on the same scale as the correlation length.

Without the high frequency limit, no rigorous mathematical statement about the validity of the kinetic limit for waves in a continuous, weakly random medium is available as of now.

To put this theory on a solid footing is the goal of this thesis.

1.2. Results of the thesis and related work

Generally, a linear Boltzmann equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t(x,k) = -\nabla_k\omega(k)\cdot\nabla_x\mu_t(x,k) + \int_{\mathbb{R}^d}\mathrm{d}k'\sigma(k',k)\left[\mu_t(x,k') - \mu_t(x,k)\right] \tag{1.7}$$

consists of a transport term depending on $\omega(k)$ (the kinetic energy in the classical[1] case, or the dispersion relation for a Schrödinger or wave equation), and of a collision operator with collision kernel σ which is a non-negative (generalized) function. μ_t is typically a measure, and only constitutes a weak solution of (1.7).

For a one-particle random Schrödinger equation $i\frac{\mathrm{d}}{\mathrm{d}t}\psi = H\psi$ with Hamiltonian $H = -\Delta + \lambda V$ on $\mathbb{R}^d$, $d \geq 2$, with V a random potential, one can again ask for the time scale on which the expected number of interactions with the potential V is of order 1; this leads to the weak coupling, kinetic limit with disorder strength $\lambda = \sqrt{\varepsilon}$, and space/time scaling $(x,t) = (X/\varepsilon, T/\varepsilon)$, $(\varepsilon \to 0)$. In this limit, the appropriately rescaled and disorder-averaged Wigner transform of ψ converges to the solution of a linear Boltzmann equation; due to the weak coupling, the Boltzmann collision kernel is given by only the first Born approximation to the quantum scattering process. This was proven for short macroscopic times T and a Gaussian potential V by Spohn, [38], and extended to all times T and a larger class of random potentials with a suitably cut-off Duhamel expansion of the perturbed Schrödinger dynamics and a subsequent graph expansion by Erdős and Yau, [16].

The graph expansion technique devised in [16] was subsequently used by Chen, [9], to obtain the same result for the discrete analog, the Anderson model given by the Schrödinger operator $-\Delta_{\mathrm{nn}} + \lambda V$ on $\mathbb{Z}^3$, with the nearest-neighbor Laplacian and a random potential i.i.d. on every lattice site. Lukkarinen and Spohn, [32], showed the corresponding result for the propagation of atom displacements in a three-dimensional harmonic crystal in the presence of isotope disorder, modeled by a discrete wave equation with slightly fluctuating coefficients.

In our model, the disorder term in the random wave speed $c(x) = 1 + \sqrt{\varepsilon}\xi(x)$ is given by a stationary random field with sufficient smoothness and fast enough spatial decorrelation. For example, ξ might be obtained from a Gaussian field, or consist of local "bumps" distributed with a Poisson point process. By setting $\psi_\pm = \nabla u \pm i\frac{\partial}{\partial t}u/c$, the wave

[1] The linear Boltzmann equation was first rigorously obtained as the scaling limit of a classical particle system, namely the classical Lorentz gas in the Boltzmann-Grad (low density) limit. In this model, a single test particle with position and momentum $(q^\varepsilon(t), p^\varepsilon(t)) \in \mathbb{R}^{2d}$, $d \geq 2$ moves through a random (say Poisson), density ε^{-d+1} distribution of fixed scatterers with diameter $\varepsilon > 0$. As ε goes to zero, the number of interactions the test particle undergoes with the scatterers in a unit time interval remains of order 1, and $(q^\varepsilon(t), p^\varepsilon(t))$ converges to a stochastic process with a linear Boltzmann equation as forward equation, as shown by Gallavotti, [20] for individual times t, and by Spohn, [39, 40] in distribution on path space.

equation is then transformed into a first order system

$$
\begin{aligned}
i\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} &= H^\varepsilon \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} = (H_0+\sqrt{\varepsilon}V)\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}\\
&= \begin{pmatrix}\sqrt{-\Delta} & 0\\ 0 & -\sqrt{-\Delta}\end{pmatrix}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} + \frac{\sqrt{\varepsilon}}{2}\begin{pmatrix}\sqrt{-\Delta}\xi+\xi\sqrt{-\Delta} & -\sqrt{-\Delta}\xi+\xi\sqrt{-\Delta}\\ \sqrt{-\Delta}\xi-\xi\sqrt{-\Delta} & -\sqrt{-\Delta}\xi-\xi\sqrt{-\Delta}\end{pmatrix}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix},
\end{aligned}
\tag{1.8}
$$

which defines a unitary time evolution for $\psi \in \mathcal{H} = L^2(\mathbb{R}^d;\mathbb{C}^2)$. The Wigner transform of both components ψ_σ, $\sigma \in \{\pm\}$, of a state $\psi \in \mathcal{H}$ is given by

$$
W^\varepsilon[\psi_\sigma](x,k) = \varepsilon^{-d} W[\psi]\left(\frac{x}{\varepsilon},k\right) = \varepsilon^{-d}\int_{\mathbb{R}^d}\mathrm{d}y\overline{\psi_\sigma\left(\frac{x}{\varepsilon}+\frac{y}{2}\right)}\psi_\sigma\left(\frac{x}{\varepsilon}-\frac{y}{2}\right)e^{2\pi i y\cdot k}. \tag{1.9}
$$

In space dimension $d \geq 2$, for any sequence of initial states $(\psi_0^\varepsilon)_{\varepsilon>0}$ with bounded energy $\|\psi_0^\varepsilon\|_{\mathcal{H}}^2 \leq C$ and fulfilling certain tightness assumptions, it is shown in Theorem 3.1 that

$$
\lim_{\varepsilon\to 0}\mathbb{E}\left[\int_{\mathbb{R}^{2d}}\mathrm{d}x\mathrm{d}k W^\varepsilon\left[\left(e^{-iH^\varepsilon T/\varepsilon}\psi_0^\varepsilon\right)_\sigma\right](x,k)a(x,k)\right] = \int_{\mathbb{R}^{2d}}\mu_{\sigma,T}(\mathrm{d}x,\mathrm{d}k)a(x,k) \tag{1.10}
$$

for $\sigma \in \{\pm\}$, a suitable class of test functions a and all $T \geq 0$. Here, $\mu_{\pm,T}$ is a time-dependent Borel measure on phase space $\mathbb{R}^{2d}$ solving the linear Boltzmann equation (1.6). The detailed statement of Theorem 3.1 will actually allow for *multi-time* measurements.

The very similar results for the continuous ($\mathbb{R}^d$) and discrete ($\mathbb{Z}^d$) case of the Schrödinger or wave equation are what one would expect from a physical point of view; after all, the behavior on long space and time scales is investigated and the microscopic details of the underlying space should have little impact, [17]. The respective proofs, however, differ in several details. In the lattice setting, the more complicated geometry of the level sets of the dispersion relation make resolvent estimates and oscillatory integral arguments much more difficult, [9, 31]. For the continuous wave equation, the unbounded momentum space ($\mathbb{R}^d$ as opposed to $[0,1)^d$ for the lattice) makes the perturbation V an unbounded operator, and more caution is required when performing the Duhamel expansion.

While the last few examples, including the topic of this thesis, all concern the weak coupling limit, this is by no means the only situation in which a random dynamics on the microscopic scale converges to a linear Boltzmann equation on macroscopic phase space. In the low density limit, the potential of the Hamiltonian $H = -\Delta + \sum_n V_{y_n}$ is given by a random (density $\varepsilon \ll 1$) configuration of scatterers with shape V_0 and center y_n. Again, on space and time scales $(x,t) = (X/\varepsilon, T/\varepsilon)$ and for spatial dimension $d \geq 3$, the disorder-averaged Wigner transform of the wave function converges to the solution of a linear Boltzmann equation, [13, 18], but now with the full quantum scattering cross section appearing in the Boltzmann collision kernel.

All results for the Schrödinger and wave equation mentioned so far only involve the *disorder-averaged* Wigner transform. In most applications, one is actually interested in the transport properties of a concrete, single realization of the medium, and would thus desire to control the typical deviations from the average, ideally improving the convergence of the expectation to an *almost sure convergence*, a phenomenon referred to as *self-averaging*.

For the classical Lorentz gas in the low density limit, convergence in probability, [38], and almost surely (Boldrighini, Bunimovich, Sinai, [6]) was shown for the evolution of absolutely continuous particle densities, in other words, under the assumption that the initial coordinates of the particle are randomized independently of the configuration of the medium. For the Schrödinger or wave equation, the initial state always carries some randomness by Heisenberg uncertainty. In fact, for the weak coupling, kinetic limit of the Anderson model in $\mathbb{Z}^3$, convergence of the Wigner function has been proven in probability, [8], and almost surely, [7], under assumptions only slightly stronger than those for the disorder-averaged result, [9].

In the case of the random wave equation at hand, a similar statement holds. Under somewhat more restrictive conditions on ξ and the initial states ψ_0^ε, we establish in Theorem 3.3 that for *almost all realizations of the medium* ξ,

$$\lim_{\varepsilon\to 0}\sup_{\tau\in[0,T]}\left|\int_{\mathbb{R}^{2d}}\mathrm{d}x\mathrm{d}k W^\varepsilon\left[\left(e^{-iH^\varepsilon\tau/\varepsilon}\psi_0^\varepsilon\right)_\pm\right](x,k)a(x,k)-\int_{\mathbb{R}^{2d}}\mu_{\pm,\tau}(\mathrm{d}x,\mathrm{d}k)a(x,k)\right|=0 \tag{1.11}$$

for all times $T\geq 0$ and test functions a. Thus we have shown that the convergence of the Wigner transform to a linear Boltzmann equation occurs almost surely, and uniformly on compact time intervals.

As mentioned earlier, the kinetic scale is the *shortest* time scale with a nontrivial dynamics. For the discrete or continuous Anderson model $-\Delta+\lambda V$ in $d\geq 3$, $\lambda\ll 1$, Erdős, Salmhofer and Yau, [14, 15], showed that on the larger, *diffusive* scale $x=\lambda^{-2-\kappa/2}\tilde{X}$, $t=\lambda^{-2-\kappa}\tilde{T}$, with a small $\kappa=\kappa(d)>0$, the disorder-averaged Wigner function converges to the solution of a heat equation in the $\tilde{X}$ variable. While the diffusion coefficient of this heat equation can be calculated formally by taking the diffusive limit of the linear Boltzmann equation, it needs to be pointed out that taking the rigorous diffusive limit of the Anderson model is *much more demanding* than the formal two-stage argument of taking the kinetic limit first, and the diffusive limit of the linear Boltzmann equation later. A comparable diffusive limit should be expected for the weakly random wave equation. We leave this as an open problem.

Another possible direction to go beyond the results of this thesis is to stay on the kinetic scale, but to prove a central limit theorem rather than our almost sure, law of large numbers type of result. One would have to identify the exact scale of the random fluctuations in (1.11), and rescale them to obtain a nontrivial limit. For a weakly random Schrödinger equation (however, under the assumption of rapid decorrelation of the potential in time) such a limit exists, and is in fact a solution of the same linear Boltzmann equation, but with random initial data [28]. At this point, it is not clear whether or how these results carry over to the case of time-independent random potentials or wave speeds.

1.3. Mathematical methods and outline of the thesis

To be able to state our results, we first specify the requirements on the random field ξ in Section 2.1 and give two examples for admissible choices of ξ. The first task is then to verify that (1.8) actually defines a unitary time evolution, and how this evolution relates to an evolution with disorder ξ_R limited to a ball of radius R in $\mathbb{R}^d$. This is accomplished by fairly standard PDE arguments in Section 2.2.1.

Next, we compare the dynamics generated by H^ε to the free H_0 dynamics with a Duhamel series, which, however, cannot be fully expanded because of the combinatorial factors incurred from the moments of ξ (this is what necessitated the restriction to short kinetic times in [38]). Instead, as in [16, 32], we only expand up to $\overline{N} = \mathcal{O}\left(|\log\varepsilon|/|\log|\log\varepsilon||\right)$ scattering events. This cut-off is $\overline{N}$ is small enough to ensure that combinatorical terms roughly of size $\overline{N}^{\overline{N}}$ can still be bounded by small positive powers of ε; yet, $\overline{N} \to \infty$ as $\varepsilon \to 0$, so we still have a chance to arrive at the full Dyson series for the solution of the linear Boltzmann equation.

After reaching the threshold $\overline{N}$, one needs to find a "soft" way to stop the expansion. For the continuous random wave equation, this problem is exacerbated by the fact that the presence of arbitrarily high powers of V from (1.8) in the expansion would require ξ to be C^∞, an assumption we would like to avoid. Instead V is split up into "well-behaved" and "uncontrolled" momentum changes, and after encountering an uncontrolled momentum change, the expansion has to "fade out" in only finitely many steps (as opposed to further $\mathcal{O}(|\log\varepsilon|/|\log|\log\varepsilon|)$ steps in [32]). A modified Duhamel expansion that accommodates for these issues is derived in Section 2.2.2, its precise paramters are fixed in Section 4.5.

The linear Boltzmann equation and the Wigner transform are introduced in a standard fashion, together with appropriate spaces of test functions. As the dispersion relation for the wave equation, $\omega(k) \sim |k|$ is not differentiable at the origin (an acoustic singularity), the linear Boltzmann dynamics is not well-defined for Wigner limit measures with mass on $\{k = 0\}$. This problem was avoided for the discrete case, [32], by the somewhat unphysical condition $|\omega(0)| > 0$. Here, however, we gain a better resolution near the acoustic singularity by enlarging the space of test functions, generalizing a construction by Harris, Lukkarinen, Teufel and Theil, [23].

We can then state our main theorems in Section 3, Theorem 3.1 about the convergence of the averaged Wigner transform, Theorem 3.2 about the vanishing variance of the Wigner function, and Theorem 3.3 concerning almost sure convergence.

Theorem 3.1 is shown in Section 4. After performing our version of the Duhamel expansion, we can use the methods laid out in [16] and [32], that is, a cumulant expansion of the expectation of each term in the Duhamel series, representing the unitary free propagators in resolvent form, and visualizing each cumulant as a Feynman graph.

As in [32], we verify that the contributions (amplitudes) of higher order partitions, crossing pairings and nested pairings as well as the graphs with "too many" scattering events converge to zero sufficiently fast as $\varepsilon \to 0$ to beat the growth of the number of graphs as $\overline{N} \to \infty$. A main ingredient are the bounds for resolvent integrals and oscillatory integrals presented in Appendix B and C. Moreover, as explained above, one has to show that the uncontrolled momentum jumps vanish in the $\varepsilon \to 0$ limit, as do the amplitudes of "non-markovian" graphs, a new feature arising from the inclusion of multiple measurements. Finally, the contributions of the simple, markovian "ladder graphs" are shown to converge to the Dyson series for the linear Boltzmann equation.

As for Theorem 3.2, we see in Section 5.1 that all contributions to the variance are given by Feynman graphs that connect the scattering processes (the "one-particle lines") of two particles. The amplitudes of such graphs vanish by the arguments developed in [8] and refined in [7].

For the proof of Theorem 3.3, assume it is already known that almost sure convergence holds along a subsequence of $\varepsilon_n \searrow 0$. To interpolate between the ε_n, we observe that, at least morally,

$$\frac{\partial}{\partial \varepsilon} \int_{\mathbb{R}^{2d}} \mathrm{d}x \mathrm{d}k W^{\varepsilon} \left[\left(e^{-iH^{\varepsilon}\tau/\varepsilon} \psi_0^{\varepsilon} \right)_{\pm} \right] (x,k) a(x,k) \sim \varepsilon^{-\beta} \tag{1.12}$$

for $\beta > 0$ possibly large, but finite. Thus, filling in the gaps between the ε_n produces little error if $\varepsilon_n \sim n^{-\alpha}$ for a fixed, tiny $\alpha > 0$. To still be able to prove almost sure convergence by a Borel-Cantelli argument along the sequence (ε_n), one thus needs to control very high ($\sim 1/\alpha$) moments of

$$\begin{split} &\int_{\mathbb{R}^{2d}} \mathrm{d}x \mathrm{d}k W^{\varepsilon} \left[\left(e^{-iH^{\varepsilon}\tau/\varepsilon} \psi_0^{\varepsilon} \right)_{\pm} \right] (x,k) a(x,k) \\ &\qquad - \mathbb{E} \int_{\mathbb{R}^{2d}} \mathrm{d}x \mathrm{d}k W^{\varepsilon} \left[\left(e^{-iH^{\varepsilon}\tau/\varepsilon} \psi_0^{\varepsilon} \right)_{\pm} \right] (x,k) a(x,k). \end{split} \tag{1.13}$$

In analogy to the variance case, this is achieved in Section 6.1 by analyzing Feynman graphs that may span multiple one-particle lines. Those Feynman graphs are then systematically reduced to "stars", structures in which a Feynman graph connects a center one-particle line to a number of periphery one-particle lines. The hardest case to estimate is a "star with only one ray", i.e. two connected one-particle lines. This brings us back to Theorem 3.2.

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2. Preliminaries

This chapter aims to introduce the main protagonists of the thesis. In Section 2.1, we state the exact requirements we have for the disorder term ξ in our random wave speed $c(x) = 1 + \sqrt{\varepsilon}\xi(x)$, show a few basic properties, and, to make sure we do not prove a theorem about the empty set, give two examples. The whole thesis heavily relies on unitarity, Duhamel expansion and support propagation properties of the perturbed wave dynamics; we establish all those in Section 2.2.

To be actually able to recognize the limit dynamics at the end of our efforts, we devote Section 2.3 to the linear Boltzmann equation and the corresponding semigroups, and Section 2.4 to Wigner transforms and their limit measures.

Two conventions will be applied throughout — the definition of the Fourier transform as the unitary continuation of

$$\widehat{f}(k) = (\mathcal{F}f)(k) = \int_{\mathbb{R}^d} \mathrm{d}x f(x) e^{-2\pi i k \cdot x}, \tag{2.1}$$

from Schwartz functions $f \in \mathcal{S}(\mathbb{R}^d)$ to $f \in L^2\left(\mathbb{R}^d\right)$, and the shorthand

$$\langle x \rangle = \left(1 + |x|^2\right)^{1/2} \tag{2.2}$$

for $x \in \mathbb{R}^d$.

2.1. The random medium

2.1.1. Basic properties

Let $(\Omega, \mathfrak{S}, \mathbb{P})$ be a probability space and

$$\begin{aligned} \xi : \Omega \times \mathbb{R}^d &\to \mathbb{R} \\ (\omega, x) &\mapsto \xi_\omega(x) \end{aligned} \tag{2.3}$$

be a map such that $\omega \mapsto \xi_\omega(x)$ is measurable for all $x \in \mathbb{R}^d$. Thus, ξ is a random field, [22]. Throughout this thesis, we will denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$, i.e. the average over all realizations of the medium.

We require ξ_ω to be continuous on $\mathbb{R}^d$ for $\mathbb{P}$-almost all $\omega \in \Omega$. Therefore, one can particularly choose Ω to be the space of continuous functions on $\mathbb{R}^d$, and $\mathfrak{S}$ to be the

Borel sets on Ω with respect to the topology of locally uniform convergence. We will only once make use of this explicit choice of Ω — when we construct a counterexample to Theorem 3.3 in Section 6.3.

Furthermore, we assume ξ to be stationary,

$$(\xi(x_1+y),...,\xi(x_n+y)) \sim (\xi(x_1),...,\xi(x_n)) \tag{2.4}$$

for all $n \in \mathbb{N}$, $x_1, ...x_n, y \in \mathbb{R}^d$, to be bounded from below,

$$\inf_{x\in\mathbb{R}^d} \xi(x) \geq -C \tag{2.5}$$

with a deterministic constant $C < \infty$ and to grow at most linearly at infinity, i.e. there is an almost surely finite random variable M_ω such that

$$\xi_\omega(x) \leq M_\omega(1+|x|). \tag{2.6}$$

To allow for the graph expansion, we furthermore need that all moments of ξ are finite,

$$\mathbb{E}\left[|\xi(x)|^q\right] < \infty \tag{2.7}$$

for all $q \in [0,\infty)$, and that the first m derivatives of ξ are almost surely continuous with

$$\max_{|\alpha|\leq m} \mathbb{E}\left[|\partial_x^\alpha \xi(x)|^q\right] < \infty \tag{2.8}$$

for all $q \in [0,\infty)$ as well. Finally, ξ has expectation zero,

$$\mathbb{E}\left[\xi(x)\right] = 0. \tag{2.9}$$

A key object in our analysis are the *cumulants* of ξ, with the n-th cumulant defined as a function $\zeta_n : \mathbb{R}^{dn} \to \mathbb{R}$ such that for any finite index set I

$$\mathbb{E}\left[\prod_{j\in I} \xi(x_j)\right] = \sum_{S\in\pi(I)} \prod_{A\in S} \zeta_{|A|}\left(x_l : l \in A\right), \tag{2.10}$$

with the sum running over all partitions S of the set I and $A \in S$ denoting an individual cluster in a given partition S, compare [37], Chapter II, §12, equation (46). The ζ_n are easily shown to exist by mathematical induction based on (2.7), and, due to stationarity, (2.4),

$$\zeta_n(x_1,...,x_n) = g_n(x_2-x_1,...,x_n-x_1), \tag{2.11}$$

with $g_n : \mathbb{R}^{d(n-1)} \to \mathbb{R}$. As $\mathbb{E}\left[\xi(x)\right] = 0$, ζ_1 and g_1 vanish identically, and (2.10) simplifies to

$$\mathbb{E}\left[\prod_{j\in I} \xi(x_j)\right] = \sum_{S\in\pi^*(I)} \prod_{A\in S} \zeta_{|A|}\left(x_l : l \in A\right), \tag{2.12}$$

with $\pi^*(I)$ comprising only those partitions of I that do not contain clusters with only one element. Because of (2.8), the ζ_n are m times continuously differentiable *with respect to every argument*, so

$$\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \zeta_n(x_1, ..., x_n) \tag{2.13}$$

is a continuous function of the x_j as long $|\alpha_j| \leq m$ for all j. Thus, for $n \geq 2$ one can define the quantity

$$\begin{aligned} \|g_n\|_m &= \max_{|\alpha_l| \leq m} \int_{\mathbb{R}^{d(n-1)}} \mathrm{d}x_2 ... \mathrm{d}x_n \left|\partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n} \zeta_n(x_1, ..., x_n)\right| \\ &= \max_{|\alpha_l| \leq m} \int_{\mathbb{R}^{d(n-1)}} \mathrm{d}y_1 ... \mathrm{d}y_{n-1} \left|\partial_{y_1}^{\alpha_1} ... \partial_{y_{n-1}}^{\alpha_{n-1}} \left(\partial_{y_1} + ... + \partial_{y_{n-1}}\right)^{\alpha_m} g_n(y_1, ..., y_{n-1})\right| \end{aligned} \tag{2.14}$$

which we assume to be finite for all $n \geq 2$, with a bound

$$\|g_n\|_m \leq C n^{Cn} \tag{2.15}$$

for some $C < \infty$. By standard Fourier calculus, (2.14) ensures a decay

$$|\widehat{g_n}(p_1, ..., p_{n-1})| \leq C_d^{mn} \|g_n\|_m \langle p_1 + ... + p_{n-1} \rangle^{-m} \prod_{j=1}^{n-1} \langle p_j \rangle^{-m}, \tag{2.16}$$

with a constant C_d only depending on dimension d. For g_2, (2.16) reads

$$|\widehat{g_2}(p)| \leq C_d^{2m} \langle p \rangle^{-2m}. \tag{2.17}$$

However, we have to assume that even

$$\sup_p \max_{|\beta| \leq \overline{\beta}} \left| \frac{\partial^\beta}{\partial p^\beta} \widehat{g_2}(p) \right| \langle p \rangle^{2m} < \infty, \tag{2.18}$$

for some $\overline{\beta} \in \mathbb{N}$, which is the case whenever

$$x \mapsto \max_{|\alpha| \leq 2m} \max_{|\beta| \leq \overline{\beta}} \left| \frac{\partial^\alpha}{\partial x^\alpha} \left(x^\beta g_2(x) \right) \right| \tag{2.19}$$

is integrable.

Definition 2.1. A random field with all properties listed above is called a field of class $(m, \overline{\beta})$.

We now want to understand the Fourier transform of ξ. As ξ is typically not integrable, we use the cutoff version

$$\xi_R(x) = \chi\left(\frac{x}{R}\right) \xi(x) \tag{2.20}$$

with $R > 0$ and $\chi : \mathbb{R}^d \to [0,1]$ smooth, $\chi(x) = 1$ for $|x| < 1$ and $\chi(x) = 0$ for $|x| > 2$. By relaxing the cutoff, one obtains

Lemma 2.1. *Fix any finite index set I and consider a continuous function $f : \mathbb{R}^{|I|d} \to \mathbb{C}$ with*

$$|f(\theta)| \le C \prod_{j\in I} \langle\theta_j\rangle^q \,, \tag{2.21}$$

$q \ge 0$. Then, if $m > d + q$,

$$\begin{aligned} &\lim_{R\to\infty} \mathbb{E}\left[\int_{\mathbb{R}^{|I|d}} \mathrm{d}\theta f(\theta) \prod_{j\in I} \widehat{\xi_R}(\theta_j)\right] \\ &\quad = \sum_{S\in\pi^*(I)} \int_{\mathbb{R}^{|I|d}} \mathrm{d}\theta f(\theta) \prod_{A\in S} \delta\left(\sum_{j\in A}\theta_j\right) \widehat{g_{|A|}}\left(\theta_j : j\in A^\sharp\right). \end{aligned} \tag{2.22}$$

Here, $A^\sharp$ is the set A with an arbitrary element removed. The choice of this element is irrelevant because of translation invariance of ξ and the delta function in (2.22).

Proof. For all $R \in (0,\infty)$,

$$\begin{aligned} \mathbb{E}\left[\prod_{j\in I}\langle\theta_j\rangle^m \left|\widehat{\xi_R}(\theta_j)\right|\right] &\le \tilde{C}_d^{|I|m} \max_{0\le|\alpha|\le m} \mathbb{E}\left[\left(\int_{\mathbb{R}^d}\mathrm{d}y \left|\partial_y^\alpha \xi_R(y)\right|\right)^{|I|}\right] \\ &\le C_d^{|I|m} R^{|I|d} \max_{0\le|\alpha|\le m} \mathbb{E}\left[|\partial^\alpha\xi(0)|^{|I|}\right] \end{aligned} \tag{2.23}$$

so by Fubini's theorem the expectation can be pulled into the integral to obtain

$$\begin{aligned} &\mathbb{E}\left[\int_{\mathbb{R}^{|I|d}} \mathrm{d}\theta f(\theta) \prod_{j\in I}\widehat{\xi_R}(\theta_j)\right] \\ &= \int_{\mathbb{R}^{|I|d}} \mathrm{d}\theta f(\theta) \mathbb{E}\left[\int_{\mathbb{R}^{|I|d}} \mathrm{d}x \prod_{j\in I}\left(\xi_R(x_j) e^{-2\pi i x_j\cdot\theta_j}\right)\right] \\ &= \sum_{S\in\pi^*(I)} \int_{\mathbb{R}^{|I|d}} \mathrm{d}\theta f(\theta) \int_{\mathbb{R}^{|I|d}} \mathrm{d}x \prod_{l\in I}\left(\chi(x_l/R) e^{-2\pi i x_l\cdot\theta_l}\right) \prod_{A\in S} g_{|A|}\left(x_j - x_{j_A} : j\in A^\sharp\right) \end{aligned} \tag{2.24}$$

with j_A the missing element, $\{j_A\} = A\setminus A^\sharp$. By the estimate (2.16),

$$|\widehat{g_n}(p_2,\dots,p_n)|\,\delta\left(\sum_{l=1}^n p_l\right) \le C_d^{mn} \|g_n\|_m \frac{\delta(p_1+\dots+p_n)}{\langle p_1\rangle^m \langle p_2\rangle^m \dots \langle p_n\rangle^m}. \tag{2.25}$$

After applying (2.25) and $m > d+q$ to both the last lines of (2.22) and (2.24), a continuity argument shows that it suffices to consider $f \in C_c^\infty\left(\mathbb{R}^{|I|d}\right)$. But for such f,

$$\begin{aligned}
&\lim_{R\to\infty} \sum_{S\in\pi^*(I)} \int_{\mathbb{R}^{|I|d}} \mathrm{d}\theta f(\theta) \int_{\mathbb{R}^{|I|d}} \mathrm{d}x \prod_{l\in I} \left(\chi(x_l/R) e^{-2\pi i x_l\cdot\theta_l}\right) \prod_{A\in S} g_{|A|}\left(x_j - x_{j_A} : j \in A^\sharp\right) \\
&\quad = \lim_{R\to\infty} \sum_{S\in\pi^*(I)} \int_{\mathbb{R}^{|I|d}} \mathrm{d}x \widehat{f}(x) \prod_{l\in I} \chi(x_l/R) \prod_{A\in S} g_{|A|}\left(x_j - x_{j_A} : j \in A^\sharp\right) \\
&\quad = \lim_{R\to\infty} \sum_{S\in\pi^*(I)} \int_{\mathbb{R}^{|I|d}} \mathrm{d}x \widehat{f}(x) \prod_{A\in S} \left(g_{|A|}\left(x_j - x_{j_A} : j \in A^\sharp\right) \chi\left(x_{j_A}/R\right)\right).
\end{aligned} \tag{2.26}$$

If we denote for a single $A \in S$ with $|A| = n$ the θ_j, $j \in A$ by $p_1, ..., p_n$ and the x_j by $y_1, ..., y_n$ and without loss of generality assume $x_{j_A} = y_1$, we have

$$\int_{\mathbb{R}^{nd}} \mathrm{d}y g_n\left(y_2 - y_1, ..., y_n - y_1\right) \chi\left(y_1/R\right) e^{-2\pi i \sum_l y_l\cdot p_l} = R^d \widehat{g_n}\left(p_2, ..., p_n\right) \widehat{\chi}\left(R \sum_{l=1}^n p_l\right) \tag{2.27}$$

which converges to

$$\delta\left(\sum_{l=1}^n p_l\right) \widehat{g_n}\left(p_2, ..., p_n\right) \tag{2.28}$$

as a distribution as $R \to \infty$. This proves the lemma. □

2.1.2. Example: Poisson bumps

Denote by $y_n \in \mathbb{R}^d$, $n \in \mathbb{N}$ the points of a Poisson point process with intensity measure equal to the Lebesgue measure. At every Poisson point there sits an obstacle with shape $\phi : \mathbb{R}^d \to [0, \infty)$ such that

$$\sup_x \langle x\rangle^{d+1+\overline{\beta}} \max_{|\alpha|\le m} \left|\frac{\partial^\alpha}{\partial x^\alpha}\phi(x)\right| < \infty. \tag{2.29}$$

The random field ξ is then given as

$$\xi(x) = \sum_{n\in\mathbb{N}} \phi(x - y_n) - \int_{\mathbb{R}^d} \mathrm{d}z \phi(z), \tag{2.30}$$

where we subtracted the integral of ϕ to make the random field centered, $\mathbb{E}\xi(x) = 0$.

ξ is stationary by the translation invariance of the Poisson point process, bounded from below by $-\int\phi$, and grows only very slowly at infinity,

Lemma 2.2. *There is a random variable $Y \ge 0$ with $\mathbb{E}[e^Y] < \infty$ such that*

$$\sup_{|x|\le r} |\xi(x)| \le Y \log(r + 2). \tag{2.31}$$

for all $r \ge 0$.

Proof. First, assume ϕ to be supported on the cube $W = [-1/2, 1/2]^d$ and to be bounded by $0 \le \phi \le 1$. We furthermore drop the "centering" term and consider only

$$\xi^*(x) = \sum_{n\in\mathbb{N}} \phi(x - y_n). \tag{2.32}$$

Obviously, for any $x \in \mathbb{R}^d$

$$0 \le \xi^*(x) \le \sharp\{y_n \in x + W\} \sim Poi(1). \tag{2.33}$$

Fixing any $r \ge 0$, define the cube $V_r = \{x \in \mathbb{R}^d : |x|_\infty \le r\}$, which can be covered by $(2\lceil r\rceil + 1)^d$ disjoint (up to sets of measure zero) translates of W, centered around $z \in \mathbb{Z} \cap V_{\lceil r\rceil}$. Because each $x + W$, $x \in V_r$ is covered by at most 2^d of those translates,

$$\sup_{x\in V_r} \xi^*(x) \le \sup_{x\in V_r} \sharp\{y_n \in x + W\} \le 2^d \max_{z\in\mathbb{Z}\cap V_{\lceil r\rceil}} \sharp\{y_n \in z + W\} = 2^d Z_r, \tag{2.34}$$

with Z_r the maximum of $(2\lceil r\rceil + 1)^d$ independent Poisson $Poi(1)$ variables, so

$$\mathbb{P}(Z_r > l) \le \frac{(2\lceil r\rceil + 1)^d}{l!}, \tag{2.35}$$

and $\mathbb{P}(Z_r > \log(r + 2))$ decays faster than any negative power of r. Thus, there is a random variable $R \ge 0$ with all moments finite such that

$$\sup_{x\in V_r} \xi^*(x) \le 2^d \log(r + 2) \tag{2.36}$$

for all $r > R$. For $r \in [0, R]$, on the other hand,

$$\sup_{x\in V_r} \xi^*(x) \le \sup_{x\in V_R} \xi^*(x) \le 2^d \log(R + 2) \le Y_0 \log(r + 2) \tag{2.37}$$

with a random variable $Y_0 = 2^d \log(R + 2)/\log(2)$, so $\mathbb{E}[e^{\beta Y_0}] < \infty$ for all $\beta > 0$. Thus, for all $r \in \mathbb{R}^d$

$$\sup_{|x|\le r} \xi^*(x) \le \sup_{x\in V_r} \xi^*(x) \le Y_0 \log(r + 2). \tag{2.38}$$

If, instead, ϕ is supported on $z + W$, $z \in \mathbb{Z}^d$, and $\phi \le b_z$, then there is a Y_z distributed identically to, but not independent of, Y_0 such that

$$\sup_{|x|\le r} \xi^*(x) \le b_z Y_z \log(r + 2) \tag{2.39}$$

for all $r \ge 0$. If ϕ is chosen generally, with $0 \le \phi \le b_z$ on $z + W$, $z \in \mathbb{Z}^d$ and

$$\sum_{z\in\mathbb{Z}^d} b_z = \beta < \infty \tag{2.40}$$

(which is definitely the case for ϕ as in (2.29)), then

$$\sup_{|x|\le r} \xi^*(x) \le \sum_z b_z Y_z \log(r + 2) = Y \log(r + 2) \tag{2.41}$$

with

$$\mathbb{E}\left[e^Y\right] = \mathbb{E}\left[\prod_z e^{b_z Y_z}\right] \leq \mathbb{E}\left[\sum_z \frac{b_z}{\beta} e^{\beta Y_z}\right] = \sum_z \frac{b_z}{\beta} \mathbb{E}\left[e^{\beta Y_0}\right] < \infty. \tag{2.42}$$

□

Lemma 2.3. *As ξ is centered, $\zeta_1 \equiv 0$, while for $n \geq 2$*

$$\begin{aligned} \zeta_n(x_1, ..., x_n) &= \int_{\mathbb{R}^d} \mathrm{d}y \prod_{l=1}^n \phi(x_l - y), \\ g_n(x_1, ..., x_{n-1}) &= \int_{\mathbb{R}^d} \mathrm{d}y \phi(-y) \prod_{l=1}^{n-1} \phi(x_l - y), \\ \widehat{g}_n(p_1, ..., p_{n-1}) &= \widehat{\phi}(-p_1 ... - p_{n-1}) \prod_{l=1}^{n-1} \widehat{\phi}(p_l), \\ \widehat{\zeta}_n(p_1, ..., p_n) &= \delta(p_1 + ... + p_n) \prod_{l=1}^n \widehat{\phi}(p_l). \end{aligned} \tag{2.43}$$

Together with (2.29), this directly implies (2.8), (2.15) and (2.18) for the random field ξ at hand, so ξ is of class $(m, \overline{\beta})$. In particular, $\widehat{g}_2 = |\widehat{\phi}|^2$.

Proof. We only check the first line of (2.43), the others then follow by standard Fourier calculus. For simplicity, we consider the cumulants ζ_n^* of the non-centered field ξ^* from (2.32), and can follow [27] in our derivation of the moment-generating function of ζ_n^*. If for some finite index set I, $f_l : \mathbb{R}^d \to \mathbb{R}$, $l \in I$ are simple functions, i.e.

$$f_l(x) = \sum_{j=1}^k \alpha_{lj} \mathbb{1}_{A_j}(x) \tag{2.44}$$

with $k \in \mathbb{N}$, $A_j \subset \mathbb{R}^d$ disjoint Borel sets of finite measure and $\alpha_{lj} \in \mathbb{R}$, the random variables

$$X_l = \sum_{n \in \mathbb{N}} f_l(y_n) = \sum_{j=1}^k \alpha_{lj} N(A_j), \tag{2.45}$$

$N(A_j)$ being the number of Poisson points in A_j, are well defined. As the $N(A_j)$ are independent for disjoint sets,

$$\mathbb{E}\left[\prod_{l \in I} e^{iX_l}\right] = \prod_{j=1}^k \mathbb{E}\left[\exp\left(i \sum_{l \in I} \alpha_{lj} N(A_j)\right)\right] = \prod_{j=1}^k \exp\left(|A_j| \left(e^{i \sum_{l \in I} \alpha_{lj}} - 1\right)\right), \tag{2.46}$$

where we denoted the Lebesgue measure of A_j by $|A_j|$ and inserted the characteristic function of a Poisson-$|A_j|$-distributed random variable. Thus,

$$\mathbb{E}\left[\prod_{l \in I} e^{iX_l}\right] = \exp\left(\int_{\mathbb{R}^d} \mathrm{d}y \left(e^{i \sum_{l \in I} f_l(y)} - 1\right)\right), \tag{2.47}$$

which by pointwise approximation also holds for general bounded and integrable functions $f_l : \mathbb{R}^d \to \mathbb{R}$. By dominated convergence,

$$\begin{aligned}\mathbb{E}\left[\prod_{l\in I} X_l\right] &= \prod_{\tilde{l}\in I}\left(-i\frac{\mathrm{d}}{\mathrm{d}t_{\tilde{l}}}\right)\mathbb{E}\left[\prod_{l\in I} e^{it_l X_l}\right]\Bigg|_{t_j=0\ \forall j\in I} \\ &= \prod_{\tilde{l}\in I}\left(-i\frac{\mathrm{d}}{\mathrm{d}t_{\tilde{l}}}\right)\exp\left(\int_{\mathbb{R}^d}\mathrm{d}y\left(e^{i\sum_{l\in I} f_l(y)}-1\right)\right)\Bigg|_{t_j=0\ \forall j\in I}.\end{aligned} \tag{2.48}$$

Without loss of generality, we assume $I = \{1, ..., |I|\}$, $I_1 = I \setminus \{1\}$, first take the t_1 derivative and then use the product rule $|I| - 1$ times to see that

$$\begin{aligned}\mathbb{E}\left[\prod_{l\in I} X_l\right] &= \prod_{\tilde{l}\in I_1}\left(-i\frac{\mathrm{d}}{\mathrm{d}t_{\tilde{l}}}\right)\left\{\exp\left(\int_{\mathbb{R}^d}\mathrm{d}y\left(e^{i\sum_{l\in I_1} f_l(y)}-1\right)\right)\int_{\mathbb{R}^d}\mathrm{d}y e^{i\sum_{l\in I_1} f_l(y)} f_1(y)\right\} \\ &= \sum_{J\in\mathcal{P}(I_1)}\mathbb{E}\left[\prod_{l\in J} X_l\right]\int_{\mathbb{R}^d}\mathrm{d}y f_1(y)\prod_{\tilde{l}\in I_1\setminus J} f_{\tilde{l}}(y).\end{aligned} \tag{2.49}$$

Mathematical induction in the size of I then yields a sum over all partitions of I,

$$\mathbb{E}\left[\prod_{l\in I} X_l\right] = \sum_{S\in\pi(I)}\prod_{A\in S}\left(\int_{\mathbb{R}^d}\mathrm{d}y\prod_{l\in A} f_l(y)\right) \tag{2.50}$$

and one can read off the cumulants

$$\mathbf{Cum}\left(X_1, ..., X_n\right) = \int_{\mathbb{R}^d}\mathrm{d}y\prod_{l=1}^{n} f_l(y). \tag{2.51}$$

With $f_l(y) = \phi(x_l - y)$, $l = 1, ..., n$, the cumulants of ξ^* equal

$$\zeta_n^*\left(x_1, ..., x_n\right) = \int_{\mathbb{R}^d}\mathrm{d}y\prod_{l=1}^{n}\phi(x_l - y). \tag{2.52}$$

For $n \geq 2$, $\zeta_n = \zeta_n^*$. □

2.1.3. Example: Disorder term derived from a Gaussian field

While the random field ξ from Section 2.1.2 exhibits a simple explicit formula for the cumulants, it may take arbitrarily high values, which makes its physical interpretation as a wave speed questionable, although it is mathematically admissible for our purposes due to its slow growth behavior. Physically more realistic are random fields ξ that are bounded and owe their nice cumulant behavior to exponential mixing. For example, let $K : \mathbb{R}^d \to \mathbb{R}$ be given by its fourier transform

$$\widehat{K}(p) = \left(1 + |2\pi p|^2\right)^{-(d+1+2m)/2}, \tag{2.53}$$

from which one can directly read off that K is continuous and bounded together with its first $2m$ derivatives. Moreover, as its Fourier transform is a non-negative, integrable function, K is positive definite by the Bochner-Schwarz theorem, [34], and by Theorem 12.1.3 in [12], there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a centered Gaussian process $(W(x))_{x\in\mathbb{R}^d}$ with covariance

$$\mathbb{E}\left[W(x)W(y)\right] = K(x-y). \tag{2.54}$$

According to Theorem 1.4.2 in [1], W is almost surely m times continuously differentiable. For $|\alpha| \leq m$, $\partial_x^\alpha W(x)$ is again a stationary Gaussian process with covariance

$$\mathbb{E}\left[\partial_x^\alpha W(x)\partial_y^\alpha W(y)\right] = (-1)^{|\alpha|}\left(\partial^{2\alpha}K\right)(x-y). \tag{2.55}$$

In particular,

$$\max_{|\alpha|\leq m} \mathbb{E}\left[|\partial_x^\alpha W(x)|^q\right] < \infty \tag{2.56}$$

for all $q \in [0,\infty)$. By Theorem 1.14 in [41] one has

$$K(x) = C_{d,m}\int_{\mathbb{R}^{2m}} \mathrm{d}y e^{-\sqrt{|x|^2+|y|^2}} \tag{2.57}$$

and therefore

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}K(x)\right| \leq C_{d,m,\alpha}e^{-|x|/2} \tag{2.58}$$

for all multiindices α and all $x \in \mathbb{R}^d$ with $|x| \geq 1$.

Now let $A, B \subset \mathbb{R}^d$ with $\mathrm{dist}(A,B) = r \geq 1$. Let H_A and H_B be the Gaussian Hilbert spaces generated by the span of $\{W(x) : x \in A\}$ and $\{W(x) : x \in B\}$ respectively, and denote the sigma algebras

$$\mathcal{A} = \sigma\left(W(x) : x \in A\right) \text{ and } \mathcal{B} = \sigma\left(W(x) : x \in B\right). \tag{2.59}$$

By Theorem 10.11 of [25], for random variables $X \in L^2(\Omega, \mathcal{A}, \mathbb{P})$, $Y \in L^2(\Omega, \mathcal{A}, \mathbb{P})$ with $\mathbb{E}[X] = \mathbb{E}[Y] = 0$ and $\mathbb{E}[X^2] = \mathbb{E}[Y^2] = 1$, one has the bound

$$|\mathbb{E}[XY]| \leq \|P(H_A, H_B)\| = \sup_{\substack{U\in H_A, V\in H_B\\ \mathbb{E}[U^2]=\mathbb{E}[V^2]=1}} |\mathbb{E}[UV]|. \tag{2.60}$$

Here, $P(H_A, H_B)$ is the projector from H_A onto H_B. To estimate the sup on the right side of (2.60), it suffices to consider finite sums

$$\begin{aligned} U &= \sum_{j=1}^n \alpha_j W(x_j) \qquad (x_j \in A, \alpha_j \in \mathbb{R}),\\ V &= \sum_{l=1}^n \beta_l W(y_l) \qquad (y_l \in B, \beta_l \in \mathbb{R}). \end{aligned} \tag{2.61}$$

In this case,

$$\begin{aligned}\mathbb{E}[UV] &= \sum_{j,l=1}^{n} \alpha_j \beta_l K(x_j - y_l) = \sum_{j,l=1}^{n} \alpha_j \beta_l \int_{\mathbb{R}^d} \mathrm{d}p \widehat{K}(p) e^{2\pi i (x_j - y_l)\cdot p} \\ &= \int_{\mathbb{R}^d} \mathrm{d}p \widehat{K}(p) \overline{u(p)} v(p)\end{aligned} \tag{2.62}$$

with

$$\begin{aligned}u(p) &= \sum_{j=1}^{n} \alpha_j e^{-2\pi i x_j \cdot p}, \\ v(p) &= \sum_{l=1}^{n} \beta_l e^{-2\pi i y_l \cdot p},\end{aligned} \tag{2.63}$$

while

$$\int_{\mathbb{R}^d} \mathrm{d}p \widehat{K}(p) |u(p)|^2 = \int_{\mathbb{R}^d} \mathrm{d}p \widehat{K}(p) |v(p)|^2 = 1. \tag{2.64}$$

Now proceeding similar as in [10], one can apply a smooth cut-off at radius r to K and obtain a $K_r : \mathbb{R}^d \to \mathbb{R}$ such that $K_r(x) = 0$ for all $|x| \geq r$, that is close to K in the sense

$$\begin{aligned}&\left\| \langle \nabla \rangle^{d+2m+1} (K - K_r) \right\|_{L^1} \leq C_{d,m} e^{-r/2}, \\ &\left| \left(\widehat{K} - \widehat{K_r} \right)(p) \right| \leq C_{d,m} \widehat{K}(p) e^{-r/2},\end{aligned} \tag{2.65}$$

where we used (2.58) and (2.53), and redefined the constant $C_{d,m}$. As $|x_j - y_l| \geq r$ for all j, l,

$$\int_{\mathbb{R}^d} \mathrm{d}p \widehat{K_r}(p) \overline{u(p)} v(p) = 0 \tag{2.66}$$

and therefore, by Cauchy-Schwarz,

$$\begin{aligned}|\mathbb{E}[UV]| &\leq \int_{\mathbb{R}^d} \mathrm{d}p \left| \left(\widehat{K} - \widehat{K_r} \right)(p) \right| |u(p) v(p)| \\ &\leq \tilde{C}_{d,m} e^{-r/2} \int_{\mathbb{R}^d} \mathrm{d}p \widehat{K}(p) |u(p) v(p)| \leq \tilde{C}_{d,m} e^{-r/2},\end{aligned} \tag{2.67}$$

thus implying exponential mixing (in the sense of exponentially vanishing strong mixing coefficient, Definition 10.5 in [25]). The same estimate is trivially true for $r < 1$.

Choose a function $\phi : \mathbb{R} \to \mathbb{R}$ that is bounded together with its first m derivatives. Furthermore, assume that

$$\mathbb{E}\left[\phi(W(0))\right] = 0, \tag{2.68}$$

for example by $\phi(w) = -\phi(-w)$. Now let

$$\xi(x) = \phi(W(x)). \tag{2.69}$$

Lemma 2.4. *For any choice of $d, m \in \mathbb{N}$, ξ is of class $(m, \overline{\beta})$ for all $\overline{\beta} \in \mathbb{N}$.*

Proof. Stationarity carries over from W, and boundedness from ϕ. The moment bound on the first m derivatives of ξ, (2.8), is due to (2.56). ξ is centered by (2.68).

The only non-trivial issue are the cumulants. With $n_1, n_2 \in \mathbb{N}$, let $x_1^{(1)}, ..., x_{n_1}^{(1)} \in A_1 \subset \mathbb{R}^d$ and $x_1^{(2)}, ..., x_{n_2}^{(2)} \in A_2 \subset \mathbb{R}^d$ such that $\mathrm{dist}(A_1, A_2) \geq r$, and define the index set

$$I = \{(1,1), ...(1,n_1), (2,1), ..., (2,n_2)\}\,. \tag{2.70}$$

For any cluster $C \subset I$, set $C_j = \{(\tilde{j}, l) \in C : \tilde{j} = j\}$, $j = 1, 2$. Then

$$\prod_{(j,l)\in C_j} \phi\left(W(x_l^{(j)})\right) - \mathbb{E}\left[\prod_{(j,l)\in C_j} \phi\left(W(x_l^{(j)})\right)\right] \tag{2.71}$$

is (with the notation analogous to (2.59)) a centered $L^2(\Omega, \mathcal{A}_j, \mathbb{P})$ variable for $j = 1, 2$, and therefore

$$\begin{aligned} &\left|\mathbb{E}\left[\prod_{(j,l)\in C} \phi\left(W(x_l^{(j)})\right)\right] - \mathbb{E}\left[\prod_{(1,l)\in C_1} \phi\left(W(x_l^{(1)})\right)\right]\mathbb{E}\left[\prod_{(2,l)\in C_2} \phi\left(W(x_l^{(2)})\right)\right]\right| \\ &\quad \leq C_{d,m} \|\phi\|_{L^\infty}^{|C|} e^{-r/2}. \end{aligned} \tag{2.72}$$

By re-expressing the cumulants in terms of moments (as in [37], Chapter II, §12, equation (47)), one has

$$\begin{aligned} &\left|\zeta_{|I|}\left(x_l^{(j)} : (j,l) \in I\right)\right| \\ &\quad = \left|\sum_{S\in\pi(I)} (|S|-1)!(-1)^{|S|-1} \prod_{C\in S} \mathbb{E}\left[\prod_{(j,l)\in C} \xi(x_l^{(j)})\right]\right| \\ &\quad \leq \sum_{S\in\pi(I)} (|S|-1)!|S|\|\phi\|_{L^\infty}^{n_1+n_2} C_{d,m} e^{-r/2} \\ &\qquad + \left|\sum_{S\in\pi(I)} (|S|-1)!(-1)^{|S|-1} \prod_{C\in S} \mathbb{E}\left[\prod_{(1,l)\in C} \xi(x_l^{(1)})\right]\mathbb{E}\left[\prod_{(2,l)\in C} \xi(x_l^{(2)})\right]\right| \\ &\quad \leq C_{d,m} \|\phi\|_{L^\infty}^{n_1+n_2} \left((n_1+n_2)!\right)^2 e^{-r/2} + 0. \end{aligned} \tag{2.73}$$

The second to last line vanishes as it looks just like the cumulant for the case that the $\xi(x_l^{(1)})$, $l = 1, ..., n_1$ are independent of the $\xi(x_l^{(2)})$, $l = 1, ..., n_2$. Cumulants for sets of random variables that decompose into two nonempty independent subsets are always zero, as can be easily shown inductively from (2.10).

For $n \geq 2$, $y_1, ..., y_{n-1} \in \mathbb{R}^d$, the set $\{0, y_1, ..., y_{n-1}\}$ can always be decomposed into two nonempty sets with distance larger or equal to $\max_j |y_j|/n$. Therefore,

$$\begin{aligned} |g_n(y_1, ..., y_{n-1})| &\leq |\zeta_n(0, y_1, ..., y_{n-1})| \leq C_{d,m} \|\phi\|_{L^\infty}^n (n!)^2 \exp\left(-\frac{\max_j |y_j|}{2n}\right) \\ &\leq C_{d,m} \|\phi\|_{L^\infty}^n n^{2n} \exp\left(-\frac{|y_1| + ... + |y_{n-1}|}{2n^2}\right), \end{aligned} \tag{2.74}$$

and

$$\int_{\mathbb{R}^{d(n-1)}} \mathrm{d}y_1...\mathrm{d}y_{n-1} \, |g_n(y_1,...,y_{n-1})| \leq C_{d,m} \left(4\|\phi\|_{L^\infty}\right)^n n^{4n}. \tag{2.75}$$

Arguing similarly, one can estimate up to m-th derivatives of ζ_n with respect to every variable, and thus obtain a constant $C = C_{d,m,\phi} < \infty$ such that

$$\|g_n\|_m \leq Cn^{Cn} \tag{2.76}$$

for all integer $n \geq 2$.

Finally, (2.18) holds for all $\overline{\beta} \in \mathbb{N}$ because $\partial_x^\alpha g_2(x)$, $|\alpha| \leq 2m$ decays exponentially in x. □

Corollary 2.5. *By the same token, any centered, stationary random field ξ is of class $(m, \overline{\beta})$ for all $\overline{\beta} \in \mathbb{N}$ if it fulfills the following two conditions:*

- *ξ is almost surely m-times continuously differentiable with bounded derivatives*

$$\mathbb{P}\left(\max_{|\alpha|\leq m}\left|\frac{\partial^\alpha}{\partial x^\alpha}\xi(x)\right| < C\right) = 1 \tag{2.77}$$

  for some deterministic finite C.

- *There is a constant $C < \infty$ such that for $A, B \subset \mathbb{R}^d$ with $\operatorname{dist}(A,B) > r$,*

$$|\mathbf{Cov}(X,Y)| \leq (\mathbb{E}[X^2]\mathbb{E}[Y^2])^{1/2} C e^{-r/C} \tag{2.78}$$

  for all $X \in L^2\left(\Omega, \sigma\left(\xi(x) : x \in A\right), \mathbb{P}\right)$, $Y \in L^2\left(\Omega, \sigma\left(\xi(x) : x \in B\right), \mathbb{P}\right)$.

2.2. The perturbed time evolution and its generator

2.2.1. Existence and unitarity

Let $c \in C^1(\mathbb{R}^d, \mathbb{R})$ with $c \geq \theta > 0$ and at most linear growth at infinity, $c(x) \leq M + M|x|$ for some $M < \infty$. We first consider the dynamic this wave speed c generates on a bounded set, and set $B = B_R(0) \subset \mathbb{R}^d$ to be an open ball of finite radius around the origin. Set $\tilde{\mathcal{H}}_B$ to be the space of functions $(u, w) \in H_0^1(B) \times L^2(B)$ (employing the usual notation for Sobolev spaces), endowed with the scalar product

$$\langle (u_1, w_1); (u_2, w_2) \rangle_B = \frac{1}{2} \int_B \left(\nabla u_1(x) \cdot \nabla u_2(x) + \frac{\overline{w_1(x)} w_2(x)}{c^2(x)} \right) \mathrm{d}x \tag{2.79}$$

which makes $\tilde{\mathcal{H}}_B$ a Hilbert space. On the dense subspace

$$\tilde{\mathcal{D}}_B = \left(H^2(B) \cap H_0^1(B) \right) \times H_0^1(B) \subset \tilde{\mathcal{H}}_B, \tag{2.80}$$

define the operator $A_R : \tilde{\mathcal{D}}_B \to \tilde{\mathcal{H}}_B$ by

$$A_R(u, w) = (w, -c^2(x)\Delta u), \tag{2.81}$$

which is symmetric with respect to the scalar product $\langle \cdot ; \cdot \rangle_B$.

Lemma 2.6. *A_R is self-adjoint.*

Proof. As $c \geq \theta > 0$ everywhere, the operator $-c^2\Delta$ is uniformly elliptic. Therefore, by [19], Chapter 6.2, Theorem 3, there is a $\gamma > 0$ such that the boundary value problem

$$-c^2\Delta u + \gamma u = g - i\sqrt{\gamma} f \qquad \text{on } B \tag{2.82}$$

$$u = 0 \qquad \text{on } \partial B \tag{2.83}$$

has a weak solution $u \in H_0^1(B)$. By the H^2 regularity theorem ([19], Chapter 6.3, Theorem 4), $u \in H^2(B)$. If we now choose $w = f - i\sqrt{\gamma}u \in H_0^1$, we see that $(u, w) \in \tilde{\mathcal{D}}_B$ and

$$A_R(u, w) + i\sqrt{\gamma}(u, w) = (f, g). \tag{2.84}$$

Thus, $\operatorname{ran}(A_R + i\sqrt{\gamma}) = \tilde{\mathcal{H}}_B$. By the same argument, $A_R - i\sqrt{\gamma}$ is onto as well, and A_R is self-adjoint. □

As a self-adjoint operator, A_R generates a strongly continuous unitary group e^{-itA_R}, $t \in \mathbb{R}$ on $\tilde{\mathcal{H}}_B$, which is strongly differentiable on $\tilde{\mathcal{D}}_B$. Thus, for $(u_0, w_0) \in \tilde{\mathcal{D}}_B$ and

$$(u(t), w(t)) = \mathrm{e}^{-itA_R}(u_0, w_0), \tag{2.85}$$

one has

$$i\frac{\mathrm{d}}{\mathrm{d}t}(u(t), w(t)) = (w(t), -c^2\Delta u(t)). \tag{2.86}$$

Setting $v = -iw$, this actually is the solution to the wave equation

$$\begin{aligned} \frac{\partial}{\partial t}u &= v \\ \frac{\partial}{\partial t}v &= c^2\Delta u, \end{aligned} \tag{2.87}$$

however only on B, and with the boundary condition

$$u = 0 \text{ on } \partial B. \tag{2.88}$$

To obtain solutions of the wave equation on the full space $\mathbb{R}^d$, we use that initially compactly supported solutions cannot travel to infinity in finite time.

Lemma 2.7. *Let $B = B_R(0)$, $R > 0$, define for $r \geq 0$ $c_r = \sup\{c(x) | x \in B_r(0)\}$ and let $r(t)$ be the solution of $\dot{r}_t = c_{r_t}$, starting from some $r(0) \in (0, R)$. If $T > 0$ such that $r(T) < R$ and $(u_0, w_0) \in \tilde{\mathcal{H}}_B$ with $u_0 = w_0 = 0$ on $B_R(0) \setminus B_{r(0)}(0)$, then for the solution of the wave equation (with operator A_R defined on $B_R(0)$ as above)*

$$(u(t), w(t)) = \mathrm{e}^{-itA_R}(u_0, w_0), \tag{2.89}$$

one has $u(t) = v(t) = 0$ on $B_R(0) \setminus B_{r(|t|)}(0)$, for all $t \in (-T, T)$.

Proof. First, consider $(u_0, v_0) \in \tilde{\mathcal{D}}_B$. Without loss of generality, concentrate on $t \in [0, T)$ and define

$$E(t) = \frac{1}{2} \int_{B_{r(t)}} \left(|\nabla u(x,t)|^2 + \left| \frac{w(x,t)}{c(x)} \right|^2 \right) \mathrm{d}x. \tag{2.90}$$

As $(u(t), w(t)) \in \tilde{\mathcal{D}}_B$ for all $t \in [0, T)$, the following calculation (which is a generalization of Theorem 6 in Chapter 2.4 of [19]) is justified due to the strong differentiability of e^{-itA_R}:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E(t) &= -\mathrm{Im} \int_{B_{r(t)}} \left(\nabla w(x,t) \cdot \nabla u(x,t) + \overline{w(x,t)} \Delta u(x,t) \right) \mathrm{d}x \\ &\quad + \frac{c_{r(t)}}{2} \int_{\partial B_{r(t)}} \left(|\nabla u(x,t)|^2 + \left| \frac{w(x,t)}{c(x)} \right|^2 \right) \mathrm{d}S(x) \\ &= -\mathrm{Im} \int_{\partial B_{r(t)}} \overline{w(x,t)} \left(\nabla u(x,t) \cdot \nu(x) \right) \mathrm{d}S(x) \\ &\quad + \frac{c_{r(t)}}{2} \int_{\partial B_{r(t)}} \left(|\nabla u(x,t)|^2 + \left| \frac{w(x,t)}{c(x)} \right|^2 \right) \mathrm{d}S(x) \\ &\geq -c_{r(t)} \int_{\partial B_{r(t)}} \left| \frac{w(x,t)}{c_{r(t)}} \right| |\nabla u(x,t)| \, \mathrm{d}S(x) \\ &\quad + \frac{c_{r(t)}}{2} \int_{\partial B_{r(t)}} \left(|\nabla u(x,t)|^2 + \left| \frac{w(x,t)}{c(x)} \right|^2 \right) \mathrm{d}S(x) \\ &\geq -\frac{c_{r(t)}}{2} \int_{\partial B_{r(t)}} \left(|\nabla u(x,t)|^2 + \left| \frac{w(x,t)}{c_{r(t)}} \right|^2 \right) \mathrm{d}S(x) \\ &\quad + \frac{c_{r(t)}}{2} \int_{\partial B_{r(t)}} \left(|\nabla u(x,t)|^2 + \left| \frac{w(x,t)}{c(x)} \right|^2 \right) \mathrm{d}S(x) \\ &\geq 0, \end{aligned} \tag{2.91}$$

where we denoted by ν the outward normal of $\partial B_{r(t)}$. As $E(t) \leq \|(u_0, w_0)\|_B^2$ by unitarity and $E(0) = \|(u_0, w_0)\|_B^2$ by assumption,

$$E(t) = \|(u_0, v_0)\|_B^2 \tag{2.92}$$

for all $t \in [0, T)$. This proves the lemma for $(u_0, v_0) \in \tilde{\mathcal{D}}_B$. For general initial states from $\tilde{\mathcal{H}}_B$, the assertion then follows as $\tilde{\mathcal{D}}_B$ is dense in $\tilde{\mathcal{H}}_B$ and e^{-itA_R} is unitary. □

Now let

$$\tilde{\mathcal{H}} = \left\{ (u, w) \in H^1\left(\mathbb{R}^d\right) \times L^2\left(\mathbb{R}^d\right) : \mathrm{spt}(u, w) \text{ compact} \right\}, \tag{2.93}$$

which is a pre-Hilbert space with the scalar product

$$\langle (u_1, w_1); (u_2, w_2) \rangle_{\tilde{\mathcal{H}}} = \frac{1}{2} \int_{\mathbb{R}^d} \left(\nabla u_1(x) \cdot \nabla u_2(x) + \frac{\overline{w_1(x)} w_2(x)}{c^2(x)} \right) \mathrm{d}x. \tag{2.94}$$

Also denote the subspace

$$\tilde{\mathcal{D}} = \left\{ (u,w) \in H^2\left(\mathbb{R}^d\right) \times H^1\left(\mathbb{R}^d\right) : \mathrm{spt}(u,w) \text{ compact } \right\}. \tag{2.95}$$

Lemma 2.8. *Let $c \in C^1(\mathbb{R}^d, \mathbb{R})$ with $c \geq \theta > 0$ and at most linear growth at infinity, $c(x) \leq M + M|x|$ for some $M < \infty$. For balls B_R with radius R around the origin, define the operator A_R as before. Then for any $t \in \mathbb{R}$ and $(u_0, w_0) \in \tilde{\mathcal{H}}$, the limit*

$$\mathcal{U}(t)(u_0, w_0) = \lim_{R\to\infty} \mathrm{e}^{-itA_R}(u_0, w_0) \tag{2.96}$$

exists. $\mathcal{U}(t) : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ is a norm-preserving, strongly continuous group of operators. It is strongly differentiable on the invariant space $\tilde{\mathcal{D}}$.

Proof. If u_0, w_0 are supported in a ball of radius $r(0)$ around the origin, then $(u_0, w_0) \in \tilde{\mathcal{H}}_{B_R}$ for all $R > r(0)$. Moreover, for $r(T)$ as defined above one has the estimate

$$r(T) \leq C(M, r(0))e^{MT}, \tag{2.97}$$

so for all $R > C(M, r(0))e^{MT}$ and all $t \in (-T, T)$,

$$\mathrm{spt}\left(\mathrm{e}^{-itA_R}(u_0, w_0)\right) \subset \mathrm{int}B_R(0). \tag{2.98}$$

Thus $\mathrm{e}^{-itA_R}(u_0, w_0)$ is independent of R for R sufficiently large, and $\mathcal{U}(t)(u_0, w_0)$ is well-defined. The stated properties of $\mathcal{U}(t)$ directly carry over from e^{-itA_R}. □

Lemma 2.9. *The map $\mathcal{E}_c : \tilde{\mathcal{H}} \to \mathcal{H} = L^2\left(\mathbb{R}^d; \mathbb{C}^2\right)$*

$$\mathcal{E}_c(u,w) = \frac{1}{2}\begin{pmatrix} \sqrt{-\Delta}u + w/c \\ \sqrt{-\Delta}u - w/c \end{pmatrix} = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \tag{2.99}$$

is norm-preserving. $\mathcal{E}_c(\tilde{\mathcal{H}})$ is dense in $\mathcal{H}$, so are $\mathcal{E}_c(\tilde{\mathcal{D}})$ and even $\mathcal{E}_c\left(C_0^\infty(\mathbb{R}^d) \times C_0^\infty(\mathbb{R}^d)\right)$.

Proof. The first assertion is immediate from (2.94). As for the second claim, it is enough to show that $\mathcal{E}_c\left(C_0^\infty(\mathbb{R}^d) \times C_0^\infty(\mathbb{R}^d)\right)$ is dense. $\psi_+ - \psi_- \in L^2\left(\mathbb{R}^d\right)$ can obviously be approximated in L^2 by w/c, with $w \in C_0^\infty(\mathbb{R}^d)$. For the approximation of $\psi_+ + \psi_-$, note that $\sqrt{-\Delta}C_0^\infty(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$ as $C_0^\infty(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$, $\sqrt{-\Delta} : \mathcal{S}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is continuous, and $\sqrt{-\Delta}\mathcal{S}(\mathbb{R}^d)$ contains all Schwartz functions with Fourier transform supported away from zero, which is a dense set in $L^2(\mathbb{R}^d)$. □

We collect our results to find

Theorem 2.10. *$\mathcal{U}(t)$ is lifted by $\mathcal{E}_c$ to a norm-preserving, strongly continuous group of operators acting on a dense subspace of $\mathcal{H}$. Therefore, there is a unique continuous extension to $\mathcal{H}$, the strongly differentiable unitary group $\mathrm{e}^{-iH_ct} : \mathcal{H} \to \mathcal{H}$, $t \in \mathbb{R}$ with generator H_c. On the invariant, dense subspace $D = \mathcal{E}_c\left(\tilde{\mathcal{D}}\right)$, e^{-iH_ct} is strongly differentiable, so D is a core for H_c.*

On D, the operator H_c is given as

$$H_c = \frac{1}{2}\begin{pmatrix} \sqrt{-\Delta}c + c\sqrt{-\Delta} & -\sqrt{-\Delta}c + c\sqrt{-\Delta} \\ \sqrt{-\Delta}c - c\sqrt{-\Delta} & -\sqrt{-\Delta}c - c\sqrt{-\Delta} \end{pmatrix}, \tag{2.100}$$

with the unperturbed wave equation with $c \equiv 1$ generated by

$$H_0 = \begin{pmatrix} \sqrt{-\Delta} & 0 \\ 0 & -\sqrt{-\Delta} \end{pmatrix}. \tag{2.101}$$

Note that H_c actually depends on the particular choice of c, while D does not.

2.2.2. Duhamel expansion with smooth cut-off

Before we start to compare the dynamics created by H_c and H_0 with a Duhamel series expansion, we have to make two observations about H_c as defined in (2.100). First, as c is acting as a multiplication operator in position space, in momentum space it will formally be a convolution with $\widehat{c}$, which is only well-defined as a distribution. We will therefore have to introduce a suitably cut-off version c_R of c to justify our calculations. Second, as $\sqrt{-\Delta}$ is a (pseudo)differential operator, iterated applications of H_c will require us to take arbitrarily high derivatives of the initial state ψ_0 (which does not pose a problem, as one can concentrate on a dense set of smooth initial states) but also of c. A straight-forward application of a Duhamel expansion as in [32] would therefore require c to be C^∞ (and actually, in order to obtain suitable bounds for all terms, fulfill very strong estimates on its derivatives). Therefore, we will have to split up the action of c into a smooth part, and a rough part that we will only have to apply finitely often, thus making the Duhamel expansion possible for c that only have finitely many derivatives. As for the first part, one has

Lemma 2.11. *Let $c(x) = c_\omega(x)$, $x \in \mathbb{R}^d$ a C^1 random field with $0 < c_\omega(x) \leq M_\omega(1+|x|)$, M_ω almost surely finite, and let $\chi : \mathbb{R}^d \to [0,1]$ be a smooth function with $\chi(x) = 1$ for $|x| \leq 1$, $\chi(x) = 0$ for $|x| \geq 2$. Let $H_\omega = H_{c_\omega}$ be the (random) self-adjoint operator associated to c_ω, and H_ω^R be the one belonging to the cut-off field*

$$c_\omega^R(x) = c_\omega(x)\chi\left(\frac{x}{R}\right). \tag{2.102}$$

Then for any fixed t and ψ_0

$$\left\|\exp\left(-iH_\omega t\right)\psi_0 - \exp\left(-iH_\omega^R t\right)\psi_0\right\|_{\mathcal{H}} \to 0 \quad (R \to \infty) \tag{2.103}$$

both almost surely and in $L^q(\mathbb{P})$ for all $q < \infty$.

Proof. First, let $\psi_0 \in D$. The corresponding (u_0, w_0) with $\mathcal{E}_{c_\omega}(u_0, w_0) = \psi_0$, are compactly supported, so as in the proof of Lemma 2.8, we find for any given $t \in \mathbb{R}$ an almost surely finite $R(\omega)$ such that

$$\mathcal{U}(t)(u_0, w_0) = \exp\left(-itA_{R(\omega)}\right)(u_0, w_0). \tag{2.104}$$

However, the definition of $A_{R(\omega)}$ is the same for c_ω and c_ω^R for all $R \geq R(\omega)$. This proves the lemma for $\psi_0 \in D$. The general case follows as $D \subset \mathcal{H}$ is dense and from unitarity. $\square$

Supressing the ω-dependence, we now look more closely at the special case of wave speeds defined by

$$c(x) = 1 + \sqrt{\varepsilon}\xi(x) \tag{2.105}$$

the resulting operators H^ε and their "cut-off" versions $H^{\varepsilon,R}$. As c_R is bounded from above and away from zero, the domain of $H^{\varepsilon,R}$ is explicitely given as the Sobolev space

$$\mathcal{D}\left(H^{\varepsilon,R}\right) = \mathcal{D}\left(H_0\right) = H^1\left(\mathbb{R}^d;\mathbb{C}^2\right). \tag{2.106}$$

Let $\xi_R(x) = \xi(x)\chi(x/R)$ denote the cut-off version of ξ and define

$$\sqrt{\varepsilon}V^R = H^{\varepsilon,R} - H_0 = \frac{\sqrt{\varepsilon}}{2}\begin{pmatrix} \sqrt{-\Delta}\xi_R + \xi_R\sqrt{-\Delta} & -\sqrt{-\Delta}\xi_R + \xi_R\sqrt{-\Delta} \\ \sqrt{-\Delta}\xi_R - \xi_R\sqrt{-\Delta} & -\sqrt{-\Delta}\xi_R - \xi_R\sqrt{-\Delta} \end{pmatrix}. \tag{2.107}$$

Now fix an $L \in (0,\infty)$. The operator $M_{\xi,R}$ multiplying $L^2\left(\mathbb{R}^d\right)$ functions by ξ_R is given by

$$\mathcal{F}\left(M_{\xi,R}f\right)(k) = \int_{\mathbb{R}^d} \mathrm{d}p\widehat{\xi_R}(k-p)\widehat{f}(p) \tag{2.108}$$

in Fourier space for any $f \in L^2\left(\mathbb{R}^d\right)$. We split it up into

$$M_{\xi,R} = M_{\xi,R,L} + M_{\xi,R,L}^{\text{rough}} = M_L + M_L^{\text{rough}}, \tag{2.109}$$

with

$$\begin{aligned} \mathcal{F}\left(M_L f\right)(k_2) = &\int_{\mathbb{R}^d} \mathrm{d}k_1\widehat{\xi_R}(k_2-k_1)\widehat{f}(k_1)\varphi\left(|k_1|-L\right) \\ &+ \int_{\mathbb{R}^d} \mathrm{d}p\widehat{\xi_R}(k_2-k_1)\widehat{f}(k_1)(1-\varphi)\left(|k_1|-L\right)\varphi\left(|k_2|-L\right) \\ = &\int_{\mathbb{R}^d} \mathrm{d}p\widehat{\xi_R}(k_2-k_1)\widehat{f}(k_1)\Phi(k_2,k_1,L). \end{aligned} \tag{2.110}$$

Here, $\varphi : \mathbb{R} \to [0,1]$ is smooth, $\varphi(s) = 1$ if $s \leq 0$, $\varphi(s) = 0$ if $s \geq 1/2$ and Φ is given as

$$\Phi(k_2,k_1,L) = \varphi\left(|k_1|-L\right) + (1-\varphi)\left(|k_1|-L\right)\varphi\left(|k_2|-L\right). \tag{2.111}$$

Roughly speaking, M_L lets the the momentum jump from "small" to (possibly) "large" momenta or the other way round, but it never maps "large" to "large" momenta - these cases occur only under M_L^{rough}. If $\xi \in C^m\left(\mathbb{R}^d;\mathbb{R}\right)$, $m \geq 1$ almost surely, one has

$$\left\|M_L f\right\|_{H^m} \leq C_{L,R,\omega}\left\|f\right\|_{L^2}, \tag{2.112}$$

with the Sobolev space $H^m\left(\mathbb{R}^d\right)$ defined by all functions g with $\widehat{g}(p)(1+|p|)^m \in L^2\left(\mathbb{R}^d\right)$. On the other hand, in that case, M_L^{rough} almost surely maps $H^r\left(\mathbb{R}^d\right)$ continuously into

$H^l\left(\mathbb{R}^d\right)$ for all $l \leq \min\{m,r\}$. (The operator norms of all these mappings depend on ω and diverge as $R, L \to \infty$, but quantitative estimates will only be required lateron). On $\mathcal{D}\left(H^{\varepsilon,R}\right)$, one can now define

$$V_L^R = \frac{1}{2}\begin{pmatrix} \sqrt{-\Delta}M_L + M_L\sqrt{-\Delta} & -\sqrt{-\Delta}M_L + M_L\sqrt{-\Delta} \\ \sqrt{-\Delta}M_L - M_L\sqrt{-\Delta} & -\sqrt{-\Delta}M_L - M_L\sqrt{-\Delta} \end{pmatrix} \tag{2.113}$$

and equivalently

$$U_L^R = \frac{1}{2}\begin{pmatrix} \sqrt{-\Delta}M_L^{\mathrm{rough}} + M_L^{\mathrm{rough}}\sqrt{-\Delta} & -\sqrt{-\Delta}M_L^{\mathrm{rough}} + M_L^{\mathrm{rough}}\sqrt{-\Delta} \\ \sqrt{-\Delta}M_L^{\mathrm{rough}} - M_L^{\mathrm{rough}}\sqrt{-\Delta} & -\sqrt{-\Delta}M_L^{\mathrm{rough}} - M_L^{\mathrm{rough}}\sqrt{-\Delta} \end{pmatrix}. \tag{2.114}$$

Almost surely, V_L^R maps $\mathcal{D}\left(H^{\varepsilon,R}\right) = H^1\left(\mathbb{R}^d;\mathbb{C}^2\right)$ continuously into $H^{m-1}\left(\mathbb{R}^d;\mathbb{C}^2\right)$, while U_L^R (and therefore V^R) maps $H^r\left(\mathbb{R}^d;\mathbb{C}^2\right)$, $r \geq 1$ continuously into $H^l\left(\mathbb{R}^d;\mathbb{C}^2\right)$, $l = \min\{r,m\} - 1$.

Lemma 2.12. (Duhamel expansion with "abrupt cut-off") *Let $R \in (0,\infty)$, $\xi_R \in C^m\left(\mathbb{R}^d\right)$ with $m \geq 2$, $\overline{N} \in \mathbb{N}$, $L = (L_1, L_2, ...) \in (0,\infty)^{\mathbb{N}}$, $\psi \in \mathcal{D}\left(H^{\varepsilon,R}\right)$ and any fixed time $t > 0$, the following expansion holds with all integrals well-defined as Riemann integrals of continuous $\mathcal{H}$-valued functions.*

$$\begin{aligned}
\mathrm{e}^{-iH^{\varepsilon,R}t}\psi &= \sum_{N=0}^{\overline{N}-1} F_N(t;R,L,\varepsilon)\psi + \sum_{N=1}^{\overline{N}-1} F_N^{\mathrm{rough}}(t;R,L,\varepsilon)\psi + R_{\overline{N}}^{\mathrm{end}}(t;R,L,\varepsilon)\psi, \\
F_N(t;R,L,\varepsilon)\psi &= \int_{\mathbb{R}_+^{N+1}} \mathrm{d}s\delta\left(t - \sum_{l=1}^{N+1} s_l\right) \mathrm{e}^{-iH_0 s_{N+1}}(-i\sqrt{\varepsilon}V_{L_N}^R)...(-i\sqrt{\varepsilon}V_{L_1}^R)\mathrm{e}^{-iH_0 s_1}\psi, \\
F_N^{\mathrm{rough}}(t;R,L,\varepsilon)\psi &= \int_{\mathbb{R}_+^{N+1}} \mathrm{d}s\delta\left(t - \sum_{l=1}^{N+1} s_l\right) \mathrm{e}^{-iH^{\varepsilon,R}s_{N+1}}(-i\sqrt{\varepsilon}U_{L_N}^R) \\
&\qquad \times \mathrm{e}^{-iH_0 s_N}(-i\sqrt{\varepsilon}V_{L_{N-1}}^R)...(-i\sqrt{\varepsilon}V_{L_1}^R)\mathrm{e}^{-iH_0 s_1}\psi, \\
R_{\overline{N}}^{\mathrm{end}}(t;R,L,\varepsilon)\psi &= \int_{\mathbb{R}_+^{\overline{N}+1}} \mathrm{d}s\delta\left(t - \sum_{l=1}^{\overline{N}+1} s_l\right) \mathrm{e}^{-iH^{\varepsilon,R}s_{\overline{N}+1}}(-i\sqrt{\varepsilon}V^R) \\
&\qquad \times \mathrm{e}^{-iH_0 s_{\overline{N}}}(-i\sqrt{\varepsilon}V_{L_{\overline{N}-1}}^R)...(-i\sqrt{\varepsilon}V_{L_1}^R)\mathrm{e}^{-iH_0 s_1}\psi.
\end{aligned} \tag{2.115}$$

Occasionally, we will use the shorthand

$$\begin{aligned}
F_{\overline{N}}^{\mathrm{main}}(t;R,L,\varepsilon)\psi &= \sum_{N=0}^{\overline{N}-1} F_N(t;R,L,\varepsilon)\psi, \\
R_{\overline{N}}(t;R,L,\varepsilon)\psi &= \sum_{N=1}^{\overline{N}-1} F_N^{\mathrm{rough}}(t;R,L,\varepsilon)\psi + R_{\overline{N}}^{\mathrm{end}}(t;R,L,\varepsilon)\psi.
\end{aligned} \tag{2.116}$$

Proof. As $\psi \in \mathcal{D}\left(H^{\varepsilon,R}\right) = \mathcal{D}\left(H_0\right)$, we can differentiate

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{+iH^{\varepsilon,R}s}\mathrm{e}^{-iH_0 s}\psi = \mathrm{e}^{+iH^{\varepsilon,R}s}(i\sqrt{\varepsilon}V^R)\mathrm{e}^{-iH_0 s}\psi. \tag{2.117}$$

Integration from 0 to t immediately yields

$$\begin{aligned}\mathrm{e}^{-iH^{\varepsilon,R}t}\psi &= \mathrm{e}^{-iH_0 t}\psi + \int_0^t \mathrm{d}s\mathrm{e}^{-iH^{\varepsilon,R}(t-s)}(-i\sqrt{\varepsilon}V^R)\mathrm{e}^{-iH_0 s}\psi \\ &= F_0(t;R,L,\varepsilon)\psi + R_1(t;R,L,\varepsilon)\psi,\end{aligned} \tag{2.118}$$

with the integrand a continuous function from $[0,t]$ to $\mathcal{H}$, proving the lemma for $\overline{N}=1$. Now assume (2.115) holds for some $\overline{N} \geq 1$. We split the V^R in the $R_{\overline{N}}^{\mathrm{end}}$ term,

$$\begin{aligned}R_{\overline{N}}^{\mathrm{end}}(t;R,L,\varepsilon)\psi &= \int_{\mathbb{R}_+^{\overline{N}+1}} \mathrm{d}s\delta\left(t - \sum_{l=1}^{\overline{N}+1} s_l\right)\mathrm{e}^{-iH^{\varepsilon,R}s_{\overline{N}+1}}(-i\sqrt{\varepsilon}V^R) \\ &\qquad \times \mathrm{e}^{-iH_0 s_{\overline{N}}}(-i\sqrt{\varepsilon}V^R_{L_{\overline{N}-1}})\ldots(-i\sqrt{\varepsilon}V^R_{L_1})\mathrm{e}^{-iH_0 s_1}\psi \\ &= F_{\overline{N}}^{\mathrm{rough}}(t;R,L,\varepsilon)\psi \\ &\quad + \int_{\mathbb{R}_+^{\overline{N}+1}} \mathrm{d}s\delta\left(t - \sum_{l=1}^{\overline{N}+1} s_l\right)\mathrm{e}^{-iH^{\varepsilon,R}s_{\overline{N}+1}} \\ &\qquad \times (-i\sqrt{\varepsilon}V^R_{L_{\overline{N}}})\mathrm{e}^{-iH_0 s_{\overline{N}}}(-i\sqrt{\varepsilon}V^R_{L_{\overline{N}-1}})\ldots(-i\sqrt{\varepsilon}V^R_{L_1})\mathrm{e}^{-iH_0 s_1}\psi.\end{aligned} \tag{2.119}$$

As $\psi \in H^1\left(\mathbb{R}^d;\mathbb{C}^2\right)$, by the above findings for the $V^R_{L_N}$ and the fact that H_0 is diagonal in Fourier space,

$$(s_1,\ldots,s_{\overline{N}}) \mapsto (-i\sqrt{\varepsilon}V^R_{L_{\overline{N}}})\mathrm{e}^{-iH_0 s_{\overline{N}}}(-i\sqrt{\varepsilon}V^R_{L_{\overline{N}-1}})\ldots(-i\sqrt{\varepsilon}V^R_{L_1})\mathrm{e}^{-iH_0 s_1}\psi \tag{2.120}$$

is a continuous function $\mathbb{R}^{\overline{N}} \to H^{m-1}\left(\mathbb{R}^d;\mathbb{C}^2\right)$. Since $m \geq 2$, the last line of (2.119) is in $\mathcal{D}\left(H^{\varepsilon,R}\right) = \mathcal{D}\left(H_0\right)$, and we can reiterate the argument leading to (2.118) to rewrite the exponential $\exp\left(-iH^{\varepsilon,R}s_{\overline{N}+1}\right)$,

$$\begin{aligned} R^{\text{end}}_{\overline{N}}(t;R,L,\varepsilon)\psi = {} & F^{\text{rough}}_{\overline{N}}(t;R,L,\varepsilon)\psi \\ & + \int_{\mathbb{R}_+^{\overline{N}+1}} \mathrm{d}s\delta\left(t-\sum_{l=1}^{\overline{N}+1}s_l\right)\mathrm{e}^{-iH_0 s_{\overline{N}+1}} \\ & \quad \times(-i\sqrt{\varepsilon}V^R_{L_{\overline{N}}})\mathrm{e}^{-iH_0 s_{\overline{N}}}(-i\sqrt{\varepsilon}V^R_{L_{\overline{N}-1}})\ldots(-i\sqrt{\varepsilon}V^R_{L_1})\mathrm{e}^{-iH_0 s_1}\psi \\ & + \int_{\mathbb{R}_+^{\overline{N}+2}} \mathrm{d}s\delta\left(t-\sum_{l=1}^{\overline{N}+2}s_l\right)\mathrm{e}^{-iH^{\varepsilon,R}s_{\overline{N}+2}}(-i\sqrt{\varepsilon}V^R) \\ & \quad \times \mathrm{e}^{-iH_0 s_{\overline{N}+1}}(-i\sqrt{\varepsilon}V^R_{L_{\overline{N}}})\ldots(-i\sqrt{\varepsilon}V^R_{L_1})\mathrm{e}^{-iH_0 s_1}\psi \\ = {} & F^{\text{rough}}_{\overline{N}}(t;R,L,\varepsilon)\psi + F_{\overline{N}}(t;R,L,\varepsilon)\psi + R^{\text{end}}_{\overline{N}+1}(t;R,L,\varepsilon)\psi. \end{aligned} \tag{2.121}$$

By mathematical induction, this finishes the proof. □

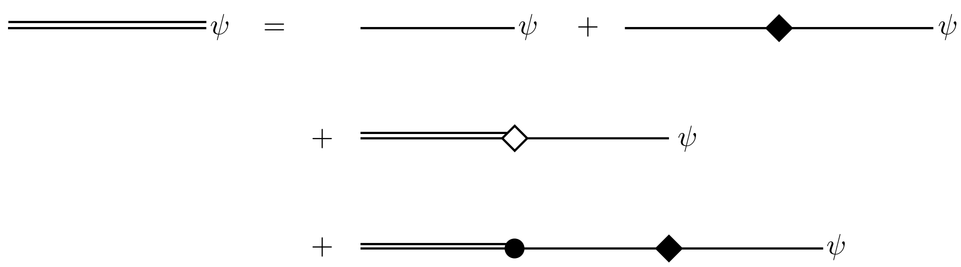

Figure 2.1.: The expansion of Lemma 2.12 for $\overline{N}=2$, with double (single) lines representing the full propagation generated by $H^{\varepsilon,R}$ (the free dynamics generated by H_0). The solid diamond denotes a scattering off $V^R_{L_1}$, while an empty diamond stands for $U^R_{L_1}$, and interaction with the full V^R is indicated by a black bullet.

The physical idea behind the Duhamel expansion in Lemma 2.12 is to interpret the perturbed dynamics as a free wave that undergoes scattering off inhomogeneities, and to expand to the $\overline{N}$-th scattering event. However, for $N<\overline{N}$, one will only continue to expand in case of a "well-behaved" scattering event (described by $V^R_{L_N}$). In case of a "bad" scattering event ($U^R_{L_N}$), the expansion is stopped immediately. Now this abrupt stopping rule has to be smoothed out to guarantee that the remainder terms $R^{\text{end}}_{\overline{N}}$ and $\sum_N F^{\text{rough}}_N$ vanish in the kinetic limit. This is accomplished as in [32] by expanding those terms a bit further, adding a weak exponential decay. However, once a $U^R_{L_N}$ has occurred, we can only expand further for finitely many scattering events - $m-2$, to be precise.

Lemma 2.13. *Let $R \in (0,\infty)$, $\xi_R \in C^m$ with $m \geq 3$, $\tilde{\psi} \in H^{m-2}\left(\mathbb{R}^d;\mathbb{C}^2\right)$, $\kappa > 0$ and $\tau > 0$. Then for all $\overline{M} \in \{1,...,m-2\}$,*

$$\begin{aligned}
\mathrm{e}^{-iH^{\varepsilon,R}\tau}\tilde{\psi} = &\sum_{M=0}^{\overline{M}-1}\int_{\mathbb{R}_+^{M+1}} \mathrm{d}s\delta\left(\tau - \sum_{l=1}^{M+1} s_l\right)\mathrm{e}^{-i(H_0-i\kappa)s_{M+1}} \\
&\qquad\qquad \times(-i\sqrt{\varepsilon}V^R)...(-i\sqrt{\varepsilon}V^R)\mathrm{e}^{-i(H_0-i\kappa)s_1}\tilde{\psi} \\
&+\kappa\sum_{M=1}^{\overline{M}}\int_{\mathbb{R}_+^{M+1}} \mathrm{d}s\delta\left(\tau - \sum_{l=1}^{M+1} s_l\right)\mathrm{e}^{-iH^{\varepsilon,R}s_{M+1}} \\
&\qquad\qquad \times \mathrm{e}^{-i(H_0-i\kappa)s_M}(-i\sqrt{\varepsilon}V^R)...(-i\sqrt{\varepsilon}V^R)\mathrm{e}^{-i(H_0-i\kappa)s_1}\tilde{\psi} \\
&+\int_{\mathbb{R}_+^{\overline{M}+1}} \mathrm{d}s\delta\left(\tau - \sum_{l=1}^{\overline{M}+1} s_l\right)\mathrm{e}^{-iH^{\varepsilon,R}s_{\overline{M}+1}}(-i\sqrt{\varepsilon}V^R)...(-i\sqrt{\varepsilon}V^R)\mathrm{e}^{-i(H_0-i\kappa)s_1}\tilde{\psi},
\end{aligned} \tag{2.122}$$

with all integrals well-defined as Riemann integrals of continuous $\mathcal{H}$-valued functions.

Proof. As $\tilde{\psi} \in H^{m-2}\left(\mathbb{R}^d;\mathbb{C}^2\right) \subset \mathcal{D}\left(H^{\varepsilon,R}\right) = \mathcal{D}\left(H_0\right)$, one can proceed as in the previous proof to obtain

$$\mathrm{e}^{-iH^{\varepsilon,R}\tau}\tilde{\psi} = \mathrm{e}^{-iH_0\tau}\tilde{\psi} + \int_0^\tau \mathrm{d}s\mathrm{e}^{-iH^{\varepsilon,R}(\tau-s)}(-i\sqrt{\varepsilon}V^R + \kappa)\mathrm{e}^{-i(H_0-i\kappa)s}\tilde{\psi}, \tag{2.123}$$

which proves the assertion for $\overline{M} = 1$. Now assume that $\overline{M} + 1 \leq m-2$ and (2.122) holds for $\overline{M}$. Consider the argument of $\mathrm{e}^{-iH^{\varepsilon,R}s_{\overline{M}+1}}$ in the last line of (2.122). Since the spaces $H^r\left(\mathbb{R}^d;\mathbb{C}^2\right)$, $r \leq m$ are conserved by the action of $\mathrm{e}^{-i(H_0-i\kappa)s_M}$, while V^R maps them to $H^{r-1}\left(\mathbb{R}^d;\mathbb{C}^2\right)$, we see that $\overline{M}$ such operations yield

$$(-i\sqrt{\varepsilon}V^R)...(-i\sqrt{\varepsilon}V^R)\mathrm{e}^{-i(H_0-i\kappa)s_1}\tilde{\psi} \in H^{m-2-\overline{M}}\left(\mathbb{R}^d;\mathbb{C}^2\right) \subset H^1\left(\mathbb{R}^d;\mathbb{C}^2\right). \tag{2.124}$$

Thus, we can once more apply (2.123) and conclude by mathematical induction. □

Lemma 2.14. (Duhamel expansion with "smooth cut-off") *Let $m \geq 3$, $R \in (0,\infty)$, $\xi_R \in C^m\left(\mathbb{R}^d\right)$, $\psi \in H^{m-1}\left(\mathbb{R}^d;\mathbb{C}^2\right)$, $L = (L_1, L_2, ...) \in (0,\infty)^{\mathbb{N}}$, $\kappa > 0$ and $t > 0$. Then for all $\overline{M} \in \{1,...,m-2\}$, and all $\overline{N} \in \mathbb{N}$, we have for the quantities from Lemma 2.12*

$$\begin{aligned}
&F_N^{\mathrm{rough}}(t;R,L,\varepsilon)\psi \\
&\quad = \sum_{M=0}^{\overline{M}-1}\left(G_{M,N}^{\mathrm{rough}}(t;R,L,\varepsilon,\kappa)\psi + \kappa\int_0^t \mathrm{d}r\mathrm{e}^{-iH^{\varepsilon,R}(t-r)}G_{M,N}^{\mathrm{rough}}(r;R,L,\varepsilon,\kappa)\psi\right) \\
&\qquad\quad + \int_0^t \mathrm{d}r\mathrm{e}^{-iH^{\varepsilon,R}(t-r)}A_{\overline{M},N}^{\mathrm{rough}}(r;R,L,\varepsilon,\kappa)\psi
\end{aligned} \tag{2.125}$$

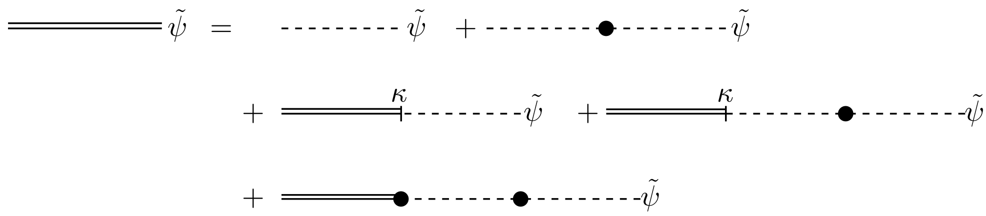

Figure 2.2.: The expansion given in Lemma 2.13 with $\overline{M} = 2$. As before, the double solid lines mean propagation by the dynamics created by $H^{\varepsilon,R}$, while dashed lines correspond to $H_0 - i\kappa$. The bullets represent V^R, while κ denotes "interaction" with the imaginary potential $-i\kappa$.

$N \in \{1, ..., \overline{N} - 1\}$, *with*

$$\begin{aligned} &G^{\text{rough}}_{M,N}(\tau; R, L, \varepsilon, \kappa)\psi \\ &= \int_{\mathbb{R}_+^{M+N+1}} \mathrm{d}s\delta\left(\tau - \sum_{l=1}^{M+N+1} s_l\right) \mathrm{e}^{-i(H_0-i\kappa)s_{M+N+1}}(-i\sqrt{\varepsilon}V^R)...(-i\sqrt{\varepsilon}V^R)\mathrm{e}^{-i(H_0-i\kappa)s_{N+1}} \\ &\qquad \times (-i\sqrt{\varepsilon}U^R_{L_N})\mathrm{e}^{-iH_0 s_N}(-i\sqrt{\varepsilon}V^R_{L_{N-1}})...(-i\sqrt{\varepsilon}V^R_{L_1})\mathrm{e}^{-iH_0 s_1}\psi \end{aligned} \tag{2.126}$$

and

$$\begin{aligned} &A^{\text{rough}}_{\overline{M},N}(\tau; R, L, \varepsilon, \kappa)\psi \\ &\quad = \int_{\mathbb{R}_+^{\overline{M}+N}} \mathrm{d}s\delta\left(\tau - \sum_{l=1}^{\overline{M}+N} s_l\right) (-i\sqrt{\varepsilon}V^R)\mathrm{e}^{-i(H_0-i\kappa)s_{\overline{M}+N}}...(-i\sqrt{\varepsilon}V^R)\mathrm{e}^{-i(H_0-i\kappa)s_{N+1}} \\ &\qquad \times (-i\sqrt{\varepsilon}U^R_{L_N})\mathrm{e}^{-iH_0 s_N}(-i\sqrt{\varepsilon}V^R_{L_{N-1}})...(-i\sqrt{\varepsilon}V^R_{L_1})\mathrm{e}^{-iH_0 s_1}\psi. \end{aligned} \tag{2.127}$$

Similarly,

$$\begin{aligned} R^{\text{end}}_{\overline{N}}(t; R, L, \varepsilon)\psi = \sum_{M=0}^{\overline{M}-1} &\left(G^{\text{end}}_{M,\overline{N}}(t; R, L, \varepsilon, \kappa)\psi + \kappa\int_0^t \mathrm{d}r\mathrm{e}^{-iH^{\varepsilon,R}(t-r)}G^{\text{end}}_{M,\overline{N}}(r; R, L, \varepsilon, \kappa)\psi\right) \\ &+ \int_0^t \mathrm{d}r\mathrm{e}^{-iH^{\varepsilon,R}(t-r)}A^{\text{end}}_{\overline{M},\overline{N}}(r; R, L, \varepsilon, \kappa)\psi, \end{aligned} \tag{2.128}$$

with

$$G^{\mathrm{end}}_{M,\overline{N}}(\tau;R,L,\varepsilon,\kappa)\psi$$
$$= \int_{\mathbb{R}_+^{M+\overline{N}+1}} \mathrm{d}s\delta\left(\tau - \sum_{l=1}^{M+\overline{N}+1} s_l\right) \mathrm{e}^{-i(H_0-i\kappa)s_{M+\overline{N}+1}}(-i\sqrt{\varepsilon}V^R)...(-i\sqrt{\varepsilon}V^R)\mathrm{e}^{-i(H_0-i\kappa)s_{\overline{N}+1}}$$
$$\times (-i\sqrt{\varepsilon}V^R)\mathrm{e}^{-iH_0 s_{\overline{N}}}(-i\sqrt{\varepsilon}V^R_{L_{\overline{N}-1}})...(-i\sqrt{\varepsilon}V^R_{L_1})\mathrm{e}^{-iH_0 s_1}\psi \tag{2.129}$$

and

$$A^{\mathrm{end}}_{M,\overline{N}}(\tau;R,L,\varepsilon,\kappa)\psi$$
$$= \int_{\mathbb{R}_+^{\overline{M}+\overline{N}}} \mathrm{d}s\delta\left(\tau - \sum_{l=1}^{\overline{M}+\overline{N}} s_l\right) (-i\sqrt{\varepsilon}V^R)\mathrm{e}^{-i(H_0-i\kappa)s_{\overline{M}+\overline{N}}}...(-i\sqrt{\varepsilon}V^R)\mathrm{e}^{-i(H_0-i\kappa)s_{\overline{N}+1}}$$
$$\times (-i\sqrt{\varepsilon}V^R)\mathrm{e}^{-iH_0 s_{\overline{N}}}(-i\sqrt{\varepsilon}V^R_{L_{\overline{N}}})...(-i\sqrt{\varepsilon}V^R_{L_1})\mathrm{e}^{-iH_0 s_1}\psi. \tag{2.130}$$

Again, all integrals are well-defined as Riemann integrals of $\mathcal{H}$-valued continuous functions.

Proof. Note that, as $\psi \in H^{m-1}\left(\mathbb{R}^d;\mathbb{C}^2\right)$, and by the properties of the $V^R_{L_N}$ and $U^R_{L_N}$, all arguments of the full unitary $\mathrm{e}^{-iH^{\varepsilon,R}s}$ on the right side of (2.115) are in $H^{m-2}\left(\mathbb{R}^d;\mathbb{C}^2\right)$. Thus Lemma 2.13 is applicable, and the claim follows. □

2.3. The linear Boltzmann equation

As soon as the distribution of the random field ξ is such that $g_2 \in L^1\left(\mathbb{R}^d\right)$ (which will clearly be the case under the much stricter conditions of Theorems 3.1, 3.2 and 3.3), $\widehat{g_2}$ will be bounded, continuous and non-negative (the latter by the Bochner-Schwartz theorem, [34]). Therefore, for each $k \in \mathbb{R}^d$, the measure $\nu_{\mathrm{sc}}(k,\cdot)$ given by

$$\nu_{\mathrm{sc}}(k,B) = \int_B \mathrm{d}k'|2\pi k|^2 \widehat{g_2}(k-k')\delta(|k|-|k'|) \tag{2.131}$$

for any Borel set $B \subseteq \mathbb{R}^d$ is non-negative, and

$$\sigma_{\mathrm{sc}}(k) = \nu_{\mathrm{sc}}\left(k,\mathbb{R}^d\right), \tag{2.132}$$

is uniformly bounded for k from compact subsets of $\mathbb{R}^d$. For either sign $\sigma \in \{\pm\}$, one can define a Markov process $(x(t),k(t))_{t\geq 0}$ with càdlàg paths on the space $\mathbb{R}^d_x \times \left(\mathbb{R}^d \setminus \{0\}\right)_k$ similar to the one in [32]: For $x(0)$, and $k(0) \neq 0$ given, $k(t) = k(0)$ for all $t \in [0,t_1)$ with t_1 an exponentially distributed waiting time with parameter $\sigma_{\mathrm{sc}}(k(0))$. The momentum $k(t_1)$

is independent of t_1 and distributed with the probability measure $\nu_{\mathrm{sc}}(k(0),\cdot)/\sigma_{\mathrm{sc}}(k(0))$. The momentum then jumps again after a waiting time t_2 (which is $\mathrm{Exp}\,(\sigma_{\mathrm{sc}}(k(t_1)))$-distributed), to a position $k(t_1+t_2)$ with distribution $\nu_{\mathrm{sc}}(k(t_1),\cdot)/\sigma_{\mathrm{sc}}(k(t_1))$. The process $k(t)$ is almost surely piecewise continuous and $|k(t)| = |k(0)| \neq 0$, so

$$x(t) = \sigma \int_0^t \mathrm{d}s \frac{k(s)}{|k(s)|} \tag{2.133}$$

is almost surely well-defined. From the continuity of $\widehat{g_2}$ and the local boundedness of σ_{sc} it is easy to see that on $C_0\left(\mathbb{R}^d \times \left(\mathbb{R}^d \setminus \{0\}\right)\right)$ (the continuous functions on $\mathbb{R}^{2d}$ vanishing at infinity and on $\{k=0\}$), this process gives rise to a probability semigroup as defined in Definition 3.4, [29]. Thus, by Theorem 3.26 in the same book, $(x(t),k(t))_{t\geq 0}$ is Feller and has a generator $\mathcal{L}_\sigma$ which is densely defined on $C_0\left(\mathbb{R}^d \times \left(\mathbb{R}^d \setminus \{0\}\right)\right)$. The smooth, compactly supported functions $a \in C_\mathrm{c}^\infty\left(\mathbb{R}^d \times \left(\mathbb{R}^d \setminus \{0\}\right)\right)$ are a core for $\mathcal{L}_\sigma$, with

$$\left(\mathcal{L}_\sigma a\right)(x,k) = \left(\int_{\mathbb{R}^d} \nu_{\mathrm{sc}}(k,\mathrm{d}k')a(x,k')\right) - \sigma_{\mathrm{sc}}(k)a(x,k) + \sigma\frac{k}{|k|}\cdot\nabla_x a(x,k) \tag{2.134}$$

for such a. For all $a \in C_0\left(\mathbb{R}^d \times \left(\mathbb{R}^d \setminus \{0\}\right)\right)$, the probability semigroup can be expanded into

$$\begin{aligned}&\left(\mathrm{e}^{\mathcal{L}_\sigma t}a\right)(x,k_0)\\ &\quad= \sum_{n=0}^\infty \int_{\mathbb{R}^{n+1}} \mathrm{d}s_0 ... \mathrm{d}s_n \delta\left(t - \sum_{l=0}^n s_l\right) \int_{\mathbb{R}^d} \nu_{\mathrm{sc}}(k_0,\mathrm{d}k_1) ... \int_{\mathbb{R}^d} \nu_{\mathrm{sc}}(k_{n-1},\mathrm{d}k_n)\\ &\qquad\quad \exp\left(-\sum_{l=0}^n \sigma_{\mathrm{sc}}(k_l)s_l\right) a\left(x + \sigma\sum_{l=0}^n s_l\frac{k_l}{|k_l|}, k_n\right).\end{aligned} \tag{2.135}$$

for $t \geq 0$, $x \in \mathbb{R}$, $k_0 \neq 0$. Note that (2.135) actually defines a (no longer strongly continuous) semigroup on the space of *all* bounded continuous functions on $\mathbb{R}^d \times \left(\mathbb{R}^d \setminus \{0\}\right)$, which, in a slight abuse of notation, we will still refer to as $\mathrm{e}^{\mathcal{L}_\sigma t}$, $t \geq 0$. In particular, this semigroup now operates on test functions of type $\mathcal{F}L^1(C^0)$, to be defined in Section 2.4.1.

Starting from any bounded Borel measure $\mu_{0,\sigma}$ on $\mathbb{R}^d \times \left(\mathbb{R}^d \setminus \{0\}\right)$ as the initial distribution of $(x(0),k(0))$, we obtain the distribution at time $t \geq 0$ by applying the adjoint of the semigroup,

$$\mu_{\sigma,t} = \left(\mathrm{e}^{\mathcal{L}_\sigma t}\right)^* \mu_{0,\sigma}. \tag{2.136}$$

The measure $\mu_{\sigma,t}$ is a weak solution of the linear Boltzmann equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_{\sigma,t}(x,k) = \int_{\mathbb{R}^d} \nu_{\mathrm{sc}}(k,\mathrm{d}k')\mu_{\sigma,t}(x,k') - \sigma_{\mathrm{sc}}(k)\mu_{\sigma,t}(x,k) - \sigma\frac{k}{|k|}\cdot\nabla_x\mu_{\sigma,t}(x,k) \tag{2.137}$$

in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\left(\mathbb{R}^d\setminus\{0\}\right)} \mu_{\sigma,t}(\mathrm{d}x,\mathrm{d}k)a(x,k) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\left(\mathbb{R}^d\setminus\{0\}\right)} \mu_{\sigma,t}(\mathrm{d}x,\mathrm{d}k)\left(\mathcal{L}_\sigma a\right)(x,k) \tag{2.138}$$

for all a from the domain of $\mathcal{L}_\sigma$.

2.4. Wigner functions

2.4.1. Definition and limit behavior

For functions $f \in L^1(\mathbb{R}^d)$, $d \in \mathbb{N}$, we define the Fourier transform $\mathcal{F}f = \widehat{f} : \mathbb{R}^d \to \mathbb{C}$ as

$$(\mathcal{F}f)(p)\widehat{f}(p) = \int_{\mathbb{R}^d} \mathrm{d}x f(x) e^{-2\pi i x\cdot p}, \tag{2.139}$$

so that the continuous extension $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is unitary. For functions defined on classical phase space, $\mathfrak{a} : \mathbb{R}^{2d} = \mathbb{R}^d_x \times \mathbb{R}^d_k \to \mathbb{C}$, $\mathcal{F}$ and $\widehat{\cdot}$ shall denote the Fourier transform only with respect to the first, position, variable,

$$(\mathcal{F}\mathfrak{a})(p,k) = \widehat{\mathfrak{a}}(p,k) \int_{\mathbb{R}^d} \mathrm{d}x \mathfrak{a}(x,k) e^{-2\pi i x\cdot p}. \tag{2.140}$$

For the moment, one may assume $\mathfrak{a}$ to be a Schwartz function. To a function $\psi \in L^2(\mathbb{R}^d)$, one may assign a function $W[\psi]$ living on phase space,

$$W[\psi](x,k) = \int_{\mathbb{R}^d} \mathrm{d}y \overline{\psi\left(x + \frac{y}{2}\right)} \psi\left(x - \frac{y}{2}\right) e^{2\pi i y\cdot k} \tag{2.141}$$

the *Wigner transform* of ψ. W takes only real values, but need not be non-negative. For a small parameter $\varepsilon > 0$, the ε-scaled Wigner transform is then obtained by a space scaling,

$$W^\varepsilon[\psi](x,k) = \varepsilon^{-d} W[\psi]\left(\frac{x}{\varepsilon}, k\right) = \varepsilon^{-d} \int_{\mathbb{R}^d} \mathrm{d}y \overline{\psi\left(\frac{x}{\varepsilon} + \frac{y}{2}\right)} \psi\left(\frac{x}{\varepsilon} - \frac{y}{2}\right) e^{2\pi i y\cdot k}. \tag{2.142}$$

$W^\varepsilon[\psi]$ acts as a tempered distribution on Schwartz functions $\mathfrak{a} \in \mathcal{S}(\mathbb{R}^{2d})$ by

$$\begin{aligned} \langle W^\varepsilon[\psi], \mathfrak{a}\rangle &= \int_{\mathbb{R}^{2d}} \mathrm{d}x \mathrm{d}k \mathfrak{a}(x,k) W^\varepsilon[\psi](x,k) \\ &= \int_{\mathbb{R}^d} \mathrm{d}p \left(\int_{\mathbb{R}^d} \mathrm{d}k \widehat{\mathfrak{a}}(p,k) \overline{\widehat{\psi}\left(k + \frac{\varepsilon p}{2}\right)} \widehat{\psi}\left(k - \frac{\varepsilon p}{2}\right)\right). \end{aligned} \tag{2.143}$$

Note the discrepancies with, for example, [16] that arise from defining $\langle W^\varepsilon[\psi], \mathfrak{a}\rangle$ as linear in $\mathfrak{a}$ instead of conjugate linear. The last line of (2.143), which follows from standard Fourier calculus, shows that we can actually understand $W^\varepsilon[\psi]$ as a continuous linear functional on the Banach space $\mathcal{F}L^1\left(\mathbb{R}^d_p; C^0(\mathbb{R}^d_k)\right) = \mathcal{F}L^1(C^0)$. This space comprises all functions $\mathfrak{a} : \mathbb{R}^{2d} \to \mathbb{C}$ for which $(p,k) \mapsto \widehat{\mathfrak{a}}(p,k)$ is continuous in the k variable and the norm

$$\|\mathfrak{a}\|_{\mathcal{F}L^1(C^0)} = \int_{\mathbb{R}^d} \mathrm{d}p \left(\sup_{k\in\mathbb{R}^d} |\widehat{\mathfrak{a}}(p,k)| \right) \tag{2.144}$$

is finite. The space $\mathcal{F}L^1(C^0)$ is slightly more general than the space defined on page 572 of [30], which also requires $\widehat{\mathfrak{a}}(p,k)$ to decay to zero as $|k| \to \infty$. The latter space, to which we will refer as $\mathfrak{X}_0$, is separable with the norm from (2.144). For $\mathfrak{a} \in \mathcal{F}L^1(C^0)$,

$$|\langle W^\varepsilon[\psi], \mathfrak{a}\rangle| \le \|\mathfrak{a}\|_{\mathcal{F}L^1(C^0)} \|\psi\|^2_{L^2}. \tag{2.145}$$

Equivalently, we can introduce the *Weyl quantization* of the observable $\mathfrak{a}$ as a bounded operator $\mathrm{Op}^\varepsilon(\mathfrak{a}) : L^2 \to L^2$ given by the expression

$$\mathcal{F}\left(\mathrm{Op}^\varepsilon(\mathfrak{a})\psi\right)(k) = \int_{\mathbb{R}^d} \mathrm{d}p \widehat{\mathfrak{a}}\left(p, k - \frac{\varepsilon p}{2}\right) \widehat{\psi}(k - \varepsilon p), \tag{2.146}$$

which is defined for Lebesgue-almost all $k \in \mathbb{R}^d$. Then

$$\langle W^\varepsilon[\psi], \mathfrak{a}\rangle = \langle \psi, \mathrm{Op}^\varepsilon(a)\psi\rangle_{L^2}. \tag{2.147}$$

A large selection of different observables (symbols), quantization procedures and scaling limits are available in the literature on pseudodifferential operators. The above Weyl quantization is a reasonable choice because of its duality with the Wigner transform, and the fact that it maps real $\mathfrak{a} \in \mathcal{F}L^1(C^0)$ to self-adjoint operators on $L^2(\mathbb{R}^d)$, [33]. A popular choice of observables are various classes of $C^\infty(\mathbb{R}^{2d})$ functions with upper (and sometimes lower, to ensure ellipticity) bounds on the growth of the derivatives, [24, 33]. We will not need to require our observables to be smooth. Finally, the scale parameter ε can enter in different fashions. As $\varepsilon \to 0$, our observables spread out in position space

$$\mathrm{Op}^\varepsilon(\mathfrak{a}) = \mathrm{Op}^1\left(\mathfrak{a}(\varepsilon x, k)\right), \tag{2.148}$$

but other authors set

$$\widetilde{\mathrm{Op}^\varepsilon}(\mathfrak{a}) = \mathrm{Op}^1\left(\mathfrak{a}(x, \varepsilon k)\right) \tag{2.149}$$

to scale the momentum variable instead, [21, 33]. While these two definitions are unitary equivalent, the former is clearly the more intuitive one for the physical setting at hand.

Returning to $\mathfrak{a}$ from the Schwartz functions for a moment, the inequality (2.145) shows that for any $C \in [0, \infty)$, the set

$$\{W^\varepsilon[\psi] : \varepsilon > 0, \|\psi\|_{L^2} \le C\} \subset \mathcal{S}'(\mathbb{R}^{2d}) \tag{2.150}$$

is contained in a polar, and thus weak-* compact subset of $\mathcal{S}'(\mathbb{R}^{2d})$ by the Alaoglu-Bourbaki theorem, [26]. As $\mathcal{S}(\mathbb{R}^{2d})$ is separable, this is actually even *sequential* weak-* compactness. Given an L^2-bounded set $(\psi^\varepsilon)_{\varepsilon>0}$, it is then interesting to ask what kind of possible limits in $\mathcal{S}'(\mathbb{R}^{2d})$ the convergent subsequences of $(W^\varepsilon[\psi^\varepsilon])_{\varepsilon>0}$ may have as $\varepsilon \to 0$.

Lemma 2.15. *For each $\varepsilon > 0$ let a $\psi^\varepsilon \in L^2(\mathbb{R}^d)$ be given such that $\sup_{\varepsilon>0} \|\psi^\varepsilon\|_{L^2}^2 < \infty$. Then one can extract sub-sequences $(\varepsilon_n)_{n\in\mathbb{N}}$, $\varepsilon_n \to 0$, such that $W^{\varepsilon_n}[\psi^{\varepsilon_n}]$ converges weak-* in $\mathcal{S}'(\mathbb{R}^{2d})$, and all limit points are non-negative Borel measures μ on $\mathbb{R}^{2d}$ which are bounded by*

$$\int_{\mathbb{R}^{2d}} \mu(\mathrm{d}x, \mathrm{d}k) \le \limsup_{n\to\infty} \|\psi^{\varepsilon_n}\|_{L^2}^2. \tag{2.151}$$

Proof. This proof follows the one presented in [30], we reproduce a version of it here to account for the different scaling of our Wigner function. As we have already seen above that such sub-sequences (ε_n) exist, assume without loss of generality that $W^\varepsilon[\psi^\varepsilon]$ converges to a limit μ in $\mathcal{S}'(\mathbb{R}^{2d})$ already before selecting a particular subsequence. With the Kernel

$$G^\varepsilon(x,k) = 2^d \varepsilon^{-d} e^{-|x|^2/\varepsilon} e^{-|2\pi k|^2/\varepsilon}, \tag{2.152}$$

we define the Husimi transform $W_H^\varepsilon[\psi^\varepsilon] : \mathbb{R}^{2d} \to \mathbb{C}$ by

$$W_H^\varepsilon[\psi^\varepsilon] = W^\varepsilon[\psi^\varepsilon] * G^\varepsilon. \tag{2.153}$$

As $G^\varepsilon \in \mathcal{S}(\mathbb{R}^{2d})$ and $\psi^\varepsilon \in L^2(\mathbb{R}^d)$, it is not hard to verify that for all $(x,k) \in \mathbb{R}^{2d}$,

$$W_H^\varepsilon[\psi^\varepsilon](x,k) = \varepsilon^{-d} \left| \int_{\mathbb{R}^d} \mathrm{d}z \varepsilon^{-d/2} \psi^\varepsilon\left(\frac{z}{\varepsilon}\right) (\pi\varepsilon)^{-d/4} \exp\left(-\frac{|x-z|^2}{2\varepsilon}\right) \exp\left(-2\pi i \frac{k\cdot z}{\varepsilon}\right) \right|^2, \tag{2.154}$$

so $W_H^\varepsilon[\psi^\varepsilon]$ is non-negative, and for all $\varepsilon > 0$

$$\int_{\mathbb{R}^{2d}} \mathrm{d}x\mathrm{d}k W_H^\varepsilon[\psi^\varepsilon](x,k) = \|\psi^\varepsilon\|_{L^2}^2 . \tag{2.155}$$

On the other hand,

$$G^\varepsilon * \mathfrak{a} \to \mathfrak{a} \qquad (\varepsilon \to 0) \tag{2.156}$$

in the topology of $\mathcal{S}(\mathbb{R}^{2d})$ for all $\mathfrak{a} \in \mathcal{S}(\mathbb{R}^{2d})$, and, as we can control the action of $W^\varepsilon[\psi^\varepsilon]$ on $\mathcal{S}$ by (2.145) uniformly in ε,

$$|\langle W_H^\varepsilon[\psi^\varepsilon] - W^\varepsilon[\psi^\varepsilon], \mathfrak{a}\rangle| = |\langle W^\varepsilon[\psi^\varepsilon], G^\varepsilon * \mathfrak{a} - \mathfrak{a}\rangle| \to 0 \tag{2.157}$$

as $\varepsilon \to 0$. Accordingly, for every Schwartz function $\mathfrak{a} \geq 0$,

$$\langle \mu, \mathfrak{a}\rangle = \lim_{\varepsilon\to 0} \langle W^\varepsilon[\psi^\varepsilon], \mathfrak{a}\rangle = \lim_{\varepsilon\to 0} \langle W_H^\varepsilon[\psi^\varepsilon], \mathfrak{a}\rangle \geq 0, \tag{2.158}$$

so μ is a non-negative tempered distribution, thus a non-negative distribution, and thus a non-negative Borel measure on $\mathbb{R}^{2d}$, [36]. Furthermore, by (2.155)

$$\int_{\mathbb{R}^{2d}} \mu(\mathrm{d}x, \mathrm{d}k) \leq \overline{\lim_{\varepsilon\to 0}} \|\psi^\varepsilon\|_{L^2}^2 . \tag{2.159}$$

□

Note that the equality (2.155) produces only an upper bound (2.159) for the limit measure. This is because energy may be lost to infinity when passing to the $\varepsilon \to 0$ limit. To avoid this from happening, in addition to

$$\sup_{\varepsilon>0} \|\psi^\varepsilon\|_{L^2}^2 < \infty \qquad \text{(Bounded energy)}, \tag{2.160}$$

we introduce two tightness conditions,

$$\lim_{R\to\infty} \limsup_{\varepsilon\to 0} \int_{|x|>R/\varepsilon} \mathrm{d}x\, |\psi^\varepsilon(x)|^2 = 0 \qquad \text{(Tightness in scaled position space)}, \tag{2.161}$$

which is the analogue to initial condition (IC2) in [32], and

$$\lim_{R\to\infty}\limsup_{\varepsilon\to 0}\int_{|k|>R}\mathrm{d}k\left|\widehat{\psi}^{\varepsilon}(k)\right|^2=0 \qquad \text{(Tightness in momentum space)}, \tag{2.162}$$

which we have to add here as the momentum space is unbouded in our case. A useful tool to identify Wigner limit measures will then be the counterpart of equations (B.17) and (B.29) in [32],

Lemma 2.16. *For $L^2(\mathbb{R}^2)$ functions $(\psi^\varepsilon)_{\varepsilon>0}$ fulfilling (2.160-2.162), as well as $W^\varepsilon[\psi^\varepsilon]\to\mu$ in $\mathcal{S}'$ as $\varepsilon\to 0$, and any continuous, bounded function $f:\mathbb{R}^d\to\mathbb{C}$, we have for all $p\in\mathbb{R}^d$ that*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\mathrm{d}k\overline{\widehat{\psi}^{\varepsilon}(k+\varepsilon p/2)}\widehat{\psi}^{\varepsilon}(k-\varepsilon p/2)f(k)=\int_{\mathbb{R}^{2d}}\mu(\mathrm{d}x,\mathrm{d}k)e^{2\pi i p\cdot x}f(k), \tag{2.163}$$

the estimate (2.159) is sharp,

$$\int_{\mathbb{R}^{2d}}\mu(\mathrm{d}x,\mathrm{d}k)=\lim_{\varepsilon\to 0}\|\psi^\varepsilon\|_{L^2}^2\,, \tag{2.164}$$

and

$$\lim_{\varepsilon\to 0}\langle W^\varepsilon[\psi^\varepsilon],\mathfrak{a}\rangle=\langle\mu,\mathfrak{a}\rangle=\int_{\mathbb{R}^{2d}}\mu(\mathrm{d}x,\mathrm{d}k)\mathfrak{a}(x,k) \tag{2.165}$$

holds for all $\mathfrak{a}\in\mathcal{F}L^1(C^0)$.

Proof. First, assume f to be a Schwartz function. The map $\mathbb{R}^d\to\mathbb{C}$,

$$q\mapsto\int_{\mathbb{R}^d}\mathrm{d}k\overline{\widehat{\psi}^{\varepsilon}(k+\varepsilon q/2)}\widehat{\psi}^{\varepsilon}(k-\varepsilon q/2)f(k) \tag{2.166}$$

is bounded and continuous for all $\varepsilon>0$, so

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\mathrm{d}k\overline{\widehat{\psi}^{\varepsilon}(k+\varepsilon p/2)}\widehat{\psi}^{\varepsilon}(k-\varepsilon p/2)f(k)\\ &\qquad=\lim_{\varepsilon\to 0}\lim_{\lambda\to 0}(\sqrt{\pi}\lambda)^{-d}\int_{\mathbb{R}^d}\mathrm{d}q e^{-(p-q)^2/\lambda^2}\int_{\mathbb{R}^d}\mathrm{d}k\overline{\widehat{\psi}^{\varepsilon}(k+\varepsilon q/2)}\widehat{\psi}^{\varepsilon}(k-\varepsilon q/2)f(k)\end{aligned} \tag{2.167}$$

provided the limit of the right side exists. If one chooses for fixed $p\in\mathbb{R}^d$, $\lambda>0$ the function $a\in\mathcal{S}(\mathbb{R}^{2d})$ such that

$$\begin{aligned}\widehat{\mathfrak{a}}(q,k)&=(\sqrt{\pi}\lambda)^{-d}e^{-(p-q)^2/\lambda^2}f(k),\\ \mathfrak{a}(x,k)&=e^{2\pi i p\cdot x}e^{-\lambda^2\pi^2x^2}f(k),\end{aligned} \tag{2.168}$$

equation (2.143) implies

$$\begin{aligned}&(\sqrt{\pi}\lambda)^{-d}\int_{\mathbb{R}^d}\mathrm{d}q e^{-(p-q)^2/\lambda^2}\int_{\mathbb{R}^d}\mathrm{d}k\overline{\widehat{\psi}^{\varepsilon}(k+\varepsilon q/2)}\widehat{\psi}^{\varepsilon}(k-\varepsilon q/2)f(k)\\ &\qquad=\varepsilon^{-d}\int_{\mathbb{R}^{3d}}\mathrm{d}x\mathrm{d}k\mathrm{d}y e^{2\pi i p\cdot x}e^{-\lambda^2\pi^2x^2}f(k)\overline{\psi^\varepsilon\left(\frac{x}{\varepsilon}+\frac{y}{2}\right)}\psi^\varepsilon\left(\frac{x}{\varepsilon}-\frac{y}{2}\right)e^{2\pi i y\cdot k}\\ &\qquad=\varepsilon^{-d}\int_{\mathbb{R}^{2d}}\mathrm{d}x\mathrm{d}y e^{2\pi i p\cdot x}e^{-\lambda^2\pi^2x^2}\widehat{f}(-y)\overline{\psi^\varepsilon\left(\frac{x}{\varepsilon}+\frac{y}{2}\right)}\psi^\varepsilon\left(\frac{x}{\varepsilon}-\frac{y}{2}\right).\end{aligned} \tag{2.169}$$

We split up the last integral into the contributions of $|x| \leq R$ and $|x| > R$, and see that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\left|\varepsilon^{-d}\int_{\mathbb{R}^{2d}}\mathrm{d}x\mathrm{d}y e^{2\pi i p\cdot x}\left(1-e^{-\lambda^2\pi^2x^2}\right)\widehat{f}(-y)\overline{\psi^\varepsilon\left(\frac{x}{\varepsilon}+\frac{y}{2}\right)}\psi^\varepsilon\left(\frac{x}{\varepsilon}-\frac{y}{2}\right)\right| \\
&\leq \left(1-e^{-\lambda^2\pi^2R^2}\right)\left\|\widehat{f}\right\|_{L^1}\limsup_{\varepsilon\to 0}\|\psi^\varepsilon\|_{L^2}^2 + 2\left\|\widehat{f}\right\|_{L^1}\limsup_{\varepsilon\to 0}\|\psi^\varepsilon\|_{L^2}\left(\int_{|z|>R/\varepsilon}|\psi^\varepsilon(z)|^2\right)^{1/2},
\end{aligned} \tag{2.170}$$

and thus obtain from (2.160) and (2.161) that

$$\begin{aligned}
\lim_{\lambda\to 0}\limsup_{\varepsilon\to 0}\Bigg|&\int_{\mathbb{R}^d}\mathrm{d}k\overline{\widehat{\psi^\varepsilon}(k+\varepsilon p/2)}\widehat{\psi^\varepsilon}(k-\varepsilon p/2)f(k) \\
&-(\sqrt{\pi}\lambda)^{-d}\int_{\mathbb{R}^d}\mathrm{d}q e^{-(p-q)^2/\lambda^2}\int_{\mathbb{R}^d}\mathrm{d}k\overline{\widehat{\psi^\varepsilon}(k+\varepsilon q/2)}\widehat{\psi^\varepsilon}(k-\varepsilon q/2)f(k)\Bigg| = 0,
\end{aligned} \tag{2.171}$$

and therefore

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\mathrm{d}k\overline{\widehat{\psi^\varepsilon}(k+\varepsilon p/2)}\widehat{\psi^\varepsilon}(k-\varepsilon p/2)f(k) \\
&=\lim_{\lambda\to 0}\lim_{\varepsilon\to 0}\varepsilon^{-d}\int_{\mathbb{R}^{3d}}\mathrm{d}x\mathrm{d}k\mathrm{d}y e^{2\pi i p\cdot x}e^{-\lambda^2\pi^2x^2}f(k)\overline{\psi^\varepsilon\left(\frac{x}{\varepsilon}+\frac{y}{2}\right)}\psi^\varepsilon\left(\frac{x}{\varepsilon}-\frac{y}{2}\right)e^{2\pi i y\cdot k} \\
&=\lim_{\lambda\to 0}\int_{\mathbb{R}^{2d}}\mu(\mathrm{d}x,\mathrm{d}k)e^{2\pi i p\cdot x}e^{-\lambda^2\pi^2x^2}f(k) \\
&=\int_{\mathbb{R}^{2d}}\mu(\mathrm{d}x,\mathrm{d}k)e^{2\pi i p\cdot x}f(k).
\end{aligned} \tag{2.172}$$

By (2.162), this generalizes to all bounded and continuous f. In particular, for $f \equiv 1$ and $p = 0$,

$$\int_{\mathbb{R}^{2d}}\mu(\mathrm{d}x,\mathrm{d}k) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\mathrm{d}k\left|\widehat{\psi^\varepsilon}(k)\right|^2 = \lim_{\varepsilon\to 0}\|\psi^\varepsilon\|_{L^2}^2. \tag{2.173}$$

For all $\mathfrak{a} \in \mathcal{F}L^1(C^0)$, (2.165) follows from (2.163) by dominated convergence in the last line of (2.143). □

2.4.2. Examples of Wigner limit measures

Standard examples for sequences of initial states $(\psi^\varepsilon)_{\varepsilon>0}$ with convergent Wigner transforms can be found, for example, in [30]. One can construct sequences which concentrate at a single point in momentum space,

$$\begin{aligned}
\psi^\varepsilon(x) &= \varepsilon^{d/2}f(\varepsilon x)e^{2\pi i k_0\cdot x}, && f\in L^2(\mathbb{R}^d), k_0\in\mathbb{R}^d, \\
\lim_{\varepsilon\to 0}\langle W^\varepsilon[\psi^\varepsilon],\mathfrak{a}\rangle &= \int_{\mathbb{R}^d}\mathrm{d}x\mathfrak{a}(x,k_0)\,|f(x)|^2 && \forall\mathfrak{a}\in\mathcal{F}L^1(C^0), \\
\mu &= |f(x)|^2\,\delta(k-k_0),
\end{aligned} \tag{2.174}$$

in position space,

$$\begin{aligned} \psi^\varepsilon(x) &= f(x - x_0/\varepsilon), \qquad f \in L^2(\mathbb{R}^d), x_0 \in \mathbb{R}^d, \\ \lim_{\varepsilon\to 0} \langle W^\varepsilon[\psi^\varepsilon], \mathfrak{a}\rangle &= \int_{\mathbb{R}^d} \mathrm{d}k \mathfrak{a}(x_0, k) \left|\widehat{f}(k)\right|^2 \qquad \forall \mathfrak{a} \in \mathcal{F}L^1(C^0), \\ \mu &= \delta(x - x_0) \left|\widehat{f}(k)\right|^2, \end{aligned} \tag{2.175}$$

or even in both variables,

$$\begin{aligned} \psi^\varepsilon(x) &= \varepsilon^{d/4} f(\sqrt{\varepsilon}x - x_0/\sqrt{\varepsilon}) e^{2\pi i x\cdot k_0}, \qquad f \in L^2(\mathbb{R}^d), x_0, k_0 \in \mathbb{R}^d, \\ \lim_{\varepsilon\to 0} \langle W^\varepsilon[\psi^\varepsilon], \mathfrak{a}\rangle &= \|f\|_{L^2}^2 \, \mathfrak{a}(x_0, k_0) \qquad \forall \mathfrak{a} \in \mathcal{F}L^1(C^0), \\ \mu &= \|f\|_{L^2}^2 \, \delta(x - x_0)\delta(k - k_0). \end{aligned} \tag{2.176}$$

A prominent case that has been studied in the context of random Schrödinger equations ([16], or [8, 9] for the discrete setting) are WKB states

$$\psi^\varepsilon(x) = \varepsilon^{d/2} f(\varepsilon x) e^{2\pi i S(\varepsilon x)/\varepsilon}, \tag{2.177}$$

with $f : \mathbb{R}^d \to \mathbb{C}$ and $S : \mathbb{R}^d \to \mathbb{R}$ Schwartz functions, for which one has

$$\begin{aligned} \lim_{\varepsilon\to 0} \langle W^\varepsilon[\psi^\varepsilon], \mathfrak{a}\rangle &= \int_{\mathbb{R}^d} \mathrm{d}x \mathfrak{a}(x, \nabla S(x)) \, |f(x)|^2 \qquad \forall \mathfrak{a} \in \mathcal{F}L^1(C^0), \\ \mu &= |f(x)|^2 \, \delta(k - \nabla S(x)). \end{aligned} \tag{2.178}$$

The limit Wigner measures of all the aforementioned examples are singular with respect to Lebesgue measure; this, however, need not be the case in general. With Schwartz functions $f_1, f_2 \in \mathcal{S}(\mathbb{R}^d)$, set

$$\psi^\varepsilon(x) = \varepsilon^{d/4} \sum_{y\in\mathbb{Z}^d} f_1\left(\sqrt{\varepsilon}y\right) f_2\left(x - \frac{y}{\sqrt{\varepsilon}}\right). \tag{2.179}$$

It is easy to show that the Wigner limit measure exists and has a density $|f_1(x)|^2 \left|\widehat{f_2}(k)\right|^2$ on phase space,

$$\begin{aligned} \lim_{\varepsilon\to 0} \langle W^\varepsilon[\psi^\varepsilon], \mathfrak{a}\rangle &= \int_{\mathbb{R}^{2d}} \mathrm{d}x\mathrm{d}k \mathfrak{a}(x, k) |f_1(x)|^2 \left|\widehat{f_2}(k)\right|^2 \qquad \forall \mathfrak{a} \in \mathcal{F}L^1(C^0), \\ \mu &= |f_1(x)|^2 \left|\widehat{f_2}(k)\right|^2. \end{aligned} \tag{2.180}$$

It is worth mentioning that all sequences $(\psi^\varepsilon)_{\varepsilon>0}$ in the examples above have bounded energy (2.160) and fulfill position and momentum tightness (2.161-2.162). In all five examples, at least after approximating $f \in L^2(\mathbb{R}^d)$ in (2.174-2.176) by Schwartz functions, the additional condition (3.6) that will be required to obtain Theorem 3.3 also holds for sufficiently small $\alpha_0 > 0$.

2.4.3. Higher resolution near the acoustic singularity

To have observables available that provide higher resolution in the neighborhood of $k = 0$, we define in analogy to Definition 2.4 in [23] the following generalization of $\mathcal{F}L^1(C^0)$

Definition 2.2. A function $\mathfrak{a} : \mathbb{R}^d_x \times \mathbb{R}^d_k \times \mathbb{R}^d_{\underline{k}} \to \mathbb{C}$ has an *admissible infra-red continuation* and we write $\mathfrak{a} \in \mathfrak{X}_{\mathrm{IR}}$, if $\widehat{\mathfrak{a}}$, the Fourier transform of $\mathfrak{a}$ in the x variable, fulfills the following

- For all $p \in \mathbb{R}^d$, the map $(k, \underline{k}) \mapsto \widehat{\mathfrak{a}}(p, k, \underline{k})$ is continuous.
- The norm
$$\|\mathfrak{a}\|_{\mathfrak{X}_{\mathrm{IR}}} = \int_{\mathbb{R}^d} \mathrm{d}p \sup_{k,\underline{k} \in \mathbb{R}^d} |\widehat{\mathfrak{a}}(p, k, \underline{k})| \tag{2.181}$$
is finite.
- There exists a function $\mathfrak{b} : \mathbb{R}^d \times \mathbb{R}^d \times S^{d-1} \to \mathbb{C}$ such that
$$\lim_{R\to\infty} \int_{\mathbb{R}^d} \mathrm{d}p \sup_{\substack{k\in\mathbb{R}^d \\ |\underline{k}| \geq R}} \left| \widehat{\mathfrak{a}}(p, k, \underline{k}) - \widehat{\mathfrak{b}}(p, k, \underline{k}/|\underline{k}|) \right| = 0. \tag{2.182}$$

For $\psi \in L^2(\mathbb{R}^d)$ and $\varepsilon > 0$, the Wigner transform $W^\varepsilon[\psi]$ constitutes a bounded linear functional on $\mathfrak{X}_{\mathrm{IR}}$ by setting

$$\begin{aligned} \langle W^\varepsilon[\psi], \mathfrak{a} \rangle_{\mathfrak{X}_{\mathrm{IR}}} &= \int_{\mathbb{R}^d} \mathrm{d}p \left(\int_{\mathbb{R}^d} \mathrm{d}k \widehat{\mathfrak{a}} \left(p, k, \frac{k}{\varepsilon} \right) \overline{\widehat{\psi} \left(k + \frac{\varepsilon p}{2} \right)} \widehat{\psi} \left(k - \frac{\varepsilon p}{2} \right) \right) \\ &= \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}k W^\varepsilon[\psi](x, k) \mathfrak{a} \left(x, k, \frac{k}{\varepsilon} \right), \end{aligned} \tag{2.183}$$

with only the first line holding for all $\mathfrak{a} \in \mathfrak{X}_{\mathrm{IR}}$, while the second line is the more intuitive formulation for nice enough $\mathfrak{a}$. Analogous to (2.146), one can introduce a bounded $L^2 \to L^2$ operator by

$$\mathcal{F} \left(\mathrm{Op}^\varepsilon_{\mathrm{IR}}(\mathfrak{a}) \psi \right)(k) = \int_{\mathbb{R}^d} \mathrm{d}p \widehat{\mathfrak{a}} \left(p, k - \frac{\varepsilon p}{2}, \frac{k}{\varepsilon} - \frac{p}{2} \right) \widehat{\psi}(k - \varepsilon p), \tag{2.184}$$

to obtain

$$\langle W^\varepsilon[\psi], \mathfrak{a} \rangle_{\mathfrak{X}_{\mathrm{IR}}} = \langle \psi, \mathrm{Op}^\varepsilon_{\mathrm{IR}}(\mathfrak{a}) \psi \rangle. \tag{2.185}$$

Following the reasoning of [23], the Lemmas 2.15 and 2.16 can be generalized to the setting at hand, as shown in Appendix A.

Lemma 2.17. *For each $\varepsilon > 0$ let a $\psi^\varepsilon \in L^2(\mathbb{R}^d)$ be given such that (2.160), (2.161) and (2.162) hold. Then one can extract sub-sequences $(\varepsilon_n)_{n\in\mathbb{N}}$, $\varepsilon_n \to 0$, such that $W^{\varepsilon_n}[\psi^{\varepsilon_n}]$ converges weak-* in ${\mathfrak{X}_{\mathrm{IR}}}^*$. For each convergent subsequence, the limit is of the form*

$$\begin{aligned} \lim_{n\to\infty} \langle W^{\varepsilon_n}[\psi^{\varepsilon_n}], \mathfrak{a} \rangle_{\mathfrak{X}_{\mathrm{IR}}} = &\int_{\mathbb{R}^d \times \mathbb{R}^d_*} \mu(\mathrm{d}x, \mathrm{d}k) \mathfrak{b} \left(x, k, \frac{k}{|k|} \right) \\ &+ \int_{\mathbb{R}^d \times S^{d-1}} \mu^{\mathrm{H}}(\mathrm{d}x, \mathrm{d}\underline{k}) \mathfrak{b}(x, 0, \underline{k}) \\ &+ \langle W[\eta], \mathfrak{a}(\cdot, 0, \cdot) \rangle. \end{aligned} \tag{2.186}$$

with μ, μ^{H} non-negative, bounded Borel measures on $\mathbb{R}^d \times \mathbb{R}^d_*$ and $\mathbb{R}^d \times S^{d-1}$, respectively, $\eta \in L^2(\mathbb{R}^d)$, and $\mathfrak{b}$ associated to $\mathfrak{a}$ by Definition 2.2. In the last line of (2.186), $W[\eta]$ is the unscaled Wigner transform of η as defined in (2.141), tested against the $\mathcal{F}L^1(C^0)$ function $(x, \underline{k}) \mapsto \mathfrak{a}(x, 0, \underline{k})$.

From Lemma 5.2, 5.4 one can directly conclude that the L^2 norm is conserved while taking the limit along any of the subsequences from Lemma 2.17,

$$\lim_{n\to\infty} \|\psi^{\varepsilon_n}\|^2_{L^2} = \mu\left(\mathbb{R}^d \times \mathbb{R}^d_*\right) + \mu^{\mathrm{H}}\left(\mathbb{R}^d \times S^{d-1}\right) + \|\eta\|^2_{L^2}\,. \tag{2.187}$$

As will become obvious in Appendix A, the μ component of the limit object $(\mu, \mu^{\mathrm{H}}, \eta)$ accounts for the part of the energy distributed over wavenumbers that stay of order 1 while $\varepsilon_n \to 0$, η represents the portion with macroscopic wavelengths, i.e. with wavenumbers vanishing like ε_n. The defect measure μ^{H} stands for the energy stored in wavenumbers much larger than ε_n but much smaller than 1.

The right side of (2.186) motivates the definition of the following functions.

$$\mathfrak{a}^{\mathrm{micro}}(x, k) = \mathfrak{b}\left(x, k, \frac{k}{|k|}\right) \tag{2.188}$$

is a bounded, continuous function of $x \in \mathbb{R}^d$ and $k \in \mathbb{R}^d \setminus \{0\}$, while

$$\mathfrak{a}^{\mathrm{meso}}(x, \underline{k}) = \mathfrak{b}\left(x, 0, \underline{k}\right) \tag{2.189}$$

is in $C_0\left(\mathbb{R}^d_x \times S^{d-1}_{\underline{k}}\right)$, i.e. a bounded, continuous function on $\mathbb{R}^d_x \times S^{d-1}_{\underline{k}}$ that vanishes as $|x| \to \infty$, and

$$\mathfrak{a}^{\mathrm{macro}}(x, \underline{k}) = \mathfrak{a}\left(x, 0, \underline{k}\right), \tag{2.190}$$

with $(x, \underline{k}) \in \mathbb{R}^d$, is a function in $\mathcal{F}L^1(C^0)$. For $\mathfrak{a}^{\mathrm{meso}} \in C_0\left(\mathbb{R}^d_x \times S^{d-1}_{\underline{k}}\right)$, $\sigma \in \{\pm\}$ and $t \in \mathbb{R}$, we set

$$\left(\mathrm{e}^{\mathcal{L}^{\mathrm{H}}_\sigma t}\mathfrak{a}^{\mathrm{meso}}\right)(x, \underline{k}) = \mathfrak{a}^{\mathrm{meso}}(x + \sigma \underline{k} t, \underline{k}), \tag{2.191}$$

which constitutes a strongly continuous group of operators on $C_0\left(\mathbb{R}^d_x \times S^{d-1}_{\underline{k}}\right)$.

Example. The arguably simplemost sequence of states $(\psi^\varepsilon)_{\varepsilon>0}$ with $W^\varepsilon\left[\psi^\varepsilon\right]$ converging in $\mathfrak{X}_{\mathrm{IR}}{}^*$ is

$$\psi^\varepsilon(x) = f(x) + \varepsilon^{d/4} g(\sqrt{\varepsilon} x) + \varepsilon^{d/2} h(\varepsilon x) \tag{2.192}$$

with $f, g, h \in L^2(\mathbb{R}^d)$. Then

$$W^\varepsilon\left[\psi^\varepsilon\right] \rightharpoonup (\mu, \mu^{\mathrm{H}}, \eta) \tag{2.193}$$

with μ a measure on $\mathbb{R}^d \times \mathbb{R}^d_*$ given by

$$\mu(\mathrm{d}x, \mathrm{d}k) = \delta(x)\left|\widehat{f}(k)\right|^2 \mathrm{d}x\mathrm{d}k, \tag{2.194}$$

the measure μ^{H} defined for $x \in \mathbb{R}^d$ and $\underline{k} \in S^{d-1}$ by

$$\mu^{\mathrm{H}}(\mathrm{d}x, \mathrm{d}\underline{k}) = \delta(x)\left(\int_0^\infty \mathrm{d}r r^{d-1}\left|\widehat{g}(r\underline{k})\right|^2\right)\mathrm{d}x\mathrm{d}\underline{k} \tag{2.195}$$

and

$$\eta = h. \tag{2.196}$$

2.4.4. Operators on $\mathcal{H}$

The operators $\mathrm{Op}^\varepsilon(\mathfrak{a})$, $\mathfrak{a} \in \mathcal{F}L^1(C^0)$ and $\mathrm{Op}^\varepsilon_{\mathrm{IR}}(\mathfrak{a})$, $\mathfrak{a} \in \mathfrak{X}_{\mathrm{IR}}$ introduced previously are bounded linear maps $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. To accommodate for the two-component structure of $\mathcal{H} = L^2\left(\mathbb{R}^d; \mathbb{C}^2\right)$, define the projections $P_{\sigma_2\sigma_1} : \mathcal{H} \to \mathcal{H}$ by setting

$$(P_{\sigma_2\sigma_1}\psi)_\sigma = \left(P_{\sigma_2\sigma_1}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}\right)_\sigma = \delta_{\sigma\sigma_2}\psi_{\sigma_1} \tag{2.197}$$

for all $\sigma_1, \sigma_2, \sigma \in \{\pm\}$ and all $\psi \in \mathcal{H}$. For $\mathfrak{a} = (\mathfrak{a}_+, \mathfrak{a}_-) : \mathbb{R}^{2d} \to \mathbb{C}^2$ such that $\mathfrak{a}_\sigma \in \mathcal{F}L^1(C^0)$ for $\sigma \in \{+,-\}$ one can thus define the bounded operator

$$Q^\varepsilon(\mathfrak{a}) = \sum_{\sigma\in\{\pm\}} \mathrm{Op}^\varepsilon\left(\mathfrak{a}_\sigma\right) P_{\sigma\sigma} = \begin{pmatrix}\mathrm{Op}^\varepsilon\left(\mathfrak{a}_+\right) & 0\\ 0 & \mathrm{Op}^\varepsilon\left(\mathfrak{a}_-\right)\end{pmatrix} \tag{2.198}$$

on $\mathcal{H}$. Analogously, for $\mathfrak{a} = (\mathfrak{a}_+, \mathfrak{a}_-) : \mathbb{R}^{3d} \to \mathbb{C}^2$ such that $\mathfrak{a}_\sigma \in \mathfrak{X}_{\mathrm{IR}}$ for $\sigma \in \{+,-\}$, we set

$$Q^\varepsilon_{\mathrm{IR}}(\mathfrak{a}) = \sum_{\sigma\in\{\pm\}} \mathrm{Op}^\varepsilon_{\mathrm{IR}}\left(\mathfrak{a}_\sigma\right) P_{\sigma\sigma} = \begin{pmatrix}\mathrm{Op}^\varepsilon_{\mathrm{IR}}\left(\mathfrak{a}_+\right) & 0\\ 0 & \mathrm{Op}^\varepsilon_{\mathrm{IR}}\left(\mathfrak{a}_-\right)\end{pmatrix}. \tag{2.199}$$

Both $Q^\varepsilon(\mathfrak{a})$ and $Q^\varepsilon_{\mathrm{IR}}(\mathfrak{a})$ are defined as diagonal with respect to the two-component structure of $\mathcal{H}$, i.e. they do not mix the ψ_+ and ψ_- components. Observables with off-diagonal components will be discussed separately in Appendix F.

In terms of Wigner transforms, our focus on diagonal observables implies that we only consider

$$W^\varepsilon[\psi_\sigma](x,k) = \varepsilon^{-d}\int_{\mathbb{R}^d} \mathrm{d}y \overline{\psi_\sigma\left(\frac{x}{\varepsilon} + \frac{y}{2}\right)}\psi_\sigma\left(\frac{x}{\varepsilon} - \frac{y}{2}\right)e^{2\pi i y\cdot k}, \tag{2.200}$$

with $\sigma \in \{\pm\}$ and exclude "cross-terms" of the form

$$W^\varepsilon[\psi_{\sigma_1}, \psi_{\sigma_2}](x,k) = \varepsilon^{-d}\int_{\mathbb{R}^d} \mathrm{d}y \overline{\psi_{\sigma_1}\left(\frac{x}{\varepsilon} + \frac{y}{2}\right)}\psi_{\sigma_2}\left(\frac{x}{\varepsilon} - \frac{y}{2}\right)e^{2\pi i y\cdot k} \tag{2.201}$$

with $\sigma_1 \neq \sigma_2$ for now.

3. Main theorems

Now, we have all definitions available to state our main results. By the duality (2.147), (2.185) of Wigner transform and Weyl quantization, both formulations can be applied equally well to describe a measurement of the propagated wave at a macroscopic time $T > 0$ with observables $\mathfrak{a}_+, \mathfrak{a}_- \in \mathfrak{X}_{\mathrm{IR}}$,

$$\left\langle \mathrm{e}^{-iH^\varepsilon T/\varepsilon} \psi_0^\varepsilon, Q_{\mathrm{IR}}^\varepsilon(\mathfrak{a}) \mathrm{e}^{-iH^\varepsilon T/\varepsilon} \psi_0^\varepsilon \right\rangle_{\mathcal{H}} = \sum_{\sigma \in \{\pm\}} \left\langle W^\varepsilon \left[\left(\mathrm{e}^{-iH^\varepsilon T/\varepsilon} \psi_0^\varepsilon \right)_\sigma \right], \mathfrak{a}_\sigma \right\rangle_{\mathfrak{X}_{\mathrm{IR}}} . \tag{3.1}$$

However, for multiple measurements at times $T^{(1)}$, $T^{(1)} + T^{(2)}$, up to $T^{(1)} + \ldots + T^{(\overline{m})}$, $\overline{m}$ being the number of measurements, the use of Weyl quantizations of the observables, i.e. of operators $Q_{\mathrm{IR}}^\varepsilon(\mathfrak{a}_j)$, $j = 1, \ldots, \overline{m}$, is much more convenient. This is how we will present our first theorem.

Theorem 3.1. *For $d \geq 2$, choose a random medium of class $(d + 185, 4)$. Let $(\psi_0^\varepsilon)_{\varepsilon > 0}$ be a sequence of initial states in $\mathcal{H}$ with components fulfilling the boundedness and tightness conditions (2.160-2.162), and assume that the Wigner transforms $W^\varepsilon \left[\psi_{0,\sigma}^\varepsilon \right]$, $\sigma \in \{\pm\}$ converge, in the sense of (2.186), to $(\mu_{0,\sigma}, \mu_{0,\sigma}^{\mathrm{H}}, \eta_{0,\sigma})$. Then for all $\overline{m} \in \mathbb{N}$, all*

$T^{(1)},...,T^{(\overline{m})}>0$ and all $\mathfrak{a}_{j,\sigma}\in\mathfrak{X}_{\mathrm{IR}}$, ($j\in\{1,...,\overline{m}\}$, $\sigma\in\{\pm\}$), we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\mathbb{E}\left[\left\langle \mathrm{e}^{-iH^\varepsilon T^{(\overline{m})}/\varepsilon}\prod_{j=1}^{\overline{m}-1}\left(Q^\varepsilon_{\mathrm{IR}}(\mathfrak{a}_j)\mathrm{e}^{-iH^\varepsilon T^{(j)}/\varepsilon}\right)\psi_0^\varepsilon,\right.\right.\\
&\qquad\qquad\left.\left. Q^\varepsilon_{\mathrm{IR}}(\mathfrak{a}_{\overline{m}})\mathrm{e}^{-iH^\varepsilon T^{(\overline{m})}/\varepsilon}\prod_{j=1}^{\overline{m}-1}\left(Q^\varepsilon_{\mathrm{IR}}(\mathfrak{a}_j)\mathrm{e}^{-iH^\varepsilon T^{(j)}/\varepsilon}\right)\psi_0^\varepsilon\right\rangle_{\mathcal{H}}\right]\\
&=\sum_{\sigma\in\{\pm\}}\int_{\mathbb{R}^d\times\mathbb{R}^d_*}\mu_{0,\sigma}(\mathrm{d}x,\mathrm{d}k)\\
&\qquad\left[e^{\mathcal{L}_\sigma T^{(1)}}\left(\left|\mathfrak{a}^{\mathrm{micro}}_{1,\sigma}\right|^2...e^{\mathcal{L}_\sigma T^{(\overline{m}-1)}}\left(\left|\mathfrak{a}^{\mathrm{micro}}_{\overline{m}-1,\sigma}\right|^2\left(e^{\mathcal{L}_\sigma T^{(\overline{m})}}\mathfrak{a}^{\mathrm{micro}}_{\overline{m},\sigma}\right)\right)\right)\right](x,k)\\
&\quad+\sum_{\sigma\in\{\pm\}}\int_{\mathbb{R}^d\times S^{d-1}}\mu^{\mathrm{H}}_{0,\sigma}(\mathrm{d}x,\mathrm{d}\underline{k})\\
&\qquad\left[e^{\mathcal{L}^{\mathrm{H}}_\sigma T^{(1)}}\left(\left|\mathfrak{a}^{\mathrm{meso}}_{1,\sigma}\right|^2...e^{\mathcal{L}^{\mathrm{H}}_\sigma T^{(\overline{m}-1)}}\left(\left|\mathfrak{a}^{\mathrm{meso}}_{\overline{m}-1,\sigma}\right|^2\left(e^{\mathcal{L}^{\mathrm{H}}_\sigma T^{(\overline{m})}}\mathfrak{a}^{\mathrm{meso}}_{\overline{m},\sigma}\right)\right)\right)\right](x,\underline{k})\\
&\quad+\left\langle \mathrm{e}^{-iH_0T^{(\overline{m})}}\prod_{j=1}^{\overline{m}-1}\left(Q^1(\mathfrak{a}^{\mathrm{macro}}_j)\mathrm{e}^{-iH_0T^{(j)}}\right)\eta_0,\right.\\
&\qquad\qquad\left. Q^1(\mathfrak{a}^{\mathrm{macro}}_{\overline{m}})\mathrm{e}^{-iH_0T^{(\overline{m})}}\prod_{j=1}^{\overline{m}-1}\left(Q^1(\mathfrak{a}^{\mathrm{macro}}_j)\mathrm{e}^{-iH_0T^{(j)}}\right)\eta_0\right\rangle_{\mathcal{H}}
\end{aligned}\tag{3.2}$$

with the objects on the right hand side defined in Sections 2.3 and 2.4.

Theorem 3.1 will be shown in Chapter 4. The limit can be understood as follows. In the first summand, the initial Wigner limit measure belonging to the microscopic wavelengths is propagated by the linear Boltzmann equation for a time $T^{(1)}$ and then multiplied by the first observable $\left|\mathfrak{a}^{\mathrm{micro}}_{1,\sigma}\right|^2$. The resulting measure continues to propagate for a time $T^{(2)}$, then picks up another observable and so on, until the last measurement with $\mathfrak{a}^{\mathrm{micro}}_{\overline{m},\sigma}$ is made after $\overline{m}$ steps. Only this part of the dynamics depends on the distribution of the random field ξ, namely by the definition of $\mathcal{L}_\sigma$.

The structure of the second summand is similar. However, as it accounts for the behavior of the mesoscopic wavelenghts much larger than the correlation length of ξ (but much shorter than the kinetic observation scale), the influence of ξ on the wave motion has completely vanished in the limit (note that the generator $\mathcal{L}^{\mathrm{H}}_\sigma$ does not depend on the distribution of ξ).

Finally, the third summand describes the propagation of the portion of the wave which exhibits wavelengths still resolvable on the observation scale. Clearly the wave nature of its dynamics is fully conserved in the limit, and only governed by the unperturbed Hamiltonian H_0.

In a first step towards self-averaging, one can then also show that the same object we have considered in Theorem 3.1 also has vanishing variance. This will require slightly better properties of the random medium, but we can relax the spacial tightness condition (2.161) for the initial states, as the following theorem does not require the existence of an initial limit Wigner measure.

Theorem 3.2. *For $d \geq 2$, choose a random field ξ of class $(d+521, 4)$ and let $(\psi_0^\varepsilon)_{\varepsilon>0}$ be a sequence of initial states in $\mathcal{H}$ fulfilling (2.160) and (2.162). Then for all $\overline{m} \in \mathbb{N}$, all $T^{(1)}, ..., T^{(\overline{m})} > 0$ and all $\mathfrak{a}_{j,\sigma} \in \mathfrak{X}_{\mathrm{IR}}$, ($j \in \{1, ..., \overline{m}\}$, $\sigma \in \{\pm\}$), we have*

$$\begin{aligned}\lim_{\varepsilon\to 0} \mathrm{Var}\Bigg(\Bigg\langle \mathrm{e}^{-iH^\varepsilon T^{(\overline{m})}/\varepsilon} \prod_{j=1}^{\overline{m}-1} \left(Q^\varepsilon_{\mathrm{IR}}(\mathfrak{a}_j)\mathrm{e}^{-iH^\varepsilon T^{(j)}/\varepsilon}\right)\psi_0^\varepsilon,\\ Q^\varepsilon_{\mathrm{IR}}(\mathfrak{a}_{\overline{m}})\mathrm{e}^{-iH^\varepsilon T^{(\overline{m})}/\varepsilon} \prod_{j=1}^{\overline{m}-1} \left(Q^\varepsilon_{\mathrm{IR}}(\mathfrak{a}_j)\mathrm{e}^{-iH^\varepsilon T^{(j)}/\varepsilon}\right)\psi_0^\varepsilon \Bigg\rangle_{\mathcal{H}}\Bigg) = 0.\end{aligned} \tag{3.3}$$

A proof of Theorem 3.2 is provided in Section 5.4.

Vanishing variance, together with convergence of the expectation from Theorem 3.1, already shows convergence in probability (and almost sure convergence along a subsequence) to the limit object on the right side of equation (3.2). For the general result on almost sure convergence presented in Theorem 3.3 below, we have to slightly improve the differentiability and decorrelation conditions for the field ξ. Moreover, at one point in the proof of Theorem 3.3, we will require a deterministic control (instead of the usual bounds on moments) of the disordered dynamics generated by H^ε. To do so, it will be necessary to assume the existence of an almost surely finite random variable $Y \geq 0$ such that

$$|\xi(x)| + |\nabla\xi(x)| \leq Y(1 + |x|) \tag{3.4}$$

for all $x \in \mathbb{R}^d$. This is a very mild requirement that in particular holds for the two examples, the Poisson bumps and the cut-off Gaussian field from Sections 2.1.2 and 2.1.3.

To be able to focus on the main ideas in the proof, given in Section 6.2, we will now limit ourselves to only one measurement, $\overline{m} = 1$, and avoid the acoustic singularity altogether by not allowing the initial states to concentrate near the origin,

$$\lim_{\lambda\to 0} \limsup_{\varepsilon\to 0} \sum_{\sigma\in\{\pm\}} \int_{|k|<\lambda} \mathrm{d}k \left|\widehat{\psi}^\varepsilon_{0,\sigma}(k)\right|^2 = 0. \tag{3.5}$$

Theorem 3.3. *Let $d \geq 2$ and the medium be of class $(d+1641, 4)$, with growth at infinity controlled by (3.4). and let $(\psi_0^\varepsilon)_{\varepsilon>0}$ be a sequence of initial states in $\mathcal{H}$ such that (2.160-2.162), and (3.5) hold. Furthermore, let there be non-negative, bounded Borel measures $\mu_{0,+}$ and $\mu_{0,-}$ on $\mathbb{R}^{2d}$ such that $W^\varepsilon\left[(\psi_0^\varepsilon)_\sigma\right]$ converges weak-* to $\mu_{0,\sigma}$ in $\mathcal{F}L^1(C^0)^*$, in the sense of (2.165). Assume finally that there is an $\alpha_0 > 0$ such that*

$$\lim_{n\to\infty} \sup_{\varepsilon\in\left[(n+1)^{-\alpha_0}, n^{-\alpha_0}\right]} \left\|\psi_0^\varepsilon - \psi_0^{n^{-\alpha_0}}\right\|_{\mathcal{H}} = 0. \tag{3.6}$$

Then there exists a set of "bad medium configurations" N_{ex} of vanishing probability, $\mathbb{P}(N_{\mathrm{ex}}) = 0$, such that for all realizations of the medium ξ_ω with $\omega \in \Omega \setminus N_{\mathrm{ex}}$, all $T > 0$ and all $\mathfrak{a}_\sigma \in \mathcal{F}L^1(C^0)$, $\sigma \in \{\pm\}$

$$\lim_{\varepsilon\to 0} \sup_{\tau\in[0,T]} \left| \left\langle W^\varepsilon \left[\left(\mathrm{e}^{-iH^\varepsilon_\omega \tau/\varepsilon} \psi^\varepsilon_0 \right)_\sigma \right], \mathfrak{a}_\sigma \right\rangle - \langle \mu_{\tau,\sigma}, \mathfrak{a}_\sigma \rangle \right| = 0, \tag{3.7}$$

with the measure $\mu_{\tau,\sigma} = \left(\mathrm{e}^{\mathcal{L}_\sigma \tau}\right)^* \mu_{0,\sigma}$ obtained from propagating $\mu_{0,\sigma}$ with the linear Boltzmann equation (2.137).

In the statement of Theorem 3.3 there is only one condition that is substantially stronger compared to Theorems 3.1 and 3.2, namely (3.6). Note that this requirement does not exclude a special type of initial states, but rather prevents the whole sequence of initial states from running through too large a part of $\mathcal{H}$. Given almost any configuration of the random medium, an initial state can be chosen such as to produce large deviations in (3.7) for exactly this choice of disorder. Although deterministic, i.e. independent of the random medium, a wild enough sequence $(\psi^\varepsilon_0)_{\varepsilon>0}$ can still turn *almost every* medium configuration into a "bad" one. An explicit example in which (3.6) and consequently the assertion of Theorem 3.3 fails is constructed in Section 6.3. However, as observed at the end of Section 2.4.2, condition (3.7) can easily be checked to hold true for all "standard" examples of initial states $(\psi^\varepsilon_0)_{\varepsilon>0}$, in particular WKB states, which have been at the center of attention in much of the related literature [8, 9, 16].

It seems appropriate to make a short remark concerning the conditions under which the above theorems hold. While the assumptions on the initial states are fairly natural, the extremely high differentiability for the random medium is somewhat unsatisfactory. It goes without saying that the existence of thousands of derivatives is not a necessary condition for the Boltzmann equation to emerge in the kinetic limit. The required medium smoothness could have been considerably reduced by noting that the number of the "worst-case graphs" often grows much slower than the overall combinatorical terms; this, however, would have made a much more detailed analysis and classification of the graphs unavoidable. Another rather wasteful choice is the use of the same stopping procedure for the Duhamel equation both for the G^{rough}, A^{rough} and the G^{end}, A^{end} case in Lemma 2.14. With these and a few other changes, the proofs could probably be tweaked to permit for a random medium that is only twenty or thirty times differentiable, but at the price of totally obscuring the main strategy. Presenting the key ideas in a slightly suboptimal but understandable fashion was clearly preferable to us.

4. Proof of Theorem 3.1

4.1. Expansion of the dynamics

Instead of directly tackling the case of general observables and initial states as defined in the assumptions of Theorem 3.1, we concentrate on a narrower class of observables in the main part of the proof, namely operators on $\mathcal{H}$ which, for a fixed $\varepsilon > 0$, are given as

$$A_j^\varepsilon = A_{j,+}^\varepsilon P_{++} + A_{j,-}^\varepsilon P_{--}. \tag{4.1}$$

Here, for each $j \in \{1, ..., 2\overline{m} - 1\}$ and $\sigma \in \{\pm\}$, $A_{j,\sigma}^\varepsilon$ acts on $f \in L^2\left(\mathbb{R}^d\right)$ by

$$\widehat{A_{j,\sigma}^\varepsilon f}(k) = a_{j,\sigma}\left(k - \varepsilon p^{(j)}/2\right) \hat{f}\left(k - \varepsilon p^{(j)}\right). \tag{4.2}$$

with the momenta $p^{(j)}$ constant rather than integration variables. The functions $a_{j,\sigma} : \mathbb{R}^d \to \mathbb{C}$ are two times differentiable, with

$$\|a_j\|_{C^2} = \sup_{k \in \mathbb{R}^d} \max_{\sigma \in \{\pm\}} \max_{|\alpha| \leq 2} \left| \frac{\partial^\alpha}{\partial k^\alpha} a_{j,\sigma}(k) \right| < \infty \tag{4.3}$$

for all $j \in \{1, ..., 2\overline{m} - 1\}$. We will use a for the collection of functions $a_{j,\sigma}$ and p for the collection of momenta $p^{(j)}$, and denote

$$C_{\text{obs}} = \max_j \left| p^{(j)} \right|. \tag{4.4}$$

Given an initial state $\psi_0^\varepsilon \in \mathcal{H}$ and a vector $T = \left(T^{(1)}, ...T^{(\overline{m})}\right)$ of observation times $T^{(j)} > 0$, consider the random variable that results from propagating the wave with the perturbed dynamics and measuring it at the times $T^{(1)}/\varepsilon, (T^{(1)} + T^{(2)})/\varepsilon, ...,$

$$\begin{aligned} \mathcal{J}^\varepsilon &= \mathcal{J}^\varepsilon\left(H^\varepsilon, \psi_0^\varepsilon, T, a, p\right) \\ &= \Big\langle \exp\left(-iH^\varepsilon \frac{T^{(\overline{m})}}{\varepsilon}\right) \left(A_{\overline{m}+1}^\varepsilon\right)^* ... \left(A_{2\overline{m}-1}^\varepsilon\right)^* \exp\left(-iH^\varepsilon \frac{T^{(1)}}{\varepsilon}\right) \psi_0^\varepsilon, \\ &\qquad A_{\overline{m}}^\varepsilon \exp\left(-iH^\varepsilon \frac{T^{(\overline{m})}}{\varepsilon}\right) ... A_1^\varepsilon \exp\left(-iH^\varepsilon \frac{T^{(1)}}{\varepsilon}\right) \psi_0^\varepsilon \Big\rangle_{\mathcal{H}}. \end{aligned} \tag{4.5}$$

Applying a spatial cut-off at $R > 0$ to produces random variables

$$\mathcal{J}_R^\varepsilon = \mathcal{J}^\varepsilon\left(H^{\varepsilon,R}, \psi_0^\varepsilon, T, a, p\right) \tag{4.6}$$

corresponding to the cut-off generators $H^{\varepsilon,R}$ discussed in and below Lemma 2.11. For any fixed ε and $\psi_0^\varepsilon \in \mathcal{H}$, one has

$$\lim_{R\to\infty} \mathcal{J}_R^\varepsilon = \mathcal{J}^\varepsilon \tag{4.7}$$

both almost surely and in all $L^q(\mathbb{P})$, $q < \infty$ by Lemma 2.11 and the boundedness of the operators $A_j^\varepsilon : \mathcal{H} \to \mathcal{H}$.

In addition to the bounded energy assumption(2.160), we also assume for the initial states $\psi_0^\varepsilon \in \mathcal{H}$ that $\widehat{\psi_{0,\pm}^\varepsilon}$ is supported in a bounded ball $\left\{k \in \mathbb{R}^d : |k| \le L^{(0)}\right\}$, with $L^{(0)}$ arbitrary but fixed, i.e. independent of $\varepsilon > 0$. In particular, $\psi_0^\varepsilon \in H^1\left(\mathbb{R}^d; \mathbb{C}^2\right) = \mathcal{D}\left(H^{\varepsilon,R}\right)$ for all $R > 0$. For an $\overline{N} \in \mathbb{N}$ to be optimized later, let

$$L_n^{(j)} = L^{(0)} + n + (j-1)\overline{N} \qquad (j \in \{1, ..., \overline{m}\},\ n \in \mathbb{N}) \tag{4.8}$$

and denote $L^{(j)} = \left(L_1^{(j)}, L_2^{(j)}, ...\right)$.

4.1.1. Duhamel expansion

We now expand the right entry of the scalar product (4.5-4.6), first applying Lemma 2.12 to

$$\begin{aligned} &A_1^\varepsilon \exp\left(-iH^{\varepsilon,R}\frac{T^{(1)}}{\varepsilon}\right)\psi_0^\varepsilon \\ &= A_1^\varepsilon\left(F_{\overline{N}}^{\text{main}}\left(T^{(1)}/\varepsilon; R, L^{(1)}, \varepsilon\right) + R_{\overline{N}}\left(T^{(1)}/\varepsilon; R, L^{(1)}, \varepsilon\right)\right)\psi_0^\varepsilon \\ &= A_1^\varepsilon\left(\sum_{N=0}^{\overline{N}-1} F_N\left(T^{(1)}/\varepsilon; R, L^{(1)}, \varepsilon\right) + R_{\overline{N}}\left(T^{(1)}/\varepsilon; R, L^{(1)}, \varepsilon\right)\right)\psi_0^\varepsilon \end{aligned} \tag{4.9}$$

Adding a second time interval, we do not touch the remainder term from the first interval, but expand the contribution of the main term further

$$\begin{aligned} &A_2^\varepsilon \exp\left(-iH^{\varepsilon,R}\frac{T^{(2)}}{\varepsilon}\right) A_1^\varepsilon \exp\left(-iH^{\varepsilon,R}\frac{T^{(1)}}{\varepsilon}\right)\psi_0^\varepsilon \\ &= \sum_{N=0}^{\overline{N}-1} A_2^\varepsilon\left(F_{\overline{N}-N}^{\text{main}}\left(T^{(2)}/\varepsilon; R, L^{(2)}, \varepsilon\right) + R_{\overline{N}-N}\left(T^{(2)}/\varepsilon; R, L^{(2)}, \varepsilon\right)\right) \\ &\qquad\qquad \times A_1^\varepsilon F_N\left(T^{(1)}/\varepsilon; R, L^{(1)}, \varepsilon\right)\psi_0^\varepsilon \\ &\quad + A_2^\varepsilon \exp\left(-iH^{\varepsilon,R}\frac{T^{(2)}}{\varepsilon}\right) A_1^\varepsilon R_{\overline{N}}\left(T^{(1)}/\varepsilon; R, L^{(1)}, \varepsilon\right)\psi_0^\varepsilon \end{aligned} \tag{4.10}$$

and finally, after $\overline{m}$ expansion steps

$$
\begin{aligned}
&\left(\prod_{j=1}^{\overline{m}} A_j^\varepsilon \exp\left(-iH^{\varepsilon,R}\frac{T^{(j)}}{\varepsilon}\right)\right)\psi_0^\varepsilon \\
&\qquad = \sum_{N^{(1)}+\ldots+N^{(\overline{m})}<\overline{N}} \prod_{j=1}^{\overline{m}} \left(A_j^\varepsilon F_{N^{(j)}}\left(T^{(j)}/\varepsilon; R, L^{(j)}, \varepsilon\right)\right)\psi_0^\varepsilon \\
&\qquad\quad + \sum_{j_0=1}^{\overline{m}} \sum_{N^{(1)}+\ldots+N^{(j_0-1)}<\overline{N}} \left[\left(\prod_{l=j_0+1}^{\overline{m}} A_l^\varepsilon \exp\left(-iH^{\varepsilon,R}\frac{T^{(l)}}{\varepsilon}\right)\right)\times\right. \\
&\qquad\qquad \left.\times A_{j_0}^\varepsilon R_{\overline{N}^{(j_0)}}\left(T^{(j_0)}/\varepsilon; R, L^{(j_0)}, \varepsilon\right)\prod_{j=1}^{j_0-1}\left(A_j^\varepsilon F_{N^{(j)}}\left(T^{(j)}/\varepsilon; R, L^{(j)}, \varepsilon\right)\right)\right]\psi_0^\varepsilon,
\end{aligned}
\tag{4.11}
$$

where it is understood that all $N^{(j)} \in \mathbb{N}_0$, and the value $\overline{N}^{(j_0)} \in \mathbb{N}$ is given as

$$\overline{N}^{(j_0)} = \overline{N} - \sum_{j=1}^{j_0-1} N^{(j)}. \tag{4.12}$$

All $\overline{m}$ applications of Lemma 2.12 were justified as long as the random field ξ is C^2, because $\psi_0^\varepsilon \in \mathcal{D}\left(H^{\varepsilon,R}\right)$ and all operators A_j^ε leave the spaces $\mathcal{D}\left(H^{\varepsilon,R}\right) = H^1(\mathbb{R}^d;\mathbb{C}^2)$ invariant. On the right side of (4.11), the dynamics has been decomposed into two parts. As the wave travels through the random medium it may either interact with the random medium less than altogether $\overline{N}$ times, with $N^{(j)}$ events in the j-th time interval, and the n-th scattering event in the j-th inteval controlled by the cut-off threshold $L_n^{(j)}$. We call this part of the wave the main part. The remainder, however, consists of all scattering processes that either lead to at least $\overline{N}$ interactions, or to a violation of one of the cut-offs Φ as defined in (2.111). The time interval in which the number of scatterings reaches $\overline{N}$ or where a scattering event outside the cut-off occurs, is given the index j_0.

For simplicity, we rewrite the main and remainder part from (4.11) as

$$\left(\prod_{j=1}^{\overline{m}} A_j^\varepsilon \exp\left(-iH^{\varepsilon,R}\frac{T^{(j)}}{\varepsilon}\right)\right)\psi_0^\varepsilon = A_{\overline{m}}^\varepsilon \Psi_1^\varepsilon + A_{\overline{m}}^\varepsilon \Psi_2^\varepsilon \tag{4.13}$$

with Ψ_1^ε (main part) and Ψ_2^ε (remainder) two random elements of $\mathcal{H}$. They implicitely depend on the observables A_j^ε, the momenta $p^{(j)}$, on the times $T^{(j)}$, on ε, the field ξ (thus the randomness), R and $\overline{N}$. After an analogous expansion of the left argument of the scalar product in (4.5-4.6) yields Ψ'^ε_1 and Ψ'^ε_2, we have by (4.7)

$$
\begin{aligned}
&\left|\mathbb{E}\left[\mathcal{J}^\varepsilon\right] - \lim_{R\to\infty}\mathbb{E}\left[\langle \Psi'^\varepsilon_1, A_{\overline{m}}^\varepsilon \Psi_1^\varepsilon\rangle\right]\right| \\
&\qquad \le \limsup_{R\to\infty}\mathbb{E}\left[\left|\langle \Psi'^\varepsilon_1, A_{\overline{m}}^\varepsilon \Psi_2^\varepsilon\rangle\right| + \left|\langle \Psi'^\varepsilon_2, A_{\overline{m}}^\varepsilon \Psi_1^\varepsilon\rangle\right| + \left|\langle \Psi'^\varepsilon_2, A_{\overline{m}}^\varepsilon \Psi_2^\varepsilon\rangle\right|\right],
\end{aligned}
\tag{4.14}
$$

provided those limits exist.

On the left hand side of (4.14), we compare $\mathcal{J}^{\varepsilon}$ to the main part, which is is expected to converge to the limit object of Theorem 3.1 (modulo the fact that we currently only work with a reduced class of observables). We hope that the right hand side of (4.14), which accounts for the impact of the remainder terms, vanishes in the kinetic limit.

4.1.2. Amplitudes for the main part

To deal with the main contribution $\mathbb{E}\left[\langle {\Psi'}^{\varepsilon}_1, A^{\varepsilon}_{\overline{m}} \Psi^{\varepsilon}_1\rangle\right]$, we need to understand

$$\begin{aligned}\Bigg\langle F_{N'^{(\overline{m})}}\left(t^{(\overline{m})}; R, L^{(\overline{m})}, \varepsilon\right)\left(\prod_{j=1}^{\overline{m}-1}\left(A^{\varepsilon}_{2\overline{m}-j}\right)^* F_{N'^{(j)}}\left(t^{(j)}; R, L^{(j)}, \varepsilon\right)\right)\psi^{\varepsilon}_0,& \\ \left(\prod_{j=1}^{\overline{m}} A^{\varepsilon}_j F_{N^{(j)}}\left(t^{(j)}; R, L^{(j)}, \varepsilon\right)\right)\psi^{\varepsilon}_0\Bigg\rangle_{\mathcal{H}},&\end{aligned} \tag{4.15}$$

with $N^{(j)}, N'^{(j)} \in \mathbb{N}_0$. Here, we have replaced the rescaled $\frac{T^{(j)}}{\varepsilon}$ by a vector $t \in \mathbb{R}^{\overline{m}}_+$ of general times $t^{(j)} > 0$, and denote $|t| = t^{(1)} + ... + t^{(\overline{m})}$. To bring all operators on the right side of the scalar product, set $N^{(j)}$, $j \in \{1, ..., \overline{m}\}$ to be the number of scatterings in the j-th time interval on the *right* side of the scalar product, and $N^{(j)}$, $j \in \{\overline{m}+1, ..., 2\overline{m}\}$ to be the number $N'^{(2\overline{m}+1-j)}$ of scatterings in the $2\overline{m}+1-j$-th time interval on the *left* side of the scalar product, yielding a vector $N \in \mathbb{N}_0^{2\overline{m}}$. So, for example, $N^{(1)}$ and $N^{(2\overline{m})}$ belong to the same (the first) time interval, but they are typically not equal. The overall number of scattering events is

$$|N| = N^{(1)} + ... + N^{(2\overline{m})} \leq 2\overline{N} - 2. \tag{4.16}$$

Further, set for $j \in \{\overline{m}+1, ..., 2\overline{m}\}$

$$\begin{aligned} L^{(j)}_n &= L^{(2\overline{m}+1-j)}_{N^{(j)}+1-n}, \qquad n \in \{1, ..., N^{(j)}\}, \\ t^{(j)} &= t^{(2\overline{m}+1-j)}, \end{aligned} \tag{4.17}$$

and let $*(j)$ indicate adjoint operators whenever $j \in \{\overline{m}+1, ..., 2\overline{m}\}$. Thus, (4.15) equals

$$\left\langle \psi^{\varepsilon}_0, F_{N^{(2\overline{m})}}\left(t^{(2\overline{m})}; R, L^{(2\overline{m})}, \varepsilon\right)^* \left(\prod_{j=1}^{2\overline{m}-1} A^{\varepsilon}_j F_{N^{(j)}}\left(t^{(j)}; R, L^{(j)}, \varepsilon\right)^{*(j)}\right)\psi^{\varepsilon}_0\right\rangle_{\mathcal{H}}. \tag{4.18}$$

In time interval j, the wave undergoes $N^{(j)}$ scatterings, so all $|N|$ scattering events can be labeled by the index set

$$I(N) = \left\{(j, n) : j \in \{1, ..., 2\overline{m}\}, n \in \{1, ..., N^{(j)}\}\right\}. \tag{4.19}$$

Those scatterings cause $N^{(j)}$ momentum changes in the j-th time interval, so there are $N^{(j)}+1$ momentum variables for each time interval, and one can collect all momenta in all time intervals into $\left\{k_n^{(j)} : (j,n) \in I_0(N)\right\}$, with index set

$$I_0(N) = \left\{(j,n) : j \in \{1, ..., 2\overline{m}\}, n \in \{0, ..., N^{(j)}\}\right\}. \tag{4.20}$$

We introduce an ordering of the index set by defining for $(j_1, n_1), (j_2, n_2) \in \mathbb{Z}^2$ the lexicographic order $\prec$, that is

$$(j_1, n_1) \prec (j_2, n_2) \Leftrightarrow \begin{cases} j_1 < j_2 \text{ or} \\ j_1 = j_2 \text{ and } n_1 < n_2. \end{cases} \tag{4.21}$$

The 2×2 matrix structure of all operators is accounted for by the signs $\sigma_n^{(j)}$, $(j,n) \in I_0(N)$. After encoding the conjugation $*(j)$ in a sign

$$\tau^{(j)} = \begin{cases} +1 \text{ if } j \in \{1, ..., \overline{m}\}, \\ -1 \text{ if } j \in \{\overline{m}+1, ..., 2\overline{m}\}, \end{cases} \tag{4.22}$$

one obtains the Duhamel expansion of the main term,

$$\begin{aligned}
&\Bigg\langle F_{N'(\overline{m})}\left(t^{(\overline{m})}; R, L^{(\overline{m})}, \varepsilon\right) \left(\prod_{j=1}^{\overline{m}-1} \left(A^{\varepsilon}_{2\overline{m}-j}\right)^* F_{N'(j)}\left(t^{(j)}; R, L^{(j)}, \varepsilon\right)\right) \psi_0^{\varepsilon}, \\
&\qquad \left(\prod_{j=1}^{\overline{m}} A_j^{\varepsilon} F_{N(j)}\left(t^{(j)}; R, L^{(j)}, \varepsilon\right)\right) \psi_0^{\varepsilon} \Bigg\rangle_{\mathcal{H}} \\
= &\int_{\mathbb{R}^{d(N^{(2\overline{m})}+1)}} \mathrm{d}k^{(2\overline{m})} \sum_{\sigma^{(2\overline{m})} \in \{\pm\}^{N^{(2\overline{m})}+1}} \cdots \int_{\mathbb{R}^{d(N^{(1)}+1)}} \mathrm{d}k^{(1)} \sum_{\sigma^{(1)} \in \{\pm\}^{N^{(1)}+1}} \\
&\prod_{j=1}^{2\overline{m}-1} \left(a_{j, \sigma^{(j)}_{N^{(j)}}} \left(\frac{k_0^{(j+1)} + k^{(j)}_{N^{(j)}}}{2} \right) \delta\left(k_0^{(j+1)} - k^{(j)}_{N^{(j)}} - \varepsilon p^{(j)}\right) \delta\left(\sigma^{(j)}_{N^{(j)}}, \sigma_0^{(j+1)}\right)\right) \\
&\prod_{j=1}^{2\overline{m}} \left[\int_{\mathbb{R}_+^{N^{(j)}+1}} \mathrm{d}s^{(j)} \delta\left(\sum_{n=0}^{N^{(j)}} s_n^{(j)} - t^{(j)}\right)\right] \\
&\prod_{(j,n)\in I_0(N)} \exp\left(-2\pi i \left|k_n^{(j)}\right| \sigma_n^{(j)} \tau^{(j)} s_n^{(j)}\right) \\
&\prod_{(j,n)\in I(N)} \left[(-i\tau^{(j)}\sqrt{\varepsilon}\pi)\left(\left|k_n^{(j)}\right| \sigma_{n-1}^{(j)} + \left|k_{n-1}^{(j)}\right| \sigma_n^{(j)}\right) \widehat{\xi_R}\left(k_n^{(j)} - k_{n-1}^{(j)}\right)\right. \\
&\qquad\qquad \left. \times \Phi\left(k_n^{(j)}, k_{n-1}^{(j)}, L_n^{(j)}\right)\right] \\
&\widehat{\psi^{\varepsilon}_{0,\sigma_0^{(1)}}}\left(k_0^{(1)}\right) \overline{\widehat{\psi}^{\varepsilon}_{0,\sigma^{(2\overline{m})}_{N^{(2\overline{m})}}}} \left(k^{(2\overline{m})}_{N^{(2\overline{m})}}\right).
\end{aligned} \tag{4.23}$$

The deltas for the k and σ variables in (4.23) indicate that the observables A^ε_j add an $-\varepsilon p^{(j)}$ shift to the momentum k, while they are diagonal with respect to the two-component structure of $\mathcal{H}$. Now we first want to take the expectation and then the $R \to \infty$ limit of (4.23). To this end, note that only $k_0^{(1)}$ and $k_n^{(j)}$, $(j,n) \in I(N)$ are actual integration variables, while $k_0^{(j)}$, $j > 1$ are already determined by the deltas. We switch the integration variables $k_n^{(j)}$, $(j,n) \in I(N)$ to

$$\theta_n^{(j)} = k_n^{(j)} - k_{n-1}^{(j)} \qquad (j \in \{1, ..., \overline{m}\},\ \ n \in \{1, ..., N^{(j)}\}). \tag{4.24}$$

One thus has to evaluate a limit of the form

$$\lim_{R\to\infty} \mathbb{E} \sum_{\sigma \in \{\pm\}^{|N|+2\overline{m}}} \int_{\mathbb{R}^d} \mathrm{d}k_0^{(1)} \int_{\mathbb{R}^{|N|d}} \mathrm{d}\theta f\left(k_0^{(1)}, p, \sigma, \theta\right) \prod_{(j,n)\in I(N)} \widehat{\xi_R}(\theta_n^{(j)}), \tag{4.25}$$

with f a function continuous in the θ variables such that

$$\begin{aligned} \left| f\left(k_0^{(1)}, p, \sigma, \theta\right)\right| \le C_{\varepsilon,t} &\left| \widehat{\psi^\varepsilon}\left(k_0^{(1)}\right) \widehat{\psi^\varepsilon}\left(k_0^{(1)} + \varepsilon \sum_{j=1}^{2\overline{m}-1} p_\sigma^{(j)} + \sum_{(j,n)\in I(N)} \theta_n^{(j)}\right)\right| \\ &\times \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^0} \prod_{(j,n)\in I(N)} \left(2L_n^{(j)} + 1 + \left|\theta_n^{(j)}\right|\right). \end{aligned} \tag{4.26}$$

Here, the definition of the cut-off function Φ was used to estimate

$$\begin{aligned} &\left|\left|k_n^{(j)}\right| \sigma_{n-1}^{(j)} + \left|k_{n-1}^{(j)}\right| \sigma_n^{(j)}\right| \Phi\left(k_n^{(j)}, k_{n-1}^{(j)}, L_n^{(j)}\right) \\ &\qquad \le 2\min\left\{\left|k_{n-1}^{(j)}\right|, \left|k_n^{(j)}\right|\right\} \Phi\left(k_n^{(j)}, k_{n-1}^{(j)}, L_n^{(j)}\right) + \left|\theta_n^{(j)}\right| \le 2L_n^{(j)} + 1 + \left|\theta_n^{(j)}\right|. \end{aligned} \tag{4.27}$$

Thus, by dominated convergence, Lemma 2.1 is applicable (with exponent $q = 1$) to the θ integral if $m > d + 1$, and we can state our findings in the following

Lemma 4.1. *With all definitions made as above, for a random field ξ of class $(m, 0)$, with $m > d + 1$, one has*

$$\begin{aligned} &\lim_{R\to\infty} \mathbb{E} \left\langle \psi_0^\varepsilon, F_{N^{(2\overline{m})}}\left(t^{(2\overline{m})}; R, L^{(2\overline{m})}, \varepsilon\right)^* \left(\prod_{j=1}^{2\overline{m}-1} A_j^\varepsilon F_{N^{(j)}}\left(t^{(j)}; R, L^{(j)}, \varepsilon\right)^{*(j)}\right) \psi_0^\varepsilon \right\rangle_{\mathcal{H}} \\ &\qquad = \sum_{S \in \pi^*(I(N))} \mathcal{K}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right), \end{aligned} \tag{4.28}$$

where $\pi^*\left(I(N)\right)$ denotes partitions of the set $I(N)$ without single-element clusters. The amplitude *of the partition S is given as*

$$
\begin{aligned}
&\mathcal{K}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right) \\
&= (\sqrt{\varepsilon}\pi)^{|N|} \int_{\mathbb{R}^{d(N^{(2\overline{m})}+1)}} \mathrm{d}k^{(2\overline{m})} \sum_{\sigma^{(2\overline{m})} \in \{\pm\}^{N^{(2\overline{m})}+1}} \dots \int_{\mathbb{R}^{d(N^{(1)}+1)}} \mathrm{d}k^{(1)} \sum_{\sigma^{(1)} \in \{\pm\}^{N^{(1)}+1}} \\
&\quad \prod_{A \in S} \left\{ \delta\left(\sum_{(j,n)\in A} \theta_n^{(j)} \right) \widehat{g_{|A|}} \left(\theta_n^{(j)} : (j,n) \in A^\sharp \right) \right\} \\
&\quad \prod_{j=1}^{2\overline{m}-1} \left(a_{j,\sigma^{(j)}_{N^{(j)}}} \left(\frac{k_0^{(j+1)} + k_{N^{(j)}}^{(j)}}{2} \right) \delta\left(k_0^{(j+1)} - k_{N^{(j)}}^{(j)} - \varepsilon p^{(j)} \right) \delta\left(\sigma^{(j)}_{N^{(j)}}, \sigma_0^{(j+1)} \right) \right) \\
&\quad \prod_{j=1}^{2\overline{m}} \left[\int_{\mathbb{R}_+^{N^{(j)}+1}} \mathrm{d}s^{(j)} \delta\left(\sum_{n=0}^{N^{(j)}} s_n^{(j)} - t^{(j)} \right) \right] \\
&\quad \prod_{(j,n)\in I_0(N)} \exp\left(-2\pi i \left| k_n^{(j)} \right| \sigma_n^{(j)} \tau^{(j)} s_n^{(j)} \right) \\
&\quad \prod_{(j,n)\in I(N)} \left[(-i\tau^{(j)}) \left(\left| k_n^{(j)} \right| \sigma_{n-1}^{(j)} + \left| k_{n-1}^{(j)} \right| \sigma_n^{(j)} \right) \varPhi\left(k_n^{(j)}, k_{n-1}^{(j)}, L_n^{(j)} \right) \right] \\
&\quad \widehat{\psi^\varepsilon_{0,\sigma_0^{(1)}}} \left(k_0^{(1)} \right) \overline{\widehat{\psi}^\varepsilon_{0,\sigma^{(2\overline{m})}_{N^{(2\overline{m})}}}} \left(k_0^{(1)} + \varepsilon \sum_{j=1}^{2\overline{m}-1} p^{(j)} \right).
\end{aligned}
\tag{4.29}
$$

For better readability, we have simultaneously used the k and θ variables as introduced in (4.24).

The next step is to rewrite the unperturbed time propagation between the scattering events in resolvent formalism. As in [32], we take $l \in \mathbb{N}_0$, $\tau \geq 0$, $(w_0, ..., w_l) = w \in \mathbb{C}^{l+1}$ with $\mathrm{Im} w_l \leq 0$ and define

$$K_l\left(w, \tau\right) = \int_{\mathbb{R}_+{}^{l+1}} \mathrm{d}s \delta\left(\tau - \sum_{n=0}^{l} s_l \right) \exp\left(-i \sum_{n=0}^{l} w_l s_l \right). \tag{4.30}$$

By standard Fourier calculus,

Lemma 4.2. *Let $\tau \geq 0$, $\gamma > 0$, and $(w_0, ..., w_l) = w \in \mathbb{C}^{l+1}$ with $\mathrm{Im} w_l \leq 0$. While for $l = 0$*

$$e^{-\gamma\tau} K_0(w, \tau) = \int_{\mathbb{R}} \frac{\mathrm{d}\alpha}{2\pi} e^{-i\alpha\tau} \frac{i}{\alpha - w_0 + i\gamma} \tag{4.31}$$

is only true in a in $L^2_\alpha \to L^2_\tau$ sense,

$$K_l\left(w, \tau\right) = e^{\gamma\tau} \int_{\mathbb{R}} \frac{\mathrm{d}\alpha}{2\pi} e^{-i\alpha\tau} \prod_{n=0}^{l} \left(\frac{i}{\alpha - w_n + i\gamma} \right) \tag{4.32}$$

holds pointwise in $\tau \in [0, \infty)$ whenever $l \geq 1$.

To have a unified notation, we will formally use (4.32) also for $l = 0$ and will only comment on this special case if the corresponding contributions really have to be treated differently.

By Lemma 4.1 and 4.2 and an application of Fubini's theorem, one obtains the representation of the amplitude we will use for a large part of our analysis

Lemma 4.3. *Under the conditions and with the notation of Lemma 4.1, and for any* $\gamma > 0$,

$$\begin{aligned}
&\mathcal{K}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right) \\
&= (\sqrt{\varepsilon}\pi)^{|N|} \int_{\mathbb{R}^{d(N^{(2\overline{m})}+1)}} \mathrm{d}k^{(2\overline{m})} \sum_{\sigma^{(2\overline{m})} \in \{\pm\}^{N^{(2\overline{m})}+1}} \cdots \int_{\mathbb{R}^{d(N^{(1)}+1)}} \mathrm{d}k^{(1)} \sum_{\sigma^{(1)} \in \{\pm\}^{N^{(1)}+1}} \\
&\quad \prod_{A \in S} \left\{ \delta\left(\sum_{(j,n)\in A} \theta_n^{(j)} \right) \widehat{g_{|A|}}\left(\theta_n^{(j)} : (j,n) \in A^\sharp\right) \right\} \\
&\quad \prod_{j=1}^{2\overline{m}-1} \left(a_{j,\sigma^{(j)}_{N^{(j)}}} \left(\frac{k_0^{(j+1)} + k_{N^{(j)}}^{(j)}}{2} \right) \delta\left(k_0^{(j+1)} - k_{N^{(j)}}^{(j)} - \varepsilon p^{(j)}\right) \delta\left(\sigma_{N^{(j)}}^{(j)}, \sigma_0^{(j+1)}\right) \right) \\
&\quad \prod_{j=1}^{2\overline{m}} \left[e^{\gamma t^{(j)}} \int_{\mathbb{R}} \frac{\mathrm{d}\alpha^{(j)}}{2\pi} e^{-i\alpha^{(j)} t^{(j)}} \right] \\
&\quad \prod_{(j,n)\in I_0(N)} \left(\frac{i}{\alpha^{(j)} - 2\pi \sigma_n^{(j)} \tau^{(j)} |k_n^{(j)}| + i\gamma} \right) \\
&\quad \prod_{(j,n)\in I(N)} \left[(-i\tau^{(j)}) \left(\left|k_n^{(j)}\right| \sigma_{n-1}^{(j)} + \left|k_{n-1}^{(j)}\right| \sigma_n^{(j)} \right) \Phi\left(k_n^{(j)}, k_{n-1}^{(j)}, L_n^{(j)}\right) \right] \\
&\quad \widehat{\psi^\varepsilon_{0,\sigma_0^{(1)}}}\left(k_0^{(1)}\right) \overline{\widehat{\psi}^\varepsilon_{0,\sigma^{(2\overline{m})}_{N^{(2\overline{m})}}}} \left(k_0^{(1)} + \varepsilon \sum_{j=1}^{2\overline{m}-1} p^{(j)} \right).
\end{aligned} \tag{4.33}$$

Here, the $\alpha^{(j)}$ *integrations can be interchanged with the* k *integrals for all* j *such that* $N^{(j)} \geq 1$.

4.1.3. Amplitudes for the remainder

Next, an analogue to Lemma 4.1 for the remainder term, i.e. the right side of equation (4.14) has to be found. Assume that "remainder-type" $R_{\overline{N}^{(j_0)}}$ scattering occurs for the first time in the j_0-th time interval, $j_0 \in \{1, ..., \overline{m}\}$. To apply the smoothed Duhamel expansion from Lemma 2.14, pick an $\overline{M} \in \mathbb{N}$ and a $\kappa > 0$ to be optimized later. We need bounds on all terms occuring in this expansion and have to estimate

$$\mathbb{E} \left\| G^{\text{rough}}_{M,N_{\text{fin}}}\left(t^{(j_0)}; R, L^{(j_0)}, \varepsilon\right) \left(\prod_{j<j_0} A_j^\varepsilon F_{N^{(j)}}\left(t^{(j)}; R, L^{(j)}, \varepsilon\right) \right) \psi_0^\varepsilon \right\|_{\mathcal{H}}^2, \tag{4.34}$$

for $0 \leq M < \overline{M}$, $1 \leq N_{\text{fin}} < \overline{N}^{(j_0)}$, as well as the same expression (4.34) with $G^{\text{rough}}_{M,N_{\text{fin}}}$ replaced by $A^{\text{rough}}_{\overline{M},N_{\text{fin}}}$, $G^{\text{end}}_{M,\overline{N}^{(j_0)}}$ or $A^{\text{end}}_{\overline{M},\overline{N}^{(j_0)}}$, each as defined in Lemma 2.14. In order to avoid redefining most of the notation from the expansion of the main part, we will without loss of generality proof all estimates in the case $j_0 = \overline{m}$. Even in this case, we have to accommodate for the fact that the $\overline{m}$-th time interval now has a more complicated scattering structure. There are $N^{(j)} \in \mathbb{N}_0$ scattering processes in the j-th, $j < \overline{m}$, time interval, so denote

$$N_< = \left(N^{(1)}, ..., N^{(\overline{m}-1)}\right), \tag{4.35}$$

with

$$\sum_{j=1}^{\overline{m}-1} N^{(j)} < \overline{N} \tag{4.36}$$

so that

$$\overline{N}^{(\overline{m})} = \overline{N} - \sum_{j=1}^{\overline{m}-1} N^{(j)} \tag{4.37}$$

is positive. In the last time interval, the wave scatters off the medium $N^{(\overline{m})}$ times, with

$$N^{(\overline{m})} = \begin{cases} N_{\text{fin}} + M & \text{in the } G^{\text{rough}} \text{ case,} \\ N_{\text{fin}} + \overline{M} & \text{in the } A^{\text{rough}} \text{ case,} \\ \overline{N}^{(\overline{m})} + M & \text{in the } G^{\text{end}} \text{ case,} \\ \overline{N}^{(\overline{m})} + \overline{M} & \text{in the } A^{\text{end}} \text{ case.} \end{cases} \tag{4.38}$$

This time, define $N \in \mathbb{N}_0^{2\overline{m}}$ by the above $N^{(j)}$ for $j \in \{1, ..., \overline{m}\}$, and $N^{(j)} = N^{(2\overline{m}+1-j)}$ for $j \in \{\overline{m}+1, ..., 2\overline{m}\}$, so, compared to the expansion of the main part, the vector N is now always symmetric due to the quadratic structure of (4.34). The index sets $I(N)$, $I_0(N)$ as well as $k_n^{(j)}$, with $(j,n) \in I_0(N)$, and $\theta_n^{(j)}$ for $(j,n) \in I(N)$, are given as usually, so are the cutoff parameters $L_n^{(j)}$ and the signs $\sigma_n^{(j)}$ and $\tau^{(j)}$. However, again due to the quadratic structure of (4.34), the observables are slightly different, while $a_{j,\sigma}$ and $p^{(j)}$ are defined as before for $j \in \{1, ..., \overline{m}-1\}$, we now have set for $j \in \{\overline{m}+1, ..., 2\overline{m}-1\}$ that

$$\begin{aligned} a_{j,\pm} &= \overline{a_{2\overline{m}-j,\pm}}, \\ p^{(j)} &= -p^{(2\overline{m}-j)}. \end{aligned} \tag{4.39}$$

Finally, as the central observable is missing (or rather, the identity on $\mathcal{H}$) in this case, $a_{\overline{m}} \equiv 1$, $p^{(\overline{m})} = 0$.

It is important to note that both between the main and remainder part, as well as between the different contributions to the remainder, the $N^{(j)}$, $p^{(j)}$, $a_{j,\sigma}$ variables mean slightly different, but structurally closely related objects. To be able to exploit similarities between the cases without rewording whole paragraphs, we use the same variables for them as it will always be clear which case is currently under consideration, so no ambiguities will arise.

Lemma 4.4. *For any $M \in \{0, ..., \overline{M}-1\}$, a random field ξ of class $(m, 0)$ with $m > d+(2+M)$, $N_< \in \mathbb{N}_0^{\overline{m}-1}$ obeying (4.36) and $N_{\text{fin}} \in \{1, ..., \overline{N}^{(\overline{m})}-1\}$,*

$$\begin{aligned} &\lim_{R\to\infty} \mathbb{E} \left\| G^{\text{rough}}_{M,N_{\text{fin}}} \left(t^{(\overline{m})}; R, L^{(\overline{m})}, \varepsilon\right) \left(\prod_{j<\overline{m}} A^\varepsilon_j F_{N^{(j)}} \left(t^{(j)}; R, L^{(j)}, \varepsilon\right)\right) \psi^\varepsilon_0 \right\|^2_{\mathcal{H}} \\ &\quad = \sum_{S\in\pi^*(I(N))} \mathcal{R}\left(G^{\text{rough}}, \psi^\varepsilon_0, \varepsilon, \kappa, a, p, L^{(0)}, t, N_<, N_{\text{fin}}, M, S\right) \end{aligned} \tag{4.40}$$

with $N^{(\overline{m})} = N_{\text{fin}} + M$ and the amplitude $\mathcal{R}\left(G^{\text{rough}}, ..., S\right)$ of each partition S given as

$$\begin{aligned} &\mathcal{R}\left(G^{\text{rough}}, \psi^\varepsilon_0, \varepsilon, \kappa, a, p, L^{(0)}, t, N_<, N_{\text{fin}}, M, S\right) \\ &= (\sqrt{\varepsilon}\pi)^{|N|} \int_{\mathbb{R}^{d(N^{(2\overline{m})}+1)}} \mathrm{d}k^{(2\overline{m})} \sum_{\sigma^{(2\overline{m})}\in\{\pm\}^{N^{(2\overline{m})}+1}} \cdots \int_{\mathbb{R}^{d(N^{(1)}+1)}} \mathrm{d}k^{(1)} \sum_{\sigma^{(1)}\in\{\pm\}^{N^{(1)}+1}} \\ &\quad \prod_{A\in S} \left\{ \delta\left(\sum_{(j,n)\in A} \theta^{(j)}_n \right) \widehat{g_{|A|}} \left(\theta^{(j)}_n : (j,n) \in A^\sharp\right)\right\} \\ &\quad \prod_{j=1}^{2\overline{m}-1} \left(a_{j,\sigma^{(j)}_{N^{(j)}}} \left(\frac{k^{(j+1)}_0 + k^{(j)}_{N^{(j)}}}{2} \right) \delta\left(k^{(j+1)}_0 - k^{(j)}_{N^{(j)}} - \varepsilon p^{(j)}\right) \delta\left(\sigma^{(j)}_{N^{(j)}}, \sigma^{(j+1)}_0\right)\right) \\ &\quad \prod_{j=1}^{2\overline{m}} \left[e^{\gamma t^{(j)}} \int_{\mathbb{R}} \frac{\mathrm{d}\alpha^{(j)}}{2\pi} e^{-i\alpha^{(j)} t^{(j)}} \right] \\ &\quad \prod_{(j,n)\in I_0(N)} \left(\frac{i}{\alpha^{(j)} - 2\pi\sigma^{(j)}_n \tau^{(j)} |k^{(j)}_n| + i\gamma + i\kappa^{(j)}_n} \right) \\ &\quad \prod_{(j,n)\in I(N)} \left[(-i\tau^{(j)}) \left(\left|k^{(j)}_n\right| \sigma^{(j)}_{n-1} + \left|k^{(j)}_{n-1}\right| \sigma^{(j)}_n \right) \Phi^{(j)}_n \left(k^{(j)}_n, k^{(j)}_{n-1}, L^{(j)}_n\right)\right] \\ &\quad \widehat{\psi^\varepsilon_{0,\sigma^{(1)}_0}} \left(k^{(1)}_0\right) \overline{\widehat{\psi}^\varepsilon_{0,\sigma^{(2\overline{m})}_{N^{(2\overline{m})}}}} \left(k^{(1)}_0\right), \end{aligned} \tag{4.41}$$

with the cutoff function

$$\Phi^{(j)}_n = \begin{cases} \Phi & \text{for } (j,n) \prec (\overline{m}, N_{\text{fin}}) \text{ or } (j,n) \succ (\overline{m}+1, N^{(\overline{m}+1)} + 1 - N_{\text{fin}}) \\ 1-\Phi & \text{for } (j,n) = (\overline{m}, N_{\text{fin}}) \text{ or } (j,n) = (\overline{m}+1, N^{(\overline{m}+1)} + 1 - N_{\text{fin}}) \\ 1 & \text{for } (j,n) \succ (\overline{m}, N_{\text{fin}}) \text{ and } (j,n) \prec (\overline{m}+1, N^{(\overline{m}+1)} + 1 - N_{\text{fin}}) \end{cases} \tag{4.42}$$

and the damping parameter

$$\kappa^{(j)}_n = \begin{cases} 0 & \text{for } (j,n) \prec (\overline{m}, N_{\text{fin}}) \text{ or } (j,n) \succ (\overline{m}+1, N^{(\overline{m}+1)} - N_{\text{fin}}) \\ \kappa & \text{for } (j,n) \succeq (\overline{m}, N_{\text{fin}}) \text{ and } (j,n) \preceq (\overline{m}+1, N^{(\overline{m}+1)} - N_{\text{fin}}). \end{cases} \tag{4.43}$$

Proof. This proof is very similar to the reasoning leading up to Lemmas 4.1 and 4.3, the only substantial difference being that starting from the N_{fin}-th interaction in the $\overline{m}$-th interval, the scattering is no longer controlled by the cutoff function $\varPhi$, and the bound (4.27) is not available for the last $M+1$ scattering events. Instead, if $N_{\text{fin}} > 1$ (which is in particular always the case if $\overline{m} = 1$ due to the support properties of $\widehat{\psi_0^\varepsilon}$), consider the last controlled momentum change

$$\left|k^{(\overline{m})}_{N_{\text{fin}}-1}\right| + \left|k^{(\overline{m})}_{N_{\text{fin}}-2}\right| \leq 2L^{(\overline{m})}_{N_{\text{fin}}-1} + 1 + \left|\theta^{(\overline{m})}_{N_{\text{fin}}-1}\right| \tag{4.44}$$

and find for $n \in \left\{N_{\text{fin}}, ..., N^{(\overline{m})}\right\}$

$$\left|k^{(\overline{m})}_{n}\right| + \left|k^{(\overline{m})}_{n-1}\right| \leq 2L^{(\overline{m})}_{N_{\text{fin}}-1} + 1 + \left|\theta^{(\overline{m})}_{N_{\text{fin}}-1}\right| + \sum_{l=N_{\text{fin}}}^{n} \left(\left|\theta^{(\overline{m})}_{l}\right| + \left|\theta^{(\overline{m})}_{l-1}\right|\right) \tag{4.45}$$

to obtain the estimate

$$\prod_{n=N_{\text{fin}}-1}^{N_{\text{fin}}+M} \left(\left|k^{(\overline{m})}_{n}\right| + \left|k^{(\overline{m})}_{n-1}\right|\right) \leq (C(|N| + \langle L^{(0)}\rangle))^{M+2} M! \sup_{\sum e_n = M+2} \prod_{n=N_{\text{fin}}-1}^{N_{\text{fin}}+M} \left\langle \theta^{(\overline{m})}_{n}\right\rangle^{e_n} \tag{4.46}$$

with some universal $C < \infty$. For all remaining $k^{(j)}_n$, $j \leq \overline{m}$ variables, one can still utilize the original (4.27). By the same argument for the $k^{(j)}_n$, $j > \overline{m}$, variables, we have an estimate analogous to (4.26) with the only difference that some of the θ variables may now come with exponents up to $M+2$, so Lemma 2.1 applies whenever $m > d + (M+2)$. If $N_{\text{fin}} = 1$, one has to replace (4.45) by

$$\begin{aligned} &\left(\left|k^{(\overline{m}-1)}_{N^{(\overline{m}-1)}}\right| + \left|k^{(\overline{m}-1)}_{N^{(\overline{m}-1)}-1}\right|\right) \prod_{n=1}^{M+1} \left(\left|k^{(\overline{m})}_{n}\right| + \left|k^{(\overline{m})}_{n-1}\right|\right) \\ &\quad \leq (C(|N| + \langle L^{(0)}\rangle))^{M+2} M! \left\langle \theta^{(\overline{m}-1)}_{N^{(\overline{m}-1)}}\right\rangle^{M+2} \left\langle \varepsilon p^{(\overline{m}-1)}\right\rangle^{M+1} \sup_{\sum e_n = M+2} \prod_{n=1}^{M+1} \left\langle \theta^{(\overline{m})}_{n}\right\rangle^{e_n}, \end{aligned} \tag{4.47}$$

to see that $m > d + (M+2)$ also suffices in this case. The resolvent formulation is then a straightforward consequence of Lemma 4.2. □

By the same argument,

Lemma 4.5. *For any $M \in \{0, ..., \overline{M} - 1\}$, any random field ξ of class $(m, 0)$ with $m > d + (2+M)$ and $N_< \in \mathbb{N}_0^{\overline{m}-1}$ obeying (4.36)*

$$\begin{aligned} &\lim_{R\to\infty} \mathbb{E} \left\| G^{\text{end}}_{M,\overline{N}^{(\overline{m})}}\left(t^{(\overline{m})}; R, L^{(\overline{m})}, \varepsilon\right) \left(\prod_{j<\overline{m}} A^\varepsilon_j F_{N^{(j)}}\left(t^{(j)}; R, L^{(j)}, \varepsilon\right)\right) \psi_0^\varepsilon \right\|^2_{\mathcal{H}} \\ &\quad = \sum_{S\in\pi^*(I(N))} \mathcal{R}\left(G^{\text{end}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, t, N_<, \overline{N}, M, S\right). \end{aligned} \tag{4.48}$$

Here, $N^{(\overline{m})} = \overline{N}^{(\overline{m})} + M$ *and the amplitude* $\mathcal{R}\left(G^{\mathrm{end}}, ..., S\right)$ *of each partition* S *given almost like* $\mathcal{R}\left(G^{\mathrm{rough}}, ..., S\right)$ *in equation (4.41), only with a cutoff function*

$$\Phi_n^{(j)} = \begin{cases} \Phi & \text{for } (j,n) \prec \left(\overline{m}, \overline{N}^{(\overline{m})}\right) \text{ or } (j,n) \succ \left(\overline{m}+1, N^{(\overline{m}+1)} + 1 - \overline{N}^{(\overline{m})}\right) \\ 1 & \text{for } (j,n) \succeq \left(\overline{m}, \overline{N}^{(\overline{m})}\right) \text{ or } (j,n) \preceq \left(\overline{m}+1, N^{(\overline{m}+1)} + 1 - \overline{N}^{(\overline{m})}\right) \end{cases} \tag{4.49}$$

and the damping parameter

$$\kappa_n^{(j)} = \begin{cases} 0 & \text{for } (j,n) \prec \left(\overline{m}, \overline{N}^{(\overline{m})}\right) \text{ or } (j,n) \succ \left(\overline{m}+1, N^{(\overline{m}+1)} - \overline{N}^{(\overline{m})}\right) \\ \kappa & \text{for } (j,n) \succeq \left(\overline{m}, \overline{N}^{(\overline{m})}\right) \text{ or } (j,n) \preceq \left(\overline{m}+1, N^{(\overline{m}+1)} - \overline{N}^{(\overline{m})}\right). \end{cases} \tag{4.50}$$

Now, one is left with estimates for the $A^{\mathrm{rough}}_{\overline{M},N_{\mathrm{fin}}}$ and $A^{\mathrm{end}}_{\overline{M},\overline{N}^{(\overline{m})}}$. The main difference to the G^{rough} and G^{end} terms is the absence of a propagator after the last interaction with the medium, so the amplitudes are “amputated”, like the $\mathcal{K}^{(\mathrm{amp})}$ amplitudes in [32]. This results in two missing resolvents in Lemmas 4.6 and 4.7, as can be seen in equation (4.52).

Lemma 4.6. *For* ξ *of class* $(m,0)$, $m > d + (2+\overline{M})$, $N_< \in \mathbb{N}_0^{\overline{m}-1}$ *obeying (4.36), and any* $N_{\mathrm{fin}} \in \{1, ..., \overline{N}^{(\overline{m})} - 1\}$,

$$\begin{aligned} &\lim_{R\to\infty} \mathbb{E} \left\| A^{\mathrm{rough}}_{\overline{M},N_{\mathrm{fin}}} \left(t^{(\overline{m})}; R, L^{(\overline{m})}, \varepsilon\right) \left(\prod_{j<\overline{m}} A_j^{\varepsilon} F_{N^{(j)}} \left(t^{(j)}; R, L^{(j)}, \varepsilon\right)\right) \psi_0^{\varepsilon} \right\|_{\mathcal{H}}^2 \\ &\quad = \sum_{S\in\pi^*(I(N))} \mathcal{R}\left(A^{\mathrm{rough}}, \psi_0^{\varepsilon}, \varepsilon, \kappa, a, p, L^{(0)}, t, N_<, N_{\mathrm{fin}}, \overline{M}, S\right) \end{aligned} \tag{4.51}$$

with $N^{(\overline{m})} = N_{\mathrm{fin}} + \overline{M}$ *and the amplitude* $\mathcal{R}\left(A^{\mathrm{rough}}, ..., S\right)$ *of each partition* S *given as*

$$
\begin{aligned}
&\mathcal{R}\left(A^{\mathrm{rough}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, t, N_<, N_{\mathrm{fin}}, \overline{M}, S\right) \\
&= (\sqrt{\varepsilon}\pi)^{|N|} \int_{\mathbb{R}^{d(N^{(2\overline{m})}+1)}} \mathrm{d}k^{(2\overline{m})} \sum_{\sigma^{(2\overline{m})} \in \{\pm\}^{N^{(2\overline{m})}+1}} \dots \int_{\mathbb{R}^{d(N^{(1)}+1)}} \mathrm{d}k^{(1)} \sum_{\sigma^{(1)} \in \{\pm\}^{N^{(1)}+1}} \\
&\quad \prod_{A \in S} \left\{ \delta\left(\sum_{(j,n)\in A} \theta_n^{(j)}\right) \widehat{g_{|A|}}\left(\theta_n^{(j)} : (j,n) \in A^\sharp\right)\right\} \\
&\quad \prod_{j=1}^{2\overline{m}-1} \left(a_{j,\sigma^{(j)}_{N^{(j)}}} \left(\frac{k_0^{(j+1)} + k^{(j)}_{N^{(j)}}}{2}\right) \delta\left(k_0^{(j+1)} - k^{(j)}_{N^{(j)}} - \varepsilon p^{(j)}\right) \delta\left(\sigma^{(j)}_{N^{(j)}}, \sigma_0^{(j+1)}\right)\right) \\
&\quad \prod_{j=1}^{2\overline{m}} \left[e^{\gamma t^{(j)}} \int_{\mathbb{R}} \frac{\mathrm{d}\alpha^{(j)}}{2\pi} e^{-i\alpha^{(j)} t^{(j)}} \right] \\
&\quad \prod_{\substack{(j,n)\in I_0(N) \\ (j,n)\neq(\overline{m}, N^{(\overline{m})}), (\overline{m}+1,0)}} \left(\frac{i}{\alpha^{(j)} - 2\pi\sigma_n^{(j)}\tau^{(j)}|k_n^{(j)}| + i\gamma + i\kappa_n^{(j)}} \right) \\
&\quad \prod_{(j,n)\in I(N)} \left[(-i\tau^{(j)}) \left(\left|k_n^{(j)}\right| \sigma_{n-1}^{(j)} + \left|k_{n-1}^{(j)}\right| \sigma_n^{(j)} \right) \varPhi_n^{(j)} \left(k_n^{(j)}, k_{n-1}^{(j)}, L_n^{(j)}\right)\right] \\
&\quad \widehat{\psi^\varepsilon_{0,\sigma_0^{(1)}}}\left(k_0^{(1)}\right) \overline{\widehat{\psi}^\varepsilon_{0,\sigma^{(2\overline{m})}_{N^{(2\overline{m})}}}} \left(k_0^{(1)}\right),
\end{aligned}
\tag{4.52}
$$

with the cutoff function

$$
\varPhi_n^{(j)} = \begin{cases} \varPhi & \text{for } (j,n) \prec (\overline{m}, N_{\mathrm{fin}}) \text{ or } (j,n) \succ (\overline{m}+1, N^{(\overline{m}+1)} + 1 - N_{\mathrm{fin}}) \\ 1 - \varPhi & \text{for } (j,n) = (\overline{m}, N_{\mathrm{fin}}) \text{ or } (j,n) = (\overline{m}+1, N^{(\overline{m}+1)} + 1 - N_{\mathrm{fin}}) \\ 1 & \text{for } (j,n) \succ (\overline{m}, N_{\mathrm{fin}}) \text{ and } (j,n) \prec (\overline{m}+1, N^{(\overline{m}+1)} + 1 - N_{\mathrm{fin}}) \end{cases} \tag{4.53}
$$

and the damping parameter

$$
\kappa_n^{(j)} = \begin{cases} 0 & \text{for } (j,n) \prec (\overline{m}, N_{\mathrm{fin}}) \text{ or } (j,n) \succ (\overline{m}+1, N^{(\overline{m}+1)} - N_{\mathrm{fin}}) \\ \kappa & \text{for } (j,n) \succeq (\overline{m}, N_{\mathrm{fin}}) \text{ and } (j,n) \preceq (\overline{m}+1, N^{(\overline{m}+1)} - N_{\mathrm{fin}}). \end{cases} \tag{4.54}
$$

Last but not least,

Lemma 4.7. *For* ξ *in class* $(m, 0)$ *with* $m > d + (2 + \overline{M})$ *and* $N_< \in \mathbb{N}_0^{\overline{m}-1}$ *such that (4.36) holds,*

$$
\begin{aligned}
&\lim_{R\to\infty} \mathbb{E} \left\| A^{\mathrm{end}}_{\overline{M}, N^{(\overline{m})}} \left(t^{(\overline{m})}; R, L^{(\overline{m})}, \varepsilon\right) \left(\prod_{j<\overline{m}} A_j^\varepsilon F_{N^{(j)}} \left(t^{(j)}; R, L^{(j)}, \varepsilon\right)\right) \psi_0^\varepsilon \right\|_{\mathcal{H}}^2 \\
&\quad = \sum_{S\in\pi^*(I(N))} \mathcal{R}\left(A^{\mathrm{end}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, t, N_<, \overline{N}, \overline{M}, S\right)
\end{aligned}
\tag{4.55}
$$

with $N^{(\overline{m})} = \overline{N}^{(\overline{m})} + \overline{M}$ *and the amplitude* $\mathcal{R}\left(A^{\mathrm{end}}, ..., S\right)$ *of each partition* S *given as in equation (4.52), only with and cut-off functions and damping parameters defined as in Lemma 4.5,*

$$\Phi_n^{(j)} = \begin{cases} \Phi & \text{for } (j,n) \prec \left(\overline{m}, \overline{N}^{(\overline{m})}\right) \text{ or } (j,n) \succ \left(\overline{m}+1, N^{(\overline{m}+1)} + 1 - \overline{N}^{(\overline{m})}\right) \\ 1 & \text{for } (j,n) \succeq \left(\overline{m}, \overline{N}^{(\overline{m})}\right) \text{ or } (j,n) \preceq \left(\overline{m}+1, N^{(\overline{m}+1)} + 1 - \overline{N}^{(\overline{m})}\right) \end{cases} \tag{4.56}$$

$$\kappa_n^{(j)} = \begin{cases} 0 & \text{for } (j,n) \prec \left(\overline{m}, \overline{N}^{(\overline{m})}\right) \text{ or } (j,n) \succ \left(\overline{m}+1, N^{(\overline{m}+1)} - \overline{N}^{(\overline{m})}\right) \\ \kappa & \text{for } (j,n) \succeq \left(\overline{m}, \overline{N}^{(\overline{m})}\right) \text{ or } (j,n) \preceq \left(\overline{m}+1, N^{(\overline{m}+1)} - \overline{N}^{(\overline{m})}\right). \end{cases} \tag{4.57}$$

4.1.4. Graph representation

To estimate the amplitudes given in Lemmas 4.3 and 4.4-4.7, we will represent the contribution stemming from a partition $S \in \pi^*(I(N))$ by a graph similar to the ones introduced in [16] and also employed in [32]. Our graphs and their classification will be more complicated due to the more detailed structure of the Duhamel expansion and the multi-time measurements. As shown in Figures 4.1 and 4.2, the wave function ψ^ε is propagated in from the left and the right, with the solid lines denoting the resolvents regularized only by $i\gamma$, and the dashed lines the resolvents regularized by $i(\gamma+\kappa)$. The graphs are oriented like a scalar product, i.e. with $\widehat{\psi}_0^\varepsilon$ on the right and $\overline{\widehat{\psi}_0^\varepsilon}$ on the left, so with respect to our ordering $\prec$, indices increase from right to left. The measurements by observables A_j^ε are indicated by empty squares. Following the notation of Figures 2.1 and 2.2, a solid diamond denotes an interaction with the medium when a cut-off Φ is present, an empty diamond represents scattering with $1-\Phi$, and a black bullet a full scattering. All three kinds of interactions cause a momentum change and thus a θ variable. Those variables are grouped together by the delta functions induced by the different clusters $A \in S$, so we connect all interactions belonging to the same cluster. Note that there are no one-element clusters as only partitions from $\pi^*(I(N))$, i.e. partitions without isolated elements, contribute.

Definition 4.1. A partition $S \in \pi^*(I(N))$ is called

- **higher order**, if it contains an $A \in S$ with $|A| > 2$. Otherwise it is called a **pairing**, and
- a **crossing pairing**, if there are pairs $\{(j_1,n_1),(j_2,n_2)\} \in S$ and $\{(\tilde{j}_1,\tilde{n}_1),(\tilde{j}_2,\tilde{n}_2)\} \in S$ with $(j_1,n_1) \prec (\tilde{j}_1,\tilde{n}_1) \prec (j_2,n_2) \prec (\tilde{j}_2,\tilde{n}_2)$,
- a **nested pairing** if it is a non-crossing pairing for which a $j \in \{1,...,2\overline{m}\}$ as well as two pairs $\{(j,n_1),(j,n_2)\} \in S$ and $\{(j,\tilde{n}_1),(j,\tilde{n}_2)\} \in S$ exist, such that $n_1 < \tilde{n}_1 < \tilde{n}_2 < n_2$,
- a **non-markovian simple pairing** if it is not crossing or nested and there is a pair $\{(j_1,n_1),(j_2,n_2)\} \in S$ such that neither $j_1 = j_2$, nor $j_1 = 2\overline{m}+1-j_2$.
- a **markovian simple pairing** otherwise.

A new feature compared to the classification in [32], is the occurence of non-markovian simple pairings. These account for the correlation of scattering events across *different* time intervals.

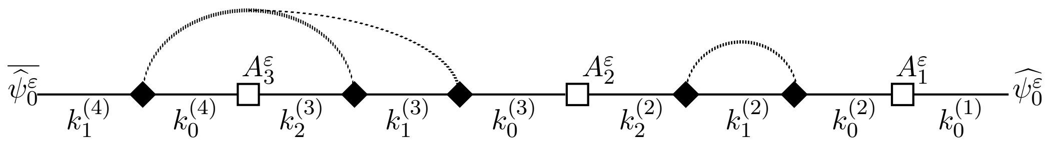

Figure 4.1.: An example of a graph. It belongs to a $\mathcal{K}$ amplitude in the sense of Lemma 4.3, with $\overline{m} = 2$, $N = (0, 2, 2, 1)$ and a (higher order) partition S consisting of one pair and one triple.

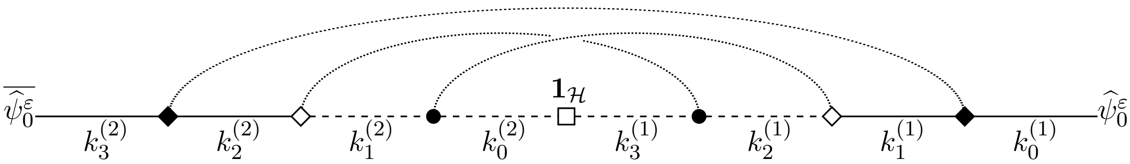

Figure 4.2.: A graph belonging to an $\mathcal{R}(G^{\text{rough}}, ...)$ amplitude, with $\overline{m} = 1$, $N_{\text{fin}} = 2$, $M = 1$. The partition S is a crossing pairing. Note that in contrast to Figure 4.1, the observable in the middle is just the identity operator $\mathbf{1}_{\mathcal{H}}$ on $\mathcal{H}$.

4.2. Basic estimates

As in [16, 32], the key ingredient of the proof of Theorem 3.1 is to find increasingly sharp bounds on the size of all $\mathcal{K}$ and $\mathcal{R}$ amplitudes, depending on which category of Definition 4.1 the partition S falls into. The first, basic estimates presented in Lemmas 4.8 and 4.9 will suffice for higher order partitions S; the idea for estimates on the contributions of pairings S will be to improve the proof of the two Lemmas below by exploiting special structures of S.

Lemma 4.8. (Basic estimate, $\mathcal{K}$ amplitudes.) *For a random field ξ of class $(d+2, 0)$ and any $\gamma \in (0, 1/2]$,*

$$\begin{aligned}
&\left|\mathcal{K}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right)\right| \\
&\qquad \leq C^{|N|+2\overline{m}} \left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{2|N|} \prod_{A\in S} \left\|g_{|A|}\right\|_{d+2} \\
&\qquad\qquad \times e^{2\gamma|t|} \varepsilon^{|N|/2} \gamma^{-|S|} \left|\log\gamma\right|^{|N|+2\overline{m}} \|\psi_0^\varepsilon\|_{\mathcal{H}}^2 \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^0}\,,
\end{aligned} \tag{4.58}$$

with a $C < \infty$ only depending on dimension d.

Proof. Because of the delta functions induced by the clusters $A \in S$, we have for $j \in \{1, ..., 2\overline{m}\}$ and $n \in \{0, ..., N^{(j)}\}$

$$k_n^{(j)} = k_0^{(1)} + \varepsilon \sum_{\tilde{\jmath}<j} p^{(\tilde{\jmath})} + \sum_{\substack{(\tilde{\jmath},\tilde{n})\in I(N): \\ (\tilde{\jmath},\tilde{n})\preceq(j,n) \\ (j,n)\prec \max A(\tilde{\jmath},\tilde{n})}} \theta_{\tilde{n}}^{(\tilde{\jmath})}, \tag{4.59}$$

where $A(\tilde{\jmath}, \tilde{n})$ is the cluster $A \in S$ that contains $(\tilde{\jmath}, \tilde{n})$, and max is defined with respect to $\prec$. Now each k variable is either of the form $k_0^{(j)}$, $j \in \{1, ..., 2\overline{m}\}$, or $k_n^{(j)}$, $(j, n) \in I(N)$. In the latter case, $(j, n) \in I(N)$ is called "free", if $(j, n) \prec \max A(j, n)$ (as in this case, $\theta_n^{(j)}$ is really a new integration variable) and "dependent" if $(j, n) = \max A(j, n)$ (because then, the value of $\theta_n^{(j)}$ is already determined by the $\theta_{\tilde{n}}^{(\tilde{\jmath})}$ with $(\tilde{\jmath}, \tilde{n}) \prec (j, n)$ and the delta functions). Assume for the moment that $N^{(j)} \geq 1$ for all $j \in \{1, ..., 2\overline{m}\}$ and note that by the estimate

$$\left\langle k_n^{(j)} \right\rangle \Phi\left(k_n^{(j)}, k_{n-1}^{(j)}, L_n^{(j)}\right) \leq \left\langle \frac{1}{2} + L_n^{(j)} + \left|\theta_n^{(j)}\right| \right\rangle, \qquad (n \in \{1, ..., N^{(j)}\}), \tag{4.60}$$

a medium smoothness $m \geq d + 2$ yields the existence of an only d-dependent constant $C < \infty$ such that

$$\begin{aligned}
&\left|\mathcal{K}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T, N, S\right)\right| \\
&\leq C^{|N|}\left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{2|N|} \sqrt{\varepsilon}^{|N|} e^{2\gamma(t^{(1)}+...+t^{(\overline{m})})} \prod_{A\in S} \left\|g_{|A|}\right\|_{d+2} \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^0} \\
&\sum_{\sigma\in\{\pm\}^{|N|+2\overline{m}}} \int_{\mathbb{R}^d} \mathrm{d}k_0^{(1)} \left|\widehat{\psi}^\varepsilon_{0,\sigma_0^{(1)}}\left(k_0^{(1)}\right)\right| \left|\widehat{\psi}^\varepsilon_{0,\sigma^{(2\overline{m})}_{N(2\overline{m})}}\left(k_0^{(1)} + \varepsilon \sum_{j=1}^{2\overline{m}-1} p^{(j)}\right)\right| \\
&\prod_{(j,n)\in I(N)} \left(\int_{\mathbb{R}^d} \mathrm{d}\theta_n^{(j)}\right) \int_{\mathbb{R}^{2\overline{m}d}} \mathrm{d}\alpha^{(1)}...\mathrm{d}\alpha^{(2\overline{m})} \prod_{A\in S} \left\{\delta\left(\sum_{(j,n)\in A} \theta_n^{(j)}\right)\right\} \\
&\quad \prod_{(j,n)\in I_0(N)} \frac{1}{\left|\alpha^{(j)} - 2\pi\sigma_n^{(j)}\tau^{(j)}|k_n^{(j)}| + i\gamma\right|} \\
&\quad \prod_{(j,n)\in I(N)} \left(\left\langle\theta_n^{(j)}\right\rangle^{-d} \left\langle k_n^{(j)}\right\rangle^{-1}\right),
\end{aligned} \tag{4.61}$$

where it is understood that all k variables are calculated from the p and θ variables by (4.59). For a choice of signs σ and a fixed $k_0^{(1)}$, focus on the last three lines of (4.61). Altogether, there are $2\overline{m}+|N|$ resolvents, each belonging to one k variable. For resolvents associated with the dependent k variables, take the L^∞ estimate

$$\sup_{k\in\mathbb{R}^d} \frac{1}{|\alpha \pm 2\pi|k| + i\gamma| \langle k \rangle} \leq \frac{C}{\gamma \langle \alpha \rangle}. \tag{4.62}$$

For the $k_n^{(j)}$ with free $(j, n) \in I(N)$, and the $k_0^{(j)}$, we iterate the following procedure (which is a straightforward generalization of Lemma 4.23 in [32]) until there are no more resolvents left: Take the largest (with respect to $\prec$) remaining (j, n).

- if $(j,n) \in I(N)$, $k_n^{(j)} = \theta_n^{(j)} + ...$ is by construction and (4.59) the only remaining k variable depending on $\theta_n^{(j)}$, so integrating out $\theta_n^{(j)}$ generates a factor
$$\int_{\mathbb{R}^d} \mathrm{d}\theta_n^{(j)} \frac{1}{\left|\alpha^{(j)} - 2\pi\sigma_n^{(j)}\tau^{(j)}|k_n^{(j)}| + i\gamma\right| \left\langle k_n^{(j)}\right\rangle \left\langle \theta_n^{(j)}\right\rangle^d} \leq \frac{C|\log\gamma|}{\langle \alpha^{(j)}\rangle}, \tag{4.63}$$
C only depending on d.
- if $n = 0$, there is already a factor $\left\langle \alpha^{(j)}\right\rangle^{-N^{(j)}}$, $N^{(j)} \geq 1$ from the estimates (4.62) and (4.63), and one can integrate out $\alpha^{(j)}$,
$$\int_{\mathbb{R}} \mathrm{d}\alpha^{(j)} \frac{1}{\left|\alpha^{(j)} - 2\pi\sigma_0^{(j)}\tau^{(j)}|k_0^{(j)}| + i\gamma\right| \langle \alpha^{(j)}\rangle} \leq C|\log\gamma|. \tag{4.64}$$

In total, there are $|S|$ factors of type (4.62), $|N| - |S|$ factors of type (4.63) and $2\overline{m}$ estimates (4.64). Now only $k_0^{(1)}$ remains to be integrated over, which is trivial thanks to $\psi_0^\varepsilon \in \mathcal{H}$. The sum over all σ can be accounted for by a factor $2^{|N|+2\overline{m}}$. After possibly redefining C, the claim follows. In case that $N^{(j)} = 0$ for some j, there are no $\alpha^{(j)}$ or $\theta_n^{(j)}$ variables to be integrated over, and one is left with only a factor 1 from the L^∞ estimate of
$$k_0^{(j)} \mapsto \exp\left(-2\pi i \left|k_0^{(j)}\right| \sigma_0^{(j)} t^{(j)}\right) \tag{4.65}$$
for these particular j, while all $k_n^{(\tilde{\jmath})}$ belonging to $\tilde{\jmath}$ with $N^{(\tilde{\jmath})} \geq 1$ are treated as before. □

Lemma 4.9. (Basic estimate, $\mathcal{R}$ amplitudes.) *For ξ in class $(d + 2(\overline{M}+2), 0)$, $\gamma \in (0, 1/2]$, $\kappa > 0$, $M \in \{0, ..., \overline{M}-1\}$, $N_< \in \mathbb{N}_0^{\overline{m}-1}$, obeying* (4.36)*, $N_{\mathrm{fin}} \in \{1, ..., \overline{N}^{(\overline{m})} - 1\}$, and $S \in \pi^*(I(N))$*
$$\begin{aligned}\left|\mathcal{R}\left(G^{\mathrm{rough}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, t, N_<, N_{\mathrm{fin}}, M, S\right)\right| \\ \leq C^{|N|+2\overline{m}} \left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{2|N|+4\overline{M}+4} \langle \varepsilon C_{\mathrm{obs}}\rangle^{4\overline{M}+4} \prod_{A\in S} \left\|g_{|A|}\right\|_{d+2(\overline{M}+2)} \\ \times e^{2\gamma|t|} \varepsilon^{|N|/2} \gamma^{-|S|} |\log\gamma|^{|N|+2\overline{m}} \|\psi_0^\varepsilon\|_{\mathcal{H}}^2 \prod_{j=1}^{\overline{m}-1} \|a_j\|_{C^0}^2,\end{aligned} \tag{4.66}$$
where C is a constant only depending on dimension d. The same bound also holds for the amplitudes $\mathcal{R}\left(G^{\mathrm{end}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, t, N_<, \overline{N}, M, S\right)$,
$\mathcal{R}\left(A^{\mathrm{rough}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, t, N_<, N_{\mathrm{fin}}, \overline{M}, S\right)$, and
$\mathcal{R}\left(A^{\mathrm{end}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, t, N_<, \overline{N}, \overline{M}, S\right)$,
with $N^{(\overline{m})}$, and thus N appropriately defined by (4.38) in each respective case.

Proof. For the most part, one can follow the proof of Lemma 4.8. The stronger requirement $m \geq d + 2(\overline{M}+2)$ and the C_{obs} factor account for the up to $\overline{M}+1$ "uncontrolled" jumps starting at $(\overline{m}, N_{\mathrm{fin}})$ or $(\overline{m}, \overline{N}^{(\overline{m})})$, respectively, as already observed in the proof of Lemma 4.4. □

4.3. Improved estimates

4.3.1. Amputated graphs

In the last proof, we ignored the better estimates that could be obtained from the resolvents which contain the larger regularizing parameter $\gamma+\kappa$ instead of γ, or are completely missing (for the amputated $\mathcal{R}(A^{\text{end}},...)$ and $\mathcal{R}(A^{\text{rough}},...)$). We will utilize those improvements in the next Lemma 4.10. If κ scales like ε^{ϑ}, $\vartheta<1$, the L^∞ bounds (4.62) become much smaller, the L^1 estimates (4.63), however, do not change much due to their logarithmic dependence on γ. Therefore, one essentially has to count the number of L^∞ estimates in the "fade-out" portion of the Duhamel expansion. In [32], this was done rather rather coarsely, necessitating a fade-out expansion of length $\mathcal{O}\left(\overline{N}(\varepsilon)\right)$, which diverges as $\varepsilon\to 0$. Because of the unbounded momentum space, we need to be able to stop the expansion in only finitely many steps after the first interaction of the wave with the "rough" part of the random field, which we achieve by a symmetry argument, effectively swapping the L^1 and L^∞ bounds.

Lemma 4.10. (Improved estimate for amputated amplitudes.) *Let ξ be of class $(d+2(\overline{M}+2),0)$, $\gamma\in(0,1/2]$, $\kappa\in(0,1]$, $N_<\in\mathbb{N}_0^{\overline{m}-1}$ such that (4.36) holds, $N_{\text{fin}}\in\{1,...,\overline{N}^{(\overline{m})}-1\}$ and $S\in\pi^*(I(N))$. Then the estimate (4.66) is still valid after the right hand side has been multiplied by $(\frac{\gamma}{\gamma+\kappa})^{\overline{M}+|S|-|N|/2}$, so*

$$\begin{aligned}\left|\mathcal{R}\left(A^{\text{rough}},\psi_0^\varepsilon,\varepsilon,\kappa,a,p,L^{(0)},t,N_<,N_{\text{fin}},\overline{M},S\right)\right|\\ \leq C^{|N|+2\overline{m}}\left(\langle L^{(0)}\rangle+\overline{m}\overline{N}\right)^{2|N|+4\overline{M}+4}\langle\varepsilon C_{\text{obs}}\rangle^{4\overline{M}+4}\prod_{A\in S}\left\|g_{|A|}\right\|_{d+2(\overline{M}+2)}\\ e^{2\gamma|t|}(\frac{\gamma}{\gamma+\kappa})^{\overline{M}}(\frac{\varepsilon}{\gamma})^{|N|/2}(\gamma+\kappa)^{|N|/2-|S|}\left|\log\gamma\right|^{|N|+2\overline{m}}\|\psi_0^\varepsilon\|_{\mathcal{H}}^2\prod_{j=1}^{\overline{m}-1}\|a_j\|^2.\end{aligned} \tag{4.67}$$

The analogous estimate holds for $\mathcal{R}\left(A^{\text{end}},\psi_0^\varepsilon,\varepsilon,\kappa,a,L^{(0)},t,N_<,\overline{N},\overline{M},S\right)$, with the $S\in\pi^*(I(N))$, and N appropriately defined.

Proof. For the $\mathcal{R}(A^{\text{rough}},...)$ case, take into account the improved resolvent estimates ignored in the proof of Lemma 4.9. If one defines for each $A\in S$ the elements $\min A$ and $\max A$ with respect to $\prec$, $r=|N|-2|S|$ denotes the number of "extra" elements in $I(N)$ that are neither "minimal" nor "maximal", that is neither $\min A$ nor $\max A$ for any $A\in S$. In the sense of the proof of Lemma 4.8, minimal or extra (j,n) are free, while maximal (j,n) are dependent. On the other hand, exactly the $2\overline{M}+2$ resolvents belonging to $k_n^{(j)}$, $(\overline{m},N_{\text{fin}})\preceq(j,n)\preceq(\overline{m}+1,\overline{M})$ are "κ resolvents" or (for $k^{(\overline{m})}_{N_{\text{fin}}+\overline{M}}$ and $k_0^{(\overline{m}+1)}$) are in fact no resolvents at all, but factors of 1. We thus can improve their L^∞ estimate (4.62), which yields an overall improvement by a factor $(\frac{\gamma}{\gamma+\kappa})^{l_1}$, with l_1 the number of "maximal"

(j,n) with $(\overline{m}, N_{\text{fin}}) \preceq (j,n) \preceq (\overline{m}+1, \overline{M})$. By symmetry (the choice to integrate out the resolvents in decreasing $\prec$ order was arbitrary!), we can also obtain a factor $(\frac{\gamma}{\gamma+\kappa})^{l_2}$, l_2 the number of "minimal" (j,n) with $(\overline{m}, N_{\text{fin}}+1) \preceq (j,n) \preceq (\overline{m}+1, \overline{M}+1)$. However, as each $(j,n) \in I(N)$ with $(\overline{m}, N_{\text{fin}}+1) \preceq (j,n) \preceq (\overline{m}+1, \overline{M})$ has to be "minimal", "maximal" or "extra",

$$l_1 + l_2 \geq 2\overline{M} - r, \tag{4.68}$$

and we pick the larger exponent for $\frac{\gamma}{\gamma+\kappa}$, which is

$$\max\{l_1, l_2\} \geq \overline{M} + |S| - |N|/2. \tag{4.69}$$

The improved estimate for $\mathcal{R}(A^{\text{end}}, ...)$ follows analogously. □

4.3.2. Crossing pairings

The last two lemmas will provide sufficiently good estimates for higher order partitions and amputated amplitudes, so in the following, only partitions S that are pairings, and amplitudes of the types $\mathcal{K}$, $\mathcal{R}(G^{\text{end}}, ...)$ and $\mathcal{R}(G^{\text{rough}}, ...)$ will be considered. We start with an estimate for crossing pairings, as defined in Definition 4.1.

Lemma 4.11. (Improved estimate for crossing pairings.) *For ξ in class $(d+3, 0)$, $S \in \pi^*(I(N))$ a crossing pairing, and $\gamma \in [2\varepsilon C_{\text{obs}}\overline{m}, 1/2]$ there is a $C < \infty$ only depending on dimension d such that*

$$\begin{aligned}
&\left|\mathcal{K}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right)\right| \\
&\quad\leq C^{|N|+2\overline{m}} \left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{2|N|+3} \|g_2\|_{d+3}^{|N|/2} \\
&\qquad\times e^{2\gamma|t|} \sqrt{\varepsilon}^{|N|} \gamma^{-|S|+1} |\log\gamma|^{|N|+2\overline{m}+1} \|\psi_0^\varepsilon\|_{\mathcal{H}}^2 \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^0},
\end{aligned} \tag{4.70}$$

for $d \geq 3$, and

$$\begin{aligned}
&\left|\mathcal{K}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right)\right| \\
&\quad\leq C^{|N|+2\overline{m}} \left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{2|N|+3} \|g_2\|_{d+3}^{|N|/2} \\
&\qquad\times e^{2\gamma|t|} \sqrt{\varepsilon}^{|N|} \gamma^{-|S|+1/2} |\log\gamma|^{|N|+2\overline{m}} \|\psi_0^\varepsilon\|_{\mathcal{H}}^2 \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^0}
\end{aligned} \tag{4.71}$$

for $d = 2$.

Proof. We can follow the proof of Lemma 4.16 in [32] word by word to find the existence of a "loose" crossing. This crossing consists of two pairs $\{(j_1,n_1), (j_2,n_2)\} \in S$ and $\{(\tilde{j}_1,\tilde{n}_1), (\tilde{j}_2,\tilde{n}_2)\} \in S$ with $(j_1,n_1) \prec (\tilde{j}_1,\tilde{n}_1) \prec (j_2,n_2) \prec (\tilde{j}_2,\tilde{n}_2)$, and is loose in the sense that any other pair $\{(\hat{j}_1,\hat{n}_1), (\hat{j}_2,\hat{n}_2)\} \in S$ fulfills $(\tilde{j}_1,\tilde{n}_1) \prec (\hat{j}_1,\hat{n}_1) \prec (j_2,n_2)$ if and only if $(\tilde{\jmath}_1, \tilde{n}_1) \prec (\hat{\jmath}_2, \hat{n}_2) \prec (j_2, n_2)$, so the the "crossing interval" between $(\tilde{\jmath}_1, \tilde{n}_1)$ and (j_2, n_2) is not connected to the outside by a third pairing. Also, we require that the crossing we consider is "minimal" in the sense that no further crossing of pairs occurs *within* the crossing interval.

Looseness implies that the θ variables inside the crossing interval cancel, and thus,

$$\begin{aligned} k_{n_2}^{(j_2)} &= k_{\tilde{n}_1}^{(\tilde{\jmath}_1)} + \varepsilon \sum_{j=\tilde{\jmath}_1}^{j_2-1} p^{(j)} + \theta_{n_2}^{(j_2)} = k_{\tilde{n}_1}^{(\tilde{\jmath}_1)} + \varepsilon \sum_{j=\tilde{\jmath}_1}^{j_2-1} p^{(j)} - \theta_{n_1}^{(j_1)} \\ &= k_{\tilde{n}_1}^{(\tilde{\jmath}_1)} - k_{n_1}^{(j_1)} + \varepsilon \sum_{j=\tilde{\jmath}_1}^{j_2-1} p^{(j)} + k_{n_1-1}^{(j_1)}. \end{aligned} \tag{4.72}$$

Note that the "highest" θ variable that $k_{n_2}^{(j_2)}$ depends on is $\theta_{\tilde{n}_1}^{(\tilde{\jmath}_1)}$. Just as in the proof of Lemma 4.8, one has to evaluate the last three lines of (4.61), but with a slightly different order of integration. Also, as ξ is of class $(d+3,0)$, we can utilize a $\left\langle k_n^{(j)} \right\rangle^{-2}$ instead of a $\left\langle k_n^{(j)} \right\rangle^{-1}$ decay for $(j,n) = (j_1, n_1)$, $(j,n) = (\tilde{\jmath}_1, \tilde{n}_1)$ and $(j,n) = (j_2, n_2)$. From now on, assume that all $N^{(j)} \geq 1$, $j \in \{1, ..., 2\overline{m}\}$, so that the resolvent expansion is applicable. This is certainly true for $j \in \{j_1, j_2, \tilde{\jmath}_1, \tilde{\jmath}_2\}$. All other values of j do not play a particular role in the proof and can be treated as in the remark at the end of the proof of Lemma 4.8. Compared to the proof in [32], the different possible patterns in the succession of scatterings and multiple measurements lead one to distinguish three cases that have to be treated differently.

i) $\tilde{\jmath}_1 = j_2$,

ii) $\tilde{\jmath}_1 < j_2$ and there is $\{(\hat{\jmath}_1, \hat{n}_1), (j_2, \hat{n}_2)\} \in S$ such that $(\tilde{\jmath}_1, \tilde{n}_1) \prec (\hat{\jmath}_1, \hat{n}_1) \prec (j_2, 0) \prec (j_2, \hat{n}_2) \prec (j_2, n_2)$, or

iii) neither i) nor ii), so $\tilde{\jmath}_1 < j_2$ and the "highest" θ variable that $k_0^{(j_2)}$ depends on according to (4.72) is $\theta_{\tilde{n}_1}^{(\tilde{\jmath}_1)}$.

In **case i)**, first take L^∞ estimates of all resolvents belonging to dependent $k_n^{(j)}$, *except* for the resolvent for $k_{n_2}^{(j_2)}$, which we keep for the moment. For the resolvents belonging to free $k_n^{(j)}$ or to $k_0^{(j)}$, one iterates the estimates (4.63) or (4.64), respectively, until arriving at $(\tilde{\jmath}_1, \tilde{n}_1) = (j_2, \tilde{n}_1)$, which is free. Here, a change of integration variable from $\theta_{\tilde{n}_1}^{(\tilde{\jmath}_1)}$ to

$k_{\tilde n_1}^{(\tilde j_1)}$ and the dependence structure (4.72) leave us with

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{d}k_{\tilde n_1}^{(\tilde j_1)} \frac{1}{\left|\alpha^{(j_2)} - 2\pi\sigma_{n_2}^{(j_2)}\left|k_{\tilde n_1}^{(\tilde j_1)} - \theta_{n_1}^{(j_1)}\right| + i\gamma\right|\left|\alpha^{(\tilde j_1)} - 2\pi\sigma_{\tilde n_1}^{(\tilde j_1)}\left|k_{\tilde n_1}^{(\tilde j_1)}\right| + i\gamma\right|} \\
&\quad\times \left\langle k_{\tilde n_1}^{(\tilde j_1)} - \theta_{n_1}^{(j_1)}\right\rangle^{-2}\left\langle k_{\tilde n_1}^{(\tilde j_1)}\right\rangle^{-2}\left\langle k_{\tilde n_1}^{(\tilde j_1)} + h(p, \theta_{\prec(\tilde j_1,\tilde n_1)})\right\rangle^{-d} \\
&\le \begin{cases} \frac{C_d}{\left|\theta_{n_1}^{(j_1)}\right|}\frac{\langle\log\gamma\rangle^2}{\langle\alpha^{(\tilde j_1)}\rangle} \text{ if } d \ge 3, \\ \frac{C_2}{\sqrt{\gamma\left|\theta_{n_1}^{(j_1)}\right|}}\frac{\langle\log\gamma\rangle}{\langle\alpha^{(\tilde j_1)}\rangle} \text{ if } d = 2 \end{cases},
\end{aligned} \tag{4.73}$$

in which h is a function of the p variables and the $\theta_n^{(j)}$ with $(j,n) \prec (\tilde j_1, \tilde n_1)$; the estimate is due to Lemma B.1 and independent of h. Now, continue iterating (4.63) or (4.64) for all (j,n) which are free or have $n = 0$, with $(j_1, n_1) \prec (j,n) \prec (\tilde j_1, \tilde n_1)$. As the right side of (4.73) only depends on $\theta_{n_1}^{(j_1)}$, it will just be carried along as a factor in the θ integrals. In case that $\tilde j_1 > j_1$, the denominator $\left\langle\alpha^{(\tilde j_1)}\right\rangle$ from the right hand side of (4.73) allows to evaluate the $\alpha^{(\tilde j_1)}$ integral in the same way as in (4.64). Having arrived at the $\theta_{n_1}^{(j_1)}$ integral, one changes to the integration variable $k_{n_1}^{(j_1)}$, and obtains, by Lemma B.2,

$$\begin{aligned}
&C_d\langle\log\gamma\rangle^2\int_{\mathbb{R}^d}\mathrm{d}k_{n_1}^{(j_1)}\frac{1}{\left|k_{n_1}^{(j_1)} - k_{n_1-1}^{(j_1)}\right|\left|\alpha^{(j_1)} - 2\pi\sigma_{n_1}^{(j_1)}\left|k_{n_1}^{(j_1)}\right| + i\gamma\right|} \\
&\qquad\times\frac{1}{\left\langle k_{n_1}^{(j_1)}\right\rangle^2\left\langle k_{n_1}^{(j_1)} + q(p,\theta_{\prec(j_1,p_1)})\right\rangle^d} \\
&\le \frac{\tilde C_d\langle\log\gamma\rangle^3}{\sqrt{\langle\alpha^{(j_1)}\rangle}}
\end{aligned} \tag{4.74}$$

for $d \ge 3$, and

$$\begin{aligned}
&\frac{C_2\langle\log\gamma\rangle}{\sqrt{\gamma}}\int_{\mathbb{R}^2}\mathrm{d}k_{n_1}^{(j_1)}\frac{1}{\sqrt{\left|k_{n_1}^{(j_1)} - k_{n_1-1}^{(j_1)}\right|}\left|\alpha^{(j_1)} - 2\pi\sigma_{n_1}^{(j_1)}\left|k_{n_1}^{(j_1)}\right| + i\gamma\right|\left\langle k_{n_1}^{(j_1)}\right\rangle^2} \\
&\le \frac{\tilde C_2\langle\log\gamma\rangle^2}{\sqrt{\gamma}\langle\alpha^{(j_1)}\rangle}
\end{aligned} \tag{4.75}$$

for $d = 2$. For all remaining resolvents belonging to $k_n^{(j)}$, $(j,n) \prec (j_1, n_1)$, one can directly follow the proof of Lemma 4.8. The only difference is the resolvent belonging to $k_0^{(j_1)}$, which is possibly only regularized by a denominator $\sqrt{\langle\alpha^{(j_1)}\rangle}$ instead of the $\left\langle\alpha^{(j_1)}\right\rangle$ appearing in (4.64), so we employ

$$\int_{\mathbb{R}}\mathrm{d}\alpha^{(j)}\frac{1}{\left|\alpha^{(j)} - 2\pi\sigma_0^{(j)}|k_0^{(j)}| + i\gamma\right|\langle\alpha^{(j)}\rangle^\delta} \le C_\delta|\log\gamma| \quad (\delta > 0) \tag{4.76}$$

with $\delta = 1/2$ for $j = j_1$. Altogether, two L^1 and one L^∞ estimate have been replaced by a factor $C_d \langle \log \gamma \rangle^3$ (for $d \geq 3$) or $C_2 \langle \log \gamma \rangle^2 / \sqrt{\gamma}$ (for $d = 2$). This yields an overall improvement by a factor $C\gamma \langle \log \gamma \rangle$ ($d \geq 3$) or $C\sqrt{\gamma}$ ($d = 2$), and proves the Lemma for case i).

Suppose the crossing is such as described in **case ii)**. The pair $\{(\hat{\jmath}_1, \hat{n}_1), (\jmath_2, \hat{n}_2)\}$ that exists by assumption may not be unique, but we now pick one and do not change it during the proof. One can start again by taking L^∞ estimates of all dependent $k_n^{(j)}$ except for $k_{n_2}^{(j_2)}$. Then, integrate out all $\theta_n^{(j)}$ for free (j,n) and all $\alpha^{(j)}$, but only for $(j,n) \succ (j_2, n_2)$, and $j > j_2$, respectively. Next, call the free (j,n) with $(\hat{\jmath}_1, \hat{n}_1) \preceq (j,n) \prec (j_2, \hat{n}_2)$ *protected*, and for those protected (j,n), switch the integration variables $\theta_n^{(j)}$ to the variables $k_n^{(j)}$. Those $k_n^{(j)}$ as well as the $k_0^{(j)}$ with $\hat{\jmath}_1 < j \leq j_2$ do no longer depend on any of the other remaining free θ variables (here, the assumption of a minimal crossing is important), so one can continue to evaluate the integrals as follows — integrate over all $\theta_n^{(j)}$, (j,n) free, but not protected, with $(\tilde{\jmath}_1, \tilde{n}_1) \prec (j,n) \prec (j_2, n_2)$, and over all $\alpha^{(j)}$, $\tilde{\jmath}_1 < j < \hat{\jmath}_1$ in the usual decreasing $\prec$ order, using the estimates (4.63), (4.64). Do not touch any of the $\alpha^{(j)}$ with $\hat{\jmath}_1 < j \leq j_2$ or the protected $k_n^{(j)}$ for now. In the $\theta_{\tilde{n}_1}^{(\tilde{\jmath}_1)}$ integral, which only involves the $k_{\tilde{n}_1}^{(\tilde{\jmath}_1)}$ and $k_{n_2}^{(j_2)}$ resolvents, switch to the integration variable $k_{\tilde{n}_1}^{(\tilde{\jmath}_1)}$ to obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{d}k_{\tilde{n}_1}^{(\tilde{\jmath}_1)} \frac{1}{\left|\alpha^{(j_2)} - 2\pi\sigma_{n_2}^{(j_2)} \left|k_{\tilde{n}_1}^{(\tilde{\jmath}_1)} - \theta_{n_1}^{(j_1)} + f(p)\right| + i\gamma\right| \left|\alpha^{(\tilde{\jmath}_1)} - 2\pi\sigma_{\tilde{n}_1}^{(\tilde{\jmath}_1)} \left|k_{\tilde{n}_1}^{(\tilde{\jmath}_1)}\right| + i\gamma\right|} \\
&\quad \times \left\langle k_{\tilde{n}_1}^{(\tilde{\jmath}_1)} - \theta_{n_1}^{(j_1)} + f(p)\right\rangle^{-2} \left\langle k_{\tilde{n}_1}^{(\tilde{\jmath}_1)}\right\rangle^{-2} \left\langle k_{\tilde{n}_1}^{(\tilde{\jmath}_1)} + h(p, \theta_{\prec(\tilde{\jmath}_1, \tilde{n}_1)})\right\rangle^{-d} \\
&\leq \begin{cases} \frac{C_d}{\left|\theta_{n_1}^{(j_1)} - f(p)\right|} \frac{\langle \log \gamma \rangle^2}{\sqrt{\langle \alpha^{(\tilde{\jmath}_1)} \rangle \langle \alpha^{(j_2)} \rangle}} & \text{if } d \geq 3, \\ \frac{C_2}{\sqrt{\gamma \left|\theta_{n_1}^{(j_1)} - f(p)\right|}} \frac{\langle \log \gamma \rangle}{\sqrt{\langle \alpha^{(\tilde{\jmath}_1)} \rangle \langle \alpha^{(j_2)} \rangle}} & \text{if } d = 2 \end{cases} ,
\end{aligned} \tag{4.77}
$$

as before, just with and additional function f of the p variables. Now, in decreasing $\prec$ order, integrate out all resolvents belonging to $k_n^{(j)}$, with (j,n) protected, or $(j,n) = (j,0)$ with $\hat{\jmath}_1 < j \leq j_2$, and then for all (j,n) which are free and fulfill $(j_1, n_1) \prec (j,n) \prec (\tilde{\jmath}_1, \tilde{n}_1)$ and all $(j,n) = (j,0)$ with $j_1 < j \leq \tilde{\jmath}_1$. This can be done with the usual (4.63) and (4.64) estimates, and by noting that the denominator $\sqrt{\langle \alpha^{(\tilde{\jmath}_1)} \rangle \langle \alpha^{(j_2)} \rangle}$ in (4.77) allows for the (4.76), $\delta = 1/2$ bound of the $\alpha^{(\tilde{\jmath}_1)}$ and $\alpha^{(j_2)}$ integrals. (Here we assume $\tilde{\jmath}_1 > j_1$. If $\tilde{\jmath}_1 = j_1$, the $\alpha^{(\tilde{\jmath}_1)} = \alpha^{(j_1)}$ integral will only be evaluated *after* (4.78) or (4.79)). Once again, one can evaluate the $\theta_{n_1}^{(j_1)}$ integral by changing to $k_{n_1}^{(j_1)}$ and finding

$$\begin{aligned} & C_d \left\langle \log \gamma \right\rangle^2 \int_{\mathbb{R}^d} \mathrm{d}k_{n_1}^{(j_1)} \frac{1}{\left| k_{n_1}^{(j_1)} - k_{n_1-1}^{(j_1)-f(p)} \right| \left| \alpha^{(j_1)} - 2\pi\sigma_{n_1}^{(j_1)} \left| k_{n_1}^{(j_1)} \right| + i\gamma \right|} \\ & \qquad \times \frac{1}{\left\langle k_{n_1}^{(j_1)} \right\rangle^2 \left\langle k_{n_1}^{(j_1)} + q(p, \theta_{\prec(j_1,p_1)}) \right\rangle^d} \\ & \leq \frac{\tilde{C}_d \left\langle \log \gamma \right\rangle^3}{\sqrt{\langle \alpha^{(j_1)} \rangle}} \end{aligned} \tag{4.78}$$

for $d \geq 3$, and

$$\begin{aligned} & \frac{C_2 \left\langle \log \gamma \right\rangle}{\sqrt{\gamma}} \int_{\mathbb{R}^2} \mathrm{d}k_{n_1}^{(j_1)} \frac{1}{\sqrt{\left| k_{n_1}^{(j_1)} - k_{n_1-1}^{(j_1)} - f(p) \right|} \left| \alpha^{(j_1)} - 2\pi\sigma_{n_1}^{(j_1)} \left| k_{n_1}^{(j_1)} \right| + i\gamma \right| \left\langle k_{n_1}^{(j_1)} \right\rangle^2} \\ & \leq \frac{\tilde{C}_2 \left\langle \log \gamma \right\rangle^2}{\sqrt{\gamma} \langle \alpha^{(j_1)} \rangle} \end{aligned} \tag{4.79}$$

for $d = 2$. For the remaining integrals, one can follow the proof of the standard estimate to end up with the same improved estimate as in case i).

In **case iii)**, keep $k_{n_2}^{(j_2)}$ again and take L^∞ estimates of all other $k_n^{(j)}$ resolvents with dependent (j, n). Then, integrate out all $\theta_n^{(j)}$ with free $(j, n) \succ (\tilde{j}_1, \tilde{n}_1)$ and all $\alpha^{(j)}$ with $j > \tilde{j}_1$ *except* for $\alpha^{(j_2)}$. Now, there are three resovents left which depend on $\theta_{\tilde{n}_1}^{(\tilde{j}_1)}$, namely those belonging to $k_{\tilde{n}_1}^{(\tilde{j}_1)} = \theta_{\tilde{n}_1}^{(\tilde{j}_1)} - h(p, \theta_{\prec(\tilde{j}_1,\tilde{n}_1)})$, as well as $k_0^{(j_2)} = k_{\tilde{n}_1}^{(\tilde{j}_1)} + f(p)$ and $k_{n_2}^{(j_2)} = k_{\tilde{n}_1}^{(\tilde{j}_1)} - \theta_{n_1}^{(j_1)} + f(p)$. Here, as before, h is a function of the p variables and the $\theta_n^{(j)}$ variables with $(j, n) \prec (\tilde{j}_1, \tilde{n}_1)$, while f depends only on the p variables, and by our assumptions fulfills

$$|f(p)| = \varepsilon \left| \sum_{j=\tilde{j}_1}^{j_2-1} p^{(j)} \right| \leq 2\varepsilon C_{\mathrm{obs}} \overline{m} \tag{4.80}$$

for all p from the support of the observables. Therefore, for $\gamma \geq 2\varepsilon C_{\mathrm{obs}} \overline{m}$,

$$\left| \alpha^{(j_2)} - 2\pi\sigma_0^{(j_2)} \left| k_{\tilde{n}_1}^{(\tilde{j}_1)} + f(p) \right| + i\gamma \right|^{-1} \leq C \left| \alpha^{(j_2)} - 2\pi\sigma_0^{(j_2)} \left| k_{\tilde{n}_1}^{(\tilde{j}_1)} \right| + i\gamma \right|^{-1}, \tag{4.81}$$

with a factor C only depending on dimension d, and one can estimate the $\theta_{\tilde{n}_1}^{(\tilde{j}_1)}$ integral by

$$\begin{aligned} & C \int_{\mathbb{R}^d} \mathrm{d}k_{\tilde{n}_1}^{(\tilde{j}_1)} \frac{1}{\left| \alpha^{(\tilde{j}_1)} - 2\pi\sigma_{\tilde{n}_1}^{(\tilde{j}_1)} \left| k_{\tilde{n}_1}^{(\tilde{j}_1)} \right| + i\gamma \right| \left| \alpha^{(j_2)} - 2\pi\sigma_0^{(j_2)} \left| k_{\tilde{n}_1}^{(\tilde{j}_1)} \right| + i\gamma \right|} \\ & \quad \times \frac{1}{\left| \alpha^{(j_2)} - 2\pi\sigma_{n_2}^{(j_2)} \left| k_{\tilde{n}_1}^{(\tilde{j}_1)} + f(p) - \theta_{n_1}^{(j_1)} \right| + i\gamma \right|} \\ & \quad \times \left\langle k_{\tilde{n}_1}^{(\tilde{j}_1)} \right\rangle^{-2} \left\langle k_{\tilde{n}_1}^{(\tilde{j}_1)} + f(p) - \theta_{n_1}^{(j_1)} \right\rangle^{-2} \left\langle k_{\tilde{n}_1}^{(\tilde{j}_1)} + h(p, \theta_{\prec(\tilde{j}_1,\tilde{n}_1)}) \right\rangle^{-d}. \end{aligned} \tag{4.82}$$

For $d \geq 3$, the substitution (B.3) with $u = \theta_{n_1}^{(j_1)} - f(p)$ yields the estimate

$$\begin{aligned}
&\frac{C \cdot C_d'}{\left|\theta_{n_1}^{(j_1)} - f(p)\right|} \int_0^\infty \mathrm{d}\rho_1 \frac{1}{\left|\alpha^{(\tilde{\jmath}_1)} - 2\pi\sigma_{\tilde{n}_1}^{(\tilde{\jmath}_1)}\rho_1 + i\gamma\right| \left|\alpha^{(j_2)} - 2\pi\sigma_0^{(j_2)}\rho_1 + i\gamma\right| \langle\rho_1\rangle} \\
&\qquad\qquad \times \int_0^\infty \mathrm{d}\rho_2 \frac{1}{\left|\alpha^{(j_2)} - 2\pi\sigma_{n_2}^{(j_2)}\rho_2 + i\gamma\right| \langle\rho_2\rangle} \\
&\leq \frac{\eta_d\left(\alpha^{(\tilde{\jmath}_1)}, \alpha^{(j_2)}, \gamma\right)}{\left|\theta_{n_1}^{(j_1)} - f(p)\right|},
\end{aligned} \tag{4.83}$$

with

$$\begin{aligned}
&\eta_d\left(\alpha^{(\tilde{\jmath}_1)}, \alpha^{(j_2)}, \gamma\right) \\
&\quad \leq \frac{C\langle\log\gamma\rangle}{\sqrt{\langle\alpha^{(j_2)}\rangle}} \min\left(\frac{1}{\gamma\sqrt{\langle\alpha^{(\tilde{\jmath}_1)}\rangle\langle\alpha^{(j_2)}\rangle}}, \frac{\langle\log\gamma\rangle}{\left||\alpha^{(\tilde{\jmath}_1)}| - |\alpha^{(j_2)}|\right|}\left(\frac{1}{\sqrt{\langle\alpha^{(j_2)}\rangle}} + \frac{1}{\sqrt{\langle\alpha^{(\tilde{\jmath}_1)}\rangle}}\right)\right),
\end{aligned} \tag{4.84}$$

with a new C depending only on dimension d. One can now take the $\alpha^{(j_2)}$ integral (there are no $\alpha^{(j_2)}$-dependent resolvents left!) to obtain a factor

$$\int_{\mathbb{R}} \mathrm{d}\alpha^{(j_2)} \eta_d\left(\alpha^{(\tilde{\jmath}_1)}, \alpha^{(j_2)}, \gamma\right) \leq \frac{C'\langle\log\gamma\rangle^3}{\sqrt{\langle\alpha^{(\tilde{\jmath}_1)}\rangle}}. \tag{4.85}$$

For $d = 2$, the estimate for (4.82) is of the form

$$\frac{\eta_2\left(\alpha^{(\tilde{\jmath}_1)}, \alpha^{(j_2)}, \gamma\right)}{\sqrt{\left|\theta_{n_1}^{(j_1)} - f(p)\right|}} \tag{4.86}$$

with

$$\int_{\mathbb{R}} \mathrm{d}\alpha^{(j_2)} \eta_2\left(\alpha^{(\tilde{\jmath}_1)}, \alpha^{(j_2)}, \gamma\right) \leq \frac{C'\langle\log\gamma\rangle^2}{\sqrt{\gamma}\sqrt{\langle\alpha^{(\tilde{\jmath}_1)}\rangle}}. \tag{4.87}$$

Now one can integrate out in the usual order all free $k_n^{(j)}$, $(j_1, n_1) \prec (j, n) \prec (\tilde{\jmath}_1, \tilde{n}_1)$ and all $\alpha^{(j)}$, $\tilde{\jmath}_1 < j \leq \tilde{\jmath}_1$, by the standard estimates (4.62), (4.63), and, for the $\alpha^{(\tilde{\jmath}_1)}$ integral, (4.76) with $\delta = 1/2$. The rest of the resolvent integrals is taken as in case ii). In the case at hand, we then have an avoided two L^1, one L^∞ resolvent estimate, as well as the estimate (4.64) for $\alpha^{(j_2)}$, and replaced them by factors $C\langle\log\gamma\rangle^4$ (if $d \geq 3$) or $C\langle\log\gamma\rangle^3/\sqrt{\gamma}$ ($d = 2$). The overall gained factors, again, are $C\gamma\langle\log\gamma\rangle$ ($d \geq 3$) or $C\sqrt{\gamma}$ ($d = 2$), with C depending only on dimension d. This finishes the proof of Lemma 4.11. □

Lemma 4.12. (Improved estimate for crossing pairings, $\mathcal{R}(G^{\text{rough}}, ...)$ and $\mathcal{R}(G^{\text{end}}, ...)$ amplitudes.) *For ξ of class $(d+2(\overline{M}+2)+3, 0)$, $\gamma \in [2\varepsilon C_{\text{obs}}\overline{m}, 1/2]$, $\kappa > 0$, $M \in \{0, ..., \overline{M}-1\}$, $N_< \in \mathbb{N}_0^{\overline{m}-1}$, such that (4.36) holds, $N_{\text{fin}} \in \{1, ..., \overline{N}^{(\overline{m})}-1\}$, and $S \in \pi^*(I(N))$ a crossing pairing,*

$$\begin{aligned} &\left|\mathcal{R}\left(G^{\text{rough}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, t, N_<, N_{\text{fin}}, M, S\right)\right| \\ &\qquad \le C^{|N|+2\overline{m}} \left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{2|N|+4\overline{M}+7} \langle \varepsilon C_{\text{obs}}\rangle^{4\overline{M}+7} \|g_2\|_{d+2(\overline{M}+2)+3}^{|N|/2} \\ &\qquad\qquad \times e^{2\gamma|t|}\varepsilon^{|N|/2}\gamma^{-|S|+1} |\log\gamma|^{|N|+2\overline{m}+1} \|\psi_0^\varepsilon\|_{\mathcal{H}}^2 \prod_{j=1}^{\overline{m}-1} \|a_j\|_{C^0}^2 , \end{aligned} \tag{4.88}$$

for $d \ge 3$,

$$\begin{aligned} &\left|\mathcal{R}\left(G^{\text{rough}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, t, N_<, N_{\text{fin}}, M, S\right)\right| \\ &\qquad \le C^{|N|+2\overline{m}} \left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{2|N|+4\overline{M}+7} \langle \varepsilon C_{\text{obs}}\rangle^{4\overline{M}+7} \|g_2\|_{d+2(\overline{M}+2)+3}^{|N|/2} \\ &\qquad\qquad \times e^{2\gamma|t|}\varepsilon^{|N|/2}\gamma^{-|S|+1/2} |\log\gamma|^{|N|+2\overline{m}} \|\psi_0^\varepsilon\|_{\mathcal{H}}^2 \prod_{j=1}^{\overline{m}-1} \|a_j\|_{C^0}^2 , \end{aligned} \tag{4.89}$$

for $d = 2$, where C is a constant only depending on dimension d. The analogous bound also holds for $\mathcal{R}\left(G^{\text{end}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, L^{(0)}, t, N_<, \overline{N}, M, S\right)$, if $N^{(\overline{m})}$ and thus N are appropriately defined.

Proof. The proof is exactly the same as for Lemma 4.11. For the necessary $\left\langle k_n^{(j)}\right\rangle^{-2}$ instead of a $\left\langle k_n^{(j)}\right\rangle^{-1}$ decay for $(j,n) = (j_1, n_1)$, $(j,n) = (\tilde{j}_1, \tilde{n}_1)$ and $(j,n) = (j_2, n_2)$, one has to assume $m \ge d + 2(\overline{M}+2) + 3$. As in the proof of Lemma 4.9, the improved estimates stemming from the resolvents with an additional κ are ignored. □

4.4. Non-crossing pairings

4.4.1. Suppression of jumps outside the cut-off Φ

Thanks to Lemma 4.11 and 4.12, we can from now on focus on non-crossing pairings. Next, we will show that the interaction of the wave with the "rough" part of the medium is supressed, i.e. that a violation of the cut-off Φ will produce amplitudes that vanish in the $\varepsilon \to 0$ limit.

Lemma 4.13. (Improved estimate for $\mathcal{R}(G^{\text{rough}}, ...)$ amplitudes, non-crossing pairings.) *For ξ of class $(d+2(\overline{M}+2), 0)$, $\varepsilon > 0$ such that $\varepsilon\overline{m}C_{\text{obs}} < \frac{1}{4}$, $\gamma \in (0, 1/2]$, $\kappa > 0$,*

$M \in \{0, ..., \overline{M} - 1\}$, $N_< \in \mathbb{N}_0^{\overline{m}-1}$ with (4.36), $N_{\text{fin}} \in \{1, ..., \overline{N}^{(\overline{m})} - 1\}$, and $S \in \pi^*(I(N))$ a non-crossing pairing,

$$\begin{aligned}
&\left|\mathcal{R}\left(G^{\text{rough}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, L^{(0)}, t, N_<, N_{\text{fin}}, M, S\right)\right| \\
&\quad \le C^{|N|+2\overline{m}} \left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{2|N|+4\overline{M}+4} \langle \varepsilon C_{\text{obs}}\rangle^{4\overline{M}+4} \|g_2\|_{d+2(\overline{M}+2)}^{|N|/2} \\
&\qquad \times e^{2\gamma|t|} \varepsilon^{|N|/2} \gamma^{-|S|+1} |\log\gamma|^{|N|+2\overline{m}} \|\psi^\varepsilon\|_{\mathcal{H}}^2 \prod_{j=1}^{\overline{m}-1} \|a_j\|_{C^0}^2 ,
\end{aligned} \tag{4.90}$$

with C a constant depending only on dimension d.

Proof. We can follow the proof of Lemmas 4.8 and 4.9 and find a constant C only depending on dimension d such that

$$\begin{aligned}
&\left|\mathcal{R}\left(G^{\text{rough}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, t, N_<, N_{\text{fin}}, M, S\right)\right| \\
&\le C^{|N|+2\overline{m}} \left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{2|N|+4\overline{M}+4} \langle \varepsilon C_{\text{obs}}\rangle^{4\overline{M}+4} \varepsilon^{|N|/2} e^{2\gamma|t|} \|g_2\|_{d+2(\overline{M}+2)}^{|N|/2} \prod_{j=1}^{\overline{m}-1} \|a_j\|_{C^0}^2 \\
&\sum_\sigma \int_{\mathbb{R}^d} \mathrm{d}k_0^{(1)} \left|\widehat{\psi}^\varepsilon_{0,\sigma_0^{(1)}}\left(k_0^{(1)}\right)\right| \left|\widehat{\psi}^\varepsilon_{0,\sigma^{(2\overline{m})}_{N^{(2\overline{m})}}}\left(k_0^{(1)}\right)\right| \\
&\prod_{(j,n)\in I(N)} \left(\int_{\mathbb{R}^d} \mathrm{d}\theta_n^{(j)}\right) \int_{\mathbb{R}^{2\overline{m}d}} \mathrm{d}\alpha^{(1)} ... \mathrm{d}\alpha^{(2\overline{m})} \prod_{A\in S} \left\{\delta\left(\sum_{(j,n)\in A} \theta_n^{(j)}\right)\right\} \\
&\quad \prod_{(j,n)\in I_0(N)} \frac{1}{\left|\alpha^{(j)} - 2\pi\sigma_n^{(j)}\tau^{(j)}|k_n^{(j)}| + i\gamma\right|} \\
&\quad \prod_{(j,n)\in I(N)} \left(\left\langle\theta_n^{(j)}\right\rangle^{-d} \left\langle k_n^{(j)}\right\rangle^{-1}\right) \\
&\quad (1-\varPhi)\left(k_{\hat{n}-1}^{(\hat{\jmath})}, k_{\hat{n}}^{(\hat{\jmath})}, L_{\hat{n}}^{(\hat{\jmath})}\right) \prod_{\substack{(j,n)\in I(N)\\(j,n)\succ(\hat{\jmath},\hat{n})}} \varPhi(k_{n-1}^{(j)}, k_n^{(j)}, L_n^{(j)}).
\end{aligned} \tag{4.91}$$

Note that this time, we have kept the $\varPhi$ and $1-\varPhi$ cut-off functions, at least for the $k^{(j)}$, $j > \overline{m}$ variables, and used the shorthand $(\hat{\jmath}, \hat{n}) = (\overline{m}+1, N^{(\overline{m}+1)} + 1 - N_{\text{fin}})$. The key observation is that $(\hat{\jmath}, \hat{n})$ is the $\prec$-largest $(j,n) \in I(N)$ for which both $k_{n-1}^{(j)}$ and $k_n^{(j)}$ are large. One can distinguish the following types of partitions.

- First, assume that S does not contain pairs $\{(j_1, n_1), (j_2, n_2)\}$ with $(j_1, n_1) \preceq (\hat{\jmath}, \hat{n})$ and $(j_2, n_2) \succ (\hat{\jmath}, \hat{n})$. In that case,

$$\sum_{(j,n)\succ(\hat{\jmath},\hat{n})} \theta_n^{(j)} = 0, \tag{4.92}$$

so

$$k^{(\overline{m})}_{N^{(2\overline{m})}} = k^{(1)}_0 + \varepsilon \sum_{j=1}^{2\overline{m}-1} p^{(j)} = k^{(\hat{\jmath})}_{\hat{n}} + \varepsilon \sum_{j=\overline{m}+1}^{2\overline{m}-1} p^{(j)}. \tag{4.93}$$

Observe that for $k_1, k_2 \in \mathbb{R}^d$, $L \in \mathbb{R}$,

$$1 - \Phi(k_1, k_2, L) = (1-\varphi)(k_1, k_2, L)(1-\varphi)(k_1, k_2, L) \tag{4.94}$$

is different from zero only if $|k_1|, |k_2| \geq L$, so the integrand in (4.91) is nonzero only if $\left|k^{(\hat{\jmath})}_{\hat{n}}\right| \geq L^{(\hat{\jmath})}_{\hat{n}} \geq L^{(0)} + 1$. On the other hand, by the support properties of $\widehat{\psi^\varepsilon}$, the integrand is also zero if $\left|k^{(\overline{m})}_{N^{(2\overline{m})}}\right| > L^{(0)}$, and thus, whenever $\varepsilon C_{\text{obs}} \overline{m} < 1$, the amplitudes of partitions S with the aforementioned property are always exactly zero.

- Conversely, if S does contain a pair $\{(j_1, n_1), (j_2, n_2)\}$ with $(j_1, n_1) \preceq (\hat{\jmath}, \hat{n})$ and $(j_2, n_2) \succ (\hat{\jmath}, \hat{n})$, choose the smallest (with respect to $\prec$) such (j_2, n_2). Define $(\tilde{\jmath}_2, \tilde{n}_2)$ to be the direct precursor of (j_2, n_2) in $I(N)$ with respect to $\prec$, and assume for the moment that $(\tilde{\jmath}_2, \tilde{n}_2) \succ (\hat{\jmath}, \hat{n})$. If $n_2 = 1$, this means $\tilde{\jmath}_2 = j_2 - 1$, $\tilde{n}_2 = N^{(\tilde{\jmath}_2)}$, while $(\tilde{\jmath}_2, \tilde{n}_2) = (j_2, n_2 - 1)$ if $n_2 > 1$. In any case, it is easy to see that, by the non-crossing property of S, both (j_2, n_2) and $(\tilde{\jmath}_2, \tilde{n}_2)$ are dependent in the sense of the proof of Lemma 4.8, and that, by construction

$$\sum_{(\hat{\jmath},\hat{n}) \prec (j,n) \preceq (j_2, n_2 - 1)} \theta^{(j)}_n = \sum_{(\hat{\jmath},\hat{n}) \prec (j,n) \preceq (\tilde{\jmath}_2, \tilde{n}_2)} \theta^{(j)}_n = 0, \tag{4.95}$$

and thus both

$$\left|k^{(\tilde{\jmath}_2)}_{\tilde{n}_2}\right| \geq \left|k^{(\hat{\jmath})}_{\hat{n}}\right| - \varepsilon \overline{m} C_{\text{obs}} \geq L^{(\hat{\jmath})}_{\hat{n}} - \varepsilon \overline{m} C_{\text{obs}} \tag{4.96}$$

and

$$\left|k^{(j_2)}_{n_2 - 1}\right| \geq L^{(\hat{\jmath})}_{\hat{n}} - \varepsilon \overline{m} C_{\text{obs}} \geq L^{(\hat{\jmath})}_{\hat{n}} - 1/4 \geq L^{(j_2)}_{n_2} + 3/4. \tag{4.97}$$

Thus, whenever $\Phi\left(k^{(j_2)}_{n_2-1}, k^{(j_2)}_{n_2}, L^{(j_2)}_{n_2}\right) \neq 0$, one has

$$\left|k^{(j_2)}_{n_2}\right| \leq L^{(j_2)}_{n_2} + \frac{1}{2} \leq L^{(\hat{\jmath})}_{\hat{n}} - \frac{1}{2} \leq \left|k^{(\tilde{\jmath}_2)}_{\tilde{n}_2}\right| - \frac{1}{2} + \varepsilon \overline{m} C_{\text{obs}} \leq \left|k^{(\tilde{\jmath}_2)}_{\tilde{n}_2}\right| - \frac{1}{4}. \tag{4.98}$$

One can therefore take the L^∞ estimates for the $k^{(\tilde{\jmath}_2)}_{\tilde{n}_2}$ and $k^{(j_2)}_{n_2}$ resolvents simultaneously,

$$\begin{aligned} &\sup_{|k^{(j_2)}_{n_2}| \leq |k^{(\tilde{\jmath}_2)}_{\tilde{n}_2}| - \frac{1}{4}} \left(\frac{1}{\left|\alpha^{(\tilde{\jmath}_2)} \pm 2\pi |k^{(\tilde{\jmath}_2)}_{\tilde{n}_2}| + i\gamma\right| \left\langle k^{(\tilde{\jmath}_2)}_{\tilde{n}_2} \right\rangle \cdot \left|\alpha^{(j_2)} \pm 2\pi |k^{(j_2)}_{n_2}| + i\gamma\right| \left\langle k^{(j_2)}_{n_2} \right\rangle} \right) \\ &\qquad \leq \frac{C}{\gamma \left\langle \alpha^{(\tilde{\jmath}_2)} \right\rangle \left\langle \alpha^{(j_2)} \right\rangle}, \end{aligned} \tag{4.99}$$

gaining an overall improvement of the estimate by a factor γ.

- Let (j_2, n_2) be defined as in the last case, but suppose S is such that the precursor of (j_2, n_2) is $(\hat{\jmath}, \hat{n})$, and even that $\{(\hat{\jmath}, \hat{n}), (j_2, n_2)\} \in S$ is a pair. Then $|k_{n_2}^{(j_2)}| \geq |k_{\hat{n}-1}^{(\hat{\jmath})}| - \varepsilon \overline{m} C_{\text{obs}}$ and $|k_{n_2-1}^{(j_2)}| \geq |k_{\hat{n}}^{(\hat{\jmath})}| - \varepsilon \overline{m} C_{\text{obs}}$ but for the integrand of (4.91) to be different from zero, both

$$\min\left\{\left|k_{\hat{n}-1}^{(\hat{\jmath})}\right|, \left|k_{\hat{n}}^{(\hat{\jmath})}\right|\right\} \geq L_{\hat{n}}^{(\hat{\jmath})} \tag{4.100}$$

and

$$\min\left\{\left|k_{n_2-1}^{(j_2)}\right|, \left|k_{n_2}^{(j_2)}\right|\right\} \leq L_{n_2}^{(j_2)} + \frac{1}{2} \leq L_{\hat{n}}^{(\hat{\jmath})} - \frac{1}{2} \tag{4.101}$$

need to hold, which is impossible for $\overline{m}\varepsilon C_{\text{obs}} < \frac{1}{2}$.

- The last possibility is that, with (j_2, n_2) defined as before, the precursor of (j_2, n_2) is $(\hat{\jmath}, \hat{n})$, but $\{(\hat{\jmath}, \hat{n}), (j_2, n_2\} \notin S$. In that case, for the integrand of (4.91) to be different from zero,

$$\begin{aligned}\min\left\{\left|k_{\hat{n}}^{(\hat{\jmath})} - \varepsilon \overline{m} C_{\text{obs}}\right|, \left|k_{n_2}^{(j_2)}\right|\right\} &\leq \min\left\{\left|k_{n_2-1}^{(j_2)}\right|, \left|k_{n_2}^{(j_2)}\right|\right\} \\ &\leq L_{n_2}^{(j_2)} + \frac{1}{2} \leq L_{\hat{n}}^{(\hat{\jmath})} - \frac{1}{2} \leq \left|k_{\hat{n}}^{(\hat{\jmath})}\right| - \frac{1}{2},\end{aligned} \tag{4.102}$$

which implies $|k_{n_2}^{(j_2)}| \leq |k_{\hat{n}}^{(\hat{\jmath})}| - \frac{1}{2}$ whenever $\varepsilon \overline{m} C_{\text{obs}} < \frac{1}{2}$. By the structure of S, $k_{\hat{n}}^{(\hat{\jmath})}$ and $k_{n_2}^{(j_2)}$ are both dependent k variables, so one should take L^∞ estimates of their resolvents. As the singularities of the resolvents do not overlap, one has

$$\begin{aligned}&\sup_{|k_{n_2}^{(j_2)}| \leq |k_{\hat{n}}^{(\hat{\jmath})}| - \frac{1}{2}} \left(\frac{1}{\left|\alpha^{(\hat{\jmath})} \pm 2\pi |k_{\hat{n}}^{(\hat{\jmath})}| + i\gamma\right| \left\langle k_{\hat{n}}^{(\hat{\jmath})}\right\rangle \cdot \left|\alpha^{(j_2)} \pm 2\pi |k_{n_2}^{(j_2)}| + i\gamma\right| \left\langle k_{n_2}^{(j_2)}\right\rangle}\right) \\ &\qquad \leq \frac{C}{\gamma \left\langle \alpha^{(\hat{\jmath})}\right\rangle \left\langle \alpha^{(j_2)}\right\rangle},\end{aligned} \tag{4.103}$$

gaining a factor of γ again.

□

4.4.2. Decoupling of + and − components

Recall that the remaining partitions are non-crossing pairings S. For given N, and a pairing $S \in \pi^*(I(N))$, write a pair as $A = \left\{(j_A, n_A), (j^A, n^A)\right\}$ with $(j_A, n_A) \prec (j^A, n^A)$. A *gate* is characterized as a pair $A \in S$ such that $(j_A, n_A + 1) = (j^A, n^A)$. Note that the two elements of the pair are not only required to be consecutive, but the two scatterings have to occur in the same time interval, on the same side of the scalar product, so $j_A = j^A$. For fixed S, denote the set of gates and their right endpoints as

$$\begin{aligned}S_{\text{gate}} &= \{A \in S : A \text{ is a gate}\}, \\ I_{\text{gate}} &= \{(j_A, n_A) : A \in S_{\text{gate}}\}.\end{aligned} \tag{4.104}$$

Until now, although the observables A_j^ε and the unperturbed time evolution generated by H_0 are diagonal with respect to the σ variables, the time evolution of the ψ_+^ε component and the ψ_-^ε component of the wave are still coupled due to the off-diagonal elements of the disorder term V, (2.107). To prove that those two objects decouple in the kinetic limit, define for a non-crossing pairing $S \in I(N)$ the amplitude

$$\begin{aligned}
&\mathcal{K}_+\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right)\\
&= (\sqrt{\varepsilon}\pi)^{|N|} \int_{\mathbb{R}^{d(N^{(2\overline{m})}+1)}} \mathrm{d}k^{(2\overline{m})} \sum_{\sigma^{(2\overline{m})} \in \{\pm\}^{N^{(2\overline{m})}+1}} \dots \int_{\mathbb{R}^{d(N^{(1)}+1)}} \mathrm{d}k^{(1)} \sum_{\sigma^{(1)} \in \{\pm\}^{N^{(1)}+1}}\\
&\quad \mathbb{1}\left\{\sigma_n^{(j)} = +1 \forall (j,n) \in I_0(N) \setminus I_{\mathrm{gate}}\right\}\\
&\quad \prod_{A \in S} \left\{\delta\left(\theta_{n^A}^{(j^A)} + \theta_{n_A}^{(j_A)}\right) \widehat{g_2}\left(\theta_{n_A}^{(j_A)}\right)\right\}\\
&\qquad \prod_{j=1}^{2\overline{m}-1} \left(a_{j,+}\left(\frac{k_0^{(j+1)} + k_{N^{(j)}}^{(j)}}{2}\right) \delta\left(k_0^{(j+1)} - k_{N^{(j)}}^{(j)} - \varepsilon p^{(j)}\right)\right)\\
&\qquad \prod_{j=1}^{2\overline{m}} \left[e^{\gamma t^{(j)}} \int_{\mathbb{R}} \frac{\mathrm{d}\alpha^{(j)}}{2\pi} e^{-i\alpha^{(j)} t^{(j)}}\right]\\
&\qquad \prod_{(j,n) \in I_0(N)} \left(\frac{i}{\alpha^{(j)} - 2\pi\sigma_n^{(j)}\tau^{(j)}|k_n^{(j)}| + i\gamma}\right)\\
&\qquad \prod_{(j,n) \in I(N)} \left[(-i\tau^{(j)})\left(\left|k_n^{(j)}\right| \sigma_{n-1}^{(j)} + \left|k_{n-1}^{(j)}\right| \sigma_n^{(j)}\right) \Phi\left(k_n^{(j)}, k_{n-1}^{(j)}, L_n^{(j)}\right)\right]\\
&\qquad \widehat{\psi_{0,+}^\varepsilon}\left(k_0^{(1)}\right) \overline{\widehat{\psi}_{0,+}^\varepsilon}\left(k_0^{(1)} + \varepsilon \sum_{j=1}^{2\overline{m}-1} p^{(j)}\right).
\end{aligned} \tag{4.105}$$

Once again, for the sake of better readability, k and θ variables were employed simultaneously. Compared to the original amplitude $\mathcal{K}$ introduced in Lemma 4.3, the only difference is the indicator function in the third line of (4.105) ensuring that the set of sign constellations to be summed over is much smaller. The only resolvents that still come with both signs $\sigma_n^{(j)} \in \{\pm 1\}$ are those "contained" within a gate, $(j,n) \in I_{\mathrm{gate}}$. All other $\sigma_n^{(j)}$ are set to $+1$, and consequently, only the $+$ components of the observables a_j, and of the initial wave function ψ_0^ε contribute. In the same manner, define $\mathcal{K}_-\left(\psi_0^\varepsilon, \varepsilon, a, b, L^{(0)}, t, N, S\right)$ by replacing

$$\begin{aligned}
\widehat{\psi}_{0,+}^\varepsilon &\to \widehat{\psi}_{0,-}^\varepsilon\\
a_{j,+} &\to a_{j,-}\\
\mathbb{1}\left\{\sigma_n^{(j)} = +1 \forall (j,n) \in I_0(N) \setminus I_{\mathrm{gate}}\right\} &\to \mathbb{1}\left\{\sigma_n^{(j)} = -1 \forall (j,n) \in I_0(N) \setminus I_{\mathrm{gate}}\right\}
\end{aligned} \tag{4.106}$$

in (4.105). With this definition,

Lemma 4.14. (Decoupling of dynamics for + and − components, $\mathcal{K}$ amplitudes.) *Assume that dimension $d \geq 2$, and let the random field ξ be of class $(d+2,0)$, and suppose that g_2 furthermore fulfills the conditions of Lemma D.1. (Requiring ξ to be of class $(d+2,3)$ is sufficient). Then, for all $\delta > 0$, there is a constant C depending only on δ, dimension d and g_2 such that for all $\gamma \in (0,1]$ and all non-crossing pairings $S \in \pi^*(I(N))$,*

$$\begin{aligned}\Big|&\mathcal{K}_+\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right) + \mathcal{K}_-\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right) \\ &\quad - \mathcal{K}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right)\Big| \\ &\leq C^{|N|+2\overline{m}}\left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{2|N|}\|g_2\|_{d+2}^{|N|/2} \\ &\quad \times e^{2\gamma|t|}(\varepsilon/\gamma)^{|N|/2}\gamma^{+1-\delta}\langle\log\gamma\rangle^{|N|+2\overline{m}}\|\psi_0^\varepsilon\|_{\mathcal{H}}^2\prod_{j=1}^{2\overline{m}-1}\|a_j\|_{C^0}.\end{aligned} \tag{4.107}$$

Proof. The right side of the representation of $\mathcal{K}$ in resolvent form, (4.33), is a sum over all possible choices of $|N| + 2\overline{m}$ signs. Consider a summand belonging to one fixed choice of signs $\sigma \in \{\pm\}^{|N|+2\overline{m}}$, and assume there is an $A \in S \setminus S_{\text{gate}}$ such that $\sigma_{n^A-1}^{(j^A)} \neq \sigma_{n^A}^{(j^A)}$. For this particular σ and A, set $(j,n) = (j^A, n^A)$, and observe

$$\begin{aligned}&\frac{i}{\alpha^{(j)} - 2\pi\sigma_n^{(j)}\tau^{(j)}|k_n^{(j)}| + i\gamma} - \frac{i}{\alpha^{(j)} - 2\pi\sigma_{n-1}^{(j)}\tau^{(j)}|k_{n-1}^{(j)}| + i\gamma} \\ =&\frac{2\pi\sigma_n^{(j)}\tau^{(j)}i\left(|k_{n-1}^{(j)}| + |k_n^{(j)}|\right)}{\left(\alpha^{(j)} - 2\pi\sigma_n^{(j)}\tau^{(j)}|k_n^{(j)}| + i\gamma\right)\left(\alpha^{(j)} - 2\pi\sigma_{n-1}^{(j)}\tau^{(j)}|k_{n-1}^{(j)}| + i\gamma\right)},\end{aligned} \tag{4.108}$$

and thus

$$\begin{aligned}&\frac{2\pi\left|\sigma_n^{(j)}|k_{n-1}^{(j)}| + \sigma_{n-1}^{(j)}|k_n^{(j)}|\right|}{\left|\alpha^{(j)} - 2\pi\sigma_n^{(j)}\tau^{(j)}|k_n^{(j)}| + i\gamma\right|\left|\alpha^{(j)} - 2\pi\sigma_{n-1}^{(j)}\tau^{(j)}|k_{n-1}^{(j)}| + i\gamma\right|} \\ \leq&\frac{1}{\left|\alpha^{(j)} - 2\pi\sigma_n^{(j)}\tau^{(j)}|k_n^{(j)}| + i\gamma\right|} + \frac{1}{\left|\alpha^{(j)} - 2\pi\sigma_{n-1}^{(j)}\tau^{(j)}|k_{n-1}^{(j)}| + i\gamma\right|}.\end{aligned} \tag{4.109}$$

Therefore, while estimating (4.33), one can replace the product of those two resolvents (and the $\sigma_n^{(j)}|k_{n-1}^{(j)}| + \sigma_{n-1}^{(j)}|k_n^{(j)}|$ "interaction term" between them) by their sum. Following the proof of Lemma 4.8, there are three different possibilites.

- Either, $n > 1$. Then, by construction, both $(j, n-1)$ and (j,n) are classified as dependent, and we have to take L^∞ estimates of their respective resolvents by (4.62). The standard proof would thus yield a factor $\left(C/(\gamma\left\langle\alpha^{(j)}\right\rangle)\right)^2$, which now reduces to $\left(C/(\gamma\left\langle\alpha^{(j)}\right\rangle)\right)$, gaining a factor γ. The loss of decay in $\alpha^{(j)}$ is not important as $N^{(j)} > 1$ in this case and (4.64) can still be performed.

- Or, $n = 1$, but still $N^{(j)} > 1$. In this case, for both summands on the right of (4.109), we can take the estimate (4.64), making use of an $\alpha^{(j)}$ decay stemming from previous estimates thanks to $N^{(j)} > 1$. Again, a factor γ is gained.
- Finally, for $n = N^{(j)} = 1$ there are only two resolvents, so the right side of the estimate (4.109) is no longer $\alpha^{(j)}$-integrable. Still, for $\delta \in (0,1)$, the bound

$$\begin{aligned}&\frac{2\pi\left|\sigma_n^{(j)}|k_{n-1}^{(j)}|+\sigma_{n-1}^{(j)}|k_n^{(j)}|\right|}{\left|\alpha^{(j)}-2\pi\sigma_n^{(j)}\tau^{(j)}|k_n^{(j)}|+i\gamma\right|\left|\alpha^{(j)}-2\pi\sigma_{n-1}^{(j)}\tau^{(j)}|k_{n-1}^{(j)}|+i\gamma\right|}\\ \leq&\frac{\left(2\pi\left|\sigma_n^{(j)}|k_{n-1}^{(j)}|+\sigma_{n-1}^{(j)}|k_n^{(j)}|\right|\right)^\delta}{\left|\alpha^{(j)}-2\pi\sigma_n^{(j)}\tau^{(j)}|k_n^{(j)}|+i\gamma\right|^\delta\left|\alpha^{(j)}-2\pi\sigma_{n-1}^{(j)}\tau^{(j)}|k_{n-1}^{(j)}|+i\gamma\right|}\\ &+\frac{\left(2\pi\left|\sigma_n^{(j)}|k_{n-1}^{(j)}|+\sigma_{n-1}^{(j)}|k_n^{(j)}|\right|\right)^\delta}{\left|\alpha^{(j)}-2\pi\sigma_n^{(j)}\tau^{(j)}|k_n^{(j)}|+i\gamma\right|\left|\alpha^{(j)}-2\pi\sigma_{n-1}^{(j)}\tau^{(j)}|k_{n-1}^{(j)}|+i\gamma\right|^\delta}\end{aligned} \tag{4.110}$$

  improves the standard estimate by $C_\delta\gamma^{1-\delta}$, $\delta > 0$ arbitrarily small.

By symmetry, summands belonging to one fixed choice of signs $\sigma \in \{\pm\}^{|N|+2\overline{m}}$, with an $A \in S \setminus S_{\mathrm{gate}}$ such that $\sigma_{n_A-1}^{(j_A)} \neq \sigma_{n_A}^{(j_A)}$ are estimated in the same fashion.

Now, consider a choice $\sigma \in \{\pm\}^{|N|+2\overline{m}}$ of signs such that there is a gate $A \in S_{\mathrm{gate}}$ with $\sigma_{n_A-1}^{(j_A)} \neq \sigma_{n_A+1}^{(j_A)}$. We write $(j,n) = (j_A, n_A)$, and assume without loss of generality that $j \leq \overline{m}$, so $\tau^{(j)} = 1$. In (4.33), one can take the integral over $k_n^{(j)}$ and the sum over $\sigma_n^{(j)}$ to obtain, as $k_{n+1}^{(j)} = k_{n-1}^{(j)}$,

$$\begin{aligned}i\pi^2\sum_{\sigma_n^{(j)}}\int_{\mathbb{R}^d}\mathrm{d}k_n^{(j)}&\frac{\widehat{g}_2(k_{n+1}^{(j)}-k_n^{(j)})}{\alpha^{(j)}-2\pi\sigma_n^{(j)}\left|k_n^{(j)}\right|+i\gamma}\left(\sigma_n^{(j)}\left|k_{n+1}^{(j)}\right|+\left|k_n^{(j)}\right|\right)\left(\sigma_n^{(j)}\left|k_{n+1}^{(j)}\right|-\left|k_n^{(j)}\right|\right)\\ &\times\Phi(k_{n+1}^{(j)},k_n^{(j)},L_n^{(j)})\\ =h_{+-}&\left(k_{n+1}^{(j)},\alpha^{(j)}+i\gamma;L_n^{(j)}\right).\end{aligned} \tag{4.111}$$

(Note that $\Phi(k_{n+1}^{(j)}, k_n^{(j)}, L_{n+1}^{(j)}) = 1$ on the support of $\Phi(k_{n+1}^{(j)}, k_n^{(j)}, L_n^{(j)})$.) By Lemma D.1 and Lemma D.2,

$$\begin{aligned}\left|h_{+-}\left(k_{n+1}^{(j)},\alpha^{(j)}+i\gamma;L_n^{(j)}\right)\right|\leq&\left|h_{+-}\left(k_{n+1}^{(j)},2\pi\sigma_{n+1}^{(j)}\left|k_{n+1}^{(j)}\right|+i\gamma;L_n^{(j)}\right)\right|\\ &+C_{g_2}\left\langle L_n^{(j)}\right\rangle^2\min\left(\langle\log\gamma\rangle\left|\alpha^{(j)}-2\pi\sigma_{n+1}^{(j)}\left|k_{n+1}^{(j)}\right|\right|,2\right)\\ \leq&\tilde{C}_{g_2}\left(\gamma+\left|k_{n+1}^{(j)}\right|+\left\langle L_n^{(j)}\right\rangle^2\min\left(\langle\log\gamma\rangle\left|\alpha^{(j)}-2\pi\sigma_{n+1}^{(j)}\left|k_{n+1}^{(j)}\right|\right|,2\right)\right).\end{aligned} \tag{4.112}$$

For each of the three summands in the last term, one can follow the proof of the basic estimate Lemma 4.8, with the following improvements. For the **first summand**, everything stays the same, except for the factor γ we obtain instead of an $C\langle\log\gamma\rangle / \left\langle\alpha^{(j)}\right\rangle$ from the usual (4.63) estimate for the gate (the momentum $k_n^{(j)}$ inside the gate is free). The now missing $1/\left\langle\alpha^{(j)}\right\rangle$ decay does not prevent a later application of an (4.64) estimate, as $N^{(j)} \geq 2$ (presence of a gate implies at least two scattering events) provides at least one more such decay factor. This yields an improvement by $C\gamma/\langle\log\gamma\rangle$. Next, for the **second summand**, note that $(j, n+1)$ is dependent, so in the standard proof, there would be an L^∞ estimate for the $k_{n+1}^{(j)}$ resolvent, but the fate of the $k_{n-1}^{(j)}$ resolvent depends on the structure of the graph—

- if $n = 1$, the $k_0^{(j)}$ resolvent would be estimated by (4.64), and the three $k_0^{(j)}$, $k_1^{(j)}$, $k_2^{(j)}$ resolvents would yield a factor $C\langle\log\gamma\rangle^2/\gamma$. Now, instead, take L^∞ estimates of all resolvents belonging to $k_{n'}^{(j')}$, (j', n') dependent, *except* for $k_2^{(j)}$. Then follow the standard algorithm until the highest remaining indices are $(j, 2)$ and $(j, 0)$ ($k_1^{(j)}$ has already been accounted for by (4.112)). Now, performing the $\alpha^{(j)}$ integral yields, with $k = k_2^{(j)} = k_0^{(j)}$ and $\alpha = \alpha^{(j)}$,

$$\sup_k \int_{\mathbb{R}} \mathrm{d}\alpha \frac{|k|}{|\alpha - 2\pi|k| + i\gamma|\,|\alpha + 2\pi|k| + i\gamma|\,\langle k\rangle} \leq C\langle\log\gamma\rangle\,. \tag{4.113}$$

  Thereafter, one can follow the standard program again, gaining an overall improvement of $C\gamma/\langle\log\gamma\rangle$.

- if $n > 1$, and the indices $(j, n+1)$ and $(j, n-1)$ are both dependent, one would usually estimate both respective resolvents with (4.62) type L^∞ estimates for a factor γ^{-2}, and a factor $\langle\log\gamma\rangle$ from the $k_n^{(j)}$ resolvent. Now, instead, with $k = k_{n+1}^{(j)} = k_{n-1}^{(j)}$ and $\alpha = \alpha^{(j)}$,

$$\sup_k \frac{|k|}{|\alpha - 2\pi|k| + i\gamma|\,|\alpha + 2\pi|k| + i\gamma|\,\langle k\rangle} \leq \frac{C}{\langle\alpha\rangle\gamma}, \tag{4.114}$$

  yielding an overall improvement by a factor $C\gamma$.

- if $n > 1$, with index $(j, n+1)$ dependent but $(j, n-1)$ free, instead of taking an L^∞ estimate of the $k_{n+1}^{(j)}$ resolvent, pull it into the $\theta_{n-1}^{(j)} = k_{n-1}^{(j)} - q$ integral, to obtain

$$\begin{aligned} &\int_{\mathbb{R}^d} \mathrm{d}k \frac{|k|}{|\alpha - 2\pi|k| + i\gamma|\,|\alpha + 2\pi|k| + i\gamma|\,\langle k\rangle\langle k-q\rangle^d} \\ &\quad \leq \int_{\mathbb{R}^d} \mathrm{d}k \frac{1}{||\alpha| - 2\pi|k| + i\gamma|\,\langle k\rangle\langle k-q\rangle^d} \leq \frac{C\langle\log\gamma\rangle}{\langle\alpha\rangle}, \end{aligned} \tag{4.115}$$

  obtaining an improvement by $\gamma/\langle\log\gamma\rangle$ again.

Finally, for the **third summand** in (4.112), all resolvents are treated in the standard way except for omitting the usual L^1 estimate for the $k_n^{(j)}$ resolvent "inside" the gate itself (thus gaining a factor $\langle \log \gamma \rangle$) and an improved L^∞ estimate for the $k_{n+1}^{(j)}$ resolvent, namely

$$\sup_{k_{n+1}^{(j)}} \frac{\min\left(\langle \log \gamma \rangle \left| \alpha^{(j)} - 2\pi \sigma_{n+1}^{(j)} \left| k_{n+1}^{(j)} \right| \right|, 2\right)}{\left| \alpha^{(j)} - 2\pi \sigma_{n+1}^{(j)} \left| k_{n+1}^{(j)} \right| + i\gamma \right| \left\langle k_{n+1}^{(j)} \right\rangle} \leq \frac{C \langle \log \gamma \rangle}{\langle \alpha^{(j)} \rangle}. \tag{4.116}$$

In this case the overall improvement is $C\gamma$.

As the observables are diagonal, (4.1), all sign changes have to originate from a scattering event, i.e. from one of the cases analyzed above. So we have now estimated all contributions of $\mathcal{K}$ except for $\mathcal{K}_+$ and $\mathcal{K}_-$, thus proving the lemma. □

Lemma 4.15. (Decoupling of dynamics for $+$ and $-$ components, $\mathcal{R}(G^{\text{end}}, ...)$ amplitudes.) *For dimension $d \geq 2$, assume that ξ is of class $(d + 2(\overline{M} + 2), 0)$, and let g_2 furthermore fulfill the conditions of Lemma D.1. (This holds if ξ is of class $(d{+}2(\overline{M}{+}2), 3)$). Choose $M \in \{0, ..., \overline{M} - 1\}$, $N_< \in \mathbb{N}_0^{\overline{m}-1}$ bounded by (4.36) and define N from $\overline{N}$, $N_<$, M as previously. Then, for all $\delta > 0$, there is a constant C depending only on δ, g_2 and dimension d such that for all $\gamma \in (0, 1]$, $\kappa > 0$, and all non-crossing pairings $S \in \pi^*(I(N))$,*

$$\begin{aligned} &\Big| \mathcal{R}\left(G^{\text{end}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, t, N_<, \overline{N}, M, S\right) \\ &\qquad - \mathcal{R}_+\left(G^{\text{end}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, t, N_<, \overline{N}, M, S\right) \\ &\qquad - \mathcal{R}_-\left(G^{\text{end}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, t, N_<, \overline{N}, M, S\right) \Big| \\ &\leq C^{|N| + 2\overline{m}} \left(\langle L^{(0)} \rangle + \overline{m}\overline{N}\right)^{2|N| + 4\overline{M} + 4} \langle \varepsilon C_{\text{obs}} \rangle^{4\overline{M}+4} \|g_2\|_{d+2(\overline{M}+2)}^{|N|/2} \\ &\qquad \times e^{2\gamma|t|} (\varepsilon/\gamma)^{|N|/2} \gamma^{1-\delta} \langle \log \gamma \rangle^{|N| + 2\overline{m}} \|\psi_0^\varepsilon\|_{\mathcal{H}}^2 \prod_{j=1}^{\overline{m}-1} \|a_j\|_{C^0}^2 . \end{aligned} \tag{4.117}$$

Here, the definition of $\mathcal{R}_+$ and $\mathcal{R}_-$ is analogous to that of the $\mathcal{K}_+$ and $\mathcal{K}_-$ amplitudes in (4.105) and (4.106).

Proof. One can make the same improvements on the proof of Lemma 4.9 as were made for Lemma 4.8 in the previous proof. For the scattering events without cut-off function Φ occurring in this case, one has to use Lemmas D.1 and D.2 for the non-cutoff $h_{\sigma_1 \sigma_2}(k, w)$ instead of $h_{\sigma_1 \sigma_2}(k, w; L)$. □

4.4.3. The $\mathcal{K}_\sigma^{(\text{main})}$ and $\mathcal{R}_\sigma^{(\text{main})}$ amplitudes

All estimates for $\mathcal{K}$ and $\mathcal{R}$ amplitudes so far have been derived by using a resolvent representation. For controlling the contribution of nested and non-markovian pairings, and eventually identifying the linear Boltzmann equation as the limit, the resolvents will no longer be practical; instead, we transform the $\alpha^{(j)}$ integrals back into time integrals. We will formulate our arguments only for the $\mathcal{K}_+$ and $\mathcal{R}_+$ amplitudes, the "$-$" amplitudes can be treated in the same fashion.

Definition 4.2. Let $S \in I(N)$ be a non-crossing pairing. The *parent* $\check{A}$ of a pair $A \in S$ is the "narrowest" pair in S embracing A, that is, from all $B \in S$ with $\min B \prec \min A \prec \max A \prec \max B$ we choose $\check{A}$ to be the one with $\min B$ maximal. For A that do not have a parent in S, we set its parent to be $\check{A} = \{(1,0),(2\overline{m}, N^{(2\overline{m})})\} =: \mathbf{0}$. We thus have a map

$$\check{\cdot} : S \to S \cup \{\mathbf{0}\} = S_0, \tag{4.118}$$

which is not onto and typically not injective, so the set of *children*

$$\underline{B} = \left\{A \in S : \check{A} = B\right\} \tag{4.119}$$

of an $B \in S_0$ may be either empty or contain several elements.

For $(j,n) \in I_0(N)$, also define the momenta

$$q_n^{(j)} = k_0^{(1)} + \sum_{\substack{(\tilde{\jmath},\tilde{n}) \in I(N) \\ (\tilde{\jmath},\tilde{n}) \preceq (j,n)}} \theta_{\tilde{n}}^{(\tilde{\jmath})}, \tag{4.120}$$

which coincide with the $k_n^{(j)}$ up to the $\mathcal{O}(\varepsilon)$ contribution of the p variables.
If there are $G^{(j)}$ gates on the j-th time interval, there remain $\tilde{N}^{(j)} + 1$ indices after integrating out the gates, with $\tilde{N}^{(j)} = N^{(j)} - G^{(j)}$, and there is an bijective map

$$\begin{aligned} \left\{0, ..., N^{(j)}\right\} \setminus \{n : (j,n) \in I_{\text{gate}}\} &\to \left\{0, ..., \tilde{N}^{(j)}\right\}, \\ n &\mapsto l(j,n) = n - \sharp\left\{n' < n : (j,n') \in I_{\text{gate}}\right\}. \end{aligned} \tag{4.121}$$

The size of the reduced index set $I(N) \setminus I_{\text{gate}}$ is $\left|\tilde{N}\right| = \tilde{N}^{(1)} + ... + \tilde{N}^{(2\overline{m})}$. The non-gate scattering events in the j-th interval is $R^{(j)} = N^{(j)} - 2G^{(j)} = \tilde{N}^{(j)} - G^{(j)}$, and the number of non-gate pairs $A \in S \setminus S_{\text{gate}}$ is

$$\frac{1}{2}\sum_{j=1}^{2\overline{m}} R^{(j)} = \frac{|N|}{2} - \sum_{j=1}^{2\overline{m}} G^{(j)} = \left|\tilde{N}\right| - \frac{|N|}{2}. \tag{4.122}$$

For $q \in \mathbb{R}^d \setminus \{0\}$, $q \neq 0$, one can set

$$\widehat{q} = \frac{q}{|q|}, \tag{4.123}$$

and define everywhere except on a set of Lebesgue measure zero

$$\begin{aligned} P^{(j)} &= \sum_{\tilde{\jmath}=1}^{j-1} p^{(\tilde{\jmath})} \qquad (j \in \{1, ..., 2\overline{m}\}), \\ w_l^{(j)} &= 2\pi\tau^{(j)}\left(|q_{n(j,l)}^{(j)}| + \varepsilon\widehat{q}_{n(j,l)}^{(j)} \cdot P^{(j)}\right) \qquad (j \in \{1, ..., 2\overline{m}\}, l \in \{0, ..., \tilde{N}^{(j)}\}). \end{aligned} \tag{4.124}$$

The contribution of the signs $\tau^{(j)}$ will show as a factor ρ_A for every $A \in S \setminus S_{\text{gate}}$,

$$\rho_A = (-i\tau^{(j_A)})(-i\tau^{(j^A)}), \tag{4.125}$$

so ρ_A is negative if both scatterings belonging to A occur on the same side of the scalar product, and positive otherwise. This sign will later give rise to the gain and loss terms in the linear Boltzmann equation. Employing the above notation, denote as the *main part* of the $\mathcal{K}_+$ amplitude belonging to a non-crossing pairing S the quantity

$$\begin{aligned}
&\mathcal{K}_+^{(\text{main})}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right) \\
&= \varepsilon^{|N|/2} \int_{\mathbb{R}^d} \mathrm{d}q_0^{(1)} \left|2\pi q_0^{(1)}\right|^{2|\tilde{N}|-|N|} \overline{\widehat{\psi}_{0,+}^\varepsilon} \left(q_0^{(1)} + \varepsilon \sum_{j=1}^{2\overline{m}-1} p^{(j)}\right) \widehat{\psi}_{0,+}^\varepsilon \left(q_0^{(1)}\right) \\
&\quad \int_{\mathbb{R}^{d|\tilde{N}|}} \prod_{(j,n)\in I(N)\setminus I_{\text{gate}}} \left(\mathrm{d}q_n^{(j)}\right) \prod_{A\in S\setminus S_{\text{gate}}} \left(\rho_A \delta\left(\theta_{n^A}^{(j^A)} + \theta_{n_A}^{(j_A)}\right) \widehat{g}_2\left(\theta_{n_A}^{(j_A)}\right)\right) \\
&\quad \prod_{j=1}^{2\overline{m}-1} a_{j,+}\left(q_{N^{(j)}}^{(j)}\right) \prod_{j=1}^{2\overline{m}} K_{\tilde{N}^{(j)}}\left(w^{(j)}, t^{(j)}\right) \\
&\quad \prod_{(j,n)\in I(N)\setminus I_{\text{gate}}} \Phi\left(q_n^{(j)}, q_{n-1}^{(j)}, L_n^{(j)}\right) \\
&\quad \prod_{(j,n)\in I_{\text{gate}}} \left(-\Theta_{\tau^{(j)}}\left(\widehat{q}_{n+1}^{(j)} \left|q_0^{(1)}\right|\right) \delta\left(q_{n+1}^{(j)} - q_{n-1}^{(j)}\right)\right),
\end{aligned} \tag{4.126}$$

with the propagators K defined as in (4.30). For $j \in \{1, ..., 2\overline{m} - 1\}$, we have set $q_0^{(j+1)} = q_{N^{(j)}}^{(j)}$.

Lemma 4.16. *Let $d \geq 2$, ξ be of class $(d+2, 0)$ and suppose g_2 fulfills the conditions of Lemma D.1. Assume furthermore that there exists a $\lambda \in (0, 1]$ such that $\widehat{\psi}_0^\varepsilon(k) = 0$ for $|k| < \lambda$* independently *of $\varepsilon > 0$. For all $\delta > 0$, there is a constant C depending only on δ, d and g_2 such that*

$$\begin{aligned}
&\left|\mathcal{K}_+\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right) - \mathcal{K}_+^{(\text{main})}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right)\right| \\
&\quad \leq C^{|N|+2\overline{m}} \left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{2|N|} \|g_2\|_{d+2}^{|N|/2} \langle C_{\text{obs}}\rangle^2 \overline{m}^2 \\
&\quad\quad \times e^{2\gamma|t|} (\varepsilon/\gamma)^{|N|/2} \langle \log\gamma\rangle^{|N|+2\overline{m}} \|\psi_0^\varepsilon\|_{\mathcal{H}}^2 \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^1} \\
&\quad\quad \times \left((\varepsilon^2|t| + \gamma)\lambda^{-1} + \gamma^{1-\delta}\right)
\end{aligned} \tag{4.127}$$

for all non-crossing pairings $S \in \pi^*(I(N))$, and all $\gamma \in [4\varepsilon \langle C_{\text{obs}}\rangle \overline{m}, 1]$.

Proof. First, we will replace the $\pi(|k_n^{(j)}| + |k_{n-1}^{(j)}|)$ factors by $2\pi|k_0^{(1)}| = 2\pi|q_0^{(1)}|$ for all scattering events belonging to non-gate pairs $A \in S \setminus S_{\text{gate}}$. There are $|\tilde{N}| - |N|/2$ such pairs, which explains the exponent $2|\tilde{N}|-|N|$ in the first line of (4.126). It is enough to show how to replace the factor $|k_{n_1}^{(j_1)}|+|k_{n_1-1}^{(j_1)}|$ for $(j_1,n_1)=\max A_1$ for an $A_1\in S\setminus S_{\mathrm{gate}}$, the min A_1 case is analogous. There is an $r\le |N|/2$ such that $A_l=\check{A}_{l-1}$, $(l\in\{2,...r\})$ until $A_r=\{(1,0),(2\overline{m},N^{(2\overline{m})})\}$. Therefore, with the notation $\max A_l=(j_l,n_l)$,

$$\begin{aligned}
\Big||k_{n_1-1}^{(j_1)}|+|k_{n_1}^{(j_1)}|-2|k_0^{(1)}|\Big| &\le \Big||k_{n_1-1}^{(j_1)}|-|k_{n_1}^{(j_1)}|\Big|+2\Big||k_{n_1}^{(j_1)}|-|k_{N^{(2\overline{m})}}^{(2\overline{m})}|\Big|+4\varepsilon\overline{m}C_{\mathrm{obs}}\\
&\le \Big||k_{n_1-1}^{(j_1)}|-|k_{n_1}^{(j_1)}|\Big|+2\sum_{l=2}^{r}\Big||k_{n_{l-1}}^{(j_{l-1})}|-|k_{n_l}^{(j_l)}|\Big|+4\varepsilon\overline{m}C_{\mathrm{obs}}\\
&\le 2\sum_{l=1}^{r-1}\Big||k_{n_l-1}^{(j_l)}|-|k_{n_l}^{(j_l)}|\Big|+2\sum_{l=2}^{r-1}\Big||k_{n_{l-1}}^{(j_{l-1})}|-|k_{n_l-1}^{(j_l)}|\Big|+2\Big||k_{n_{r-1}}^{(j_{r-1})}|-|k_{N^{(2\overline{m})}}^{(2\overline{m})}|\Big|+4\varepsilon\overline{m}C_{\mathrm{obs}}\\
&\le 2\sum_{l=1}^{r-1}\Big||k_{n_l-1}^{(j_l)}|-|k_{n_l}^{(j_l)}|\Big|+8\varepsilon\overline{m}C_{\mathrm{obs}}.
\end{aligned}\tag{4.128}$$

Now assume all replacements of this type for pairs $A\in S\setminus S_{\mathrm{gate}}$ with $\max A\succ\max A_1$ have already been accomplished. Then, without needing better decay conditions for g_2 than in Lemma 4.8, one can follow the proof of that lemma, and observe that the error from the replacement causes r error terms, a trivial one with an improvement factor $\varepsilon\overline{m}C_{\mathrm{obs}}$ compared to the basic bound, and $r-1$ more complicated ones, originating from the sum in the last line of (4.128). We can utilize each of those summands to estimate

$$\begin{aligned}
&\frac{2\pi\Big||k_{n_l-1}^{(j_l)}|-|k_{n_l}^{(j_l)}|\Big|}{\Big|\alpha^{(j_l)}-2\pi\tau^{(j_l)}|k_{n_l}^{(j_l)}|+i\gamma\Big|\Big|\alpha^{(j_l)}-2\pi\tau^{(j_l)}|k_{n_l-1}^{(j_l)}|+i\gamma\Big|}\\
\le&\frac{1}{\Big|\alpha^{(j_l)}-2\pi\tau^{(j_l)}|k_{n_l}^{(j_l)}|+i\gamma\Big|}+\frac{1}{\Big|\alpha^{(j_l)}-2\pi\tau^{(j_l)}|k_{n_l-1}^{(j_l)}|+i\gamma\Big|},
\end{aligned}\tag{4.129}$$

from where one can continue as in the proof of Lemma 4.14 after estimate (4.109). Therefore, one can replace all factors of the type $|k_n^{(j)}|+|k_{n-1}^{(j)}|$ in the definition of the $\mathcal{K}_+$ amplitude by $2|k_0^{(1)}|$ with an error of the form

$$(\text{Basic estimate from Lemma 4.8})\times C|N|(|N|\gamma^{1-\delta}+\varepsilon\overline{m}C_{\mathrm{obs}})\tag{4.130}$$

with $\delta>0$ arbitrarily small and C depending only on dimension d and δ.

Next, one has to approximate the contribution of the gates by the Θ_σ functions defined in Lemma D.3. Consider a gate, $A\in S_{\mathrm{gate}}$, with $\min A=(j,n)$. First, assume, that $j\le\overline{m}$, so $\tau^{(j)}=1$. In this case, the factors

$$\left(\left|k_n^{(j)}\right|+\sigma_n^{(j)}\left|k_{n-1}^{(j)}\right|\right)=\left(\left|k_n^{(j)}\right|+\sigma_n^{(j)}\left|k_{n+1}^{(j)}\right|\right)\tag{4.131}$$

have not been replaced by $|k_0^{(1)}|$ in the previous step. Instead, we now integrate out all factors in (4.105) that depend on $k_n^{(j)}$ or $\sigma_n^{(j)}$, to obtain

$$i\pi^2 \sum_{\sigma_n^{(j)}} \int_{\mathbb{R}^d} \mathrm{d}k_n^{(j)} \frac{\widehat{g}_2\left(k_n^{(j)} - k_{n+1}^{(j)}\right)}{\alpha^{(j)} - 2\pi\sigma_n^{(j)}\left|k_n^{(j)}\right| + i\gamma} \left(\left|k_n^{(j)}\right| + \sigma_n^{(j)}\left|k_{n+1}^{(j)}\right|\right)^2 \Phi(k_{n+1}^{(j)}, k_n^{(j)}, L_n^{(j)}) = h_{++}\left(k_{n+1}^{(j)}, \alpha^{(j)} + i\gamma; L_n^{(j)}\right), \tag{4.132}$$

and estimate

$$\begin{aligned}
&\left|h_{++}\left(k_{n+1}^{(j)}, \alpha^{(j)} + i\gamma; L_n^{(j)}\right) - \Theta_+\left(\widehat{q}_{n+1}^{(j)}|k_0^{(1)}|\right)\right| \\
&\quad \le \left|h_{++}\left(k_{n+1}^{(j)}, \alpha^{(j)} + i\gamma; L_n^{(j)}\right) - h_{++}\left(k_{n+1}^{(j)}, 2\pi|k_{n+1}^{(j)}| + i\gamma; L_n^{(j)}\right)\right| \\
&\qquad + \left|h_{++}\left(k_{n+1}^{(j)}, 2\pi|k_{n+1}^{(j)}| + i\gamma; L_n^{(j)}\right) - h_{++}\left(q_{n+1}^{(j)}, 2\pi|q_{n+1}^{(j)}| + i\gamma; L_n^{(j)}\right)\right| \\
&\qquad + \left|h_{++}\left(q_{n+1}^{(j)}, 2\pi|q_{n+1}^{(j)}| + i\gamma; L_n^{(j)}\right) - h_{++}\left(\widehat{q}_{n+1}^{(j)}|k_0^{(1)}|, 2\pi|k_0^{(1)}| + i\gamma\right)\right| \\
&\qquad + \left|h_{++}\left(\widehat{q}_{n+1}^{(j)}|k_0^{(1)}|, 2\pi|k_0^{(1)}| + i\gamma\right) - \Theta_+\left(\widehat{q}_{n+1}^{(j)}|k_0^{(1)}|\right)\right| \\
&\le C\left\langle L_n^{(j)}\right\rangle^2 \Big[\min\left(\langle\log\gamma\rangle\,|\alpha^{(j)} - 2\pi|k_{n+1}^{(j)}||, 1\right) + 2\overline{m}C_{\mathrm{obs}}\varepsilon\,\langle\log\varepsilon\rangle \\
&\qquad + \min\left(\langle\log\gamma\rangle\,||k_0^{(1)}| - |k_{n+1}^{(j)}||\left\langle\log(||k_0^{(1)}| - |k_{n+1}^{(j)}||)\right\rangle, 1\right) \\
&\qquad + \langle C_{\mathrm{obs}}\overline{m}\rangle^2\,\varepsilon\,\langle\log\varepsilon\rangle + \gamma\,\langle\log\gamma\rangle\Big],
\end{aligned} \tag{4.133}$$

where we used Lemma D.1 and Lemma D.3. After replacing the argument $k_{n+1}^{(j)}$ of the gate functions h_{++} by $\widehat{q}_{n+1}^{(j)}|k_0^{(1)}| = \widehat{q}_{n+1}^{(j)}|q_0^{(1)}|$, we can also shed the cut-off threshold $L_n^{(j)}$ for all gates, as $|k_0^{(1)}| \le L^{(0)} \le L_n^{(j)}$. Now, one can again insert the above estimate into the proof of Lemma 4.8. This way, the last two lines of (4.133) yield the following improvements. From some summands, one directly obtains a factor $\varepsilon\,\langle\log\varepsilon\rangle$ or $\gamma\,\langle\log\gamma\rangle$. We have already seen in the proof of Lemma 4.14 how the summand

$$\min\left(\langle\log\gamma\rangle\,|\alpha^{(j)} - 2\pi|k_{n+1}^{(j)}||, 1\right) \tag{4.134}$$

gives an improvement factor γ. Finally, the summand

$$\min\left(\langle\log\gamma\rangle\,||k_0^{(1)}| - |k_{n+1}^{(j)}||\left\langle\log(||k_0^{(1)}| - |k_{n+1}^{(j)}||)\right\rangle, 1\right) \tag{4.135}$$

can be estimated as in (4.128), with an only slightly worse improvement factor of the form

$$\left(|N|\gamma^{1-\delta} + \langle C_{\mathrm{obs}}\overline{m}\rangle^2\,\varepsilon^{1-\delta}\right), \tag{4.136}$$

$\delta > 0$ arbitrarily small. To see how to replace gates by Θ functions for $j > \overline{m}$, observe that $\tau^{(j)} = -1$ and

$$i\pi^2 \sum_{\sigma_n^{(j)}} \int_{\mathbb{R}^d} \mathrm{d}k_n^{(j)} \frac{\widehat{g}_2\left(k_n^{(j)} - k_{n+1}^{(j)}\right)}{\alpha^{(j)} + 2\pi\sigma_n^{(j)}\left|k_n^{(j)}\right| + i\gamma} \left(\left|k_n^{(j)}\right| + \sigma_n^{(j)}\left|k_{n+1}^{(j)}\right|\right)^2 \Phi(k_{n+1}^{(j)}, k_n^{(j)}, L_{n+1}^{(j)}) = h_{--}\left(k_{n+1}^{(j)}, \alpha^{(j)} + i\gamma; L_{n+1}^{(j)}\right), \tag{4.137}$$

which one can replace by $\Theta_-\left(\widehat{q}_{n+1}^{(j)}|k_0^{(1)}|\right)$ with the same error as before. As we have already started with the Θ_σ functions, we want to change from $k_n^{(j)}$ to $q_n^{(j)}$ arguments. First, note that this easily achieved for the argument of the $\widehat{g}_2$ functions, as

$$\widehat{g}_2\left(\theta_n^{(j)}\right) = \widehat{g}_2\left(k_n^{(j)} - k_{n-1}^{(j)}\right) = \widehat{g}_2\left(q_n^{(j)} - q_{n-1}^{(j)}\right) \tag{4.138}$$

for all $(j,n) \in I(N)$. Using the differentiability of the observables,

$$\left| a_{j,+}\left(\frac{k_0^{(j+1)} + k_{N(j)}^{(j)}}{2}\right) - a_{j,+}\left(q_{N(j)}^{(j)}\right)\right| \leq 2\varepsilon C_{\mathrm{obs}}\overline{m}\left\|a_j\right\|_{C^1}, \tag{4.139}$$

so all k arguments in the observables can be replaced with an error proportional to the basic estimate times $2\varepsilon C_{\mathrm{obs}}\overline{m}^2$. Finally, for the cut-off functions,

$$\left|\Phi\left(k_{n-1}^{(j)}, k_n^{(j)}, L_n^{(j)}\right) - \Phi\left(q_{n-1}^{(j)}, q_n^{(j)}, L_n^{(j)}\right)\right| \leq 2\varepsilon C C_{\mathrm{obs}}\overline{m}\Phi\left(k_{n-1}^{(j)}, k_n^{(j)}, L_n^{(j)} + 1\right) \tag{4.140}$$

and one can therefore replace the arguments of all Φ functions with an error consisting of the standard estimate from Lemma 4.8 times $2|N|\varepsilon C_{\mathrm{obs}}\overline{m}$. After finishing those replacements, one can always borrow a decay $\left\langle q_n^{(j)}\right\rangle^{-1}$ from $\widehat{g}_2$ and Φ just as we got a $\left\langle k_n^{(j)}\right\rangle^{-1}$ decay in the proof of Lemma 4.8.

The only task left is to estimate the error of replacing the arguments of the resolvents. Note for all $(j,n) \in I_0(N)$,

$$k_n^{(j)} = q_n^{(j)} + \varepsilon\sum_{\tilde{j}<j} p^{(\tilde{j})} = q_n^{(j)} + \varepsilon P^{(j)} \tag{4.141}$$

with $|P^{(j)}| \leq 2\overline{m}C_{\mathrm{obs}}\varepsilon$, and therefore for $|q_n^{(j)}| > 4\overline{m}C_{\mathrm{obs}}\varepsilon$

$$\left||k_n^{(j)}| - |q_n^{(j)}| - \varepsilon\widehat{q}_n^{(j)} \cdot P^{(j)}\right| \leq \frac{1}{2(|q_n^{(j)}| - \varepsilon|P^{(j)}|)}|\varepsilon P^{(j)}|^2 \leq \frac{(2\overline{m}C_{\mathrm{obs}}\varepsilon)^2}{|q_n^{(j)}|}. \tag{4.142}$$

For $0 < |q_n^{(j)}| < 4\overline{m}C_{\mathrm{obs}}\varepsilon$,

$$\left||k_n^{(j)}| - |q_n^{(j)}| - \varepsilon\widehat{q}_n^{(j)} \cdot P^{(j)}\right| \leq \frac{(4\overline{m}C_{\mathrm{obs}}\varepsilon)^2}{|q_n^{(j)}|} \tag{4.143}$$

holds trivially, and one has

$$\begin{aligned}&\left|\frac{1}{\alpha^{(j)}-2\pi\tau^{(j)}\left|q_n^{(j)}+\varepsilon P^{(j)}\right|+i\gamma}-\frac{1}{\alpha^{(j)}-2\pi\tau^{(j)}\left(|q_n^{(j)}|+\varepsilon\widehat{q}_n^{(j)}\cdot P^{(j)}\right)+i\gamma}\right|\\&\leq\frac{C(C_{\mathrm{obs}}\overline{m}\varepsilon)^2}{\left|\alpha^{(j)}-2\pi\tau^{(j)}\left|q_n^{(j)}+\varepsilon P^{(j)}\right|+i\gamma\right|\left|\alpha^{(j)}-2\pi\tau^{(j)}\left(|q_n^{(j)}|+\varepsilon\widehat{q}_n^{(j)}\cdot P^{(j)}\right)+i\gamma\right||q_n^{(j)}|},\end{aligned}\tag{4.144}$$

for all $q_n^{(j)}\neq 0$. Now we follow the proof of Lemma 4.8 with minor changes. First, the gate resolvents no longer have to be estimated, as the gates have already be taken care of. As each of those resolvents would have contributed a factor $\langle\log\gamma\rangle$, this only yiels a neglible improvement of the basic estimate. For resolvents belonging to free $(j,n)\in I(N)\setminus I_{\mathrm{gate}}$, we no longer use $k_n^{(j)}$ but $q_n^{(j)}$ as integration variable (this is just a translation). As mentioned above, we can use $\left\langle q_n^{(j)}\right\rangle^{-1}$ instead of $\left\langle k_n^{(j)}\right\rangle^{-1}$ decays for all $(j,n)\in I(N)$. Finally we fix a certain $(\tilde{\jmath},\tilde{n})\in I_0(N)\setminus I_{\mathrm{gate}}$, for which we want to replace

$$\frac{1}{\alpha^{(\tilde{\jmath})}-2\pi\tau^{(\tilde{\jmath})}\left|q_{\tilde{n}}^{(\tilde{\jmath})}+\varepsilon P^{(\tilde{\jmath})}\right|+i\gamma}\to\frac{1}{\alpha^{(\tilde{\jmath})}-2\pi\tau^{(\tilde{\jmath})}\left(|q_{\tilde{n}}^{(\tilde{\jmath})}|+\varepsilon\widehat{q}_{\tilde{n}}^{(\tilde{\jmath})}\cdot P^{(\tilde{\jmath})}\right)+i\gamma}.\tag{4.145}$$

We assume that the same replacement has already been conducted for $(j,n)\succ(\tilde{\jmath},\tilde{n})$, but not yet for $(j,n)\prec(\tilde{\jmath},\tilde{n})$. For $(j,n)\succ(\tilde{\jmath},\tilde{n})$, note that the assumption $4\langle C_{\mathrm{obs}}\rangle\,\overline{m}\varepsilon\leq\gamma\leq 1$ implies that analogues of (4.62), (4.63), (4.64) can be used without difficulty, namely

$$\begin{aligned}\sup_{q_n^{(j)}}\frac{1}{\left|\alpha^{(j)}-2\pi\tau^{(j)}\left(|q_n^{(j)}|+\varepsilon\widehat{q}_n^{(j)}\cdot P^{(j)}\right)+i\gamma\right|\left\langle q_n^{(j)}\right\rangle}&\leq\frac{C}{\left\langle\alpha^{(j)}\right\rangle\gamma},\\\int_{\mathbb{R}^d}\mathrm{d}q_n^{(j)}\frac{1}{\left|\alpha^{(j)}-2\pi\tau^{(j)}\left(|q_n^{(j)}|+\varepsilon\widehat{q}_n^{(j)}\cdot P^{(j)}\right)+i\gamma\right|\left\langle q_n^{(j)}\right\rangle\left\langle q_n^{(j)}-q_{n-1}^{(j)}\right\rangle^d}&\leq\frac{C\left\langle\log\gamma\right\rangle}{\left\langle\alpha^{(j)}\right\rangle},\\\int_{\mathbb{R}}\mathrm{d}\alpha^{(j)}\frac{1}{\left|\alpha^{(j)}-2\pi\tau^{(j)}\left(|q_n^{(j)}|+\varepsilon\widehat{q}_n^{(j)}\cdot P^{(j)}\right)+i\gamma\right|\left\langle\alpha^{(j)}\right\rangle}&\leq\frac{C\left\langle\log\gamma\right\rangle}{\left\langle\alpha^{(j)}\right\rangle},\end{aligned}\tag{4.146}$$

with a C depending only on dimension d. Arriving at $(\tilde{\jmath},\tilde{n})$, suppose that there is an $A\in S\setminus S_{\mathrm{gate}}$ such that $\min A\preceq(\tilde{\jmath},\tilde{n})\prec\max A$, and choose A with $\min A$ maximal. Note that $q_{n_A}^{(j_A)}=q_{\tilde{n}}^{(\tilde{\jmath})}$. First, assume that $(\tilde{\jmath},\tilde{n})$ is dependent, and derive from (4.144) the estimate

$$\begin{aligned}&\left|\frac{1}{\alpha^{(\tilde{\jmath})}-2\pi\tau^{(\tilde{\jmath})}\left|q_{\tilde{n}}^{(\tilde{\jmath})}+\varepsilon P^{(\tilde{\jmath})}\right|+i\gamma}-\frac{1}{\alpha^{(\tilde{\jmath})}-2\pi\tau^{(\tilde{\jmath})}\left(|q_{\tilde{n}}^{(\tilde{\jmath})}|+\varepsilon\widehat{q}_{\tilde{n}}^{(\tilde{\jmath})}\cdot P^{(\tilde{\jmath})}\right)+i\gamma}\right|\left\langle q_{\tilde{n}}^{(\tilde{\jmath})}\right\rangle^{-1}\\&\qquad\leq\frac{C}{\left\langle\alpha^{(\tilde{\jmath})}\right\rangle|q_{\tilde{n}}^{(\tilde{\jmath})}|}\end{aligned}\tag{4.147}$$

with a C dependent only on d. We then follow the original proof of Lemma 4.8 again, carrying the $|q_{\tilde{n}}^{(\tilde{\jmath})}| = |q_{n_A}^{(j_A)}|$ denominator along until we integrate over $q_{n_A}^{(j_A)}$ (which is free), to obtain

$$\int_{\mathbb{R}^d} \frac{\mathrm{d}q_{n_A}^{(j_A)}}{\left|\alpha^{(j_A)} - 2\pi\tau^{(j_A)}\left|q_{n_A}^{(j_A)} + \varepsilon P^{(j_A)}\right| + i\gamma\right| |q_{n_A}^{(j_A)}| \left\langle q_{n_A}^{(j_A)} \right\rangle \left\langle q_{n_A}^{(j_A)} - q_{n_A-1}^{(j_A)} \right\rangle^d} \le C \langle \log \gamma \rangle , \tag{4.148}$$

because $d \ge 2$. This yields the basic estimate with an additional factor $C\gamma$. Second, assume $(\tilde{\jmath}, \tilde{n})$ to be free. In that case, the standard proof calls for a $q_{\tilde{n}}^{(\tilde{\jmath})}$ integral, into which we plug (4.144) and have

$$\begin{aligned} \int_{\mathbb{R}^d} \mathrm{d}q_{\tilde{n}}^{(\tilde{\jmath})} & \frac{C(C_{\mathrm{obs}}\overline{m}\varepsilon)^2}{\left|\alpha^{(\tilde{\jmath})} - 2\pi\tau^{(\tilde{\jmath})}\left|q_{\tilde{n}}^{(\tilde{\jmath})} + \varepsilon P^{(\tilde{\jmath})}\right| + i\gamma\right| \left|\alpha^{(\tilde{\jmath})} - 2\pi\tau^{(\tilde{\jmath})}\left(|q_{\tilde{n}}^{(\tilde{\jmath})}| + \varepsilon \widehat{q}_{\tilde{n}}^{(\tilde{\jmath})} \cdot P^{(\tilde{\jmath})}\right) + i\gamma\right|} \\ & \times \frac{1}{|q_{\tilde{n}}^{(\tilde{\jmath})}| \left\langle q_{\tilde{n}}^{(\tilde{\jmath})} \right\rangle \left\langle q_{\tilde{n}}^{(\tilde{\jmath})} - q_{\tilde{n}-1}^{(\tilde{\jmath})} \right\rangle^d} \\ \le & \frac{\tilde{C} C_{\mathrm{obs}}\overline{m}\varepsilon}{\langle \alpha^{(\tilde{\jmath})} \rangle}, \end{aligned} \tag{4.149}$$

gaining a factor ε over the basic estimate again. Third, if $\tilde{n} = 0$ plug (4.144) into the $\alpha^{(\tilde{\jmath})}$ integral,

$$\begin{aligned} \int_{\mathbb{R}} \mathrm{d}\alpha^{(\tilde{\jmath})} & \frac{C(C_{\mathrm{obs}}\overline{m}\varepsilon)^2}{\left|\alpha^{(\tilde{\jmath})} - 2\pi\tau^{(\tilde{\jmath})}\left|q_{\tilde{n}}^{(\tilde{\jmath})} + \varepsilon P^{(\tilde{\jmath})}\right| + i\gamma\right| \left|\alpha^{(\tilde{\jmath})} - 2\pi\tau^{(\tilde{\jmath})}\left(|q_{\tilde{n}}^{(\tilde{\jmath})}| + \varepsilon \widehat{q}_{\tilde{n}}^{(\tilde{\jmath})} \cdot P^{(\tilde{\jmath})}\right) + i\gamma\right| |q_{\tilde{n}}^{(\tilde{\jmath})}|} \\ \le & \frac{\tilde{C} C_{\mathrm{obs}}\overline{m}\varepsilon}{|q_{\tilde{n}}^{(\tilde{\jmath})}|}. \end{aligned} \tag{4.150}$$

The denominator $|q_{\tilde{n}}^{(\tilde{\jmath})}|$ is dealt with as before, and again, an extra factor ε is gained.

Conversely, assume now that there is no $A \in S \setminus S_{\mathrm{gate}}$ such that $\min A \preceq (\tilde{\jmath}, \tilde{n}) \prec \max A$. In this case, $(\tilde{\jmath}, \tilde{n})$ cannot be free, but for $(\tilde{\jmath}, \tilde{n})$ dependent or $(\tilde{\jmath}, \tilde{n}) = (\tilde{\jmath}, 0)$, the estimates (4.147) and (4.150) still apply. Their right hand sides can be controlled by

$$\frac{1}{|q_{\tilde{n}}^{(\tilde{\jmath})}|} \le \frac{1}{\lambda}, \tag{4.151}$$

as we now have $q_{\tilde{n}}^{(\tilde{\jmath})} = k_0^{(1)}$, with $|k_0^{(1)}| \ge \lambda$ on the support of $\widehat{\psi}_{0,+}^{\varepsilon}$.

A short remark concerning the case $N^{(\tilde{\jmath})} = 0$ is in order. Here, the resolvent representation is only valid formally, and the change of arguments for the resolvent actually means a change of arguments in the unitary, which is controlled by

$$\begin{aligned}&\left|\exp\left(-2\pi i\tau^{(\tilde{\jmath})}\left|q_{\tilde{n}}^{(\tilde{\jmath})}+\varepsilon P^{(\tilde{\jmath})}\right|t^{(\tilde{\jmath})}\right)-\exp\left(-2\pi i\tau^{(\tilde{\jmath})}\left(|q_{\tilde{n}}^{(\tilde{\jmath})}|+\varepsilon\widehat{q}_{\tilde{n}}^{(\tilde{\jmath})}\cdot P^{(\tilde{\jmath})}\right)t^{(\tilde{\jmath})}\right)\right|\\&\qquad\le\frac{C(C_{\mathrm{obs}}\overline{m}\varepsilon)^2}{|q_{\tilde{n}}^{(\tilde{\jmath})}|}t^{(\tilde{\jmath})}.\end{aligned}\tag{4.152}$$

As before, the $|q_{\tilde{n}}^{(\tilde{\jmath})}|$ is then either integrated over or estimated by $1/\lambda$.

To prove the lemma, one now only has to return to expressing the propagation by unitaries instead of resolvents, applying Lemma 4.2 in the reverse direction compared to the introduction of the resolvent representation. The graph expansion started with propagators $K_{N(j)}$ on the j-th inteval, but we now only recover $K_{\tilde{N}(j)}$, as the gates have been integrated out, so there are no time variables corresponding to "time spent in a gate" anymore. □

The definition of the main part of the $\mathcal{R}_+(G^{\mathrm{end}},...)$ amplitudes is rather similar to (4.126),

$$\begin{aligned}&\mathcal{R}_+^{(\mathrm{main})}\left(G^{\mathrm{end}},\psi_0^\varepsilon,\varepsilon,\kappa,a,p,L^{(0)},t,N_<,\overline{N},M,S\right)\\&=\varepsilon^{|N|/2}\int_{\mathbb{R}^d}\mathrm{d}q_0^{(1)}\left|2\pi q_0^{(1)}\right|^{2|\tilde{N}|-|N|}\left|\widehat{\psi_{0,+}^\varepsilon}(q_0^{(1)})\right|^2\\&\qquad\int_{\mathbb{R}^{d|\tilde{N}|}}\prod_{(j,n)\in I(N)\setminus I_{\mathrm{gate}}}\left(\mathrm{d}q_n^{(j)}\right)\prod_{A\in S\setminus S_{\mathrm{gate}}}\left(\rho_A\delta\left(\theta_{n^A}^{(j^A)}+\theta_{n_A}^{(j_A)}\right)\widehat{g}_2\left(\theta_{n_A}^{(j_A)}\right)\right)\\&\qquad\prod_{j=1}^{2\overline{m}-1}a_{j,+}\left(q_{N(j)}^{(j)}\right)\prod_{j=1}^{2\overline{m}}K_{\tilde{N}(j)}\left(w^{(j)},t^{(j)}\right)\\&\qquad\prod_{(j,n)\in I(N)\setminus I_{\mathrm{gate}}}\Phi_n^{(j)}\left(q_n^{(j)},q_{n-1}^{(j)},L_n^{(j)}\right)\\&\qquad\prod_{(j,n)\in I_{\mathrm{gate}}}\left(-\Theta_{\tau^{(j)}}\left(\widehat{q}_{n+1}^{(j)}\left|q_0^{(1)}\right|\right)\delta\left(q_{n+1}^{(j)}-q_{n-1}^{(j)}\right)\right).\end{aligned}\tag{4.153}$$

For convenience, we again have set $q_0^{(j+1)}=q_{N(j)}^{(j)}$ for all $j\in\{1,...,2\overline{m}-1\}$. The only differences of (4.153) and (4.126) are the different definition of N, with $N^{(\overline{m})}=\overline{N}^{(\overline{m})}+M=\overline{N}+M-N^{(1)}...-N^{(\overline{m}-1)}$, the missing cutoff function for the last scattering events

$$\Phi_n^{(j)}=\begin{cases}\Phi & \text{for } (j,n)\prec\left(\overline{m},\overline{N}^{(\overline{m})}\right)\text{ or }(j,n)\succ\left(\overline{m}+1,N^{(\overline{m}+1)}+1-\overline{N}^{(\overline{m})}\right)\\1 & \text{for } (j,n)\succeq\left(\overline{m},\overline{N}^{(\overline{m})}\right)\text{ or }(j,n)\preceq\left(\overline{m}+1,N^{(\overline{m}+1)}+1-\overline{N}^{(\overline{m})}\right),\end{cases}\tag{4.154}$$

the decay parameter κ added to the frequencies in the last unitaries

$$w_l^{(j)}=2\pi\tau^{(j)}\left(|q_{n(j,l)}^{(j)}|+\varepsilon\widehat{q}_{n(j,l)}^{(j)}\cdot P^{(j)}\right)-i\kappa_{n(j,l)}^{(j)},\tag{4.155}$$

with

$$\kappa_n^{(j)} = \begin{cases} 0 & \text{for } (j,n) \prec \left(\overline{m}, \overline{N}^{(\overline{m})}\right) \text{ or } (j,n) \succ \left(\overline{m}+1, N^{(\overline{m}+1)} - \overline{N}^{(\overline{m})}\right) \\ \kappa & \text{for } (j,n) \succeq \left(\overline{m}, \overline{N}^{(\overline{m})}\right) \text{ or } (j,n) \preceq \left(\overline{m}+1, N^{(\overline{m}+1)} - \overline{N}^{(\overline{m})}\right), \end{cases} \tag{4.156}$$

and the slightly different definition of the observables, (4.39). Following the same reasoning as in the proof of Lemma 4.16, one concludes

Lemma 4.17. *Let $d \geq 2$, let ξ be a random field of class $(d + 2(\overline{M} + 2), 0)$ and with g_2 fulfilling the conditions of Lemma D.1. Assume furthermore that there exists a $\lambda \in (0, 1]$ such that $\widehat{\psi}_0^\varepsilon(k) = 0$ for $|k| < \lambda$* independently *of $\varepsilon > 0$, and that $M \in \{0, ..., \overline{M} - 1\}$ and $N_< \in \mathbb{N}_0^{\overline{m}-1}$ is bounded by (4.36). For $\delta > 0$ arbitrarily small, there is a constant C depending only on δ, d and g_2 such that for all $\gamma, \kappa > 0$ with $4\varepsilon \langle C_{\text{obs}} \rangle \overline{m} < \gamma \leq 1$ and all non-crossing pairings $S \in \pi^*(I(N))$*

$$\begin{aligned} &\Big| \mathcal{R}_+ \left(G^{\text{end}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, t, N_<, \overline{N}, M, S \right) \\ &\qquad - \mathcal{R}_+^{(\text{main})} \left(G^{\text{end}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, t, N_<, \overline{N}, M, S \right) \Big| \\ &\leq C^{|N|+2\overline{m}} \left(\langle L^{(0)} \rangle + \overline{m}\overline{N} \right)^{2|N|+4\overline{M}+4} \|g_2\|_{d+2(\overline{M}+2)}^{|N|/2} \langle \varepsilon C_{\text{obs}} \rangle^{4\overline{M}+6} \overline{m}^2 \\ &\quad \times e^{2\gamma|t|} (\varepsilon/\gamma)^{|N|/2} \langle \log \gamma \rangle^{|N|+2\overline{m}} \|\psi_0^\varepsilon\|_{\mathcal{H}}^2 \prod_{j=1}^{\overline{m}-1} \|a_j\|_{C^1}^2 \\ &\quad \times \left((\varepsilon^2|t| + \gamma)\lambda^{-1} + \gamma^{1-\delta} \right). \end{aligned} \tag{4.157}$$

4.4.4. Transforming the time integrals

We will now rewrite $\mathcal{K}_+^{(\text{main})}$ and $\mathcal{R}_+^{(\text{main})}(G^{\text{end}}, ...)$ in a fashion that facilitates their analysis for large values of $|N|$. Due to the presence of the momentum delta functions, the integral over the $q_n^{(j)}$, $(j,n) \in I(N) \setminus I_{\text{gate}}$ in (4.126) and (4.153) has only $|\tilde{N}| - |N|/2$ free variables, one for each pair that is not a gate. To evaluate the integral, one can choose for each $A = \{(j_A, n_A), (j^A, n^A)\} \in S \setminus S_{\text{gate}}$ an integration variable

$$q_A = q_{n_A}^{(j_A)}. \tag{4.158}$$

Then for each index $(j,n) \in I(N) \setminus I_{\text{gate}}$ with $(j_A, n_A) \preceq (j,n) \prec (j^A, n^A)$ that is not "contained" within a further pair $B \in \underline{A}$ (in the sense that $\min B \preceq (j,n) \prec \max B$), the delta functions enforce $q_n^{(j)} = q_A$. Consequently, for such (j,n) and $l = l(j,n)$, the frequency of the propagator is

$$w_l^{(j)} = 2\pi\tau^{(j)} \left(|q_A| + \varepsilon \widehat{q}_A \cdot P^{(j)} \right), \tag{4.159}$$

for the $\mathcal{K}_+^{(\text{main})}$ amplitudes, or

$$w_l^{(j)} = 2\pi\tau^{(j)} \left(|q_A| + \varepsilon \widehat{q}_A \cdot P^{(j)} \right) - i\kappa_n^{(j)}, \tag{4.160}$$

for the $\mathcal{R}_+^{(\mathrm{main})}(G^{\mathrm{end}},...)$ case. After defining $\boldsymbol{q_0} = q_0^{(1)}$, we see that for every $(j,n) \in I(N) \setminus I_{\mathrm{gate}}$ which is not straddled by any pair at all, the momentum equals $q_n^{(j)} = \boldsymbol{q_0}$, and the frequency in this case is $w_l^{(j)} = 2\pi\tau^{(j)}\left(|\boldsymbol{q_0}| + \varepsilon\widehat{\boldsymbol{q_0}} \cdot P^{(j)}\right)$. As any remaining, i.e. non-gate, scattering event is a transition from a parent to a child pair or vice versa, the arguments of the $\widehat{g}_2$ functions can be rewritten as

$$\widehat{g}_2\left(\theta_{n_A}^{(j_A)}\right) = \widehat{g}_2\left(q_A - q_{\check{A}}\right), \tag{4.161}$$

$A \in S \setminus S_{\mathrm{gate}}$. Likewise, one can replace the product of cutoff functions $\varPhi$ in (4.126) and (4.153) by

$$\prod_{A \in S \setminus S_{\mathrm{gate}}} \left[\varPhi_{n_A}^{(j_A)}\left(q_A, q_{\check{A}}, L_{n_A}^{(j_A)}\right)\varPhi_{n^A}^{(j^A)}\left(q_A, q_{\check{A}}, L_{n^A}^{(j^A)}\right)\right], \tag{4.162}$$

the product of $\varTheta$ functions by

$$\prod_{B \in S_{\mathrm{gate}}} \left(-\varTheta_{\tau^{(j_B)}}\left(\widehat{q}_{\check{B}} \cdot |\boldsymbol{q_0}|\right)\right) \tag{4.163}$$

and finally the observables by

$$a_{j,+}\left(q_{A(j)}\right) \tag{4.164}$$

with $A(j)$ the "narrowest" pair in $A \in S \setminus S_{\mathrm{gate}}$ such that $\min A \preceq (j, N^{(j)}) \prec \max A$.

Before transforming the time integral as well, we will plug in for the time variables $t^{(j)}$ the scaled macroscopic times $T^{(j)}/\varepsilon$, with $T^{(1)}, ..., T^{(\overline{m})} > 0$, $T^{(j)} = T^{(2\overline{m}+1-j)}$ for $j \in \{\overline{m}+1, ..., 2\overline{m}\}$ and $|T| = T^{(1)} + ... + T^{(\overline{m})}$. Note that the time integral in (4.126) or (4.153), from all the propagators taken together,

$$\begin{aligned}&\prod_{j=1}^{2\overline{m}} K_{\tilde{N}^{(j)}}\left(w^{(j)}, T^{(j)}/\varepsilon\right)\\ &\quad= \prod_{j=1}^{2\overline{m}} \int_{\mathbb{R}_+^{\tilde{N}^{(j)}+1}} \mathrm{d}\tilde{s}_0^{(j)} ... \mathrm{d}\tilde{s}_{\tilde{N}^{(j)}}^{(j)} \delta\left(\sum_{l=1}^{\tilde{N}^{(j)}} \tilde{s}_l^{(j)} - T^{(j)}/\varepsilon\right) \prod_{l=0}^{\tilde{N}^{(j)}} \exp\left(-iw_l^{(j)}\tilde{s}_l^{(j)}\right)\end{aligned} \tag{4.165}$$

is $|\tilde{N}|$-dimensional. Instead of simply using the times $\tilde{s}_l^{(j)}$ between two consecutive scatterings as integration variables, it is advantageous to introduce time variables that better represent the structure of the pairing S.
If $A \in S \setminus S_{\mathrm{gate}}$, $A = \{(j_1, n_1), (j_2, n_2)\}$ with $(j_1, n_1) \prec (j_2, n_2)$, define

$$s_A^+ = \begin{cases} \sum_{j<j_1} T^{(j)} + \varepsilon \sum_{l=0}^{l(j_1,n_1)-1} \tilde{s}_l^{(j_1)} & \text{for } j_1 \in \{1, ..., \overline{m}\} \\ 2|T| - \sum_{j<j_1} T^{(j)} - \varepsilon \sum_{l=0}^{l(j_1,n_1)-1} \tilde{s}_l^{(j_1)} & \text{for } j_1 \in \{\overline{m}+1, ..., 2\overline{m}\} \end{cases} \tag{4.166}$$

and

$$s_A^- = \begin{cases} \sum_{j<j_2} T^{(j)} + \varepsilon \sum_{l=0}^{l(j_2,n_2)-1} \tilde{s}_l^{(j_2)} & \text{for } j_2 \in \{1, ..., \overline{m}\} \\ 2|T| - \sum_{j<j_2} T^{(j)} - \varepsilon \sum_{l=0}^{l(j_2,n_2)-1} \tilde{s}_l^{(j_2)} & \text{for } j_2 \in \{\overline{m}+1, ..., 2\overline{m}\} \end{cases}. \tag{4.167}$$

The (signed) time that is spent "within" this pair A, but not in one of the children pairs $B \in \underline{A}$ as defined in Definition 4.2 is equal to

$$\begin{aligned} b_A &= \frac{1}{\varepsilon}\left[\left(s_A^- - s_A^+\right) - \sum_{B \in \underline{A} \cap S \setminus S_{\text{gate}}} \left(s_B^- - s_B^+\right)\right] \\ &= \frac{1}{\varepsilon}\left(s_A^- - s_A^+\right) - \sum_{B \in (\underline{A} \cup \underline{\underline{A}} \ldots) \cap S \setminus S_{\text{gate}}} b_B. \end{aligned} \tag{4.168}$$

We allow b_A to take both positive or negative values to account for the complex conjugation in the scalar product, (so far we used the sign $\tau^{(j)}$ for the same purpose).
On the other hand, for gates $A \in S_{\text{gate}}$, $A = \{(j_A, n_A), (j_A, n_A + 1)\}$, let

$$r_A = \begin{cases} \sum_{j<j_A} T^{(j)} + \varepsilon \sum_{l=0}^{l(j_A, n_A+1)-1} \tilde{s}_l^{(j_A)} & \text{for } j_A \in \{1, ..., \overline{m}\} \\ 2|T| - \sum_{j<j_A} T^{(j)} - \varepsilon \sum_{l=0}^{l(j_A, n_A+1)-1} \tilde{s}_l^{(j_A)} & \text{for } j_A \in \{\overline{m}+1, ..., 2\overline{m}\} \end{cases}. \tag{4.169}$$

We will use the $|\tilde{N}| - |N|/2$ variables of s^-, the $|\tilde{N}| - |N|/2$ different b and $|N| - |\tilde{N}|$ different r variables as new integration variables. Compared to the $\tilde{s}$ variables, $|N|/2$ of them have been scaled with a factor ε (the s^- and r), while $|\tilde{N}| - |N|/2$ of them (the b variables) are not. This implies that

$$\prod_{j=1}^{2\overline{m}} \int_{\mathbb{R}_+^{\tilde{N}^{(j)}+1}} \mathrm{d}\tilde{s}_0^{(j)} \ldots \mathrm{d}\tilde{s}_{\tilde{N}^{(j)}}^{(j)} \delta\left(\sum_{l=1}^{\tilde{N}^{(j)}} \tilde{s}_l^{(j)} - T^{(j)}/\varepsilon\right) \to \int_{\mathcal{Q}_{S,T}^{\varepsilon}} \prod_{A \in S \setminus S_{\text{gate}}} \left(\mathrm{d}s_A^- \mathrm{d}b_A\right) \prod_{B \in S_{\text{gate}}} \mathrm{d}r_B \tag{4.170}$$

is a transformation with Jacobian $\varepsilon^{|N|/2}$, which cancels with the prefactor of (4.126) or (4.153). The integration domain $\mathcal{Q}_{S,T}^{\varepsilon}$ on the right hand side of (4.170) depends both on the structure of the pairing S as well as on the lenghts of the single time intervals encoded in T. While its structure may be rather intricate, the following observations are straightforward.

$$\mathcal{Q}_{S,T}^{\varepsilon} \subset \left\{(s^-, b, r) \in \mathbb{R}^{|\tilde{N}|} : (s^-, r) \in \mathcal{Q}_{S,T}\right\}, \tag{4.171}$$

with $\mathcal{Q}_{S,T} \subset [0, T]^{|N|/2}$ such that

$$\int_{\mathcal{Q}_{S,T}} \mathrm{d}s^- \mathrm{d}r \leq \frac{(2|T|)^{|N|/2}}{(|N|/2)!} \tag{4.172}$$

and, for any given $A, A' \in S \setminus S_{\text{gate}}$, $B \in S_{\text{gate}}$ and $x \in \mathbb{R}$,

$$\begin{aligned} \int_{\mathcal{Q}_{S,T}} \mathrm{d}s^- \mathrm{d}r \delta\left(s_A^- - x\right) &\leq \frac{(2|T|)^{|N|/2-1}}{(|N|/2-1)!}, \\ \int_{\mathcal{Q}_{S,T}} \mathrm{d}s^- \mathrm{d}r \delta\left(s_A^- - s_{A'}^- - x\right) &\leq \frac{(2|T|)^{|N|/2-1}}{(|N|/2-1)!}, \\ \int_{\mathcal{Q}_{S,T}} \mathrm{d}s^- \mathrm{d}r \delta\left(s_A^- - r_B - x\right) &\leq \frac{(2|T|)^{|N|/2-1}}{(|N|/2-1)!}. \end{aligned} \tag{4.173}$$

To see that (4.172) is true, note that for each of the altogether $|N|/2$ different s^- and r variables there is a $j \in \{1, ...\overline{m}\}$ such that the variable in question is confined to an interval of length smaller or equal to $T^{(j)}$. If one allows each variable to live on the full $2|T| = T^{(1)} + ... + T^{(2\overline{m})}$ instead, but keeps the order of the time variables (that carries over from $\prec$ and the structure of the particular S) intact, the estimate is immediate. The bound (4.173) comes from the same argument with one degree of freedom removed. So far, we have only transformed the integration *domain* of (4.165). The *integrand*, that is the phase

$$\prod_{j=1}^{2\overline{m}} \prod_{l=0}^{\tilde{N}^{(j)}} \exp\left(-iw_l^{(j)} \tilde{s}_l^{(j)}\right) \tag{4.174}$$

transforms into

$$\prod_{A \in S \setminus S_{\text{gate}}} \left[\exp\left(-2\pi i b_A |q_A|\right) h_A \left(s^-, b, r, p, \widehat{q}_A, \kappa\right)\right] h_{\mathbf{0}} \left(s^-, b, r, p, q_{\mathbf{0}}, \kappa\right). \tag{4.175}$$

The continuous functions $h_{\mathbf{0}}, h_A$ are bounded by $|h| \leq 1$. They may depend on all time variables s^-, b, r as well as $p^{(j)}$, $j \in \{1, ..., 2\overline{m} - 1\}$ and (if appropriate) the damping parameter $\kappa > 0$; regarding the q variables, $h_{\mathbf{0}}$ is a function of $q_{\mathbf{0}}$, however, the h_A for $A \in S \setminus S_{\text{gate}}$ only depend on the *normalized* $\widehat{q_A} = q_A/|q_A|$. So h_A is independent of the absolute values $|q_A|$ when $A \neq \mathbf{0}$. The last statement is immediate from (4.159) and from the definition of b_A in (4.168). In the new variables q_A and s^-, b, r, and after an application of Fubini's theorem to interchange the q_A and time integrals, the amplitudes $\mathcal{K}_+^{(\text{main})}$ and $\mathcal{R}_+^{(\text{main})}$ read

$$\begin{aligned}
&\mathcal{K}_+^{(\text{main})}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, N, S\right) \\
&= \int_{\mathbb{R}^d} \mathrm{d}q_{\mathbf{0}} \, |2\pi q_{\mathbf{0}}|^{2|\tilde{N}|-|N|} \, \widehat{\psi_{0,+}^\varepsilon}(q_{\mathbf{0}}) \overline{\widehat{\psi}_{0,+}^\varepsilon}\left(q_{\mathbf{0}} + \varepsilon \sum_{j=1}^{2\overline{m}-1} p^{(j)}\right) \\
&\qquad \int_{\mathcal{Q}_{S,T}^\varepsilon} \prod_{A \in S \setminus S_{\text{gate}}} \left(\mathrm{d}s_A^- \mathrm{d}b_A\right) \prod_{B \in S_{\text{gate}}} \mathrm{d}r_B \\
&\qquad \int_{\mathbb{R}^{(|\tilde{N}|-|N|/2)d}} \prod_{A \in S \setminus S_{\text{gate}}} \mathrm{d}q_A \\
&\qquad\quad \prod_{A \in S \setminus S_{\text{gate}}} \left[\rho_A \widehat{g}_2(q_A - q_{\check{A}}) \varPhi\left(q_A, q_{\check{A}}, L_{n_A}^{(j_A)}\right) \varPhi\left(q_A, q_{\check{A}}, L_{n^A}^{(j^A)}\right)\right] \\
&\qquad\quad \prod_{j=1}^{2\overline{m}-1} \left[a_{j,+}\left(q_{A(j)}\right)\right] \prod_{B \in S_{\text{gate}}} \left(-\varTheta_{\tau^{(j_B)}}\left(\widehat{q}_{\check{B}} \cdot |q_{\mathbf{0}}|\right)\right) \\
&\qquad\quad \prod_{A \in S \setminus S_{\text{gate}}} \left[\exp\left(-2\pi i b_A |q_A|\right) h_A\left(s^-, b, r, p, \widehat{q}_A, 0\right)\right] h_{\mathbf{0}}\left(s^-, b, r, p, q_{\mathbf{0}}, 0\right)
\end{aligned} \tag{4.176}$$

and

$$
\begin{aligned}
&\mathcal{R}_{+}^{(\mathrm{main})}\left(G^{\mathrm{end}}, \psi_0^{\varepsilon}, \varepsilon, \kappa, a, p, L^{(0)}, T/\varepsilon, N_<, \overline{N}, M, S\right) \\
&= \int_{\mathbb{R}^d} \mathrm{d}q_{\mathbf{0}} \, |2\pi q_{\mathbf{0}}|^{2|\tilde{N}|-|N|} \left|\widehat{\psi_{0,+}^{\varepsilon}}(q_{\mathbf{0}})\right|^2 \\
&\qquad \int_{\mathcal{Q}_{S,T}^{\varepsilon}} \prod_{A\in S\setminus S_{\mathrm{gate}}} \left(\mathrm{d}s_A^- \mathrm{d}b_A\right) \prod_{B\in S_{\mathrm{gate}}} \mathrm{d}r_B \\
&\qquad \int_{\mathbb{R}^{(|\tilde{N}|-|N|/2)d}} \prod_{A\in S\setminus S_{\mathrm{gate}}} \mathrm{d}q_A \\
&\qquad\quad \prod_{A\in S\setminus S_{\mathrm{gate}}} \left[\rho_A \widehat{g}_2(q_A - q_{\check{A}}) \Phi_{n_A}^{(j_A)}\left(q_A, q_{\check{A}}, L_{n_A}^{(j_A)}\right) \Phi_{n^A}^{(j^A)}\left(q_A, q_{\check{A}}, L_{n^A}^{(j^A)}\right)\right] \\
&\qquad\quad \prod_{j=1}^{2\overline{m}-1} \left[a_{j,+}\left(q_{A(j)}\right)\right] \prod_{B\in S_{\mathrm{gate}}} \left(-\Theta_{\tau(j_B)}\left(\widehat{q}_{\check{B}} \cdot |q_{\mathbf{0}}|\right)\right) \\
&\qquad\quad \prod_{A\in S\setminus S_{\mathrm{gate}}} \left[\exp\left(-2\pi i b_A |q_A|\right) h_A\left(s^-, b, r, p, \widehat{q}_A, \kappa\right)\right] h_{\mathbf{0}}\left(s^-, b, r, p, q_{\mathbf{0}}, \kappa\right),
\end{aligned}
\tag{4.177}
$$

where we used the definitions (4.154) and (4.155) in the case of $\mathcal{R}_{+}^{(\mathrm{main})}\left(G^{\mathrm{end}}, ...\right)$.

4.4.5. Basic estimates — obtaining the $1/\overline{N}!$ factor

In the Dyson series (2.135) for the linear Boltzmann equation, the time integral in the n-th summand is taken over set of n-dimensional volume $t^n/n!$. We aim to show the convergence of the Wigner transform to a solution of a linear Boltzmann equation, so we should expect that at some point a bound like $1/n!$, n being the number of scatterings, should become available for our amplitudes as well. This will prove crucial to our ability to actually break off the Duhamel expansion at a suitable cut-off $\overline{N}$, and to resummate the $\mathcal{K}_{\sigma}^{(\mathrm{main})}$ amplitudes in Section 4.6.

Lemma 4.18. (Basic estimate of the $\mathcal{K}_{+}^{(\mathrm{main})}$ and $\mathcal{R}_{+}^{(\mathrm{main})}\left(G^{\mathrm{end}}, ...\right)$ amplitudes.) *Let $d \geq 2$, $\varepsilon > 0$, $\kappa > 0$, $\widehat{\psi}_0^{\varepsilon}(k) = 0$ for $|k| > L^{(0)}$, and $S \in \pi^*(I(N))$ be a non-crossing pairing, with N appropriately defined (that is, for $\mathcal{R}_{+}^{(\mathrm{main})}\left(G^{\mathrm{end}}, ...\right)$ amplitudes, $N_< \in N \in \mathbb{N}_0^{\overline{m}-1}$ with (4.36), $M \in \{0, ..., \overline{M}-1\}$, $N^{(\overline{m})} = \overline{N}^{(\overline{m})} + M$, N symmetric). Furthermore assume that g_2 is such that*

$$
\max_{0\leq|\alpha|\leq 4} \left|\frac{\partial^\alpha}{\partial q^\alpha} \widehat{g}_2(q)\right| \leq C_{g_2} \langle q\rangle^{-d-3}, \tag{4.178}
$$

with $C_{g_2} < \infty$. Then there is a $C_{g_2,d}$ only depending on g_2 and d such that

$$
\left|\mathcal{K}_{+}^{(\mathrm{main})}\left(\psi_0^{\varepsilon}, \varepsilon, a, p, L^{(0)}, T/\varepsilon, N, S\right)\right| \leq \frac{\left(C_{g_2,d} L^{(0)} \left\langle L^{(0)}\right\rangle |T|\right)^{|N|/2}}{(|N|/2)!} \|\psi_0^{\varepsilon}\|_{\mathcal{H}}^2 \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^2} \tag{4.179}
$$

and

$$\begin{aligned}&\left|\mathcal{R}_{+}^{(\mathrm{main})}\left(G^{\mathrm{end}},\psi_0^{\varepsilon},\varepsilon,\kappa,a,p,L^{(0)},T/\varepsilon,N_<,\overline{N},M,S\right)\right|\\&\qquad\leq\frac{\left(C_{g_2,d}L^{(0)}\left\langle L^{(0)}\right\rangle|T|\right)^{|N|/2}}{(|N|/2)!}\|\psi_0^{\varepsilon}\|_{\mathcal{H}}^2\prod_{j=1}^{\overline{m}-1}\|a_j\|_{C^2}^2.\end{aligned}\tag{4.180}$$

Proof. Without loss of generality, one can concentrate on estimating $\mathcal{K}_{+}^{(\mathrm{main})}$. Fixing $q_{\mathbf{0}}$, the p and s^{-},r,b variables, the $\mathrm{d}q_A$ integrals can be estimated as shown in Lemma C.3. We set $\phi=\widehat{g}_2$, $n=|\tilde{N}|-|N|/2$ and

$$\begin{aligned}f_A\left(q_A,q_{\check{A}}\right)=\rho_A\Phi&\left(q_A,q_{\check{A}},L_{n_A}^{(j_A)}\right)\Phi\left(q_A,q_{\check{A}},L_{n^A}^{(j^A)}\right)h_A\left(s^{-},b,r,p,\widehat{q}_A,0\right)\\&\prod_{\substack{j=1\\A(j)=A}}^{2\overline{m}-1}\left[a_{j,+}\left(q_A\right)\right]\prod_{B\in S_{\mathrm{gate}}\cap\underline{A}}\left(-\Theta_{\tau(j_B)}\left(\widehat{q}_A\cdot|q_{\mathbf{0}}|\right)\right),\end{aligned}\tag{4.181}$$

and thus have for the last four lines of (4.176) the bound

$$(C_dC_\phi)^{|\tilde{N}|-|N|/2}\prod_{A\in S\setminus S_{\mathrm{gate}}}\left(C_{f_A}\left\langle b_A\right\rangle^{-2}\right)\tag{4.182}$$

in which

$$\begin{aligned}C_\phi&=\max_{|\alpha|\leq4}\sup_{q\in\mathbb{R}^d}\left(\langle q\rangle^{d+1}\left|\frac{\partial^\alpha}{\partial q^\alpha}\widehat{g}_2(q)\right|\right),\\C_{f_A}&=C_d\left(C_{g_2,d}L^{(0)}\left\langle L^{(0)}\right\rangle\right)^{|S_{\mathrm{gate}}\cap\underline{A}|}\prod_{\substack{j=1\\A(j)=A}}^{2\overline{m}-1}\left[\max_{|\alpha|\leq2}\sup_{q\in\mathbb{R}^d}\left|\frac{\partial^\alpha}{\partial q^\alpha}a_{j,+}\left(q\right)\right|\right],\end{aligned}\tag{4.183}$$

with a factor $C_{g_2,d}$ only depending on dimension and g_2. Here, we used the estimate from Lemma D.3 for the Θ functions. With the decay

$$\prod_{A\in S\setminus S_{\mathrm{gate}}}\langle b_A\rangle^{-2}\tag{4.184}$$

and (4.172), the time integrals evaluate to

$$\int_{\mathcal{Q}_{S,T}^{\varepsilon}}\mathrm{d}s^{-}\mathrm{d}r\mathrm{d}b\prod_{A\in S\setminus S_{\mathrm{gate}}}\langle b_A\rangle^{-2}\leq C^{|\tilde{N}|-|N|/2}\int_{\mathcal{Q}_{S,T}}\mathrm{d}s^{-}\mathrm{d}r\leq\frac{(\tilde{C}|T|)^{|N|/2}}{(|N|/2)!}.\tag{4.185}$$

The remaining $q_{\mathbf{0}}$ integral yields a factor $\|\psi_0^{\varepsilon}\|_{\mathcal{H}}^2$. Concerning the prefactor, note that every gate contributes a factor of the form $L^{(0)}\left\langle L^{(0)}\right\rangle$, while every non-gate pair contributes $(L^{(0)})^2$, so in the worst case (all gates), the $L^{(0)}$ dependence is

$$\left(L^{(0)}\left\langle L^{(0)}\right\rangle\right)^{|N|/2}.\tag{4.186}$$

This proves the lemma. □

Corollary 4.19. (Basic estimate of the $\mathcal{K}_+$ and $\mathcal{R}_+\left(G^{\text{end}}, ...\right)$ amplitudes.) *Under the conditions of Lemma 4.18, with the additional assumptions* $0 < 4\varepsilon \langle C_{\text{obs}} \rangle \overline{m} \leq 1$ *and*

$$\sup_{q \in \mathbb{R}^d} \left| \widehat{g}_2(q) \langle q \rangle^{2d+4(\overline{M}+2)} \right| < \infty, \tag{4.187}$$

there is for any arbitrarily small $\delta > 0$ *a constant* $C < \infty$ *depending only on* δ, d *and* g_2 *such that*

$$\begin{aligned} &\left| \mathcal{K}_+ \left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, N, S \right) \right| \\ &\quad \leq \left[(\langle L^{(0)} \rangle + \overline{m}\overline{N})^{2|N|} \langle C_{\text{obs}} \rangle^3 \overline{m}^3 e^{8\langle C_{\text{obs}} \rangle \overline{m} |T|} \langle \log \varepsilon \rangle^{|N|+2\overline{m}} \varepsilon^{1-\delta} \right. \\ &\qquad \left. + \frac{\left(L^{(0)} \left\langle L^{(0)} \right\rangle |T| \right)^{|N|/2}}{(|N|/2)!} \right] C^{|N|+2\overline{m}} \|\psi_0^\varepsilon\|_{\mathcal{H}}^2 \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^2}, \end{aligned} \tag{4.188}$$

and

$$\begin{aligned} &\left| \mathcal{R}_+ \left(G^{\text{end}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, T/\varepsilon, N_<, \overline{N}, M, S \right) \right| \\ &\quad \leq \left[(\langle L^{(0)} \rangle + \overline{m}\overline{N})^{2|N|+4\overline{M}+4} \langle C_{\text{obs}} \rangle^{4\overline{M}+7} \overline{m}^3 e^{8\langle C_{\text{obs}} \rangle \overline{m} |T|} \langle \log \varepsilon \rangle^{|N|+2\overline{m}} \varepsilon^{1-\delta} \right. \\ &\qquad \left. + \frac{\left(L^{(0)} \left\langle L^{(0)} \right\rangle |T| \right)^{|N|/2}}{(|N|/2)!} \right] C^{|N|+2\overline{m}} \|\psi_0^\varepsilon\|_{\mathcal{H}}^2 \prod_{j=1}^{\overline{m}-1} \|a_j\|_{C^2}^2, \end{aligned} \tag{4.189}$$

with N *appropriately defined in both cases, and* $S \in \pi^*(N)$ *being any non-crossing pairing.*

Proof. Again, we focus on the $\mathcal{K}_+$ case. As opposed to Lemmas 4.16 and 4.17 the statement at hand does *not* require $\widehat{\psi_0^\varepsilon}$ to be supported away from zero, the second summand in the estimate actually even improves if the support of $\widehat{\psi_0^\varepsilon}$ concentrates at zero. The reason is that in the proof of Lemmas 4.16 and 4.17, the infrared cutoff λ was only used to replace

$$\left| q_{\mathbf{0}} + \varepsilon P^{(j)} \right| \to |q_{\mathbf{0}}| + \varepsilon \widehat{q_{\mathbf{0}}} \cdot P^{(j)}, \tag{4.190}$$

but was not necessary for the treatment of any of the q_A, $A \neq \mathbf{0}$. Accordingly, by setting $\gamma = 4\varepsilon \langle C_{\text{obs}} \rangle \overline{m}$, one can repeat the proof of Lemma 4.16, omitting only the arguments utilizing the cutoff λ, and obtain

$$\begin{aligned}&\left|\mathcal{K}_{+}\left(\psi_0^\varepsilon,\varepsilon,a,b,L^{(0)},t,N,S\right)-\mathcal{K}_{+}^{(\mathrm{aux})}\left(\psi_0^\varepsilon,\varepsilon,a,b,L^{(0)},t,N,S\right)\right|\\&\quad\le C^{|N|+2\overline{m}}(\langle L^{(0)}\rangle+\overline{m}\overline{N})^{2|N|}\left\langle C_{\mathrm{obs}}\right\rangle^3\overline{m}^3\\&\qquad\times e^{8\langle C_{\mathrm{obs}}\rangle\overline{m}|T|}\left\langle\log\varepsilon\right\rangle^{|N|+2\overline{m}}\|\psi_0^\varepsilon\|_{\mathcal{H}}^2\prod_{j=1}^{2\overline{m}-1}\|a_j\|_{C^1}\varepsilon^{1-\delta}.\end{aligned}\tag{4.191}$$

Here $\delta>0$ can be chosen arbitrarily small, C depends only on d, g_2 and δ, and the amplitude $\mathcal{K}_{+}^{(\mathrm{aux})}$ is still given as in (4.176), only with a slightly different function $h_{\mathbf{0}}$, which is still bounded by 1. As the key argument in the proof of Lemma 4.18 was the estimate for the q_A, $A\neq\mathbf{0}$ integrals, while for the $q_{\mathbf{0}}$ integral only $|h_{\mathbf{0}}|\le 1$ was used, the bound for $\mathcal{K}_{+}^{(\mathrm{main})}$ holds just as well for $\mathcal{K}_{+}^{(\mathrm{aux})}$. □

4.4.6. Nested and non-markovian graphs

A closer look at the geometry of $\mathcal{Q}_{S,T}^{\varepsilon}$ will result in an improved estimate for the $\mathcal{K}_{+}^{(\mathrm{main})}$ amplitudes associated with nested or non-markovian pairings S.

Lemma 4.20. (Improved estimates of $\mathcal{K}_{+}^{(\mathrm{main})}$ for nested and non-markovian pairings.) *In addition to the assumptions of Lemma 4.18, assume that $S\in\pi^*\left(I(N)\right)$ is a nested or nonmarkovian simple pairing. Then there is a $C_{g_2,d}<\infty$ only depending on g_2 and d such that*

$$\begin{aligned}&\left|\mathcal{K}_{+}^{(\mathrm{main})}\left(\psi_0^\varepsilon,\varepsilon,a,p,L^{(0)},T/\varepsilon,N,S\right)\right|\\&\qquad\le\frac{\left(C_{g_2,d}L^{(0)}\left\langle L^{(0)}\right\rangle|T|\right)^{|N|/2}}{(|N|/2-1)!}\frac{\langle\log|T|\rangle\,\varepsilon\,\langle\log\varepsilon\rangle}{|T|}\|\psi_0^\varepsilon\|_{\mathcal{H}}^2\prod_{j=1}^{2\overline{m}-1}\|a_j\|_{C^2}.\end{aligned}\tag{4.192}$$

Proof. First, assume that S is a nested pairing. This is exactly the case if there exists a pair $E\in S\setminus S_{\mathrm{gate}}$ such that $j_E=j^E$, with non-empty children set $\underline{E}\neq\emptyset$. Furthermore, one can always choose E minimal in the sense that it straddles only gates, $\underline{E}\subset S_{\mathrm{gate}}$. After selecting any gate $G\in\underline{E}$, one observes in Figure 4.3 that

$$|b_E|\ge\frac{1}{\varepsilon}\left|s_E^{-}-r_G\right|\tag{4.193}$$

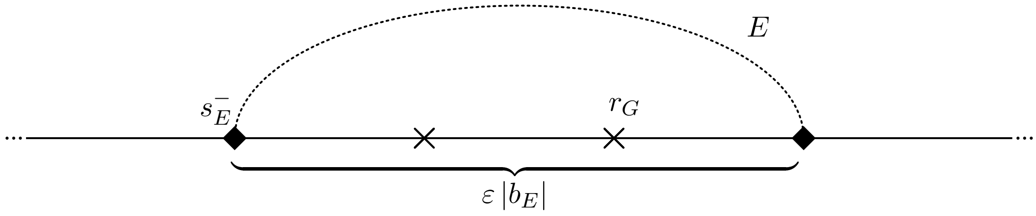

Figure 4.3.: A detail of a graph belonging to a nested pairing. Crosses are gates that have already been integrated over, black diamonds indicate scattering with cutoff function.

on $\mathcal{Q}^{\varepsilon}_{S,T}$. We follow the proof of Lemma 4.18, but can improve (4.185)

$$\begin{aligned}
&\int_{\mathcal{Q}^{\varepsilon}_{S,T}} \mathrm{d}s^{-}\mathrm{d}r\mathrm{d}b \prod_{A\in S\setminus S_{\text{gate}}} \langle b_A\rangle^{-2} \\
&\quad\leq \int_{\mathcal{Q}_{S,T}} \mathrm{d}s^{-}\mathrm{d}r \prod_{\substack{A\in S\setminus S_{\text{gate}}\\ A\neq E}} \left(\int_{\mathbb{R}} \mathrm{d}b_A \langle b_A\rangle^{-2}\right) \int_{|b_E|\geq\frac{1}{\varepsilon}\left|s_E^- - r_G\right|} \mathrm{d}b_E \langle b_E\rangle^{-2} \\
&\quad\leq C^{|N|/2} \int_{\mathcal{Q}_{S,T}} \mathrm{d}s^{-}\mathrm{d}r \min\left(1, \frac{\varepsilon}{\left|s_E^- - r_G\right|}\right) \\
&\quad\leq C^{|N|/2} \int_{-|T|}^{|T|} \mathrm{d}x \min\left(1, \frac{\varepsilon}{|x|}\right) \int_{\mathcal{Q}_{S,T}} \mathrm{d}s^{-}\mathrm{d}r\delta\left(s_E^- - r_G - x\right) \\
&\quad\leq C^{|N|/2} \frac{(2|T|)^{|N|/2-1}}{(|N|/2-1)!} 2\varepsilon\left(1+\log|T| + |\log\varepsilon|\right),
\end{aligned} \tag{4.194}$$

employing (4.173) in the last step.

Now, let S be simple, but non-markovian, and first suppose that there is an $E\in S\setminus S_{\text{gate}}$ with either $j_E < j^E \leq \overline{m}$ or $\overline{m} < j_E < j^E$ (i.e. the "culprit" lies completely on one side of the scalar product). Either, E is just as shown in Figure 4.4, so that there is no $F\in \underline{E}\cap(S\setminus S_{\text{gate}})$. Or, if the is such an F, as F is not a gate and S is not a nested pairing, F also has to straddle an observable, so $j^E \geq j^F > j_F \geq j_E$. We take the F as our new E. After finitely many updates of E, there are no longer any $F\in E\cap(S\setminus S_{\text{gate}})$. In this case, with

$$T_E = \sum_{j=1}^{j^E-1} \tau^{(j)} T^{(j)}, \tag{4.195}$$

one can easily see from Figure 4.4 that

$$|b_E| \geq \frac{1}{\varepsilon}\left|s_E^- - T_E\right|. \tag{4.196}$$

The combination of (4.173) and (4.196) yields exactly the same estimate (4.194) as before.

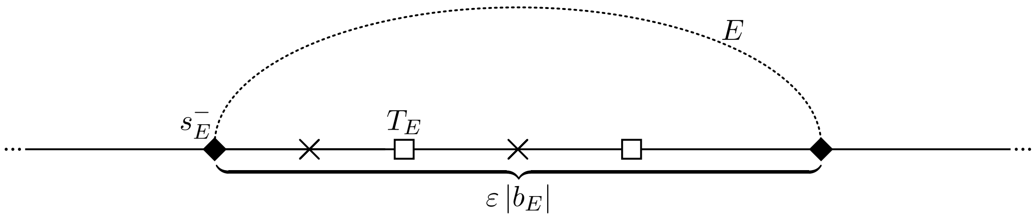

Figure 4.4.: An example of a non-markovian pair E connecting two scattering events "on the same side of the scalar product". Here, $j^E = j_E + 2$. The presence or absence of gates does not influence the validity of the argument; the empty squares, however, are essential, as they indicate the observables, and thus separate the time intervals $T^{(j)}$.

Finally, the other kind of a simple, non-markovian pairing is an S without non-markovian scattering "on one side of the scalar product". In this case, there has to be an $E \in S \setminus S_{\text{gate}}$ such that $j_E \leq \overline{m}$, $j^E > \overline{m}$ and $j_E + j^E \neq 2\overline{m} + 1$. Without loss of generality, we assume $j_E + j^E > 2\overline{m} + 1$, as shown in example Figure 4.5. With T_E defined as before, and using the notation from Figure 4.5,

$$\left| b_E + \sum_{F \in (\underline{E} \cup \underline{\underline{E}} \ldots) \cap S \setminus S_{\text{gate}}} b_F \right| = \frac{x_1 - x_2}{\varepsilon} \geq \frac{1}{\varepsilon} \left| s_E^- - T_E \right|. \tag{4.197}$$

For any $n \in \mathbb{N}$, $y > 0$,

$$\begin{aligned} \int_{|b_1 + \ldots + b_n| \geq y} \prod_{l=1}^{n} \mathrm{d}b_l \, \langle b_l \rangle^{-2} &\leq \int_{\|b\|_1 \geq y} \prod_{l=1}^{n} \mathrm{d}b_l \, \langle b_l \rangle^{-2} \leq \int_{\|b\|_\infty \geq y/n} \prod_{l=1}^{n} \mathrm{d}b_l \, \langle b_l \rangle^{-2} \\ &\leq n \int_{|b_1| \geq y/n} \prod_{l=1}^{n} \mathrm{d}b_l \, \langle b_l \rangle^{-2} \leq n C^n \min(1, \frac{n}{y}) \leq \tilde{C}^n \min(1, \frac{1}{y}), \end{aligned} \tag{4.198}$$

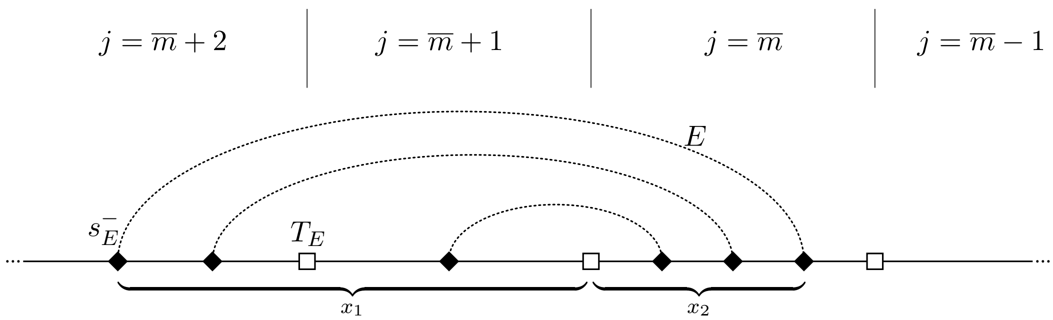

Figure 4.5.: An example of a non-markovian pair E involving two scattering events "on different sides of the scalar product", with $j_E = \overline{m}$, $j^E = \overline{m} + 2$ and two non-gate pairs in the "offspring" of E. Again, gates may exist but are unimportant.

so again, there is a universal constant $\tilde{C} < \infty$ such that

$$\begin{aligned}\int_{\mathcal{Q}^{\varepsilon}_{S,T}} \mathrm{d}s^- \mathrm{d}r \mathrm{d}b \prod_{A \in S \setminus S_{\mathrm{gate}}} \langle b_A \rangle^{-2} &\leq \tilde{C}^{|N|/2} \int_{\mathcal{Q}_{S,T}} \mathrm{d}s^- \mathrm{d}r \min\left(1, \frac{\varepsilon}{\left|s^-_E - T_E\right|}\right) \\ &\leq \tilde{C}^{|N|/2} \int_{-|T|}^{|T|} \mathrm{d}x \min\left(1, \frac{\varepsilon}{|x|}\right) \int_{\mathcal{Q}_{S,T}} \mathrm{d}s^- \mathrm{d}r \delta\left(s^-_E - T_E - x\right) \\ &\leq \tilde{C}^{|N|/2} \frac{(2|T|)^{|N|/2-1}}{(|N|/2-1)!} 2\varepsilon \left(1 + \log|T| + |\log\varepsilon|\right).\end{aligned} \tag{4.199}$$

□

4.5. Collecting the error estimates

So far we have found estimates for single contributions to the graph expansion of main part and remainder, i.e. for a fixed number of scatterings $N \in \mathbb{N}_0^{2\overline{m}}$ and a given partition $S \in \pi^*(I(N))$. We now have to make sure that the bounds for individual contributions suffice to control the combinatorical factors which result from taking the sum over all possible N and S by specifying the parameters $\overline{N}$, $\overline{M}$ and κ of the expansion.

4.5.1. The main part

From now on, let the random field ξ be of class $(d + 2\overline{M} + 7, 4)$, and, in addition to the energy bound (2.160), assume that the initial states $(\psi_0^\varepsilon)_{\varepsilon>0}$ have Fourier transforms $\widehat{\psi}_0^\varepsilon(k)$ which vanish whenever $|k| < \lambda$ or $|k| > L^{(0)}$, with ε-independent $0 < \lambda < L^{(0)} < \infty$. Therefore all Lemmas presented in Chapter 4 up to this point are applicable. Returning to the notation of (4.13-4.14), only the amplitudes of simple markovian pairings $S \in \pi_{\mathrm{sm}}(I(N))$ contribute to the main term, up to an error of

$$\Bigg| \lim_{R\to\infty} \mathbb{E}\left[\langle \Psi'^{\varepsilon}_1, A^{\varepsilon}_{\overline{m}} \Psi^{\varepsilon}_1 \rangle\right] - \sum_{\substack{N \in \mathbb{N}_0^{2\overline{m}} \\ N^{(1)}+\ldots+N^{(\overline{m})} < \overline{N}(\varepsilon) \\ N^{(\overline{m}+1)}+\ldots+N^{(2\overline{m})} < \overline{N}(\varepsilon)}} \sum_{S \in \pi_{\mathrm{sm}}(I(N))} \mathcal{K}^{(\mathrm{main})}_{+}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, N, S\right)$$

$$- \sum_{\substack{N \in \mathbb{N}_0^{2\overline{m}} \\ N^{(1)}+\ldots+N^{(\overline{m})} < \overline{N}(\varepsilon) \\ N^{(\overline{m}+1)}+\ldots+N^{(2\overline{m})} < \overline{N}(\varepsilon)}} \sum_{S \in \pi_{\mathrm{sm}}(I(N))} \mathcal{K}^{(\mathrm{main})}_{-}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, N, S\right) \Bigg|$$

$$\leq \sum_{\substack{N\in\mathbb{N}_0^{2\overline{m}} \\ N^{(1)}+\ldots+N^{(\overline{m})}<\overline{N}(\varepsilon) \\ N^{(\overline{m}+1)}+\ldots+N^{(2\overline{m})}<\overline{N}(\varepsilon)}} \left[\sum_{\substack{S\in\pi^*(I(N)) \\ \text{higher order}}} \left|\mathcal{K}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, N, S\right)\right| \right.$$
$$+ \sum_{\substack{S\in\pi^*(I(N)) \\ \text{crossing pairing}}} \left|\mathcal{K}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, N, S\right)\right|$$
$$+ \sum_{\substack{S\in\pi^*(I(N)) \\ \text{non-crossing pairing}}} \left|\mathcal{K} - \mathcal{K}_+ - \mathcal{K}_-\right| \left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, N, S\right)$$
$$+ \sum_{\substack{S\in\pi^*(I(N)) \\ \text{non-crossing pairing}}} \left|\mathcal{K}_+ - \mathcal{K}_+^{(\text{main})}\right| \left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, N, S\right)$$
$$+ \sum_{\substack{S\in\pi^*(I(N)) \\ \text{non-crossing pairing}}} \left|\mathcal{K}_- - \mathcal{K}_-^{(\text{main})}\right| \left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, N, S\right)$$
$$+ \sum_{\substack{S\in\pi^*(I(N)) \\ \text{nested/non-markovian}}} \left|\mathcal{K}_+^{(\text{main})}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, N, S\right)\right|$$
$$\left. + \sum_{\substack{S\in\pi^*(I(N)) \\ \text{nested/non-markovian}}} \left|\mathcal{K}_-^{(\text{main})}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, N, S\right)\right| \right]. \tag{4.200}$$

For $\varepsilon \in (0, 1/(4\langle C_{\text{obs}}\rangle \overline{m})]$, plugging the appropriate values of γ into Lemma 4.8 ($\gamma = \varepsilon$), Lemma 4.11 ($\gamma = 2\varepsilon\langle C_{\text{obs}}\rangle \overline{m}$), Lemma 4.14 ($\gamma = \varepsilon$), and Lemma 4.16 ($\gamma = 4\varepsilon\langle C_{\text{obs}}\rangle \overline{m}$), as well as an application of Lemma 4.20 produce a bound

$$\begin{aligned}
(4.200) \leq & C^{\overline{N}+\overline{m}} \binom{\overline{N}-1+\overline{m}}{\overline{m}}^2 \|\psi_0^\varepsilon\|_{\mathcal{H}}^2 \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^1} \\
& \times \left(\left\langle L^{(0)}\right\rangle + \overline{m}\overline{N}\right)^{4\overline{N}+1} e^{8\langle C_{\text{obs}}\rangle \overline{m}|T|} \left|\log\varepsilon\right|^{2\overline{N}+2\overline{m}} (2\overline{N})! \\
& \times \left(\max_{D\in\{1,\ldots,2\overline{N}-4\}} \left\{\varepsilon^{D/2} D^{CD}\right\} + \frac{\varepsilon|T|}{\lambda}\right) \\
& + C^{\overline{N}} \binom{\overline{N}-1+\overline{m}}{\overline{m}}^2 \|\psi_0^\varepsilon\|_{\mathcal{H}}^2 \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^2} \\
& \times \left\langle L^{(0)}\right\rangle^{2\overline{N}} |T|^{\overline{N}-2} \langle \log|T|\rangle \varepsilon|\log\varepsilon|,
\end{aligned} \tag{4.201}$$

with C a finite constant depending only on the statistics of the random field ξ and the dimension d. In (4.201), we have omitted the ε-dependence of $\overline{N}(\varepsilon)$, the binomial coefficient accounts for the number of possible choices of N, and the factor $(2\overline{N})!$ is an upper bound for the cardinality of $I(N)$. The latter is missing in the second summand of

(4.201), which only provides the bound for the last two lines of (4.200) — the number of non-crossing pairings of $I(N)$ is not larger than $4^{\overline{N}}$. The contributions of all other terms are collected in the first summand; by inspection, we notice that the bounds for higher order partitions and the replacement of $\mathcal{K}_\sigma$ by $\mathcal{K}_\sigma^{(\mathrm{main})}$ dominate. The only non-trivial observation entering (4.201) is that (2.15) implies for any higher order partition S

$$\begin{aligned} \varepsilon^{|N|/2-|S|} \prod_{A\in S} \left\| g_{|A|} \right\|_{d+2} &\leq \varepsilon^{D/2} C^{|N|} \prod_{A\in S} |A|^{C|A|} \leq \varepsilon^{D/2} \tilde{C}^{|N|} \prod_{A\in S} (|A|-2)^{C(|A|-2)} \\ &\leq \varepsilon^{D/2} \tilde{C}^{|N|} D^{CD}, \end{aligned} \tag{4.202}$$

where $D = |N| - 2|S| \in \{1, ..., 2\overline{N} - 4\}$, $C, \tilde{C} < \infty$ are constants only depending on d and the distribution of ξ, and the last inequality follows from a convexity argument.

Setting

$$\overline{N} = \overline{N}(\varepsilon) = \left\lceil a \frac{|\log \varepsilon|}{|\log|\log \varepsilon||} \right\rceil, \tag{4.203}$$

with $a \in (0, \infty)$ to be fixed later, one can now find a $C < \infty$ depending only on $\overline{m}$, $|T|$, the distribution of ξ, d, $L^{(0)}$ and C_{obs} such that

$$(4.200) \leq C^{\overline{N}(\varepsilon)+1} \, \|\psi_0^\varepsilon\|_{\mathcal{H}}^2 \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^2} \, \overline{N}(\varepsilon)^{6\overline{N}(\varepsilon)} \, |\log \varepsilon|^{2\overline{N}+2\overline{m}} \left(\sqrt{\varepsilon} + \frac{\varepsilon}{\lambda} \right) \tag{4.204}$$

for sufficiently small $\varepsilon > 0$.

4.5.2. The remainder

For the remainder part, on the other hand, the bound

$$\begin{aligned} &\|\Psi_2^\varepsilon\|_{\mathcal{H}}^2 \\ &\quad \leq \overline{m} \sum_{j_0=1}^{\overline{m}} \binom{\overline{N}(\varepsilon) - 2 + j_0}{j_0 - 1} \sum_{\substack{N \in \mathbb{N}_0^{j_0-1} \\ N^{(1)}+\ldots+N^{(j_0-1)} < \overline{N}(\varepsilon)}} \\ &\qquad \prod_{l=j_0}^{\overline{m}-1} \|a_l\|_{C^0}^2 \left\| R_{\overline{N}^{(j_0)}} \left(T^{(j_0)}/\varepsilon; R, L^{(j_0)}, \varepsilon \right) \prod_{j=1}^{j_0-1} \left(A_j^\varepsilon F_{N^{(j)}} \left(T^{(j)}/\varepsilon; R, L^{(j)}, \varepsilon \right) \right) \psi_0^\varepsilon \right\|_{\mathcal{H}}^2, \end{aligned} \tag{4.205}$$

with $\overline{N}^{(j_0)} = \overline{N}(\varepsilon) - N^{(1)} - ... - N^{(j_0-1)}$, follows from (4.11), and Lemma 2.12 provides us with the estimate

$$\begin{aligned}
&\left\| R_{\overline{N}^{(j_0)}} \left(T^{(j_0)}/\varepsilon; R, L^{(j_0)}, \varepsilon \right) \prod_{j=1}^{j_0-1} \left(A_j^\varepsilon F_{N^{(j)}} \left(T^{(j)}/\varepsilon; R, L^{(j)}, \varepsilon \right) \right) \psi_0^\varepsilon \right\|_{\mathcal{H}}^2 \\
&\quad \leq \overline{N}^{(j_0)} \left(\sum_{N_{\text{fin}}=1}^{\overline{N}^{(j_0)}-1} \left\| F_{N_{\text{fin}}}^{\text{rough}} \left(T^{(j_0)}/\varepsilon; R, L^{(j_0)}, \varepsilon \right) \prod_{j=1}^{j_0-1} \left(A_j^\varepsilon F_{N^{(j)}} \left(T^{(j)}/\varepsilon; R, L^{(j)}, \varepsilon \right) \right) \psi_0^\varepsilon \right\|_{\mathcal{H}}^2 \right. \\
&\qquad \left. + \left\| R_{\overline{N}^{(j_0)}}^{\text{end}} \left(T^{(j_0)}/\varepsilon; R, L^{(j_0)}, \varepsilon \right) \prod_{j=1}^{j_0-1} \left(A_j^\varepsilon F_{N^{(j)}} \left(T^{(j)}/\varepsilon; R, L^{(j)}, \varepsilon \right) \right) \psi_0^\varepsilon \right\|_{\mathcal{H}}^2 \right).
\end{aligned} \tag{4.206}$$

For each $N_{\text{fin}} \in \left\{ 1, ..., \overline{N}^{(j_0)} - 1 \right\}$, Lemma 2.14 and several applications of the Cauchy-Schwarz inequality yield

$$\begin{aligned}
&\mathbb{E}\left[\left\| F_{N_{\text{fin}}}^{\text{rough}} \left(T^{(j_0)}/\varepsilon; R, L^{(j_0)}, \varepsilon \right) \prod_{j=1}^{j_0-1} \left(A_j^\varepsilon F_{N^{(j)}} \left(T^{(j)}/\varepsilon; R, L^{(j)}, \varepsilon \right) \right) \psi_0^\varepsilon \right\|_{\mathcal{H}}^2 \right] \\
&\quad \leq 2 \left(T^{(j_0)}/\varepsilon \right)^2 \sup_{r \in [0, T^{(j_0)}/\varepsilon]} \\
&\qquad \mathbb{E}\left[\left\| A_{\overline{M}, N_{\text{fin}}}^{\text{rough}} \left(r; R, L^{(j_0)}, \varepsilon \right) \prod_{j=1}^{j_0-1} \left(A_j^\varepsilon F_{N^{(j)}} \left(T^{(j)}/\varepsilon; R, L^{(j)}, \varepsilon \right) \right) \psi_0^\varepsilon \right\|_{\mathcal{H}}^2 \right] \\
&\quad + 2 \left(1 + \left(\kappa T^{(j_0)}/\varepsilon \right)^2 \right) \overline{M} \sum_{M=0}^{\overline{M}-1} \sup_{r \in [0, T^{(j_0)}/\varepsilon]} \\
&\qquad \mathbb{E}\left[\left\| G_{M, N_{\text{fin}}}^{\text{rough}} \left(r; R, L^{(j_0)}, \varepsilon \right) \prod_{j=1}^{j_0-1} \left(A_j^\varepsilon F_{N^{(j)}} \left(T^{(j)}/\varepsilon; R, L^{(j)}, \varepsilon \right) \right) \psi_0^\varepsilon \right\|_{\mathcal{H}}^2 \right],
\end{aligned} \tag{4.207}$$

and, likewise,

$$\begin{aligned}
&\mathbb{E}\left[\left\| R^{\text{end}}_{\overline{N}^{(j_0)}}\left(T^{(j_0)}/\varepsilon; R, L^{(j_0)}, \varepsilon\right) \prod_{j=1}^{j_0-1}\left(A^{\varepsilon}_j F_{N^{(j)}}\left(T^{(j)}/\varepsilon; R, L^{(j)}, \varepsilon\right)\right)\psi^{\varepsilon}_0\right\|^2_{\mathcal{H}}\right] \\
&\quad \leq 2\left(T^{(j_0)}/\varepsilon\right)^2 \sup_{r\in[0,T^{(j_0)}/\varepsilon]} \\
&\qquad \mathbb{E}\left[\left\| A^{\text{end}}_{\overline{M},\overline{N}^{(j_0)}}\left(r; R, L^{(j_0)}, \varepsilon\right) \prod_{j=1}^{j_0-1}\left(A^{\varepsilon}_j F_{N^{(j)}}\left(T^{(j)}/\varepsilon; R, L^{(j)}, \varepsilon\right)\right)\psi^{\varepsilon}_0\right\|^2_{\mathcal{H}}\right] \\
&\quad + 2\left(1+\left(\kappa T^{(j_0)}/\varepsilon\right)^2\right)\overline{M}\sum_{M=0}^{\overline{M}-1} \sup_{r\in[0,T^{(j_0)}/\varepsilon]} \\
&\qquad \mathbb{E}\left[\left\| G^{\text{end}}_{M,\overline{N}^{(j_0)}}\left(r; R, L^{(j_0)}, \varepsilon\right) \prod_{j=1}^{j_0-1}\left(A^{\varepsilon}_j F_{N^{(j)}}\left(T^{(j)}/\varepsilon; R, L^{(j)}, \varepsilon\right)\right)\psi^{\varepsilon}_0\right\|^2_{\mathcal{H}}\right].
\end{aligned} \tag{4.208}$$

By Lemma 4.6 and Lemma 4.10 (into which we plug the more general choice j_0 instead of $\overline{m}$, as well as $\varepsilon \in (0,1)$, and $\gamma = \varepsilon < \kappa$), we obtain for all $N_{\text{fin}} \in \{1, ..., \overline{N}-1\}$

$$\begin{aligned}
&\lim_{R\to\infty} \sup_{r\in[0,T^{(j_0)}/\varepsilon]} \mathbb{E}\left[\left\| A^{\text{rough}}_{\overline{M},N_{\text{fin}}}\left(r; R, L^{(j_0)}, \varepsilon\right) \prod_{j=1}^{j_0-1}\left(A^{\varepsilon}_j F_{N^{(j)}}\left(T^{(j)}/\varepsilon; R, L^{(j)}, \varepsilon\right)\right)\psi^{\varepsilon}_0\right\|^2_{\mathcal{H}}\right] \\
&\leq (2\overline{N}+6\overline{M})! C^{\overline{N}+\overline{M}+j_0}\left(\langle L^{(0)}\rangle + j_0\overline{N}\right)^{4\overline{N}+6\overline{M}} \langle \varepsilon C_{\text{obs}}\rangle^{4\overline{M}+4} e^{2|T|}\,\|\psi^{\varepsilon}_0\|^2_{\mathcal{H}} \prod_{j=1}^{j_0-1}\|a_j\|^2_{C^0} \\
&\qquad \left(\frac{\varepsilon}{\kappa}\right)^{\overline{M}} |\log\varepsilon|^{2\overline{N}+2\overline{M}+2j_0} \sup_{D\in\{0,\ldots,2\overline{N}+2\overline{M}\}} \kappa^{D/2} D^{CD},
\end{aligned} \tag{4.209}$$

with a $C < \infty$ only depending on d, $\overline{M}$ and the distribution of ξ. The same bound (4.209) is valid for

$$\lim_{R\to\infty} \sup_{r\in[0,T^{(j_0)}/\varepsilon]} \mathbb{E}\left[\left\| A^{\text{end}}_{\overline{M},\overline{N}^{(j_0)}}\left(r; R, L^{(j_0)}, \varepsilon\right) \prod_{j=1}^{j_0-1}\left(A^{\varepsilon}_j F_{N^{(j)}}\left(T^{(j)}/\varepsilon; R, L^{(j)}, \varepsilon\right)\right)\psi^{\varepsilon}_0\right\|^2_{\mathcal{H}}\right]. \tag{4.210}$$

According to Lemmas 4.4, 4.9, 4.12 and 4.13, (in which we always set j_0 instead of $\overline{m}$, choose $\varepsilon \in (0, 1/(4\langle C_{\text{obs}}\rangle j_0)$, and $\gamma = 2\varepsilon\langle C_{\text{obs}}\rangle j_0)$ for all $M \in \{0, ...\overline{M}-1\}$, all

$N_{\text{fin}} \in \{1, ..., \overline{N} - 1\}$, and all $r \in [0, T^{(j_0)}/\varepsilon]$ the estimate

$$\lim_{R\to\infty} \mathbb{E}\left[\left\|G^{\text{rough}}_{M,N_{\text{fin}}}\left(r; R, L^{(j_0)}, \varepsilon\right)\prod_{j=1}^{j_0-1}\left(A_j^\varepsilon F_{N^{(j)}}\left(T^{(j)}/\varepsilon; R, L^{(j)}, \varepsilon\right)\right)\psi_0^\varepsilon\right\|_{\mathcal{H}}^2\right]$$
$$\leq \sum_{\substack{S\in\pi^*(I(N))\\ \text{higher order}}} \left|\mathcal{R}\left(G^{\text{rough}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, \left(T^{(1)}/\varepsilon, ..., r\right), N_<, N_{\text{fin}}, M, S\right)\right|$$
$$+ \sum_{\substack{S\in\pi^*(I(N))\\ \text{crossing pairing}}} \left|\mathcal{R}\left(G^{\text{rough}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, \left(T^{(1)}/\varepsilon, ..., r\right), N_<, N_{\text{fin}}, M, S\right)\right|$$
$$+ \sum_{\substack{S\in\pi^*(I(N))\\ \text{non-crossing pairing}}} \left|\mathcal{R}\left(G^{\text{rough}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, \left(T^{(1)}/\varepsilon, ..., r\right), N_<, N_{\text{fin}}, M, S\right)\right|$$
$$\leq (2\overline{N} + 6\overline{M})! C^{\overline{N}+\overline{M}+j_0}\left(\langle L^{(0)}\rangle + j_0\overline{N}\right)^{4\overline{N}+6\overline{M}+7}\langle\varepsilon C_{\text{obs}}\rangle^{4\overline{M}+7}$$
$$e^{2\langle C_{\text{obs}}\rangle j_0|T|}\,\|\psi_0^\varepsilon\|_{\mathcal{H}}^2\prod_{j=1}^{j_0-1}\|a_j\|^2\,|\log\varepsilon|^{2\overline{N}+2\overline{M}+2j_0}$$
$$\sup_{D\in\{1,...,2\overline{N}+2\overline{M}\}}\varepsilon^{D/2}D^{CD} \tag{4.211}$$

holds. C is a finite constant only depending on dimension d, $\overline{M}$ and the distribution of ξ.

Finally, due to Lemma 4.5, for all $r \in [0, T^{(j_0)}/\varepsilon]$

$$\lim_{R\to\infty} \mathbb{E}\left[\left\|G^{\text{end}}_{M,\overline{N}^{(j_0)}}\left(r; R, L^{(j_0)}, \varepsilon\right)\prod_{j=1}^{j_0-1}\left(A_j^\varepsilon F_{N^{(j)}}\left(T^{(j)}/\varepsilon; R, L^{(j)}, \varepsilon\right)\right)\psi_0^\varepsilon\right\|_{\mathcal{H}}^2\right]$$
$$\leq \sum_{\substack{S\in\pi^*(I(N))\\ \text{higher order}}} \left|\mathcal{R}\left(G^{\text{end}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, \left(T^{(1)}/\varepsilon, ..., r\right), N_<, \overline{N}, M, S\right)\right|$$
$$+ \sum_{\substack{S\in\pi^*(I(N))\\ \text{crossing pairing}}} \left|\mathcal{R}\left(G^{\text{end}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, \left(T^{(1)}/\varepsilon, ..., r\right), N_<, \overline{N}, M, S\right)\right|$$
$$+ \sum_{\substack{S\in\pi^*(I(N))\\ \text{non-crossing}\\ \text{pairing}}} |\mathcal{R} - \mathcal{R}_+ - \mathcal{R}_-|\left(G^{\text{end}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, \left(T^{(1)}/\varepsilon, ..., r\right), N_<, \overline{N}, M, S\right)$$
$$+ \sum_{\substack{S\in\pi^*(I(N))\\ \text{non-crossing}\\ \text{pairing}}} |\mathcal{R}_+ + \mathcal{R}_-|\left(G^{\text{end}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, \left(T^{(1)}/\varepsilon, ..., r\right), N_<, \overline{N}, M, S\right) \tag{4.212}$$

so if we plug in Lemma 4.9, 4.12, 4.15 and Corollary 4.19,

$$\lim_{R\to\infty} \sup_{r\in[0,T^{(j_0)}/\varepsilon]} \mathbb{E}\left[\left\| G^{\mathrm{end}}_{M,\overline{N}^{(j_0)}}\left(r;R,L^{(j_0)},\varepsilon\right) \prod_{j=1}^{j_0-1} \left(A^\varepsilon_j F_{N^{(j)}}\left(T^{(j)}/\varepsilon;R,L^{(j)},\varepsilon\right)\right)\psi^\varepsilon_0 \right\|^2_{\mathcal{H}}\right]$$
$$\leq \Bigg[(2\overline{N}+6\overline{M})!(\langle L^{(0)}\rangle + \overline{m}\overline{N})^{4\overline{N}+6\overline{N}+7} \langle C_{\mathrm{obs}}\rangle^{4\overline{M}+10} j_0^3 e^{8\langle C_{\mathrm{obs}}\rangle \overline{m}|T|} |\log\varepsilon|^{2\overline{N}+2\overline{M}+2j_0+1}$$
$$\max_{D\in\{1,2\overline{N}+2\overline{M}\}} \varepsilon^{D/2} D^{CD}$$
$$+\frac{\left(L^{(0)}\left\langle L^{(0)}\right\rangle |T|\right)^{\overline{N}+\overline{M}}}{\overline{N}!}\Bigg] C^{\overline{N}+\overline{M}+j_0} \|\psi^\varepsilon_0\|^2_{\mathcal{H}} \prod_{j=1}^{j_0-1} \|a_j\|^2_{C^2}, \tag{4.213}$$

for all $\varepsilon \in (0, 1/4\langle C_{\mathrm{obs}}\rangle j_0)$, with a C depending only on the distribution of ξ, on dimension d and on $\overline{M}$. We have set $\gamma = 2\varepsilon\langle C_{\mathrm{obs}}\rangle j_0$ in Lemmas 4.9, 4.12, 4.15 and made use of the fact that there are no more than $4^{\overline{N}+\overline{M}}$ non-crossing pairings of $I(N)$ whenever $|N| \leq 2\overline{N}+2\overline{M}$. When applying Corollary 4.19, we estimated the denominator in the last line of (4.189) by $\overline{N}!$, because $|N| \geq 2\overline{N}$.

After inserting the value from (4.203) for $\overline{N}$ and

$$\kappa = \kappa(\varepsilon) = \varepsilon^\vartheta, \tag{4.214}$$

for κ, with $\vartheta \in (0,1)$ to be optimized later, we conclude from equations (4.205-4.213), for sufficiently small $\varepsilon > 0$

$$\lim_{R\to\infty} \mathbb{E}\left[\|\Psi^\varepsilon_2\|^2_{\mathcal{H}}\right]$$
$$\leq C^{\overline{N}(\varepsilon)+1} \|\psi^\varepsilon_0\|^2_{\mathcal{H}} \prod_{j=1}^{\overline{m}-1} \|a_j\|^2_{C^2}$$
$$\times\left(\overline{N}(\varepsilon)^{6\overline{N}(\varepsilon)} |\log\varepsilon|^{2\overline{N}(\varepsilon)+2\overline{M}+2\overline{m}} \left(\left(\frac{\kappa}{\varepsilon}\right)^2\sqrt{\varepsilon} + \left(\frac{\varepsilon}{\kappa}\right)^{\overline{M}} \varepsilon^{-2}\right) + \left(\frac{\kappa}{\varepsilon}\right)^2 \overline{N}(\varepsilon)^{-\overline{N}(\varepsilon)}\right), \tag{4.215}$$

with a $C < \infty$ depending on $\overline{m}$, $\overline{M}$, $|T|$, the distribution of ξ, d, $L^{(0)}$ and C_{obs}, but not on ε, ψ^ε_0 or the functions a_j.

4.5.3. Picking the right parameters

With the choice of parameters

$$\begin{aligned} \overline{M} &= 89, \\ \theta &= \frac{141}{145}, \\ a &= \frac{1}{18}, \end{aligned} \tag{4.216}$$

in equations (4.204) and (4.215), there exists an $\varepsilon_0 > 0$ depending only on $\overline{m}$, $|T|$, the statistics of ξ, dimension d, $L^{(0)}$ and C_{obs} such that for all $0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned} &\Bigg| \lim_{R\to\infty} \mathbb{E}\left[\langle {\Psi'}_1^{\varepsilon}, A^{\varepsilon}_{\overline{m}} \Psi_1^{\varepsilon}\rangle\right] - \sum_{\substack{N\in\mathbb{N}_0^{2\overline{m}} \\ N^{(1)}+\ldots+N^{(\overline{m})}<\overline{N}(\varepsilon) \\ N^{(\overline{m}+1)}+\ldots+N^{(2\overline{m})}<\overline{N}(\varepsilon)}} \sum_{S\in\pi_{\text{sm}}(I(N))} \mathcal{K}_+^{(\text{main})}\left(\psi_0^{\varepsilon}, \varepsilon, a, p, L^{(0)}, T/\varepsilon, N, S\right) \\ &- \sum_{\substack{N\in\mathbb{N}_0^{2\overline{m}} \\ N^{(1)}+\ldots+N^{(\overline{m})}<\overline{N}(\varepsilon) \\ N^{(\overline{m}+1)}+\ldots+N^{(2\overline{m})}<\overline{N}(\varepsilon)}} \sum_{S\in\pi_{\text{sm}}(I(N))} \mathcal{K}_-^{(\text{main})}\left(\psi_0^{\varepsilon}, \varepsilon, a, p, L^{(0)}, T/\varepsilon, N, S\right) \Bigg| \\ &\leq \left(1 + \frac{1}{\lambda}\right) \|\psi_0^{\varepsilon}\|_{\mathcal{H}}^2 \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^2}\, \varepsilon^{1/19}, \end{aligned} \tag{4.217}$$

and

$$\lim_{R\to\infty} \mathbb{E}\left[\|\Psi_2^{\varepsilon}\|_{\mathcal{H}}^2\right] \leq \|\psi_0^{\varepsilon}\|_{\mathcal{H}}^2 \prod_{j=1}^{\overline{m}-1} \|a_j\|_{C^2}^2\, \varepsilon^{1/2700}. \tag{4.218}$$

Analogously, for ${\Psi'}_2^{\varepsilon}$ as defined below equation (4.13)

$$\lim_{R\to\infty} \mathbb{E}\left[\|{\Psi'}_2^{\varepsilon}\|_{\mathcal{H}}^2\right] \leq \|\psi_0^{\varepsilon}\|_{\mathcal{H}}^2 \prod_{j=\overline{m}+1}^{2\overline{m}-1} \|a_j\|_{C^2}^2\, \varepsilon^{1/2700}. \tag{4.219}$$

The extremely tiny exponent in (4.218) is due to choosing $\overline{M}$ as small as possible.

From (4.14), (4.217) and (4.218), we conclude

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \mathbb{E}\left[\mathcal{J}^{\varepsilon}\left(H^{\varepsilon}, \psi_0^{\varepsilon}, T, a, p\right)\right] \\
&= \lim_{\varepsilon\to 0} \sum_{\substack{N\in\mathbb{N}_0^{2\overline{m}} \\ N^{(1)}+\ldots+N^{(\overline{m})}<\overline{N}(\varepsilon) \\ N^{(\overline{m}+1)}+\ldots+N^{(2\overline{m})}<\overline{N}(\varepsilon)}} \sum_{S\in\pi_{\mathrm{sm}}(I(N))} \mathcal{K}_{+}^{(\mathrm{main})}\left(\psi_0^{\varepsilon}, \varepsilon, a, p, L^{(0)}, T/\varepsilon, N, S\right) \\
&+ \lim_{\varepsilon\to 0} \sum_{\substack{N\in\mathbb{N}_0^{2\overline{m}} \\ N^{(1)}+\ldots+N^{(\overline{m})}<\overline{N}(\varepsilon) \\ N^{(\overline{m}+1)}+\ldots+N^{(2\overline{m})}<\overline{N}(\varepsilon)}} \sum_{S\in\pi_{\mathrm{sm}}(I(N))} \mathcal{K}_{-}^{(\mathrm{main})}\left(\psi_0^{\varepsilon}, \varepsilon, a, p, L^{(0)}, T/\varepsilon, N, S\right)
\end{aligned} \tag{4.220}$$

provided the limits exist.

4.6. Resummation of the ladder graphs

In this section, we still consider initial states which, in addition to the assumptions of Theorem 3.1, have a Fourier transform $\widehat{\psi}_0^{\varepsilon}(k)$ that vanishes whenever $|k| \notin [\lambda, L^{(0)}]$. Also, we assume that the observables a_j are bounded up to the second derivative.

4.6.1. Notation

We now want to resummate the $\mathcal{K}_{\sigma}^{(\mathrm{main})}$ ($\sigma \in \{+,-\}$) amplitudes which belong to the simple, markovian pairings, $S \in \pi_{\mathrm{sm}}(I(N))$ and calculate

$$\lim_{\varepsilon\to 0} \sum_{\substack{N\in\mathbb{N}_0^{2\overline{m}} \\ N^{(1)}+\ldots+N^{(\overline{m})}<\overline{N}(\varepsilon) \\ N^{(\overline{m}+1)}+\ldots+N^{(2\overline{m})}<\overline{N}(\varepsilon)}} \sum_{S\in\pi_{\mathrm{sm}}(I(N))} \mathcal{K}_{\sigma}^{(\mathrm{main})}\left(\psi_0^{\varepsilon}, \varepsilon, a, p, L^{(0)}, T/\varepsilon, N, S\right). \tag{4.221}$$

To this end, recall that actually the j-th and $2\overline{m}+1-j$ time intervals are physically the same. This can be visualized as in Figure 4.6 by bending the right half of the graphs used so far by 180°, so that corresponding time intervals come to lie parallel to each other. Simple markovian pairings S now appear as the "ladder graphs" described in [16] and [32], with the only types of pairs being gates (which have already been integrated over and appear as crosses) and "rungs", which always connect opposite time intervals (all other possibilities would be either nested or non-markovian). For the j-th time interval (with j from now on only ranging from 1 to $\overline{m}$), there are $R^{(j)} \in \mathbb{N}_0$ rungs, and between two consecutive rungs (or a rung and an observable), there may be $G_{l,\rho}^{(j)} \in \mathbb{N}_0$ gates. Here $l \in \{0, ..., R^{(j)}\}$ represents the time slot after the l-th rung in the j-th interval, the time slot before the first rung being indicated by $l = 0$, while $\rho \in \{\pm\}$ is $+$ for the lower line

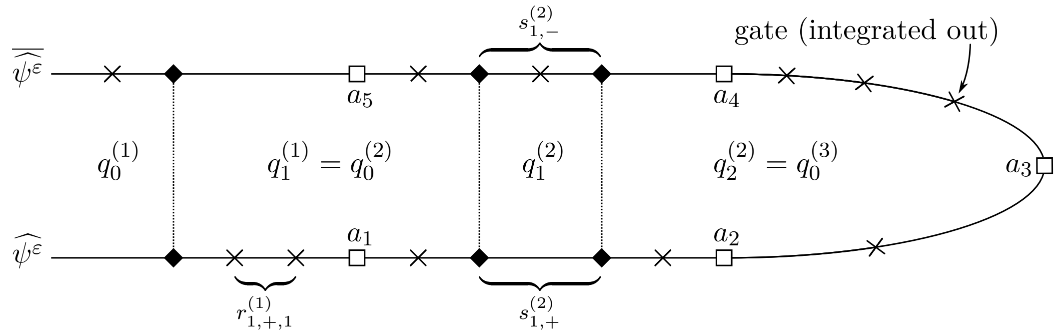

Figure 4.6.: A ladder graph with $\overline{m} = 3$, $R^{(1)} = 1$, $R^{(2)} = 2$, $R^{(3)} = 0$. Rungs are indicated by dotted lines, while crosses represent gates (which have already been integrated over). For example, $G^{(3)}_{0,-} = 3$ and $G^{(2)}_{2,+} = 1$. The momenta $q^{(j)}_l$ are the same for the $-$ (top) and $+$ (bottom) line. Time variables of type $s^{(j)}_{l,\rho}$ and $r^{(j)}_{l,\rho,\nu}$ give the spacing of rungs and gates, respectively. The black diamonds at the ends of the rungs mean that these scattering events occur with a cut-off function present.

in Figure 4.6 and $-$ for the upper one. With the short hand

$$
\begin{aligned}
\mathbf{R} &= \left(R^{(j)}\right)_{j\in\{1,\dots\overline{m}\}} \\
\mathbf{G} &= \left(G^{(j)}_{l,\rho}\right)_{\substack{j\in\{1,\dots\overline{m}\}\\ l\in\{0,\dots,R^{(j)}\}\\ \rho\in\{\pm\}}} \\
|\mathbf{R}| &= \sum_{j=1}^{\overline{m}} R^{(j)}, \\
|\mathbf{G}| &= \sum_{j=1}^{\overline{m}} \sum_{l=0}^{R^{(j)}} \sum_{\rho\in\{\pm\}} G^{(j)}_{l,\rho},
\end{aligned}
\tag{4.222}
$$

there is a bijective mapping $(N, S) \leftrightarrow (\mathbf{R}, \mathbf{G})$. The overall number of pairs is of course the sum of rungs and gates,

$$
|N|/2 = |\mathbf{R}| + |\mathbf{G}|. \tag{4.223}
$$

Accordingly, one can rewrite the sum

$$
\begin{aligned}
&\sum_{\substack{N\in\mathbb{N}_0^{2\overline{m}}\\ N^{(1)}+\dots+N^{(\overline{m})}<\overline{N}(\varepsilon)\\ N^{(\overline{m}+1)}+\dots+N^{(2\overline{m})}<\overline{N}(\varepsilon)}} \sum_{S\in\pi_{\mathrm{sm}}(I(N))} \mathcal{K}^{(\mathrm{main})}_\sigma\left(\psi^\varepsilon_0, \varepsilon, a, p, L^{(0)}, T/\varepsilon, N, S\right) \\
&= \sum_{\mathbf{R}\in\mathbb{N}_0^{\overline{m}}} \sum_{\mathbf{G}\in\Gamma_\varepsilon(\mathbf{R})} \mathcal{K}^{(\mathrm{main})}_\sigma\left(\psi^\varepsilon_0, \varepsilon, a, p, L^{(0)}, T/\varepsilon, \mathbf{R}, \mathbf{G}\right),
\end{aligned}
\tag{4.224}
$$

with the last sum running over all $\mathbf{G}$ in

$$\varGamma_\varepsilon(\mathbf{R}) = \left\{ \mathbf{G} : G_{l,\rho}^{(j)} \in \mathbb{N}_0 \forall j,l,\rho; 2\sum_{j=1}^{\overline{m}} \sum_{l=0}^{R^{(j)}} G_{l,\rho}^{(j)} < \overline{N}(\varepsilon) - |\mathbf{R}| \forall \rho \right\}. \tag{4.225}$$

The sum over $\mathbf{R}$ in (4.224) has only finitely many non-zero summands, as the set $\varGamma_\varepsilon(\mathbf{R})$ will be empty if any of the components of $\mathbf{R}$ exceeds $\overline{N}(\varepsilon)$. However, as $\varepsilon \searrow 0$, $\overline{N}(\varepsilon) \to \infty$ and thus $\varGamma_\varepsilon(\mathbf{R})$ approaches

$$\varGamma_0(\mathbf{R}) = \left\{ \mathbf{G} : G_{l,\rho}^{(j)} \in \mathbb{N}_0 \forall j \in \{1,...,\overline{m}\}, l \in \{0,...,R^{(j)}\}, \rho \in \{\pm\} \right\}. \tag{4.226}$$

4.6.2. Taking the limit

From Lemma 4.18, one can conclude that

$$\sum_{\mathbf{G} \in \varGamma_0(\mathbf{R})} \mathcal{K}_\sigma^{(\mathrm{main})} \left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, \mathbf{R}, \mathbf{G} \right) \tag{4.227}$$

is well-defined for any $\varepsilon > 0$, with

$$\begin{aligned} &\left| \sum_{\mathbf{G} \in \varGamma_\varepsilon(\mathbf{R})} \mathcal{K}_\sigma^{(\mathrm{main})} \left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, \mathbf{R}, \mathbf{G} \right) \right. \\ &\qquad \left. - \sum_{\mathbf{G} \in \varGamma_0(\mathbf{R})} \mathcal{K}_\sigma^{(\mathrm{main})} \left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, \mathbf{R}, \mathbf{G} \right) \right| \\ &\qquad\qquad \leq \frac{C^{\overline{m}+|\mathbf{R}|+\overline{N}(\varepsilon)}}{|\mathbf{R}|! \left(\overline{N}(\varepsilon) - |\mathbf{R}| \right)_+ !} \|\psi_0^\varepsilon\|_{\mathcal{H}}^2 \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^2} \to 0 \quad (\varepsilon \to 0) \end{aligned} \tag{4.228}$$

and

$$\left| \sum_{\mathbf{G} \in \varGamma_\varepsilon(\mathbf{R})} \mathcal{K}_\sigma^{(\mathrm{main})} \left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, \mathbf{R}, \mathbf{G} \right) \right| \leq \frac{C^{\overline{m}+|\mathbf{R}|}}{|\mathbf{R}|!} \|\psi_0^\varepsilon\|_{\mathcal{H}}^2 \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^2} \tag{4.229}$$

with a constant $C < \infty$ depending on d, g_2, $L^{(0)}$ and T, but not on $\mathbf{R}$ and ε. If one assumes for a moment that there is a limit

$$\lim_{\varepsilon \to 0} \sum_{\mathbf{G} \in \varGamma_0(\mathbf{R})} \mathcal{K}_\sigma^{(\mathrm{main})} \left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, \mathbf{R}, \mathbf{G} \right) \tag{4.230}$$

for all $\mathbf{R} \in \mathbb{N}_0^{\overline{m}}$, dominated convergence applied to the $\mathbf{R}$ sum in (4.224) yields

$$\begin{aligned} &\lim_{\varepsilon \to 0} \sum_{\mathbf{R} \in \mathbb{N}_0^{\overline{m}}} \sum_{\mathbf{G} \in \varGamma_\varepsilon(\mathbf{R})} \mathcal{K}_\sigma^{(\mathrm{main})} \left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, \mathbf{R}, \mathbf{G} \right) \\ &\qquad = \sum_{\mathbf{R} \in \mathbb{N}_0^{\overline{m}}} \lim_{\varepsilon \to 0} \sum_{\mathbf{G} \in \varGamma_0(\mathbf{R})} \mathcal{K}_\sigma^{(\mathrm{main})} \left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, \mathbf{R}, \mathbf{G} \right). \end{aligned} \tag{4.231}$$

So the only task remaining is to identify the limit (4.230). For now, we fix a single configuration of rungs and gates, $(\mathbf{R}, \mathbf{G})$ (and thus a particular choice of (N, S)). By placing $R^{(j)}$ rungs, the j-th time interval is divided into $R^{(j)} + 1$ subintervals, both on the $+$ and on the $-$ side of the graph, giving rise to time variables $s_{l,\rho}^{(j)} \geq 0$ with $l \in \{0, ..., R^{(j)}\}$ and $\rho \in \{\pm\}$ and requirement

$$\sum_{l=0}^{R^{(j)}} s_{l,\rho}^{(j)} = T^{(j)} \tag{4.232}$$

for all j, ρ. In turn, by adding the gates between two consecutive rungs, every subinterval of length $s_{l,\rho}^{(j)}$ is divided into $G_{l,\rho}^{(j)} + 1$ parts of length $r_{l,\rho,\nu}^{(j)} \geq 0$, $\nu \in \{0, ..., G_{l,\rho}^{(j)}\}$, with

$$\sum_{\nu=0}^{G_{l,\rho}^{(j)}} r_{l,\rho,\nu}^{(j)} = s_{l,\rho}^{(j)}. \tag{4.233}$$

Similar to [32], but with an extra index j accounting for the multiple measurements, we define the time variables

$$\begin{aligned} s_l^{(j)} &= \frac{s_{l,+}^{(j)} + s_{l,-}^{(j)}}{2}, \\ b_l^{(j)} &= \frac{s_{l,-}^{(j)} - s_{l,+}^{(j)}}{\varepsilon}. \end{aligned} \tag{4.234}$$

The momentum variables in consideration are still q_A, $A \in S \backslash S_{\text{gate}}$, but as all A are rungs now, we write them as $q_l^{(j)}$, $j \in \{1, ..., \overline{m}\}$, $l \in \{0, ..., R^{(j)}\}$. We will use the variables $q_{R^{(j)}}^{(j)} = q_0^{(j+1)}$ simultaneously to simplify notation. Note that these $q_l^{(j)}$ are *not* indexed in the same way as the $q_n^{(j)}$ introduced before: j now only runs from 1 to $\overline{m}$ instead of 1 to $2\overline{m}$, and l gets only updated after scattering events belonging to rungs, not, as before, also after gates. Accordingly, (while the cutoff for gates has already dropped out in the proof of Lemma 4.16) every rung comes with a cutoff function

$$\begin{aligned} \Upsilon\left(q_l^{(j)}, q_{l-1}^{(j)}, j, l, \mathbf{G}\right) &= \Phi\left(q_l^{(j)}, q_{l-1}^{(j)}, L_{n_+}^{(j)}\right) \Phi\left(q_l^{(j)}, q_{l-1}^{(j)}, L_{n_-}^{(j)}\right), \\ n_\rho &= \sum_{\tilde{l}=0}^{l-1} \left(1 + 2G_{l,\rho}^{(j)}\right). \end{aligned} \tag{4.235}$$

Finally, as all $A \in S \setminus S_{\text{gate}}$ are rungs, the signs ρ_A defined as in (4.125) all equal $+1$. For a simple, markovian pairing with rungs $\mathbf{R}$ and gates $\mathbf{G}$, the amplitude in (4.176) can thus be rewritten as

$$\begin{aligned}
&\mathcal{K}_\sigma^{(\mathrm{main})}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, \mathbf{R}, \mathbf{G}\right)\\
&= \int_{\mathbb{R}^d} \mathrm{d}q_0^{(1)} \left|2\pi q_0^{(1)}\right|^{2|\mathbf{R}|} \widehat{\psi}_{0,\sigma}^\varepsilon\left(q_0^{(1)}\right) \overline{\widehat{\psi}}_{0,\sigma}^\varepsilon\left(q_0^{(1)} + \varepsilon P^{(2\overline{m})}\right)\\
&\prod_{j=1}^{\overline{m}} \prod_{l=0}^{R^{(j)}} \left(\int_{\mathbb{R}_+} \mathrm{d}s_l^{(j)} \int_{\mathbb{R}} \mathrm{d}b_l^{(j)} \prod_{\nu=0}^{G_{l,+}^{(j)}} \int_{\mathbb{R}_+} \mathrm{d}r_{l,+,\nu}^{(j)} \prod_{\nu'=0}^{G_{l,-}^{(j)}} \int_{\mathbb{R}_+} \mathrm{d}r_{l,-,\nu'}^{(j)}\right) \int_{\mathbb{R}^{|\mathbf{R}|d}} \prod_{j=1}^{\overline{m}} \prod_{l=1}^{R^{(j)}} \mathrm{d}q_l^{(j)}\\
&\qquad \prod_{j=1}^{\overline{m}} \left[\delta\left(T^{(j)} - \sum_{l=0}^{R^{(j)}} s_l^{(j)}\right) \delta\left(\sum_{l=0}^{R^{(j)}} b_l^{(j)}\right)\right.\\
&\qquad\qquad \left.\prod_{l=0}^{R^{(j)}} \left(\delta\left(s_{l,+}^{(j)} - \sum_{\nu=0}^{G_{l,+}^{(j)}} r_{l,+,\nu}^{(j)}\right) \delta\left(s_{l,-}^{(j)} - \sum_{\nu=0}^{G_{l,-}^{(j)}} r_{l,-,\nu}^{(j)}\right) \mathbb{1}\left\{\left|b_l^{(j)}\right| \le 2s_l^{(j)}/\varepsilon\right\}\right)\right]\\
&\qquad \prod_{j=1}^{\overline{m}} \prod_{l=0}^{R^{(j)}} \left[\left(\widehat{g}_2\left(q_l^{(j)} - q_{l-1}^{(j)}\right) \Upsilon\left(q_l^{(j)}, q_{l-1}^{(j)}, j, l, \mathbf{G}\right)\right)^{1(l\neq 0)}\right.\\
&\qquad\qquad \times \left(-\Theta_{-\sigma}\left(\widehat{q}_l^{(j)} \left|q_0^{(1)}\right|\right)\right)^{G_{l,-}^{(j)}} \exp\left(2\pi i\sigma\left(\left|q_l^{(j)}\right| + \varepsilon \widehat{q}_l^{(j)} \cdot P^{(2\overline{m}+1-j)}\right) s_{l,-}^{(j)}/\varepsilon\right)\\
&\qquad\qquad \left.\times \left(-\Theta_{\sigma}\left(\widehat{q}_l^{(j)} \left|q_0^{(1)}\right|\right)\right)^{G_{l,+}^{(j)}} \exp\left(-2\pi i\sigma\left(\left|q_l^{(j)}\right| + \varepsilon \widehat{q}_l^{(j)} \cdot P^{(j)}\right) s_{l,+}^{(j)}/\varepsilon\right)\right]\\
&\qquad \prod_{j=1}^{\overline{m}} a_{j,\sigma}\left(q_{R^{(j)}}^{(j)}\right) \prod_{j=\overline{m}+1}^{2\overline{m}-1} a_{j,\sigma}\left(q_{R^{(2\overline{m}-j)}}^{(2\overline{m}-j)}\right).
\end{aligned} \tag{4.236}$$

The phase associated with a pair (j, l) equals

$$\begin{aligned}
&\exp\left(-2\pi i\sigma\left(\left|q_l^{(j)}\right| + \varepsilon \widehat{q}_l^{(j)} \cdot P^{(j)}\right) s_{l,+}^{(j)}/\varepsilon\right) \exp\left(2\pi i\sigma\left(\left|q_l^{(j)}\right| + \varepsilon \widehat{q}_l^{(j)} \cdot P^{(2\overline{m}+1-j)}\right) s_{l,-}^{(j)}/\varepsilon\right)\\
&\quad = \exp\left(2\pi i\sigma\left|q_l^{(j)}\right| b_l^{(j)}\right) \exp\left(-2\pi i\sigma \widehat{q}_l^{(j)} \cdot (P^{(j)} - P^{(2\overline{m}+1-j)}) s_l^{(j)}\right)\\
&\qquad\quad \cdot \exp\left(\varepsilon\pi i\sigma \widehat{q}_l^{(j)} \cdot (P^{(j)} + P^{(2\overline{m}+1-j)}) b_l^{(j)}\right).
\end{aligned} \tag{4.237}$$

In view of (4.237), the $\prod_{j=1}^{\overline{m}} \prod_{l=1}^{R^{(j)}} \mathrm{d}q_l^{(j)}$ integral of the last four lines in (4.236) can be bounded by

$$\begin{aligned}
&C_{d,g_2}^{|\mathbf{R}|} \prod_{j=1}^{2\overline{m}-1} \left(\|a_j\|_{C^2}\right) \left\langle L^{(0)}\right\rangle^{2|\mathbf{G}|}\\
&\qquad \prod_{j=1}^{\overline{m}-1} \left[\left\langle b_{R^{(j)}}^{(j)} + \sum_{l'=1}^{R^{(j+1)}} b_{l'}^{(j+1)}\right\rangle^{-2} \prod_{l=1}^{R^{(j)}-1} \left\langle b_l^{(j)}\right\rangle^{-2}\right] \prod_{l''=1}^{R^{(\overline{m})}} \left\langle b_{l''}^{(\overline{m})}\right\rangle^{-2},
\end{aligned} \tag{4.238}$$

where Lemma C.3 was used with the trivial ancestry relation $\check{\cdot}$ that comes with the ordering of the rungs. Therefore, the $b_l^{(j)}$ integrals are bounded independently of ε, while from the geometry of the integration domain, the dsdr integral is less or equal to

$$|T|^{|\mathbf{G}|+|\mathbf{R}|}\left(\prod_{j=1}^{\overline{m}}\left(R^{(j)}!\prod_{l=1}^{R^{(j)}}\prod_{\rho\in\{\pm\}}G_{l,\rho}^{(j)}!\right)\right)^{-1}. \tag{4.239}$$

The $q_0^{(1)}$ and p integrals then yield the bound

$$\left(C_{d,g_2}\left\langle L^{(0)}\right\rangle^2|T|\right)^{|\mathbf{G}|+|\mathbf{R}|}\left(\prod_{j=1}^{\overline{m}}\left(R^{(j)}!\prod_{l=1}^{R^{(j)}}\prod_{\rho\in\{\pm\}}G_{l,\rho}^{(j)}!\right)\right)^{-1}\|\psi_0^\varepsilon\|_{\mathcal{H}}^2\prod_{j=1}^{2\overline{m}-1}\|a_j\|_{C^2} \tag{4.240}$$

for the amplitude, which is clearly summable over $\mathbf{G}$. Therefore, when plugging (4.236) into

$$\lim_{\varepsilon\to 0}\sum_{\mathbf{G}\in\Gamma_0(\mathbf{R})}\mathcal{K}_\sigma^{(\mathrm{main})}\left(\psi_0^\varepsilon,\varepsilon,a,b,L^{(0)},T/\varepsilon,\mathbf{R},\mathbf{G}\right), \tag{4.241}$$

dominated convergence and Fubini's theorem are applicable to all sums and integrals as long as one makes sure that the $\prod_{j=1}^{\overline{m}}\prod_{l=1}^{R^{(j)}}\mathrm{d}q_l^{(j)}$ integral is executed before the $\prod_{j=1}^{\overline{m}}\prod_{l=1}^{R^{(j)}}\mathrm{d}b_l^{(j)}$ integral. Thanks to the $\left\langle b_l^{(j)}\right\rangle^{-2}$ decay for $l>0$ and the identity $b_0^{(j)}=-b_1^{(j)}-\ldots-b_{R^{(j)}}^{(j)}$, one can use dominated convergence for the $b_l^{(j)}$ integrals, eliminate the last factor in (4.237) by observing

$$\left|\exp\left(\varepsilon\pi i\sigma\hat{q}_l^{(j)}\cdot(P^{(j)}+P^{(2\overline{m}+1-j)})b_l^{(j)}\right)-1\right|\le\min\left(1,2\pi\overline{m}C_{\mathrm{obs}}\varepsilon\left|b_l^{(j)}\right|\right). \tag{4.242}$$

and remove the restriction

$$\mathbb{1}\left\{\left|b_l^{(j)}\right|\le 2s_l^{(j)}/\varepsilon\right\}\to 1 \tag{4.243}$$

on the set of full measure $\{s_l^{(j)}\neq 0\}$ for all j,l. In the same fashion, dominated convergence for the $b_l^{(j)}$ integrals and the $\mathbf{G}$ sum allows to modify the delta functions in the fifth line of (4.236), because

$$\begin{aligned}&\left|\prod_{j,l,\rho}\int_{\mathbb{R}_+^{G_{l,\rho}^{(j)}+1}}\mathrm{d}r_{l,\rho}^{(j)}\delta\left(s_{l,\rho}^{(j)}-\sum_{\nu=0}^{G_{l,\rho}^{(j)}}r_{l,\rho,\nu}^{(j)}\right)-\prod_{j,l,\rho}\int_{\mathbb{R}_+^{G_{l,\rho}^{(j)}+1}}\mathrm{d}r_{l,\rho}^{(j)}\delta\left(s_l^{(j)}-\sum_{\nu=0}^{G_{l,\rho}^{(j)}}r_{l,\rho,\nu}^{(j)}\right)\right|\\&\quad\le|\mathbf{G}||T|^{|\mathbf{G}|-1}\min\left(|T|/|\mathbf{G}|,\varepsilon\max_{j,l}|b_{j,l}|\right)\left(\prod_{j,l,\rho}G_{l,\rho}^{(j)}!\right)^{-1}.\end{aligned} \tag{4.244}$$

We then can integrate out the r variables and have, in case the limit of the right side exists,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{\mathbf{G}\in\varGamma_0(\mathbf{R})}\mathcal{K}_\sigma^{(\mathrm{main})}\left(\psi_0^\varepsilon,\varepsilon,a,p,L^{(0)},T/\varepsilon,\mathbf{R},\mathbf{G}\right)\\
&=\lim_{\varepsilon\to 0}\sum_{\mathbf{G}\in\varGamma_0(\mathbf{R})}\int_{\mathbb{R}^d}\mathrm{d}q_0^{(1)}\left|2\pi q_0^{(1)}\right|^{2|\mathbf{R}|}\widehat{\psi}^\varepsilon_{0,\sigma}\left(q_0^{(1)}\right)\overline{\widehat{\psi}^\varepsilon_{0,\sigma}}\left(q_0^{(1)}+\varepsilon P^{(2\overline{m})}\right)\\
&\quad\int_{\mathbb{R}_+^{|\mathbf{R}|+\overline{m}}}\prod_{j=1}^{\overline{m}}\prod_{l=0}^{R^{(j)}}\left(\mathrm{d}s_l^{(j)}\right)\int_{\mathbb{R}^{|\mathbf{R}|}}\prod_{j=1}^{\overline{m}}\prod_{l=1}^{R^{(j)}}\left(\mathrm{d}b_l^{(j)}\right)\int_{\mathbb{R}^{|\mathbf{R}|d}}\prod_{j=1}^{\overline{m}}\prod_{l=1}^{R^{(j)}}\mathrm{d}q_l^{(j)}\\
&\quad\prod_{j=1}^{\overline{m}}\left[\delta\left(T^{(j)}-\sum_{l=0}^{R^{(j)}}s_l^{(j)}\right)\prod_{l=0}^{R^{(j)}}\left\{\prod_{\rho\in\{\pm\}}\left(-\varTheta_{\rho\sigma}\left(\widehat{q}_l^{(j)}\left|q_0^{(1)}\right|\right)s_l^{(j)}\right)^{G_{l,\rho}^{(j)}}\left(G_{l,\rho}^{(j)}!\right)^{-1}\right.\right.\\
&\qquad\left(\widehat{g}_2\left(q_l^{(j)}-q_{l-1}^{(j)}\right)\varUpsilon\left(q_l^{(j)},q_{l-1}^{(j)},j,l,\mathbf{G}\right)\exp\left(2\pi i\sigma\left(\left|q_l^{(j)}\right|-\left|q_0^{(j)}\right|\right)b_l^{(j)}\right)\right)^{1(l\neq 0)}\\
&\qquad\left.\left.\left(\exp\left(-2\pi i\sigma\widehat{q}_l^{(j)}\cdot(P^{(j)}-P^{(2\overline{m}+1-j)})s_l^{(j)}\right)\right)\right\}\right]\\
&\quad\prod_{j=1}^{\overline{m}}a_{j,\sigma}\left(q_{R^{(j)}}^{(j)}\right)\prod_{j=\overline{m}+1}^{2\overline{m}-1}a_{j,\sigma}\left(q_{R^{(2\overline{m}-j)}}^{(2\overline{m}-j)}\right).
\end{aligned}\tag{4.245}$$

Now we apply Lemma C.4 to the $b_l^{(j)}$, $q_l^{(j)}$ integrals and obtain delta functions

$$\prod_{j=1}^{\overline{m}}\prod_{l=1}^{R^{(j)}}\delta\left(\left|q_l^{(j)}\right|-\left|q_0^{(j)}\right|\right)=\prod_{j=1}^{\overline{m}}\prod_{l=1}^{R^{(j)}}\delta\left(\left|q_l^{(j)}\right|-\left|q_{l-1}^{(j)}\right|\right).\tag{4.246}$$

These delta functions ensure that $\left|q_l^{(j)}\right|=\left|q_0^{(1)}\right|\leq L^{(0)}$, so all cut-off functions $\varUpsilon$ are equal to 1. Only now can one take the sum over $\mathbf{G}$ (with the $\varUpsilon$ still present, the implicit dependence of the cutoff thresholds on $\mathbf{G}$ would have complicated matters), which is an exponential series for each value of j,l,ρ. These series converge uniformly due to boundedness of their argument resulting from the compact support of $\widehat{\psi}^\varepsilon_{0,\sigma}$ and (4.246). Furthermore, on the support of the deltas,

$$\widehat{q}_l^{(j)}\left|q_0^{(1)}\right|=q_l^{(j)},\tag{4.247}$$

and thus, for each j,l, there is a factor

$$\exp\left(-\varTheta_+\left(q_l^{(j)}\right)s_l^{(j)}\right)\exp\left(-\varTheta_-\left(q_l^{(j)}\right)s_l^{(j)}\right)=\exp\left(-2\mathrm{Re}\varTheta_+\left(q_l^{(j)}\right)s_l^{(j)}\right)\tag{4.248}$$

by Lemma D.4.

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \sum_{\mathbf{G}\in\Gamma_0(\mathbf{R})} \mathcal{K}_\sigma^{(\mathrm{main})}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, \mathbf{R}, \mathbf{G}\right) \\
&= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} \mathrm{d}q_0^{(1)} \widehat{\psi}^\varepsilon_{0,\sigma}\left(q_0^{(1)}\right) \overline{\widehat{\psi}^\varepsilon_{0,\sigma}}\left(q_0^{(1)} + \varepsilon P^{(2\overline{m})}\right) \\
&\qquad \int_{\mathbb{R}_+^{|\mathbf{R}|+\overline{m}}} \prod_{j=1}^{\overline{m}} \left[\prod_{l=0}^{R^{(j)}} \left(\mathrm{d}s_l^{(j)}\right) \delta\left(T^{(j)} - \sum_{l=0}^{R^{(j)}} s_l^{(j)}\right)\right] \int_{\mathbb{R}^{|\mathbf{R}|d}} \prod_{j=1}^{\overline{m}} \prod_{l=1}^{R^{(j)}} \mathrm{d}q_l^{(j)} \\
&\qquad\quad \prod_{j=1}^{\overline{m}} \prod_{l=0}^{R^{(j)}} \left\{ \left(\left|2\pi q_l^{(j)}\right|^2 \widehat{g}_2\left(q_l^{(j)} - q_{l-1}^{(j)}\right) \delta\left(\left|q_l^{(j)}\right| - \left|q_{l-1}^{(j)}\right|\right)\right)^{1(l\neq 0)} \right. \\
&\qquad\qquad\quad \left. \exp\left(-2\mathrm{Re}\Theta_+\left(q_l^{(j)}\right) s_l^{(j)}\right) \exp\left(-2\pi i\sigma \widehat{q}_l^{(j)} \cdot (P^{(j)} - P^{(2\overline{m}+1-j)}) s_l^{(j)}\right) \right\} \\
&\qquad\quad \prod_{j=1}^{\overline{m}} a_{j,\sigma}\left(q^{(j)}_{R^{(j)}}\right) \prod_{j=\overline{m}+1}^{2\overline{m}-1} a_{j,\sigma}\left(q^{(2\overline{m}-j)}_{R^{(2\overline{m}-j)}}\right).
\end{aligned} \tag{4.249}$$

We recall (4.231), the definitions (2.131), (2.132) of the measure ν_{sc} and the cross-section σ_{sc} as well as the representation (D.25) of σ_{sc}, and have by dominated convergence

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \sum_{\mathbf{R}\in\mathbb{N}_0^{\overline{m}}} \sum_{\mathbf{G}\in\Gamma_\varepsilon(\mathbf{R})} \mathcal{K}_\sigma^{(\mathrm{main})}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, \mathbf{R}, \mathbf{G}\right) \\
&= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} \mathrm{d}q_0^{(1)} \widehat{\psi}^\varepsilon_{0,\sigma}\left(q_0^{(1)}\right) \overline{\widehat{\psi}^\varepsilon_{0,\sigma}}\left(q_0^{(1)} + \varepsilon P^{(2\overline{m})}\right) \\
&\qquad \sum_{R^{(1)}=0}^{\infty} \dots \sum_{R^{(\overline{m})}=0}^{\infty} \int_{\mathbb{R}_+^{|\mathbf{R}|+\overline{m}}} \prod_{j=1}^{\overline{m}} \left[\prod_{l=0}^{R^{(j)}} \left(\mathrm{d}s_l^{(j)}\right) \delta\left(T^{(j)} - \sum_{l=0}^{R^{(j)}} s_l^{(j)}\right)\right] \\
&\qquad\quad \int_{\mathbb{R}^{|\mathbf{R}|d}} \prod_{j=1}^{\overline{m}} \prod_{l=0}^{R^{(j)}} \left\{ \left(\nu_{\mathrm{sc}}\left(q_{l-1}^{(j)}, \mathrm{d}q_l^{(j)}\right)\right)^{1(l\neq 0)} \right. \\
&\qquad\qquad\quad \left. \exp\left(-\sigma_{\mathrm{sc}}\left(q_l^{(j)}\right) s_l^{(j)}\right) \exp\left(-2\pi i\sigma \widehat{q}_l^{(j)} \cdot (P^{(j)} - P^{(2\overline{m}+1-j)}) s_l^{(j)}\right) \right\} \\
&\qquad\quad \prod_{j=1}^{\overline{m}} a_{j,\sigma}\left(q^{(j)}_{R^{(j)}}\right) \prod_{j=\overline{m}+1}^{2\overline{m}-1} a_{j,\sigma}\left(q^{(2\overline{m}-j)}_{R^{(2\overline{m}-j)}}\right)
\end{aligned} \tag{4.250}$$

in which all sums and integrals may be taken in arbitrary order according to Fubini's theorem. Away from $q_0^{(1)} = 0$, the last four lines of (4.250) are a bounded, continuous function of $q_0^{(1)}$. Because we assume that $\widehat{\psi}_0^\varepsilon$ is supported outside a ball of radius $\lambda > 0$ around the origin, and that $W^\varepsilon\left[\widehat{\psi}^\varepsilon_{0,\sigma}\right]$ converges in $\mathfrak{X}_{\mathrm{IR}}{}^*$ (and thus in $\mathcal{F}L^1(C^0)^*$), Lemma

2.16 is applicable. Therefore, the $\varepsilon \to 0$ limit of the right side exists and equals

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \sum_{\mathbf{R}\in\mathbb{N}_0^{\overline{m}}} \sum_{\mathbf{G}\in\Gamma_\varepsilon(\mathbf{R})} \mathcal{K}_\sigma^{(\mathrm{main})}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, T/\varepsilon, \mathbf{R}, \mathbf{G}\right) \\
&= \int_{\mathbb{R}^{2d}} \mu_{0,\sigma}\left(\mathrm{d}x, \mathrm{d}q_0^{(1)}\right) e^{2\pi i P^{(2m)}\cdot x} \\
&\qquad \sum_{R^{(1)}=0}^{\infty} \dots \sum_{R^{(\overline{m})}=0}^{\infty} \int_{\mathbb{R}_+^{|\mathbf{R}|+\overline{m}}} \prod_{j=1}^{\overline{m}} \left[\prod_{l=0}^{R^{(j)}} \left(\mathrm{d}s_l^{(j)}\right) \delta\left(T^{(j)} - \sum_{l=0}^{R^{(j)}} s_l^{(j)}\right)\right] \\
&\qquad \int_{\mathbb{R}^{|\mathbf{R}|d}} \prod_{j=1}^{\overline{m}} \prod_{l=0}^{R^{(j)}} \left\{\left(\nu_{\mathrm{sc}}\left(q_{l-1}^{(j)}, \mathrm{d}q_l^{(j)}\right)\right)^{1(l\neq 0)}\right. \\
&\qquad\qquad \left. \exp\left(-\sigma_{\mathrm{sc}}\left(q_l^{(j)}\right) s_l^{(j)}\right) \exp\left(-2\pi i\sigma \widehat{q}_l^{(j)} \cdot (P^{(j)} - P^{(2\overline{m}+1-j)}) s_l^{(j)}\right)\right\} \\
&\qquad\qquad \prod_{j=1}^{\overline{m}} a_{j,\sigma}\left(q_{R^{(j)}}^{(j)}\right) \prod_{j=\overline{m}+1}^{2\overline{m}-1} a_{j,\sigma}\left(q_{R^{(2\overline{m}-j)}}^{(2\overline{m}-j)}\right).
\end{aligned} \tag{4.251}$$

The last equation is essentially already the statement of Theorem 3.1, however with a restricted class of initial states (namely, with an infrared cut-off λ and an ultraviolet cut-off $L^{(0)}$ still in place) and observables (the functions a_j are C^2 instead of merely continuous, and only defined on k space instead of (x,k) momentum space). We will remove these restrictions in the two sections below.

4.7. Extending the space of test functions

Assume for a moment that $\overline{m} = 1$, $p^{(1)} = 0$ and define the operator $A_{1,\pm}^\varepsilon$ by (4.2) with

$$a_{1,+} = a_{1,-} : \mathbb{R}^d \to [0,1] \tag{4.252}$$

smooth with bounded derivatives such that $a_{1,\pm}(k) = 0$ for $|k| \le L^{(0)}$, and $a_{1,\pm}(k) = 1$ for $|k| \ge L^{(0)} + 1/2$. If we set $Q : \mathcal{H} \to \mathcal{H}$ to be the projection on the subspace of functions with Fourier transform supported only on $k \in \mathbb{R}^d$ with $|k| \ge L^{(0)} + 1/2$, we have

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \mathbb{E}\left[\left\|Q \exp\left(-iH^\varepsilon \frac{T^{(1)}}{\varepsilon}\right)\psi_0^\varepsilon\right\|_{\mathcal{H}}^2\right] \\
&\quad \le \limsup_{\varepsilon\to 0} \mathbb{E}\left[\left\langle \exp\left(-iH^\varepsilon \frac{T^{(1)}}{\varepsilon}\right)\psi_0^\varepsilon, A_1^\varepsilon \exp\left(-iH^\varepsilon \frac{T^{(1)}}{\varepsilon}\right)\psi_0^\varepsilon\right\rangle_{\mathcal{H}}\right] = 0,
\end{aligned} \tag{4.253}$$

where we have used (4.220), the support properties of $\widehat{\psi_0^\varepsilon}$ and the fact that ν_{sc} in (4.251) conserves the absolute value of the momentum. Let now $p^{(j)} \in \mathbb{R}^d$, $j \in \{1, \dots, 2\overline{m} - 1\}$

take arbitrary values, and assume that the functions $a_{j,\pm} : \mathbb{R}^d \to \mathbb{C}$ are only bounded and continuous. Then for any $\delta > 0$, there are functions $a^\delta_{j,\pm} : \mathbb{R}^d \to \mathbb{C}$ which are bounded up to the second derivatives, and fulfill

$$\begin{aligned} \max_{|k|\le L^{(0)}+1} \left| a_j(k) - a^\delta_j(k) \right| &\le \delta \, \|a_j\|_{C^0} \,, \\ \left\| a^\delta_j \right\|_{C^0} &\le \|a_j\|_{C^0} \,. \end{aligned} \tag{4.254}$$

If the operators A^ε_1 and $A^{\varepsilon,\delta}_1$ are defined from a_1 and a^δ_1 by (4.2), equation (4.253) immediately implies

$$\limsup_{\varepsilon\to 0} \mathbb{E}\left[\left\| \left(A^\varepsilon_1 - A^{\varepsilon,\delta}_1 \right) \exp\left(-iH^\varepsilon \frac{T^{(1)}}{\varepsilon} \right) \psi^\varepsilon_0 \right\|^2_{\mathcal{H}} \right] \le C\delta^2 \overline{\lim_{\varepsilon\to 0}} \, \|\psi^\varepsilon_0\|^2_{\mathcal{H}} \, \|a_1\|^2_{C^0} \,. \tag{4.255}$$

Iterating this argument for all j from 1 up to $\overline{m}$ and from $2\overline{m}-1$ down to $\overline{m}$, one can see that there is a constant C such that for all $\delta \in (0, 1]$

$$\overline{\lim_{\varepsilon\to 0}} \mathbb{E}\left[\left| \mathcal{J}^\varepsilon\left(H^\varepsilon, \psi^\varepsilon_0, T, a, p \right) - \mathcal{J}^\varepsilon\left(H^\varepsilon, \psi^\varepsilon_0, T, a^\delta, p \right) \right| \right] \le C\overline{m}\sqrt{\delta} \overline{\lim_{\varepsilon\to 0}} \, \|\psi^\varepsilon_0\|^2_{\mathcal{H}} \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^0} \,. \tag{4.256}$$

But because (4.251) is applicable to a^δ for all $\delta > 0$, and its right side converges as $a^\delta_{j,\pm}(k) \to a_{j,\pm}(k)$ uniformly for $|k| \le L^{(0)} + 1$, we actually have

$$\begin{aligned} &\lim_{\varepsilon\to 0} \mathbb{E}\left[\mathcal{J}^\varepsilon\left(H^\varepsilon, \psi^\varepsilon_0, T, a, p \right) \right] \\ &= \sum_{\sigma\in\{\pm\}} \int_{\mathbb{R}^{2d}} \mu_{0,\sigma}\left(\mathrm{d}x, \mathrm{d}q^{(1)}_0 \right) e^{2\pi i P^{(2m)}\cdot x} \\ &\qquad \sum_{R^{(1)}=0}^{\infty} \cdots \sum_{R^{(\overline{m})}=0}^{\infty} \int_{\mathbb{R}_+^{|\mathbf{R}|+\overline{m}}} \prod_{j=1}^{\overline{m}} \left[\prod_{l=0}^{R^{(j)}} \left(\mathrm{d}s^{(j)}_l \right) \delta\left(T^{(j)} - \sum_{l=0}^{R^{(j)}} s^{(j)}_l \right) \right] \\ &\qquad \int_{\mathbb{R}^{|\mathbf{R}|d}} \prod_{j=1}^{\overline{m}} \prod_{l=0}^{R^{(j)}} \left\{ \left(\nu_{\mathrm{sc}}\left(q^{(j)}_{l-1}, \mathrm{d}q^{(j)}_l \right) \right)^{1(l\neq 0)} \right. \\ &\qquad\qquad \left. \exp\left(-\sigma_{\mathrm{sc}}\left(q^{(j)}_l \right) s^{(j)}_l \right) \exp\left(-2\pi i \sigma \widehat{q}^{(j)}_l \cdot (P^{(j)} - P^{(2\overline{m}+1-j)}) s^{(j)}_l \right) \right\} \\ &\qquad\quad \prod_{j=1}^{\overline{m}} a_{j,\sigma}\left(q^{(j)}_{R^{(j)}} \right) \prod_{j=\overline{m}+1}^{2\overline{m}-1} a_{j,\sigma}\left(q^{(2\overline{m}-j)}_{R^{(2\overline{m}-j)}} \right) \end{aligned} \tag{4.257}$$

for all bounded and continuous functions $a_{j,\pm}$, $j \in \{1, ..., 2\overline{m} - 1\}$.

Next, consider observables which no longer merely live on k-space alone, $a_{j,\pm} : \mathbb{R}^d \to \mathbb{C}$, but are functions on phase space, $\mathfrak{a}_{j,\pm} : \mathbb{R}^{2d} \to \mathbb{C}$, with $\mathfrak{a}_{j,\pm} \in \mathcal{F}L^1(C^0)$ for all $j \in \{1, ..., \overline{m}\}$. Operators $Q^\varepsilon(\mathfrak{a}_j) : \mathcal{H} \to \mathcal{H}$ can then be defined as in (2.198). In view of the definition of the + and − components of $Q^\varepsilon(\mathfrak{a}_j)$, (2.146), the momenta $p^{(1)},...,p^{(2\overline{m}-1)}$ become integration variables, and by (4.257) and dominated convergence, one can see that,

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathbb{E}\Bigg[\Bigg\langle &\mathrm{e}^{-iH^\varepsilon T^{(\overline{m})}/\varepsilon}\prod_{j=1}^{\overline{m}-1}\left(Q^\varepsilon(\mathfrak{a}_j)\mathrm{e}^{-iH^\varepsilon T^{(j)}/\varepsilon}\right)\psi_0^\varepsilon,\\ &Q^\varepsilon(\mathfrak{a}_{\overline{m}})\mathrm{e}^{-iH^\varepsilon T^{(\overline{m})}/\varepsilon}\prod_{j=1}^{\overline{m}-1}\left(Q^\varepsilon(\mathfrak{a}_j)\mathrm{e}^{-iH^\varepsilon T^{(j)}/\varepsilon}\right)\psi_0^\varepsilon\Bigg\rangle_{\mathcal{H}}\Bigg]\\ =\int_{\mathbb{R}^{d(2\overline{m}-1)}}&\mathrm{d}p^{(1)}...\mathrm{d}p^{(2\overline{m}-1)}\lim_{\varepsilon\to 0}\mathbb{E}\left[\mathcal{J}^\varepsilon\left(H^\varepsilon,\psi_0^\varepsilon,T,\widehat{\mathfrak{a}}(p,\cdot),p\right)\right],\end{aligned}\tag{4.258}$$

with the k-space-only observables entering in the second line of (4.258) being given for fixed $p\in\mathbb{R}^{d(2\overline{m}-1)}$ as

$$\begin{aligned}a_{j,\pm}(k)&=\widehat{\mathfrak{a}}_{j,\pm}(p^{(j)},k)\qquad(j\in\{1,...,\overline{m}\}),\\ a_{j,\pm}(k)&=\overline{\widehat{\mathfrak{a}}}_{2\overline{m}-j,\pm}(-p^{(j)},k)\qquad(j\in\{\overline{m}+1,...,2\overline{m}\}),\end{aligned}\tag{4.259}$$

for all $k\in\mathbb{R}^d$. We replace $p^{(1)},...,p^{(2\overline{m}-1)}$ by $P^{(2)},...,P^{(2\overline{m})}$, while $P^{(1)}=0$ by definition. Then, (4.257) and (4.258) yield

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathbb{E}\Bigg[\Bigg\langle &\mathrm{e}^{-iH^\varepsilon T^{(\overline{m})}/\varepsilon}\prod_{j=1}^{\overline{m}-1}\left(Q^\varepsilon(\mathfrak{a}_j)\mathrm{e}^{-iH^\varepsilon T^{(j)}/\varepsilon}\right)\psi_0^\varepsilon,\\ &Q^\varepsilon(\mathfrak{a}_{\overline{m}})\mathrm{e}^{-iH^\varepsilon T^{(\overline{m})}/\varepsilon}\prod_{j=1}^{\overline{m}-1}\left(Q^\varepsilon(\mathfrak{a}_j)\mathrm{e}^{-iH^\varepsilon T^{(j)}/\varepsilon}\right)\psi_0^\varepsilon\Bigg\rangle_{\mathcal{H}}\Bigg]\\ =\sum_{\sigma\in\{\pm\}}&\int_{\mathbb{R}^{d(2\overline{m}-1)}}\mathrm{d}P^{(2)}...\mathrm{d}P^{(2\overline{m})}\int_{\mathbb{R}^{2d}}\mu_{0,\sigma}\left(\mathrm{d}x,\mathrm{d}q_0^{(1)}\right)e^{2\pi iP^{(2m)}\cdot x}\\ &\sum_{R^{(1)}=0}^{\infty}...\sum_{R^{(\overline{m})}=0}^{\infty}\int_{\mathbb{R}_+^{|\mathbf{R}|+\overline{m}}}\prod_{j=1}^{\overline{m}}\left[\prod_{l=0}^{R^{(j)}}\left(\mathrm{d}s_l^{(j)}\right)\delta\left(T^{(j)}-\sum_{l=0}^{R^{(j)}}s_l^{(j)}\right)\right]\\ &\int_{\mathbb{R}^{|\mathbf{R}|d}}\prod_{j=1}^{\overline{m}}\prod_{l=0}^{R^{(j)}}\Bigg\{\left(\nu_{\mathrm{sc}}\left(q_{l-1}^{(j)},\mathrm{d}q_l^{(j)}\right)\right)^{1(l\neq 0)}\\ &\qquad\exp\left(-\sigma_{\mathrm{sc}}\left(q_l^{(j)}\right)s_l^{(j)}\right)\exp\left(-2\pi i\sigma\widehat{q}_l^{(j)}\cdot(P^{(j)}-P^{(2\overline{m}+1-j)})s_l^{(j)}\right)\Bigg\}\\ &\prod_{j=1}^{\overline{m}}\widehat{\mathfrak{a}}_{j,\sigma}\left(P^{(j+1)}-P^{(j)},q_{R^{(j)}}^{(j)}\right)\prod_{j=\overline{m}+1}^{2\overline{m}-1}\overline{\widehat{\mathfrak{a}}}_{2\overline{m}-j,\sigma}\left(-P^{(j+1)}+P^{(j)},q_{R^{(2\overline{m}-j)}}^{(2\overline{m}-j)}\right).\end{aligned}\tag{4.260}$$

As $\mu_{0,\pm}$ is supported away from $\left\{q_0^{(1)}=0\right\}$, the deltas ensure that all $q_l^{(j)}\neq 0$ on the support of the integrand, and we can therefore restrict all observables to $\mathbb{R}^d\times\left(\mathbb{R}^d\setminus\{0\}\right)$, so the semigroups $\left(e^{\mathcal{L}_\pm t}\right)_{t\geq 0}$ and their expansion (2.135) are applicable. One can first single out

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{d}P^{(\overline{m}+1)} \int_{\mathbb{R}^d} \mathrm{d}P^{(\overline{m})} \\
&\quad \overline{\widehat{\mathfrak{a}}}_{\overline{m}-1,\sigma} \left(-P^{(\overline{m}+2)} + P^{(\overline{m}+1)}, q^{(\overline{m}-1)}_{R^{(\overline{m}-1)}}\right) \widehat{\mathfrak{a}}_{\overline{m}-1,\sigma} \left(P^{(\overline{m})} - P^{(\overline{m}-1)}, q^{(\overline{m}-1)}_{R^{(\overline{m}-1)}}\right) \\
&\qquad \sum_{R^{(\overline{m})}=0}^{\infty} \int_{\mathbb{R}_+^{R^{(\overline{m})}+1}} \mathrm{d}s_0^{(\overline{m})} ... \mathrm{d}s^{(\overline{m})}_{R^{(\overline{m})}} \delta\left(T^{(\overline{m})} - \sum_{l=0}^{R^{(\overline{m})}} s_l^{(\overline{m})}\right) \\
&\qquad \int_{\mathbb{R}^d} \nu_{\mathrm{sc}}\left(q_0^{(\overline{m})}, \mathrm{d}q_1^{(\overline{m})}\right) ... \int_{\mathbb{R}^d} \nu_{\mathrm{sc}}\left(q^{(\overline{m})}_{R^{(\overline{m}-1)}}, \mathrm{d}q^{(\overline{m})}_{R^{(\overline{m})}}\right) \\
&\qquad\quad \prod_{l=0}^{R^{(\overline{m})}} \exp\left(\left(-\sigma_{\mathrm{sc}}\left(q_l^{(\overline{m})}\right) + 2\pi i \sigma \widehat{q}_l^{(\overline{m})} \cdot \left(P^{(\overline{m}+1)} - P^{(\overline{m})}\right)\right) s_l^{(\overline{m})}\right) \\
&\qquad\quad \widehat{\mathfrak{a}}_{\overline{m},\sigma}\left(P^{(\overline{m}+1)} - P^{(\overline{m})}, q^{(\overline{m})}_{R^{(\overline{m})}}\right) \\
&= \mathcal{F}\left[\mathfrak{a}_{\overline{m}-1,\sigma}\left(e^{\mathcal{L}_\sigma T^{(\overline{m})}} \mathfrak{a}_{\overline{m},\sigma}\right) \overline{\mathfrak{a}}_{\overline{m}-1,\sigma}\right] \left(P^{(\overline{m}+2)} - P^{(\overline{m}-1)}, q^{(\overline{m}-1)}_{R^{(\overline{m}-1)}}\right),
\end{aligned} \tag{4.261}$$

where $\mathcal{F}$ denotes Fourier transform in the first variable. The function in the square brackets of the last line is obtained by pointwise multiplication in (x, k) phase space and is again an element of $\mathcal{F}L^1(C^0)$. Plugging it back into (4.260) therefore produces another instance of (4.260) with $\overline{m}$ reduced to $\overline{m}-1$ and the "central observable" no longer being $\mathfrak{a}_{\overline{m},\sigma}$ but

$$\mathfrak{a}_{\overline{m}-1,\sigma}\left(e^{\mathcal{L}_\sigma T^{(\overline{m})}} \mathfrak{a}_{\overline{m},\sigma}\right) \overline{\mathfrak{a}}_{\overline{m}-1,\sigma}. \tag{4.262}$$

Iterating this procedure altogether $\overline{m}-1$ times and resubstituting gives us the result

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \mathbb{E}\left[\left\langle \mathrm{e}^{-iH^\varepsilon T^{(\overline{m})}/\varepsilon} \prod_{j=1}^{\overline{m}-1} \left(Q^\varepsilon(\mathfrak{a}_j) \mathrm{e}^{-iH^\varepsilon T^{(j)}/\varepsilon}\right) \psi_0^\varepsilon, \right.\right. \\
&\qquad\qquad \left.\left. Q^\varepsilon(\mathfrak{a}_{\overline{m}}) \mathrm{e}^{-iH^\varepsilon T^{(\overline{m})}/\varepsilon} \prod_{j=1}^{\overline{m}-1} \left(Q^\varepsilon(\mathfrak{a}_j) \mathrm{e}^{-iH^\varepsilon T^{(j)}/\varepsilon}\right) \psi_0^\varepsilon \right\rangle_{\mathcal{H}}\right] \\
&= \sum_{\sigma\in\{\pm\}} \int_{\mathbb{R}^{2d}} \mu_{0,\sigma}\left(\mathrm{d}x, \mathrm{d}q_0^{(1)}\right) \int_{\mathbb{R}^d} \mathrm{d}P^{(2\overline{m})} e^{2\pi i P^{(2\overline{m})}\cdot x} \\
&\qquad \mathcal{F}\left[e^{\mathcal{L}_\sigma T^{(1)}}\left(|\mathfrak{a}_{1,\sigma}|^2 ... e^{\mathcal{L}_\sigma T^{(\overline{m}-1)}}\left(|\mathfrak{a}_{\overline{m}-1,\sigma}|^2 \left(e^{\mathcal{L}_\sigma T^{(\overline{m})}} \mathfrak{a}_{\overline{m},\sigma}\right)\right)\right)\right] \left(P^{(2\overline{m})}, q_0^{(1)}\right) \\
&= \sum_{\sigma\in\{\pm\}} \int_{\mathbb{R}^{2d}} \mu_{0,\sigma}(\mathrm{d}x, \mathrm{d}k) \\
&\qquad\qquad \left[e^{\mathcal{L}_\sigma T^{(1)}}\left(|\mathfrak{a}_{1,\sigma}|^2 ... e^{\mathcal{L}_\sigma T^{(\overline{m}-1)}}\left(|\mathfrak{a}_{\overline{m}-1,\sigma}|^2 \left(e^{\mathcal{L}_\sigma T^{(\overline{m})}} \mathfrak{a}_{\overline{m},\sigma}\right)\right)\right)\right](x, k)
\end{aligned} \tag{4.263}$$

4.8. Removal of the cut-off for large and small momenta

4.8.1. Decomposing the initial states

We will first show how to remove the infra-red cut-off, the removal of the ultra-violet cut-off is then eays, cf. Section 4.8.4. For clarity, we will limit ourselves to one measurement, $\overline{m} = 1$. Assume that there is a sequence of initial states $\psi_0^\varepsilon \in \mathcal{H}$, $(\varepsilon > 0)$ such that both components $\psi_{0,\pm}^\varepsilon$ obey (2.160), (2.161), and (2.162). Moreover, assume that for $\sigma \in \{\pm\}$, $W^\varepsilon\left[\psi_{0,\sigma}^\varepsilon\right]$ converge in $\mathfrak{X}_{\mathrm{IR}}{}^*$ to a limit object $(\mu_{0,\sigma}, \mu_{0,\sigma}^{\mathrm{H}}, \eta_{0,\sigma})$, in the sense of (2.186), as $\varepsilon \to 0$. By testing the right side of (2.186) against suitable functions $\mathfrak{a} \in \mathfrak{X}_{\mathrm{IR}}$, one can observe that $\mu_{0,\sigma}$, $\mu_{0,\sigma}^{\mathrm{H}}$ and $W[\eta_{0,\sigma}]$ are uniquely determined, while $\eta_{0,\sigma}$ is determined up to a constant phase factor. It is clear, that $W^\varepsilon\left[\psi_{0,\sigma}^\varepsilon\right]$, if tested only against $\mathcal{F}L^1(C^0)$ functions, converges to a measure on the entire phase space, namely

$$\tilde{\mu}_{0,\sigma}(\mathrm{d}x, \mathrm{d}k) = \mu_{0,\sigma}(\mathrm{d}x, \mathrm{d}k)\mathbb{1}(k \neq 0) + \mu_{0,\sigma}^{\mathrm{H}}(\mathrm{d}x, S^{d-1})\delta(k)\mathrm{d}k + |\eta(x)|^2\,\delta(k)\mathrm{d}x\mathrm{d}k. \tag{4.264}$$

Moreover, there is a subsequence S of $\varepsilon \to 0$ such that

$$\varepsilon^{-d/2}\psi_{0,\sigma}^\varepsilon\left(\frac{\cdot}{\varepsilon}\right) \rightharpoonup \zeta_{0,\sigma} \ni L^2(\mathbb{R}^d), \qquad (S \ni \varepsilon \to 0), \tag{4.265}$$

weakly in $L^2(\mathbb{R}^d)$ for $\sigma \in \{\pm\}$, where $\zeta_{0,\sigma}$ does not need to be equal to $\eta_{0,\sigma}$. For $\varphi : [0,\infty) \to [0,1]$ smooth with $\varphi([0,1]) = \{1\}$ and $\varphi([2,\infty)) = \{0\}$, and any $0 < 2\lambda < L < \infty$, and $\sigma = \pm$, set

$$\begin{aligned}
\widehat{\psi}_{>,0,\sigma}^{\varepsilon,\lambda,L}(k) &= (1 - \varphi(|k|/\lambda))\,\varphi(|k|/L)\widehat{\psi}_{0,\sigma}^\varepsilon(k),\\
\widehat{\psi}_{<,0,\sigma}^{\varepsilon,\lambda}(k) &= \varphi(|k|/\lambda)\left(\widehat{\psi}_{0,\sigma}^\varepsilon(k) - \varepsilon^{-d/2}\widehat{\zeta}_{0,\sigma}\left(k/\varepsilon\right)\right),\\
\widehat{\zeta}_{0,\sigma}^{\varepsilon,\lambda}(k) &= \varepsilon^{-d/2}\varphi(|k|/\lambda)\widehat{\zeta}_{0,\sigma}\left(k/\varepsilon\right).
\end{aligned} \tag{4.266}$$

The only difference to Appendix A is the additional cut-off for large momenta. Just as shown in Appendix A, one can extract subsequences S'' of $\lambda \to 0$ and $S' \subset S$ of $\varepsilon \to 0$, such that for all $\mathfrak{a} \in \mathfrak{X}_{\mathrm{IR}}$

$$\begin{aligned}
\lim_{S' \ni \varepsilon \to 0}\left\langle W^\varepsilon[\psi_{>,0,\sigma}^{\varepsilon,\lambda,L}], \mathfrak{a}\right\rangle_{\mathfrak{X}_{\mathrm{IR}}} &= \int_{\mathbb{R}^{2d}} \mu_{0,\sigma}(\mathrm{d}x, \mathrm{d}k)\mathfrak{b}\left(x, k, \frac{k}{|k|}\right)(1 - \varphi(|k|/\lambda))^2\,\varphi(|k|/L)^2\\
&:= \int_{\mathbb{R}^{2d}} \mu_{0,\sigma}^{\lambda,L}(\mathrm{d}x, \mathrm{d}k)\mathfrak{b}\left(x, k, \frac{k}{|k|}\right) = \int_{\mathbb{R}^{2d}} \mu_{0,\sigma}^{\lambda,L}(\mathrm{d}x, \mathrm{d}k)\mathfrak{a}^{\mathrm{micro}}\left(x, k\right)
\end{aligned} \tag{4.267}$$

for λ, L fixed, as well as

$$\lim_{S' \ni \varepsilon \to 0}\left\langle W^\varepsilon\left[\zeta_{0,\sigma}^{\varepsilon,\lambda}\right], \mathfrak{a}\right\rangle_{\mathfrak{X}_{\mathrm{IR}}} = \langle W[\zeta_{0,\sigma}], \mathfrak{a}(\cdot, 0, \cdot)\rangle = \langle W[\eta_{0,\sigma}], \mathfrak{a}(\cdot, 0, \cdot)\rangle = \langle W[\eta_{0,\sigma}], \mathfrak{a}^{\mathrm{macro}}\rangle \tag{4.268}$$

for λ fixed, and

$$\begin{aligned}\lim_{S''\ni\lambda\to 0}\lim_{S'\ni\varepsilon\to 0}\left\langle W^\varepsilon\left[\psi^{\varepsilon,\lambda}_{<,0,\sigma}\right],\mathfrak{a}\right\rangle_{\mathfrak{X}_{\mathrm{IR}}} &= \int_{\mathbb{R}^d\times S^{d-1}}\mu^{\mathrm{H}}_{0,\sigma}(\mathrm{d}x,\mathrm{d}\underline{k})\mathfrak{b}(x,0,\underline{k}) \\ &= \int_{\mathbb{R}^d\times S^{d-1}}\mu^{\mathrm{H}}_{0,\sigma}(\mathrm{d}x,\mathrm{d}\underline{k})\mathfrak{a}^{\mathrm{meso}}(x,\underline{k}),\end{aligned} \tag{4.269}$$

with all cross terms vanishing in the $\lim_{S''\ni\lambda\to 0}\lim_{S'\ni\varepsilon\to 0}$ limit. It is important to note that the $\mu_{0,\sigma}$, $\mu^{\mathrm{H}}_{0,\sigma}$ and $W[\eta_{0,\sigma}]$ are really the original quantities, due to their above-mentioned uniqueness. In particular, as $W[\zeta_{0,\sigma}] = W[\eta_{0,\sigma}]$, there has to be a $z\in\mathbb{C}$ with $|z|=1$ that $\zeta_{0,\sigma} = z\eta_{0,\sigma}$.

Now let any $T>0$ be given, and consider the time-evolved, and thus random, states in $\mathcal{H}$,

$$\psi^{\varepsilon,\lambda,L}_{>,T} = \mathrm{e}^{-iH^\varepsilon T/\varepsilon}\psi^{\varepsilon,\lambda,L}_{>,0}, \qquad \psi^{\varepsilon,\lambda}_{<,T} = \mathrm{e}^{-iH^\varepsilon T/\varepsilon}\psi^{\varepsilon,\lambda}_{<,0}, \qquad \zeta^{\varepsilon,\lambda}_T = \mathrm{e}^{-iH^\varepsilon T/\varepsilon}\zeta^{\varepsilon,\lambda}_0. \tag{4.270}$$

Also, for $\sigma\in\{\pm\}$, we denote by $\mu_{T,\sigma} = \mathrm{e}^{\mathcal{L}_\sigma T}\mu_{0,\sigma}$ and $\mu^{\lambda,L}_{T,\sigma} = \mathrm{e}^{\mathcal{L}_\sigma T}\mu^{\lambda,L}_{0,\sigma}$ the measures on phase space obtained by propagating $\mu_{0,\sigma}$ and $\mu^{\lambda,L}_{0,\sigma}$ with the linear Boltzmann dynamics.

4.8.2. Large wavenumbers

For fixed λ, L, the sequence $\left(\psi^{\varepsilon,\lambda,L}_{>,0,\sigma}\right)_{\varepsilon\in S'}$ fulfills all assumptions made for initial states up to Section 4.7 (to be specific, there is and ε-independent infrared and ultraviolet cut-off, and (4.267) shows that $W^\varepsilon[\psi^{\varepsilon,\lambda,L}_{>,0,\sigma}]$ converges weak-* in $\mathcal{F}L^1(C^0)^*$). Therefore, the convergence (4.257) (with $\overline{m}=1$, $p^{(1)}=0$ and $a_{1,\sigma}(k) = \varphi\left(|2k|/\lambda\right)$) holds for $\left(\psi^{\varepsilon,\lambda,L}_{>,0,\sigma}\right)_{\varepsilon\in S'}$ with fixed λ, L

$$\begin{aligned}\lim_{S'\ni\varepsilon\to 0}\mathbb{E}\left[\int_{\mathbb{R}^d}\mathrm{d}k\varphi\left(\frac{|2k|}{\lambda}\right)\left|\widehat{\psi}^{\varepsilon,\lambda,L}_{>,T,\sigma}\right|^2\right] &= \int_{\mathbb{R}^{2d}}\mu^{\lambda,L}_{T,\sigma}(\mathrm{d}x,\mathrm{d}k)\varphi\left(\frac{|2k|}{\lambda}\right) \\ = \int_{\mathbb{R}^{2d}}\mu_{0,\sigma}(\mathrm{d}x,\mathrm{d}k)\varphi\left(\frac{|2k|}{\lambda}\right)&\left(1-\varphi(|k|/\lambda)\right)^2\varphi(|k|/L)^2 = 0,\end{aligned} \tag{4.271}$$

and thus, employing (2.182),

$$\begin{aligned}&\lim_{S'\ni\varepsilon\to 0}\mathbb{E}\left\langle W^\varepsilon[\psi^{\varepsilon,\lambda,L}_{>,T,\sigma}],\mathfrak{a}\right\rangle_{\mathfrak{X}_{\mathrm{IR}}} \\ &= \lim_{S'\ni\varepsilon\to 0}\mathbb{E}\int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}k\widehat{\mathfrak{a}}(p,k,k/\varepsilon)\overline{\widehat{\psi}^{\varepsilon,\lambda,L}_{>,T,\sigma}(k+\varepsilon p/2)}\widehat{\psi}^{\varepsilon,\lambda,L}_{>,T,\sigma}(k-\varepsilon p/2) \\ &= \lim_{S'\ni\varepsilon\to 0}\mathbb{E}\int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}k\widehat{\mathfrak{a}}(p,k,k/\varepsilon)(1-\varphi(4|k|/\lambda))\overline{\widehat{\psi}^{\varepsilon,\lambda,L}_{>,T,\sigma}(k+\varepsilon p/2)}\widehat{\psi}^{\varepsilon,\lambda,L}_{>,T,\sigma}(k-\varepsilon p/2) \\ &= \lim_{S'\ni\varepsilon\to 0}\mathbb{E}\int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}k\widehat{\mathfrak{b}}(p,k,k/|k|)(1-\varphi(4|k|/\lambda))\overline{\widehat{\psi}^{\varepsilon,\lambda,L}_{>,T,\sigma}(k+\varepsilon p/2)}\widehat{\psi}^{\varepsilon,\lambda,L}_{>,T,\sigma}(k-\varepsilon p/2) \\ &= \int_{\mathbb{R}^{2d}}\mu_{T,\sigma}(\mathrm{d}x,\mathrm{d}k)\left(1-\varphi(|k|/\lambda)\right)^2\varphi(|k|/L)^2\mathfrak{b}(x,k,k/|k|).\end{aligned} \tag{4.272}$$

In the last equality, the fact that $(x,k) \mapsto \mathfrak{b}(x,k,k/|k|)(1-\varphi(4|k|/\lambda))$ is an $\mathcal{F}L^1(C^0)$ function was used. Obviously, then

$$\begin{aligned} \lim_{S'' \ni \lambda \to 0} \lim_{S' \ni \varepsilon \to 0} \mathbb{E} \left\langle W^\varepsilon[\psi^{\varepsilon,\lambda,L}_{>,T,\sigma}], \mathfrak{a} \right\rangle_{\mathfrak{X}_{\mathrm{IR}}} &= \int_{\mathbb{R}^{2d}} \mu_{T,\sigma}(\mathrm{d}x,\mathrm{d}k) \mathfrak{b}(x,k,k/|k|) \varphi(|k|/L)^2 \\ &= \int_{\mathbb{R}^{2d}} \mu_{T,\sigma}(\mathrm{d}x,\mathrm{d}k) \mathfrak{a}^{\mathrm{micro}}(x,k) \varphi(|k|/L)^2. \end{aligned} \tag{4.273}$$

4.8.3. Small and intermediate wavenumbers

Next, let $f^{\varepsilon,\lambda}_{0,\sigma}$ be either $\psi^{\varepsilon,\lambda}_{<,0,\sigma}(k)$ or $\zeta^{\varepsilon,\lambda}_{0,\sigma}(k)$. Therefore, its Fourier transform $\widehat{f}^{\varepsilon,\lambda}_{0,\sigma}$ is supported in a ball of radius 2λ around the origin. Recall that

$$f^{\varepsilon,\lambda}_T = \mathrm{e}^{-iH^\varepsilon T/\varepsilon} f^{\varepsilon,\lambda}_0 = \lim_{R\to\infty} (\Psi^\varepsilon_1 + \Psi^\varepsilon_2) \tag{4.274}$$

in $\mathcal{H}$, for fixed $\varepsilon, \lambda > 0$ and with $\Psi^\varepsilon_1, \Psi^\varepsilon_2$ defined in (4.13) (with $\overline{m} = 1$ and initial state $f^{\varepsilon,\lambda}_0$). One has

$$\mathbb{E}\left[\left\| f^{\varepsilon,\lambda}_T - \mathrm{e}^{-iH_0 T/\varepsilon} f^{\varepsilon,\lambda}_0 \right\|^2_{\mathcal{H}} \right] \le 2\mathbb{E}\left[\left\| \Psi^\varepsilon_1 - \mathrm{e}^{-iH_0 T/\varepsilon} f^{\varepsilon,\lambda}_0 \right\|^2_{\mathcal{H}} \right] + 2\mathbb{E}\left[\|\Psi^\varepsilon_2\|^2_{\mathcal{H}} \right]. \tag{4.275}$$

With the same parameters as in (4.216), the analogue of estimate (4.218) still holds for the second summand. For the first, however, we have to proceed slightly different than in Section 4.6, as we can no longer use Lemma 4.16 because $\widehat{f}^{\varepsilon,\lambda}_{0,\sigma}$ does not vanish near the origin. Instead, with $\overline{m} = 1$, $p = p^{(1)} = 0$, the observable $a = a^{(1)} : \mathbb{R}^d \to \mathbb{C}$ such that $a \equiv 1$, $N = \left(N^{(1)}, N^{(2)}\right)$ and the $\mathcal{K}$ amplitudes still defined as in in (4.29),

$$\begin{aligned} &\lim_{R\to 0} \mathbb{E}\left[\left\| \Psi^\varepsilon_1 - \mathrm{e}^{-iH_0 T/\varepsilon} f^{\varepsilon,\lambda}_0 \right\|^2_{\mathcal{H}} \right] \\ &= \sum_{N^{(1)}=1}^{\overline{N}-1} \sum_{N^{(2)}=1}^{\overline{N}-1} \sum_{S\in\pi^*(I(N))} \mathcal{K}\left(f^{\varepsilon,\lambda}_0, \varepsilon, a\equiv 1, p=0, 2\lambda, T/\varepsilon, N, S \right) \\ &\le \left\| f^{\varepsilon,\lambda}_0 \right\|^2_{\mathcal{H}} \varepsilon^{1/19} \\ &\qquad + \sum_{N^{(1)}=1}^{\overline{N}-1} \sum_{N^{(2)}=1}^{\overline{N}-1} \sum_{\substack{S\in\pi^*(I(N)) \\ s \text{ non-crossing}}} \sum_{\sigma=\pm} \left| \mathcal{K}_\sigma\left(f^{\varepsilon,\lambda}_0, \varepsilon, a\equiv 1, p=0, 2\lambda, T/\varepsilon, N, S \right) \right| \end{aligned} \tag{4.276}$$

for all $\lambda \in (0,1)$ and $\varepsilon \in (0,\varepsilon_0)$, with $\varepsilon_0 > 0$ only depending on $|T|$, the distribution of ξ, and dimension d. The last line can then be estimated with Corollary 4.19 as

$$\begin{aligned} &\sum_{N^{(1)}=1}^{\overline{N}-1} \sum_{N^{(2)}=1}^{\overline{N}-1} \sum_{\substack{S\in\pi^*(I(N)) \\ s \text{ non-crossing}}} \sum_{\sigma=\pm} \left| \mathcal{K}_\sigma\left(f^{\varepsilon,\lambda}_0, \varepsilon, a\equiv 1, p=0, 2\lambda, T/\varepsilon, N, S \right) \right| \\ &\le \left\| f^{\varepsilon,\lambda}_0 \right\|^2_{\mathcal{H}} \left(\varepsilon^{1/19} + \lambda \sum_{N^{(1)},N^{(2)}=1}^{\infty} \frac{C^{N^{(1)}+N^{(2)}}}{((N^{(1)}+N^{(2)})/2)!} \right). \end{aligned} \tag{4.277}$$

Therefore, controlling $\left\|f_0^{\varepsilon,\lambda}\right\|_{\mathcal{H}}^2$ with (2.160),

$$\lim_{S''\ni\lambda\to 0}\limsup_{S'\ni\varepsilon\to 0}\mathbb{E}\left[\left\|f_T^{\varepsilon,\lambda}-\mathrm{e}^{-iH_0T/\varepsilon}f_0^{\varepsilon,\lambda}\right\|_{\mathcal{H}}^2\right]=0. \tag{4.278}$$

In the Wigner transform, the cross-terms between $f_T^{\varepsilon,\lambda}$ and $\psi_{>,T}^{\varepsilon,\lambda,L}$ can now be estimated by combining (4.271) and (4.278),

$$\begin{aligned}
&\lim_{S''\ni\lambda\to 0}\limsup_{S'\ni\varepsilon\to 0}\mathbb{E}\left|\int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}k\widehat{\mathfrak{a}}(p,k,k/\varepsilon)\overline{\widehat{f}_{T,\sigma}^{\varepsilon,\lambda}(k+\varepsilon p/2)}\widehat{\psi}_{>,T,\sigma}^{\varepsilon,\lambda,L}(k-\varepsilon p/2)\right|^2\\
\le&\|\mathfrak{a}\|_{\mathfrak{X}_{\mathrm{IR}}}\left\|f_0^{\varepsilon,\lambda}\right\|_{L^2}^2\lim_{S''\ni\lambda\to 0}\limsup_{S'\ni\varepsilon\to 0}\int_{\mathbb{R}^d}\mathrm{d}p\sup_{k',k''}|\widehat{\mathfrak{a}}(p,k',k'')|\\
&\qquad\mathbb{E}\int_{\mathbb{R}^d}\mathrm{d}k\mathbb{1}(\lambda\le|k|\le 2\lambda+\varepsilon|p|)\left|\widehat{\psi}_{>,T,\sigma}^{\varepsilon,\lambda,L}(k)\right|^2\\
\le&4\|\mathfrak{a}\|_{\mathfrak{X}_{\mathrm{IR}}}^2\|\psi_0^\varepsilon\|_{\mathcal{H}}^2\lim_{S''\ni\lambda\to 0}\limsup_{S'\ni\varepsilon\to 0}\mathbb{E}\int_{\mathbb{R}^d}\mathrm{d}k\mathbb{1}(\lambda\le|k|\le 3\lambda)\left|\widehat{\psi}_{>,T,\sigma}^{\varepsilon,\lambda,L}(k)\right|^2\\
\le&4\|\mathfrak{a}\|_{\mathfrak{X}_{\mathrm{IR}}}^2\|\psi_0^\varepsilon\|_{\mathcal{H}}^2\lim_{S''\ni\lambda\to 0}\int_{\mathbb{R}^{2d}}\mu_{T,\sigma}^{\lambda,L}(\mathrm{d}x,\mathrm{d}k)\mathbb{1}(\lambda\le|k|\le 3\lambda)\\
=&0,
\end{aligned} \tag{4.279}$$

where we used the already established convergence (4.257) from the third last to second last line.

For the Wigner transform of the small wave-numbers, observe that for $\sigma\in\{\pm\}$

$$\begin{aligned}
&\lim_{S''\ni\lambda\to 0}\lim_{S'\ni\varepsilon\to 0}\mathbb{E}\left\langle W^\varepsilon\left[\zeta_{T,\sigma}^{\varepsilon,\lambda}\right],\mathfrak{a}\right\rangle_{\mathfrak{X}_{\mathrm{IR}}}\\
=&\lim_{S''\ni\lambda\to 0}\lim_{S'\ni\varepsilon\to 0}\int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}k\widehat{\mathfrak{a}}(p,k,k/\varepsilon)\overline{\widehat{\zeta}_{0,\sigma}^{\varepsilon,\lambda}(k+\varepsilon p/2)}\widehat{\zeta}_{0,\sigma}^{\varepsilon,\lambda}(k-\varepsilon p/2)\\
&\qquad\exp\left(2\pi i\sigma T(|k+\varepsilon p/2|-|k-\varepsilon p/2|)/\varepsilon\right)\\
=&\lim_{S''\ni\lambda\to 0}\lim_{S'\ni\varepsilon\to 0}\int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}\underline{k}\widehat{\mathfrak{a}}(p,\varepsilon\underline{k},\underline{k})\overline{\widehat{\zeta}_{0,\sigma}(\underline{k}+p/2)}\widehat{\zeta}_{0,\sigma}(\underline{k}-p/2)\varphi\left(\frac{|\varepsilon\underline{k}+\varepsilon p/2|}{\lambda}\right)\\
&\qquad\varphi\left(\frac{|\varepsilon\underline{k}-\varepsilon p/2|}{\lambda}\right)\exp\left(2\pi i\sigma T(|\underline{k}+p/2|-|\underline{k}-p/2|)\right)\\
=&\int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}\underline{k}\widehat{\mathfrak{a}}(p,0,\underline{k})\overline{\widehat{\zeta}_{0,\sigma}(\underline{k}+p/2)}\widehat{\zeta}_{0,\sigma}(\underline{k}-p/2)\exp\left(2\pi i\sigma T(|\underline{k}+p/2|-|\underline{k}-p/2|)\right)\\
=&\left\langle W\left[\left(\mathrm{e}^{-iH_0T}\eta_0\right)_\sigma\right],\mathfrak{a}^{\mathrm{macro}}\right\rangle_{\mathcal{F}L^1(C^0)}.
\end{aligned} \tag{4.280}$$

As for the cross-terms between small and intermediate wave-numbers,

$$\begin{aligned}
&\lim_{S''\ni\lambda\to 0}\lim_{S'\ni\varepsilon\to 0}\mathbb{E}\int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}k\widehat{\mathfrak{a}}(p,k,k/\varepsilon)\overline{\widehat{\zeta}^{\varepsilon,\lambda}_{T,\sigma}(k+\varepsilon p/2)}\widehat{\psi}^{\varepsilon,\lambda}_{<,T,\sigma}(k-\varepsilon p/2)\\
&=\lim_{S''\ni\lambda\to 0}\lim_{S'\ni\varepsilon\to 0}\varepsilon^{d/2}\int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}\underline{k}\widehat{\mathfrak{a}}(p,\varepsilon\underline{k},\underline{k})\overline{\widehat{\zeta}_{0,\sigma}(\underline{k}+p/2)}\widehat{\psi}^{\varepsilon,\lambda}_{<,0,\sigma}(\varepsilon(\underline{k}-p/2))\\
&\qquad\varphi\left(\frac{|\varepsilon\underline{k}+\varepsilon p/2|}{\lambda}\right)\exp\left(2\pi i\sigma T(|\underline{k}+p/2|-|\underline{k}-p/2|)\right)\\
&=0,
\end{aligned}\tag{4.281}$$

because for any fixed $p\in\mathbb{R}^d$, $\lambda>0$ and $T\geq 0$,

$$\begin{aligned}
&\exp\left(2\pi i\sigma T|\underline{k}+p/2|\right)\widehat{\mathfrak{a}}(p,\varepsilon\underline{k},\underline{k})\overline{\widehat{\zeta}_{0,\sigma}(\underline{k}+p/2)}\varphi\left(\frac{|\varepsilon\underline{k}+\varepsilon p/2|}{\lambda}\right)\\
&\quad\to\exp\left(2\pi i\sigma T|\underline{k}+p/2|\right)\widehat{\mathfrak{a}}(p,0,\underline{k})\overline{\widehat{\zeta}_{0,\sigma}(\underline{k}+p/2)}\qquad(\varepsilon\to 0)
\end{aligned}\tag{4.282}$$

strongly in $L^2(\mathbb{R}^d_{\underline{k}})$, while

$$\varepsilon^{d/2}\exp\left(2\pi i\sigma T|\underline{k}-p/2|\right)\widehat{\psi}^{\varepsilon,\lambda}_{<,0,\sigma}(\varepsilon(\underline{k}-p/2))\rightharpoonup 0\qquad(\varepsilon\to 0)\tag{4.283}$$

weakly in $L^2(\mathbb{R}^d_{\underline{k}})$.

For intermediate wave-numbers, we consider

$$\begin{aligned}
&\mathbb{E}\left\langle W^\varepsilon\left[\widehat{\psi}^{\varepsilon,\lambda}_{<,T,\sigma}\right],\mathfrak{a}\right\rangle_{\mathfrak{X}_{\mathrm{IR}}}\\
&\quad=\int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}k\widehat{\mathfrak{a}}(p,k,k/\varepsilon)\overline{\widehat{\psi}^{\varepsilon,\lambda}_{<,0,\sigma}(k+\varepsilon p/2)}\widehat{\psi}^{\varepsilon,\lambda}_{<,0,\sigma}(k-\varepsilon p/2)\\
&\qquad\exp\left(2\pi i\sigma T(|k/\varepsilon+p/2|-|k/\varepsilon-p/2|)\right)
\end{aligned}\tag{4.284}$$

and can follow the arguments of [23], laid out in Appendix A up to equation (A.22) to see that for fixed $\lambda>0$, and with the cut-off function φ as defined at the beginning of Section 4.8.1,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\Bigg|\int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}k\widehat{\mathfrak{a}}(p,k,k/\varepsilon)\overline{\widehat{\psi}^{\varepsilon,\lambda}_{<,0,\sigma}(k+\varepsilon p/2)}\widehat{\psi}^{\varepsilon,\lambda}_{<,0,\sigma}(k-\varepsilon p/2)\\
&\qquad\exp\left(2\pi i\sigma T(|k/\varepsilon+p/2|-|k/\varepsilon-p/2|)\right)\\
&\quad-\int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}k\widehat{\mathfrak{a}}(p,k,k/\varepsilon)\overline{\widehat{\psi}^{\varepsilon,\lambda}_{<,0,\sigma}(k+\varepsilon p/2)}\widehat{\psi}^{\varepsilon,\lambda}_{<,0,\sigma}(k-\varepsilon p/2)\\
&\qquad\exp\left(2\pi i\sigma T\frac{k}{|k|}\cdot p\right)(1-\varphi(|k|/\varepsilon))\Bigg|\\
&=0
\end{aligned}\tag{4.285}$$

For any $T\geq 0$, the function $\mathfrak{a}_T$ with

$$\widehat{\mathfrak{a}}_T(p,k,\underline{k})=\widehat{\mathfrak{a}}(p,k,\underline{k})(1-\varphi(|\underline{k}|))\exp\left(2\pi i\sigma T\frac{\underline{k}}{|\underline{k}|}\cdot p\right)\tag{4.286}$$

is a $\mathfrak{X}_{\mathrm{IR}}$ function with

$$\mathfrak{a}_T^{\mathrm{meso}} = \mathrm{e}^{\mathcal{L}_\sigma^{\mathrm{H}} T} \mathfrak{a}^{\mathrm{meso}}. \tag{4.287}$$

Therefore, by (4.269),

$$\begin{aligned}\lim_{S''\ni\lambda\to 0}\lim_{S'\ni\varepsilon\to 0}\left\langle W^\varepsilon\left[\psi^{\varepsilon,\lambda}_{<,T,\sigma}\right],\mathfrak{a}\right\rangle_{\mathfrak{X}_{\mathrm{IR}}} &= \lim_{S''\ni\lambda\to 0}\lim_{S'\ni\varepsilon\to 0}\left\langle W^\varepsilon\left[\psi^{\varepsilon,\lambda}_{<,0,\sigma}\right],\mathfrak{a}_T\right\rangle_{\mathfrak{X}_{\mathrm{IR}}} \\ &= \int_{\mathbb{R}^d\times S^{d-1}} \mu^{\mathrm{H}}_{0,\sigma}(\mathrm{d}x,\mathrm{d}\underline{k})\left(\mathrm{e}^{\mathcal{L}_\sigma^{\mathrm{H}} T}\mathfrak{a}^{\mathrm{meso}}\right)(x,\underline{k}),\end{aligned} \tag{4.288}$$

and we have verified the convergence

$$\begin{aligned}\lim_{S'\ni\varepsilon\to 0}\mathbb{E}\left[\left\langle W^\varepsilon\left[\psi^{\varepsilon}_{<,T,\sigma}\right],\mathfrak{a}\right\rangle_{\mathfrak{X}_{\mathrm{IR}}}\right] &= \int_{\mathbb{R}^d\times\mathbb{R}^d_*}\mu_{0,\sigma}(\mathrm{d}x,\mathrm{d}k)\mathrm{e}^{\mathcal{L}_\sigma T}\mathfrak{a}^{\mathrm{micro}}(x,k)\varphi(|k|/L)^2 \\ &+ \int_{\mathbb{R}^d\times S^{d-1}}\mu^{\mathrm{H}}_{0,\sigma}(\mathrm{d}x,\mathrm{d}\underline{k})\mathrm{e}^{\mathcal{L}^{\mathrm{H}}_\sigma T}\mathfrak{a}^{\mathrm{meso}}(x,\underline{k}) \\ &+ \left\langle W\left[\left(\mathrm{e}^{-iH_0T}\eta_0\right)_\sigma\right],\mathfrak{a}^{\mathrm{macro}}\right\rangle_{\mathcal{F}L^1(C^0)}\end{aligned} \tag{4.289}$$

along the subsequence S'. But such a subsequence S' can be extracted from *any* sequence of $\varepsilon_n \to 0$, and the respective limit always has to coincide with the one on the right side of (4.289). Thus, we have verified (3.2) as ε goes to zero *continuously*, at least for the case $\overline{m} = 1$ and with large-wave-number cut-off L still present. The generalization to multiple measurements ($\overline{m} > 1$) is tedious, but straightforward.

4.8.4. UV cut-off

To finish the proof of Theorem 3.1, one now only has to remove the cut-off L. From (2.162), we observe that

$$\lim_{L\to\infty}\limsup_{\varepsilon\to 0}\int_{\mathbb{R}^d}\mathrm{d}k\left|\widehat{\psi}^\varepsilon_0(k)\right|^2(1-\varphi(|k|/L))^2 = 0, \tag{4.290}$$

while

$$\lim_{L\to\infty}\int_{\mathbb{R}^d\times\mathbb{R}^d_*}\mu_{0,\sigma}(\mathrm{d}x,\mathrm{d}k)\varphi(|k|/L)^2\mathfrak{a}^{\mathrm{micro}}(x,k) = \int_{\mathbb{R}^d\times\mathbb{R}^d_*}\mu_{0,\sigma}(\mathrm{d}x,\mathrm{d}k)\mathfrak{a}^{\mathrm{micro}}(x,k) \tag{4.291}$$

for all $\mathfrak{a}^{\mathrm{micro}} \in \mathcal{F}L^1(C^0)$. Thus, both sides of (3.2) converge to their non-cut-off limits as $L \to 0$.

5. Vanishing variance

5.1. Graph expansion

5.1.1. Amplitudes for the main part

As in the beginning of Chapter 4, we start out with initial states $\psi_0^\varepsilon \in \mathcal{H}$ with bounded energy $\sup_{\varepsilon>0} \|\psi_0^\varepsilon\|_{\mathcal{H}}^2 < \infty$, and Fourier transforms supported in a ball of radius $L^{(0)}$ around the origin uniform in $\varepsilon > 0$. The operators A_j^ε, $j \in \{1, ..., 2\overline{m} - 1\}$ are given by (4.1-4.2), again with functions $a_{j,\sigma} : \mathbb{R}^d \to \mathbb{C}$ bounded up to their second derivative. The variance of the random variabe $\mathcal{J}_R^\varepsilon$ as defined in (4.6) naturally contains contributions both of the main part and the remainder of the Duhamel expansion (4.11). We will see in Section 5.4 that one can re-use all estimates for the remainder. In the Section 5.1 at hand, we therefore focus on controlling the variance of (4.18) for any given $N \in \mathbb{N}_0^{2\overline{m}}$ with

$$\begin{aligned} N^{(1)} + ... + N^{(\overline{m})} &< \overline{N}, \\ N^{(\overline{m}+1)} + ... + N^{(2\overline{m})} &< \overline{N}. \end{aligned} \tag{5.1}$$

To find such an estimate, we follow the notation of [8] and [7], but have to pay attention to the more complicated structure due to multiple measurements, the "+" and "−" components of the wave function ψ^ε, and, as the random field ξ is not Gaussian, the presence of higher-order partitions. To motivate the notation, first assume we wanted to calculate

$$\lim_{R\to\infty} \mathbb{E} \left| \left\langle \psi_0^\varepsilon, F_{N^{(2\overline{m})}} \left(t^{(2\overline{m})}; R, L^{(2\overline{m})}, \varepsilon\right)^* \left(\prod_{j=1}^{2\overline{m}-1} A_j^\varepsilon F_{N^{(j)}} \left(t^{(j)}; R, L^{(j)}, \varepsilon\right)^{*(j)} \right) \psi_0^\varepsilon \right\rangle_{\mathcal{H}} \right|^2 . \tag{5.2}$$

To write out the square inside the expectation, we can fall back on the notation introduced in Section 4.1.2. As the same time intervals $t^{(j)}$ and cut-off thresholds $L_n^{(j)}$ appear in both factors of the square, the definition (4.17) can still be employed, but for most other variables we have to introduce a new index $r \in \{1, 2\}$, with $r = 1$ standing for the contributions of the original (4.18), and $r = 2$ for those of its complex conjugate. As one now has to keep track of $2|N|$ scattering events, we index them with a set $I(N, N)$, which is just a union of two disjoint copies of the original $I(N)$ from (4.19),

$$I(N, N) = \left\{(j, n, r) : j \in \{1, ..., 2\overline{m}\}, n \in \{1, ..., N^{(j)}\}, r \in \{1, 2\}\right\}. \tag{5.3}$$

As before, we refer to the set of partitions of $I(N,N)$ as $\pi\left(I(N,N)\right)$, and $\pi^*\left(I(N,N)\right)$ is the set of partitions without isolated elements, i.e. clusters of size 1. Finally, the set $\pi^*_{\text{conn}}\left(I(N,N)\right)$ is the set of all partitions that do not contain isolated elements and *connect* the first and second one-particle line. Thus an $S \in \pi^*_{\text{conn}}\left(I(N,N)\right)$ always contains a cluster $A \in S$ such that there are $(j_1,n_1),(j_2,n_2) \in I(N)$ with $(j_1,n_1,1),(j_2,n_2,2) \in A$. From the interpretation of $I(N,N)$ as two copies of $I(N)$, it is clear that the set of partitions that connect the two one-particle lines is just the set of all partitions, except for those that decompose into partitions of the first and second $I(N)$,

$$\pi^*_{\text{conn}}\left(I(N,N)\right) = \pi^*\left(I(N,N)\right) \setminus \left(\pi^*(I(N)) \times \pi^*(I(N))\right). \tag{5.4}$$

Between consecutive scatterings, as shown in Figure 5.1, the wave in the r-th one-particle line travels at a momentum $k^{(j)}_{n,r}$, indexed by $(j,n,r) \in I_0(N,N)$, with

$$I_0(N,N) = \left\{(j,n,r) : j \in \{1,...,2\overline{m}\}, n \in \{0,...,N^{(j)}\}, r \in \{1,2\}\right\}, \tag{5.5}$$

and has a "+" and "−" component denoted by $\sigma^{(j)}_{n,r}$, again with $(j,n,r) \in I_0(N,N)$. Finally, the complex conjugation for $r=2$ can be represented by a sign $\tau_r = -(-1)^r$ which combines with (4.22) to $\tau^{(j)}_r = \tau^{(j)}\tau_r$. For each $(j,n,r) \in I(N,N)$, the momentum change at the corresponding scattering event is

$$\theta^{(j)}_{n,r} = \tau_r\left(k^{(j)}_{n,r} - k^{(j)}_{n-1,r}\right). \tag{5.6}$$

We will later comment on the different choice of signs for $r=1$ and $r=2$.

If, for $r \in \{1,2\}$, k_r denotes the vector of momenta $k^{(j)}_{n,r}$, $(j,n) \in I_0(N)$, and σ_r is the collection of all signs $\sigma^{(j)}_{n,r}$, the product of all observables reads

$$\begin{aligned}&\mathcal{A}_1\left(k_1,\sigma_1,p,\varepsilon\right)\\ &= \prod_{j=1}^{2\overline{m}-1}\left(a_{j,\sigma^{(j+1)}_{0,1}}\left(\frac{k^{(j+1)}_{0,1}+k^{(j)}_{N^{(j)},1}}{2}\right)\delta\left(k^{(j+1)}_{0,1}-k^{(j)}_{N^{(j)},1}-\varepsilon p^{(j)}\right)\delta\left(\sigma^{(j)}_{N^{(j)},1},\sigma^{(j+1)}_{0,1}\right)\right)\end{aligned} \tag{5.7}$$

on the first one-particle line, and

$$\begin{aligned}&\mathcal{A}_2\left(k_2,\sigma_2,p,\varepsilon\right)\\ &= \prod_{j=1}^{2\overline{m}-1}\left(\overline{a_{j,\sigma^{(j+1)}_{0,2}}}\left(\frac{k^{(j+1)}_{0,2}+k^{(j)}_{N^{(j)},2}}{2}\right)\delta\left(k^{(j+1)}_{0,2}-k^{(j)}_{N^{(j)},2}-\varepsilon p^{(j)}\right)\delta\left(\sigma^{(j)}_{N^{(j)},2},\sigma^{(j+1)}_{0,2}\right)\right)\end{aligned} \tag{5.8}$$

on the second one-particle line.

To represent the propagation of the wave in resolvent formulation, for each index $j \in \{1,...,2\overline{m}\}$, $r \in \{1,2\}$, there will be a resolvent integral with parameter $\alpha^{(j)}_r$, and

α_r denotes the collection of all such parameters for the r-th one-particle line. The propagators in resolvent form are then given as

$$\begin{aligned}
&\mathcal{P}_r\left(k_r, \sigma_r, \alpha_r, \gamma\right) \\
&\quad = \prod_{(j,n)\in I_0(N)} \left(\frac{i}{\alpha_r^{(j)} - 2\pi\sigma_{n,r}^{(j)}\tau_r^{(j)}|k_{n,r}^{(j)}| + i\gamma} \right) \\
&\qquad \times \prod_{(j,n)\in I(N)} \left[(-i\tau_r^{(j)}) \left(\left|k_{n,r}^{(j)}\right| \sigma_{n-1,r}^{(j)} + \left|k_{n-1,r}^{(j)}\right| \sigma_{n,r}^{(j)} \right) \Phi\left(k_{n,r}^{(j)}, k_{n-1,r}^{(j)}, L_n^{(j)}\right) \right]
\end{aligned} \tag{5.9}$$

on the r-th one-particle line, where the parameter $\gamma > 0$ can be chosen freely, and will again be optimized later.

Lemma 5.1. *For a random field ξ of class $(m, 0)$, $m > d+1$, and $N \in \mathbb{N}_0^{2\overline{m}}$ fulfilling (5.1),*

$$\begin{aligned}
&\lim_{R\to\infty} \operatorname{Var} \left\langle \psi_0^\varepsilon, F_{N^{(2\overline{m})}}\left(t^{(2\overline{m})}; R, L^{(2\overline{m})}, \varepsilon\right)^* \left(\prod_{j=1}^{2\overline{m}-1} A_j^\varepsilon F_{N^{(j)}}\left(t^{(j)}; R, L^{(j)}, \varepsilon\right)^{*(j)} \right) \psi_0^\varepsilon \right\rangle_{\mathcal{H}} \\
&\quad = \sum_{S\in\pi^*_{\mathrm{conn}}(I(N,N))} \mathcal{V}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right),
\end{aligned} \tag{5.10}$$

with the amplitude $\mathcal{V}$ of a single partition $S \in \pi^*(I(N,N))$ given by the formula

$$\begin{aligned}
&\mathcal{V}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right) \\
&\quad = \varepsilon^{|N|}\pi^{2|N|} \sum_{\substack{\sigma_{n,r}^{(j)}\in\{\pm\} \\ \forall (j,n,r)\in I_0(N,N)}} \int_{\mathbb{R}^{(2|N|+4\overline{m})d}} \prod_{(j,n,r)\in I_0(N,N)} \mathrm{d}k_{n,r}^{(j)} \\
&\qquad\qquad \prod_{A\in S} \left(\delta\left(\sum_{(j,n,r)\in A} \theta_{n,r}^{(j)} \right) \widehat{g}_{|A|}\left(\theta_{n,r}^{(j)} : (j,n,r) \in A^\sharp\right) \right) \\
&\qquad\qquad \times \mathcal{A}_1\left(k_1, \sigma_1, p, \varepsilon\right) \mathcal{A}_2\left(k_2, \sigma_2, p, \varepsilon\right) \\
&\qquad\qquad \times \int_{\mathbb{R}^{4\overline{m}}} \prod_{r=1}^{2} \prod_{j=1}^{2\overline{m}} \left(e^{\gamma t^{(j)}} \frac{\mathrm{d}\alpha_r^{(j)}}{2\pi} e^{-i\alpha_r^{(j)} t^{(j)}} \right) \\
&\qquad\qquad\qquad \mathcal{P}_1\left(k_1, \sigma_1, \alpha_1, \gamma\right) \mathcal{P}_2\left(k_2, \sigma_2, \alpha_2, \gamma\right) \\
&\qquad\qquad \times \widehat{\psi}^\varepsilon_{0,\sigma_{0,1}^{(1)}}\left(k_{0,1}^{(1)}\right) \overline{\widehat{\psi}^\varepsilon_{0,\sigma^{(2\overline{m})}_{N^{(2\overline{m})},1}}} \left(k^{(2\overline{m})}_{N^{(2\overline{m})},1}\right) \\
&\qquad\qquad \times \overline{\widehat{\psi}^\varepsilon_{0,\sigma_{0,2}^{(1)}}}\left(k_{0,2}^{(1)}\right) \widehat{\psi}^\varepsilon_{0,\sigma^{(2\overline{m})}_{N^{(2\overline{m})},2}} \left(k^{(2\overline{m})}_{N^{(2\overline{m})},2}\right),
\end{aligned} \tag{5.11}$$

in which $\gamma > 0$ can be chosen arbitrarily. As in the remark after Lemma 4.2, the resolvent representation of the propagator for the j-th time interval, both for $r = 1$ and $r = 2$, is *only valid in case* $N^{(j)} \geq 1$*. For* j *with* $N^{(j)} = 0$*, the respective* $\alpha_r^{(j)}$ *integral should be interpreted as the unitary*

$$\exp\left(-2\pi i \sigma_{n,r}^{(j)} \tau_r^{(j)} |k_{n,r}^{(j)}| t^{(j)}\right), \qquad (r = 1, 2). \tag{5.12}$$

On the other hand, whenever $N^{(j)} \geq 1$*, the* $\alpha_1^{(j)}$ *and* $\alpha_2^{(j)}$ *integral can be interchanged with the* k *integrals.*

Proof. One can follow the proof of Lemma 4.1 and 4.3 (with the same caveats for the indices j with $N^{(j)} = 0$) to verify the equation

$$\begin{aligned}\lim_{R\to\infty} \mathbb{E} &\left| \left\langle \psi_0^\varepsilon, F_{N^{(2\overline{m})}}\left(t^{(2\overline{m})}; R, L^{(2\overline{m})}, \varepsilon\right)^* \left(\prod_{j=1}^{2\overline{m}-1} A_j^\varepsilon F_{N^{(j)}}\left(t^{(j)}; R, L^{(j)}, \varepsilon\right)^{*(j)}\right) \psi_0^\varepsilon \right\rangle_{\mathcal{H}} \right|^2 \\ &= \sum_{S \in \pi^*(I(N,N))} \mathcal{V}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right)\end{aligned} \tag{5.13}$$

together with all the remarks below (5.11). The only difference to Lemma 4.1 is the complex conjugation on the second one-particle line. For $r = 2$, the scattering events therefore involve a convolution with $\overline{\widehat{\xi_R}}$, so we have to evaluate terms of the form

$$\overline{\widehat{\xi_R}}\left(k_{n,2}^{(j)} - k_{n-1,2}^{(j)}\right) = \widehat{\xi_R}\left(-k_{n,2}^{(j)} + k_{n-1,2}^{(j)}\right) = \widehat{\xi_R}\left(\theta_{n,2}^{(j)}\right), \tag{5.14}$$

for $(j, n) \in I(N)$, where we have used that the random field ξ takes only real values. This explaines the choice of signs in (5.6).
Directly by taking the square of Lemma 4.1 and 4.3 one has that

$$\begin{aligned}\lim_{R\to\infty} &\left| \mathbb{E} \left\langle \psi_0^\varepsilon, F_{N^{(2\overline{m})}}\left(t^{(2\overline{m})}; R, L^{(2\overline{m})}, \varepsilon\right)^* \left(\prod_{j=1}^{2\overline{m}-1} A_j^\varepsilon F_{N^{(j)}}\left(t^{(j)}; R, L^{(j)}, \varepsilon\right)^{*(j)}\right) \psi_0^\varepsilon \right\rangle_{\mathcal{H}} \right|^2 \\ &= \sum_{S \in \pi^*(I(N)) \times \pi^*(I(N))} \mathcal{V}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right)\end{aligned} \tag{5.15}$$

with the sum running over all partitions $S \in \pi^*(I(N, N))$ that decompose into a partition of the first and of the second particle line. The lemma then follows from (5.4). □

5.1.2. Graph classification

For every $S \in \pi^*_{\text{conn}}(I(N, N))$, the structure $I(N, N) = I(N) \dot{\cup} I(N)$ of the index set gives rise to the notion of *internal clusters*, which are $A \in S$ such that either all elements $(j, n, r) \in A$ have $r = 1$ (an internal cluster on the first one-particle line) or all elements

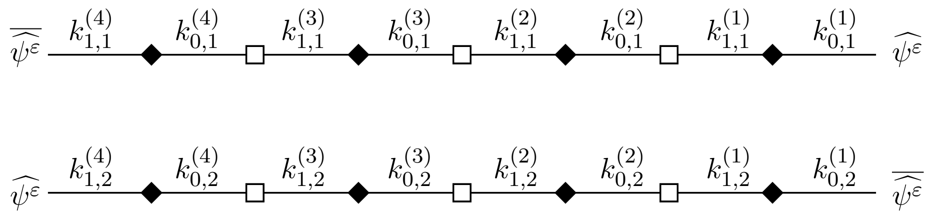

Figure 5.1.: Scatterings (black diamonds) and measurements (empty squares) for the case $\overline{m} = 2$, $N = (1, 1, 1, 1)$. The solid lines indicate the propagation of the wave at a current momentum of $k_{n,r}^{(j)}$. Two one-particle lines are needed because of the quadratic structure of the variance. $I(N, N)$ can be identified with the set of all black diamonds in the graph.

have $r = 2$ (an internal cluster on the second one-particle line). Clusters $A \in S$ that contain both elements $(j_1, n_1, 1) \in A$ and $(j_2, n_2, 2) \in A$ are *transfer clusters.* If $|A| = 2$, we speak of internal pairs or transfer pairs, respectively. Accordingly, the partition S decomposes into $S = S_1 \dot\cup S_2 \dot\cup S_{\mathrm{tr}}$, with S_1, S_2 being the sets of internal clusters on the first and second one-particle line, respectively, and S_{tr} being the set of all transfer clusters. If every $A \in S$ is indicated in Figure 5.1 by connecting the respective scattering events (black diamonds) by dotted lines, this visualization of partitions $S \in \pi^*_{\mathrm{conn}}(I(N, N))$ gives rise to the following

Definition 5.1. We classify the partitions in $\pi^*_{\mathrm{conn}}(I(N, N))$ similarly to Definition 1 of [7].

- $S \in \pi^*_{\mathrm{conn}}(I(N, N))$ is called **higher order**, if there exists an $A \in S$ with $|A| > 2$, and otherwise, that is, if all clusters $A \in S$ are pairs, a **pairing**.
- A pairing $S \in \pi^*_{\mathrm{conn}}(I(N, N))$ has a **generalized crossing** on the r-th one-particle line if there is an internal pair $\{(j_1, n_1, r), (j_2, n_2, r)\} \in S$ on the r-th particle line, and a second pair $\{(\tilde{j}_1, \tilde{n}_1, r), (\tilde{j}_2, \tilde{n}_2, r')\} \in S$ such that, employing the ordering $\prec$ of $I(N)$, $(j_1, n_1) \prec (\tilde{j}_1, \tilde{n}_1) \prec (j_2, n_2)$ and
  - either $r \neq r'$, as in Figure 5.2,
  - or $r = r'$ and $(j_2, n_2) \prec (\tilde{j}_2, \tilde{n}_2)$.
- A pairing $S \in \pi^*_{\mathrm{conn}}(I(N, N))$ without generalized crossing has **parallel transfer pairs** if for *every* possible choice of two transfer pairs

$$\{(j_1, n_1, 1), (j_2, n_2, 2)\}, \{(\tilde{j}_1, \tilde{n}_1, 1), (\tilde{j}_2, \tilde{n}_2, 2)\} \in S, \tag{5.16}$$

  the ordering $(j_1, n_1) \prec (\tilde{j}_1, \tilde{n}_1)$ implies $(j_2, n_2) \prec (\tilde{j}_2, \tilde{n}_2)$, as is the case in Figure 5.6

- A pairing $S \in \pi^*_{\mathrm{conn}}(I(N,N))$ without generalized crossing has **anti-parallel transfer pairs** if for *every* possible choice of two transfer pairs
$$\{(j_1,n_1,1),(j_2,n_2,2)\},\{(\tilde{j}_1,\tilde{n}_1,1),(\tilde{j}_2,\tilde{n}_2,2)\} \in S, \tag{5.17}$$
the ordering $(j_1,n_1) \prec (\tilde{j}_1,\tilde{n}_1)$ implies $(j_2,n_2) \succ (\tilde{j}_2,\tilde{n}_2)$, as in Figure 5.7. A pairing S without generalized crossing and only one transfer pair is thus classified both as having parallel and anti-parallel transfer pairs.
- If a pairing $S \in \pi^*_{\mathrm{conn}}(I(N,N))$ does neither exhibit a generalized crossing nor (anti-)parallel transfer pairs, we say that it has **crossing transfer pairs**, for example the pairing shown in Figure 5.3.

5.2. Basic estimate

In [7, 8], the next step would be a factorization lemma (Lemma 5.3 or Lemma 4 respectively), that essentially factorizes $\mathcal{V}$ from (5.11) into the contributions of the first and second one-particle line. To do so, one would now assign a transfer momentum u to every transfer cluster $A \in S_{\mathrm{tr}}$, and rewrite the delta function
$$\delta\left(\sum_{(j,n,r)\in A} \theta^{(j)}_{n,r}\right) = \int_{\mathbb{R}^d} \mathrm{d}u \delta\left(u - \sum_{(j,n,1)\in A} \theta^{(j)}_{n,1}\right)\delta\left(u + \sum_{(j,n,2)\in A} \theta^{(j)}_{n,2}\right). \tag{5.18}$$
For estimates in the spirit of Lemma 4.8, the momentum u could then be considered as an additional free momentum on, say, the first one-particle line, but as a dependent momentum on the second one-particle line. In our case, things are a bit more difficult; while the random potential V from [7, 8] was a Gaussian random field on $\mathbb{Z}^3$ with a Fourier transform $\widehat{V}$ such that
$$\mathbb{E}\left[\widehat{V}(\theta_1)\widehat{V}(\theta_2)\right] = \delta(\theta_1+\theta_2) \tag{5.19}$$
on the momentum space $[0,1)^3$, our analogue of (5.19) always comes with a decay factor $\widehat{g}_{|A|}$ which does not factor as nicely as (5.18). We will therefore not directly make use of the notion of transfer momenta; however, (5.18) will influence our definition of free and dependent indices (j,n,r) in the subsequent lemmas, starting with the following basic estimate, similar to Lemma 5.5 in [8].

Lemma 5.2. (Basic estimate, $\mathcal{V}$ amplitudes.) *For ξ of class $(d+2,0)$ and $\gamma \in (0,1/2]$,*
$$\begin{aligned}&\left|\mathcal{V}\left(\psi^\varepsilon_0,\varepsilon,a,p,L^{(0)},t,N,S\right)\right| \\ &\quad\le C^{|N|+2\overline{m}}\left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{4|N|} e^{4\gamma|t|}\,\|\psi^\varepsilon_0\|^4_{\mathcal{H}} \prod_{j=1}^{2\overline{m}-1}\|a_j\|^2_{C^0}\prod_{A\in S}\left\|g_{|A|}\right\|_{d+2} \\ &\qquad\times \varepsilon^{|N|}\gamma^{-|S|}|\log\gamma|^{2|N|+4\overline{m}},\end{aligned} \tag{5.20}$$
with $C<\infty$ only depending on dimension d.

Proof. Assume for the moment that $N^{(j)} \geq 1$ for all $j \in \{1, ..., 2\overline{m}\}$, so the resolvent representation of propagators is applicable. We start from

$$\begin{aligned}
&\left|\mathcal{V}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right)\right| \\
&\quad \leq \varepsilon^{|N|}\pi^{2|N|}e^{4\gamma|t|}\prod_{j=1}^{2\overline{m}-1}\|a_j\|_{C^0}^2 \sum_{\substack{\sigma_{n,r}^{(j)}\in\{\pm\} \\ \forall (j,n,r)\in I_0(N,N)}} \\
&\qquad \int_{\mathbb{R}^{(2|N|+4\overline{m})d}} \prod_{(j,n,r)\in I_0(N,N)} \mathrm{d}k_{n,r}^{(j)} \\
&\qquad\quad \prod_{A\in S}\left(\delta\left(\sum_{(j,n,r)\in A}\theta_{n,r}^{(j)}\right)\left|\widehat{g}_{|A|}\left(\theta_{n,r}^{(j)} : (j,n,r)\in A^\sharp\right)\right|\right) \\
&\qquad\quad \times \prod_{j=1}^{2\overline{m}-1}\left(\delta\left(k_{0,1}^{(j+1)} - k_{N^{(j)},1}^{(j)} - \varepsilon p^{(j)}\right)\delta\left(k_{0,2}^{(j+1)} - k_{N^{(j)},2}^{(j)} - \varepsilon p^{(j)}\right)\right) \\
&\qquad\quad \times \int_{\mathbb{R}^{4\overline{m}}}\prod_{r=1}^{2}\prod_{j=1}^{2\overline{m}}\left(\frac{\mathrm{d}\alpha_r^{(j)}}{2\pi}\right)\left|\mathcal{P}_1\left(k_1,\sigma_1,\alpha_1,\gamma\right)\mathcal{P}_2\left(k_2,\sigma_2,\alpha_2,\gamma\right)\right| \\
&\qquad\quad \times \left(\left|\widehat{\psi}_0^\varepsilon\left(k_{0,1}^{(1)}\right)\right|^2 + \left|\widehat{\psi}_0^\varepsilon\left(k_{N^{(2\overline{m})},1}^{(2\overline{m})}\right)\right|^2\right)\left(\left|\widehat{\psi}_0^\varepsilon\left(k_{0,2}^{(1)}\right)\right|^2 + \left|\widehat{\psi}_0^\varepsilon\left(k_{N^{(2\overline{m})},2}^{(2\overline{m})}\right)\right|^2\right).
\end{aligned} \tag{5.21}$$

Multiplying out the last line of (5.21) produces four different summands, and we will concentrate on the one containing $\left|\widehat{\psi}_0^\varepsilon\left(k_{0,1}^{(1)}\right)\right|^2\left|\widehat{\psi}_0^\varepsilon\left(k_{0,2}^{(1)}\right)\right|^2$. In this case, fix a choice of signs σ and switch the integration variables $k_{n,r}^{(j)}$, $(j,n,r) \in I_0(N,N)$ to $k_{0,1}^{(1)}$, $k_{0,2}^{(1)}$ as well as $\theta_{n,r}^{(j)}$, $(j,n,r) \in I(N,N)$. Then an argument along the lines of the estimates leading up to (4.61) provides us with a $C < \infty$ depending only on dimension d, such that the last five lines of (5.21) are bounded by

$$\begin{aligned}
&C^{|N|}\left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{4|N|}\prod_{A\in S}\left\|g_{|A|}\right\|_{d+2} \\
&\quad \int_{\mathbb{R}^d}\mathrm{d}k_{0,1}^{(1)}\int_{\mathbb{R}^d}\mathrm{d}k_{0,2}^{(1)}\left|\widehat{\psi}_0^\varepsilon\left(k_{0,1}^{(1)}\right)\right|^2\left|\widehat{\psi}_0^\varepsilon\left(k_{0,2}^{(1)}\right)\right|^2 \\
&\qquad \int_{\mathbb{R}^{2|N|d}}\prod_{(j,n,r)\in I(N,N)}\left(\mathrm{d}\theta_{n,r}^{(j)}\right)\prod_{A\in S}\left(\delta\left(\sum_{(j,n,r)\in A}\theta_{n,r}^{(j)}\right)\right) \\
&\qquad \int_{\mathbb{R}^{4\overline{m}}}\mathrm{d}\alpha_1^{(1)}...\mathrm{d}\alpha_1^{(2\overline{m})}\mathrm{d}\alpha_2^{(1)}...\mathrm{d}\alpha_2^{(2\overline{m})} \\
&\qquad\quad \prod_{(j,n,r)\in I_0(N,N)}\left|\frac{1}{\alpha_r^{(j)} - 2\pi\sigma_{n,r}^{(j)}\tau_r^{(j)}|k_{n,r}^{(j)}| + i\gamma}\right| \\
&\qquad\quad \prod_{(j,n,r)\in I(N,N)}\left(\left\langle\theta_{n,r}^{(j)}\right\rangle^{-d}\left\langle k_{n,r}^{(j)}\right\rangle^{-1}\right),
\end{aligned} \tag{5.22}$$

as long as we restrict ourselves to the summand $\left|\widehat{\psi}_0^\varepsilon\left(k_{0,1}^{(1)}\right)\right|^2 \left|\widehat{\psi}_0^\varepsilon\left(k_{0,2}^{(1)}\right)\right|^2$. For notational simplicity, the k variables are still used, but understood as functions of the integration variables by

$$k_{n,r}^{(j)} = k_{0,r}^{(1)} + \varepsilon \sum_{\tilde{\jmath}=1}^{j-1} p^{(\tilde{\jmath})} + \tau_r \sum_{\substack{(\tilde{\jmath},\tilde{n})\in I(N) \\ (\tilde{\jmath},\tilde{n})\preceq(j,n)}} \theta_{\tilde{n},r}^{(\tilde{\jmath})}. \tag{5.23}$$

As in the proof of Lemma 4.8, all indices in $I(N,N)$ will now be classified as free or dependent. For a cluster $A \in S$, let $\max_r A$ be the largest (with respect to $\prec$) $(j,n) \in I(N)$ such that $(j,n,r) \in A$.

Definition 5.2. *(Definition of free and dependent indices, if both one-particle lines are to be integrated out from left to right.)* The index $(j,n,r) \in I(N,N)$ is *dependent*,

i) if $(j,n,r) \in A$, with A an internal cluster of the r-th one-particle line, r arbitrary, and $(j,n) = \max_r A$, or

ii) if $(j,n,r) \in A$, with A a transfer cluster, $r = 2$, and $(j,n) = \max_2 A$,

and *free* otherwise.

This way, every cluster A consists of exactly one dependent and $|A| - 1$ free elements, so one can can replace the third line of (5.22) by

$$\int_{\mathbb{R}^{(2|N|-|S|)d}} \prod_{\substack{(j,n,r)\in I(N,N) \\ (j,n,r) \text{ free}}} \left(\mathrm{d}\theta_{n,r}^{(j)}\right), \tag{5.24}$$

if one plugs into the integrand

$$\theta_{n,r}^{(j)} = - \sum_{\substack{(\tilde{\jmath},\tilde{n},\tilde{r})\in A(j,n,r) \\ (\tilde{\jmath},\tilde{n},\tilde{r}) \text{ free}}} \theta_{\tilde{n},\tilde{r}}^{(\tilde{\jmath})} \tag{5.25}$$

for dependent (j,n,r). Here we have used the notation $A(j,n,r)$ for the unique cluster in S containing (j,n,r). The substitution (5.25) yields the appropriate analogue of (4.59), namely

$$k_{n,1}^{(j)} = k_{0,1}^{(1)} + \varepsilon \sum_{\tilde{\jmath}=1}^{j-1} p^{(\tilde{\jmath})} + \sum_{\substack{A\in S_1 \\ (j,n)\prec\max_1 A}} \sum_{\substack{(\tilde{\jmath},\tilde{n},1)\in A \\ (\tilde{\jmath},\tilde{n})\preceq(j,n)}} \theta_{\tilde{n},1}^{(\tilde{\jmath})} + \sum_{A\in S_{\mathrm{tr}}} \sum_{\substack{(\tilde{\jmath},\tilde{n},1)\in A \\ (\tilde{\jmath},\tilde{n})\preceq(j,n)}} \theta_{\tilde{n},1}^{(\tilde{\jmath})} \tag{5.26}$$

on the first one-particle line and

$$k_{n,2}^{(j)} = k_{0,2}^{(1)} + \varepsilon \sum_{\tilde{\jmath}=1}^{j-1} p^{(\tilde{\jmath})} - \sum_{\substack{A\in S_2\cup S_{\mathrm{tr}} \\ (j,n)\prec\max_2 A}} \sum_{\substack{(\tilde{\jmath},\tilde{n},2)\in A \\ (\tilde{\jmath},\tilde{n})\preceq(j,n)}} \theta_{\tilde{n},2}^{(\tilde{\jmath})} + \sum_{\substack{A\in S_{\mathrm{tr}} \\ (j,n)\succeq\max_2 A}} \sum_{(\tilde{\jmath},\tilde{n},1)\in A} \theta_{\tilde{n},1}^{(\tilde{\jmath})} \tag{5.27}$$

on the second one-particle line, the last term in (5.27) representing the momentum transfer between the two one-particle lines caused by the transfer clusters. To estimate the last four lines of (5.22), one can proceed just as in the proof for Lemma 4.8. First, for $k_{n,r}^{(j)}$ with dependent $(j,n,r) \in I(N,N)$, the L^∞ bound (4.62) is immediately applicable, producing factors $C/\left(\gamma\left\langle\alpha_r^{(j)}\right\rangle\right)$. Then, we integrate out all $\theta_{n,2}^{(j)}$, $(j,n,2) \in I(N,N)$ free and $\alpha_2^{(j)}$, $j \in \{1,...,2\overline{m}\}$ by an iteration of the following steps (for $r=2$)

Iteration 5.1. *(Integrating out the r-th one-particle line, from left to right, i.e. decreasing in $\prec$.)*

- Of all remaining free indices $(j,n,r) \in I(N,N)$, and all remaining $(j,0,r)$, $j \in \{1,...2\overline{m}\}$, pick the one with the largest (j,n) with respect to $\prec$.
- If $n \neq 0$, and thus $(j,n,r) \in I(N,N)$, one can check in (5.26-5.27) that $k_{n,r}^{(j)}$ is the only remaining k variable depending on $\theta_{n,r}^{(j)}$. Integrating over $\theta_{n,r}^{(j)}$ produces a factor
$$\int_{\mathbb{R}^d} \mathrm{d}\theta_{n,r}^{(j)} \frac{1}{\left|\alpha_r^{(j)} - 2\pi\sigma_{n,r}^{(j)}\tau_r^{(j)}|k_{n,r}^{(j)}| + i\gamma\right| \left\langle k_{n,r}^{(j)}\right\rangle \left\langle \theta_{n,r}^{(j)}\right\rangle^d} \leq \frac{C|\log\gamma|}{\left\langle\alpha_r^{(j)}\right\rangle}, \tag{5.28}$$
with a constant $C < \infty$ depending only on d.
- In case $(j,n,r) = (j,0,r)$, we have made sure by previous steps of our integration that the only resolvent depending on $\alpha_r^{(j)}$ is the one belonging to $k_{0,r}^{(j)}$, while there is a factor $\left\langle\alpha_r^{(j)}\right\rangle^{-N^{(j)}}$, $N^{(j)} \geq 1$ stemming from the (4.62) and (5.28) bounds, and we obtain a factor
$$\int_{\mathbb{R}} \mathrm{d}\alpha_r^{(j)} \frac{1}{\left|\alpha_r^{(j)} - 2\pi\sigma_{0,r}^{(j)}\tau_r^{(j)}|k_{0,r}^{(j)}| + i\gamma\right| \left\langle\alpha_r^{(j)}\right\rangle} \leq C|\log\gamma| \tag{5.29}$$
from the $\alpha_r^{(j)}$ integral, C only depending on d.

After this procedure, the remaining integrand does no longer depend on any of the $k_{\cdot,2}^{(\cdot)}$, so we do not need to worry about the $\theta_{\cdot,1}^{(\cdot)}$ dependence in (5.27) anymore. Consequentely, we now can apply the analogous procedure to the first one-particle line, i.e. plug in $r=1$ into the above Iteration 5.1, thus integrating out all $\theta_{n,1}^{(j)}$, $(j,n,1) \in I(N,N)$ free, and $\alpha_1^{(j)}$, $j \in \{1,...,2\overline{m}\}$. Collecting all factors so far, there are $|S|$ contributions from dependent resolvents, each $C\gamma^{-1}$, and $2|N|+4\overline{m}-|S|$ factors $C|\log\gamma|$ from the integrals over the α and independent θ variables. After taking the $k_{0,1}^{(1)}$ and $k_{0,2}^{(1)}$ integrals, one has the bound
$$C^{|N|+2\overline{m}} \left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{4|N|} \prod_{A\in S} \left\|g_{|A|}\right\|_{d+2} \|\psi_0^\varepsilon\|_{\mathcal{H}}^4 \gamma^{-|S|} |\log\gamma|^{2|N|+4\overline{m}} \tag{5.30}$$
for (5.22), where the constant C has been redefined, but still only depends on d. So far, this is only a estimate for the last five lines of (5.21) for the choice of the summand $\left|\widehat{\psi}_0^\varepsilon\left(k_{0,1}^{(1)}\right)\right|^2 \left|\widehat{\psi}_0^\varepsilon\left(k_{0,2}^{(1)}\right)\right|^2$. For the contributions of the other summands, we

would also need to integrate out the one-particle lines from the "open end" to the $\left|\widehat{\psi}_0^\varepsilon\right|^2$, thus defining free and dependent indices according to $\succ$ rather than $\prec$ on the first one-particle line (for the case involving $\left|\widehat{\psi}_0^\varepsilon\left(k_{N^{(2\overline{m})},1}^{(2\overline{m})}\right)\right|^2\left|\widehat{\psi}_0^\varepsilon\left(k_{0,2}^{(1)}\right)\right|^2$) or second one-particle line (for $\left|\widehat{\psi}_0^\varepsilon\left(k_{0,1}^{(1)}\right)\right|^2\left|\widehat{\psi}_0^\varepsilon\left(k_{N^{(2\overline{m})},2}^{(2\overline{m})}\right)\right|^2$) one-particle line, or even on both lines (for $\left|\widehat{\psi}_0^\varepsilon\left(k_{N^{(2\overline{m})},1}^{(2\overline{m})}\right)\right|^2\left|\widehat{\psi}_0^\varepsilon\left(k_{N^{(2\overline{m})},2}^{(2\overline{m})}\right)\right|^2$). The assertion follows after absorbing the sum over σ in (5.21) into the constant C.

In case $N^{(j)}=0$ for one or several $j\in\{1,...,2\overline{m}\}$, the remark at the end of the proof of Lemma 4.8 applies. □

5.3. Improved bounds

5.3.1. Crossing estimates

As in Chapter 4, the basic estimate from Lemma 5.2 suffices for higher order partitions S, and we can turn to pairings, first tackling those with generalized crossings, like Figure 5.2.

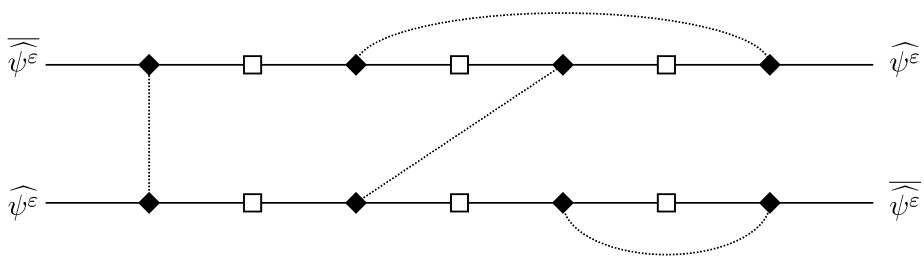

Figure 5.2.: For $\overline{m}=2$ and $N=(1,1,1,1)$, a pairing $S\in\pi^*_{\mathrm{conn}}\left(I(N,N)\right)$ that exhibits a generalized crossing.

Lemma 5.3. (Bound for amplitude $\mathcal{V}$ of pairings with generalized crossings.) *For ξ of class $(d+3,0)$, $S\in\pi^*_{\mathrm{conn}}\left(I(N,N)\right)$ a pairing with a generalized crossing on one of its one-particle lines, and $\gamma\in[2\varepsilon C_{\mathrm{obs}}\overline{m},1/2]$, there is a C only depending on dimension $d\geq 2$ such that*

$$\begin{aligned}&\left|\mathcal{V}\left(\psi_0^\varepsilon,\varepsilon,a,p,L^{(0)},t,N,S\right)\right|\\&\qquad\leq C^{|N|+2\overline{m}}\left(\langle L^{(0)}\rangle+\overline{m}\overline{N}\right)^{4|N|+3}e^{4\gamma|t|}\,\|g_2\|_{d+3}^{|N|}\,\|\psi_0^\varepsilon\|_{\mathcal{H}}^4\prod_{j=1}^{2\overline{m}-1}\|a_j\|_{C^0}^2\\&\qquad\quad\times\varepsilon^{|N|}\gamma^{-|N|+1}|\log\gamma|^{2|N|+4\overline{m}+1},\end{aligned}\tag{5.31}$$

for $d \geq 3$, *and*

$$
\begin{aligned}
&\left|\mathcal{V}\left(\psi_0^{\varepsilon}, \varepsilon, a, p, L^{(0)}, t, N, S\right)\right| \\
&\quad \leq C^{|N|+2\overline{m}} \left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{4|N|+3} e^{4\gamma|t|} \left\|g_2\right\|_{d+3}^{|N|} \left\|\psi_0^{\varepsilon}\right\|_{\mathcal{H}}^{4} \prod_{j=1}^{2\overline{m}-1} \left\|a_j\right\|_{C^0}^{2} \\
&\qquad \times \varepsilon^{|N|}\gamma^{-|N|+1/2}|\log\gamma|^{2|N|+4\overline{m}},
\end{aligned}
\tag{5.32}
$$

for $d = 2$.

Proof. Without loss of generality, let a generalized crossing occur on the first one-particle line. As in the proof of the previous lemma, we will focus on the summand $\left|\widehat{\psi}_0^{\varepsilon}\left(k_{0,1}^{(1)}\right)\right|^2 \left|\widehat{\psi}_0^{\varepsilon}\left(k_{0,2}^{(1)}\right)\right|^2$, the other cases being similar. Let the generalized crossing consist of two pairs denoted by $\{(j_1, n_1, 1), (j_2, n_2, 1)\}$ and $\{(\tilde{\jmath}_1, \tilde{n}_1, 1), (\tilde{\jmath}_2, \tilde{n}_2, r')\}$; in addition to the requirements of Definition 5.1 we can assume (after finitely many reduction steps) that the crossing interval, i.e. the set

$$
\{(j, n, 1) \in I(N, N) : (\tilde{\jmath}_1, \tilde{n}_1) \preceq (j, n) \preceq (j_2, n_2)\}
\tag{5.33}
$$

does not contain the crossing interval of another generalized crossing as a proper subset. While we certainly have $N^{(j)} \geq 1$ for the relevant indices $j = j_1, j_2, \tilde{\jmath}_1, \tilde{\jmath}_2$, we assume for simplicity, to make the resolvent expansion work, that $N^{(j)} \geq 1$ for all $j \in \{1, ..., 2\overline{m}\}$. We invoke $m \geq d + 3$ and the fact that S is a pairing to obtain an improved version of (5.22)

$$
\begin{aligned}
&C^{|N|}\left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{4|N|+3} \left\|g_2\right\|_{d+3}^{|N|} \\
&\quad \int_{\mathbb{R}^d} \mathrm{d}k_{0,1}^{(1)} \int_{\mathbb{R}^d} \mathrm{d}k_{0,2}^{(1)} \left|\widehat{\psi}_0^{\varepsilon}\left(k_{0,1}^{(1)}\right)\right|^2 \left|\widehat{\psi}_0^{\varepsilon}\left(k_{0,2}^{(1)}\right)\right|^2 \\
&\qquad \int_{\mathbb{R}^{2|N|d}} \prod_{(j,n,r)\in I(N,N)} \left(\mathrm{d}\theta_{n,r}^{(j)}\right) \prod_{A\in S} \left(\delta\left(\sum_{(j,n,r)\in A} \theta_{n,r}^{(j)}\right)\right) \\
&\qquad \int_{\mathbb{R}^{4\overline{m}}} \mathrm{d}\alpha_1^{(1)}...\mathrm{d}\alpha_1^{(2\overline{m})}\mathrm{d}\alpha_2^{(1)}...\mathrm{d}\alpha_2^{(2\overline{m})} \\
&\qquad\quad \prod_{(j,n,r)\in I_0(N,N)} \left|\frac{1}{\alpha_r^{(j)} - 2\pi\sigma_{n,r}^{(j)}\tau_r^{(j)}|k_{n,r}^{(j)}| + i\gamma}\right| \\
&\qquad\quad \prod_{(j,n,r)\in I(N,N)} \left(\left\langle\theta_{n,r}^{(j)}\right\rangle^{-d}\left\langle k_{n,r}^{(j)}\right\rangle^{-1}\right) \\
&\qquad\quad \left\langle k_{n_1,1}^{(j_1)}\right\rangle^{-1}\left\langle k_{\tilde{n}_1,1}^{(\tilde{\jmath}_1)}\right\rangle^{-1}\left\langle k_{n_2,1}^{(j_2)}\right\rangle^{-1},
\end{aligned}
\tag{5.34}
$$

which we estimate as follows. Adopting the Definition 5.2 of dependent and free indices $(j, n, r) \in I(N, N)$ from the proof of Lemma 5.2, one can take the L^{∞} estimates of all resolvents belonging to $k_{n,r}^{(j)}$, (j, n, r) dependent, *with the exception* of the resolvent belonging to $k^{(j_2)}_{n_2,1}$, which we keep for now. The $\alpha^{(j)}_2$ and free $\theta^{(j)}_{n,2}$ variables on the second one-particle line are integrated out as in the previous proof. For the first one-particle line, we are then exactly in the setting of Lemma 4.11, and obtain the same improvement factor $C\gamma|\log\gamma|$ (if $d \geq 3$) or $C\sqrt{\gamma}$ (if $d = 2$) over the basic estimate. $\square$

Lemma 5.4. (Bound for amplitude $\mathcal{V}$ of pairings with crossing transfer pairs.) *For ξ of class $(d+3,0)$, $S \in \pi^*_{\mathrm{conn}}(I(N,N))$ a pairing with crossing transfer pairs, and $\gamma \in (0,1/2]$, there is a C only depending on dimension $d \geq 2$ such that*

$$\begin{aligned}\left|\mathcal{V}\left(\psi^\varepsilon_0, \varepsilon, a, p, L^{(0)}, t, N, S\right)\right| & \\ \leq C^{|N|+2\overline{m}} \left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{4|N|+3} & e^{4\gamma|t|} \, \|g_2\|^{|N|}_{d+3} \, \|\psi^\varepsilon_0\|^4_{\mathcal{H}} \prod_{j=1}^{2\overline{m}-1} \|a_j\|^2_{C^0} \\ \times \, \varepsilon^{|N|}\gamma^{-|N|+1}|\log\gamma|^{2|N|+4\overline{m}+1}, &\end{aligned} \tag{5.35}$$

for $d \geq 3$, and

$$\begin{aligned}\left|\mathcal{V}\left(\psi^\varepsilon_0, \varepsilon, a, p, L^{(0)}, t, N, S\right)\right| & \\ \leq C^{|N|+2\overline{m}} \left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{4|N|+3} & e^{4\gamma|t|} \, \|g_2\|^{|N|}_{d+3} \, \|\psi^\varepsilon_0\|^4_{\mathcal{H}} \prod_{j=1}^{2\overline{m}-1} \|a_j\|^2_{C^0} \\ \times \, \varepsilon^{|N|}\gamma^{-|N|+1/2}|\log\gamma|^{2|N|+4\overline{m}}, &\end{aligned} \tag{5.36}$$

for $d = 2$.

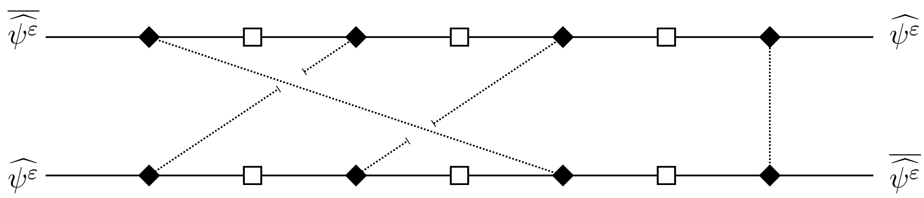

Figure 5.3.: A pairing $S \in \pi^*_{\mathrm{conn}}(I(N,N))$ with crossing transfer pairs. Rotating the lower one-particle line by 180° will dissolve the present crossings, but create new ones.

Proof. To make full use of the resolvent expansion, we only present the proof for the case that all $N^{(j)} \geq 1$. Again, we differentiate between the four different summands from the last line in (5.21), and first choose $\left|\widehat{\psi^\varepsilon_0}\left(k^{(1)}_{0,1}\right)\right|^2 \left|\widehat{\psi^\varepsilon_0}\left(k^{(1)}_{0,2}\right)\right|^2$. As the transfer pairs of S are not parallel, there exist two transfer pairs

$$\{(j_1, n_1, 1), (j_2, n_2, 2)\}, \{(\tilde{j}_1, \tilde{n}_1, 1), (\tilde{j}_2, \tilde{n}_2, 2)\} \in S \tag{5.37}$$

such that $(j_1, n_1) \prec (\tilde{\jmath}_1, \tilde{n}_1)$ and $(j_2, n_2) \succ (\tilde{\jmath}_2, \tilde{n}_2)$. As in the beginning of the proof of Lemma 4.11, in case there are several such structures, we have to make a convenient choice. A way to do so is laid out in Lemma 8 of [7] — select the smallest possible index $(\tilde{\jmath}_2, \tilde{n}_2)$ with respect to $\prec$, which also determines $(\tilde{\jmath}_1, \tilde{n}_1)$. Then, of all admissible $(j_1, n_1) \prec (\tilde{\jmath}_1, \tilde{n}_1)$, choose the $\prec$-largest one. For this choice of indices, proceed as in the

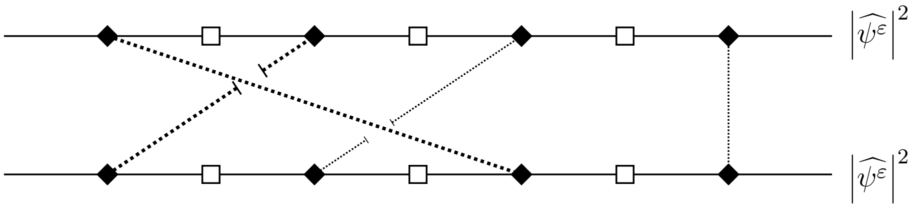

Figure 5.4.: The pairing S from Figure 5.3. The wave-functions are in "cis" constellation, and the two pairs $\{(j_1, n_1, 1), (j_2, n_2, 2)\}$ and $\{(\tilde{\jmath}_1, \tilde{n}_1, 1), (\tilde{\jmath}_2, \tilde{n}_2, 2)\}$ are shown in bold.

proof of Lemma 5.3 to find an improvement of (5.22),

$$
\begin{aligned}
&C^{|N|} \left(\langle L^{(0)} \rangle + \overline{m}\overline{N} \right)^{4|N|+3} \|g_2\|_{d+3}^{|N|} \\
&\quad \int_{\mathbb{R}^d} \mathrm{d}k_{0,1}^{(1)} \int_{\mathbb{R}^d} \mathrm{d}k_{0,2}^{(1)} \left| \widehat{\psi_0^{\varepsilon}} \left(k_{0,1}^{(1)} \right) \right|^2 \left| \widehat{\psi_0^{\varepsilon}} \left(k_{0,2}^{(1)} \right) \right|^2 \\
&\qquad \int_{\mathbb{R}^{2|N|d}} \prod_{(j,n,r) \in I(N,N)} \left(\mathrm{d}\theta_{n,r}^{(j)} \right) \prod_{A \in S} \left(\delta \left(\sum_{(j,n,r) \in A} \theta_{n,r}^{(j)} \right) \right) \\
&\qquad \int_{\mathbb{R}^{4\overline{m}}} \mathrm{d}\alpha_1^{(1)} \ldots \mathrm{d}\alpha_1^{(2\overline{m})} \mathrm{d}\alpha_2^{(1)} \ldots \mathrm{d}\alpha_2^{(2\overline{m})} \\
&\qquad\quad \prod_{(j,n,r) \in I_0(N,N)} \left| \frac{1}{\alpha_r^{(j)} - 2\pi \sigma_{n,r}^{(j)} \tau_r^{(j)} |k_{n,r}^{(j)}| + i\gamma} \right| \\
&\qquad\quad \prod_{(j,n,r) \in I(N,N)} \left(\left\langle \theta_{n,r}^{(j)} \right\rangle^{-d} \left\langle k_{n,r}^{(j)} \right\rangle^{-1} \right) \\
&\qquad\quad \left\langle k_{n_1,1}^{(j_1)} \right\rangle^{-1} \left\langle k_{\tilde{n}_1,1}^{(\tilde{\jmath}_1)} \right\rangle^{-1} \left\langle k_{\tilde{n}_2,1}^{(\tilde{\jmath}_2)} \right\rangle^{-1} .
\end{aligned}
\tag{5.38}
$$

One can now argue as in case i) of the proof in Lemma 4.11; case ii) and case iii), which accounted for the possiblity of an observable within the crossing interval, have no equivalent here. We classify the indices of $I(N, N)$ into dependent and free ones as in Definition 5.2, and take the L^∞ estimates of all $k_{n,r}^{(j)}$ resolvents, (j, n, r) dependent, except that we keep the one belonging to $(\tilde{\jmath}_2, \tilde{n}_2, 2)$. The following, modified program will then integrate out the $\theta_{n,r}^{(j)}$ variables associated with free indices (j, n, r) and the $\alpha_r^{(j)}$.

Instead of integrating out the full second one-particle line at once, we now only integrate over all $\theta_{n,2}^{(j)}$ with $(j,n) \succ (\tilde{\jmath}_2, \tilde{n}_2)$ and $(j,n,2)$ free, and the $\alpha_2^{(j)}$ with $j > \tilde{\jmath}_2$ according to the Iteration 5.1. The $\prec$-largest remaining k momentum on the second one-particle line is thus $k_{\tilde{n}_2,2}^{(\tilde{\jmath}_2)}$, so the leftover k momenta on the second one-particle line can by (5.27) and choice of $(\tilde{\jmath}_2, \tilde{n}_2)$ merely depend on $\theta_{n,1}^{(j)}$ with $(j,n) \preceq (\tilde{\jmath}_1, \tilde{n}_1)$. Therefore, switching to $r = 1$, the Iteration 5.1 can already be used to integrate out all $\theta_{n,1}^{(j)}$ as long $(j,n,1)$ is free and $(j,n) \succ (\tilde{\jmath}_1, \tilde{n}_1)$ and all $\alpha_1^{(j)}$ with $j > \tilde{\jmath}_1$. After reaching this point of the integration procedure, $\theta_{\tilde{n}_1,1}^{(\tilde{\jmath}_1)}$ only enters the definition of $k_{\tilde{n}_1,1}^{(\tilde{\jmath}_1)}$ and $k_{\tilde{n}_2,2}^{(\tilde{\jmath}_2)}$ by

$$
\begin{aligned}
k_{\tilde{n}_1,1}^{(\tilde{\jmath}_1)} &= \theta_{\tilde{n}_1,1}^{(\tilde{\jmath}_1)} + k_{n_1,1}^{(j_1)} + f(p), \\
k_{\tilde{n}_2,2}^{(\tilde{\jmath}_2)} &= \theta_{\tilde{n}_1,1}^{(\tilde{\jmath}_1)} + k_{\tilde{n}_2-1,2}^{(\tilde{\jmath}_2)},
\end{aligned}
\tag{5.39}
$$

with $f(p)$ only a function of the p variables.Our choice of $(j_1, n_1, 1)$ was used in the first line. The integration over $\theta_{\tilde{n}_1,1}^{(\tilde{\jmath}_1)}$ therefore produces a factor

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{d}\theta_{\tilde{n}_1,1}^{(\tilde{\jmath}_1)} \frac{1}{\left|\alpha_1^{(\tilde{\jmath}_1)} - 2\pi\sigma_{\tilde{n}_1,1}^{(\tilde{\jmath}_1)}\tau_1^{(\tilde{\jmath}_1)}\left|\theta_{\tilde{n}_1,1}^{(\tilde{\jmath}_1)} + k_{n_1,1}^{(j_1)} + f(p)\right| + i\gamma\right|} \\
&\quad \times \frac{1}{\left|\alpha_2^{(\tilde{\jmath}_2)} - 2\pi\sigma_{\tilde{n}_2,2}^{(\tilde{\jmath}_2)}\tau_2^{(\tilde{\jmath}_2)}\left|\theta_{\tilde{n}_1,1}^{(\tilde{\jmath}_1)} + k_{\tilde{n}_2-1,2}^{(\tilde{\jmath}_2)}\right| + i\gamma\right|} \\
&\quad \times \left\langle \theta_{\tilde{n}_1,1}^{(\tilde{\jmath}_1)} + k_{n_1,1}^{(j_1)} + f(p)\right\rangle^{-2} \left\langle \theta_{\tilde{n}_1,1}^{(\tilde{\jmath}_1)} + k_{\tilde{n}_2-1,2}^{(\tilde{\jmath}_2)}\right\rangle^{-2} \left\langle \theta_{\tilde{n}_1,1}^{(\tilde{\jmath}_1)}\right\rangle^{-d} \\
&\quad \leq \begin{cases} \dfrac{C_d |\log\gamma|^2}{\left|k_{n_1,1}^{(j_1)} + f(p) - k_{\tilde{n}_2-1,2}^{(\tilde{\jmath}_2)}\right| \sqrt{\left\langle \alpha_1^{(\tilde{\jmath}_1)}\right\rangle \left\langle \alpha_2^{(\tilde{\jmath}_2)}\right\rangle}} & \text{for } d \geq 3 \\[2ex] \dfrac{C_2 |\log\gamma|}{\sqrt{\gamma \left|k_{n_1,1}^{(j_1)} + f(p) - k_{\tilde{n}_2-1,2}^{(\tilde{\jmath}_2)}\right|} \sqrt{\left\langle \alpha_1^{(\tilde{\jmath}_1)}\right\rangle \left\langle \alpha_2^{(\tilde{\jmath}_2)}\right\rangle}} & \text{for } d = 2 \end{cases}
\end{aligned}
\tag{5.40}
$$

by Lemma B.1. One can now continue Iteration 5.1 on the first one-particle line, for free $\theta_{n,1}^{(j)}$ with $(j_1, n_1) \prec (j,n) \prec (\tilde{\jmath}_1, \tilde{n}_1)$ and for all $\alpha_1^{(j)}$ with $j_1 < j \leq \tilde{\jmath}_1$. By choice of $(\tilde{\jmath}_2, \tilde{n}_2, 2)$ and $(j_1, n_1, 1)$, the expression

$$
\left|k_{n_1,1}^{(j_1)} + f(p) - k_{\tilde{n}_2-1,2}^{(\tilde{\jmath}_2)}\right|
\tag{5.41}
$$

on the right side of (5.40) does not depend on any of the integration variables on this section of the iteration, and is just carried along as a constant. The $\theta_{n_1,1}^{(j_1)}$ integral can finally be estimated by switching the integration variable to $k_{n_1,1}^{(j_1)} = \theta_{n_1,1}^{(j_1)} + q$, q some linear combination of all momenta not integrated over yet, and applying Lemma B.2,

$$\begin{aligned}\sup_q C_d|\log\gamma|^2 \int_{\mathbb{R}^d} \mathrm{d}k_{n_1,1}^{(j_1)} & \frac{1}{\left|k_{n_1,1}^{(j_1)} + f(p) - k_{\tilde{n}_2-1,2}^{(\tilde{j}_2)}\right| \left|\alpha_1^{(j_1)} - 2\pi\sigma_{n_1,1}^{(j_1)}\tau_1^{(j_1)}\left|k_{n_1,1}^{(j_1)}\right| + i\gamma\right|} \\ & \times \left\langle k_{n_1,1}^{(j_1)}\right\rangle^{-2} \left\langle k_{n_1,1}^{(j_1)} - q\right\rangle^{-d} \\ & \leq \frac{\tilde{C}_d|\log\gamma|^3}{\sqrt{\left\langle \alpha_1^{(j_1)}\right\rangle}}\end{aligned} \tag{5.42}$$

for $d \geq 3$ and

$$\begin{aligned}\sup_q \frac{C_2|\log\gamma|}{\sqrt{\gamma}} \int_{\mathbb{R}^2} \mathrm{d}k_{n_1,1}^{(j_1)} & \frac{1}{\sqrt{\left|k_{n_1,1}^{(j_1)} + f(p) - k_{\tilde{n}_2-1,2}^{(\tilde{j}_2)}\right|} \left|\alpha_1^{(j_1)} - 2\pi\sigma_{n_1,1}^{(j_1)}\tau_1^{(j_1)}\left|k_{n_1,1}^{(j_1)}\right| + i\gamma\right|} \\ & \times \left\langle k_{n_1,1}^{(j_1)}\right\rangle^{-2} \\ & \leq \frac{\tilde{C}_d|\log\gamma|^2}{\sqrt{\gamma}\sqrt{\left\langle \alpha_1^{(j_1)}\right\rangle}}\end{aligned} \tag{5.43}$$

for $d = 2$. The remainder of both one-particle lines can then be handled as in the derivation of the standard bound, Lemma 5.2. Thus, one L^∞ and two L^1 resolvent estimates have been replaced by a factor $|\log\gamma|^3$ (for $d \geq 3$) or $|\log\gamma|^2/\sqrt{\gamma}$ (for $d = 2$), yielding an improvement of $\gamma|\log\gamma|$ or $\sqrt{\gamma}$, respectively.

So far, we have only paid attention to the term including $\left|\widehat{\psi}_0^\varepsilon\left(k_{0,1}^{(1)}\right)\right|^2 \left|\widehat{\psi}_0^\varepsilon\left(k_{0,2}^{(1)}\right)\right|^2$, i.e. both squared wave-functions sit on the right of the graph. The case of both squared wave-functions on the left, represented by the term $\left|\widehat{\psi}_0^\varepsilon\left(k_{N^{(2\overline{m})},1}^{(2\overline{m})}\right)\right|^2 \left|\widehat{\psi}_0^\varepsilon\left(k_{N^{(2\overline{m})},2}^{(2\overline{m})}\right)\right|^2$ is entirely analogous after replacing $\prec$ by $\succ$ in the definition of freeness, dependence and integration order. However, the cross-over situation deserves a closer look. Without loss of generality, consider

$$\left|\widehat{\psi}_0^\varepsilon\left(k_{0,1}^{(1)}\right)\right|^2 \left|\widehat{\psi}_0^\varepsilon\left(k_{N^{(2\overline{m})},2}^{(2\overline{m})}\right)\right|^2 \tag{5.44}$$

(wave-function on the right of the first, but on the left of the second one-particle line). As the transfer pairs of S are not anti-parallel, either, there are two transfer pairs

$$\{(j_1, n_1, 1), (j_2, n_2, 2)\}, \{(\tilde{j}_1, \tilde{n}_1, 1), (\tilde{j}_2, \tilde{n}_2, 2)\} \in S \tag{5.45}$$

such that $(j_1, n_1) \prec (\tilde{j}_1, \tilde{n}_1)$ and $(j_2, n_2) \prec (\tilde{j}_2, \tilde{n}_2)$. Given the choice of several such structures, select the $\prec$-largest possible $(\tilde{j}_2, \tilde{n}_2)$, which also defines $(\tilde{j}_1, \tilde{n}_1)$; then choose the $\prec$-largest possible $(j_1, n_1) \prec (\tilde{j}_1, \tilde{n}_1)$. With this choice, the $\left|\widehat{\psi}_0^\varepsilon\left(k_{0,1}^{(1)}\right)\right|^2 \left|\widehat{\psi}_0^\varepsilon\left(k_{N^{(2\overline{m})},2}^{(2\overline{m})}\right)\right|^2$

contribution to the last five lines of (5.21) can be bounded by

$$
\begin{aligned}
&C^{|N|}\left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{4|N|+3}\|g_2\|_{d+3}^{|N|} \\
&\quad \int_{\mathbb{R}^d} \mathrm{d}k_{0,1}^{(1)} \int_{\mathbb{R}^d} \mathrm{d}k_{N^{(2\overline{m})},2}^{(2\overline{m})} \left|\widehat{\psi}_0^{\varepsilon}\left(k_{0,1}^{(1)}\right)\right|^2 \left|\widehat{\psi}_0^{\varepsilon}\left(k_{N^{(2\overline{m})},2}^{(2\overline{m})}\right)\right|^2 \\
&\qquad \int_{\mathbb{R}^{2|N|d}} \prod_{(j,n,r)\in I(N,N)} \left(\mathrm{d}\theta_{n,r}^{(j)}\right) \prod_{A\in S}\left(\delta\left(\sum_{(j,n,r)\in A}\theta_{n,r}^{(j)}\right)\right) \\
&\qquad \int_{\mathbb{R}^{4\overline{m}}} \mathrm{d}\alpha_1^{(1)}...\mathrm{d}\alpha_1^{(2\overline{m})}\mathrm{d}\alpha_2^{(1)}...\mathrm{d}\alpha_2^{(2\overline{m})} \\
&\qquad\quad \prod_{(j,n,r)\in I_0(N,N)}\left|\frac{1}{\alpha_r^{(j)} - 2\pi\sigma_{n,r}^{(j)}\tau_r^{(j)}|k_{n,r}^{(j)}| + i\gamma}\right| \\
&\qquad\quad \prod_{(j,n)\in I(N)}\left(\left\langle\theta_{n,1}^{(j)}\right\rangle^{-d}\left\langle k_{n,1}^{(j)}\right\rangle^{-1}\left\langle\theta_{n,2}^{(j)}\right\rangle^{-d}\left\langle k_{n-1,2}^{(j)}\right\rangle^{-1}\right) \\
&\qquad\quad \left\langle k_{n_1,1}^{(j_1)}\right\rangle^{-1}\left\langle k_{\tilde{n}_1,1}^{(\tilde{j}_1)}\right\rangle^{-1}\left\langle k_{\tilde{n}_2-1,1}^{(\tilde{j}_2)}\right\rangle^{-1}.
\end{aligned}
\tag{5.46}
$$

Instead of Definition 5.2, we employ

Definition 5.3. *(Definition of free and dependent indices, if the first one-particle line is to be integrated out from left to right, but the second one from right to left.)* The index $(j,n,r)\in I(N,N)$ is *dependent*,

i) if $(j,n,r)=(j,n,1)\in A$, with A an internal cluster of first one-particle line and $(j,n)=\max_1 A$, or

ii) if $(j,n,r)=(j,n,2)\in A$, with A an internal cluster of second one-particle line and $(j,n)=\min_1 A$, or

iii) if $(j,n,r)\in A$, with A a transfer cluster, $r=2$, and $(j,n)=\min_2 A$,

and *free* otherwise.

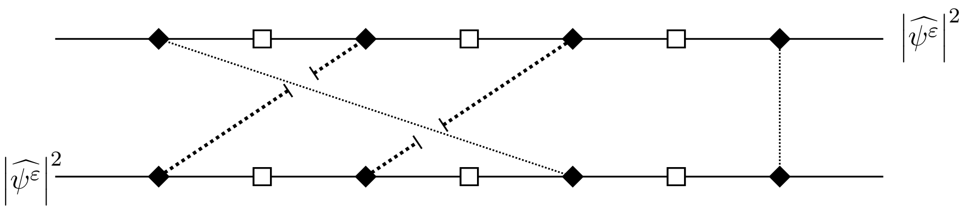

Figure 5.5.: The pairing S from Figure 5.3. The wave-functions are in "cross-over" constellation, the two pairs $\{(j_1,n_1,1),(j_2,n_2,2)\}$ and $\{(\tilde{j}_1,\tilde{n}_1,1),(\tilde{j}_2,\tilde{n}_2,2)\}$ as selected in the proof are shown in bold.

While the $k_{\cdot,1}^{(\cdot)}$ are still given by (5.26), the k momenta on the second one-particle line can be calculated via

$$k_{n,2}^{(j)} = k_{N^{(2\overline{m})},2}^{(2\overline{m})} - \varepsilon \sum_{\tilde{\jmath}=j}^{2\overline{m}-1} p^{(\tilde{\jmath})} + \sum_{\substack{A \in S_2 \cup S_{\mathrm{tr}} \\ (j,n) \succeq \min_2 A}} \sum_{\substack{(\tilde{\jmath},\tilde{n},2) \in A \\ (\tilde{\jmath},\tilde{n}) \succ (j,n)}} \theta_{\tilde{n},2}^{(\tilde{\jmath})} - \sum_{\substack{A \in S_{\mathrm{tr}} \\ (j,n) \prec \min_2 A}} \sum_{(\tilde{\jmath},\tilde{n},1) \in A} \theta_{\tilde{n},1}^{(\tilde{\jmath})}. \quad (5.47)$$

The L^∞ estimate (4.62) is taken for the resolvent belonging to $k_{n-1,2}^{(j)}$ for all dependent $(j,n,2) \in I(N,N)$, except for $(\tilde{\jmath}_2,\tilde{n}_2,2)$, and for all resolvents belonging to $k_{n,1}^{(j)}$, $(j,n,1) \in I(N,N)$ dependent. One can then integrate out all free $\theta_{n,2}^{(j)}$ as long as $(j,n) \prec (\tilde{\jmath}_2,\tilde{n}_2)$ and all $\alpha_2^{(j)}$, $j < \tilde{\jmath}_2$ by plugging $r = 2$ into the following algorithm

Iteration 5.2. *(Integrating out the r-th one-particle line, from right to left, i.e. increasing in $\prec$.)*

- Of all remaining free $(j,n,r) \in I(N,N)$, and all remaining $(j,N^{(j)}+1,r)$, $j \in \{1,...2\overline{m}\}$, pick the one with the smallest (j,n) with respect to $\prec$.
- If $n < N^{(j)}+1$, and thus $(j,n,r) \in I(N,N)$, one can check in (5.47) that $k_{n-1,r}^{(j)}$ is the only remaining k variable depending on $\theta_{n,r}^{(j)}$. Integrating over $\theta_{n,r}^{(j)}$ produces a factor

$$\int_{\mathbb{R}^d} \mathrm{d}\theta_{n,r}^{(j)} \frac{1}{\left|\alpha_r^{(j)} - 2\pi\sigma_{n-1,r}^{(j)}\tau_r^{(j)}|k_{n-1,r}^{(j)}| + i\gamma\right| \left\langle k_{n-1,r}^{(j)}\right\rangle \left\langle \theta_{n,r}^{(j)}\right\rangle^d} \leq \frac{C|\log\gamma|}{\left\langle \alpha_r^{(j)}\right\rangle}, \quad (5.48)$$

  with a constant $C < \infty$ depending only on d.
- In case $(j,n,r) = (j,N^{(j)}+1,r)$, we have made sure by previous steps of our integration that the only resolvent depending on $\alpha_r^{(j)}$ is the one belonging to $k_{N^{(j)},r}^{(j)}$, while there is a decay $\left\langle \alpha_r^{(j)}\right\rangle^{-N^{(j)}}$, $N^{(j)} \geq 1$ stemming from the (4.62) and (5.48) bounds, and we obtain a factor

$$\int_{\mathbb{R}} \mathrm{d}\alpha_r^{(j)} \frac{1}{\left|\alpha_r^{(j)} - 2\pi\sigma_{N^{(j)},r}^{(j)}\tau_r^{(j)}|k_{N^{(j)},r}^{(j)}| + i\gamma\right| \left\langle \alpha_r^{(j)}\right\rangle} \leq C|\log\gamma| \quad (5.49)$$

  from the $\alpha_r^{(j)}$ integral, C only depending on d.

The $\prec$-smallest remaining k momentum on the second one-particle line is thus the momentum $k_{\tilde{n}_2-1,2}^{(\tilde{\jmath}_2)}$, so the leftover k momenta on the second one-particle line can by (5.47) and choice of $(\tilde{\jmath}_2,\tilde{n}_2)$ merely depend on $\theta_{n,1}^{(j)}$ with $(j,n) \preceq (\tilde{\jmath}_1,\tilde{n}_1)$. Therefore, switching to $r = 1$, the Iteration 5.1 can already be used to integrate out all $\theta_{n,1}^{(j)}$ as long $(j,n,1)$ is free and $(j,n) \succ (\tilde{\jmath}_1,\tilde{n}_1)$ and all $\alpha_1^{(j)}$ with $j > \tilde{\jmath}_1$. Now, integrating out $\theta_{\tilde{n}_1,1}^{(\tilde{\jmath}_1)}$

involves only the resolvents belonging to $k^{(\tilde{\jmath}_1)}_{\tilde{n}_1,1}$ and $k^{(\tilde{\jmath}_2)}_{\tilde{n}_2-1,2}$, producing an estimate like (5.40), namely

$$\begin{aligned} &\frac{C_d|\log\gamma|^2}{\left|k^{(j_1)}_{n_1,1}+f(p)+k^{(\tilde{\jmath}_2)}_{\tilde{n}_2,2}\right|\sqrt{\left\langle\alpha^{(\tilde{\jmath}_1)}_1\right\rangle\left\langle\alpha^{(\tilde{\jmath}_2)}_2\right\rangle}} \text{ for } d\geq 3 \\ &\frac{C_2|\log\gamma|}{\sqrt{\gamma\left|k^{(j_1)}_{n_1,1}+f(p)+k^{(\tilde{\jmath}_2)}_{\tilde{n}_2,2}\right|}\sqrt{\left\langle\alpha^{(\tilde{\jmath}_1)}_1\right\rangle\left\langle\alpha^{(\tilde{\jmath}_2)}_2\right\rangle}} \text{ for } d=2 \end{aligned} \tag{5.50}$$

which does not interfere with ours integrating out the $\theta^{(j)}_{n,1}$, $(j_1,n_1)\prec(j,n)\prec(\tilde{\jmath}_1,\tilde{n}_1)$, and $\alpha^{(j)}_1$, $j_1<j\leq\tilde{\jmath}_1$, variables by Iteration 5.1. The $\theta^{(j_1)}_{n_1,1}$ integral then yields the same bound as (5.42-5.43) did. After all leftover $\theta^{(\cdot)}_{\cdot,2}$ and $\alpha^{(\cdot)}_2$ variables have been taken care of by completing Iteration 5.2, an application of Iteration 5.1 to the rest of the first one-particle line finishes the proof for the cross-over case, with the same improvement over the basic Lemma 5.2. □

5.3.2. (Anti-)parallel transfer pairs

In the last proof, the improvement in comparison to the standard estimate originated from two different structures: For the "cis" constellation of wave functions (both $\left|\widehat{\psi^\varepsilon_0}\right|^2$ on the left of the graph, or both of the right), we utilized the existence of two intersecting transfer pairs, Figure 5.4, while in the "cross-over case", (one $\left|\widehat{\psi^\varepsilon_0}\right|^2$ on the left, one on the right) two non-intersecting transfer pairs, as highlighted in Figure 5.5 were central to the argument. The equivalence of those two cases is obvious when the "cross-over" graph is brought into "cis" form by rotating the second one-particle line by 180° — the previously non-intersecting pairs then cross each other. At least one of those two arguments fails when dealing with (anti-)parallel transfer pairs; there is always at least one rotation of the second one-particle line that makes all intersections vanish, and a different approach is needed for the proof of the following Lemma.

Lemma 5.5. (Bound for amplitude $\mathcal{V}$ for pairings with (anti-)parallel transfer pairs.) *Let ξ be of class $(d+3,0)$, $\gamma\in(0,1/2]$ and $S\in\pi^*_{\text{conn}}(I(N,N))$ be a pairing with parallel or anti-parallel transfer pairs. There is a C depending only on dimension $d\geq 2$ such that*

$$\begin{aligned} &\left|\mathcal{V}\left(\psi^\varepsilon_0,\varepsilon,a,p,L^{(0)},t,N,S\right)\right| \\ &\quad\leq C^{|N|+2\overline{m}}\left(\langle L^{(0)}\rangle+\overline{m}\overline{N}\right)^{4|N|+2}e^{4\gamma|t|}\,\|g_2\|^{|N|}_{d+3}\,\|\psi^\varepsilon_0\|^4_{\mathcal{H}}\prod_{j=1}^{2\overline{m}-1}\|a_j\|^2_{C^0} \\ &\qquad\times\varepsilon^{|N|}\gamma^{-|N|}|\log\gamma|^{2|N|+4\overline{m}+1} \\ &\qquad\times\begin{cases}\gamma^{\frac{d-1}{d+1}} & (d\geq 3),\\ \gamma^{1/5} & (d=2).\end{cases} \end{aligned} \tag{5.51}$$

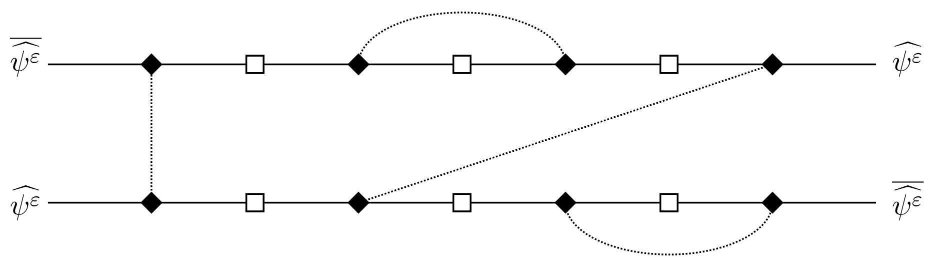

Figure 5.6.: A pairing with two parallel transfer pairs.

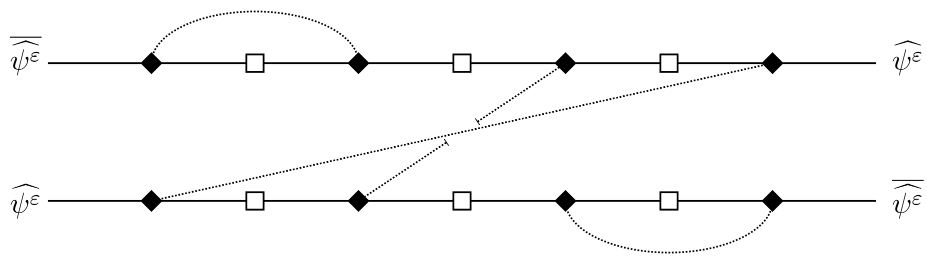

Figure 5.7.: A pairing with two anti-parallel transfer pairs. Rotating the lower one-particle by 180° makes all intersections disappear.

Proof. For the whole proof, assume for simplicity that all $N^{(j)} \geq 1$, $j \in \{1, ..., 2\overline{m}\}$, so that the resolvent representation is fully applicable. First, let $S \in \pi^*_{\text{conn}}(I(N, N))$ be a pairing with *parallel transfer pairs*, and choose $\{(j_1, n_1, 1), (j_2, n_2, 2)\} \in S$ to be the transfer pair with (j_1, n_1) (and thus (j_2, n_2)) $\prec$-maximal. For $\eta > 0$ to be optimized later, define

$$\mathcal{B} = \left\{ k \in \mathbb{R}^{(2|N|+4\overline{m})d} : \left| k^{(j_1)}_{n_1,1} - k^{(j_2)}_{n_2,2} \right| < \eta \right\} \tag{5.52}$$

to split up $\mathcal{V}$ into

$$\begin{aligned} \left| \mathcal{V}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right) \right| \leq & \left| \mathcal{V}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S, \mathcal{B}\right) \right| \\ & + \left| \mathcal{V}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S, \mathcal{B}^c\right) \right|, \end{aligned} \tag{5.53}$$

with $\mathcal{V}(..., \mathcal{B})$ and $\mathcal{V}(..., \mathcal{B}^c)$ defined by the right side of (5.11), but with the k integration domain $\mathbb{R}^{(2|N|+4\overline{m})d}$ replaced by $\mathcal{B}$ or $\mathcal{B}^c$, respectively. To understand the contribution of $\mathcal{V}(..., \mathcal{B})$, first note that, as S has parallel transfer pairs, the arguments of the two wave-functions on the left of the graph in Figure 5.6 are equal to

$$\begin{aligned} k^{(2\overline{m})}_{N^{(2\overline{m})},1} &= k^{(j_1)}_{n_1-1,1} + \theta^{(j_1)}_{n_1,1} + \varepsilon \sum_{j=j_1}^{2\overline{m}-1} p^{(j)}, \\ k^{(2\overline{m})}_{N^{(2\overline{m})},2} &= k^{(j_2)}_{n_2-1,1} + \theta^{(j_1)}_{n_1,1} + \varepsilon \sum_{j=j_2}^{2\overline{m}-1} p^{(j)}. \end{aligned} \tag{5.54}$$

On the other hand, on the set $\mathcal{B}$, as a consequence of the delta functions induced by S, the arguments of the wave-functions on the right obey

$$\left| k^{(1)}_{0,1} - k^{(1)}_{0,2} + p^* \right| < \eta, \tag{5.55}$$

with $p^* = \varepsilon \sum_{j=1}^{j_1-1} p^{(j)} - \varepsilon \sum_{j=1}^{j_2-1} p^{(j)}$. This time, an estimate of a product of wave functions as a sum of squares is too coarse, and we rather write

$$\begin{aligned} &\left| \mathcal{V}\left(\psi^{\varepsilon}_0, \varepsilon, a, p, L^{(0)}, t, N, S, \mathcal{B}\right) \right| \\ &\quad \leq (C\varepsilon)^{|N|} \left(\langle L^{(0)} \rangle + \overline{m}\overline{N} \right)^{4|N|} e^{4\gamma|t|} \prod_{j=1}^{2\overline{m}-1} \|a_j\|^2_{C^0} \prod_{A\in S} \left\| g_{|A|} \right\|_{d+2} \sum_{\substack{\sigma^{(j)}_{n,r}\in\{\pm\} \\ \forall (j,n,r)\in I_0(N,N)}} \\ &\qquad \int_{\left| k^{(1)}_{0,1} - k^{(1)}_{0,2} + p^* \right| < \eta} \mathrm{d}k^{(1)}_{0,1} \mathrm{d}k^{(1)}_{0,2} \left| \widehat{\psi}^{\varepsilon}_0 \left(k^{(1)}_{0,1} \right) \right| \left| \widehat{\psi}^{\varepsilon}_0 \left(k^{(1)}_{0,2} \right) \right| \\ &\qquad \int_{\mathbb{R}^{2|N|d}} \prod_{(j,n,r)\in I(N,N)} \left(\mathrm{d}\theta^{(j)}_{n,r} \right) \prod_{A\in S} \left(\delta \left(\sum_{(j,n,r)\in A} \theta^{(j)}_{n,r} \right) \right) \\ &\qquad \int_{\mathbb{R}^{4\overline{m}}} \mathrm{d}\alpha^{(1)}_1 ... \mathrm{d}\alpha^{(2\overline{m})}_1 \mathrm{d}\alpha^{(1)}_2 ... \mathrm{d}\alpha^{(2\overline{m})}_2 \\ &\qquad \prod_{(j,n,r)\in I_0(N,N)} \left| \frac{1}{\alpha^{(j)}_r - 2\pi\sigma^{(j)}_{n,r}\tau^{(j)}_r |k^{(j)}_{n,r}| + i\gamma} \right| \left(\left\langle \theta^{(j)}_{n,r} \right\rangle^{-d} \left\langle k^{(j)}_{n,r} \right\rangle^{-1} \right)^{(n\neq 0)} \\ &\qquad \left| \widehat{\psi}^{\varepsilon}_0 \left(k^{(j_1)}_{n_1-1,1} + \theta^{(j_1)}_{n_1,1} + \varepsilon \sum_{j=j_1}^{2\overline{m}-1} p^{(j)} \right) \right| \left| \widehat{\psi}^{\varepsilon}_0 \left(k^{(j_2)}_{n_2-1,1} + \theta^{(j_1)}_{n_1,1} + \varepsilon \sum_{j=j_2}^{2\overline{m}-1} p^{(j)} \right) \right| \end{aligned} \tag{5.56}$$

with a constant C depending only on dimension d. To estimate the last four lines of (5.56), we modify Definition 5.2 in such a manner that also $(j_1, n_1, 1)$ is dependent instead of free, and take the usual L^∞ bounds (4.62) of all resolvents belonging to $k^{(j)}_{n,r}$, $(j, n, r) \in I(N, N)$ dependent, which now yields one more factor γ^{-1} than usual. Iteration 5.1 can then be applied to all $\theta^{(j)}_{n,1}$, $(j, n, 1) \in I(N, N)$ free with $(j, n) \succ (j_1, n_1)$ and all $\alpha^{(j)}_1$, $j > j_1$, and likewise to all $\theta^{(j)}_{n,2}$, $(j, n, 2) \in I(N, N)$ free with $(j, n) \succ (j_2, n_2)$ and all $\alpha^{(j)}_2$, $j > j_2$. Next, the integral over $\theta^{(j_1)}_{n_1,1}$ will simply integrate out the last line of (5.56), for a factor of at most $\|\psi_0^\varepsilon\|_{\mathcal{H}}^2$. We then follow the proof of the standard estimate, integrating out the remaining α and free θ variables, on the second and then on the first one-particle line, with Iteration 5.1. The last four lines of (5.56) have been completely taken care of, and by Cauchy-Schwarz, the $k_{0,1}^{(1)}$, $k_{0,2}^{(1)}$ integral is smaller or equal to $\eta^d \|\psi_0^\varepsilon\|_{\mathcal{H}}^2$. We therefore have gained an additional factor $\eta^d\gamma^{-1}$ compared to the standard estimate, and there is an only d-dependent constant $C < \infty$ such that

$$\begin{aligned}&\left|\mathcal{V}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S, \mathcal{B}\right)\right| \\ &\quad \leq C^{|N|+2\overline{m}} \left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{4|N|} e^{4\gamma|t|} \|g_2\|_{d+2}^{|N|} \|\psi_0^\varepsilon\|_{\mathcal{H}}^4 \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^0}^2 \\ &\qquad \times \varepsilon^{|N|}\gamma^{-|N|-1}|\log\gamma|^{2|N|+4\overline{m}}\eta^d.\end{aligned} \tag{5.57}$$

The contribution of $\mathcal{V}(..., \mathcal{B}^c)$ can be controlled by the analogue of (5.21), only with the k integral running only over $\mathcal{B}^c$ instead of $\mathbb{R}^{(2|N|+4\overline{m})d}$. Again, we have to distinguish the four different summands arising from the last line of (5.21). First, consider the "cis" case, with both wave functions on the left or both on the right. Without loss of generality, we only treat the summand containing

$$\left|\widehat{\psi}_0^\varepsilon\left(k_{0,1}^{(1)}\right)\right|^2 \left|\widehat{\psi}_0^\varepsilon\left(k_{0,2}^{(1)}\right)\right|^2, \tag{5.58}$$

and thus have to find a bound for

$$\begin{aligned}&C^{|N|}\left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{4|N|+2} \|g_2\|_{d+3}^{|N|} \\ &\int_{\mathbb{R}^d} \mathrm{d}k_{0,1}^{(1)} \int_{\mathbb{R}^d} \mathrm{d}k_{0,2}^{(1)} \left|\widehat{\psi}_0^\varepsilon\left(k_{0,1}^{(1)}\right)\right|^2 \left|\widehat{\psi}_0^\varepsilon\left(k_{0,2}^{(1)}\right)\right|^2 \\ &\quad \int_{\mathbb{R}^{2|N|d}} \prod_{(j,n,r)\in I(N,N)} \left(\mathrm{d}\theta_{n,r}^{(j)}\right) \prod_{A\in S}\left(\delta\left(\sum_{(j,n,r)\in A}\theta_{n,r}^{(j)}\right)\right) \\ &\quad \int_{\mathbb{R}^{4\overline{m}}} \mathrm{d}\alpha_1^{(1)}...\mathrm{d}\alpha_1^{(2\overline{m})}\mathrm{d}\alpha_2^{(1)}...\mathrm{d}\alpha_2^{(2\overline{m})} \\ &\qquad \prod_{(j,n,r)\in I_0(N,N)} \left|\frac{1}{\alpha_r^{(j)} - 2\pi\sigma_{n,r}^{(j)}\tau_r^{(j)}|k_{n,r}^{(j)}| + i\gamma}\right| \left(\left\langle\theta_{n,r}^{(j)}\right\rangle^{-d}\left\langle k_{n,r}^{(j)}\right\rangle^{-1}\right)^{n\neq 0} \\ &\qquad \left\langle k_{n_1,1}^{(j_1)}\right\rangle^{-1}\left\langle k_{n_2,2}^{(j_2)}\right\rangle^{-1} \mathbb{1}\left(\left|k_{n_1,1}^{(j_1)} - k_{n_2,2}^{(j_2)}\right| \geq \eta\right).\end{aligned} \tag{5.59}$$

Now we define dependent and free indices (j,n,r) as in Definition 5.2 and take L^∞ estimates of all resolvents belonging to $k_{n,r}^{(j)}$, $(j,n,r) \in I(N,N)$ dependent except that we keep the $k_{n_2,2}^{(j_2)}$ resolvent for now. By Iteration 5.1, all $\theta_{n,2}^{(j)}$ variables with $(j,n,2) \in I(N,N)$ free and $(j,n) \succ (j_2,n_2)$ and all $\alpha_2^{(j)}$, $j > j_2$ are integrated out, the same procedure is applied to the $\theta_{n,1}^{(j)}$ with $(j,n,1) \in I(N,N)$ free and $(j,n) \succ (j_1,n_1)$ and all $\alpha_1^{(j)}$, $j > j_1$.

By (5.26-5.27), the only remaining k variables that still depend on $\theta_{n_1,1}^{(j_1)}$ are

$$\begin{aligned} k_{n_1,1}^{(j_1)} &= \theta_{n_1,1}^{(j_1)} + k_{n_1-1,1}^{(j_1)}, \\ k_{n_2,2}^{(j_2)} &= \theta_{n_1,1}^{(j_1)} + k_{n_2-1,2}^{(j_2)}. \end{aligned} \tag{5.60}$$

On $\mathcal{B}^c$, one also has $\left|k_{n_1-1,1}^{(j_1)} - k_{n_2-1,2}^{(j_2)}\right| \geq \eta$, and Lemma B.1 lets us bound the $\theta_{n_1,1}^{(j_1)}$ integral by

$$\begin{aligned} \int_{\mathbb{R}^d} \mathrm{d}\theta_{n_1,1}^{(j_1)} & \frac{1}{\left|\alpha_1^{(j_1)} - 2\pi\sigma_{n_1,1}^{(j_1)}\tau_1^{(j_1)}\left|\theta_{n_1,1}^{(j_1)} + k_{n_1-1,1}^{(j_1)}\right| + i\gamma\right|} \\ & \times \frac{1}{\left|\alpha_2^{(j_2)} - 2\pi\sigma_{n_2,2}^{(j_2)}\tau_2^{(j_2)}\left|\theta_{n_1,1}^{(j_1)} + k_{n_2-1,2}^{(j_2)}\right| + i\gamma\right|} \\ & \times \left\langle \theta_{n_1,1}^{(j_1)} + k_{n_1-1,1}^{(j_1)} \right\rangle^{-2} \left\langle \theta_{n_1,1}^{(j_1)} + k_{n_2-1,2}^{(j_2)} \right\rangle^{-2} \left\langle \theta_{n_1,1}^{(j_1)} \right\rangle^{-d} \\ & \leq \begin{cases} \frac{C_d |\log\gamma|^2}{\eta} \left\langle \alpha_1^{(j_1)} \right\rangle^{-1/2} \left\langle \alpha_2^{(j_2)} \right\rangle^{-1/2} & \text{for } d \geq 3, \\ \frac{C_2 |\log\gamma|}{\sqrt{\gamma\eta}} \left\langle \alpha_1^{(j_1)} \right\rangle^{-1/2} \left\langle \alpha_2^{(j_2)} \right\rangle^{-1/2} & \text{for } d = 2. \end{cases} \end{aligned} \tag{5.61}$$

One can then integrate out the remainder of the second and then the first one-particle lines by Iteration 5.1, just as in the proof of the basic estimate, which has thus been improved by a factor $\gamma|\log\gamma|/\eta$ ($d \geq 3$), or $\sqrt{\gamma/\eta}$ ($d = 2$). The last remaining contribution to $\mathcal{V}(..., \mathcal{B}^c)$ stems from the wave-functions in "cross-over" position. In case that the partition S in consideration contains *more than one transfer pair*, it is straightforward to just bound $\mathbb{1}(\mathcal{B}^c) \leq 1$ and apply the "cross-over" portion of the proof of Lemma 5.4 to obtain and improvement factor $\gamma|\log\gamma|$ (for $d \geq 3$) or $\sqrt{\gamma}$ (for $d = 2$). Therefore, for S with parallel transfer pairings and $|S_{\mathrm{tr}}| > 1$, the basic estimate Lemma 5.2 still holds if multiplied by a factor

$$C\left(\eta^d\gamma^{-1} + \gamma|\log\gamma|\eta^{-1} + \gamma|\log\gamma|\right) \quad (d \geq 3), \tag{5.62}$$

$$C\left(\eta^2\gamma^{-1} + \sqrt{\frac{\gamma}{\eta}} + \sqrt{\gamma}\right) \quad (d = 2), \tag{5.63}$$

C only depending on dimension d. Optimizing η as a function of γ proves the result in this case.

For $S \in \pi^*_{\mathrm{conn}}(I(N,N))$ with *several transfer pairs*, which are *anti-parallel*, choose a transfer pair $\{(j_1, n_1, 1), (j_2, n_2, 2)\}$ and define the set

$$\mathcal{C} = \left\{ k \in \mathbb{R}^{(2|N|+4\overline{m})d} : \left|k_{n_1,1}^{(j_1)} + k_{n_2-1,2}^{(j_2)}\right| < \eta \right\}. \tag{5.64}$$

Then with analogous definitions to the previous case, one has

$$\begin{aligned} & \left|\mathcal{V}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S, \mathcal{C}\right)\right| \\ & \quad \leq C^{|N|+2\overline{m}} \left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{4|N|} e^{4\gamma|t|} \|g_2\|_{d+2}^{|N|} \|\psi_0^\varepsilon\|_{\mathcal{H}}^4 \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^0}^2 \\ & \qquad \times \varepsilon^{|N|}\gamma^{-|N|-1}|\log\gamma|^{2|N|+4\overline{m}}\eta^d. \end{aligned} \tag{5.65}$$

The amplitude stemming from the complement $\mathcal{C}^c$, on the other hand, is

$$\begin{aligned}
&\left|\mathcal{V}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S, \mathcal{C}^c\right)\right| \\
&\quad\le C^{|N|+2\overline{m}}\left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{4|N|+2} e^{4\gamma|t|}\,\|g_2\|_{d+3}^{|N|}\,\|\psi_0^\varepsilon\|_{\mathcal{H}}^4 \prod_{j=1}^{2\overline{m}-1}\|a_j\|_{C^0}^2 \\
&\qquad\times \varepsilon^{|N|}\gamma^{-|N|}|\log\gamma|^{2|N|+4\overline{m}} \\
&\qquad\times \begin{cases}\gamma|\log\gamma|(1+1/\eta) & (d\ge 3),\\ \sqrt{\gamma}(1+1/\sqrt{\eta}) & (d=2),\end{cases}
\end{aligned} \tag{5.66}$$

where we used Lemma B.1 for the contribution of “cross-over” wave-functions, and argued as in the proof of Lemma 5.4 for the “cis” wave functions. Optimization of η yields the same results as before.

Finally, for the last case $|S_{\mathrm{tr}}| = 1$, which is classified as both parallel and anti-parallel, the Lemma follows by splitting $\mathcal{V}$ into

$$\begin{aligned}
&\left|\mathcal{V}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S,\right)\right| \\
&\quad\le \left|\mathcal{V}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S, \mathcal{B}\right)\right| + \left|\mathcal{V}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S, \mathcal{C}\right)\right| \\
&\qquad+ \left|\mathcal{V}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S, \mathcal{B}^c\cap\mathcal{C}^c\right)\right|,
\end{aligned} \tag{5.67}$$

and applying (5.57) and (5.65) to the first two summands, while controlling both the “cis” and “cross-over” contributions to the third summand by Lemma B.1. □

5.4. Proof of Theorem 3.2

5.4.1. Collecting the bounds

First, suppose that $(\psi_0^\varepsilon)_{\varepsilon>0}$ is still a sequence of initial states obeying (2.160), such that the Fourier transform $\widehat{\psi}_{0,\sigma}^\varepsilon(k)$, $\sigma\in\{\pm\}$, vanishes for all $k\in\mathbb{R}^d$ with $|k| > L^{(0)}$ for some $L^{(0)} < \infty$, uniformly in $\varepsilon > 0$. Thanks to (4.7) and (4.13),

$$\begin{aligned}
&\mathbb{E}\left[\left|\mathcal{J}^\varepsilon\left(H^\varepsilon, \psi^\varepsilon, T, a, p\right) - \mathbb{E}\mathcal{J}^\varepsilon\left(H^\varepsilon, \psi^\varepsilon, T, a, p\right)\right|\right] \\
&= \lim_{R\to\infty}\mathbb{E}\left[\left|\mathcal{J}^\varepsilon\left(H^{\varepsilon,R}, \psi^\varepsilon, T, a, p\right) - \mathbb{E}\mathcal{J}^\varepsilon\left(H^{\varepsilon,R}, \psi^\varepsilon, T, a, p\right)\right|\right] \\
&\le \lim_{R\to\infty}\left(\mathrm{Var}\left(\langle \Psi'^\varepsilon_1, A^\varepsilon_{\overline{m}}\Psi^\varepsilon_1\rangle\right)\right)^{1/2} \\
&\quad+ \lim_{R\to\infty} 2\,\|a_{\overline{m}}\|_{C^0}\,\mathbb{E}\left[\|\Psi'^\varepsilon_1\|_{\mathcal{H}}\|\Psi^\varepsilon_2\|_{\mathcal{H}} + \|\Psi'^\varepsilon_2\|_{\mathcal{H}}\|\Psi^\varepsilon_1\|_{\mathcal{H}} + \|\Psi'^\varepsilon_2\|_{\mathcal{H}}\|\Psi^\varepsilon_2\|_{\mathcal{H}}\right].
\end{aligned} \tag{5.68}$$

For the object in the second last line of (5.68), by the Cauchy-Schwarz inequality and Lemma 5.1

$$\begin{aligned}
&\lim_{R\to\infty} \operatorname{Var}\left(\left\langle {\Psi'}^{\varepsilon}_1, A^{\varepsilon}_{\overline{m}} \Psi^{\varepsilon}_1 \right\rangle\right) \\
&\leq \lim_{R\to\infty} \binom{\overline{N}(\varepsilon)-1+\overline{m}}{\overline{m}}^2 \sum_{\substack{N\in\mathbb{N}_0^{2\overline{m}} \\ N^{(1)}+\ldots+N^{(\overline{m})}<\overline{N}(\varepsilon) \\ N^{(\overline{m}+1)}+\ldots+N^{(2\overline{m})}<\overline{N}(\varepsilon)}} \\
&\quad \operatorname{Var}\left\langle \psi^{\varepsilon}_0, F_{N^{(2\overline{m})}}\left(T^{(2\overline{m})}/\varepsilon; R, L^{(2\overline{m})}, \varepsilon\right)^* \left(\prod_{j=1}^{2\overline{m}-1} A^{\varepsilon}_j F_{N^{(j)}}\left(T^{(j)}/\varepsilon; R, L^{(j)}, \varepsilon\right)^{*(j)}\right) \psi^{\varepsilon}_0 \right\rangle_{\mathcal{H}} \\
&= \binom{\overline{N}(\varepsilon)-1+\overline{m}}{\overline{m}}^2 \sum_{\substack{N\in\mathbb{N}_0^{2\overline{m}} \\ N^{(1)}+\ldots+N^{(\overline{m})}<\overline{N}(\varepsilon) \\ N^{(\overline{m}+1)}+\ldots+N^{(2\overline{m})}<\overline{N}(\varepsilon)}} \sum_{S\in\pi^*_{\mathrm{conn}}(I(N,N))} \mathcal{V}\left(\psi^{\varepsilon}_0, \varepsilon, a, p, L^{(0)}, T/\varepsilon, N, S\right).
\end{aligned} \tag{5.69}$$

Whenever ξ is of class $(d+3,0)$, by Lemmas 5.2-5.5 and because $|\pi^*(I(N,N))| \leq (2|N|)! \leq (4\overline{N})!$, there is a C depending only on $\overline{m}$, $|T|$, $L^{(0)}$, C_{obs} and the statistics of ξ such that

$$\begin{aligned}
&\lim_{R\to\infty} \operatorname{Var}\left(\left\langle {\Psi'}^{\varepsilon}_1, A^{\varepsilon}_{\overline{m}} \Psi^{\varepsilon}_1 \right\rangle\right) \\
&\leq C^{\overline{N}+1}\overline{N}^{8\overline{N}} \|\psi^{\varepsilon}_0\|^4_{\mathcal{H}} \prod_{j=1}^{2\overline{m}-1} \|a_j\|^2_{C^0} \\
&\qquad |\log\varepsilon|^{4(\overline{N}+\overline{m})} \left(\varepsilon^{1/5} + \overline{N}^{4\overline{N}} \max_{D\in\{1,\ldots,4\overline{N}\}} \varepsilon^D D^{CD}\right)
\end{aligned} \tag{5.70}$$

for all $\varepsilon < 1/(4\langle C_{\mathrm{obs}}\rangle \overline{m})$. In case ξ is even of class $(d+2\overline{M}+7,4)$ one can also use all estimates from Section 4.5.2 for the remainder Ψ^{ε}_2. We have to choose $\overline{N}(\varepsilon) = \lceil a|\log\varepsilon|/\log|\log\varepsilon|\rceil$, as well as $\kappa = \varepsilon^{1-\vartheta}$ and $\overline{M}$ (the latter two only influence the estimate (4.215) for Ψ^{ε}_2) such that both (4.215) and (5.70) vanish in the $\varepsilon \to 0$ limit. To this end, we have to identify $a > 0$, $\vartheta \in (0,1)$ and $\overline{M} \in \mathbb{N}$ such that

$$\begin{aligned}
\frac{1}{5} - 12a &> 0 \\
\frac{1}{2} - 2(1-\vartheta) - 8a &> 0 \\
-2 + \overline{M}(1-\vartheta) - 8a &> 0 \\
-2(1-\vartheta) + a &> 0
\end{aligned} \tag{5.71}$$

simultaneously hold. Such a triple $(a, \vartheta, \overline{M})$ can be found whenever $\overline{M} \geq 257$. If $\overline{M}$ can be chosen arbitrarily large, and $(1-\vartheta)$ arbitrarily small, the optimal choice of a,

$a = 1/65$ yields a decay rate

$$\begin{aligned}
&\mathrm{Var}\left(\mathcal{J}^\varepsilon\left(H^\varepsilon, \psi^\varepsilon, T, a, p\right)\right) \\
\leq &2\left(\|\psi_0^\varepsilon\|_{\mathcal{H}}^2 \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^2}\right) \mathbb{E}\left[\left|\mathcal{J}^\varepsilon\left(H^\varepsilon, \psi^\varepsilon, T, a, p\right) - \mathbb{E}\mathcal{J}^\varepsilon\left(H^\varepsilon, \psi^\varepsilon, T, a, p\right)\right|\right] \\
\leq &C\left(\|\psi_0^\varepsilon\|_{\mathcal{H}}^4 \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^2}^2\right) \varepsilon^{1/131},
\end{aligned} \tag{5.72}$$

with a finite constant C that only depends on $\overline{m}$, $|T|$, $L^{(0)}$, C_{obs} and the statistics of ξ.

5.4.2. Extension to general initial data and test functions

Now that we have established

$$\lim_{\varepsilon\to 0} \mathrm{Var}\left(\mathcal{J}^\varepsilon\left(H^\varepsilon, \psi^\varepsilon, T, a, p\right)\right) = 0 \tag{5.73}$$

for initial states with uniformly compactly supported Fourier transforms, and functions $a_{j,\pm} : \mathbb{R}^d \to \mathbb{C}$ with two bounded derivatives, we can first relax the observables to be only bounded and continuous by invoking (4.256). In this step, however, we lose the explicit control on the convergence speed, (5.72). Next, the boundedness of the functions $a_{j,\pm}$ and the unitarity of the time evolution $\mathrm{e}^{-iH^\varepsilon t}$ imply that (5.73) holds for all sequences of initial states $(\psi_0^\varepsilon)_{\varepsilon>0}$ in $\mathcal{H}$ that fulfill (2.160) and (2.162).

To further generalize this result to observables $\mathfrak{a}_{j,\pm} \in \mathcal{F}L^1(C^0)$, recall that

$$\begin{aligned}
&\Big\langle \mathrm{e}^{-iH^\varepsilon T^{(\overline{m})}/\varepsilon} \prod_{j=1}^{\overline{m}-1} \left(Q^\varepsilon(\mathfrak{a}_j)\mathrm{e}^{-iH^\varepsilon T^{(j)}/\varepsilon}\right) \psi_0^\varepsilon, \\
&\qquad Q^\varepsilon(\mathfrak{a}_{\overline{m}})\mathrm{e}^{-iH^\varepsilon T^{(\overline{m})}/\varepsilon} \prod_{j=1}^{\overline{m}-1} \left(Q^\varepsilon(\mathfrak{a}_j)\mathrm{e}^{-iH^\varepsilon T^{(j)}/\varepsilon}\right) \psi_0^\varepsilon \Big\rangle_{\mathcal{H}} \\
&= \int_{\mathbb{R}^{d(2\overline{m}-1)}} \mathrm{d}p^{(1)} \ldots \mathrm{d}p^{(2\overline{m}-1)} \mathcal{J}^\varepsilon\left(H^\varepsilon, \psi_0^\varepsilon, T, \widehat{\mathfrak{a}}(p,\cdot), p\right),
\end{aligned} \tag{5.74}$$

where (5.73) helps us to control the variance of the integrand on the right side for given values of $p^{(j)}$. Thus,

$$\begin{aligned}
&\mathrm{Var}\Big\langle \mathrm{e}^{-iH^\varepsilon T^{(\overline{m})}/\varepsilon} \prod_{j=1}^{\overline{m}-1} \left(Q^\varepsilon(\mathfrak{a}_j)\mathrm{e}^{-iH^\varepsilon T^{(j)}/\varepsilon}\right) \psi_0^\varepsilon, \\
&\qquad Q^\varepsilon(\mathfrak{a}_{\overline{m}})\mathrm{e}^{-iH^\varepsilon T^{(\overline{m})}/\varepsilon} \prod_{j=1}^{\overline{m}-1} \left(Q^\varepsilon(\mathfrak{a}_j)\mathrm{e}^{-iH^\varepsilon T^{(j)}/\varepsilon}\right) \psi_0^\varepsilon \Big\rangle_{\mathcal{H}} \\
&\leq \left(\int_{\mathbb{R}^{d(2\overline{m}-1)}} \mathrm{d}p^{(1)} \ldots \mathrm{d}p^{(2\overline{m}-1)} \left[\mathrm{Var}\left(\mathcal{J}^\varepsilon\left(H^\varepsilon, \psi_0^\varepsilon, T, \widehat{\mathfrak{a}}(p,\cdot), p\right)\right)\right]^{1/2}\right)^2 \to 0
\end{aligned} \tag{5.75}$$

as $\varepsilon \to 0$ by dominated convergence for the p integral, because

$$[\mathrm{Var}\left(\mathcal{J}^{\varepsilon}\left(H^{\varepsilon}, \psi_0^{\varepsilon}, T, \widehat{\mathfrak{a}}(p,\cdot), p\right)\right)]^{1/2} \le \prod_{j=1}^{\overline{m}} \sup_{k,\sigma} \left|\widehat{\mathfrak{a}}_{j,\sigma}(p^{(j)},k)\right| \prod_{j=\overline{m}+1}^{2\overline{m}-1} \sup_{k,\sigma} \left|\widehat{\mathfrak{a}}_{2\overline{m}-j,\sigma}(p^{(j)},k)\right|, \tag{5.76}$$

with the right side ε-independent and integrable in the p variables. The only remaining task is to replace observables $\mathfrak{a}_{j,\sigma} \in \mathcal{F}L^1(C^0)$ by general $\mathfrak{a}_{j,\sigma} \in \mathfrak{X}_{\mathrm{IR}}$, and Q^{ε} by $Q^{\varepsilon}_{\mathrm{IR}}$ in equation (5.75). This is achieved in a fashion very similar to Section 4.8; we introduce a small cut-off parameter $\lambda > 0$ as in (4.266), to obtain

$$\begin{aligned} \widehat{\psi}^{\varepsilon,\lambda}_{>,0,\sigma}(k) &= \left(1 - \varphi(|k|/\lambda)\right) \widehat{\psi}^{\varepsilon}_{0,\sigma}(k), \\ \widehat{f}^{\varepsilon,\lambda}_{0,\sigma}(k) &= \varphi(|k|/\lambda) \widehat{\psi}^{\varepsilon}_{0,\sigma}(k). \end{aligned} \tag{5.77}$$

The functions

$$\mathfrak{a}^{\mathrm{micro}}_{j,\sigma,\lambda}(x,k) = \mathfrak{a}^{\mathrm{micro}}_{j,\sigma}(x,k)\left(1 - \varphi(4|k|/\lambda)\right) \tag{5.78}$$

are in $\mathcal{F}L^1(C^0)$ for $\lambda > 0$, and by a straightforward generalization of (4.271) to multiple observation times, as well as (5.75), we have for the large wave-numbers that

$$\begin{aligned} &\lim_{\varepsilon\to 0} \mathrm{Var}\Bigg(\Bigg\langle \mathrm{e}^{-iH^{\varepsilon}T^{(\overline{m})}/\varepsilon} \prod_{j=1}^{\overline{m}-1} \left(Q^{\varepsilon}_{\mathrm{IR}}(\mathfrak{a}_j) \mathrm{e}^{-iH^{\varepsilon}T^{(j)}/\varepsilon}\right) \psi^{\varepsilon,\lambda}_{>,0}, \\ &\qquad Q^{\varepsilon}_{\mathrm{IR}}(\mathfrak{a}_{\overline{m}}) \mathrm{e}^{-iH^{\varepsilon}T^{(\overline{m})}/\varepsilon} \prod_{j=1}^{\overline{m}-1} \left(Q^{\varepsilon}_{\mathrm{IR}}(\mathfrak{a}_j) \mathrm{e}^{-iH^{\varepsilon}T^{(j)}/\varepsilon}\right) \psi^{\varepsilon,\lambda}_{>,0} \Bigg\rangle_{\mathcal{H}}\Bigg) \\ =&\lim_{\varepsilon\to 0} \mathrm{Var}\Bigg(\Bigg\langle \mathrm{e}^{-iH^{\varepsilon}T^{(\overline{m})}/\varepsilon} \prod_{j=1}^{\overline{m}-1} \left(Q^{\varepsilon}(\mathfrak{a}^{\mathrm{micro}}_{j,\lambda}) \mathrm{e}^{-iH^{\varepsilon}T^{(j)}/\varepsilon}\right) \psi^{\varepsilon,\lambda}_{>,0}, \\ &\qquad Q^{\varepsilon}(\mathfrak{a}^{\mathrm{micro}}_{\overline{m},\lambda}) \mathrm{e}^{-iH^{\varepsilon}T^{(\overline{m})}/\varepsilon} \prod_{j=1}^{\overline{m}-1} \left(Q^{\varepsilon}(\mathfrak{a}^{\mathrm{micro}}_{j,\lambda}) \mathrm{e}^{-iH^{\varepsilon}T^{(j)}/\varepsilon}\right) \psi^{\varepsilon,\lambda}_{>,0} \Bigg\rangle_{\mathcal{H}}\Bigg) \\ =&0 \end{aligned} \tag{5.79}$$

for any fixed $\lambda > 0$. For the small and intermediate (a distinction is not necessary here) wave-numbers, on the other hand, iteratively applying (4.278) shows that both

$$\begin{aligned} &\lim_{\lambda\to 0} \limsup_{\varepsilon\to 0} \mathrm{Var}\Bigg(\Bigg\langle \mathrm{e}^{-iH^{\varepsilon}T^{(\overline{m})}/\varepsilon} \prod_{j=1}^{\overline{m}-1} \left(Q^{\varepsilon}_{\mathrm{IR}}(\mathfrak{a}_j) \mathrm{e}^{-iH^{\varepsilon}T^{(j)}/\varepsilon}\right) f^{\varepsilon,\lambda}_0, \\ &\qquad Q^{\varepsilon}_{\mathrm{IR}}(\mathfrak{a}_{\overline{m}}) \mathrm{e}^{-iH^{\varepsilon}T^{(\overline{m})}/\varepsilon} \prod_{j=1}^{\overline{m}-1} \left(Q^{\varepsilon}_{\mathrm{IR}}(\mathfrak{a}_j) \mathrm{e}^{-iH^{\varepsilon}T^{(j)}/\varepsilon}\right) f^{\varepsilon,\lambda}_0 \Bigg\rangle_{\mathcal{H}}\Bigg) = 0 \end{aligned} \tag{5.80}$$

and cross-terms of the form

$$\lim_{\lambda\to 0}\limsup_{\varepsilon\to 0}\operatorname{Var}\Bigg(\Bigg\langle \mathrm{e}^{-iH^\varepsilon T^{(\overline{m})}/\varepsilon}\prod_{j=1}^{\overline{m}-1}\Big(Q^\varepsilon_{\mathrm{IR}}(\mathfrak{a}_j)\mathrm{e}^{-iH^\varepsilon T^{(j)}/\varepsilon}\Big)\,\psi^{\varepsilon,\lambda}_{>,0}, \\ Q^\varepsilon_{\mathrm{IR}}(\mathfrak{a}_{\overline{m}})\mathrm{e}^{-iH^\varepsilon T^{(\overline{m})}/\varepsilon}\prod_{j=1}^{\overline{m}-1}\Big(Q^\varepsilon_{\mathrm{IR}}(\mathfrak{a}_j)\mathrm{e}^{-iH^\varepsilon T^{(j)}/\varepsilon}\Big)\,f^{\varepsilon,\lambda}_0\Bigg\rangle_{\mathcal{H}}\Bigg)=0 \tag{5.81}$$

vanish. This proves Theorem 3.2.

6. Higher moments and almost sure convergence

6.1. Higher moment estimates

As in the previous chapters, instead of directly considering observables in $\mathcal{F}L^1(C^0)$, we start out from operators $A_j^\varepsilon : \mathcal{H} \to \mathcal{H}$, $j \in \{1, ..., 2\overline{m} - 1\}$ defined by (4.2) from C_b^2 functions $a_j : \mathbb{R}^d \to \mathbb{C}^2$. Also, we consider initial states $(\psi_0^\varepsilon)_{\varepsilon>0}$ with bounded $\mathcal{H}$-norm (2.160) and Fourier transform vanishing outside a ball around the origin of radius $L^{(0)}$. Instead of the variance

$$\mathbb{E}\left[|\mathcal{J}^\varepsilon - \mathbb{E}[\mathcal{J}^\varepsilon]|^2\right] \tag{6.1}$$

of observables $\mathcal{J}^\varepsilon$ as defined in (4.5), we now want to control

$$\mathbb{E}\left[|\mathcal{J}^\varepsilon - \mathbb{E}[\mathcal{J}^\varepsilon]|^{2l}\right] \tag{6.2}$$

for arbitrary large $l \in \mathbb{N}$. Such an estimate has already been found for the discrete random Schrödinger equation in Theorem 2.2 of [8] — essentially, a bound of type ε^C was derived for (6.2), with $C > 0$ independent of l. This, however, is a direct consequence of the $l = 1$ variance case together with the boundedness of the observable. But for reasonably nice random variables $\mathcal{J}^\varepsilon$, one should actually hope for (6.2) to be of order ε^{Cl}; an improvement that will prove crucial for the establishment of almost sure convergence, Section 6.2. In [7], such a scaling of the higher moments was derived from the assumption of a Gaussian random potential, which made the terms of the Duhamel expansion polynomials of Gaussian variables. Then, the hypercontractivity properties of the normal distribution, [25], provided a control on higher moments (6.2) in terms of lower moments (6.1). Here, the random fluctuations ξ of the wave speed cannot be Gaussian, as they have to be bounded from below; even-degree polynomials of a suitable Gaussian field would be admissible on purely mathematical grounds, but unbounded and thus physically still hard to justify as a wave speed. If we want to stick with our quite general choice of ξ, we cannot "a posteriori" upgrade a variance estimate to higher moment bounds by mere probabilistic methods, but rather have to undertake the full graph expansion of (6.2). This expansion was already performed in [8], so we can proceed somewhat similar to define the graphs in the expansion, but will then have to find much finer estimates of their respective contributions.

6.1.1. Amplitudes for the main terms

First, we want to calculate the $R \to \infty$ limit of

$$\mathbb{E}\left|\left\langle \psi_0^\varepsilon, F_{N(2\overline{m})}\left(t^{(2\overline{m})}; R, L^{(2\overline{m})}, \varepsilon\right)^* \left(\prod_{j=1}^{2\overline{m}-1} A_j^\varepsilon F_{N^{(j)}}\left(t^{(j)}; R, L^{(j)}, \varepsilon\right)^{*(j)}\right) \psi_0^\varepsilon \right\rangle_{\mathcal{H}} \right.$$
$$\left. -\mathbb{E}\left\langle \psi_0^\varepsilon, F_{N(2\overline{m})}\left(t^{(2\overline{m})}; R, L^{(2\overline{m})}, \varepsilon\right)^* \left(\prod_{j=1}^{2\overline{m}-1} A_j^\varepsilon F_{N^{(j)}}\left(t^{(j)}; R, L^{(j)}, \varepsilon\right)^{*(j)}\right) \psi_0^\varepsilon \right\rangle_{\mathcal{H}} \right|^{2l} . \tag{6.3}$$

Analogous to Section 5.1, the wave now travels in from the left and from the right of each of the $2l$ individual one-particle lines, undergoing the same number of scatterings on each line; therefore, the scattering events can be indexed by the index set $I(N;2l)$, which consists of all (j,n,r) with $(j,n) \in I(N)$ and $r \in \{1,...,2l\}$. As in [8], define the *two-connected partitions* as the set of all partitions of $I(N;2l)$, so that every one-particle line is connected to at least one other one-particle line (note that this does not coincide with the standard definition of k-connectedness for graphs, [11]); as always, the star indicates that we do not consider partitions containing isolated elements

$$\begin{aligned} \pi^*_{\text{conn}}(I(N;2l)) = \{S \in \pi(I(N;2l)) : \\ &|A| \geq 2 \ \forall A \in S, \\ &\text{and } \forall r \in \{1,...,2l\} \ \exists r' \neq r, A \in S : A_r \neq \emptyset \neq A_{r'} \\ &\} . \end{aligned}$$

Here we have denoted for a subset $A \subset I(N;2l)$ the restriction of A_r of A to the r-th one-particle line,

$$A_r = \{(j,n) \in I(N) : (j,n,r) \in A\} . \tag{6.4}$$

There is a bijection $S \leftrightarrow (P,(S_B)_{B\in P})$ that maps $S \in \pi^*_{\text{conn}}(I(N;2l))$ to a $P \in \pi^*(\{1,...,2l\})$ (which is a partition of $\{1,...,2l\}$ withouth isolated elements) and a collection $(S_B)_{B\in P}$, with each S_B a partition from $\pi^*_{\text{full}}(I(N;B))$. Here

$$I(N;B) = \{(j,n,r) : (j,n) \in I(N), r \in B\} , \tag{6.5}$$

and $\pi^*_{\text{full}}(I(N;B))$ comprises all partitions of the set $I(N;B)$ which

- do not contain one-element clusters,
- do not decompose the set of involved one-particle-lines, or equivalentely, the set B, into several connectivity components.

In analogy to the $l = 1$ (variance) case, the momentum and sign after the n-th scattering event in the j-th time interval on the r-th one-particle line are denoted by $k_{n,r}^{(j)}$ and $\sigma_{n,r}^{(j)}$, respectively, indexed by

$$I_0(N;2l) = \{(j,n,r) : (j,n) \in I_0(N), r \in \{1,...,2l\}\} , \tag{6.6}$$

and to account for complex conjugation in every second of the $2l$ factors of (6.3), we define

$$\begin{aligned}\tau_r &= -(-1)^r \quad (r \in \{1, ..., 2l\}) \\ \theta_{n,r}^{(j)} &= \tau_r \left(k_{n,r}^{(j)} - k_{n-1,r}^{(j)}\right) \quad ((j,n,r) \in I_0(N;2l)) \\ \tau_r^{(j)} &= \tau_r \tau^{(j)}.\end{aligned} \tag{6.7}$$

With the usual caveats in case $N^{(j)} = 0$ for some j, we extend the notation of the propagator $\mathcal{P}_r(k_r, \sigma_r, \alpha_r, \gamma)$ from (5.9) to all $r \in \{1, ..., 2l\}$, and denote by $\mathcal{A}_r(k_r, \sigma_r, p, \varepsilon)$ the analogue of (5.7) for r odd, and (5.8) for r even.

Lemma 6.1. *Whenever ξ is of class $(d+1, 0)$, $m > d+1$, and $N \in \mathbb{N}_0^{2\overline{m}}$ such that (5.1) holds, one has the representation*

$$\begin{aligned}&\lim_{R\to\infty} \mathbb{E}\left|\left\langle \psi_0^\varepsilon, F_{N^{(2\overline{m})}}\left(t^{(2\overline{m})}; R, L^{(2\overline{m})}, \varepsilon\right)^* \left(\prod_{j=1}^{2\overline{m}-1} A_j^\varepsilon F_{N^{(j)}}\left(t^{(j)}; R, L^{(j)}, \varepsilon\right)^{*(j)}\right) \psi_0^\varepsilon\right\rangle_{\mathcal{H}}\right. \\ &\qquad \left. - \mathbb{E}\left\langle \psi_0^\varepsilon, F_{N^{(2\overline{m})}}\left(t^{(2\overline{m})}; R, L^{(2\overline{m})}, \varepsilon\right)^* \left(\prod_{j=1}^{2\overline{m}-1} A_j^\varepsilon F_{N^{(j)}}\left(t^{(j)}; R, L^{(j)}, \varepsilon\right)^{*(j)}\right) \psi_0^\varepsilon\right\rangle_{\mathcal{H}}\right|^{2l} \\ &= \sum_{S \in \pi^*_{\mathrm{conn}}(I(N;2l))} \mathcal{V}_{2l}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right) \\ &= \sum_{P \in \pi^*(\{1,...,2l\})} \prod_{B \in P} \left(\sum_{S_B \in \pi^*_{\mathrm{full}}(I(N;B))} \mathcal{V}_B\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S_B\right)\right),\end{aligned} \tag{6.8}$$

with the amplitudes $\mathcal{V}_B$ given by

$$\begin{aligned}
&\mathcal{V}_B\left(\psi_0^\varepsilon,\varepsilon,a,p,L^{(0)},t,N,S_B\right)\\
&=\varepsilon^{|N||B|/2}\pi^{|N||B|}\sum_{\substack{\sigma_{n,r}^{(j)}\in\{\pm\}\\ \forall(j,n,r)\in I_0(N;B)}}\int_{\mathbb{R}^{(|N|+2\overline{m})|B|d}}\prod_{(j,n,r)\in I_0(N;B)}\mathrm{d}k_{n,r}^{(j)}\\
&\qquad\prod_{A\in S_B}\left(\delta\left(\sum_{(j,n,r)\in A}\theta_{n,r}^{(j)}\right)\widehat{g}_{|A|}\left(\theta_{n,r}^{(j)}:(j,n,r)\in A^\sharp\right)\right)\\
&\qquad\times\prod_{r\in B}\mathcal{A}_r\left(k_r,\sigma_r,p,\varepsilon\right)\\
&\qquad\times\int_{\mathbb{R}^{2\overline{m}|B|}}\prod_{r\in B}\prod_{j=1}^{2\overline{m}}\left(e^{\gamma t^{(j)}}\frac{\mathrm{d}\alpha_r^{(j)}}{2\pi}e^{-i\alpha_r^{(j)}t^{(j)}}\right)\\
&\qquad\quad\prod_{r\in B}\mathcal{P}_r\left(k_r,\sigma_r,\alpha_r,\gamma\right)\\
&\qquad\times\prod_{\substack{r\in B\\ r\ odd}}\widehat{\psi}^\varepsilon_{0,\sigma_{0,r}^{(1)}}\left(k_{0,r}^{(1)}\right)\overline{\widehat{\psi}^\varepsilon_{0,\sigma^{(2\overline{m})}_{N^{(2\overline{m})},r}}}\left(k^{(2\overline{m})}_{N^{(2\overline{m})},r}\right)\\
&\qquad\times\prod_{\substack{r\in B\\ r\ even}}\overline{\widehat{\psi}^\varepsilon_{0,\sigma_{0,r}^{(1)}}}\left(k_{0,r}^{(1)}\right)\widehat{\psi}^\varepsilon_{0,\sigma^{(2\overline{m})}_{N^{(2\overline{m})},r}}\left(k^{(2\overline{m})}_{N^{(2\overline{m})},r}\right).
\end{aligned}\tag{6.9}$$

instead of the special case $\mathcal{V}$ from (5.11). If $S\in\pi^*_{\text{conn}}(I(N;2l))$ corresponds to $P\in\pi^*(\{1,...,2\overline{m}\})$ and partitions $S_B\in\pi^*_{\text{full}}(I(N;B))$, $B\in P$, its amplitude is determined from the $\mathcal{V}_B$ by

$$\mathcal{V}_{2l}\left(\psi_0^\varepsilon,\varepsilon,a,p,L^{(0)},t,N,S\right)=\prod_{B\in P}\mathcal{V}_B\left(\psi_0^\varepsilon,\varepsilon,a,p,L^{(0)},t,N,S_B\right).\tag{6.10}$$

This lemma applies to cases with one or several $N^{(j)}=0$ in the same sense as Lemma 5.1.

Proof. The idea behind the structure of the individual amplitudes $\mathcal{V}_B$ is the same as for Lemma 5.1; however, one has to make sure that the sum of amplitudes is taken over the correct set $\pi^*_{\text{conn}}(I(N;2l))$ of partitions S. In fact, for the discrete, Gaussian case, it has already been observed in equation (56) of [8] that the two-connected pairings are the right choice. For our more general case, introduce random variables $X_1=X_3=...X_{2l-1}$ and $X_2=X_4=...=X_{2l}$ with

$$X_1=\overline{X_2}=\left\langle\psi_0^\varepsilon,F_{N^{(2\overline{m})}}\left(t^{(2\overline{m})};R,L^{(2\overline{m})},\varepsilon\right)^*\left(\prod_{j=1}^{2\overline{m}-1}A_j^\varepsilon F_{N^{(j)}}\left(t^{(j)};R,L^{(j)},\varepsilon\right)^{*(j)}\right)\psi_0^\varepsilon\right\rangle_{\mathcal{H}}.\tag{6.11}$$

As the $X_r - \mathbb{E}X_r$ are centered, the moments-to-cumulants formula, [37], Chapter II, §12, equation (46), implies

$$\mathbb{E}\left[\prod_{r=1}^{2l}(X_r - \mathbb{E}X_r)\right] = \sum_{P\in\pi^*(\{1,...,2l\})}\prod_{B\in P}\mathbf{Cum}(X_r : r \in B), \tag{6.12}$$

$\pi^*(\{1, ..., 2l\})$ being the partitions of $\{1, ..., 2l\}$ without isolated elements. In particular, all sets B in (6.12) contain at least two elements, $|B| \geq 2$. It follows by an easy induction argument that

$$\mathbf{Cum}(X_r : r \in B) = \sum_{S_B\in\pi^*_{\text{full}}(I(N);B)} \mathcal{V}_B\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S_B\right), \tag{6.13}$$

with $\pi^*_{\text{full}}(I(N); B)$ defined as above. This proves the lemma. □

6.1.2. Star decomposition

As shown in Figure 6.1, from a given partition $S \in \pi^*_{\text{conn}}(I(N; 2l))$, one can draw a graph $\mathscr{G} = \mathscr{G}(S)$ with vertex set $\{1, ..., 2l\}$ by connecting two vertices $r \neq r'$ by an edge if and only if there is an $A \in S$ with $A_r \neq \emptyset \neq A_{r'}$. The graph $\mathscr{G}$ comprises a set P of different connectivity components $\mathscr{B} \in P$; if we write $B = V(\mathscr{B})$ for the vertex set of $\mathscr{B}$, these P and B just correspond to the ones from the last line of (6.8)). Each connectivity component includes at least two vertices, $|B| \geq 2$. For a fixed $\mathscr{B} \in P$, pick an arbitrary spanning tree $\mathscr{T}$ like in Figure 6.2, let $b \in B$ be a leaf of $\mathscr{T}$, and assign to all $r \in B$ the rank

$$w(r) = \text{dist}(r, b), \tag{6.14}$$

with dist denoting the distance in $\mathscr{T}$. This gives rise to the two disjoint sets of edges in $\mathscr{T}$,

$$\begin{aligned} E_{odd} &= \{xy \in E(\mathscr{T}) : 0 < w(x) < w(y), w(x) \text{ odd}\}, \\ E_{even} &= \{xy \in E(\mathscr{T}) : 0 < w(x) < w(y), w(x) \text{ even}\}. \end{aligned} \tag{6.15}$$

As b is a leaf, we have $|E_{odd}| + |E_{even}| = |B| - 2$; we choose E_+ to be the larger of those two sets, in case of equality, we arbitrarily set $E_+ = E_{even}$. We call all edges in E_+ as well as the one edge starting at b *strong*, all other edges of $\mathscr{T}$ are *weak*. There are then at least $|B|/2$ strong edges, and removing the weak edges will reduce $\mathscr{T}$ to a collection of $\bar{s}$ stars, each consisting of a center vertex $c_s \in B$ $s \in \{1, ..., \bar{s}\}$, which is connected (by strong edges) to $e_s \in \mathbb{N}_0$ periphery vertices, which do not have (strong) edges among each other. Each periphery vertex uniquely corresponds to a strong edge, so

$$\sum_{s=1}^{\bar{s}} e_s \geq |B|/2. \tag{6.16}$$

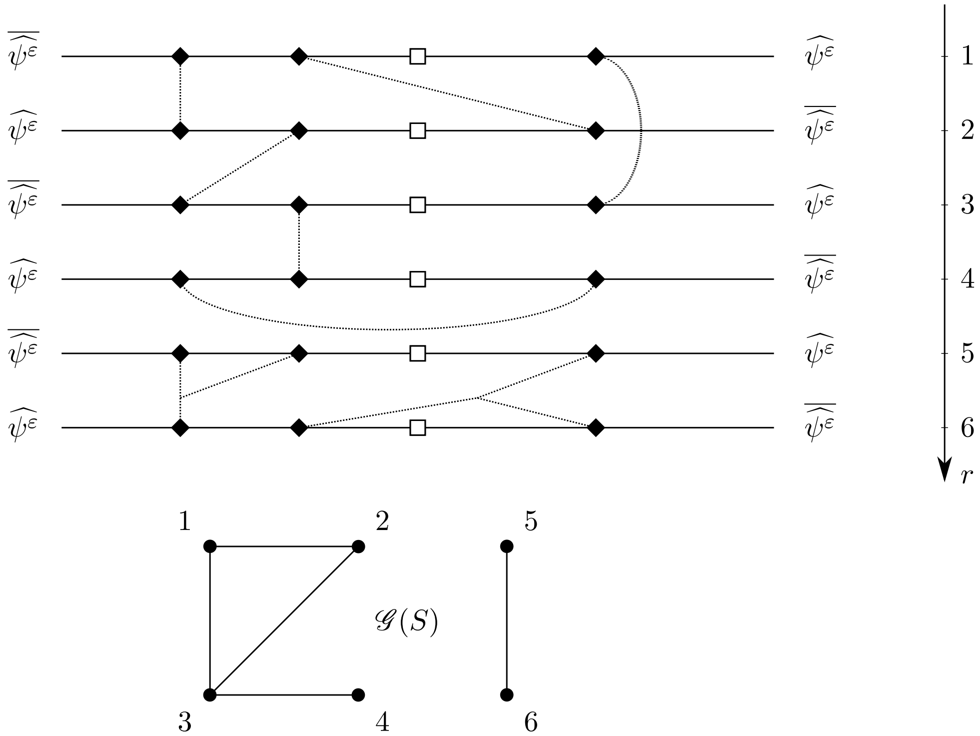

Figure 6.1.: *Top:* For $\overline{m} = 1$ and $N = (1, 2)$ and $l = 3$, this is a graph resulting from a partition $S \in \pi^*_{\text{conn}}(I(N; 2l))$ which consists of pairs and triplets. As in previous figures, the observable is indicated by an empty square, the black diamonds are scattering events with cut-off functions present. All scattering events in the same cluster of S are connected by dotted lines. The partition of $\{1, ..., 6\}$ assigned to S is $P = \{\{1, 2, 3, 4\}, \{5, 6\}\}$. *Below,* the graph $\mathscr{G}(S)$ is obtained by shrinking each one-particle line into a vertex (dark bullets) and connecting vertices whenever a cluster from S connects the corresponding one-particle lines.

The definition of center and periphery is unique whenever $e_s \geq 2$, for $e_s = 1$ (when the star is actually just a pair), one can arbitrarily pick one vertex as the "center". There may also be isolated centers without a periphery, $e_s = 0$. We then have a decomposition of B into

$$B = \bigcup_{s=1}^{\overline{s}} K_s, \tag{6.17}$$

each K_s containing all vertices of a star.

We factorize the right hand side of

$$\left| \mathcal{V}_B \left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S_B \right) \right|$$

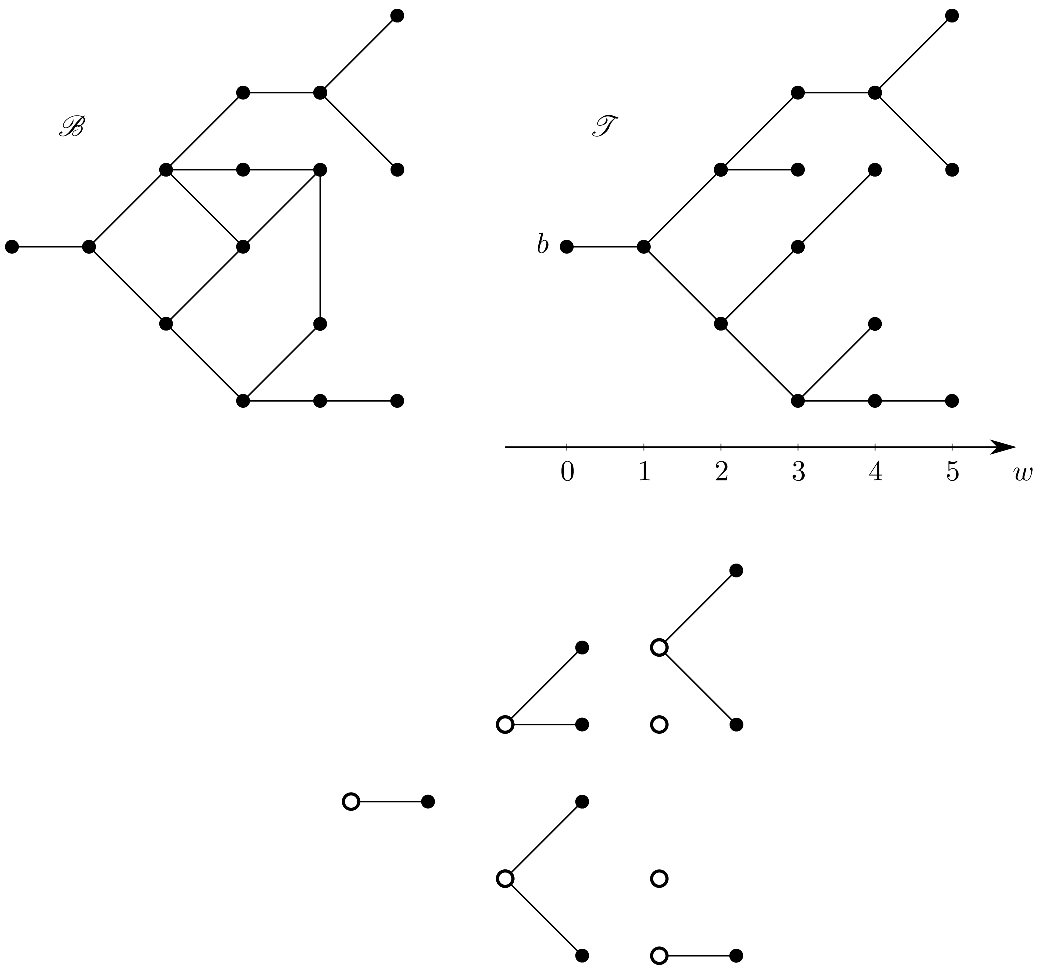

Figure 6.2.: *Top left:* A connectivity component $\mathscr{B}$ with $|B| = 15$ vertices. Every dark bullet symbolizes a one-particle-line. *Top right:* A spanning tree $\mathscr{T}$ has been extracted, we have chosen a leaf b and assigned a rank w to each vertex. In this case, $|E_{odd}| = 6$, $|E_{even}| = 7$, so $E_+ = E_{even}$. *Bottom:* We have removed all but the strong edges and thus isolated $\overline{s} = 7$ stars, with the center vertices indicated by circles. Note that two of the stars consist only of their respective center. The overall number of periphery vertices is $\sum_{s=1}^{\overline{s}} e_s = 8 \geq |B|/2$.

$$
\begin{aligned}
&\leq \varepsilon^{|N||B|/2}\pi^{|B||N|}e^{2|B|\gamma|t|}\prod_{j=1}^{2\overline{m}-1}\|a_j\|_{C^0}^{|B|} \sum_{\substack{\sigma_{n,r}^{(j)}\in\{\pm\} \\ \forall (j,n)\in I_0(N), r\in B}} \\
&\qquad \int_{\mathbb{R}^{|B|(|N|+2\overline{m})d}} \prod_{(j,n)\in I_0(N), r\in B} \mathrm{d}k_{n,r}^{(j)} \\
&\qquad\qquad \prod_{A\in S_B}\left(\delta\left(\sum_{(j,n,r)\in A}\theta_{n,r}^{(j)}\right)\left|\widehat{g}_{|A|}\left(\theta_{n,r}^{(j)} : (j,n,r)\in A^\sharp\right)\right|\right) \\
&\qquad\qquad \times \prod_{j=1}^{2\overline{m}-1}\prod_{r\in B}\left(\delta\left(k_{0,r}^{(j+1)} - k_{N^{(j)},r}^{(j)} - \varepsilon p^{(j)}\right)\right) \\
&\qquad\qquad \times \int_{\mathbb{R}^{2|B|\overline{m}}}\prod_{r\in B}\prod_{j=1}^{2\overline{m}}\left(\frac{\mathrm{d}\alpha_r^{(j)}}{2\pi}\right) \\
&\qquad\qquad\qquad \prod_{r\in B}\left|\mathcal{P}_r\left(k_r,\sigma_r,\alpha_r,\gamma\right)\right| \\
&\qquad\qquad \times \prod_{r\in B}\left(\left|\widehat{\psi}_0^\varepsilon\left(k_{0,r}^{(1)}\right)\right|\left|\widehat{\psi}_0^\varepsilon\left(k_{N^{(2\overline{m})},r}^{(2\overline{m})}\right)\right|\right)
\end{aligned} \tag{6.18}
$$

into the contributions of the individual stars by defining

Definition 6.1. For a given B, and $s \in \{1, ..., \overline{s}\}$, define the set of clusters in S_B that "reach up to but no further" than the star K_s

$$S_B(s) = \{A \in S_B : \exists r \in K_s : A_r \neq \emptyset \wedge A_{r'} = \emptyset \forall r' \in K_{s'}, \forall s' > s\}, \tag{6.19}$$

and assign to K_s the two values

$$
\begin{aligned}
d_s &= \sharp S_B(s), \\
f_s &= |N|(e_s + 1) - d_s.
\end{aligned} \tag{6.20}
$$

For a cluster $A \in S_B$, set the sub-cluster

$$A(s) = \{(j,n,r) \in A : r \in K_s\}. \tag{6.21}$$

In case $A \in S_B(s)$, it has the transfer momentum

$$u_A = \sum_{\substack{(j,n,r)\in A \\ r\in K_{s'}, s'<s}} \theta_{n,r}^{(j)} \tag{6.22}$$

associated to it. For K_s, there are d_s transfer momenta in total, which we collect in

$$\mathfrak{u}(s) = (u_A : A \in S_B(s)) \in \mathbb{R}^{d\cdot d_s}. \tag{6.23}$$

For $s \in \{1, ..., \overline{s}\}$ and $\mathfrak{u} \in \mathbb{R}^{d \cdot d_s}$, set

$$\begin{aligned}
&\mathcal{V}_{K_s}\left(\psi_0^\varepsilon, \varepsilon, \gamma, p, L^{(0)}, t, N, S_B, \mathfrak{u}\right) \\
&= \sup_{\sigma \in \{\pm\}^{(|N|+2\overline{m})(e_s+1)}} \\
&\qquad \int_{\mathbb{R}^{(e_s+1)(|N|+2\overline{m})d}} \prod_{(j,n)\in I_0(N), r\in K_s} \mathrm{d}k_{n,r}^{(j)} \prod_{\substack{(j,n)\in I(N)\\ r\in K_s}} \left\langle \theta_{n,r}^{(j)} \right\rangle^{-d-3} \\
&\qquad\qquad \prod_{A\in S_B(s)} \left(\delta\left(u_A + \sum_{(j,n,r)\in A(s)} \theta_{n,r}^{(j)}\right)\right) \\
&\qquad\qquad \times \prod_{j=1}^{2\overline{m}-1} \prod_{r\in K_s} \left(\delta\left(k_{0,r}^{(j+1)} - k_{N^{(j)},r}^{(j)} - \varepsilon p^{(j)}\right)\right) \\
&\qquad\qquad \times \int_{\mathbb{R}^{2(e_s+1)\overline{m}}} \prod_{r\in K_s} \prod_{j=1}^{2\overline{m}} \left(\frac{\mathrm{d}\alpha_r^{(j)}}{2\pi}\right) \\
&\qquad\qquad\qquad \prod_{r\in K_s} \left|\mathcal{P}_r\left(k_r, \sigma_r, \alpha_r, \gamma\right)\right| \\
&\qquad\qquad \times \prod_{r\in K_s} \left(\left|\widehat{\psi}_0^\varepsilon\left(k_{0,r}^{(1)}\right)\right| \left|\widehat{\psi}_0^\varepsilon\left(k_{N^{(2\overline{m})},r}^{(2\overline{m})}\right)\right|\right)
\end{aligned} \tag{6.24}$$

One can argue similar to equation (107) of [8] to find that there is a constant $C < \infty$ only depending on dimension d such that

$$\begin{aligned}
&\left|\mathcal{V}_B\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S_B\right)\right| \\
&\quad \le \varepsilon^{|B||N|/2} C^{|B|(|N|+2\overline{m})} \varepsilon^{2|B||\gamma|t|} \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^0}^{|B|} \prod_{A\in S_B} \left\|g_{|A|}\right\|_{d+3} \\
&\qquad \times \prod_{s=1}^{\overline{s}} \sup_{\mathfrak{u}\in\mathbb{R}^{d\cdot d_s}} \mathcal{V}_{K_s}\left(\psi_0^\varepsilon, \varepsilon, \gamma, p, L^{(0)}, t, N, S_B, \mathfrak{u}\right).
\end{aligned} \tag{6.25}$$

This estimate corresponds to integrating out $\mathcal{V}_B$, starting from the star $K_{\overline{s}}$ down to K_1, bounding the factor contributed by each star K_s by maximizing $\mathcal{V}_{K_s}$ over all possible transfer momenta $\mathfrak{u} = \mathfrak{u}(s)$ coming in from "lower" stars $K_{s'}$, $s' < s$.

6.1.3. Bound for a single star

To refer to the one-particle lines within a star, we will no longer use the index $r \in K_s \subset B \subset \{1, ..., 2l\}$, but rather $\rho \in \{0, ..., e_s\}$. The index r can then be uniquely recovered as $r = r(\rho, s, B)$. For every K_s, we utilize the following nomenclature, which is also illustrated in Figure 6.3:

- $\rho = 0$ refers to the center of the star.

- the largest values of ρ, say $\rho \in \{\underline{e}_s + 1, ..., e_s\}$, with $\underline{e}_s \in \{0, ..., e_s\}$, are reserved for those peripheral one-particle lines which are connected to the center (i.e. 0-th) one-particle line by at least one cluster of size ≥ 3. This cluster may not be a subset of K_s, but may have its third (fourth,...) member in a different star!

- the peripheral one-particle lines with $\rho \in \{1, ..., \underline{e}_s\}$ are consequently connected to the center one-particle line *only by pairs* (but at least one pair, due to the definition of the star).

In a slight abuse of notation, we simply overload phrases like "the ρ-th one-particle line", indices (j, n, ρ), signs τ_ρ, momenta $k_{n,\rho}^{(j)}$, etc. to refer to the obvious objects in the context of K_s.

Lemma 6.2. *Let $d \geq 2$, $\gamma, \varepsilon \in (0, 1/2]$, and K_s be a star with $s \in \{1, ..., \overline{s}\}$, and the parameters $e_s, \underline{e}_s, d_s$ and f_s defined as above. Its contribution in (6.25) is bounded by*

$$\begin{aligned} &\sup_{\mathfrak{u} \in \mathbb{R}^{d \cdot d_s}} \mathcal{V}_{K_s}\left(\psi_0^\varepsilon, \varepsilon, \gamma, p, L^{(0)}, t, N, S_B, \mathfrak{u}\right) \\ &\quad \leq C^{(|N| + 2\overline{m})(e_s+1)} \left(\langle L^{(0)} \rangle + \overline{m}|N|\right)^{3|N|(e_s+1)} \|\psi_0^\varepsilon\|_{\mathcal{H}}^{2(e_s+1)} \\ &\qquad \times |\log \gamma|^{f_s + 2\overline{m}(e_s+1)} \gamma^{-d_s} \left(\gamma |\log \gamma|\right)^{\left\lceil \frac{\underline{e}_s - 1}{2} \right\rceil} \end{aligned} \tag{6.26}$$

for $d \geq 3$, and

$$\begin{aligned} &\sup_{\mathfrak{u} \in \mathbb{R}^{d \cdot d_s}} \mathcal{V}_{K_s}\left(\psi_0^\varepsilon, \varepsilon, \gamma, p, L^{(0)}, t, N, S_B, \mathfrak{u}\right) \\ &\quad \leq C^{(|N| + 2\overline{m})(e_s+1)} \left(\langle L^{(0)} \rangle + \overline{m}|N|\right)^{3|N|(e_s+1)} \|\psi_0^\varepsilon\|_{\mathcal{H}}^{2(e_s+1)} \\ &\qquad \times |\log \gamma|^{f_s + 2\overline{m}(e_s+1)} \gamma^{-d_s} \left(\sqrt{\gamma}\right)^{\left\lceil \frac{\underline{e}_s - 1}{2} \right\rceil} \end{aligned} \tag{6.27}$$

for $d = 2$, with a constant $C < \infty$ only depending on d.

Proof. As always, we simplify matters by assuming $N^{(j)} \geq 1$ for all $j \in \{1, ..., 2\overline{m}\}$. For each $\rho \in \{0, ..., e_s\}$, one can bound

$$\left|\widehat{\psi}_0^\varepsilon\left(k_{0,\rho}^{(1)}\right)\right| \left|\widehat{\psi}_0^\varepsilon\left(k_{N^{(2\overline{m})},\rho}^{(2\overline{m})}\right)\right| \leq \frac{1}{2}\left(\left|\widehat{\psi}_0^\varepsilon\left(k_{0,\rho}^{(1)}\right)\right|^2 + \left|\widehat{\psi}_0^\varepsilon\left(k_{N^{(2\overline{m})},\rho}^{(2\overline{m})}\right)\right|^2\right) \tag{6.28}$$

We shall only consider the contribution of $\left|\widehat{\psi}_0^\varepsilon\left(k_{0,\rho}^{(1)}\right)\right|^2$ to the last six lines of (6.24),

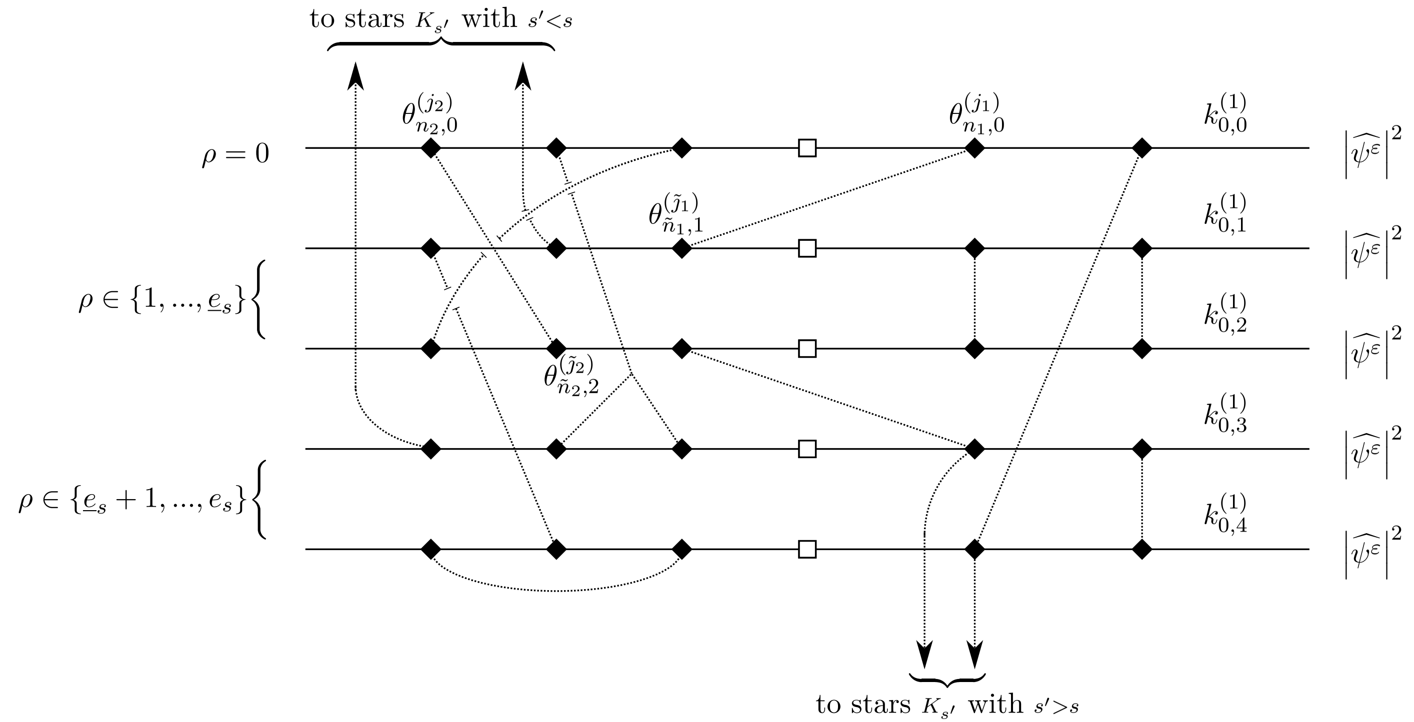

Figure 6.3.: The graph belonging to a single star K_s with $e_s = 4$ (so there are five one-particle lines) and $\underline{e}_s = 2$. The number of measurements is $\overline{m} = 1$, the number of scatterings $N = (2, 3)$. We have already invoked (6.28) and consider the case with the factors $|\widehat{\psi^\varepsilon}|^2$ sitting on the right side of each one-particle line. The uppermost one-particle line is the center of the star, $\rho = 0$. Next come the periphery lines connected to the center only by pairs, $\rho \in \{1, ..., \underline{e}_s\}$. The lowest part of the graph are the periphery one-particle lines that are connected to the center by at least one cluster of size ≥ 3. In addition to the center-periphery connections, there are also clusters connecting different periphery lines, and even clusters that run from K_s to other stars $K_{s'}$, $s' \neq s$. We only display the θ momenta that correspond to the "special" pairs $\{(j_\rho, n_\rho, 0), (\tilde{j}_\rho, \tilde{n}_\rho, \rho)\}$, $\rho \in \{1, ..., \underline{e}_s\}$, selected as in (6.30-6.31).

which, for fixed σ and $\mathfrak{u}$, and "in ρ notation", is bounded by

$$
\begin{aligned}
&C^{|N|(e_s+1)} \left(\langle L^{(0)}\rangle + \overline{m}|N|\right)^{3|N|(e_s+1)} \\
&\int_{\mathbb{R}^{(e_s+1)d}} \prod_{\rho=0}^{e_s} \left(\mathrm{d}k_{0,\rho}^{(1)} \left|\widehat{\psi}_0^{\varepsilon}\left(k_{0,\rho}^{(1)}\right)\right|^2 \right) \\
&\quad \int_{\mathbb{R}^{(e_s+1)|N|d}} \prod_{(j,n)\in I(N)} \prod_{\rho=0}^{e_s} \left(\mathrm{d}\theta_{n,\rho}^{(j)} \left\langle \theta_{n,\rho}^{(j)} \right\rangle^{-d} \left\langle k_{n,\rho}^{(j)} \right\rangle^{-2} \right) \\
&\quad\quad \int_{\mathbb{R}^{2(e_s+1)\overline{m}}} \prod_{\rho=0}^{e_s} \prod_{j=1}^{2\overline{m}} \left(\frac{\mathrm{d}\alpha_\rho^{(j)}}{2\pi} \right) \\
&\quad\quad\quad \prod_{A\in S_B(s)} \left(\delta\left(u_A + \sum_{(j,n,\rho)\in A(s)} \theta_{n,\rho}^{(j)} \right) \right) \\
&\quad\quad\quad\quad \prod_{(j,n)\in I_0(N)} \prod_{\rho=0}^{e_s} \left(\frac{1}{\left|\alpha_\rho^{(j)} - 2\pi\sigma_{n,\rho}^{(j)}\tau_\rho^{(j)}|k_{n,\rho}^{(j)}| + i\gamma\right|} \right),
\end{aligned}
\tag{6.29}
$$

with C an only dimension-dependent constant. Possibly after reordering the set $\{1, ..., \underline{e}_s\}$, one arrives at the situation shown in Figure 6.3 — for each $\rho \in \{1, ..., \underline{e}_s\}$ there is a pair

$$\{(j_\rho, n_\rho, 0); (\tilde{\jmath}_\rho, \tilde{n}_\rho, \rho)\} \in S_B \tag{6.30}$$

such that $(j_\rho, n_\rho) \succ (j_{\rho'}, n_{\rho'})$ for all $0 < \rho' < \rho$ and all pairs

$$\{(j_{\rho'}, n_{\rho'}, 0); (\tilde{\jmath}_{\rho'}, \tilde{n}_{\rho'}, \rho')\} \in S_B. \tag{6.31}$$

Of all (6.30) admissible in that sense, select the pair for which $(\tilde{\jmath}_\rho, \tilde{n}_\rho)$ is smallest with respect to $\prec$.

Definition 6.2. Let the index (j, n, r), $(j, n) \in I(N)$, $r \in K_s$ have K_s-internal index (j, n, ρ), $\rho \in \{0, ..., e_s\}$. We call (j, n, ρ) *dependent* if and only if $(j, n, \rho) \in A \in S_B(s)$, defined in Definition 6.1, and this A fulfills all of the following

i. $A_{\rho'}(s) = \emptyset$ for all $\rho' > \rho$, and

ii. $(j, n) = \max_\rho A(s)$.

Otherwise, (j, n, ρ) is called *free*. The *unique* dependent index belonging to $A \in S_B(s)$ is denoted (j^A, n^A, ρ^A). This way, the set $I(N) \times \{0, ..., e_s\}$ decomposes into d_s dependent and f_s free indices.

The $k_{n,\rho}^{(j)}$ in (6.29) are then functions of the integration variables and $\mathfrak{u}$ by

$$\begin{aligned} k_{n,\rho}^{(j)} = k_{0,\rho}^{(1)} + \varepsilon \sum_{\hat{\jmath}=1}^{j-1} p^{(\hat{\jmath})} &+ \tau_\rho \sum_{A \in S_B \setminus S_B(s)} \sum_{\substack{(\hat{\jmath},\hat{n},\rho) \in A(s): \\ (\hat{\jmath},\hat{n}) \preceq (j,n)}} \theta_{\hat{n},\rho}^{(\hat{\jmath})} \\ &+ \tau_\rho \sum_{\substack{A \in S_B(s): \\ \rho^A > \rho \vee (j^A, n^A) \succ (j,n)}} \sum_{\substack{(\hat{\jmath},\hat{n},\rho) \in A(s): \\ (\hat{\jmath},\hat{n}) \preceq (j,n)}} \theta_{\hat{n},\rho}^{(\hat{\jmath})} \\ &- \tau_\rho \sum_{\substack{A \in S_B(s): \\ \rho^A = \rho \wedge (j^A, n^A) \preceq (j,n)}} \left(u_A + \sum_{\substack{(\hat{\jmath},\hat{n},\hat{\rho}) \in A(s): \\ \hat{\rho} < \rho}} \theta_{\hat{n},\hat{\rho}}^{(\hat{\jmath})} \right). \end{aligned} \tag{6.32}$$

In (6.29), one can first take L^∞ bounds like (4.62) on every resolvent belonging to $k_{n,\rho}^{(j)}$, $(j,n) \in I(N)$ and $\rho \in \{0, ..., e_s\}$, with (j,n,ρ) dependent, except for $(\tilde{\jmath}_\rho, \tilde{n}_\rho, \rho)$, $\rho \in \{1, ..., \underline{e}_s\}$. Next, we completely integrate out the e_s-th to $\underline{e}_s + 1$-th one-particle lines by the analogue of Iteration 5.1. One is then left with the ρ-th one-particle lines, $\rho \in \{1, ..., \underline{e}_s\}$, which can be taken care of by starting from $\rho = \underline{e}_s$ and iterating

Iteration 6.1. *("Integrating out a star.")*

i. Assume we have not touched any of the $\tilde{\rho}$-th one-particle lines, $\tilde{\rho} \in \{1, ...\rho - 1\}$ yet, but that the ρ'-th one-particle lines, $\rho' \in \{\rho + 1, ...\underline{e}_s\}$ have been completely integrated out. The ρ-th one-particle line itself may have been integrated out partially, but all $\alpha_\rho^{(j)}$ with $j \leq \tilde{\jmath}_\rho$ and all free $\theta_{n,\rho}^{(j)}$, $(j,n) \preceq (\tilde{\jmath}_\rho, \tilde{n}_\rho)$ have not been integrated over yet.

ii. Apply the analogue of Iteration 5.1 to all remaining free $\theta_{n,\rho}^{(j)}$, $(j,n) \in I(N)$ with $(j,n) \succ (\tilde{\jmath}_\rho, \tilde{n}_\rho)$, and all $\alpha_\rho^{(j)}$, $j > \tilde{\jmath}_\rho$.

iii. Apply the analogue of Iteration 5.1 to all remaining free $\theta_{n,0}^{(j)}$, $(j,n) \in I(N)$ with $(j,n) \succ (j_\rho, n_\rho)$, and all $\alpha_0^{(j)}$, $j > j_\rho$.

iv. In case $\rho = 1$, just take the L^∞ estimate (4.62) of the resolvent belonging to $k_{\tilde{n}_1,1}^{(\tilde{\jmath}_1)}$, and the L^1 estimate (4.63) of the $k_{n_1,0}^{(j_1)}$ resolvent. Iteration 5.1 finishes the rest of the 1-st and 0-th one-particle line, and the integration is completed.

v. In case $\rho > 1$, the only k momenta still depending on $\theta_{n_\rho,0}^{(j_\rho)}$ are

$$\begin{aligned} k_{n_\rho,0}^{(j_\rho)} &= \tau_0 \theta_{n_\rho,0}^{(j_\rho)} + k_{n_\rho-1,0}^{(j_\rho)}, \\ k_{\tilde{n}_\rho,\rho}^{(\tilde{\jmath}_\rho)} &= -\tau_\rho \theta_{n_\rho,0}^{(j_\rho)} + k_{\tilde{n}_\rho-1,\rho}^{(\tilde{\jmath}_\rho)}, \end{aligned} \tag{6.33}$$

so taking the $\theta^{(j_\rho)}_{n_\rho,0}$ integral produces a factor

$$\int_{\mathbb{R}^d} \mathrm{d}\theta^{(j_\rho)}_{n_\rho,0} \frac{1}{\left|\alpha^{(j_\rho)}_0 - 2\pi\sigma^{(j_\rho)}_{n_\rho,0}\tau^{(j_\rho)}_0 \left|\tau_0\theta^{(j_\rho)}_{n_\rho,0} + k^{(j_\rho)}_{n_\rho-1,0}\right| + i\gamma\right|}
\times \frac{1}{\left|\alpha^{(\tilde{\jmath}_\rho)}_\rho - 2\pi\sigma^{(\tilde{\jmath}_\rho)}_{\tilde{n}_\rho,\rho}\tau^{(\tilde{\jmath}_\rho)}_\rho \left|-\tau_\rho\theta^{(j_\rho)}_{n_\rho,0} + k^{(\tilde{\jmath}_\rho)}_{\tilde{n}_\rho-1,\rho}\right| + i\gamma\right|}
\times \left\langle \tau_0\theta^{(j_\rho)}_{n_\rho,0} + k^{(j_\rho)}_{n_\rho-1,0}\right\rangle^{-2} \left\langle -\tau_\rho\theta^{(j_\rho)}_{n_\rho,0} + k^{(\tilde{\jmath}_\rho)}_{\tilde{n}_\rho-1,\rho}\right\rangle^{-2} \left\langle \theta^{(j_\rho)}_{n_\rho,0}\right\rangle^{-d}
\leq \begin{cases} \frac{C_d|\log\gamma|^2}{\left|\tau_0 k^{(j_\rho)}_{n_\rho-1,0} + \tau_\rho k^{(\tilde{\jmath}_\rho)}_{\tilde{n}_\rho-1,\rho}\right|} \left\langle\alpha^{(j_\rho)}_0\right\rangle^{-1/2} \left\langle\alpha^{(\tilde{\jmath}_\rho)}_\rho\right\rangle^{-1/2} & \text{for } d \geq 3, \\ \frac{C_2|\log\gamma|}{\sqrt{\gamma\left|\tau_0 k^{(j_\rho)}_{n_\rho-1,0} + \tau_\rho k^{(\tilde{\jmath}_\rho)}_{\tilde{n}_\rho-1,\rho}\right|}} \left\langle\alpha^{(j_\rho)}_0\right\rangle^{-1/2} \left\langle\alpha^{(\tilde{\jmath}_\rho)}_\rho\right\rangle^{-1/2} & \text{for } d = 2, \end{cases} \tag{6.34}$$

by Lemma B.1.

vi. Choose $(j^*, n^*) \in I(N)$ as $\prec$-large as possible such that $(j^*, n^*, 0)$ is free, and $k^{(j_\rho)}_{n_\rho-1,0}$ depends on $\theta^{(j^*)}_{n^*,0}$ via (6.32). Because $\rho > 1$, such a (j^*, n^*) certainly exists, and we even have $(j^*, n^*) \succeq (j_{\rho-1}, n_{\rho-1})$.

vii. In case $(j^*, n^*, 0) \in A \in S$, with A a pair with second element $(j_*, n_*, \rho - 1)$ in the $\rho - 1$-th one-particle line of the same star K_s, first apply the analogue of Iteration 5.1 to the remainder of the ρ-th one-particle line, and then to the $\rho-1$-th one-particle line down to, but not including, $(j_*, n_*, \rho - 1)$. During that process, as soon as we run into a free $\theta^{(j)}_{n,r}$ variable on which $k^{(\tilde{\jmath}_\rho)}_{\tilde{n}_\rho-1,\rho}$ depends by (6.32),

$$k^{(\tilde{\jmath}_\rho)}_{\tilde{n}_\rho-1,\rho} = k' \pm \theta^{(j)}_{n,r}, \tag{6.35}$$

k' being a function of "earlier" θ variables, one has by Lemma B.2

$$\int_{\mathbb{R}^d} \mathrm{d}\theta^{(j)}_{n,r} \frac{1}{\left|\tau_0 k^{(j_\rho)}_{n_\rho-1,0} + \tau_\rho(k' \pm \theta^{(j)}_{n,r})\right| \left|\alpha^{(j)}_r \pm 2\pi\left|k^{(j)}_{n-1,r} + \tau_r\theta^{(j)}_{n,r}\right| + i\gamma\right|}
\times \left\langle k^{(j)}_{n-1,r} + \tau_r\theta^{(j)}_{n,r}\right\rangle^{-2} \left\langle\theta^{(j)}_{n,r}\right\rangle^{-d}
\leq \frac{\tilde{C}_d \langle\log\gamma\rangle}{\sqrt{\left\langle\alpha^{(j)}_r\right\rangle}} \tag{6.36}$$

if $d \geq 3$, and

$$\int_{\mathbb{R}^2} \frac{\mathrm{d}\theta^{(j)}_{n,r}}{\sqrt{\left|\tau_0 k^{(j_\rho)}_{n_\rho-1,0} + \tau_\rho(k' \pm \theta^{(j)}_{n,r})\right|} \left|\alpha^{(j)}_r \pm 2\pi\left|k^{(j)}_{n-1,r} + \tau_r\theta^{(j)}_{n,r}\right| + i\gamma\right| \left\langle k^{(j)}_{n-1,r} + \tau_r\theta^{(j)}_{n,r}\right\rangle^2}
\leq \frac{\tilde{C}_2 \langle\log\gamma\rangle}{\left\langle\alpha^{(j)}_r\right\rangle} \tag{6.37}$$

if $d = 2$. We then set $\rho_{\text{next}} = \rho - 1$.

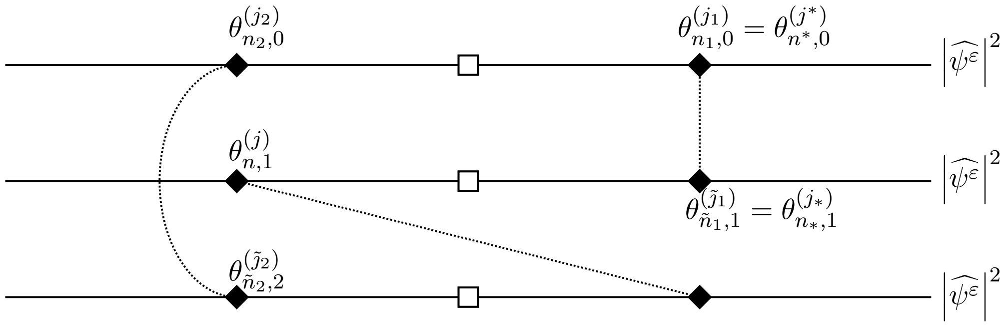

Figure 6.4.: At this instant of integrating out a star (that originally may have been larger), $\rho = 2$. Iteration 6.1 would jump to vii.

viii. If, as in vii., $(j^*, n^*, 0) \in A \in S$, with A a pair with second element $(j_*, n_*, \rho - 1)$ in the $\rho - 1$-th one-particle line of the same star K_s, but, we do *not* encounter such a $\theta^{(j)}_{n,r}$, take (in case this has not happened yet) the L^∞ estimate of the resolvent belonging to $k^{(j_*)}_{n_*,\rho-1}$, apply Iteration 5.1 to the 0-th one-particle line down to, but not including $(j^*, n^*, 0)$, and observe that

$$\begin{aligned} k^{(j_\rho)}_{n_\rho-1,0} &= \tau_0 \theta^{(j^*)}_{n^*,0} + k'', \\ k^{(j^*)}_{n^*,0} &= \tau_0 \theta^{(j^*)}_{n^*,0} + k^{(j^*)}_{n^*-1,0} \end{aligned} \tag{6.38}$$

are the only remaining k variables depending on $\theta^{(j^*)}_{n^*,0}$. Again by Lemma B.2

$$\begin{aligned} \int_{\mathbb{R}^d} \mathrm{d}\theta^{(j^*)}_{n^*,0} & \frac{1}{\left|\theta^{(j^*)}_{n^*,0} + \tau_0 k'' + \tau_\rho k^{(\tilde{j}_\rho)}_{\tilde{n}_\rho-1,\rho}\right| \left|\alpha^{(j^*)}_0 \pm 2\pi \left|\tau_0 \theta^{(j^*)}_{n^*,0} + k^{(j^*)}_{n^*-1,0}\right| + i\gamma\right|} \\ & \quad \times \left\langle \tau_0 \theta^{(j^*)}_{n^*,0} + k^{(j^*)}_{n^*-1,0} \right\rangle^{-2} \left\langle \theta^{(j^*)}_{n^*,0} \right\rangle^{-d} \\ & \leq \frac{\tilde{C}_d \left\langle \log \gamma \right\rangle}{\sqrt{\left\langle \alpha^{(j^*)}_0 \right\rangle}} \end{aligned} \tag{6.39}$$

if $d \geq 3$, and

$$\begin{aligned} \int_{\mathbb{R}^2} & \frac{\mathrm{d}\theta^{(j^*)}_{n^*,0}}{\sqrt{\left|\theta^{(j^*)}_{n^*,0} + \tau_0 k'' + \tau_\rho k^{(\tilde{j}_\rho)}_{\tilde{n}_\rho-1,\rho}\right|} \left|\alpha^{(j^*)}_0 \pm 2\pi \left|\tau_0 \theta^{(j^*)}_{n^*,0} + k^{(j^*)}_{n^*-1,0}\right| + i\gamma\right|} \\ & \quad \times \left\langle \tau_0 \theta^{(j^*)}_{n^*,0} + k^{(j^*)}_{n^*-1,0} \right\rangle^{-2} \\ & \leq \frac{\tilde{C}_2 \left\langle \log \gamma \right\rangle}{\left\langle \alpha^{(j)}_r \right\rangle} \end{aligned} \tag{6.40}$$

if $d = 2$. In case $(j^*, n^*) \neq (j_{\rho-1}, n_{\rho-1})$, we set $\rho_{\text{next}} = \rho - 1$. If $(j^*, n^*) = (j_{\rho-1}, n_{\rho-1})$, we have "lost" the pair $\{(j_{\rho-1}, n_{\rho-1}, 0), (\tilde{j}_{\rho-1}, \tilde{n}_{\rho-1}, \rho-1)\}$, so we just can integrate out whatever remains of the $\rho - 1$-th one-particle line and set $\rho_{\text{next}} = \rho - 2$.

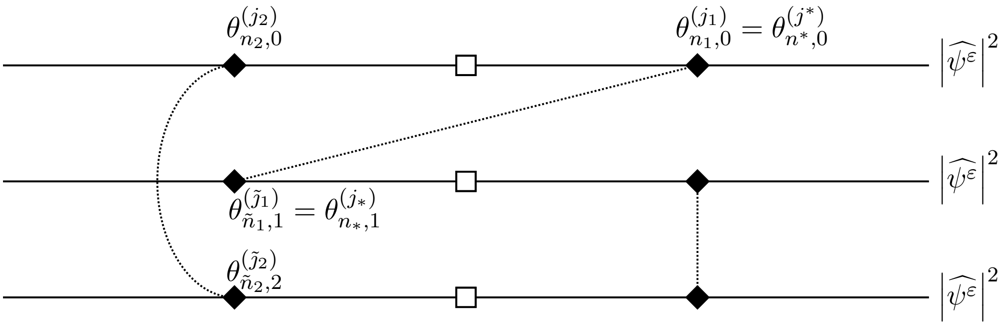

Figure 6.5.: In this example, again with a current value of $\rho = 2$, Iteration 6.1 would jump to viii.

ix. Conversely, if instead of vii. or viii., the cluster $A \in S$ that contains $(j^*, n^*, 0)$ is not a pair, or is a pair connecting $(j^*, n^*, 0)$ to any index not on the $\rho - 1$-th one-particle line of the same star, we have by construction that (6.38) and (6.39-6.40) hold again, but this time, as opposed to viii., we do not have to sacrifice $\{(j_\rho, n_\rho, 0), (\tilde{j}_\rho, \tilde{n}_\rho, \rho)\}$. We integrate out the remainder of the ρ-th one-particle line and set $\rho_{\text{next}} = \rho - 1$.

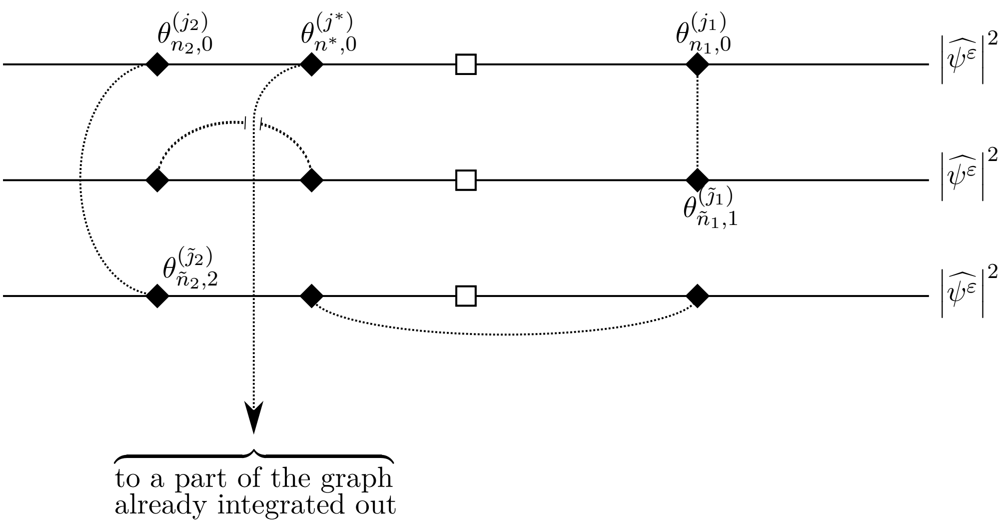

Figure 6.6.: In this situation, Iteration 6.1 would jump to ix.

x. If $\rho_{\text{next}} \geq 1$, we are now just in the setting of i., with ρ replaced by ρ_{next}. For $\rho_{\text{next}} = 0$, however, we apply Iteration 5.1 to the remainder of the 0-th one-particle line, and the integration is finished.

Altogether, there is a factor of $|\log\gamma|$ for each of the f_s free indices (j,n,r), and $2(e_s+1)\overline{m}$ further such factors from the $\alpha_r^{(j)}$ integrals. From each of the d_s dependent (j,n,r), we incurred a factor γ^{-1} except for all those $(\tilde{\jmath}_\rho, \tilde{n}_\rho, \rho)$ with $\rho \in \{2, ..., \underline{e}_s\}$ that were treated in step v. to ix.. of the above iteration. They only contribute a factor $|\log\gamma|$ (for $d \geq 3$) or $1/\sqrt{\gamma}$ (for $d = 2$), and there are (due to the updating rule $\rho_{\mathrm{next}} \geq \rho - 2$) *at least*

$$\left\lceil \frac{\underline{e}_s - 1}{2} \right\rceil \tag{6.41}$$

such cases. Taking the $k_{0,\rho}^{(1)}$ integrals, $\rho \in \{0, ..., e_s\}$, and redefining the constant C finishes the proof. □

In case $\underline{e}_s \leq 1$, the above lemma essentially is just a complicated way to rewrite the basic estimates from Lemma 4.8 and 5.2. In case $\underline{e}_s = 0$, this is enough, but for $\underline{e}_s = 1$ (the *star with only one ray*), one can argue as in Section 5.3 to obtain

Lemma 6.3. *With all other conditions as in Lemma 6.2, let now K_s be a star with $\underline{e}_s = 1$. Then, there is a constant $C < \infty$ depending only on $d \geq 2$ such that*

$$\begin{aligned}&\sup_{\mathfrak{u}\in\mathbb{R}^{d\cdot d_s}} \mathcal{V}_{K_s}\left(\psi_0^\varepsilon, \varepsilon, \gamma, p, L^{(0)}, t, N, S_B, \mathfrak{u}\right)\\ &\leq C^{(|N|+2\overline{m})(e_s+1)}\left(\langle L^{(0)}\rangle + \overline{m}|N|\right)^{3|N|(e_s+1)} \|\psi_0^\varepsilon\|_{\mathcal{H}}^{2(e_s+1)} |\log\gamma|^{(|N|+2\overline{m})(e_s+1)+1}\gamma^{-d_s}\gamma^{\frac{d-1}{d+1}}\end{aligned} \tag{6.42}$$

for $d \geq 3$, and

$$\begin{aligned}&\sup_{\mathfrak{u}\in\mathbb{R}^{d\cdot d_s}} \mathcal{V}_{K_s}\left(\psi_0^\varepsilon, \varepsilon, \gamma, p, L^{(0)}, t, N, S_B, \mathfrak{u}\right)\\ &\leq C^{(|N|+2\overline{m})(e_s+1)}\left(\langle L^{(0)}\rangle + \overline{m}|N|\right)^{3|N|(e_s+1)} \|\psi_0^\varepsilon\|_{\mathcal{H}}^{2(e_s+1)} |\log\gamma|^{(|N|+2\overline{m})(e_s+1)+1}\gamma^{-d_s}\gamma^{1/5}\end{aligned} \tag{6.43}$$

for $d = 2$.

6.1.4. Bound for the full amplitudes

Corollary 6.4. *For each $d \geq 2$, and a random field of class $(d+3, 0)$, there is a $C < \infty$ such that for $\varepsilon \in (0, 1/2]$ and all $S \in \pi^*_{\mathrm{conn}}(I(N; 2l))$*

$$\begin{aligned}&\left|\mathcal{V}_{2l}\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S\right)\right|\\ &\qquad\leq C^{2l(|N|+2\overline{m})}\left(\langle L^{(0)}\rangle + \overline{m}|N|\right)^{6|N|l} \varepsilon^{4l\varepsilon|t|}\|\psi_0^\varepsilon\|_{\mathcal{H}}^{4l}\prod_{j=1}^{2\overline{m}-1}\|a_j\|_{C^0}^{2l}\prod_{A\in S}\left\|g_{|A|}\right\|_{d+3}\\ &\qquad\quad\times|\log\varepsilon|^{2(2\overline{m}+|N|+1)l}\varepsilon^{(l|N|-|S|)/5}\begin{cases}\varepsilon^{l/6} & (d=2),\\ \varepsilon^{l/5} & (d\geq 3).\end{cases}\end{aligned} \tag{6.44}$$

Proof. Inserting Lemma 6.2 and 6.3 in (6.25), we obtain for each connectivitiy component $B \in P$

$$\begin{aligned} &\left|\mathcal{V}_B\left(\psi_0^\varepsilon, \varepsilon, a, p, L^{(0)}, t, N, S_B\right)\right| \\ &\qquad \le C^{|B|(|N|+2\overline{m})}\left(\langle L^{(0)}\rangle + \overline{m}|N|\right)^{3|B||N|} \varepsilon^{2|B|\gamma|t|} \|\psi_0^\varepsilon\|_{\mathcal{H}}^{2|B|} \prod_{j=1}^{2\overline{m}-1} \|a_j\|_{C^0}^{|B|} \prod_{A\in S_B} \left\|g_{|A|}\right\|_{d+3} \\ &\qquad\quad \times |\log\gamma|^{(2\overline{m}+|N|+1)|B|}\varepsilon^{|B||N|/2} \prod_{s=1}^{\overline{s}} \gamma^{h_s-d_s}, \end{aligned} \tag{6.45}$$

where the exponent h_s is given as a function of $\underline{e}_s \in \mathbb{N}_0$ and $d \ge 2$ by the table

	$\underline{e}_s = 0$	$\underline{e}_s = 1$	$\underline{e}_s \ge 2$
$d = 2$	0	$\frac{1}{5}$	$\frac{1}{2}\left\lceil\frac{\underline{e}_s-1}{2}\right\rceil$
$d \ge 3$	0	$\frac{1}{2}$	$\left\lceil\frac{\underline{e}_s-1}{2}\right\rceil$.

We observe

$$\begin{aligned} \sum_{s=1}^{\overline{s}} d_s &= |S_B|, \\ \sum_{s=1}^{\overline{s}} (e_s - \underline{e}_s) &\le 2(|B||N| - 2|S_B|) =: 2D_B, \end{aligned} \tag{6.46}$$

so after setting $\gamma = \varepsilon$, the last line of (6.45) is bounded by

$$|\log\varepsilon|^{(2\overline{m}+|N|+1)|B|} \prod_{s=1}^{\overline{s}} \varepsilon^{(e_s-\underline{e}_s)/5+h_s+D_B/10}. \tag{6.47}$$

One can easily verify

$$\frac{e_s - \underline{e}_s}{5} + h_s \ge \begin{cases} e_s/6 & (d=2) \\ e_s/5 & (d \ge 3) \end{cases} \tag{6.48}$$

for all $e_s \in \mathbb{N}_0$, so, by (6.16),

$$(6.47) \le |\log\varepsilon|^{(2\overline{m}+|N|+1)|B|}\varepsilon^{D_B/10} \times \begin{cases} \varepsilon^{|B|/12} & (d=2), \\ \varepsilon^{|B|/10} & (d \ge 3). \end{cases} \tag{6.49}$$

With $\sum_{B\in P} |B| = 2l$, taking the product (6.10) over all $B \in P$ proves the corollary. □

So far, we have obtained higher-order estimates for the contributions to the main term of the Duhamel expansion. The G^{end} and G^{rough} parts of the remainder can be bounded in exactly the same fashion, only with notation from Section 4.1.3 for the variables N, $N_<$, N_{fin}, and $\overline{N}^{(\overline{m})}$.

Corollary 6.5. *For $d \geq 2$, ξ of class $(d + 3(\overline{M} + 2), 0)$, $N_< \in \mathbb{N}_0^{\overline{m}-1}$ obeying (4.36), $\overline{N}^{(\overline{m})}$ given by (4.37) and $N_{\text{fin}} \in \{1, ..., \overline{N}^{(\overline{m})} - 1\}$, the identities*

$$\lim_{R\to\infty} \mathbb{E}\left[\left|\left\| G^{\text{rough}}_{M,N_{\text{fin}}}\left(t^{(\overline{m})}; R, L^{(\overline{m})}, \varepsilon\right) \left(\prod_{j<\overline{m}} A^\varepsilon_j F_{N^{(j)}}\left(t^{(j)}; R, L^{(j)}, \varepsilon\right)\right) \psi^\varepsilon_0 \right\|^2_{\mathcal{H}} \right.\right.$$
$$\left.\left. - \mathbb{E}\left\| G^{\text{rough}}_{M,N_{\text{fin}}}\left(t^{(\overline{m})}; R, L^{(\overline{m})}, \varepsilon\right) \left(\prod_{j<\overline{m}} A^\varepsilon_j F_{N^{(j)}}\left(t^{(j)}; R, L^{(j)}, \varepsilon\right)\right) \psi^\varepsilon_0 \right\|^2_{\mathcal{H}} \right|^{2l}\right]$$
$$= \sum_{S\in\pi^*_{\text{conn}}(I(N;2l))} \mathcal{V}_{2l}\left(G^{\text{rough}}, \psi^\varepsilon_0, \varepsilon, \kappa, a, p, L^{(0)}, t, N_<, N_{\text{fin}}, M, S\right), \tag{6.50}$$

and

$$\lim_{R\to\infty} \mathbb{E}\left[\left|\left\| G^{\text{end}}_{M,\overline{N}^{(\overline{m})}}\left(t^{(\overline{m})}; R, L^{(\overline{m})}, \varepsilon\right) \left(\prod_{j<\overline{m}} A^\varepsilon_j F_{N^{(j)}}\left(t^{(j)}; R, L^{(j)}, \varepsilon\right)\right) \psi^\varepsilon_0 \right\|^2_{\mathcal{H}} \right.\right.$$
$$\left.\left. - \mathbb{E}\left\| G^{\text{end}}_{M,\overline{N}^{(\overline{m})}}\left(t^{(\overline{m})}; R, L^{(\overline{m})}, \varepsilon\right) \left(\prod_{j<\overline{m}} A^\varepsilon_j F_{N^{(j)}}\left(t^{(j)}; R, L^{(j)}, \varepsilon\right)\right) \psi^\varepsilon_0 \right\|^2_{\mathcal{H}} \right|^{2l}\right]$$
$$= \sum_{S\in\pi^*_{\text{conn}}(I(N;2l))} \mathcal{V}_{2l}\left(G^{\text{end}}, \psi^\varepsilon_0, \varepsilon, \kappa, a, L^{(0)}, t, N_<, \overline{N}, M, S\right) \tag{6.51}$$

hold. Here, the individual $\mathcal{V}_{2l}$ amplitudes are bounded by

$$\begin{aligned} &C^{(|N|+2\overline{m})l} e^{4l\varepsilon|t|} \left\langle \varepsilon\overline{M}\right\rangle^{12(\overline{M}+1)l} \left\langle L^{(0)} + \overline{m}\overline{N}\right\rangle^{6|N|l+12(\overline{M}+1)l} \\ &\times \|\psi^\varepsilon_0\|^{4l}_{\mathcal{H}} \prod_{A\in S} \left\|g_{|A|}\right\|_{d+3(\overline{M}+2)} \prod_{j=1}^{\overline{m}-1} \|a_j\|^{4l}_{C^0} \\ &\times |\log\varepsilon_n|^{2(2\overline{m}+|N|+1)l} \varepsilon_n^{(l|N|-|S|)/5} \begin{cases} \varepsilon^{l/6} & (d=2), \\ \varepsilon^{l/5} & (d\geq 3). \end{cases} \end{aligned} \tag{6.52}$$

in each case, $C < \infty$ only depending on d.

6.1.5. Amplitudes for amputated graphs

As we now want to consider remainder terms of A^{rough} type, we need to modify the observables $\mathcal{A}_r(k_r, \sigma_r, p, \varepsilon)$ by setting them to

$$\prod_{j=1}^{2\overline{m}-1}\left(a_{j,\sigma_{0,r}^{(j+1)}}\left(\frac{k_{0,r}^{(j+1)}+k_{N^{(j)},r}^{(j)}}{2}\right)\delta\left(k_{0,r}^{(j+1)}-k_{N^{(j)},r}^{(j)}-\varepsilon p^{(j)}\right)\delta\left(\sigma_{N^{(j)},r}^{(j)},\sigma_{0,r}^{(j+1)}\right)\right) \tag{6.53}$$

for r odd, and its complex conjugate, for r even, but $a_{j,\pm}$ now given as in (4.39). Also, the propagators have to account for the extra decay stemming from κ, and the presence or absence of cutoff functions now,

$$\begin{aligned}&\mathcal{P}_r(k_r,\sigma_r,\alpha_r,\gamma)\\ &\quad=\prod_{\substack{(j,n)\in I_0(N)\\ (j,n)\neq(\overline{m},N^{(\overline{m})}),(\overline{m}+1,0)}}\left(\frac{i}{\alpha_r^{(j)}-2\pi\sigma_{n,r}^{(j)}\tau_r^{(j)}|k_{n,r}^{(j)}|+i\gamma+i\kappa_n^{(j)}}\right)\\ &\qquad\times\prod_{(j,n)\in I(N)}\left[\left(-i\tau_r^{(j)}\right)\left(\left|k_{n,r}^{(j)}\right|\sigma_{n-1,r}^{(j)}+\left|k_{n-1,r}^{(j)}\right|\sigma_{n,r}^{(j)}\right)\Phi_n^{(j)}\left(k_{n,r}^{(j)},k_{n-1,r}^{(j)},L_n^{(j)}\right)\right]\end{aligned} \tag{6.54}$$

with $\Phi_n^{(j)}$ and $\kappa_n^{(j)}$ defined by (4.53) and (4.54) respectively. The index set $I(N;2l)$, however, is (up to the new definition of N) the same as before, and $\pi^*(I(N;2l))$ denotes the set of partitions of $I(N;2l)$ without clusters of size 1. We can rewrite every $S\in\pi^*(I(N;2l))$ as a disjoint union

$$S = S_{\text{h}}\cup S_{\text{tr}}\cup\bigcup_{r=1}^{2l}S_{\text{int}}(r), \tag{6.55}$$

with all clusters $A\in S_{\text{h}}$ having more than two elements $|A|>2$, all $A\in S_{\text{tr}}$ being transfer pairs as defined before, i.e. pairs with members belonging to different one-particle lines, and finally $S_{\text{int}}(r)$ being the set of all internal pairs of the r-th one-particle line. Similar to Lemma 4.6

Lemma 6.6. *For ξ of class $(m,0)$, with $m>d+(2+\overline{M})$, $N_<\in\mathbb{N}_0^{\overline{m}-1}$ obeying (4.36), and any $N_{\text{fin}}\in\{1,...,\overline{N}^{(\overline{m})}-1\}$,*

$$\begin{aligned}&\lim_{R\to\infty}\mathbb{E}\left\|A^{\text{rough}}_{\overline{M},N_{\text{fin}}}\left(t^{(\overline{m})};R,L^{(\overline{m})},\varepsilon\right)\left(\prod_{j<\overline{m}}A_j^{\varepsilon}F_{N^{(j)}}\left(t^{(j)};R,L^{(j)},\varepsilon\right)\right)\psi_0^{\varepsilon}\right\|_{\mathcal{H}}^{4l}\\ &\quad=\sum_{S\in\pi^*(I(N;2l))}\mathcal{V}_{2l}\left(A^{\text{rough}},\psi_0^{\varepsilon},\varepsilon,\kappa,a,p,L^{(0)},t,N_<,N_{\text{fin}},\overline{M},S\right),\end{aligned} \tag{6.56}$$

with the amplitude of each partition S given for an arbitrary choice of $\gamma > 0$ by

$$\begin{aligned}
&\mathcal{V}_{2l}\left(A^{\text{rough}}, \psi_0^\varepsilon, \varepsilon, \kappa, a, p, L^{(0)}, t, N_<, N_{\text{fin}}, \overline{M}, S\right) \\
&= \varepsilon^{|N|l}\pi^{2|N|l} \sum_{\substack{\sigma_{n,r}^{(j)} \in \{\pm\} \\ \forall (j,n,r)\in I_0(N;2l)}} \int_{\mathbb{R}^{(2|N|+4\overline{m})ld}} \prod_{(j,n,r)\in I_0(N;2l)} \mathrm{d}k_{n,r}^{(j)} \\
&\qquad \prod_{A\in S}\left(\delta\left(\sum_{(j,n,r)\in A}\theta_{n,r}^{(j)}\right)\widehat{g}_{|A|}\left(\theta_{n,r}^{(j)} : (j,n,r)\in A^\sharp\right)\right) \\
&\qquad \times \prod_{r=1}^{2l}\mathcal{A}_r\left(k_r, \sigma_r, p, \varepsilon\right) \\
&\qquad \times \int_{\mathbb{R}^{4\overline{m}l}} \prod_{r=1}^{2l}\prod_{j=1}^{2\overline{m}}\left(e^{\gamma t^{(j)}}\frac{\mathrm{d}\alpha_r^{(j)}}{2\pi}e^{-i\alpha_r^{(j)}t^{(j)}}\right) \\
&\qquad\qquad \prod_{r=1}^{2l}\mathcal{P}_r\left(k_r, \sigma_r, \alpha_r, \gamma, \kappa\right) \\
&\qquad \times \prod_{\substack{r=1\\ r\text{ odd}}}^{2l} \widehat{\psi}^\varepsilon_{0,\sigma_{0,r}^{(1)}}\left(k_{0,r}^{(1)}\right)\overline{\widehat{\psi}^\varepsilon_{0,\sigma^{(2\overline{m})}_{N^{(2\overline{m})},r}}}\left(k^{(2\overline{m})}_{N^{(2\overline{m})},r}\right) \\
&\qquad \times \prod_{\substack{r=1\\ r\text{ even}}}^{2l} \overline{\widehat{\psi}^\varepsilon_{0,\sigma_{0,r}^{(1)}}}\left(k_{0,r}^{(1)}\right)\widehat{\psi}^\varepsilon_{0,\sigma^{(2\overline{m})}_{N^{(2\overline{m})},r}}\left(k^{(2\overline{m})}_{N^{(2\overline{m})},r}\right),
\end{aligned} \tag{6.57}$$

where the resolvent integrals are understood as a formal way to write the unperturbed propagator for all j with $N^{(j)} = 0$.

To have a handle on the resolvents regularized by $\gamma + \kappa$ instead of only γ, we set the *central part* of the index set $I(N; 2l)$ to be

$$\mathscr{C} = \left\{(j,n,r) \in I(N;2l) : (\overline{m}, N_{\text{fin}} + 1) \preceq (j,n) \preceq (\overline{m}+1, \overline{M})\right\}. \tag{6.58}$$

For any $S \in \pi^*(I(N;2l))$ the central part $\mathscr{C}$ then decomposes into

$$\mathscr{C} = \mathscr{C}_{\text{h}} \cup \mathscr{C}_{\text{tr}}^{+} \cup \mathscr{C}_{\text{tr}}^{-} \cup \bigcup_{r=1}^{2l}\left(\mathscr{C}_{\text{int}}(+,r) \cup \mathscr{C}_{\text{int}}(-,r)\right). \tag{6.59}$$

Here, for $\sigma \in \{\pm\}$,

$$\begin{aligned}
\mathscr{C}_{\text{h}} &= \left\{(j,n,r) \in \mathscr{C} : \exists A \in S_{\text{h}} : (j,n,r) \in A\right\}, \\
\mathscr{C}_{\text{tr}}^{\sigma} &= \left\{(j,n,r) \in \mathscr{C} : \exists A \in S_{\text{tr}} : (j,n,r) \in A, A = \{(j,n,r);(j',n',r')\}, \sigma r > \sigma r'\right\}, \\
\mathscr{C}_{\text{int}}(\sigma,r) &= \{(j,n,r) \in \mathscr{C} : \exists A \in S_{\text{int}}(r) : (j,n,r) \in A, A = \{(j,n,r);(j',n',r)\}, \\
&\qquad\qquad (\sigma j, \sigma n) \succ (\sigma j', \sigma n')\}.
\end{aligned} \tag{6.60}$$

Lemma 6.7. (Higher moment bound for A^{rough} remainder terms.) *If ξ is of class $(d+2(\overline{M}+2),0)$, there is a $C<\infty$ depending only on dimension $d\geq 2$, such that for any $S\in\pi^*(I(N;2l))$, and all $\varepsilon,\gamma\in(0,1/2]$, $\kappa>0$*

$$\begin{aligned}
&\left|\mathcal{V}_{2l}\left(A^{\text{rough}},\psi_0^\varepsilon,\varepsilon,\kappa,a,p,L^{(0)},t,N_<,N_{\text{fin}},\overline{M},S\right)\right| \\
&\quad\leq e^{4\gamma l|t|}C^{2(|N|+2\overline{m})l}\left(\langle L^{(0)}\rangle+\overline{m}\overline{N}\right)^{4|N|l+(8\overline{M}+8)l}\langle\varepsilon C_{\text{obs}}\rangle^{(8\overline{M}+8)l} \\
&\quad\times\|\psi_0^\varepsilon\|_{\mathcal{H}}^{4l}\prod_{j=1}^{\overline{m}-1}\|a_j\|^{4l}\prod_{A\in S}\left\|g_{|A|}\right\|_{d+2(\overline{M}+2)} \\
&\quad\times\varepsilon^{|N|l}|\log\gamma|^{2l(|N|+2\overline{m})-|S|}\gamma^{-|S|}\left(\frac{\gamma}{\gamma+\kappa}\right)^{2\overline{M}l-\frac{1}{2}|\mathscr{C}_{\text{h}}|}.
\end{aligned} \tag{6.61}$$

Setting $\gamma=\varepsilon$, the last line of (6.61) is bounded by

$$|\log\varepsilon|^{2l(|N|+2\overline{m})-|S|}\varepsilon^{|N|l-|S|}\left(\frac{\varepsilon}{\kappa}\right)^{\left(2\overline{M}l-3(|N|l-|S|)\right)_+}. \tag{6.62}$$

Proof. Assume that $N^{(j)}\geq 1$ for all $j\in\{1,...,2\overline{m}\}$; suppose furthermore, without loss of generality, that $|\mathscr{C}_{\text{tr}}^+|\geq|\mathscr{C}_{\text{tr}}^-|$. Instead of (6.28), we employ

$$\left|\widehat{\psi}_0^\varepsilon\left(k_{0,r}^{(1)}\right)\right|\left|\widehat{\psi}_0^\varepsilon\left(k_{N^{(2\overline{m})},r}^{(2\overline{m})}\right)\right|\leq\frac{1}{2}\left(b_r\left|\widehat{\psi}_0^\varepsilon\left(k_{0,r}^{(1)}\right)\right|^2+b_r^{-1}\left|\widehat{\psi}_0^\varepsilon\left(k_{N^{(2\overline{m})},r}^{(2\overline{m})}\right)\right|^2\right) \tag{6.63}$$

for each $r\in\{1,...,2l\}$, with a parameter $b_r\in(0,\infty)$ to be optimized later. Then

$$\begin{aligned}
&\left|\mathcal{V}_{2l}\left(A^{\text{rough}},\psi_0^\varepsilon,\varepsilon,\kappa,a,p,L^{(0)},t,N_<,N_{\text{fin}},\overline{M},S\right)\right| \\
&\quad\leq 2^{-2l}\sum_{\varpi\in\{\pm\}^{2l}}\left(\prod_{r=1}^{2l}b_r^{\varpi_r}\right)\mathcal{V}_{2l}\left(A^{\text{rough}},\varpi,\psi_0^\varepsilon,\varepsilon,\gamma,\kappa,a,p,L^{(0)},t,N_<,N_{\text{fin}},\overline{M},S\right)
\end{aligned} \tag{6.64}$$

with

$$
\begin{aligned}
&\mathcal{V}_{2l}\left(A^{\mathrm{rough}}, \varpi, \psi_0^\varepsilon, \varepsilon, \gamma, \kappa, a, p, L^{(0)}, t, N_<, N_{\mathrm{fin}}, \overline{M}, S\right) \\
&\quad = \varepsilon^{|N|l} e^{4\gamma l|t|} C^{2(|N|+2\overline{m})l} \left(\langle L^{(0)}\rangle + \overline{m}\overline{N}\right)^{4|N|l} \langle \varepsilon C_{\mathrm{obs}}\rangle^{(8\overline{M}+8)l} \\
&\qquad \times \prod_{j=1}^{\overline{m}-1} \|a_j\|^{4l} \prod_{A\in S} \left\|g_{|A|}\right\|_{d+2(\overline{M}+2)} \\
&\qquad \max_{\sigma\in\{\pm\}^{2l(|N|+2\overline{m})}} \int_{\mathbb{R}^{(2|N|+4\overline{m})ld}} \prod_{(j,n,r)\in I_0(N;2l)} \mathrm{d}k_{n,r}^{(j)} \\
&\qquad \prod_{A\in S}\left(\delta\left(\sum_{(j,n,r)\in A} \theta_{n,r}^{(j)}\right)\right) \\
&\qquad \times \prod_{r=1}^{2l}\prod_{j=1}^{2\overline{m}-1}\left(\delta\left(k_{0,r}^{(j+1)} - k_{N^{(j)},r}^{(j)} - \varepsilon p^{(j)}\right)\right) \\
&\qquad \times \int_{\mathbb{R}^{4\overline{m}l}} \prod_{r=1}^{2l}\prod_{j=1}^{2\overline{m}}\left(e^{\gamma t^{(j)}}\frac{\mathrm{d}\alpha_r^{(j)}}{2\pi} e^{-i\alpha_r^{(j)}t^{(j)}}\right) \\
&\qquad\quad \prod_{r=1}^{2l}\left|\prod_{\substack{(j,n)\in I_0(N)\\ (j,n)\neq(\overline{m},N^{(\overline{m})}),(\overline{m}+1,0)}} \frac{1}{\alpha_r^{(j)} - 2\pi\sigma_{n,r}^{(j)}\tau_r^{(j)}|k_{n,r}^{(j)}| + i\gamma + i\kappa_n^{(j)}}\right| \\
&\qquad \times \prod_{\substack{r=1\\ \varpi_r=1}}^{2l}\left(\left|\widehat{\psi}^\varepsilon_{0,\sigma_{0,r}^{(1)}}\left(k_{0,r}^{(1)}\right)\right|^2 \prod_{(j,n)\in I(N)} \left\langle\theta_{n,r}^{(j)}\right\rangle^{-d}\left\langle k_{n,r}^{(j)}\right\rangle^{-1}\right) \\
&\qquad \times \prod_{\substack{r=1\\ \varpi_r=-1}}^{2l}\left(\left|\widehat{\psi}^\varepsilon_{0,\sigma_{N^{(2\overline{m})},r}^{(2\overline{m})}}\left(k_{N^{(2\overline{m})},r}^{(2\overline{m})}\right)\right|^2 \prod_{(j,n)\in I(N)} \left\langle\theta_{n,r}^{(j)}\right\rangle^{-d}\left\langle k_{n-1,r}^{(j)}\right\rangle^{-1}\right),
\end{aligned}
\tag{6.65}
$$

with a finite C depending only on dimension d. The k integrals are then transformed into θ integrals in the usual fashion, and, starting from $r = 2l$, down to $r = 1$, each of the one-particle-lines is integrated out, in decreasing $\prec$ order if $\varpi_r = 1$, and increasing in $\prec$ if $\varpi_r = -1$. Each of the altogether $4l\overline{m}$ integrals over α variables will produce factors $C|\log\gamma|$, as will the L^1 resolvent estimates, $|A|-1$ of them for each cluster $A \in S$. However, of the $|S|$ resolvent estimates of L^∞ type, a few can be improved from $C\gamma^{-1}$ to $C(\gamma+\kappa)^{-1}$ (or even C, for the missing, "amputated resolvents") — this is in certainly the case

- for the $k_{n,r}^{(j)}$ resolvent if $\varpi_r = 1$, and $(j,n,r) \in \mathscr{C}_{\mathrm{tr}}^+ \cup \mathscr{C}_{\mathrm{int}}(+,r)$, and
- for the $k_{n-1,r}^{(j)}$ resolvent if $\varpi_r = -1$, and $(j,n,r) \in \mathscr{C}_{\mathrm{tr}}^+ \cup \mathscr{C}_{\mathrm{int}}(-,r)$.

Thus, there is a constant $C < \infty$ only depending on d such that the last seven lines of

(6.65) are bounded by

$$C^{2l(|N|+2\overline{m})}|\log\gamma|^{2l(|N|+2\overline{m})-|S|}\gamma^{-|S|}\left(\frac{\gamma}{\gamma+\kappa}\right)^{c(\varpi,S)}, \tag{6.66}$$

with the exponent

$$c(\varpi,S)=\left|\mathscr{C}_{\mathrm{tr}}^{+}\right|+\sum_{r=1}^{2l}|\mathscr{C}_{\mathrm{int}}(\varpi_r,r)|, \tag{6.67}$$

and one can optimize

$$2^{-2l}\sum_{\varpi\in\{\pm\}^{2l}}\left(\prod_{r=1}^{2l}b_r^{\varpi_r}\left(\frac{\gamma}{\gamma+\kappa}\right)^{|\mathscr{C}_{\mathrm{int}}(\varpi_r,r)|}\right)=\prod_{r=1}^{2l}\left(\frac{\gamma}{\gamma+\kappa}\right)^{(|\mathscr{C}_{\mathrm{int}}(+,r)|+|\mathscr{C}_{\mathrm{int}}(-,r)|)/2}, \tag{6.68}$$

with each b_r chosen appropriately. From $|\mathscr{C}_{\mathrm{tr}}^{+}|\geq|\mathscr{C}_{\mathrm{tr}}^{-}|$, we conclude

$$|\mathscr{C}_{\mathrm{tr}}^{+}|+\frac{1}{2}\sum_{r=1}^{2l}(|\mathscr{C}_{\mathrm{int}}(+,r)|+|\mathscr{C}_{\mathrm{int}}(-,r)|)\geq\frac{1}{2}(|\mathscr{C}|-|\mathscr{C}_{\mathrm{h}}|)=2\overline{M}l-\frac{1}{2}|\mathscr{C}_{\mathrm{h}}|, \tag{6.69}$$

which proves equation (6.61). Equation (6.62) then follows from $|\mathscr{C}_{\mathrm{h}}|\leq 4\overline{M}l$ and

$$|\mathscr{C}_{\mathrm{h}}|\leq\sum_{A\in S_{\mathrm{h}}}|A|\leq 3\sum_{A\in S_{\mathrm{h}}}(|A|-2)=6\,(|N|l-|S|)\,. \tag{6.70}$$

□

By the same reasoning, with the appropriate redefinition of $N\in\mathbb{N}_0^{2\overline{m}}$ for type A^{end} remainder terms

Corollary 6.8. (Higher moment estimate for A^{end} remainder terms.) *For dimension $d\geq 2$, a random field ξ of class $(d+2(\overline{M}+2),0)$, $N_<\in\mathbb{N}_0^{\overline{m}-1}$ obeying (4.36), $\overline{N}^{(\overline{m})}$ given by (4.37), we have*

$$\begin{aligned}\lim_{R\to\infty}\mathbb{E}&\left\|A^{\mathrm{end}}_{\overline{M},\overline{N}^{(\overline{m})}}\left(t^{(\overline{m})};R,L^{(\overline{m})},\varepsilon\right)\left(\prod_{j<\overline{m}}A_j^{\varepsilon}F_{N^{(j)}}\left(t^{(j)};R,L^{(j)},\varepsilon\right)\right)\psi_0^{\varepsilon}\right\|_{\mathcal{H}}^{4l}\\&=\sum_{S\in\pi^*(I(N;2l))}\mathcal{V}_{2l}\left(A^{\mathrm{end}},\psi_0^{\varepsilon},\varepsilon,\kappa,a,p,L^{(0)},t,N_<,\overline{N},\overline{M},S\right),\end{aligned} \tag{6.71}$$

and there is a $C<\infty$ only depending on dimension d such that for all $S\in\pi^*(I(N;2l))$, $\varepsilon\in(0,1/2]$, and all $\kappa>0$,

$$\begin{aligned}&\left|\mathcal{V}_{2l}\left(A^{\mathrm{end}},\psi_0^{\varepsilon},\varepsilon,\kappa,a,p,L^{(0)},t,N_<,\overline{N},\overline{M},S\right)\right|\\&\quad\leq e^{4\varepsilon l|t|}C^{2(|N|+2\overline{m})l}\left(\langle L^{(0)}\rangle+\overline{m}\overline{N}\right)^{4|N|l+(8\overline{M}+8)l}\langle\varepsilon C_{\mathrm{obs}}\rangle^{(8\overline{M}+8)l}\\&\quad\times\|\psi_0^{\varepsilon}\|_{\mathcal{H}}^{4l}\prod_{j=1}^{\overline{m}-1}\|a_j\|^{4l}\prod_{A\in S}\left\|g_{|A|}\right\|_{d+2(\overline{M}+2)}\\&\quad\times|\log\varepsilon|^{2l(|N|+2\overline{m})-|S|}\varepsilon^{|N|l-|S|}\left(\frac{\varepsilon}{\kappa}\right)^{\left(2\overline{M}l-3(|N|l-|S|)\right)_+}.\end{aligned} \tag{6.72}$$

6.2. Proof of Theorem 3.3

6.2.1. Collecting the amplitude bounds

Set the number of measurements to $\overline{m} = 1$ and let $(\psi_0^\varepsilon)_{\varepsilon>0}$ be a sequence of initial states in $\mathcal{H}$ fulfilling all conditions of Theorem 3.3, and additionally assume that $\widehat{\psi_{0,\pm}^\varepsilon}(k)$ vanishes for $|k| > L^{(0)}$. We discretize the continuous parameter $\varepsilon > 0$ by a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$, $\varepsilon_n = n^{-\alpha}$, with an α to be chosen later. We also fix a time $T \in (0,\infty)$. To be able to vary the strength of disorder independently of space and time scaling, introduce another parameter $\varepsilon' > 0$, and denote by $H^{\varepsilon'}$ the random operator $H_0 + \sqrt{\varepsilon'}V$. For $\varepsilon > 0$, $\varepsilon' > 0$, $t \in \mathbb{R}$ and $\sigma \in \{\pm\}$, define the random state

$$\psi[\varepsilon,\varepsilon',t] = \exp(-iH^{\varepsilon'}t)\psi_0^\varepsilon, \qquad \psi[\varepsilon,\varepsilon',t,\sigma] = \left(\exp(-iH^{\varepsilon'}t)\psi_0^\varepsilon\right)_\sigma \tag{6.73}$$

in $\mathcal{H}$, and, for any $\overline{\tau} \in [0,T]$, $p \in \mathbb{R}^d$ and any bounded and continuous observable $a_\pm : \mathbb{R}^d \to \mathbb{C}$, consider the random variable

$$\begin{aligned}
&X(a,\overline{\tau},p,n) \\
&= \sup_{\varepsilon'\in[\varepsilon_{n+1},\varepsilon_n]} \sum_\sigma \\
&\quad \bigg| \int_{\mathbb{R}^d} \mathrm{d}k a_\sigma(k) \overline{\widehat{\psi}[\varepsilon_n,\varepsilon',\overline{\tau}/\varepsilon_n,\sigma](k+\varepsilon_n p/2)} \widehat{\psi}[\varepsilon_n,\varepsilon',\overline{\tau}/\varepsilon_n,\sigma](k-\varepsilon_n p/2) \\
&\qquad - \mathbb{E}\int_{\mathbb{R}^d} \mathrm{d}k a_\sigma(k) \overline{\widehat{\psi}[\varepsilon_n,\varepsilon',\overline{\tau}/\varepsilon_n,\sigma](k+\varepsilon_n p/2)} \widehat{\psi}[\varepsilon_n,\varepsilon',\overline{\tau}/\varepsilon_n,\sigma](k-\varepsilon_n p/2) \bigg| .
\end{aligned} \tag{6.74}$$

Note that each summand in the Duhamel expansion of $\psi[\varepsilon_n,\varepsilon',\overline{\tau}/\varepsilon_n,\sigma]$ scales like a non-negative power of ε', and can be bounded *deterministically* by setting ε' to the largest possible value ε_n. Thus, there is a constant $C_{T,d}$ only depending on T and d such that

$$\begin{aligned}
&\mathbb{E}\left[|X(a,\overline{\tau},p,n)|^{2l}\right] \\
&\leq C_{T,d}^{l} \max_{0\leq N^{(1)},N^{(2)}<\overline{N}} \max_{0\leq N_{\text{fin}}<\overline{N}} \max_{0\leq M<\overline{M}} \lim_{R\to\infty} \\
&\Bigg\{\overline{N}^{4l}\mathbb{E}\left|\left\langle \psi_0^{\varepsilon_n}, F_{N^{(2)}}\left(\overline{\tau}/\varepsilon_n; R, L^{(1)}, \varepsilon_n\right)^* A^{\varepsilon_n} F_{N^{(1)}}\left(\overline{\tau}/\varepsilon_n; R, L^{(1)}, \varepsilon_n\right)\psi_0^{\varepsilon_n}\right\rangle_{\mathcal{H}}\right. \\
&\qquad\qquad \left. -\mathbb{E}\left\langle \psi_0^{\varepsilon_n}, F_{N^{(2)}}\left(\overline{\tau}/\varepsilon_n; R, L^{(1)}, \varepsilon_n\right)^* A^{\varepsilon_n} F_{N^{(1)}}\left(\overline{\tau}/\varepsilon_n; R, L^{(1)}, \varepsilon_n\right)\psi_0^{\varepsilon_n}\right\rangle_{\mathcal{H}}\right|^{2l} \\
&\quad + \|a\|_{C^0}^{2l}\left(\frac{\overline{N}\overline{M}\kappa}{\varepsilon_n}\right)^{4l} \sup_{t\in[0,\overline{\tau}/\varepsilon]}\Bigg[\left(\mathbb{E}\left\|G_{M,N_{\text{fin}}}^{\text{rough}}\left(t;R,L^{(1)},\varepsilon_n\right)\psi_0^{\varepsilon_n}\right\|_{\mathcal{H}}^2\right)^{2l} \\
&\qquad\qquad + \mathbb{E}\left|\left\|G_{M,N_{\text{fin}}}^{\text{rough}}\left(t;R,L^{(1)},\varepsilon_n\right)\psi_0^{\varepsilon_n}\right\|_{\mathcal{H}}^2 - \mathbb{E}\left\|G_{M,N_{\text{fin}}}^{\text{rough}}\left(t;R,L^{(1)},\varepsilon_n\right)\psi_0^{\varepsilon_n}\right\|_{\mathcal{H}}^2\right|^{2l}\Bigg] \\
&\quad + \|a\|_{C^0}^{2l}\left(\frac{\overline{M}\kappa}{\varepsilon_n}\right)^{4l} \sup_{t\in[0,\overline{\tau}/\varepsilon_n]}\Bigg[\left(\mathbb{E}\left\|G_{M,\overline{N}}^{\text{end}}\left(t;R,L^{(1)},\varepsilon_n\right)\psi_0^{\varepsilon_n}\right\|_{\mathcal{H}}^2\right)^{2l} \\
&\qquad\qquad + \mathbb{E}\left|\left\|G_{M,\overline{N}}^{\text{end}}\left(t;R,L^{(1)},\varepsilon_n\right)\psi_0^{\varepsilon_n}\right\|_{\mathcal{H}}^2 - \mathbb{E}\left\|G_{M,\overline{N}}^{\text{end}}\left(t;R,L^{(1)},\varepsilon_n\right)\psi_0^{\varepsilon_n}\right\|_{\mathcal{H}}^2\right|^{2l}\Bigg] \\
&\quad + \|a\|_{C^0}^{2l}\left(\frac{\overline{N}}{\varepsilon_n}\right)^{4l} \sup_{t\in[0,\overline{\tau}/\varepsilon_n]}\mathbb{E}\left\|A_{\overline{M},N_{\text{fin}}}^{\text{rough}}\left(t;R,L^{(1)},\varepsilon_n\right)\psi_0^{\varepsilon_n}\right\|_{\mathcal{H}}^{4l} \\
&\quad + \|a\|_{C^0}^{2l}\,\varepsilon_n^{-4l} \sup_{t\in[0,\overline{\tau}/\varepsilon_n]}\mathbb{E}\left\|A_{\overline{M},\overline{N}}^{\text{end}}\left(t;R,L^{(1)},\varepsilon_n\right)\psi_0^{\varepsilon_n}\right\|_{\mathcal{H}}^{4l}\Bigg\}.
\end{aligned} \tag{6.75}$$

As in previous sections, we set (with possibly different values of b and ϑ than before)

$$\overline{N} = \overline{N}(\varepsilon_n) = \left\lceil \frac{b\,|\log\varepsilon_n|}{|\log|\log\varepsilon_n||}\right\rceil \qquad b>0, \tag{6.76}$$

$$\kappa = \kappa(\varepsilon_n) = \varepsilon_n^{1-\vartheta} \qquad \vartheta\in(0,1/3). \tag{6.77}$$

A suitable choice of $\overline{M}$ will only be determined later, but we assume from now on that the random field is of class $(d+3\overline{M}+6,4)$, making all results of Chapter 4 and Section 6.1 applicable. We therefore use a graph expansion of all expressions on the right hand side of (6.75), and note that the number of amplitudes to sum over in each case is bounded by

$$|\pi^*(I(N;2l))| \leq \left(2l(|N|+2\overline{M})\right)! \leq (4l(\overline{N}+\overline{M}))! \tag{6.78}$$

Collecting the estimates from Lemmas 6.1, 6.6 and 6.7 and Corollaries 6.4, 6.5 and 6.8, as well as equation (4.211) and (4.213), one can find a $C<\infty$ which depends on $T, d\geq 2, L^{(0)}, \overline{M}, \vartheta, b, l$ and the distribution of ξ, but not on ε_n, such that (6.75) is bounded by

$$
\begin{aligned}
&\mathbb{E}\left[|X(a,\overline{\tau},p,n)|^{2l}\right] \\
&\quad\leq C^{\overline{N}}\,\|\psi_0^{\varepsilon_n}\|_{\mathcal{H}}^{4l}\,\|a\|_{C^0}^{2l}\left(\overline{N}^{16\overline{N}l}\,|\log\varepsilon_n|^{4\overline{N}l+C}\,\varepsilon_n^{l/6}\left(\frac{\kappa}{\varepsilon_n}\right)^{4l}+\overline{N}^{-2\overline{N}l}\left(\frac{\kappa}{\varepsilon_n}\right)^{4l}\right. \\
&\qquad\qquad\left.+\overline{N}^{12\overline{N}l}\,|\log\varepsilon_n|^{4\overline{N}l+C}\,\varepsilon_n^{-4l}\left(\frac{\varepsilon_n}{\kappa}\right)^{2\overline{M}l}\right).
\end{aligned}
\tag{6.79}
$$

Here, we have already bounded the contribution of the higher cumulants of the random field in a fashion similar to Section 4.5; for example, setting $D = l|N| - |S|$ for any partition $S \in I(N;2l)$, one obtains the bound

$$
\varepsilon_n^{D/10}\prod_{A\in S}\left\|g_{|A|}\right\|_{d+3(\overline{M}+2)} \leq \max_{D\in\{0,\dots,2l(\overline{N}+\overline{M})\}}\varepsilon_n^{D/10}D^{CD} \leq K^{\overline{N}},
\tag{6.80}
$$

in Lemma 6.5, or, in Lemma 6.7, since $\vartheta \in (0,1/3)$,

$$
\varepsilon_n^{D}(\varepsilon_n/\kappa)^{-3D}\prod_{A\in S}\left\|g_{|A|}\right\|_{d+2(\overline{M}+2)} \leq \max_{D\in\{0,\dots,2l(\overline{N}+\overline{M})\}}\varepsilon_n^{D}(\varepsilon_n/\kappa)^{-3D}D^{\tilde{C}D} \leq \tilde{K}^{\overline{N}}
\tag{6.81}
$$

with $C, \tilde{C}, K, \tilde{K}$ only depending on d, $\overline{M}$, ϑ, b and the distribution of ξ.

Therefore, whenever ξ is of class $(d+1641,4)$ and one thus can choose $\overline{M} \geq 545$, there is a $\beta > 0$ such that for all $n, l \in \mathbb{N}$

$$
\mathbb{E}\left[|X(a,\overline{\tau},p,n)|^{2l}\right] \leq C_l\left(\sup_{\varepsilon>0}\|\psi_0^{\varepsilon}\|_{\mathcal{H}}^{4l}\right)\|a\|_{C^0}^{2l}\,\varepsilon_n^{\beta l}
\tag{6.82}
$$

with $C_l < \infty$ depending only on l, dimension d, the distribution of ξ, $L^{(0)}$ and T, but *not* on n and p (because we have chosen $\overline{m} = 1$, so all C_{obs} terms drop out in the Lemmas of Section 6.1).

6.2.2. Preserving tightness

Now pick and fix a small $\rho > 0$, and define $R_\rho < \infty$ such that

$$
\limsup_{\varepsilon\to 0}\int_{|x|\geq R_\rho/\varepsilon}\mathrm{d}x\,|\psi_0^{\varepsilon}(x)|^2 < \rho,
\tag{6.83}
$$

and set $R'_\rho = R_\rho + T$. Let $\chi_\rho : \mathbb{R}^d \to [0,1]$ be smooth, with $\chi_\rho(y) = 1$ for $|y| \le R'_\rho$ and $\chi_\rho(y) = 0$ for $|y| \ge R'_\rho + 1$. With these definitions, by Theorem 3.1,

$$\begin{aligned}
&\sup_{\varepsilon' \in [\varepsilon_{n+1}, \varepsilon_n]} \sum_\sigma \int_{|x| > (R'_\rho + 1)/\varepsilon_n} \mathrm{d}x \left|\psi[\varepsilon_n, \varepsilon', \overline{\tau}/\varepsilon_n, \sigma](x)\right|^2 \\
&\le \|\psi_0^{\varepsilon_n}\|_{\mathcal{H}}^2 \\
&\;\; - \inf_{\varepsilon' \in [\varepsilon_{n+1}, \varepsilon_n]} \sum_\sigma \mathbb{E} \int_{\mathbb{R}^{2d}} \mathrm{d}p\mathrm{d}k \widehat{\chi}_\rho(p) \overline{\widehat{\psi}\left[\varepsilon_n, \varepsilon', \frac{\overline{\tau}}{\varepsilon_n}, \sigma\right]\left(k + \frac{\varepsilon_n p}{2}\right)} \widehat{\psi}\left[\varepsilon_n, \varepsilon', \frac{\overline{\tau}}{\varepsilon_n}, \sigma\right]\left(k - \frac{\varepsilon_n p}{2}\right) \\
&\;\; + \int_{\mathbb{R}^d} \mathrm{d}p \, |\widehat{\chi}_\rho(p)| \, X(1, \overline{\tau}, p, n) \\
&\le \sum_\sigma \int_{|x| > R'_\rho} \mu_{T,\sigma}(\mathrm{d}x, \mathrm{d}k) + r(n, \rho) + \int_{\mathbb{R}^d} \mathrm{d}p \, |\widehat{\chi}_\rho(p)| \, X(1, \overline{\tau}, p, n) \\
&\le \sum_\sigma \int_{|x| > R_\rho} \mu_{0,\sigma}(\mathrm{d}x, \mathrm{d}k) + r(n, \rho) + \int_{\mathbb{R}^d} \mathrm{d}p \, |\widehat{\chi}_\rho(p)| \, X(1, \overline{\tau}, p, n) \\
&\le \rho + r(n, \rho) + Z(\rho, \overline{\tau}, n).
\end{aligned} \tag{6.84}$$

Here, $r(n, \rho) \ge 0$ is a deterministic quantity that can be chosen uniformly for all $\overline{\tau} \in [0, T]$, with

$$r(n, \rho) \to 0 \quad (n \to \infty) \tag{6.85}$$

for $\rho > 0$ fixed. $Z(\rho, \overline{\tau}, n)$, on the other hand, is a random variable such that for all $l \in \mathbb{N}$

$$\mathbb{E}\left[|Z(\rho, \overline{\tau}, n)|^{2l}\right] \le C_{l,\rho} \varepsilon_n^{\beta l} \tag{6.86}$$

with $C_{l,\rho} < \infty$ depending on l, dimension d, the distribution of ξ, $L^{(0)}$, T, $\sup_{\varepsilon > 0} \|\psi_0^\varepsilon\|_{\mathcal{H}}$ and ρ. By the same token,

$$\begin{aligned}
&\sup_{\varepsilon' \in [\varepsilon_{n+1}, \varepsilon_n]} \sum_\sigma \int_{|k| > L^{(0)} + 1} \mathrm{d}k \left|\widehat{\psi}[\varepsilon_n, \varepsilon', \overline{\tau}/\varepsilon_n, \sigma](k)\right|^2 \\
&\le r(n) + Z(\overline{\tau}, n),
\end{aligned} \tag{6.87}$$

with deterministic $r(n) \to 0$, and $\mathbb{E}\,|Z(\overline{\tau}, n)|^{2l} \le K_l \varepsilon_n^{\beta l}$ with some constant $K_l < \infty$ depending on the ame parameters as $C_{l,\rho}$ above (except for ρ).

6.2.3. Bounding the interpolation error

Fix a single $\overline{\tau} \in [0, T]$, a momentum bound $P_{\max} \in (0, \infty)$, a momentum $\overline{p} \in \mathbb{R}^d$, $|\overline{p}| \le P_{\max}$ as well as a continuous and bounded function $a_\pm : \mathbb{R}^d \to \mathbb{C}$. To $n \in \mathbb{N}$, assign a time spacing $\delta_n > 0$, a momentum spacing $h_n > 0$, both to be specified later, as well as

$$\nu_n = T\left(\frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n}\right) + \frac{\delta_n}{\varepsilon_{n+1}} > 0. \tag{6.88}$$

From (6.84) and (6.87), we see by suitable cut-offs in both position and momentum space that there is a state $f[\varepsilon_n, \varepsilon', \overline{\tau}] \in \mathcal{H}$ such that $\|f[\varepsilon_n, \varepsilon', \overline{\tau}]\|_{\mathcal{H}} \leq \|\psi[\varepsilon_n, \varepsilon', \overline{\tau}/\varepsilon_n]\|_{\mathcal{H}}$ and

$$\begin{aligned} \sup_{\varepsilon' \in [\varepsilon_{n+1}, \varepsilon_n]} & \|f[\varepsilon_n, \varepsilon', \overline{\tau}] - \psi[\varepsilon_n, \varepsilon', \overline{\tau}/\varepsilon_n]\|_{\mathcal{H}}^2 \\ & \leq C\left(\rho + r(n, \rho) + Z(\rho, \overline{\tau}, n) + r(n) + Z(\overline{\tau}, n)\right) \end{aligned} \tag{6.89}$$

with an only d-dependent $C < \infty$, while

$$f[\varepsilon_n, \varepsilon', \overline{\tau}, \sigma](x) = 0 \text{ whenever } |x| > (R'_\rho + 2)/\varepsilon_n \tag{6.90}$$

and, with C only d-dependent,

$$\sup_{\varepsilon' \in [\varepsilon_{n+1}, \varepsilon_n]} \sum_\sigma \|f[\varepsilon_n, \varepsilon', \overline{\tau}, \sigma]\|_{H^1(\mathbb{R}^d)} \leq C \left\langle L^{(0)} \right\rangle \sup_{\varepsilon > 0} \|\psi_0^\varepsilon\|_{\mathcal{H}} . \tag{6.91}$$

By Lemma E.1, there exists an $\tilde{R}_\rho \in \left(R'_\rho + 2, \infty\right)$ depending only on R'_ρ, ρ, and d as well as a function $u[\varepsilon_n, \varepsilon', \overline{\tau}] : \mathbb{R}^d \to \mathbb{C}$ supported on $\left\{|x| \leq \tilde{R}_\rho/\varepsilon_n\right\}$ such that almost surely

$$\begin{aligned} \sup_{\varepsilon' \in [\varepsilon_{n+1}, \varepsilon_n]} & \left\| |\nabla| u[\varepsilon_n, \varepsilon', \overline{\tau}] - f[\varepsilon_n, \varepsilon', \overline{\tau}, +] - f[\varepsilon_n, \varepsilon', \overline{\tau}_n, -] \right\|_{L^2} \leq \rho \\ \sup_{\varepsilon' \in [\varepsilon_{n+1}, \varepsilon_n]} & \left\| |\nabla| u[\varepsilon_n, \varepsilon', \overline{\tau}] \right\|_{H^1} \leq C \left\langle L^{(0)} \right\rangle \sup_{\varepsilon > 0} \|\psi_0^\varepsilon\|_{\mathcal{H}} , \end{aligned} \tag{6.92}$$

C depending only on d. Also, the function

$$w[\varepsilon_n, \varepsilon', \overline{\tau}](x) = (1 + \sqrt{\varepsilon'}\xi(x)) \left(f[\varepsilon_n, \varepsilon', \overline{\tau}, +](x) - f[\varepsilon_n, \varepsilon', \overline{\tau}, -](x)\right) \tag{6.93}$$

is clearly supported within $\left\{|x| \leq (R'_\rho + 2)/\varepsilon_n\right\}$. Thus, using the map $\mathcal{E}$ from Lemma 2.9,

$$\begin{aligned} \tilde{f}[\varepsilon_n, \varepsilon', \overline{\tau}] &= \frac{1}{2} \begin{pmatrix} |\nabla| u[\varepsilon_n, \varepsilon', \overline{\tau}] + w[\varepsilon_n, \varepsilon', \overline{\tau}]/(1 + \sqrt{\varepsilon'}\xi) \\ |\nabla| u[\varepsilon_n, \varepsilon', \overline{\tau}] - w[\varepsilon_n, \varepsilon', \overline{\tau}]/(1 + \sqrt{\varepsilon'}\xi) \end{pmatrix} \\ &= \mathcal{E}_{1+\sqrt{\varepsilon'}\xi} \left(u[\varepsilon_n, \varepsilon', \overline{\tau}], w[\varepsilon_n, \varepsilon', \overline{\tau}]\right) , \end{aligned} \tag{6.94}$$

fulfills

$$\left\| \tilde{f}[\varepsilon_n, \varepsilon', \overline{\tau}] - f[\varepsilon_n, \varepsilon', \overline{\tau}] \right\|_{\mathcal{H}} \leq \rho. \tag{6.95}$$

By construction, $\tilde{f}[\varepsilon_n, \varepsilon', \overline{\tau}]$ is an element of $\tilde{\mathcal{D}}$ as defined in (2.95), and one can write for every $s \in [0, \nu_n]$ and every $\varepsilon' \in [\varepsilon_{n+1}, \varepsilon_n]$

$$\left\| \mathrm{e}^{-iH^{\varepsilon'} s} \tilde{f}[\varepsilon_n, \varepsilon', \overline{\tau}] - \tilde{f}[\varepsilon_n, \varepsilon', \overline{\tau}] \right\|_{\mathcal{H}} \leq \nu_n \left\| H^{\varepsilon'} \tilde{f}[\varepsilon_n, \varepsilon', \overline{\tau}] \right\|_{\mathcal{H}} . \tag{6.96}$$

But by the unitary equivalence observed in Lemma 2.9 and Theorem 2.10,

$$\left\| H^{\varepsilon'} \tilde{f}[\varepsilon_n, \varepsilon', \overline{\tau}] \right\|_{\mathcal{H}}^2 = \left\| A\left(u[\varepsilon_n, \varepsilon', \overline{\tau}], w[\varepsilon_n, \varepsilon', \overline{\tau}]\right) \right\|_B^2 , \tag{6.97}$$

the norm on the right side defined by the scalar product (2.79), with $B = B_{\tilde{R}_\rho/\varepsilon_n}(0)$, and the operator A given in (2.81), with $c(x) = 1 + \sqrt{\varepsilon'}\xi(x)$. Therefore

$$\begin{aligned}
&\left\| A\left(u[\varepsilon_n, \varepsilon', \overline{\tau}], w[\varepsilon_n, \varepsilon', \overline{\tau}]\right)\right\|_B^2 \\
&\qquad = \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{d}x \left(\left|\nabla w[\varepsilon_n, \varepsilon', \overline{\tau}]\right|^2 + (1+\sqrt{\varepsilon'}\xi(x))^2 \left|\Delta u[\varepsilon_n, \varepsilon', \overline{\tau}]\right|^2 \right) \\
&\qquad \le C\left(1 + \sqrt{\varepsilon'} Y \tilde{R}_\rho/\varepsilon_n\right)^2 \left(\sum_\sigma \left\| f[\varepsilon_n, \varepsilon', \overline{\tau}, \sigma]\right\|_{H^1}^2 + \left\| |\nabla| u[\varepsilon_n, \varepsilon', \overline{\tau}]\right\|_{H^1}^2 \right) \\
&\qquad \le C \left\langle L^{(0)} \right\rangle^2 \left(1 + \sqrt{\varepsilon'} Y \tilde{R}_\rho/\varepsilon_n\right)^2 \sup_{\varepsilon>0} \|\psi_0^\varepsilon\|_{\mathcal{H}}^2 ,
\end{aligned} \tag{6.98}$$

with an only d-dependent $C < \infty$ and an almost surely finite random variable $Y \ge 0$ such that

$$|\xi(x)| + |\nabla \xi(x)| \le Y(1 + |x|) \tag{6.99}$$

for all $x \in \mathbb{R}^d$.

Altogether, there is a constant $C < \infty$ *only* depending on d such that

$$\begin{aligned}
\sup_{\varepsilon' \in [\varepsilon_{n+1}, \varepsilon_n]} \sup_{s \in [0, \nu_n]} &\left\| \psi[\varepsilon_n, \varepsilon', \overline{\tau}/\varepsilon_n + s] - \psi[\varepsilon_n, \varepsilon', \overline{\tau}/\varepsilon_n] \right\|_{\mathcal{H}}^2 \\
&\le C\left(\rho + r(n, \rho) + Z(\rho, \overline{\tau}, n) + r(n) + Z(\overline{\tau}, n)\right) \\
&\quad + C \left\langle L^{(0)} \right\rangle^2 \nu_n^2 \left(1 + Y \tilde{R}_\rho/\sqrt{\varepsilon_n}\right)^2 \sup_{\varepsilon>0} \|\psi_0^\varepsilon\|_{\mathcal{H}}^2 .
\end{aligned} \tag{6.100}$$

Furthermore, with the state $f[\varepsilon_n, \varepsilon', \overline{\tau}]$ from before, and any bounded, continuous observable $a : \mathbb{R}^d \to \mathbb{C}^2$,

$$\begin{aligned}
&\sup_{\varepsilon'\in[\varepsilon_{n+1},\varepsilon_n]} \sup_{\substack{|p|\leq P_{\max} \\ |p-\overline{p}|_\infty\leq h_n}} \sup_{\varepsilon\in[\varepsilon_{n+1},\varepsilon_n]} \sum_\sigma \\
&\quad \Bigg| \int_{\mathbb{R}^d} \mathrm{d}k a_\sigma(k) \overline{\widehat{\psi}[\varepsilon_n,\varepsilon',\overline{\tau}/\varepsilon_n,\sigma](k+\varepsilon p/2)} \widehat{\psi}[\varepsilon_n,\varepsilon',\overline{\tau}/\varepsilon_n,\sigma](k-\varepsilon p/2) \\
&\qquad - \int_{\mathbb{R}^d} \mathrm{d}k a_\sigma(k) \overline{\widehat{\psi}[\varepsilon_n,\varepsilon',\overline{\tau}/\varepsilon_n,\sigma](k+\varepsilon_n\overline{p}/2)} \widehat{\psi}[\varepsilon_n,\varepsilon',\overline{\tau}/\varepsilon_n,\sigma](k-\varepsilon_n\overline{p}/2) \Bigg| \\
&\leq C \sup_{\varepsilon>0} \|\psi_0^\varepsilon\|_{\mathcal{H}} \|a\|_{C^0} \left(\rho + r(n,\rho) + Z(\rho,\overline{\tau},n) + r(n) + Z(\overline{\tau},n)\right)^{1/2} \\
&\quad + \sup_{\varepsilon'\in[\varepsilon_{n+1},\varepsilon_n]} \sup_{\substack{|p|\leq P_{\max} \\ |p-\overline{p}|_\infty\leq h_n}} \sup_{\varepsilon\in[\varepsilon_{n+1},\varepsilon_n]} \sum_\sigma \\
&\qquad \Bigg| \int_{\mathbb{R}^d} \mathrm{d}k a_\sigma(k) \overline{\widehat{f}[\varepsilon_n,\varepsilon',\overline{\tau},\sigma](k+\varepsilon p/2)} \widehat{f}[\varepsilon_n,\varepsilon',\overline{\tau},\sigma](k-\varepsilon p/2) \\
&\qquad\quad - \int_{\mathbb{R}^d} \mathrm{d}k a_\sigma(k) \overline{\widehat{f}[\varepsilon_n,\varepsilon',\overline{\tau},\sigma](k+\varepsilon_n\overline{p}/2)} \widehat{f}[\varepsilon_n,\varepsilon',\overline{\tau},\sigma](k-\varepsilon_n\overline{p}/2) \Bigg| \\
&\leq C \sup_{\varepsilon>0} \|\psi_0^\varepsilon\|_{\mathcal{H}} \|a\|_{C^0} \left(\rho + r(n,\rho) + Z(\rho,\overline{\tau},n) + r(n) + Z(\overline{\tau},n)\right)^{1/2} \\
&\quad + C \sup_{\varepsilon>0} \|\psi_0^\varepsilon\|_{\mathcal{H}}^2 \|a\|_{C^0} \frac{R'_\rho + 2}{\varepsilon_n} \left(\varepsilon_n h_n + (\varepsilon_n - \varepsilon_{n+1}) P_{\max}\right).
\end{aligned} \tag{6.101}$$

Here, (6.89) has been employed in the first estimate. For the second one, we have bounded

$$\left\| \nabla \widehat{f}[\varepsilon_n,\varepsilon',\overline{\tau},\sigma] \right\|_{L^2} \leq C \frac{R'_\rho + 2}{\varepsilon_n} \|\psi_0^\varepsilon\|_{\mathcal{H}} \tag{6.102}$$

by exploiting the support properties of $f[\varepsilon_n,\varepsilon',\overline{\tau},\sigma]$, and finally observed that

$$|\varepsilon p - \varepsilon_n \overline{p}| \leq \varepsilon_n h_n + (\varepsilon_n - \varepsilon_{n+1}) P_{\max} \tag{6.103}$$

for the choice of parameters in consideration.

Thus, after noting that for all $\overline{\tau} \in [0,T]$, $(\overline{\tau} + \delta_n)/\varepsilon_{n+1} \leq \overline{\tau} + \nu_n$, one obtains from

(6.100-6.101) a $C < \infty$ only depending on d such that

$$\begin{aligned}
&\sup_{\overline{\tau}\in\delta_n\mathbb{N}\cap[0,T]} \sup_{\substack{\overline{p}\in h_n\mathbb{Z}^d\\ |\overline{p}|\leq P_{\max}}} \sup_{\varepsilon'\in[\varepsilon_{n+1},\varepsilon_n]} \sup_{t\in[\overline{\tau}/\varepsilon_n,(\overline{\tau}+\delta_n)/\varepsilon_{n+1}]} \sup_{\substack{|p|\leq P_{\max}\\ |p-\overline{p}|_\infty\leq h_n}} \sup_{\varepsilon\in[\varepsilon_{n+1},\varepsilon_n]} \sum_\sigma \\
&\qquad \left| \int_{\mathbb{R}^d} \mathrm{d}k a_\sigma(k) \overline{\widehat{\psi}[\varepsilon,\varepsilon',t,\sigma](k+\varepsilon p/2)} \widehat{\psi}[\varepsilon,\varepsilon',t,\sigma](k-\varepsilon p/2) \right. \\
&\qquad\qquad \left. - \int_{\mathbb{R}^d} \mathrm{d}k a_\sigma(k) \overline{\widehat{\psi}[\varepsilon_n,\varepsilon',\overline{\tau}/\varepsilon_n,\sigma](k+\varepsilon_n\overline{p}/2)} \widehat{\psi}[\varepsilon_n,\varepsilon',\overline{\tau}/\varepsilon_n,\sigma](k-\varepsilon_n\overline{p}/2) \right| \\
&\leq C\|a\|_{C^0} \sup_{\varepsilon>0} \|\psi_0^\varepsilon\|_{\mathcal{H}} \left(\sup_{\overline{\tau}\in\delta_n\mathbb{N}\cap[0,T]} (\rho + r(n,\rho) + Z(\rho,\overline{\tau},n) + r(n) + Z(\overline{\tau},n))^{1/2} \right. \\
&\qquad + \sup_{\varepsilon>0} \|\psi_0^\varepsilon\|_{\mathcal{H}} \frac{R'_\rho+2}{\varepsilon_n} (\varepsilon_n h_n + (\varepsilon_n - \varepsilon_{n+1}) P_{\max}) \\
&\qquad + \sup_{\varepsilon>0} \|\psi_0^\varepsilon\|_{\mathcal{H}} \left\langle L^{(0)} \right\rangle \nu_n \left(1 + Y\tilde{R}_\rho/\sqrt{\varepsilon_n}\right) \\
&\qquad \left. + \sup_{\varepsilon\in[\varepsilon_{n+1},\varepsilon_n]} \|\psi_0^\varepsilon - \psi_0^{\varepsilon_n}\|_{\mathcal{H}} \right).
\end{aligned} \tag{6.104}$$

For the second to fourth lines of the right side to vanish almost surely (namely on the event $\{Y < \infty\}$) in the $n \to \infty$ limit, it is sufficient that the coefficient in $\varepsilon_n = n^{-\alpha}$ is contained in $\alpha \in (0, 2/3)$, spacing paramenters can be chosen to be $\delta_n = n^{-2\alpha}$ and any sequence $h_n \to 0$, and one has to pick $\alpha \leq \alpha_0$, so that

$$\lim_{n\to\infty} \sup_{\varepsilon\in[(n+1)^{-\alpha},n^{-\alpha}]} \left\| \psi_0^\varepsilon - \psi_0^{n^{-\alpha}} \right\|_{\mathcal{H}} = 0 \tag{6.105}$$

holds. On the first line on the right side of (6.104), $r(n) + r(\rho, n) \to 0$ deterministically. For the random variables $Z(\rho, \overline{\tau}, n)$ (and similarly $Z(\overline{\tau}, n)$), observe that the sequence

$$\begin{aligned}
\mathbb{E}\left[\left(\sup_{\overline{\tau}\in\delta_n\mathbb{N}\cap[0,T]} Z(\rho,\overline{\tau},n) \right)^{2l} \right] &\leq \frac{T}{\delta_n} \sup_{\overline{\tau}\in[0,T]} \mathbb{E}\left[Z(\rho,\overline{\tau},n)^{2l} \right] \leq TC_{l,\rho}\varepsilon_n^{\beta l}\delta_n^{-1} \\
&\leq TC_{l,\rho} n^{(2-\beta l)\alpha}
\end{aligned} \tag{6.106}$$

is summable in $n \in \mathbb{N}$ as long as we take $l \in \mathbb{N}$ large enough, no matter how small $\alpha, \beta > 0$ may be. To achieve this was the whole point of Section 6.1. By a Markov estimate and the Borel-Cantelli Lemma,

$$\lim_{n\to\infty} \sup_{\overline{\tau}\in\delta_n\mathbb{N}\cap[0,T]} Z(\rho,\overline{\tau},n) = 0 \tag{6.107}$$

almost surely for every choice of $\rho > 0$. As ρ can be arbitrarily small, one deduces that for a fixed bounded and continuous observable a, the left hand side of (6.104) converges to zero almost surely as $n \to \infty$.

6.2.4. Controlling a single node $(\varepsilon_n, \overline{\tau}, \overline{p})$, conclusion

Recalling the definition (6.74) of the random variables X, one also has, for, say, $h_n = \varepsilon_n$, that

$$\begin{aligned}\mathbb{E}\left[\left(\sup_{\overline{\tau}\in\delta_n\mathbb{N}\cap[0,T]}\ \sup_{\substack{\overline{p}\in h_n\mathbb{Z}^d\\|\overline{p}|\le P_{\max}}} X(a,\overline{\tau},\overline{p},n)\right)^{2l}\right] &\le \frac{(2P_{\max})^d T}{\delta_n h_n^d} C_l \sup_{\varepsilon>0}\|\psi_0^\varepsilon\|_{\mathcal{H}}^{4l}\,\|a\|_{C^0}^{2l}\,\varepsilon^{\beta l} \\ &\le \tilde{C}_l n^{(2+d-\beta l)\alpha},\end{aligned} \tag{6.108}$$

for all $l \in \mathbb{N}$. Again, by choosing l large enough and applying a Borel-Cantelli argument,

$$\sup_{\overline{\tau}\in\delta_n\mathbb{N}\cap[0,T]}\ \sup_{\substack{\overline{p}\in h_n\mathbb{Z}^d\\|\overline{p}|\le P_{\max}}} X(a,\overline{\tau},\overline{p},n) \to 0 \tag{6.109}$$

almost surely as $n \to \infty$. Because our disorder scaling fulfills $\sqrt{\varepsilon_{n+1}} \le \sqrt{\varepsilon'} \le \sqrt{\varepsilon_n}$ and $\varepsilon_{n+1}/\varepsilon_n \to 1$ as $n \to \infty$, and all bounds in the proof of Theorem 3.1 are uniform in $\overline{\tau} \in [0,T]$ and $|\overline{p}| \le P_{\max}$, it is clear that

$$\begin{aligned}&\lim_{n\to\infty}\sup_{\overline{\tau}\in[0,T]}\sup_{|\overline{p}|\le P_{\max}}\sup_{\varepsilon'\in[\varepsilon_{n+1},\varepsilon_n]}\\ &\qquad\Bigg|\mathbb{E}\int_{\mathbb{R}^d}\mathrm{d}k a_\sigma(k)\overline{\widehat{\psi}[\varepsilon_n,\varepsilon',\overline{\tau}/\varepsilon_n,\sigma](k+\varepsilon_n\overline{p}/2)}\widehat{\psi}[\varepsilon_n,\varepsilon',\overline{\tau}/\varepsilon_n,\sigma](k-\varepsilon_n\overline{p}/2)\\ &\qquad\qquad-\int_{\mathbb{R}^{2d}}\mu_{\overline{\tau},\sigma}(\mathrm{d}x,\mathrm{d}k)e^{2\pi i\overline{p}\cdot x}a_\sigma(k)\Bigg|\\ &=\lim_{n\to\infty}\sup_{\overline{\tau}\in[0,T]}\sup_{|\overline{p}|\le P_{\max}}\sup_{\varepsilon'\in[\varepsilon_{n+1},\varepsilon_n]}\\ &\qquad\Bigg|\mathbb{E}\int_{\mathbb{R}^d}\mathrm{d}k a_\sigma(k)\overline{\widehat{\psi}[\varepsilon_n,\varepsilon',\overline{\tau}/\varepsilon_n,\sigma](k+\varepsilon_n\overline{p}/2)}\widehat{\psi}[\varepsilon_n,\varepsilon',\overline{\tau}/\varepsilon_n,\sigma](k-\varepsilon_n\overline{p}/2)\\ &\qquad\quad-\mathbb{E}\int_{\mathbb{R}^d}\mathrm{d}k a_\sigma(k)\overline{\widehat{\psi}[\varepsilon_n,\varepsilon_n,\overline{\tau}/\varepsilon_n,\sigma](k+\varepsilon_n\overline{p}/2)}\widehat{\psi}[\varepsilon_n,\varepsilon_n,\overline{\tau}/\varepsilon_n,\sigma](k-\varepsilon_n\overline{p}/2)\Bigg|\\ &=0.\end{aligned} \tag{6.110}$$

Finally, because $h_n, \delta_n \to 0$, the limit measures μ_τ fulfill

$$\begin{aligned}&\sup_{\overline{\tau}\in[0,T]}\sup_{|\overline{p}|\le P_{\max}}\sup_{|\tau-\overline{\tau}|\le\delta_n}\sup_{|p-\overline{p}|\le h_n}\\ &\qquad\left|\int_{\mathbb{R}^{2d}}\mu_{\tau,\sigma}(\mathrm{d}x,\mathrm{d}k)e^{2\pi ip\cdot x}a_\sigma(k)-\int_{\mathbb{R}^{2d}}\mu_{\overline{\tau},\sigma}(\mathrm{d}x,\mathrm{d}k)e^{2\pi i\overline{p}\cdot x}a_\sigma(k)\right|\\ &\to 0 \quad (n\to\infty).\end{aligned} \tag{6.111}$$

Here, the locally time-uniform tightness of the μ_τ in the x variable was employed to replace $p \to \overline{p}$. The same tightness of μ_τ in the x variable, as well as the compact support (and time-uniform non-concentration at 0, due to assumption (3.5)) of μ_τ in the k variable then allows to approximate the function $(x,k)\mapsto e^{2\pi i\overline{p}\cdot x}a_\sigma(k)$ by a $C_0\left(\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\})\right)$ function as introduced in Section 2.3. Finally, the strong time continuity of the semigroup $\mathrm{e}^{\mathcal{L}_\sigma\tau}$, $\tau\geq 0$ helps to control the error when replacing $\tau\to\overline{\tau}$.

From (6.104), (6.109), (6.110) and (6.111) one has for fixed bounded and continuous $a:\mathbb{R}^d\to\mathbb{C}^2$, and fixed $P_{\max}\in[0,\infty)$

$$\lim_{\varepsilon\to 0}\sup_{\tau\in[0,T]}\sup_{|p|\leq P_{\max}}\sum_{\sigma\in\{\pm\}}\Big|\int_{\mathbb{R}^d}\mathrm{d}k a_\sigma(k)\overline{\widehat{\psi}[\varepsilon,\varepsilon,\tau/\varepsilon,\sigma](k+\varepsilon p/2)}\widehat{\psi}[\varepsilon,\varepsilon,\tau/\varepsilon,\sigma](k-\varepsilon p/2) - \int_{\mathbb{R}^{2d}}\mu_{\tau,\sigma}(\mathrm{d}x,\mathrm{d}k)e^{2\pi i p\cdot x}a_\sigma(k)\Big| = 0 \tag{6.112}$$

almost surely. The momentum bound $P_{\max}$ may then be dropped to see that almost surely

$$\lim_{\varepsilon\to 0}\sup_{\tau\in[0,T]}\sum_{\sigma\in\{\pm\}}\Big|\int_{\mathbb{R}^d}\mathrm{d}k a_\sigma(k)\overline{\widehat{\psi}[\varepsilon,\varepsilon,\tau/\varepsilon,\sigma](k+\varepsilon p/2)}\widehat{\psi}[\varepsilon,\varepsilon,\tau/\varepsilon,\sigma](k-\varepsilon p/2) - \int_{\mathbb{R}^{2d}}\mu_{\tau,\sigma}(\mathrm{d}x,\mathrm{d}k)e^{2\pi i p\cdot x}a_\sigma(k)\Big| = 0 \tag{6.113}$$

for *all* $p\in\mathbb{R}^d$. However, at this point of the proof, the "bad configurations" of the medium, the exception events $N_a\subset\Omega$ of probability zero, may still depend on a.

To amend this, recall that so far we assume that $\widehat{\psi}^\varepsilon_{0,\pm}$ is supported in a ball of radius $L^{(0)}$, and consequently, $\mu_{\tau,\pm}(\mathrm{d}x,\mathrm{d}k)$ has no mass on $\{|k|>L^{(0)}\}$. By selecting an observable $a_\pm:\mathbb{R}^d\to[0,1]$ such that $a_\pm(k)=0$ for $|k|\leq L^{(0)}$ and $a_\pm(k)=1$ for $|k|\geq L^{(0)}+1$, (6.113) implies that

$$\lim_{\varepsilon\to 0}\sup_{\tau\in[0,T]}\sum_{\sigma\in\{\pm\}}\int_{|k|\geq L^{(0)}+1}\mathrm{d}k\left|\widehat{\psi}[\varepsilon,\varepsilon,\tau/\varepsilon,\sigma](k)\right|^2 = 0 \tag{6.114}$$

almost surely. Thus, it suffices to consider bounded and continuous functions $a:\mathbb{R}^d\to\mathbb{C}^2$ which vanish at infinity, which is a separable Banach space with a countable dense set $\{a_1,a_2,...\}$. Whenever the realization of the random field is ξ_ω, $\omega\in\Omega\setminus N'$, with exception event $N'=\cup_{r=1}^\infty N_{a_r}$, $\mathbb{P}(N')=0$, the convergence (6.113) holds for all $p\in\mathbb{R}^d$ and *all* bounded and continuous observables $a_\pm$. Reasoning as in (4.258) and applying dominated convergence in the p variable we see that on $\Omega\setminus N'$,

$$\lim_{\varepsilon\to 0}\sup_{\tau\in[0,T]}\left|\left\langle W^\varepsilon\left[\left(\mathrm{e}^{-iH^\varepsilon\tau/\varepsilon}\psi^\varepsilon_0\right)_\sigma\right],\mathfrak{a}_\sigma\right\rangle-\langle\mu_{\tau,\sigma},\mathfrak{a}_\sigma\rangle\right| = 0 \tag{6.115}$$

for all $\mathfrak{a}_\pm\in\mathcal{F}L^1(C^0)$. Finally, considering a sequence of initial states $(\psi^\varepsilon_0)_{\varepsilon>0}$ that only fulfills (2.162), one can introduce the cut-off $L^{(0)}$, and control the limit $L^{(0)}\to\infty$ as in

Section 4.8.4. The event $N' = N'\left(L^{(0)}, T\right)$ depends on the choice of time interval $[0, T]$ and cut-off parameter $L^{(0)}$. Equation (3.7) then holds for all $\omega \in \Omega \setminus N_{\mathrm{ex}}$, with

$$N_{\mathrm{ex}} = \bigcup_{T \in \mathbb{N}} \bigcup_{L^{(0)} \in \mathbb{N}} N'\left(L^{(0)}, T\right), \qquad \mathbb{P}\left(N_{\mathrm{ex}}\right) = 0. \tag{6.116}$$

This finishes the proof.

6.3. A counterexample

For $d \geq 2$, fix a $T > 0$, and select $\phi_0^\varepsilon = \phi_0 \in \mathcal{H}$ to be an ε-independent state with components $\phi_{0,\sigma} \neq 0$. To simplify our argument, we furthermore assume that $\widehat{\phi}_0$ has compact support inside a ball of radius $L < \infty$ around the origin. It is clear that for $\sigma \in \{\pm\}$ and all observables $\mathfrak{a} \in \mathcal{F}L^1(C^0)$,

$$\begin{aligned} \left\langle W^\varepsilon\left[\phi_{0,\sigma}^\varepsilon\right], \mathfrak{a}\right\rangle &\to \int_{\mathbb{R}^d} \mu_0(\mathrm{d}x, \mathrm{d}k)\mathfrak{a}(x,k) = \langle \mu_{0,\sigma}, \mathfrak{a}\rangle \qquad (\varepsilon \searrow 0), \\ \mu_{0,\sigma}(\mathrm{d}x, \mathrm{d}k) &= \delta(x)\left|\widehat{\phi}_{0,\sigma}(k)\right|^2 \mathrm{d}x\mathrm{d}k. \end{aligned} \tag{6.117}$$

The initial states ϕ_0^ε trivially fulfill all assumptions of Theorem 3.3, so if we define

$$\phi_T^\varepsilon = \exp\left(+iH^\varepsilon \frac{T}{\varepsilon}\right)\phi_0^\varepsilon, \tag{6.118}$$

it is almost surely true that the random variable

$$Y_\varepsilon(\mathfrak{a}) = \max_\sigma \left|\left\langle W^\varepsilon\left[\phi_{T,\sigma}^\varepsilon\right], \mathfrak{a}\right\rangle - \left\langle \left(\mathrm{e}^{T\mathcal{L}_{-\sigma}}\right)^* \mu_{0,\sigma}, \mathfrak{a}\right\rangle\right| \tag{6.119}$$

vanishes *for all* $\mathfrak{a} \in \mathcal{F}L^1(C^0)$ in the limit $\varepsilon \searrow 0$. Here, it is important to note the sign of $\mathcal{L}_{-\sigma}$, which is a result of propagating ϕ_0^ε *backwards* in time. As $\varepsilon \searrow 0$, the quantity

$$Z_\varepsilon = \int_{|x|>(T+1)/\varepsilon} \mathrm{d}x\, |\phi_T^\varepsilon(x)|^2 + \int_{|k|>L+1} \mathrm{d}k \left|\widehat{\phi_T^\varepsilon}(k)\right|^2 \tag{6.120}$$

also converges to zero almost surely. Let $\mathfrak{X}_0$ be the space of all $\mathfrak{a} \in \mathcal{F}L^1(C^0)$ such that $\widehat{\mathfrak{a}}(p,k) \to 0$ as $|k| \to \infty$ for almost all $p \in \mathbb{R}^d$. Because $\mathfrak{X}_0$ is separable, there exists a dense, countable subset $(\mathfrak{a}_m)_{m\in\mathbb{N}}$. Furthermore, we fix a sequence of $\delta_n > 0$, $n \in \mathbb{N}$, such that $\sum_{n\in\mathbb{N}} \delta_n < \infty$. For $\varepsilon > 0$, define the event

$$\Omega_n(\varepsilon) = \left\{Y_{\tilde\varepsilon}(\mathfrak{a}_m) + Z_{\tilde\varepsilon} < \delta_n, \forall \tilde\varepsilon \in (0,\varepsilon), \forall m \in \{1, ..., n\}\right\}, \tag{6.121}$$

and observe that for any n, $\Omega_n(\varepsilon)$ increases to an event of full probability as $\varepsilon \searrow 0$. One can therefore find a decreasing sequence of $(\varepsilon_n)_{n\in\mathbb{N}}$ such that

- $\varepsilon_n \searrow 0$, as $n \to \infty$, and
- $\mathbb{P}\left(\Omega_n(\varepsilon_n)\right) > 1 - \delta_n$.

The probability space Ω only contains continuous functions $\mathbb{R}^d \to \mathbb{R}$, and therefore has at most the same cardinality as $\mathbb{R}$, allowing for a function $\underline{\omega} : (0, \varepsilon_1] \to \Omega$ which is a *surjective* map

$$(\varepsilon_{n+1}, \varepsilon_n] \to \Omega_n, \tag{6.122}$$

setting $\Omega_n = \Omega_n(\varepsilon_n)$. Now let, for $\varepsilon \in (0, \varepsilon_1]$,

$$\psi_0^\varepsilon = \exp\left(+iH^\varepsilon_{\underline{\omega}(\varepsilon)} \frac{T}{\varepsilon}\right) \phi_0^\varepsilon, \tag{6.123}$$

which is not a random state, but deterministic, as we evaluate the random generator H^ε only at a single $\omega = \underline{\omega}(\varepsilon)$. By construction, for $n \in \mathbb{N}$, $m \le n$, $\varepsilon \in (\varepsilon_{n+1}, \varepsilon_n]$,

$$\left|\left\langle W^\varepsilon\left[\psi^\varepsilon_{0,\sigma}\right], \mathfrak{a}_m\right\rangle - \left\langle \left(\mathrm{e}^{T\mathcal{L}_{-\sigma}}\right)^* \mu_{0,\sigma}, \mathfrak{a}_m\right\rangle\right| = Y_\varepsilon(\mathfrak{a}_m)\left[\underline{\omega}(\varepsilon)\right] \le \delta_n, \tag{6.124}$$

because $\underline{\omega}(\varepsilon) \in \Omega_n$. By density of the a_m in $\mathfrak{X}_0$, one has

$$\lim_{\varepsilon\to 0} \left\langle W^\varepsilon\left[\psi^\varepsilon_{0,\sigma}\right], \mathfrak{a}\right\rangle = \left\langle \left(\mathrm{e}^{T\mathcal{L}_{-\sigma}}\right)^* \mu_{0,\sigma}, \mathfrak{a}\right\rangle \tag{6.125}$$

for all $\mathfrak{a} \in \mathfrak{X}_0$. Moreover,

$$\int_{|x|>T/\varepsilon} \mathrm{d}x \, |\psi_0^\varepsilon(x)|^2 + \int_{|k|>L+1} \mathrm{d}k \left|\widehat{\psi_0^\varepsilon}(k)\right|^2 = Z_\varepsilon\left[\underline{\omega}(\varepsilon)\right] \to 0, \tag{6.126}$$

as $\varepsilon \to 0$, so (6.125) even holds for all $\mathfrak{a} \in \mathcal{F}L^1(C^0)$. Except for the condition (3.6), the sequence $(\psi_0^\varepsilon)_{\varepsilon\in(0,\varepsilon_1]}$ fulfills all requirements for initial states in Theorem 3.3. If the assertion of the theorem were still true, we would almost surely have that

$$\lim_{\varepsilon\to 0} \left\langle W^\varepsilon\left[\left(\exp\left(-iH^\varepsilon \frac{T}{\varepsilon}\right)\psi_0^\varepsilon\right)_\sigma\right], \mathfrak{a}\right\rangle = \left\langle \left(\mathrm{e}^{T\mathcal{L}_{+\sigma}}\right)^* \left(\mathrm{e}^{T\mathcal{L}_{-\sigma}}\right)^* \mu_{0,\sigma}, \mathfrak{a}\right\rangle \tag{6.127}$$

for all $a \in \mathfrak{X}$. In fact, however, as $\sum_n \delta_n < \infty$, the event

$$\Omega' = \liminf_{n\to\infty} \Omega_n \tag{6.128}$$

has probability $\mathbb{P}(\Omega') = 1$. Thus, for an arbitrary $\omega \in \Omega'$, there is a $N = N(\omega) < \infty$ such that $\omega \in \Omega_n$ for all $n \ge N$, and for all $n \ge N$ one can identify an $\varepsilon'_n = \varepsilon'_n(\omega) \in (\varepsilon_{n+1}, \varepsilon_n]$ with $\underline{\omega}(\varepsilon'_n) = \omega$. Accordingly,

$$\exp\left(-iH^{\varepsilon'_n}_\omega \frac{T}{\varepsilon'_n}\right) \psi_0^{\varepsilon'_n} = \phi_0^{\varepsilon'_n}, \tag{6.129}$$

for $n \ge N$. Concluding, for all $\omega \in \Omega'$, there is a sequence $\varepsilon'_n(\omega) \to 0$ such that along this sequence, and for this particular ω,

$$\begin{aligned} &\lim_{\varepsilon=\varepsilon'_n(\omega)\to 0} \left\langle W^\varepsilon\left[\left(\exp\left(-iH^\varepsilon \frac{T}{\varepsilon}\right)\psi_0^\varepsilon\right)_\sigma\right], \mathfrak{a}\right\rangle \\ &= \lim_{\varepsilon=\varepsilon'_n(\omega)\to 0} \left\langle W^\varepsilon\left[\phi^\varepsilon_{0,\sigma}\right], \mathfrak{a}\right\rangle \\ &= \langle \mu_{0,\sigma}, \mathfrak{a}\rangle \end{aligned} \tag{6.130}$$

for all $\mathfrak{a} \in \mathcal{F}L^1(C^0)$. As for all $T > 0$ (and non-vanishing disorder ξ)

$$\left(\mathrm{e}^{T\mathcal{L}_{+\sigma}}\right)^* \left(\mathrm{e}^{T\mathcal{L}_{-\sigma}}\right)^* \mu_{0,\sigma} \neq \mu_{0,\sigma}, \tag{6.131}$$

this constitutes a contradiction to (6.127) on a set of full probability.

A. Proof of Lemma 2.17

The proof of Lemma 2.17 given here is a modification of the one found in [23], we present it to account for the differences between the discrete model, [23], and our continuous setting. As $(\psi^\varepsilon)_{\varepsilon>0}$ fulfills conditions (2.160), (2.161) and (2.162), we have from Lemma 2.15 and Lemma 2.16 the existence of a subsequence $\varepsilon_n \to 0$ such that

$$\lim_{n\to\infty} \langle W^{\varepsilon_n}[\psi^{\varepsilon_n}], \mathfrak{a}\rangle = \int_{\mathbb{R}^{2d}} \mu(\mathrm{d}x, \mathrm{d}k)\mathfrak{a}(x,k) \tag{A.1}$$

for all $\mathfrak{a} \in \mathcal{F}L^1(C^0)$ (not yet $\mathfrak{X}_{\mathrm{IR}}$). Here, the non-negative, bounded Borel measure μ is defined on the entire phase space $\mathbb{R}^d_x \times \mathbb{R}^d_k$, and $\mu(\{k=0\})$ need not be zero. Moreover, we can assume that along the same subsequence (ε_n),

$$\varepsilon_n^{-d/2}\psi^{\varepsilon_n}\left(\frac{\cdot}{\varepsilon_n}\right) \rightharpoonup \eta \ni L^2(\mathbb{R}^d), \qquad (n\to\infty), \tag{A.2}$$

weakly in $L^2(\mathbb{R}^d)$. From now on, to keep notation simple, the subsequence (ε_n) and further subsequences to be chosen below will be referred to by (ε). For some $\lambda > 0$, performing the infra-red cutoff splits the wave function into three components

$$\psi^\varepsilon = \psi^{\varepsilon,\lambda}_> + \psi^{\varepsilon,\lambda}_< + \eta^{\varepsilon,\lambda}, \tag{A.3}$$

with

$$\begin{aligned} \widehat{\psi}^{\varepsilon,\lambda}_>(k) &= (1-\varphi(|k|/\lambda))\,\widehat{\psi}^\varepsilon(k), \\ \widehat{\psi}^{\varepsilon,\lambda}_<(k) &= \varphi(|k|/\lambda)\left(\widehat{\psi}^\varepsilon(k) - \varepsilon^{-d/2}\widehat{\eta}\,(k/\varepsilon)\right), \\ \widehat{\eta}^{\varepsilon,\lambda}(k) &= \varepsilon^{-d/2}\varphi(|k|/\lambda)\widehat{\eta}\,(k/\varepsilon)\,, \end{aligned} \tag{A.4}$$

and $\varphi : [0,\infty) \to [0,1]$ a smooth function such that $\varphi(r) = 1$ for $r \le 1$ and $\varphi(r) = 0$ for $r \ge 2$.

Now, let $\mathfrak{a} \in \mathfrak{X}_{\mathrm{IR}}$. First, consider

$$\left\langle W^\varepsilon[\psi^{\varepsilon,\lambda}_>], \mathfrak{a}\right\rangle_{\mathfrak{X}_{\mathrm{IR}}} = \int_{\mathbb{R}^d} \mathrm{d}p \left(\int_{\mathbb{R}^d} \mathrm{d}k\widehat{\mathfrak{a}}\left(p,k,\frac{k}{\varepsilon}\right)\overline{\widehat{\psi}^{\varepsilon,\lambda}_>\left(k+\frac{\varepsilon p}{2}\right)}\widehat{\psi}^{\varepsilon,\lambda}_>\left(k-\frac{\varepsilon p}{2}\right)\right), \tag{A.5}$$

and note that on the support of the integrand, $|k| \ge \lambda - \varepsilon|p|/2$, so by Definition 2.2 and dominated convergence in the p variable,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left\langle W^\varepsilon[\psi_>^{\varepsilon,\lambda}],\mathfrak{a}\right\rangle_{\mathfrak{X}_{\mathrm{IR}}}\\
=&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\mathrm{d}p\left(\int_{\mathbb{R}^d}\mathrm{d}k\widehat{\mathfrak{b}}\left(p,k,\frac{k}{|k|}\right)\overline{\widehat{\psi}_>^{\varepsilon,\lambda}\left(k+\frac{\varepsilon p}{2}\right)}\widehat{\psi}_>^{\varepsilon,\lambda}\left(k-\frac{\varepsilon p}{2}\right)\right)\\
=&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}k\widehat{\mathfrak{b}}\left(p,k,\frac{k}{|k|}\right)\\
&\qquad(1-\varphi(|k+\varepsilon p/2|/\lambda))\,(1-\varphi(|k-\varepsilon p/2|/\lambda))\,\overline{\widehat{\psi}^\varepsilon\left(k+\frac{\varepsilon p}{2}\right)}\widehat{\psi}^\varepsilon\left(k-\frac{\varepsilon p}{2}\right)\\
=&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}k\widehat{\mathfrak{b}}\left(p,k,\frac{k}{|k|}\right)(1-\varphi(|k|/\lambda))^2\,\overline{\widehat{\psi}^\varepsilon\left(k+\frac{\varepsilon p}{2}\right)}\widehat{\psi}^\varepsilon\left(k-\frac{\varepsilon p}{2}\right),
\end{aligned}\tag{A.6}$$

with the last line equal to $W^\varepsilon[\psi^\varepsilon]$ being tested against an ordinary $\mathcal{F}L^1(C^0)$ function. Thus, from (A.1),

$$\lim_{\varepsilon\to 0}\left\langle W^\varepsilon[\psi_>^{\varepsilon,\lambda}],\mathfrak{a}\right\rangle_{\mathfrak{X}_{\mathrm{IR}}}=\int_{\mathbb{R}^{2d}}\mu(\mathrm{d}x,\mathrm{d}k)\mathfrak{b}\left(x,k,\frac{k}{|k|}\right)(1-\varphi(|k|/\lambda))^2\,. \tag{A.7}$$

Next, with $f^{\varepsilon,\lambda}$ being either of $\psi_<^{\varepsilon,\lambda}$ or $\eta^{\varepsilon,\lambda}$, for cross-terms like

$$\int_{\mathbb{R}^d}\mathrm{d}p\left(\int_{\mathbb{R}^d}\mathrm{d}k\widehat{\mathfrak{a}}\left(p,k,\frac{k}{\varepsilon}\right)\overline{\widehat{\psi}_>^{\varepsilon,\lambda}\left(k+\frac{\varepsilon p}{2}\right)}\widehat{f}^{\varepsilon,\lambda}\left(k-\frac{\varepsilon p}{2}\right)\right), \tag{A.8}$$

one has on the support of the integrand

$$\lambda\le|k+\varepsilon p/2|\le 2\lambda+\varepsilon|p|, \tag{A.9}$$

and thus, again by dominated convergence in p,

$$\begin{aligned}
&\overline{\lim_{\varepsilon\to 0}}\left|\int_{\mathbb{R}^d}\mathrm{d}p\left(\int_{\mathbb{R}^d}\mathrm{d}k\widehat{\mathfrak{a}}\left(p,k,\frac{k}{\varepsilon}\right)\overline{\widehat{\psi}_>^{\varepsilon,\lambda}\left(k+\frac{\varepsilon p}{2}\right)}\widehat{f}^{\varepsilon,\lambda}\left(k-\frac{\varepsilon p}{2}\right)\right)\right|^2\\
\le&\overline{\lim_{\varepsilon\to 0}}\,\|\mathfrak{a}\|_{\mathfrak{X}_{\mathrm{IR}}}\left\|\widehat{f}^{\varepsilon,\lambda}\right\|_{L^2}^2\int_{\mathbb{R}^d}\mathrm{d}p\sup_{k',k''}|\widehat{\mathfrak{a}}\,(p,k',k'')|\int_{\mathbb{R}^d}\mathrm{d}k\mathbb{1}\left(\lambda\le|k|\le 2\lambda+\varepsilon|p|\right)\left|\widehat{\psi^\varepsilon}(k)\right|^2\\
\le&4\,\|\mathfrak{a}\|_{\mathfrak{X}_{\mathrm{IR}}}^2\,\overline{\lim_{\varepsilon\to 0}}\,\|\psi^\varepsilon\|_{L^2}^2\int_{\mathbb{R}^d}\mathrm{d}k\mathbb{1}\left(\lambda\le|k|\le 3\lambda\right)\left|\widehat{\psi^\varepsilon}(k)\right|^2\\
\le&4\,\|\mathfrak{a}\|_{\mathfrak{X}_{\mathrm{IR}}}^2\left(\overline{\lim_{\varepsilon\to 0}}\,\|\psi^\varepsilon\|_{L^2}^2\right)\int_{\mathbb{R}^{2d}}\mu(\mathrm{d}x,\mathrm{d}k)\mathbb{1}\left(\lambda\le|k|\le 3\lambda\right),
\end{aligned}\tag{A.10}$$

where we used Lemma 2.16 in the last line. Then, by dominated convergence in k,

$$\begin{aligned}
&\lim_{\lambda\to 0}\overline{\lim_{\varepsilon\to 0}}\left|\int_{\mathbb{R}^d}\mathrm{d}p\left(\int_{\mathbb{R}^d}\mathrm{d}k\widehat{\mathfrak{a}}\left(p,k,\frac{k}{\varepsilon}\right)\overline{\widehat{f}^{\varepsilon,\lambda}\left(k+\frac{\varepsilon p}{2}\right)}\widehat{\psi}_>^{\varepsilon,\lambda}\left(k-\frac{\varepsilon p}{2}\right)\right)\right|\\
=&\lim_{\lambda\to 0}\overline{\lim_{\varepsilon\to 0}}\left|\int_{\mathbb{R}^d}\mathrm{d}p\left(\int_{\mathbb{R}^d}\mathrm{d}k\widehat{\mathfrak{a}}\left(p,k,\frac{k}{\varepsilon}\right)\overline{\widehat{\psi}_>^{\varepsilon,\lambda}\left(k+\frac{\varepsilon p}{2}\right)}\widehat{f}^{\varepsilon,\lambda}\left(k-\frac{\varepsilon p}{2}\right)\right)\right|=0.
\end{aligned}\tag{A.11}$$

Now, we substitute $\underline{k} = k/\varepsilon$ in

$$\begin{aligned}\left\langle W^{\varepsilon}\left[\eta^{\varepsilon,\lambda}\right], \mathfrak{a}\right\rangle_{\mathfrak{X}_{\mathrm{IR}}} &= \varepsilon^{-d}\int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}k\widehat{\mathfrak{a}}\left(p,k,\frac{k}{\varepsilon}\right)\varphi\left(\frac{|k+\varepsilon p/2|}{\lambda}\right)\varphi\left(\frac{|k-\varepsilon p/2|}{\lambda}\right)\\ &\qquad\overline{\widehat{\eta}\left(\frac{k+\varepsilon p/2}{\varepsilon}\right)}\widehat{\eta}\left(\frac{k-\varepsilon p/2}{\varepsilon}\right)\\ &= \int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}\underline{k}\widehat{\mathfrak{a}}\left(p,\varepsilon\underline{k},\underline{k}\right)\varphi\left(\frac{|\varepsilon\underline{k}+\varepsilon p/2|}{\lambda}\right)\varphi\left(\frac{|\varepsilon\underline{k}-\varepsilon p/2|}{\lambda}\right)\\ &\qquad\overline{\widehat{\eta}\left(\underline{k}+p/2\right)}\widehat{\eta}\left(\underline{k}-p/2\right).\end{aligned} \tag{A.12}$$

For any fixed $\lambda > 0$, we thus have

$$\begin{aligned}\lim_{\varepsilon\to 0}\left\langle W^{\varepsilon}\left[\eta^{\varepsilon,\lambda}\right], \mathfrak{a}\right\rangle_{\mathfrak{X}_{\mathrm{IR}}} &= \int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}\underline{k}\widehat{\mathfrak{a}}\left(p,0,\underline{k}\right)\overline{\widehat{\eta}\left(\underline{k}+p/2\right)}\widehat{\eta}\left(\underline{k}-p/2\right)\\ &= \langle W[\eta], \mathfrak{a}(\cdot,0,\cdot)\rangle.\end{aligned} \tag{A.13}$$

In the only remaining cross-term, the one between $\psi_{<}^{\varepsilon,\lambda}$ and $\eta^{\varepsilon,\lambda}$, the same substitution $\underline{k} = k/\varepsilon$ produces

$$\begin{aligned}&\varepsilon^{-d}\int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}k\widehat{\mathfrak{a}}\left(p,k,\frac{k}{\varepsilon}\right)\varphi\left(\frac{|k+\varepsilon p/2|}{\lambda}\right)\overline{\widehat{\eta}\left(\frac{k+\varepsilon p/2}{\varepsilon}\right)}\\ &\qquad\varphi\left(\frac{|k-\varepsilon p/2|}{\lambda}\right)\left(\varepsilon^{d/2}\widehat{\psi}^{\varepsilon}\left(k-\varepsilon p/2\right)-\widehat{\eta}\left(\frac{k-\varepsilon p/2}{\varepsilon}\right)\right)\\ &= \int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}\underline{k}\widehat{\mathfrak{a}}\left(p,\varepsilon\underline{k},\underline{k}\right)\varphi\left(\frac{|\varepsilon\underline{k}+\varepsilon p/2|}{\lambda}\right)\overline{\widehat{\eta}\left(\underline{k}+p/2\right)}\\ &\qquad\varphi\left(\frac{|\varepsilon\underline{k}-\varepsilon p/2|}{\lambda}\right)\left(\varepsilon^{d/2}\widehat{\psi}^{\varepsilon}\left(\varepsilon\underline{k}-\varepsilon p/2\right)-\widehat{\eta}\left(\underline{k}-p/2\right)\right).\end{aligned} \tag{A.14}$$

For fixed $p \in \mathbb{R}^d$, and $\lambda > 0$, it is immediate from (A.2) that the function

$$\varphi\left(\frac{|\varepsilon\underline{k}-\varepsilon p/2|}{\lambda}\right)\left(\varepsilon^{d/2}\widehat{\psi}^{\varepsilon}\left(\varepsilon\underline{k}-\varepsilon p/2\right)-\widehat{\eta}\left(\underline{k}-p/2\right)\right)\rightharpoonup 0 \quad (\varepsilon\to 0) \tag{A.15}$$

weakly in $L^2(\mathbb{R}^d_{\underline{k}})$, while

$$\widehat{\mathfrak{a}}\left(p,\varepsilon\underline{k},\underline{k}\right)\varphi\left(\frac{|\varepsilon\underline{k}+\varepsilon p/2|}{\lambda}\right)\overline{\widehat{\eta}\left(\underline{k}+p/2\right)} \to \widehat{\mathfrak{a}}\left(p,0,\underline{k}\right)\overline{\widehat{\eta}\left(\underline{k}+p/2\right)} \quad (\varepsilon\to 0) \tag{A.16}$$

strongly in $L^2(\mathbb{R}^d_{\underline{k}})$. By dominated convergence for the p integral, (A.14) thus vanishes for $\varepsilon \to 0$ for every fixed positive λ. We are now left with analyzing the limit of

$$\begin{aligned}&\left\langle W^{\varepsilon}\left[\psi_{<}^{\varepsilon,\lambda}\right], \mathfrak{a}\right\rangle_{\mathfrak{X}_{\mathrm{IR}}}\\ =&\varepsilon^{d}\int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}\underline{k}\widehat{\mathfrak{a}}\left(p,\varepsilon\underline{k},\underline{k}\right)\overline{\widehat{\psi}_{<}^{\varepsilon,\lambda}(\varepsilon\underline{k}+\varepsilon p/2)}\widehat{\psi}_{<}^{\varepsilon,\lambda}(\varepsilon\underline{k}-\varepsilon p/2)\end{aligned} \tag{A.17}$$

for $\mathfrak{a} \in \mathfrak{X}_{\mathrm{IR}}$. As noted before, for every fixed $\lambda > 0$, the function

$$\widehat{w}^{\varepsilon,\lambda}(q) = \varepsilon^{d/2}\widehat{\psi}^{\varepsilon,\lambda}_{<}(\varepsilon q) = \varphi\left(\frac{|\varepsilon q|}{\lambda}\right)\left(\varepsilon^{d/2}\widehat{\psi}^{\varepsilon}\left(\varepsilon q\right) - \widehat{\eta}\left(q\right)\right) \tag{A.18}$$

converges to 0 weakly in $L^2(\mathbb{R}^d_q)$. Moreover, it is tight on position space, which can be readily seen from the representation

$$w^{\varepsilon,\lambda}(x) = \varepsilon^{-d/2}\psi^{\varepsilon,\lambda}_{<}(x/\varepsilon) = \left(\frac{\lambda}{\varepsilon}\right)^d \int_{\mathbb{R}^d} \mathrm{d}y \chi\left(\frac{\lambda|x-y|}{\varepsilon}\right)\left(\varepsilon^{-d/2}\psi^{\varepsilon}\left(y/\varepsilon\right) - \eta\left(y\right)\right), \tag{A.19}$$

with a fixed function $\chi \in \mathcal{S}(\mathbb{R}^d)$, and (2.161). From Lemma A.2, [23], we have for any subset $B \subset \mathbb{R}^d$ with bounded Lebesgue measure that

$$\lim_{\varepsilon\to\infty}\left\|\widehat{w}^{\varepsilon,\lambda}\right\|_{L^2(B)} = 0, \tag{A.20}$$

while $\lambda > 0$ remains fixed. Accordingly, for

$$\widehat{w}^{\varepsilon,\lambda,M}(q) = \left(1 - \varphi\left(\frac{|q|}{2M}\right)\right)\widehat{w}^{\varepsilon,\lambda}(q) \tag{A.21}$$

and any $\lambda, M \in (0,\infty)$ fixed,

$$\begin{aligned}\lim_{\varepsilon\to 0}\Bigg| &\int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}\underline{k}\widehat{\mathfrak{a}}\left(p,\varepsilon\underline{k},\underline{k}\right)\overline{\widehat{w}^{\varepsilon,\lambda}(\underline{k}+p/2)}\widehat{w}^{\varepsilon,\lambda}(\underline{k}-p/2)\\ &- \int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}\underline{k}\widehat{\mathfrak{a}}\left(p,\varepsilon\underline{k},\underline{k}\right)\overline{\widehat{w}^{\varepsilon,\lambda,M}(\underline{k}+p/2)}\widehat{w}^{\varepsilon,\lambda,M}(\underline{k}-p/2)\Bigg| = 0\end{aligned} \tag{A.22}$$

by (2.160). On the other hand, by definition of $\mathfrak{X}_{\mathrm{IR}}$,

$$\begin{aligned}\lim_{M\to\infty}\sup_{\varepsilon,\lambda}\Bigg| &\int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}\underline{k}\widehat{\mathfrak{a}}\left(p,\varepsilon\underline{k},\underline{k}\right)\overline{\widehat{w}^{\varepsilon,\lambda,M}(\underline{k}+p/2)}\widehat{w}^{\varepsilon,\lambda,M}(\underline{k}-p/2)\\ &- \int_{|p|\le M}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}\underline{k}\widehat{\mathfrak{a}}\left(p,\varepsilon\underline{k},\underline{k}\right)\overline{\widehat{w}^{\varepsilon,\lambda,M}(\underline{k}+p/2)}\widehat{w}^{\varepsilon,\lambda,M}(\underline{k}-p/2)\Bigg| = 0.\end{aligned} \tag{A.23}$$

On the support of the integrand in the last line of (A.23), $M \le \underline{k} \le \lambda/\varepsilon$, so

$$\begin{aligned}\lim_{\lambda\to 0}\sup_{\varepsilon,M}\Bigg| &\int_{|p|\le M}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}\underline{k}\widehat{\mathfrak{a}}\left(p,\varepsilon\underline{k},\underline{k}\right)\overline{\widehat{w}^{\varepsilon,\lambda,M}(\underline{k}+p/2)}\widehat{w}^{\varepsilon,\lambda,M}(\underline{k}-p/2)\\ &- \int_{|p|\le M}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}\underline{k}\widehat{\mathfrak{a}}\left(p,0,\underline{k}\right)\overline{\widehat{w}^{\varepsilon,\lambda,M}(\underline{k}+p/2)}\widehat{w}^{\varepsilon,\lambda,M}(\underline{k}-p/2)\Bigg| = 0,\end{aligned} \tag{A.24}$$

as well as

$$\lim_{M\to\infty}\sup_{\varepsilon,\lambda}\left|\int_{|p|\le M}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}\underline{k}\widehat{\mathfrak{a}}\left(p,0,\underline{k}\right)\overline{\widehat{w}^{\varepsilon,\lambda,M}(\underline{k}+p/2)}\widehat{w}^{\varepsilon,\lambda,M}(\underline{k}-p/2)\right.$$
$$\left.-\int_{|p|\le M}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}\underline{k}\widehat{\mathfrak{b}}\left(p,0,\frac{\underline{k}}{|\underline{k}|}\right)\overline{\widehat{w}^{\varepsilon,\lambda,M}(\underline{k}+p/2)}\widehat{w}^{\varepsilon,\lambda,M}(\underline{k}-p/2)\right|=0. \tag{A.25}$$

The last line clearly defines a bounded linear functional on the separable Banach space $\mathfrak{C}$ of all functions $\mathfrak{c}:\mathbb{R}^d\times S^{d-1}\to\mathbb{C}$ such that the Fourier transform in the first variable $\widehat{\mathfrak{c}}$ fulfills, $\widehat{\mathfrak{c}}\in L^1(\mathbb{R}^d;C^0(S^{d-1}))$, by

$$\left\langle\mathfrak{c},\Lambda^{\varepsilon,\lambda,M}\right\rangle=\int_{|p|\le M}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}\underline{k}\widehat{\mathfrak{c}}\left(p,\frac{\underline{k}}{|\underline{k}|}\right)\overline{\widehat{w}^{\varepsilon,\lambda,M}(\underline{k}+p/2)}\widehat{w}^{\varepsilon,\lambda,M}(\underline{k}-p/2). \tag{A.26}$$

The functionals $\Lambda^{\varepsilon,\lambda,M}$ are uniformly bounded by

$$\sup_{\varepsilon,\lambda,M}\left|\left\langle\mathfrak{c},\Lambda^{\varepsilon,\lambda,M}\right\rangle\right|\le 4\left(\int_{\mathbb{R}^d}\mathrm{d}p\sup_{|\underline{k}|=1}|\widehat{\mathfrak{c}}(p,\underline{k})|\right)\sup_{\varepsilon>0}\|\psi^\varepsilon\|^2_{L^2}. \tag{A.27}$$

Therefore, by the Banach-Alaoglu theorem, there is, for each choice of λ and M, a subsequence of $\varepsilon\to 0$ such that along this subsequence $\Lambda^{\varepsilon,\lambda,M}\stackrel{*}{\rightharpoonup}\Lambda^{\lambda,M}$ in $\mathfrak{C}^*$, then, for each choice of λ there is a subsequence of $M\to\infty$ such that $\Lambda^{\lambda,M}\stackrel{*}{\rightharpoonup}\Lambda^{\lambda}$ along this subsequence, and finally, along a subsequence $\lambda\to 0$, one has $\Lambda^\lambda\stackrel{*}{\rightharpoonup}\Lambda$. One can show as in [23] that Λ is given by a non-negative Borel measure μ^{H} on $\mathbb{R}^d\times S^{d-1}$. By diagonalization there is a subsequence $\lambda_m\to 0$ and a subsequence $\varepsilon_n'\to 0$ independent of it such that

$$\lim_{\lambda_m\to 0}\lim_{\varepsilon_n'\to 0}\left\langle W^{\varepsilon_n'}\left[\psi_<^{\varepsilon_n',\lambda_m}\right],\mathfrak{a}\right\rangle_{\mathfrak{X}_{\mathrm{IR}}}=\int_{\mathbb{R}^d\times S^{d-1}}\mu^{\mathrm{H}}(\mathrm{d}x,\mathrm{d}\underline{k})\mathfrak{b}(x,0,\underline{k}). \tag{A.28}$$

Together with (A.13) and the $\lambda\to 0$ limit of (A.7), this is just the right side of (2.186). So far, we have verified that there do exist subsequences $W^\varepsilon[\psi^\varepsilon]$ that converge weak-* in ${\mathfrak{X}_{\mathrm{IR}}}^*$, and that their limit point are of the form (2.186). That *all* limit points are of this form is shown by applying the same reasoning to an arbitrary convergent subsequence, and noting that there are sub-subsequences along which the limit object has to be of the above $(\mu,\mu^{\mathrm{H}},\eta)$ kind.

B. The two-resolvent integral

Lemma B.1. *For all dimensions $d \geq 2$ there is a constant C_d such that for all $\gamma \in (0,1]$, $\alpha_{1,2} \in \mathbb{R}$, and $u \in \mathbb{R}^d \setminus \{0\}$, $q \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} \mathrm{d}k \frac{1}{|\alpha_1 - 2\pi|k| + i\gamma|\,|\alpha_2 - 2\pi|k-u| + i\gamma|\,\langle k\rangle^2 \langle k-u\rangle^2 \langle k-q\rangle^{d-3}} \leq \frac{C_d}{|u|} \frac{\langle \log\gamma\rangle^2}{\sqrt{\langle\alpha_1\rangle\langle\alpha_2\rangle}}, \tag{B.1}$$

for $d \geq 3$, while for $d = 2$

$$\int_{\mathbb{R}^2} \mathrm{d}k \frac{1}{|\alpha_1 - 2\pi|k| + i\gamma|\,|\alpha_2 - 2\pi|k-u| + i\gamma|\,\langle k\rangle^2 \langle k-u\rangle^2} \leq \frac{C_2 \langle\log\gamma\rangle}{\sqrt{\gamma|u|}\sqrt{\langle\alpha_1\rangle\langle\alpha_2\rangle}}. \tag{B.2}$$

Proof. Let P be the projection on $\operatorname{span}\{u\}^\perp$, and note that

$$k \mapsto \left(|k|, |k-u|, \frac{Pk}{|Pk|}\right) = (\rho_1, \rho_2, \omega) \tag{B.3}$$

is a diffeomorphism from $\mathbb{R}^d \setminus \operatorname{span}\{u\}$ to $\{\rho_1, \rho_2 : |\rho_1 - \rho_2| < |u| < \rho_1 + \rho_2\} \times S^{d-2}$ with

$$\mathrm{d}k = \frac{\rho_1\rho_2}{|u|} y^{d-3} \mathrm{d}\rho_1 \mathrm{d}\rho_2 \mathrm{d}\omega, \tag{B.4}$$

where $y = |Pk|$. As, for $d \geq 3$,

$$y(\rho_1,\rho_2)^{d-3} \int_{S^{d-2}} \mathrm{d}\omega \,\langle k(\rho_1,\rho_2,\omega) - q\rangle^{-d+3} \leq C_d' \tag{B.5}$$

with C_d' independent of q and ρ_1, ρ_2,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{d}k \frac{1}{|\alpha_1 - 2\pi|k| + i\gamma|\,|\alpha_2 - 2\pi|k-u| + i\gamma|\,\langle k\rangle^2 \langle k-u\rangle^2 \langle k-q\rangle^{d-3}} \\
&\quad\leq \frac{C_d'}{|u|} \int_0^\infty \mathrm{d}\rho_1 \int_0^\infty \mathrm{d}\rho_2 \frac{1}{|\alpha_1 - 2\pi\rho_1 + i\gamma|\,|\alpha_2 - 2\pi\rho_2 + i\gamma|\,\langle\rho_1\rangle\langle\rho_2\rangle} \\
&\quad\leq \frac{C_d}{|u|} \frac{\langle\log\gamma\rangle^2}{\sqrt{\langle\alpha_1\rangle\langle\alpha_2\rangle}}.
\end{aligned} \tag{B.6}$$

For $d = 2$, we cut out the y^{-1} singularity and obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2, y>\delta} \mathrm{d}k \frac{1}{|\alpha_1 - 2\pi|k| + i\gamma|\,|\alpha_2 - 2\pi|k-u| + i\gamma|\,\langle k\rangle^2 \langle k-u\rangle^2} \\
&\quad\leq \frac{2}{|u|\delta} \int_0^\infty \mathrm{d}\rho_1 \int_0^\infty \mathrm{d}\rho_2 \frac{1}{|\alpha_1 - 2\pi\rho_1 + i\gamma|\,|\alpha_2 - 2\pi\rho_2 + i\gamma|\,\langle\rho_1\rangle\langle\rho_2\rangle} \\
&\quad\leq \frac{C}{|u|\delta} \frac{\langle\log\gamma\rangle^2}{\sqrt{\langle\alpha_1\rangle\langle\alpha_2\rangle}},
\end{aligned} \tag{B.7}$$

while

$$\begin{aligned}&\int_{\mathbb{R}^2, y\le\delta} \mathrm{d}k \frac{1}{|\alpha_1 - 2\pi|k| + i\gamma|^2 \langle k\rangle^4} \\ &\le C\delta \int_0^\infty \frac{\mathrm{d}\rho}{|\alpha_1 - 2\pi\rho + i\gamma|^2 \langle\rho\rangle^4} \le C\frac{\delta}{\gamma\langle\alpha_1\rangle^2}.\end{aligned} \tag{B.8}$$

By Cauchy-Schwarz, we obtain for $\delta = \sqrt{\gamma/|u|}\,\langle\log\gamma\rangle$

$$\int_{\mathbb{R}^2} \mathrm{d}k \frac{1}{|\alpha_1 - 2\pi|k| + i\gamma|\,|\alpha_2 - 2\pi|k-u| + i\gamma|\,\langle k\rangle^2 \langle k-u\rangle^2} \le \frac{C_2\langle\log\gamma\rangle}{\sqrt{\gamma|u|}\sqrt{\langle\alpha_1\rangle\langle\alpha_2\rangle}}. \tag{B.9}$$

□

Lemma B.2. *For all dimensions $d \ge 2$, there is a finite constant $\tilde{C}_d$ such that for all $\alpha \in \mathbb{R}$ and uniformly in $u, q \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} \frac{\mathrm{d}k}{|k-u|\,|\alpha - 2\pi|k| + i\gamma|\,\langle k\rangle^2 \langle k-q\rangle^{d-3}} \le \frac{\tilde{C}_d\langle\log\gamma\rangle}{\sqrt{\langle\alpha\rangle}} \tag{B.10}$$

if $d \ge 3$, and

$$\int_{\mathbb{R}^2} \frac{\mathrm{d}k}{\sqrt{|k-u|}\,|\alpha - 2\pi|k| + i\gamma|\,\langle k\rangle^2} \le \frac{\tilde{C}_2\langle\log\gamma\rangle}{\langle\alpha\rangle} \tag{B.11}$$

if $d = 2$.

Proof. For $d \ge 3$, with the substitution (B.3) for k,

$$\begin{aligned}&\int_{\mathbb{R}^d} \frac{\mathrm{d}k}{|k-u|\,|\alpha - 2\pi|k| + i\gamma|\,\langle k\rangle^2 \langle k-q\rangle^{d-3}} \\ &\le \frac{C_d'}{|u|}\int_0^\infty \mathrm{d}\rho_1 \int_{|\rho_1 - |u||}^{\rho_1+|u|} \mathrm{d}\rho_2 \frac{\rho_1\rho_2}{\rho_2\,|\alpha - 2\pi\rho_1 + i\gamma|\,\langle\rho_1\rangle^2} \\ &\le 2C_d' \int_0^\infty \frac{\mathrm{d}\rho_1}{|\alpha - 2\pi\rho_1 + i\gamma|\,\langle\rho_1\rangle} \\ &\le \frac{\tilde{C}_d\langle\log\gamma\rangle}{\sqrt{\langle\alpha\rangle}},\end{aligned} \tag{B.12}$$

while for $d = 2$, the estimate

$$\int_{\mathbb{R}^2} \frac{\mathrm{d}k}{\sqrt{|k-u|}\,|\alpha - 2\pi|k| + i\gamma|\,\langle k\rangle^2} \le C\int_0^\infty \frac{\mathrm{d}\rho}{|\alpha - 2\pi\rho + i\gamma|\,\langle\rho\rangle^{3/2}} \le \frac{\tilde{C}_2\langle\log\gamma\rangle}{\langle\alpha\rangle} \tag{B.13}$$

follows for all u directly from

$$\sup_{u\in\mathbb{R}^2}\int_{\mathbb{R}^2} \mathrm{d}k \frac{\delta(|k|-r)}{\sqrt{|k-u|}} \le C\sqrt{r}. \tag{B.14}$$

□

C. Oscillatory integrals

Lemma C.1. *For a space dimension d, let $m, r, \overline{\beta} \in \mathbb{N}$, with $\overline{\beta} \leq d$ and $m \geq d + r + 1$ and $\phi : \mathbb{R}^d \to \mathbb{C}$ such that*

$$\left| \frac{\partial^\beta}{\partial q^\beta} \phi(q) \right| \leq C_\phi \langle q \rangle^{-m} \tag{C.1}$$

for all multi-indices β with $|\beta| \leq \overline{\beta}$. Assume $f : \mathbb{R}^{2d} \to \mathbb{C}$ has radial derivatives with respect to p such that

$$\left| \frac{\partial^n}{\partial |p|^n} f(k,p) \right| \leq C_f(k) \langle k - p \rangle^r \tag{C.2}$$

for all $p, k \in \mathbb{R}^d$, and all $n \leq \overline{\beta}$. Then, there is a $C_{\overline{\beta},d}$ depending only on $\overline{\beta}, d$ such that

$$I(k) = \int_{\mathbb{R}^d} \mathrm{d}p \phi(k-p) f(k,p) e^{-2\pi i |p| s} \tag{C.3}$$

is bounded by

$$|I(k)| \leq C_{d,\overline{\beta}} C_\phi C_f(k) \langle s \rangle^{-\overline{\beta}}. \tag{C.4}$$

Proof. Observe that for

$$\tilde{I}(k,\rho) = \int_{\mathbb{R}^d} \mathrm{d}p \delta(|p| - \rho) \phi(k-p) f(k,p) \tag{C.5}$$

and all $n \leq \overline{\beta} - 1$,

$$\left| \frac{\partial^n}{\partial \rho^n} \tilde{I}(k,\rho) \right| \leq C_{d,\overline{\beta}} C_\phi C_f(k) \left(\frac{\rho}{\langle \rho \rangle} \right)^{d-1-n} \langle |k| - \rho \rangle^{-m+d+r-1/2}, \tag{C.6}$$

while for $n = \overline{\beta}$

$$\left| \frac{\partial^{\overline{\beta}}}{\partial \rho^{\overline{\beta}}} \tilde{I}(k,\rho) \right| \leq C_{d,\overline{\beta}} C_\phi C_f(k) \langle |k| - \rho \rangle^{-m+d+r-1/2}. \tag{C.7}$$

Therefore, whenever $m \geq d + r + 1$ and $s \neq 0$, one can perform $\overline{\beta} - 1$ integrations by parts on the right side of

$$I(k) = \int_0^\infty \mathrm{d}\rho \tilde{I}(k,\rho) e^{-2\pi i \rho s} \tag{C.8}$$

without obtaining boundary terms, and then an $\overline{\beta}$-th integration by parts yields the estimate

$$|I(k)| \leq C_{d,\overline{\beta}} C_\phi C_f(k) |s|^{-\overline{\beta}}. \tag{C.9}$$

Together with the trivial estimate

$$|I(k)| \leq \int_{\mathbb{R}^d} \mathrm{d}p \, |\phi(k-p) f(k,p)| \leq C_{d,\overline{\beta}} C_\phi C_f(k), \tag{C.10}$$

the lemma is proven after redefining $C_{\overline{\beta},d}$. □

Lemma C.2. *In addition to the assumptions of the previous Lemma C.1, let* $d \geq \overline{\beta} = 2$*. Then for* $\alpha \in \mathbb{R}$*,* $n \in \mathbb{N}$ *and* $0 < \gamma_1 \leq ... \leq \gamma_n \leq 1$*, one has, for all* $k \in \mathbb{R}^d$*,*

$$\sup_{\alpha\in\mathbb{R},\sigma\in\{\pm\}} \left| \int_{\mathbb{R}^d} \mathrm{d}p \frac{\phi(k-p)f(k,p)}{(\alpha + i\gamma_1 - 2\pi\sigma|p|)} \right| \leq C_d C_\phi C_f(k) \tag{C.11}$$

for $n = 1$*,*

$$\sup_{\alpha\in\mathbb{R},\sigma\in\{\pm\}} \left| \int_{\mathbb{R}^d} \mathrm{d}p \frac{\phi(k-p)f(k,p)}{(\alpha + i\gamma_1 - 2\pi\sigma|p|)(\alpha + i\gamma_2 - 2\pi\sigma|p|)} \right| \leq C_d C_\phi C_f(k) \langle \log \gamma_1 \rangle \tag{C.12}$$

for $n = 2$*, and*

$$\sup_{\alpha\in\mathbb{R},\sigma\in\{\pm\}} \left| \int_{\mathbb{R}^d} \mathrm{d}p \frac{\phi(k-p)f(k,p)}{(\alpha + i\gamma_1 - 2\pi\sigma|p|)\cdots(\alpha + i\gamma_n - 2\pi\sigma|p|)} \right| \leq C_d C_\phi C_f(k) \gamma_1^{2-n} \tag{C.13}$$

for $n \geq 3$*, with constants* C_ϕ *and* $C_f(k)$ *defined as before, and* C_d *just depending on dimension* $d \geq 2$*. The analogue estimates hold for* $0 > \gamma_1 \geq ... \geq \gamma_n$*.*

Proof. For the case of all $\gamma_l > 0$,

$$\prod_{l=1}^{n} \frac{i}{\alpha + i\gamma_l - 2\pi\sigma|p|} = \int_{\mathbb{R}_+^n} \mathrm{d}s \prod_{l=1}^{n} \exp\left((i\alpha - \gamma_l - 2\pi i\sigma|p|)s_l\right), \tag{C.14}$$

and thus, by Fubini's theorem and Lemma C.1

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} \mathrm{d}p \frac{\phi(k-p)f(k,p)}{(\alpha + i\gamma_1 - 2\pi\sigma|p|)\cdots(\alpha + i\gamma_n - 2\pi\sigma|p|)} \right| \\ &\leq C_{d,2} C_\phi C_f(k) \int_{\mathbb{R}_+^n} \mathrm{d}s \exp\left(-\gamma_1 \sum_{l=1}^{n} s_l\right) \left\langle \sum_{l=1}^{n} s_l \right\rangle^{-2} \\ &= C_{d,2} C_\phi C_f(k) \frac{1}{(n-1)!} \int_0^\infty \mathrm{d}u \frac{u^{n-1}}{\langle u \rangle^2} e^{-\gamma_1 u}. \end{aligned} \tag{C.15}$$

The integral is bounded by a constant C for $n = 1$ and by $C \langle \log \gamma_1 \rangle$ for $n = 2$. For $n \geq 3$, we can estimate it by

$$\int_0^\infty \mathrm{d}u u^{n-3} e^{-\gamma_1 u} = \gamma_1^{2-n} \int_0^\infty \mathrm{d}x x^{n-3} e^{-x} = \gamma_1^{2-n}(n-3)! \tag{C.16}$$

□

Lemma C.3. *Let $d \geq 2$ and $\phi : \mathbb{R}^d \to \mathbb{C}$ with a constant $C_\phi < \infty$ and an $m \geq d+1$ such that*

$$\left| \frac{\partial^\alpha}{\partial q^\alpha} \phi(q) \right| \leq C_\phi \langle q \rangle^{-m} \tag{C.17}$$

for all multiindices with $0 \leq |\alpha| \leq 4$. For some $n \in \mathbb{N}$, similar to Definition 4.2, let

$$\check{\cdot} : \{1, ...n\} \to \{0, ..., n\} \tag{C.18}$$

be a map free of fixed points or cycles, i.e. it can be interpreted as mapping each element of $\{1, ...n\}$ to its parent, with 0 the common ancestor of all $l \in \{1, ...n\}$. Furthermore, let functions $f_l : \left(\mathbb{R}^d \setminus \{0\}\right)^2 \to \mathbb{C}$, $l \in \{1, ...n\}$ be given which are not necessarily differentiable, but have up to second radial derivatives,

$$\sup_{x,y \neq 0} \max_{0 \leq k \leq 2} \left| \frac{\partial^k}{\partial |x|^k} f_l(x, y) \right| \leq C_{f_l} \quad (l \in \{1, ..., n\}, \ \check{l} = 0) \tag{C.19}$$

and

$$\sup_{x,y \neq 0} \max_{0 \leq j,k \leq 2} \left| \frac{\partial^j}{\partial |x|^j} \frac{\partial^k}{\partial |y|^k} f_l(x, y) \right| \leq C_{f_l} \quad (l \in \{1, ..., n\}, \ \check{l} \neq 0) \tag{C.20}$$

with constants $C_{f_l} < \infty$. Then independently of the exact structure of $\check{\cdot}$, there is a $C_d < \infty$ only depending on dimension d such that

$$\sup_{q_0 \in \mathbb{R}^d} \left| \int_{\mathbb{R}^{dn}} \mathrm{d}q_1 ... \mathrm{d}q_n \prod_{l=1}^n \left(\phi\left(q_l - q_{\check{l}}\right) f_l\left(q_l, q_{\check{l}}\right) e^{-2\pi i b_l |q_l|} \right) \right| \leq C_d^n C_\phi^n \prod_{l=1}^n \left(C_{f_l} \langle b_l \rangle^{-2} \right) \tag{C.21}$$

for all $b_1, ..., b_n \in \mathbb{R}$.

Proof. For $L \in \{0, ..., n\}$, define the set of children

$$\underline{L} = \left\{ l \in \{1, ..., n\} : \check{l} = L \right\} \tag{C.22}$$

and similarly the set of grandchildren $\underline{\underline{L}}$ and so forth. The set of all descendants of L is given as

$$K_L = \underline{L} \cup \underline{\underline{L}} \cup ... \tag{C.23}$$

After relabeling we can assume that $\underline{0} = \{1, ..., r\}$, $1 \leq r \leq n$ and observe that the integral factorizes

$$\begin{aligned} &\int_{\mathbb{R}^{dn}} \mathrm{d}q_1 ... \mathrm{d}q_n \prod_{l=1}^n \left(\phi\left(q_l - q_{\check{l}}\right) f_l\left(q_l, q_{\check{l}}\right) e^{-2\pi i b_l |q_l|} \right) \\ &\quad = \prod_{j=1}^r \int_{\mathbb{R}^d} \mathrm{d}q_j f_j\left(q_j, q_0\right) \phi\left(q_j - q_0\right) e^{-2\pi i b_j |q_j|} H_j\left(q_j\right), \end{aligned} \tag{C.24}$$

with

$$H_j\left(q_j\right)=\int_{\mathbb{R}^{d|K_j|}}\prod_{l\in K_j}\left(\mathrm{d}q_l\phi\left(q_l-q_{\check{l}}\right)f_l\left(q_l,q_{\check{l}}\right)e^{-2\pi i b_l|q_l|}\right). \tag{C.25}$$

For each $j\in\{1,...,r\}$, first integrate out the angular part of q_j to obtain for $\rho\in[0,\infty)$

$$F_j(\rho)=\int_{\mathbb{R}^d}\mathrm{d}q_j\delta\left(|q_j|-\rho\right)f_j\left(q_j,q_0\right)\phi\left(q_j-q_0\right)H_j\left(q_j\right), \tag{C.26}$$

and observe that there is a constant C_d depending only on dimension d such that for $k=0,1,2$

$$\left|\frac{\partial^k}{\partial\rho^k}F_j(\rho)\right|\leq C_dC_\phi C_{f_j}\left(\frac{\rho}{\langle\rho\rangle}\right)^{(d-1-k)_+}\langle\rho-|q_0|\rangle^{-m+d-1/2}\left(\sup_{q_j}\sum_{\beta=0}^{2}\left|\frac{\partial^\beta}{\partial|q_j|^\beta}H_j(q_j)\right|\right). \tag{C.27}$$

We therefore can conduct two integrations by parts of

$$\int_0^\infty\mathrm{d}\rho F_j(\rho)e^{-2\pi i\rho b_j}, \tag{C.28}$$

the first one leaving no boundary terms. As in the proof of Lemma C.1, this yields (with a new constant C_d)

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}\mathrm{d}q_jf_j\left(q_j,q_0\right)\phi\left(q_j-q_0\right)e^{-2\pi i b_j|q_j|}H_j\left(q_j\right)\right|\\&\qquad\leq C_dC_\phi C_{f_j}\left(\sup_{q_j}\sum_{\beta=0}^{2}\left|\frac{\partial^\beta}{\partial|q_j|^\beta}H_j(q_j)\right|\right)\langle b_j\rangle^{-2}\end{aligned} \tag{C.29}$$

for all $j\in\{1...r\}$. On the right side of the last equation one can estimate

$$\begin{aligned}&\sup_{q_j}\sum_{\beta=0}^{2}\left|\frac{\partial^\beta}{\partial|q_j|^\beta}H_j(q_j)\right|\\&\quad\leq C\sup_{q_j}\sum_{\substack{\alpha,\beta\\ \sum_l(\alpha_l+\beta_l)\leq 2}}\left|\int_{\mathbb{R}^{d|K_j|}}\prod_{l\in\underline{j}}\left(\mathrm{d}q_l\left[\partial^{\alpha_l}_{|q_j|}\phi\left(q_l-q_j\right)\right]\left[\partial^{\beta_l}_{|q_j|}f_l\left(q_l,q_j\right)\right]e^{-2\pi i b_l|q_l|}\right)\right.\\&\qquad\qquad\left.\prod_{l\in K_j\setminus\underline{j}}\left(\mathrm{d}q_l\phi\left(q_l-q_{\check{l}}\right)f_l\left(q_l,q_{\check{l}}\right)e^{-2\pi i b_l|q_l|}\right)\right|\end{aligned} \tag{C.30}$$

with a universal constant C. The number of summands in the sum on the right side is bounded by $C\left(|\underline{j}|+1\right)^2$, while the integral can be estimated in the same fashion as the original integral on the left side of (C.24). Iterating this procedure (note that only up to fourth derivatives of ϕ will be taken), one is left with the estimate

$$\left|\int_{\mathbb{R}^{dn}} \mathrm{d}q_1...\mathrm{d}q_n \prod_{l=1}^{n} \left(\phi\left(q_l - q_{\check{l}}\right) f_l\left(q_l, q_{\check{l}}\right) e^{-2\pi i b_l |q_l|}\right)\right| \leq C_d^n C_\phi^n \prod_{l=1}^{n} \left(\left(|\underline{l}|+1\right)^2 C_{f_l} \left\langle b_l\right\rangle^{-2}\right) \tag{C.31}$$

with some C_d only depending on d. This already almost looks like (C.21). The factor $\prod_l (|\underline{l}|+1)^2$ can be absorbed into the choice of C_d after deducing from

$$\sum_{l=1}^{n} |\underline{l}| \leq n \tag{C.32}$$

that

$$\prod_{l=1}^{n} \left(|\underline{l}|+1\right) \leq 2^n. \tag{C.33}$$

□

Lemma C.4. (On-shell scattering.) *Let $d \geq 2$, $\overline{m} \in \mathbb{N}$ and $R^{(1)}, ..., R^{(\overline{m})} \in \mathbb{N}_0$ be given. For functions ϕ as in (C.17) and $f_l^{(j)}$, ($j \in \{1, ..., \overline{m}\}$, $l \in \{1, ..., R^{(j)}\}$) as in (C.20), set*

$$I\left(q_0^{(1)}\right) = \int_{\mathbb{R}^{|\mathbf{R}|}} \prod_{j=1}^{\overline{m}} \prod_{l=1}^{R^{(j)}} \mathrm{d}b_l^{(j)} \int_{\mathbb{R}^{|\mathbf{R}|d}} \prod_{j=1}^{\overline{m}} \prod_{l=1}^{R^{(j)}} \mathrm{d}q_l^{(j)}$$
$$\prod_{j=1}^{\overline{m}} \prod_{l=1}^{R^{(j)}} \left(\exp\left(2\pi i \left(\left|q_l^{(j)}\right| - \left|q_0^{(j)}\right|\right) b_l^{(j)}\right) \phi\left(q_l^{(j)} - q_{l-1}^{(j)}\right) f_l^{(j)}\left(q_l^{(j)}, q_{l-1}^{(j)}\right)\right), \tag{C.34}$$

with $q_0^{(1)} \in \mathbb{R}^d \setminus \{0\}$ arbitrary and

$$q_0^{(j)} = q_{R^{(j-1)}}^{(j-1)} \tag{C.35}$$

for all $j \geq 2$. Then

$$I\left(q_0^{(1)}\right) = \int_{\mathbb{R}^{|\mathbf{R}|d}} \prod_{j=1}^{\overline{m}} \prod_{l=1}^{R^{(j)}} \left(\mathrm{d}q_l^{(j)} \delta\left(\left|q_l^{(j)}\right| - \left|q_0^{(j)}\right|\right) \phi\left(q_l^{(j)} - q_{l-1}^{(j)}\right) f_l^{(j)}\left(q_l^{(j)}, q_{l-1}^{(j)}\right)\right). \tag{C.36}$$

Proof. We only treat the case $R^{(j)} > 0$ for all j. After taking the $q_l^{(j)}$ integrals, Lemma C.3 implies a decay of the form

$$\prod_{j=1}^{\overline{m}-1} \left[\left\langle b_{R^{(j)}}^{(j)} + \sum_{l'=1}^{R^{(j+1)}} b_{l'}^{(j+1)}\right\rangle^{-2} \prod_{l=1}^{R^{(j)}-1} \left\langle b_l^{(j)}\right\rangle^{-2}\right] \prod_{l''=1}^{R^{(\overline{m})}} \left\langle b_{l''}^{(\overline{m})}\right\rangle^{-2}, \tag{C.37}$$

so by dominated convergence,

$$I\left(q_0^{(1)}\right) = \lim_{\substack{\beta_l^{(j)} \searrow 0 \\ \forall j,l}} \int_{\mathbb{R}^{|\mathbf{R}|}} \prod_{j=1}^{\overline{m}} \prod_{l=1}^{R^{(j)}} \mathrm{d}b_l^{(j)} \exp\left(-\beta_l^{(j)} \left|b_l^{(j)}\right|\right) \int_{\mathbb{R}^{|\mathbf{R}|d}} \prod_{j=1}^{\overline{m}} \prod_{l=1}^{R^{(j)}} \mathrm{d}q_l^{(j)}$$
$$\prod_{j=1}^{\overline{m}} \prod_{l=1}^{R^{(j)}} \left(\exp\left(2\pi i \left(\left|q_l^{(j)}\right| - \left|q_0^{(j)}\right|\right) b_l^{(j)}\right) \phi\left(q_l^{(j)} - q_{l-1}^{(j)}\right) f_l^{(j)}\left(q_l^{(j)}, q_{l-1}^{(j)}\right)\right), \tag{C.38}$$

where the limits can be taken in arbitrary order. Now, by Fubini's theorem, the $b_l^{(j)}$ integrals can be evaluated first, leaving us with

$$\lim_{\substack{\beta_l^{(j)} \searrow 0 \\ \forall j,l}} \int_{\mathbb{R}^{|\mathbf{R}|d}} \prod_{j=1}^{\overline{m}} \prod_{l=1}^{R^{(j)}} \left(\frac{2\beta_l^{(j)} \mathrm{d}q_l^{(j)}}{(\beta_l^{(j)})^2 + 4\pi^2 \left(\left|q_l^{(j)}\right| - \left|q_0^{(j)}\right|\right)^2} \phi\left(q_l^{(j)} - q_{l-1}^{(j)}\right) f_l^{(j)}\left(q_l^{(j)}, q_{l-1}^{(j)}\right)\right). \tag{C.39}$$

By the continuity and decay properties of the integrand, all $\beta_l^{(j)} \searrow 0$ limits can be taken to immediately obtain (C.36). □

D. The gate function

Definition D.1. For $k \in \mathbb{R}^d$, $w \in \mathbb{C} \setminus \mathbb{R}$, $\sigma_1, \sigma_2 \in \{-1, 1\}$ and $L \in \mathbb{R}$, define

$$h_{\sigma_1\sigma_2}(k, w) = i\pi^2 \sum_{\sigma' \in \{\pm 1\}} \int_{\mathbb{R}^d} \mathrm{d}k' \frac{\widehat{g}_2(k - k')}{w - 2\pi\sigma' |k'|} (\sigma'|k| + \sigma_1|k'|) (\sigma'|k| + \sigma_2|k'|) \tag{D.1}$$

and

$$h_{\sigma_1\sigma_2}(k, w; L) = i\pi^2 \sum_{\sigma' \in \{\pm 1\}} \int_{\mathbb{R}^d} \mathrm{d}k' \frac{\widehat{g}_2(k - k')}{w - 2\pi\sigma' |k'|} (\sigma'|k| + \sigma_1|k'|) (\sigma'|k| + \sigma_2|k'|) \times \Phi(k, k', L). \tag{D.2}$$

Lemma D.1. *Write $w = \alpha + i\gamma$ with $\alpha, \gamma \in \mathbb{R}$, $|\gamma| \in (0, 1]$, assume that $d \geq 2$ and that for all multi-indices ν with $0 \leq |\nu| \leq 3$,*

$$\left| \frac{\partial^\nu}{\partial q^\nu} \widehat{g}_2(q) \right| \leq \tilde{C}_{g_2} \langle q \rangle^{-d-3}. \tag{D.3}$$

Then there is a constant $C_{g_2,d}$ only depending on dimension d and the function g_2 such that

$$\sup_{\alpha,\gamma,\sigma_1,\sigma_2} |h_{\sigma_1\sigma_2}(k, \alpha + i\gamma)| \leq C_{g_2,d} \langle k \rangle^2, \tag{D.4}$$

$$\sup_{\alpha,\gamma,\sigma_1,\sigma_2} |\nabla_k h_{\sigma_1\sigma_2}(k, \alpha + i\gamma)| \leq C_{g_2,d} \langle k \rangle^2, \tag{D.5}$$

$$\sup_{\alpha,\sigma_1,\sigma_2} \left| \frac{\partial}{\partial \alpha} h_{\sigma_1\sigma_2}(k, \alpha + i\gamma) \right| \leq C_{g_2,d} \langle k \rangle^2 \langle \log \gamma \rangle, \tag{D.6}$$

and

$$\sup_{\alpha,\sigma_1,\sigma_2} \left| \frac{\partial}{\partial \gamma} h_{\sigma_1\sigma_2}(k, \alpha + i\gamma) \right| \leq C_{g_2,d} \langle k \rangle^2 \langle \log \gamma \rangle. \tag{D.7}$$

Note that the first two bounds are independent of γ as long as $\gamma \neq 0$. All estimates are valid for $h_{\sigma_1\sigma_2}(k, w; L)$ as well, with the factor $\langle k \rangle^2$ replaced by $\langle L \rangle^2$.

Proof. We only show the case without cut-off, the cut-off case being similar. All four estimates are a consequence of Lemma C.2. The role of f is played by

$$f(k, k') = (\sigma'|k| + \sigma_1|k'|) (\sigma'|k| + \sigma_2|k'|) \tag{D.8}$$

or by any of the components of the gradient,

$$f_j(k,k') = \frac{\partial}{\partial k_j} f(k,k'). \tag{D.9}$$

As

$$\begin{aligned} |\partial^n_{|k'|} f(k,k')| &\le (\langle k\rangle + \langle k'\rangle)^2, \\ |\partial^n_{|k'|} f_j(k,k')| &\le (\langle k\rangle + \langle k'\rangle), \end{aligned} \tag{D.10}$$

for all radial k' derivatives, $n \in \mathbb{N}_0$, the bound (C.2) is applicable with $C_f(k) = 4\langle k\rangle^2$ and $r = 2$. Now, (D.4) and (D.5) follow from (C.11) with $\phi = \partial^\nu \widehat{g_2}$ for any multiindices ν with $0 \le |\nu| \le 1$.

As $h_{\sigma_1\sigma_2}(k,w)$ is holomorophic in w, to estimate the left hand sides of (D.6) and (D.7) it suffices to consider the $w = \alpha + i\gamma$ derivative, which is

$$\begin{aligned} \frac{\partial}{\partial w} h_{\sigma_1\sigma_2}(k,w) &= -i\pi^2 \sum_{\sigma'\in\{\pm1\}} \int_{\mathbb{R}^d} \mathrm{d}k' \frac{\widehat{g_2}(k-k')}{(w - 2\pi\sigma' |k'|)^2} \left(\sigma'|k| + \sigma_1|k'|\right)\left(\sigma'|k| + \sigma_2|k'|\right) \\ &= -i\pi^2 \sum_{\sigma'\in\{\pm1\}} \int_{\mathbb{R}^d} \mathrm{d}k' \frac{\widehat{g_2}(k-k')}{(\alpha + i\gamma - 2\pi\sigma' |k'|)^2} \left(\sigma'|k| + \sigma_1|k'|\right)\left(\sigma'|k| + \sigma_2|k'|\right), \end{aligned} \tag{D.11}$$

and apply (C.12) with f as above and $\phi = \widehat{g_2}$. □

Lemma D.2. *If there is a finite constant C_{g_2} such that*

$$|\widehat{g_2}(q)| \le C_{g_2} \langle q\rangle^{-d-1}, \tag{D.12}$$

there is a $C < \infty$ depending only on g_2 and dimension $d \ge 2$, such that for all $k \in \mathbb{R}^d$ and $\gamma \in (0,1]$,

$$\begin{aligned} \sup_{\sigma\in\{\pm1\}} |h_{+-}(k, 2\pi\sigma|k| + i\gamma)| &\le C(|k| + \gamma) \\ \sup_{L,\sigma} |h_{+-}(k, 2\pi\sigma|k| + i\gamma; L)| &\le C(|k| + \gamma), \end{aligned} \tag{D.13}$$

and likewise for h_{-+}.

Proof. Without loss of generality, choose $\sigma = 1$ and focus on the h_{+-} terms without cutoff,

$$h_{+-}(k, 2\pi|k| + i\gamma) = 2i\pi^2(2\pi|k| + i\gamma) \int_{\mathbb{R}^d} \mathrm{d}k' \frac{4\pi^2 \widehat{g_2}(k-k')\left(|k|^2 - |k'|^2\right)}{(2\pi|k| + i\gamma)^2 - (2\pi|k'|)^2}, \tag{D.14}$$

so

$$
\begin{aligned}
|h_{+-}(k, 2\pi|k| + i\gamma)| &\leq C'(|k| + \gamma) \|\widehat{g_2}\|_{L^1} \\
&\qquad + C'\gamma(|k| + \gamma)^2 \int_{\mathbb{R}^d} \mathrm{d}k' \frac{\widehat{g_2}(k - k')}{|4\pi^2|k|^2 - \gamma^2 + 4\pi i\gamma|k| - 4\pi|k'|^2|} \\
&\leq C''(|k| + \gamma) \\
&\quad + C''\gamma(|k| + \gamma)^2 \int_0^\infty \mathrm{d}\rho \frac{\rho}{\langle\rho\rangle \langle\rho - 2\pi|k|\rangle |4\pi^2|k|^2 - \gamma^2 + 4\pi i\gamma|k| - \rho^2|} \\
&\leq C(|k| + \gamma),
\end{aligned}
\tag{D.15}
$$

with constants only depending on dimension d. A fortiori, the same estimate is valid for the cut-off case. □

While the off-diagonal components h_{+-} and h_{-+} are small in the above sense, the diagonal components of h have a non-trivial limit.

Lemma D.3. *Under the conditions of Lemma D.1, let $\sigma \in \{\pm 1\}$. There is a function $\Theta_\sigma : \mathbb{R}^d \to \mathbb{C}$ such that*

$$
\lim_{\gamma \searrow 0} h_{\sigma\sigma}(k, 2\pi\sigma|k| + i\gamma) = \Theta_\sigma(k) \tag{D.16}
$$

for all $k \in \mathbb{R}^d$. There is a constant C only depending on d and g_2 such that

$$
|\Theta_\sigma(k)| \leq C|k| \langle k \rangle \tag{D.17}
$$

and

$$
|\Theta_\sigma(k_1) - \Theta_\sigma(k_2)| \leq C(\langle k_1 \rangle^2 + \langle k_2 \rangle^2)|k_1 - k_2| \langle \log|k_1 - k_2| \rangle \tag{D.18}
$$

for all $k, k_1, k_2 \in \mathbb{R}^d$.

Proof. From (D.7), we directly obtain for $1 \geq \gamma \geq \gamma' > 0$ that

$$
\left| h_{\sigma\sigma}(k, 2\pi\sigma|k| + i\gamma) - h_{\sigma\sigma}(k, 2\pi\sigma|k| + i\gamma') \right| \leq C \langle k \rangle^2 \gamma \langle \log \gamma \rangle, \tag{D.19}
$$

which proves the existence of the limit $\Theta_\sigma(k)$ as well as the estimate

$$
|h_{\sigma\sigma}(k, 2\pi\sigma|k| + i\gamma) - \Theta_\sigma(k)| \leq C \langle k \rangle^2 \gamma \langle \log \gamma \rangle. \tag{D.20}
$$

The bound (D.17) is a direct consequence of (D.4) for large k, while for k close to zero, one has by (D.5)

$$
|h_{\sigma\sigma}(k, 2\pi\sigma|k| + i\gamma)| \leq |h_{\sigma\sigma}(0, 0 + i\gamma)| + C|k|, \tag{D.21}
$$

which yields the assertion after observing that

$$
|h_{\sigma\sigma}(0, 0 + i\gamma)| \leq C\gamma. \tag{D.22}
$$

Finally, regarding the continuity of Θ_σ, set $\gamma = |k_1 - k_2|$ in

$$\begin{aligned}|\Theta_\sigma(k_1) - \Theta_\sigma(k_2)| \leq& |h_{\sigma\sigma}(k_1, 2\pi\sigma|k_1| + i\gamma) - h_{\sigma\sigma}(k_2, 2\pi\sigma|k_2| + i\gamma)| \\ &+ C(\langle k_1\rangle + \langle k_2\rangle)^2 \gamma \langle \log\gamma\rangle \\ \leq& C(\langle k_1\rangle + \langle k_2\rangle)^2 \left(\langle\log\gamma\rangle \left||k_1| - |k_2|\right| + |k_1 - k_2| + \gamma\langle\log\gamma\rangle\right),\end{aligned} \tag{D.23}$$

in which we have used (D.5) and (D.6). □

Lemma D.4. *Under the conditions of Lemma D.1, for all $k \in \mathbb{R}^d$,*

$$\Theta_-(k) = \overline{\Theta_+(k)} \tag{D.24}$$

and

$$2\mathrm{Re}\Theta_+(k) = \Theta_+(k) + \Theta_-(k) = \sigma_{\mathrm{sc}}(k), \tag{D.25}$$

with σ_{sc} defined as in (2.132).

Proof. For the first equation,

$$\begin{aligned}\Theta_-(k) &= \lim_{\gamma\searrow 0} h_{--}(k, -2\pi|k| + i\gamma) \\ &= \lim_{\gamma\searrow 0} i\pi^2 \sum_{\sigma'\in\{\pm 1\}} \int_{\mathbb{R}^d} \mathrm{d}k' \frac{\widehat{g}_2(k-k')}{-2\pi|k| + i\gamma - 2\pi\sigma'|k'|} \left(\sigma'|k| - |k'|\right)^2 \\ &= \lim_{\gamma\searrow 0} i\pi^2 \sum_{\sigma'\in\{\pm 1\}} \int_{\mathbb{R}^d} \mathrm{d}k' \frac{\widehat{g}_2(k-k')}{-2\pi|k| + i\gamma + 2\pi\sigma'|k'|} \left(\sigma'|k| + |k'|\right)^2 \\ &= \lim_{\gamma\searrow 0} (-i)\pi^2 \sum_{\sigma'\in\{\pm 1\}} \int_{\mathbb{R}^d} \mathrm{d}k' \frac{\widehat{g}_2(k-k')}{2\pi|k| - i\gamma - 2\pi\sigma'|k'|} \left(\sigma'|k| + |k'|\right)^2 \\ &= \lim_{\gamma\searrow 0} \overline{h_{++}(k, 2\pi|k| + i\gamma)} = \overline{\Theta_+(k)},\end{aligned} \tag{D.26}$$

where we used that $\widehat{g}_2$ is real. Second,

$$2\mathrm{Re}\Theta_+(k) = \lim_{\gamma\searrow 0} \pi^2 \sum_{\sigma'\in\{\pm 1\}} \int_{\mathbb{R}^d} \mathrm{d}k' \frac{2\gamma\widehat{g}_2(k-k')}{\gamma^2 + (2\pi|k| - 2\pi\sigma'|k'|)^2} \left(\sigma'|k| + |k'|\right)^2. \tag{D.27}$$

The $\sigma' = -1$ summand is bounded by

$$\frac{\gamma}{2} \int_{\mathbb{R}^d} \mathrm{d}k' \widehat{g}_2(k-k') \to 0, \tag{D.28}$$

and thus, by the continuity and decay properties of $\widehat{g}_2$,

$$\begin{aligned}2\mathrm{Re}\Theta_+(k) &= \lim_{\gamma\searrow 0} \pi^2 \int_{\mathbb{R}^d} \mathrm{d}k' \frac{2\gamma\widehat{g}_2(k-k')}{\gamma^2 + (2\pi|k| - 2\pi|k'|)^2} \left(|k| + |k'|\right)^2 \\ &= (2\pi|k|)^2 \int_{\mathbb{R}^d} \mathrm{d}k' \widehat{g}_2(k-k') \delta\left(|k| - |k'|\right) = \nu_{\mathrm{sc}}\left(k, \mathbb{R}^d\right) = \sigma_{\mathrm{sc}}(k).\end{aligned} \tag{D.29}$$

□

E. Approximation by waves with compact support

Lemma E.1. *Let $f : \mathbb{R}^d \to \mathbb{C}$ such that $f \in H^1(\mathbb{R}^d)$, and $f(x) = 0$ for all $x \in \mathbb{R}^d$ with $|x| \geq R$, for some $R \in [1, \infty)$. Then for every $\rho \in (0, 1)$ there is a $C_{\rho,d} < \infty$ which* only *depends on ρ and dimension d, a constant C_d only depending on d, and a function $u_\rho \in H^2(\mathbb{R}^d)$ such that*

- $\| |\nabla| \, u_\rho - f \|_{L^2} \leq \rho \, \|f\|_{H^1}$*, and*
- $\| |\nabla| u_\rho \|_{H^1} \leq C_d \, \|f\|_{H^1}$*,*
- $u_\rho(x) = 0$ *for all* $|x| > C_{\rho,d} R$.

Note that u_ρ is typically not bounded in L^2 as $\rho \to 0$.

Proof. Let a $\delta \in (0, 1)$ be given, and choose a smooth function $\chi : \mathbb{R}^d \to [0, 1]$ such that $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. Observe that

$$\int_{\mathbb{R}^d} \mathrm{d}k \left| (1 - \chi(\delta k)) \widehat{f}(k) \right|^2 \leq \frac{\delta^2}{4\pi^2} \, \|f\|_{H^1}^2 \, . \tag{E.1}$$

From the support properties of f,

$$\left\| \widehat{f} \right\|_{L^\infty} \leq \|f\|_{L^1} \leq R^{d/2} \, \|f\|_{L^2} \, , \tag{E.2}$$

so

$$\int_{\mathbb{R}^d} \mathrm{d}k \left| \chi(Rk/\delta) \widehat{f}(k) \right|^2 \leq C_d \delta^d \, \|f\|_{L^2}^2 \tag{E.3}$$

with a constant $C_d < \infty$ only depending on dimension d. For the δ- and R-dependent Schwartz function $L : \mathbb{R}^d \to \mathbb{R}$ given as

$$L(k) = |2\pi k|^{-1} \, (1 - \chi(Rk/\delta)) \, \chi(\delta k) \tag{E.4}$$

we thus have

$$\int_{\mathbb{R}^d} \mathrm{d}k \left| (1 - |2\pi k| L(k)) \, \widehat{f}(k) \right|^2 \leq \left(\frac{\delta^2}{4\pi^2} + C_d \delta^d \right) \|f\|_{H^1}^2 \, . \tag{E.5}$$

The Fourier transform of χ is a Schwartz function with $\int_{\mathbb{R}^d} \widehat{\chi} = 1$, and it is not hard to see that for the the function

$$L_{\delta,R} = \left(R/\delta^2 \right)^d \widehat{\chi} \left(\frac{R}{\delta^2} \cdot \right) * L \tag{E.6}$$

one has an only dimension-dependent constant C_d such that

$$|L_{\delta,R}(k) - L(k)| \leq C_d \sqrt{\delta} |k|^{-1}. \tag{E.7}$$

From (E.5) and (E.7), one has

$$\int_{\mathbb{R}^d} \mathrm{d}k \left| (1 - |2\pi k| L_{\delta,R}(k)) \, \widehat{f}(k) \right|^2 \leq \tilde{C}_d \delta \, \|f\|_{H^1}^2 \tag{E.8}$$

with a constant $\tilde{C}_d < \infty$. Now fix a (thus only ρ- and d-dependent) $\delta \in (0,1)$ such that $\tilde{C}_d \delta \leq \rho^2$, and choose

$$\begin{aligned} u_\rho(x) &= \left(\mathcal{F}\left(L_{\delta,R}\right) * f\right)(x) \\ \widehat{u}_\rho(k) &= L_{\delta,R}(k) \widehat{f}(k). \end{aligned} \tag{E.9}$$

Then clearly

$$\|f - |\nabla| u_\rho\|_{L^2} = \left(\int_{\mathbb{R}^d} \mathrm{d}k \left| \widehat{f}(k) - |2\pi k| \widehat{u}_\rho(k) \right|^2 \right)^{1/2} \leq \rho \, \|f\|_{H^1} . \tag{E.10}$$

Furthermore, from (E.4) and (E.7) one has

$$|2\pi k \widehat{u}_\rho(k)| \leq (1 + C_d' \rho) \left| \widehat{f}(k) \right| \tag{E.11}$$

and thus, after redefining $C_d < \infty$, $\||\nabla| u_\rho\|_{H^1} \leq C_d \|f\|_{H^1}$ uniform in $\rho \in (0,1)$. Finally,

$$\mathcal{F}\left(L_{\delta,R}\right)(x) = \mathcal{F} L(x) \chi \left(\frac{\delta^2 x}{R} \right) \tag{E.12}$$

is supported in a ball of radius $2R\delta^{-2}$ around the origin, and thus

$$u_\rho(x) = \left(\mathcal{F}\left(L_{\delta,R}\right) * f\right)(x) = 0 \tag{E.13}$$

for all $|x| \geq \left(1 + 2\delta^{-2}\right) R$, and we obtain $C_{\rho,d} = \left(1 + 2\delta^{-2}\right)$, which, by the choice of δ, only depends on ρ and d. □

F. Off-diagonal observables

In this appendix, we attach a short informal discussion of observables with off-diagonal entries (or, equivalently, of the off-diagonal entries of a 2×2 matrix-valued Wigner transform). To be on the safe side, we only look at observables $\mathfrak{a}, \mathfrak{b}$ that are Schwartz functions on phase space, and initial states $(\psi_0^\varepsilon)_{\varepsilon>0} \subset \mathcal{H}$ with bounded energy, Fourier transforms vanishing for $|k| \notin [\lambda, L^{(0)}]$, with ε-independent $0 < \lambda < L^{(0)} < \infty$, and with Wigner limit measures $\mu_{0,+}$ and $\mu_{0,-}$.

The first thing we note is that the self-averaging properties of the Wigner function are just as valid for the off-diagonal components, as all estimates derived in Sections 5.3 and 6.1 only involved the absolute value of the resolvents, and were thus independent of the sign of the phase. It is therefore enough to analyze the disorder-averaged value of the off-diagonal components of the Wigner transform.

For a single measurement, $\overline{m} = 1$, we propagate the state ψ_0^ε for a microscopic time T/ε with the perturbed dynamics and then test it against $\mathrm{Op}^\varepsilon(\mathfrak{a})P_{-+}$, an off-diagonal observable; thus we are interested in the $\varepsilon \to 0$ limit of

$$\mathbb{E}\left\langle \mathrm{e}^{-iH^\varepsilon T/\varepsilon}\psi_0^\varepsilon, \mathrm{Op}^\varepsilon(\mathfrak{a})P_{-+}\mathrm{e}^{-iH^\varepsilon T/\varepsilon}\psi_0^\varepsilon\right\rangle_{\mathcal{H}} \tag{F.1}$$

It is not hard to see that all arguments in Sections 4.1 to 4.5 are still valid, and only the ladder graphs of section 4.6 contribute. But also all ladders with one or several rungs are suppressed in the limit — this is because the phases on the different sides of the ladder do not cancel, but add up, and the ladder contribution can be estimated with an oscillatory phase argument along the lines of Lemma 4.20. The only graphs left are degenerate ladders without rungs, with propagators only "decorated" with gates, so

$$\begin{aligned}\lim_{\varepsilon\to 0}\Bigg|&\mathbb{E}\left\langle \mathrm{e}^{-iH^\varepsilon T/\varepsilon}\psi_0^\varepsilon, \mathrm{Op}^\varepsilon(\mathfrak{a})P_{-+}\mathrm{e}^{-iH^\varepsilon T/\varepsilon}\psi_0^\varepsilon\right\rangle_{\mathcal{H}}\\ &-\int_{\mathbb{R}^d}\mathrm{d}p\int_{\mathbb{R}^d}\mathrm{d}k\,\widehat{\mathfrak{a}}(p,k)\overline{\widehat{\psi}_{0,-}^\varepsilon}(k+\varepsilon p/2)\widehat{\psi}_{0,+}^\varepsilon(k-\varepsilon p/2)e^{-4\pi i|k|T/\varepsilon-2\Theta_+(k)T}\Bigg| = 0.\end{aligned} \tag{F.2}$$

Accordingly, the off-diagonal component is highly oscillatory and vanishes if averaged over short macroscopic intervals,

$$\lim_{\varepsilon\to 0}\int_{T-\delta}^{T+\delta}\mathrm{d}\tau\,\mathbb{E}\left\langle \mathrm{e}^{-iH^\varepsilon\tau/\varepsilon}\psi_0^\varepsilon, \mathrm{Op}^\varepsilon(\mathfrak{a})P_{-+}\mathrm{e}^{-iH^\varepsilon\tau/\varepsilon}\psi_0^\varepsilon\right\rangle_{\mathcal{H}} = 0. \tag{F.3}$$

Multiple measurements are much more exciting. To see why, note that the operator $P_{-+} + P_{+-}$ causes a “time-reversal” of the perturbed dynamics

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} H^\varepsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -H^\varepsilon. \tag{F.4}$$

Imagine a wave is emitted and travels for a macroscopic time T through the random medium, is then time-reversed by some mechanism (in practice, a device records the wave for a short period of time and re-emits the signal in reverse order, [2]), and travels back by another period of time T, before it is measured by another device. If we model the time reversal by $\mathrm{Op}^\varepsilon(\mathfrak{b})P_{-+}$, and the final measurement by $\mathrm{Op}^\varepsilon(\mathfrak{a})P_{--}$, $\mathfrak{a}$ real, we are interested in

$$\left\langle \mathrm{e}^{-iH^\varepsilon T/\varepsilon}\mathrm{Op}^\varepsilon(\mathfrak{b})P_{-+}\mathrm{e}^{-iH^\varepsilon T/\varepsilon}\psi_0^\varepsilon, \mathrm{Op}^\varepsilon(\mathfrak{a})P_{--}\mathrm{e}^{-iH^\varepsilon T/\varepsilon}\mathrm{Op}^\varepsilon(\mathfrak{b})P_{-+}\mathrm{e}^{-iH^\varepsilon T/\varepsilon}\psi_0^\varepsilon \right\rangle_{\mathcal{H}}. \tag{F.5}$$

Again, one can perform a graph expansion, eliminate higher order partitions, crossing and nested pairings; however due to the time-reversal caused by P_{-+}, there is no such thing as “non-markovian” graphs. The contributions to the limit are more complicated than for the diagonal case, but can still be calculated by a resummation of somewhat generalized ladder graphs. We now denote by a solid line the unperturbed propagator

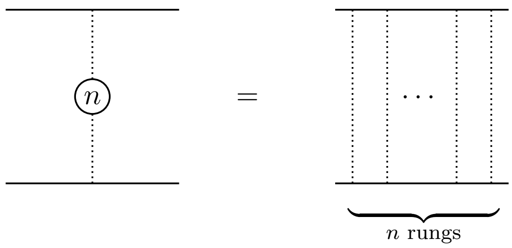

Figure F.1.: The shorthand for n parallel rungs.

already “dressed” with gates, and introduce an abbreviation for several parallel rungs in a graph, cf. Figure F.1. With this notation,

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathbb{E}&\left\langle \mathrm{e}^{-iH^\varepsilon T/\varepsilon}\mathrm{Op}^\varepsilon(\mathfrak{b})P_{-+}\mathrm{e}^{-iH^\varepsilon T/\varepsilon}\psi_0^\varepsilon, \mathrm{Op}^\varepsilon(\mathfrak{a})P_{--}\mathrm{e}^{-iH^\varepsilon T/\varepsilon}\mathrm{Op}^\varepsilon(\mathfrak{b})P_{-+}\mathrm{e}^{-iH^\varepsilon T/\varepsilon}\psi_0^\varepsilon \right\rangle_{\mathcal{H}} \\ &= \sum_{r,l,m,n=0}^{\infty}\lim_{\varepsilon\to 0}\mathcal{K}(\varepsilon,T;r,l,m,n) = \int_{\mathbb{R}^{2d}}\mu_{0,+}(\mathrm{d}x,\mathrm{d}k)\mathfrak{c}(x,k).\end{aligned} \tag{F.6}$$

Here, the amplitudes $\mathcal{K}(\varepsilon,T;r,l,m,n)$ are given as the sum of three graphs which are presented in Figure F.2. The propagated observable $\mathfrak{c}$ can be identified as

$$\begin{aligned}\mathfrak{c} =& \mathrm{e}^{\mathcal{L}_+T}\left(|\mathfrak{b}|^2\mathrm{e}^{\mathcal{L}_-T}\mathfrak{a}\right) \\ &+ 2\mathrm{Re}\int_0^T \mathrm{d}\tau \mathrm{e}^{\mathcal{L}_+(T-\tau)}\left[\left(\mathcal{L}_0\mathrm{e}^{\mathcal{L}_+\tau}\mathfrak{b}\right)\left(\mathrm{e}^{\mathcal{L}_+\tau}\overline{\mathfrak{b}}\right)\left(\mathrm{e}^{\mathcal{L}_-(T-\tau)}\mathfrak{a}\right)\right] \\ &- 2\mathrm{Re}\int_0^T \mathrm{d}\tau \mathrm{e}^{\mathcal{L}_+(T-\tau)}\left[\left(\mathrm{e}^{\mathcal{L}_+\tau}\mathfrak{b}\right)\mathcal{L}_0\left(\left(\mathrm{e}^{\mathcal{L}_+\tau}\overline{\mathfrak{b}}\right)\left(\mathrm{e}^{\mathcal{L}_-(T-\tau)}\mathfrak{a}\right)\right)\right],\end{aligned} \tag{F.7}$$

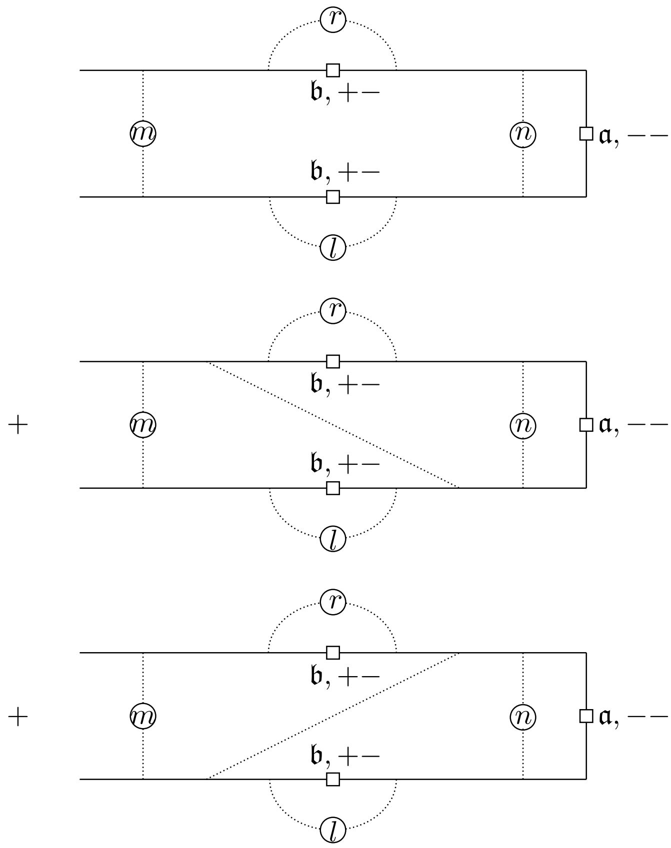

Figure F.2.: The definition of an amplitude $\mathcal{K}(\varepsilon, T; r, l, m, n)$.

with $\mathcal{L}_0 = (\mathcal{L}_+ + \mathcal{L}_-)/2$ the generator of the momentum jump process alone. Note that for a perfect time reversal ($\mathfrak{b} \equiv 1$) one has $\mathfrak{c} = \mathfrak{a}$, as expected.

List of Symbols

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