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# Essays on Stochastic Processes and their Applications

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# Introduction

In this thesis we study problems related to *random media*, *Markov process and semimartingale theory* and *mathematical finance*. In the following, we give a short overview on the topics of the thesis. More detailed introductions can be found in the beginning of each chapter. In general, all chapters are self-contained.

This thesis is based on joint work with Noam Berger [11] and the following publications:

[28] Criens, D. (2020). Lyapunov criteria for the Feller–Dynkin property of martingale problems. *Stochastic Processes and their Applications*, 130(5):2693–2736.

[29] Criens, D. (2020). No arbitrage in continuous financial markets. *Mathematics and Financial Economics*, 14(3):461–506.

[30] Criens, D. (2020). On absolute continuity and singularity of multidimensional diffusions. In revision for *Electronic Journal of Probability.*

[31] Criens, D. (2020). On the existence of semimartingales with continuous characteristics. *Stochastics*, 92(5):785–813.

[32] Criens, D. and Glau, K. (2018). Absolute continuity of semimartingales. *Electronic Journal of Probability*, 23(125):1–28.

The content of Chapters 2 – 6 is, for the most part, identical to the published articles. Two exceptions are the following: We have removed the recurrence assumption in [29] and the section on multidimensional diffusions in [32], because this setting is studied in more detail in [30]. We now describe the main results.

In the first part of this thesis (Chapter 1), which is based on joint work with Noam Berger [11], we study a question from the wider research area of random media. More precisely, we consider *random walks in balanced random environment (RWBRE)*, which is a model with two components: First, the *environment*, which is fixed throughout the time evolution, and second, the *random walk*, which is a time-homogeneous nearest-neighbor Markov chain on $\mathbb{Z}^d$ whose transition probabilities depend on the environment. As the terminology indicates, our attention lies on *balanced* random environments, which means that we consider only environments which turn the walk into a martingale. In the same manner as Brownian motion is connected to the heat equation, Markov chains can be related to certain difference equations. Let $(X_n)_{n\in\mathbb{Z}_+}$ be the nearest-neighbor Markov chain on $\mathbb{Z}^d$ with transition kernel

$$P(X_{n+1} = x + e | X_n = x) = \omega(x, e)$$

for $x \in \mathbb{Z}^d$ and $e$ being a neighbor of the origin. For a finite set $S \subset \mathbb{Z}^d$ and $N \in \mathbb{Z}_+$ a function $u\colon \overline{S} \times \{0, \dots, N + 1\} \to \mathbb{R}$ is called *caloric* on $S \times \{0, \dots, N\}$ if

$$u(x, n) = E\big[u(X_1, n + 1)\big] = \sum_{e\sim 0} \omega(x, e)u(x + e, n + 1)$$

for all $(x, n) \in S \times \{0, \ldots, N\}$. The main result (Theorem 1.2) of Chapter 1 is a parabolic Harnack inequality for non-negative caloric functions associated to the walk in the RWBRE model. We also prove an oscillation inequality and a quantitative homogenization estimate for caloric functions. Furthermore, we show that RWBRE is transient for $d \geq 3$.

In the second part (Chapter 2, based on [28]) we study a classical question from Markov process theory: For a given Markov process we ask whether it is a Feller–Dynkin process, i.e. whether its transition semigroup is a self-map on the space of continuous functions vanishing at infinity. In the first part of Chapter 2 we consider a family of solutions to an abstract martingale problem and derive Lyapunov-type conditions to affirm or reject its Feller–Dynkin property. Under additional assumptions on the input data, we extend the sufficient condition to be also necessary. In the second part we consider so-called *switching diffusions*: A continuous process $Y$ is called a *switching diffusion*, if it solves the stochastic differential equation

$$dY_t = b(Y_t, \xi_t)dt + \sigma(Y_t, \xi_t)dW_t,$$

where $W$ is a standard Brownian motion and $\xi$ is a continuous-time Markov chain whose $Q$-matrix might dependent on the present state of $Y$. Using our Lyapunov-type criteria, we prove a Khasminskii-type integral test for the Feller–Dynkin property of $(Y, \xi)$. For the state-independent case we give a complete characterization in terms of the family of diffusions with fixed regimes: We show that $(Y, \xi)$ has the Feller–Dynkin property precisely when for all $i$ in the state space of $\xi$ the solutions to the stochastic differential equations

$$dY_t^i = b(Y_t^i, i)dt + \sigma(Y_t^i, i)dW_t$$

have the Feller–Dynkin property. Based on this characterization, we also formulate Khasminskii and Feller-type integral tests for the Feller–Dynkin property of $(Y, \xi)$.

In Chapters 3 and 4 (based on [30] and [32]) we study conditions for two laws of semimartingales on random sets to be (locally) absolutely continuous and/or singular. For general (conservative) semimartingales the state of the art results for (local) absolute continuity require a strong uniqueness assumption called *local uniqueness* in [70]. Using a new generalized Girsanov theorem (Theorem 3.1), we replace the local uniqueness assumption by a usual uniqueness assumption and a local integrability condition, see Corollary 3.1. As an application, we generalize Beneš [7] linear growth condition for the martingale property of certain stochastic exponentials.

For parametric processes it is natural to ask for analytic characterizations of absolute continuity and singularity. In Chapter 4 we study this question for multidimensional, possibly explosive, diffusions, which are parameterized via a drift and a diffusion coefficient. In the main result (Corollary 4.2) we show that absolute continuity is equivalent to almost sure explosion of an auxiliary time-changed diffusion and that singularity is equivalent to its almost sure non-explosion. As an application we prove integral tests for explosion and non-explosion of time-changed Brownian motion. Our key tool is a new existence and uniqueness theorem for time-changed diffusions, see Theorem 4.1.

Chapter 5, which is based on [31], deals with a general (weak) existence question for semimartingales. More precisely, we ask for conditions implying the existence of a probability measure on the Skorokhod space turning the coordinate process into a semimartingale with semimartingale characteristics given by a certain candidate triplet. We derive continuity and growth conditions on the candidate triplet which ensure existence, see Theorems 5.1 and 5.2. Moreover, we deduce existence criteria for time-inhomogeneous jump-diffusions, see Corollary 5.1.

In the final part of this thesis we study a question from mathematical finance. For a given financial model it is important to understand the existence and absence of arbitrage opportunities. Three classical notions of no arbitrage are *no free lunch with vanishing risk, no feasible free lunch with vanishing risk* and *no relative arbitrage*. Each of these can be related to the existence of a certain probabilistic object, namely an *equivalent (local) martingale measure* or a *strict martingale density*. In Chapter 6 (based on [29]) we study the existence and non-existence of these probabilistic objects for path- and time-continuous single asset models. More precisely, we either assume that the discounted asset price process $P$ is the stochastic exponential of an Itô process, i.e.

$$dP_t = P_t dS_t, \quad dS_t = b_t dt + \sigma_t dW_t,$$

or a positive switching diffusion, i.e.

$$dP_t = b(P_t, \xi_t)dt + \sigma(P_t, \xi_t)dW_t,$$

where $W$ is a one-dimensional standard Brownian motion and $\xi$ is a continuous-time Markov chain. Our main results (Theorems 6.4 – 6.7) are analytic integral tests, which are even sufficient and necessary for Markov switching models. The proofs are based on new analytic integral tests for the true and strict local martingale property of certain stochastic exponentials, which extend some results in Chapter 3 in the sense that the drift and volatility coefficients are allowed to depend on several sources of risk.

# Contents

# 1 A Parabolic Harnack Inequality for RWRE in Balanced Environments

## 1.1 Introduction

Consider the non-divergence form operator

$$(L^a f)(x) \triangleq \sum_{i,j=1}^{d} a_{ij}(x) \frac{d^2 f}{dx_i dx_j}(x), \quad (f,x) \in C^2(\mathbb{R}^d) \times \mathbb{R}^d, \tag{1.1}$$

where $a = (a_{ij})_{i,j=1}^d$ is a measurable function from $\mathbb{R}^d$ into the set of symmetric positive definite matrices which is uniformly elliptic, i.e. there is a constant $0 < \lambda \leq 1$ such that

$$\lambda \|y\|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x) y_i y_j \leq \frac{1}{\lambda} \|y\|^2, \quad (x,y) \in \mathbb{R}^d \times \mathbb{R}^d.$$

For an open domain $D$ in $\mathbb{R}^{d+1}$ a function $u \colon D \to \mathbb{R}$ is called *caloric* if it solves the (backward) heat equation $\frac{d}{dt} u = -L^a u$. In a seminal paper, Krylov and Safonov [84] proved a *parabolic Harnack inequality (PHI)* for non-negative caloric functions on $[0, R^2] \times B_R(0)$.

More precisely, they proved the existence of a positive constant $C = C(\lambda)$ such that for any radius $R > 0$ and every (non-negative) caloric function $u$ on $[0, R^2] \times B_R(0)$ it holds that

$$\max_{Q_+} u \leq C \min_{Q_-} u, \tag{PHI}$$

where $Q_- \triangleq [0, \frac{1}{4} R^2] \times B_{R/2}(0)$ and $Q_+ \triangleq [\frac{1}{2} R^2, \frac{3}{4} R^2] \times B_{R/2}(0)$. The PHI has many important applications such as a priori estimates in parabolic Hölder spaces (see [84]) or Hölder regularity results (see [128]). PHIs for discrete uniformly elliptic heat equations can be found in [89, 111], see also [91] for its elliptic (i.e. time independent) counterpart. Remarkably, the constant in the PHI of Krylov and Safonov does not depend on the regularity of the coefficient $a$ but only on the ellipticity constant $\lambda$.

More recently, there is a growing interest in PHIs for settings which are not necessarily uniformly elliptic. We mention the paper [58], where heat equations arising from random walks on percolation clusters (RWPC) are studied, and the articles [3, 18, 41], where the PHI is proved for equations related to the random conductance model (RCM). In these works the PHI was used to prove a local limit theorem for the corresponding stochastic processes.

In contrast to (1.1), the equations associated to RWPC and RCM are in divergence form, i.e. reversible. Discrete equations in non-divergence form appear in the context of *random walks in balanced random environment (RWBRE)*. An *elliptic Harnack inequality (EHI)* for such equations in fully non-elliptic environments has recently been proven in [10]. To the best of our knowledge, this result is the first of its kind for such a degenerate framework. We refer to [43, 143] for more information on the RWRE model.

The main result in the chapter is Theorem 1.2 below, which is a PHI for random difference equations associated to non-elliptic random walks in balanced i.i.d. random environments, which is the setting from [10]. More precisely, we prove the PHI for all non-negative caloric functions which satisfy a certain exponential growth condition, which we discuss below in more detail, and we show by example that it can fail without it. As the EHI holds in full generality, our result points to an interesting difference between parabolic and elliptic frameworks. To the best of our knowledge, a comparable phenomenon has not been reported before. The Harnack constant in our PHI is optimal in the sense that it can be taken arbitrarily close to its counterpart in the PHI of the limiting Brownian motion from the corresponding invariance principle proven in [12].

As pointed out above, the PHI in Theorem 1.2 is only proven for (non-negative) caloric functions $f\colon B_R(0) \times [0, R^2] \to \mathbb{R}_+$ satisfying the growth condition given in Eq. (1.2) below, which roughly states that

$$\max f \leq e^{R^{2-\xi}} \min f,$$

for an arbitrarily small constant $\xi > 0$. We also find a counter example to the PHI satisfying

$$\max f = e^{R^2} \min f.$$

The counter example shows that the growth condition (1.2) is sharp.

In the following we comment on the growth condition: In general the growth condition is quite mild. In particular, in most applications (e.g. for local limit theorems) all functions that are considered the maximum to minimum ratio grows like a power of $R$ and therefore satisfy our growth condition.

To the best of our knowledge, such a growth condition appears here the first time in a PHI. We believe that this phenomenon, namely that a mild growth condition guarantees an otherwise false PHI, exists in a large variety of models which are not uniformly elliptic. In particular, we believe that for RCMs which are elliptic but not uniformly elliptic, and where the conductances have a thick enough tail around zero (see, e.g. [9]), a similar phenomenon could hold.

We now comment on the proof of our PHI. The basic strategy is borrowed from Fabes and Stroock [48] and their proof for the continuous uniformly elliptic case. In general, our Fabes–Stroock argument relies on two central ingredients which are of independent interest: A parabolic oscillation inequality and a parabolic quantitative homogenization estimate. The former is used for the iterative scheme in the Fabes–Stroock argument and the latter yields estimates for the exit measure of the random walker, which we use roughly the same way Fabes and Stroock [48] used heat kernel estimates. In contrast to the setting of Fabes and Stroock, our model lacks connectivity in the sense that the movement of the random walker is restricted by holes in the environment. In addition, we have to deal with local degeneracies, as the positive transition probabilities in the random environments might not be bounded away from zero. To control the sizes of the holes in the environments we use percolation estimates which use the i.i.d. structure. Due to our parabolic setting the speed of the random walker is a major issue. The growth condition ensures that the random walker reaches certain parts of the environments fast enough. The Fabes–Stroock method was also used in [10] to establish the EHI. In contrast to our setting, the issue of speed plays no role in [10], which also explains why the EHI holds in full generality.

Before we turn to the main body of this chapter, let us comment on follow up questions which are left for future research. It is interesting to compare our result to those for the RCM. In [3, 18] it was shown that the PHI holds under certain moment assumptions on the conductances, which are violated in degenerate cases. Our result suggests that also for the RCM the PHI might hold when restricted to a suitable class of functions. Conversely, the results from [3, 18] suggest that a full PHI might hold for elliptic RWBRE under suitable moment assumptions on the ellipticity constant. We think our PHI is a first step into the direction of a local limit theorem for non-elliptic RWBRE. At this point we stress that our PHI cannot be used directly to solve this question as in [3], because the method there relies on a PHI for adjoint equations. In the reversible (self-adjoint) framework from [3] it is clear that the PHI also applies to adjoint equations, but in our non-symmetric setting this is not the case.

The chapter is structured as follows. In Section 1.2 we introduce our setting and state our main results. The proofs are given in the remaining sections, whose structure is explained at the end of Section 1.2.

## 1.2 Setup and Main Results

### 1.2.1 The Setup

Let $d \geq 2$, let $\{e_i : i = 1, \ldots, d\}$ be the unit vectors in $\mathbb{Z}^d$, set $e_{d+i} \triangleq -e_i$ for $i = 1, \ldots, d$ and let $\mathcal{M}$ be the space of all probability measures on $\{e_i : i = 1, \ldots, 2d\}$ equipped with the topology of convergence in distribution. Moreover, define the product space

$$\Omega \triangleq \prod_{\mathbb{Z}^d} \mathcal{M}$$

and let $\mathcal{F} \triangleq \mathcal{B}(\Omega)$ be the Borel $\sigma$-field on $\Omega$. We call $\omega \in \Omega$ an *environment.* Take $P$ to be an i.i.d. Borel probability measure on $\Omega$, i.e.

$$P \triangleq \bigotimes_{\mathbb{Z}^d} \nu \text{ for some } \nu \in \mathcal{M}.$$

Let $D(\mathbb{Z}_+, \mathbb{Z}^d)$ be the space of all paths $\mathbb{Z}_+ \to \mathbb{Z}^d$, equipped with the product topology, and let $X = (X_n)_{n \in \mathbb{Z}_+}$ be the coordinate process on $D(\mathbb{Z}_+, \mathbb{Z}^d)$, i.e. $X_n(\alpha) = \alpha(n)$ for $\alpha \in D(\mathbb{Z}_+, \mathbb{Z}^d)$ and $n \in \mathbb{Z}_+$. For every $\omega \in \Omega$ and $x \in \mathbb{Z}^d$ let $P^x_\omega$ be (unique) Borel probability measure on $D(\mathbb{Z}_+, \mathbb{Z}^d)$ which turns $X$ into a time-homogeneous Markov chain with initial value $x$ and transition kernel $\omega$, i.e.

$$P^x_\omega(X_0 = x) = 1, \quad P^x_\omega(X_n = y + e_k | X_{n-1} = y) = \omega(y, e_k), \quad z \in \mathbb{Z}^d, k = 1, \ldots, 2d.$$

The law $P^x_\omega$ is called the *quenched law of the walk.* An environment $\omega \in \Omega$ is called *balanced* if for all $z \in \mathbb{Z}^d$ and $k = 1, \ldots, d$

$$\omega(z, e_k) = \omega(z, -e_k).$$

The set of balanced environments is denoted by $\mathsf{B}$. For $n \in \mathbb{Z}_+$ we set $\mathcal{F}_n \triangleq \sigma(X_m, m \in [n])$, where $[n] \triangleq \{0, \ldots, n\}$. All terms such as *martingale, stopping time*, etc., refer to $(\mathcal{F}_n)_{n \in \mathbb{Z}_+}$ as underlying filtration.

**Lemma 1.1.** *The walk $X$ is a $P^x_\omega$-martingale for all $x \in \mathbb{Z}^d$ if and only if $\omega \in \mathsf{B}$.*

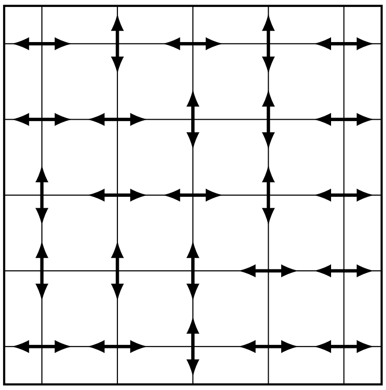

Figure 1.1: An illustration of Example 1.1 restricted to a small box.

*Proof.* The Markov property of the walk implies that $P^x_\omega$-a.s.

$$E^x_\omega\big[X_{n+1}|\mathcal{F}_n\big] = E^{X_n}_\omega\big[X_1\big] = X_n + \sum_{k=1}^{2d} e_k\omega(X_n, e_k).$$

This observation yields the claim. □

We say that $\omega \in \Omega$ is *genuinely $d$-dimensional* if for every $k = 1, \dots, 2d$ there exists a $z \in \mathbb{Z}^d$ such that $\omega(z, e_k) > 0$. We denote the set of all genuinely $d$-dimensional environments by $\mathsf{G}$.

**Example 1.1.** *An example for an environment measure $P$ with $P(\mathsf{B} \cap \mathsf{G}) = 1$ is the following:*

$$P\Big(\omega \in \Omega\colon \omega(0, e_i) = \omega(0, -e_i) = \frac{1}{2}\Big) = \frac{1}{d}, \quad i = 1, \dots, d.$$

*In this case the environment chooses uniformly at random one of the $\pm e_i$ directions, see Figure 1.1.*

For a finite set $S \subset \mathbb{Z}^d$ and $N \in \mathbb{Z}_+$ we say that a function $u\colon \overline{S} \times [N+1] \to \mathbb{R}$ is *$\omega$-caloric on $S \times [N]$* if for every $(x, m) \in S \times [N]$

$$u(x, m) = E^x_\omega\big[u(X_1, 1+m)\big] = \sum_{k=1}^{2d} \omega(x, e_k) u(x + e_k, 1 + m).$$

The following simple observation provides a probabilistic interpretation for the definition of a caloric function.

**Lemma 1.2.** *Let $\omega \in \mathsf{B}$ and $u\colon \overline{S} \times [N+1] \to \mathbb{R}$. Set*

$$\tau_m \triangleq \inf(n \in \mathbb{Z}_+\colon (X_n, n + m) \notin S \times [N]), \quad m \in [N].$$

*The following are equivalent:*

(a) *$u$ is $\omega$-caloric.*

(b) *For all $(x, m) \in S \times [N]$ the process $(u(X_{n\wedge\tau_m}, n \wedge \tau_m + m))_{n\in\mathbb{Z}_+}$ is a $P^x_\omega$-martingale.*

*Proof.* The implication (b) ⇒ (a) follows from the fact that martingales have constant expectation and $P^x_\omega$-a.s. $\tau_m \geq 1$. For the converse implication, assume that (a) holds and let $n \in \mathbb{Z}_+$. The Markov property of the walk yields that $P^x_\omega$-a.s. on $\{n < \tau_m\} = \{n+1 \leq \tau_m\} \in \mathcal{F}_n$

$$\begin{aligned} E^x_\omega\big[u(X_{(n+1)\wedge\tau_m}, (n+1) \wedge \tau_m + m)\big|\mathcal{F}_n\big] &= E^x_\omega\big[u(X_{n+1}, n+1+m)\big|\mathcal{F}_n\big] \\ &= E^{X_n}_\omega\big[u(X_1, n+1+m)\big] \\ &= u(X_n, n+m). \end{aligned}$$

Because on $\{\tau_n \leq n\}$ there is nothing to show, we conclude that (b) holds. □

We end this section with technical notation: For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ define the usual norms:

$$\|x\|_1 \triangleq \sum_{k=1}^d |x_i|, \qquad \|x\|_2 \triangleq \Big(\sum_{k=1}^d x_k^2\Big)^{\frac{1}{2}}, \qquad \|x\|_\infty \triangleq \max_{k=1,\dots,d} |x_i|.$$

For $R > 0$ and $y \in \mathbb{R}^d$ let

$$\mathbb{B}_R(y) \triangleq \big\{x \in \mathbb{R}^d \colon \|x - y\|_2 < R\big\}, \quad B_R \triangleq \mathbb{B}_R \cap \mathbb{Z}^d.$$

We also write $\mathbb{B}_R \triangleq \mathbb{B}_R(0)$ and $B_R \triangleq B_R(0)$. For a set $G \subset \mathbb{Z}^d$, we define its discrete boundary by

$$\partial G \triangleq \big\{x \in \mathbb{Z}^d \backslash G \colon \exists y \in G, \|x - y\|_\infty = 1\big\}.$$

Define

$$O_R \triangleq \big\{x \in B_R \colon \|x - y\|_\infty = 1 \Rightarrow y \in B_R\big\}.$$

In case $R > 1$, $O_R$ is the biggest subset of $B_R$ such that $\overline{O}_R \triangleq O_R \cup \partial O_R = B_R$. Furthermore, define by for $\hat{x} = (x, t) \in \mathbb{R}^d \times \mathbb{R}_+$

$$\mathbb{K}_R(\hat{x}) \triangleq \mathbb{B}_R(x) \times [t, t + R^2) \subset \mathbb{R}^d \times \mathbb{R}_+, \quad K_R(\hat{x}) \triangleq \mathbb{K}_R(\hat{x}) \cap (\mathbb{Z}^d \times \mathbb{Z}_+)$$

the continuous and discrete parabolic cylinder with radius $R > 0$ and center $\hat{x}$, respectively. We also set $\mathbb{K}_R \triangleq \mathbb{K}_R(0), K_R \triangleq K_R(0)$ and

$$\begin{aligned} \partial^p \mathbb{K}_R &\triangleq \big(\partial \mathbb{B}_R \times (0, R^2]\big) \cup \big(\mathbb{B}_R \times \{R^2\}\big), \\ \partial^p K_R &\triangleq \big(\partial B_R \times [\lceil R^2 \rceil]\big) \cup \big(B_R \times \{\lceil R^2 \rceil\}\big), \end{aligned}$$

and $\overline{\mathbb{K}}_R \triangleq \mathbb{K}_R \cup \partial^p \mathbb{K}_R, \overline{K}_R \triangleq K_R \cup \partial^p K_R$. Here, $\partial \mathbb{B}_R$ refers to the boundary of $\mathbb{B}_R$ in $\mathbb{R}^d$. We also define

$$Q_R \triangleq O_R \times [\lfloor R^2 \rfloor - 1], \qquad \partial^p Q_R \triangleq \big(\partial O_R \times [\lfloor R^2 \rfloor]\big) \cup \big(O_R \times \{\lfloor R^2 \rfloor\}\big).$$

Moreover, we set

$$K_R^- \triangleq \big(\mathbb{B}_R \times (0, R^2)\big) \cap \big(\mathbb{Z}^d \times \mathbb{Z}_+\big), \qquad K_R^+ \triangleq \big(\mathbb{B}_R \times (2R^2, 3R^3)\big) \cap \big(\mathbb{Z}^d \times \mathbb{Z}_+\big).$$

To capture parities, we define for any set $G \subseteq \mathbb{R}^d \times \mathbb{Z}_+$

$$\Theta^{o/e}(G) \triangleq \big\{(x, t) \in G \colon \|x\|_1 + t \text{ is odd/even}\big\}.$$

**Convention.** *Without explicitly mentioning it, all constants might depend on the measure $P$ and the dimension $d$. Moreover, constants might change from line to line. We denote a generic positive constant by $\mathfrak{c}$.*

### 1.2.2 Main Results

Throughout this chapter, we impose the following:

**Standing Assumption 1.1.** $P(\mathsf{B} \cap \mathsf{G}) = 1$.

We recall the following from [12]:

**Theorem 1.1.** ([12, Theorem 1.1]) *The quenched invariance principle holds with a deterministic diagonal covariance matrix $\mathfrak{A}$, that is for $P$-a.a. $\omega \in \Omega$ as $N \to \infty$ the law of the continuous $\mathbb{R}^d$-valued process*

$$\frac{1}{\sqrt{N}} X_{\lfloor tN \rfloor} + \frac{tN - \lfloor tN \rfloor}{\sqrt{N}} \big( X_{\lfloor tN \rfloor + 1} - X_{\lfloor tN \rfloor} \big), \quad t \in \mathbb{R}_+,$$

*under $P^x_\omega$ converges weakly (on $C(\mathbb{R}_+, \mathbb{R}^d)$ equipped with the local uniform topology) to the law of a Brownian motion with covariance $\mathfrak{A}$ starting at $x$.*

For $a \in (\sqrt{3}, 2]$, let $H_a \in (0, \infty)$ be the following Harnack constant for Brownian motion: For every non-negative solution $u$ to the (backward) heat equation

$$\frac{du}{dt} + \frac{1}{2} \sum_{i,j=1}^{d} \mathfrak{A}_{ij} \frac{d^2 u}{dx_i dx_j} = 0$$

in $\mathbb{K}_{aR}$ it holds that

$$\sup_{\mathbb{B}_R \times (2R^2, 3R^2)} u \leq H_a \inf_{\mathbb{B}_R \times (0, R^2)} u,$$

see [109, Theorem 1]. The following *parabolic Harnack inequality (PHI)* is the main result in this chapter.

**Theorem 1.2.** *Fix $\varepsilon \in (0, 2 - \sqrt{3}), \xi \in (0, \frac{1}{5})$ and $\mathfrak{w} > 1$. There are two constants $R^*, \delta > 0$ such that for all $R \geq R^*$ there exists a set $G \in \mathcal{F}$ such that $P(G) \geq 1 - e^{-R^\delta}$ and for every $\omega \in G$, $p \in \{o, e\}$ and every non-negative $\omega$-caloric function $u$ on $\overline{K}_{2R}$ satisfying*

$$\max_{\Theta^p(\overline{K}_{2R})} u \leq \mathfrak{w}^{R^{2-\xi}} \min_{\Theta^p(\overline{K}_{2R})} u, \tag{1.2}$$

*it holds that*

$$\max_{\Theta^p(K_R^+)} u \leq \frac{(1 + 3\varepsilon) H_{2-\varepsilon}}{(1 - \varepsilon)^2} \min_{\Theta^p(K_R^-)} u. \tag{1.3}$$

The proof of Theorem 1.2 is given in Section 1.6.

**Example 1.2.** *In the following we provide an example which shows that in non degenerate settings the PHI cannot hold in full generality without a certain growth condition. Let us consider the setting of Example 1.1 with $d = 2$. More precisely, take the environment given*

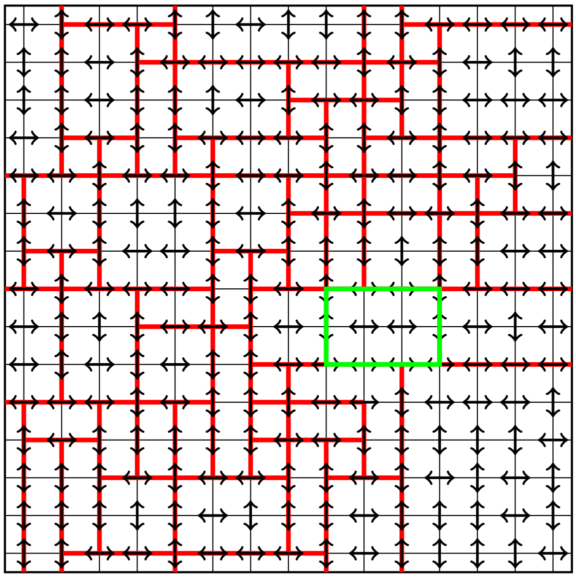

Figure 1.2: An example for the necessity of the growth condition.

*by Figure 1.2. The red part in Figure 1.2 is called the* sink. *It is easy to see that once the walk has reached the sink, it cannot exit it. Consequently, by its recursive definition, the values of a caloric function on the sink are not influenced by the values outside of it. For contradiction, assume that the PHI holds for all caloric functions $u$, i.e. there exists a constant $C > 0$ independent of $u$ and $R$ such that*

$$\max_{\Theta^p(K_R^+)} u \le C \min_{\Theta^p(K_R^+)} u.$$

*Denote the points in the green box in Figure 1.2 by $x_1$ and $x_2$. Fix $R$ large enough such that $2^{R^2} > C$ and take a non-negative caloric function $u$ on the cylinder with radius $2R$ which takes the value one on the sink and*

$$u(x_1, 4R^2) \equiv u(x_2, 4R^2) \triangleq 2^{3R^2}.$$

*Such a caloric function can be defined by recursion. We stress that $u$ does not satisfy the growth condition* (1.2)*. Using the recursive definition, we note that*

$$\max_{\Theta^p(K_R^+)} u \ge 2^{R^2},$$

*and the PHI implies*

$$2^{R^2} \le C \min_{\Theta^p(K_R^-)} u \le C,$$

*which is a contradiction. We conclude that the PHI does not hold for $u$.*

**Remark 1.1.** (i) *The Harnack constant in Theorem 1.2 is optimal in the sense that it can be taken arbitrarily close to $H_2$.*

(ii) *In uniformly elliptic settings the growth condition* (1.2) *is not needed, see [89, 111].*

(iii) *Typically PHIs are formulated for forward equations. The time substitution $t \mapsto 4R^2 - t$ transforms the PHI for backward equations into a PHI for forward equations.*

(iv) *As the following simple example illustrates, it is necessary to compare cylinders of the same parity. Let $u$ be a solution to the (backward) heat equation for the one-dimensional simple random walk in $K_R$ with terminal condition*

$$f(x,t) = \begin{cases} 1, & |x| + t \text{ odd}, \\ 0, & |x| + t \text{ even}. \end{cases}$$

*The recursive definition of a caloric function shows that $u = f$, which implies*

$$\max_{K^+} u = 1, \quad \min_{K^-} u = 0.$$

*Clearly,* (1.3) *does not hold when $\Theta^p(K_R^+)$ and $\Theta^p(K_R^-)$ are replaced by $K_R^+$ and $K_R^-$, respectively. An alternative strategy to deal with the parity issue is to formulate the PHI for*

$$\hat{u}(x,n) \triangleq u(x,n+1) + u(x,n)$$

*instead of $u$. This has been done in [58] for random walks on percolation clusters.*

The proof of Theorem 1.2 is based on arguments introduced by Fabes and Stroock [48]. A version of the Fabes–Stroock argument has also been used in [10] to prove an elliptic Harnack inequality (EHI) under Standing Assumption 1.1. Many ideas in the proof of Theorem 1.2 are borrowed from [10].

Harnack inequalities are important tools in the study of path properties of RWRE. We think that the PHI given in Theorem 1.2 might be the first step in direction of a local limit theorem. The EHI in [10] can for instance be used to prove transience of the RWBRE for $d \geq 3$ in genuinely $d$-dimensional environments. Because this result seems not to appear in the literature, we provide a statement and a proof, which is similar to those of [143, Theorem 3.3.22] and given in Section 1.7.

**Theorem 1.3.** *When $d \geq 3$, the RWRE is transient for $P$-a.a. environments.*

One key tool for the proof of Theorem 1.2 is the following oscillation inequality, which can be seen as a parabolic version of [10, Theorem 4.1].

**Theorem 1.4.** *There are constants $R', \delta > 0, \zeta > 1$ and $\gamma \in (0,1)$ such that for every $R \geq R'$ there exists a set $G \in \mathcal{F}$ such that $P(G) \geq 1 - e^{-R^\delta}$, and for every $\omega \in G$ and every $\omega$-caloric function $u$ on $\overline{K}_{\zeta R}$ it holds that*

$$\operatorname*{osc}_{\Theta^p(K_R)} u \leq \gamma \operatorname*{osc}_{\Theta^p(K_{\zeta R})} u, \qquad p \in \{o, e\},$$

*where*

$$\operatorname*{osc}_G u \triangleq \max_G u - \min_G u, \qquad G \subset \mathbb{Z}^d \times \mathbb{Z}_+ \text{ finite}.$$

The proof of Theorem 1.4 is based on the explicit construction of a coupling. Let us sketch the idea: Suppose that $\hat{X}$ and $\hat{Y}$ are coupled space-time walks in a fixed environment $\omega \in \mathsf{B}$ such that the probability that $\hat{X}$ and $\hat{Y}$ leave a subcylinder of $K_{\zeta R}$ in the same point is bounded from below by a uniform constant $1 - \gamma > 0$. Denote the corresponding hitting times of the boundary by $T$ and $S$, respectively. Then, if $\hat{X}$ starts, say, at $\hat{x} \in \Theta^p(K_R)$ and $\hat{Y}$ starts, say, at $\hat{y} \in \Theta^p(K_R)$, the optional stopping theorem yields for every $\omega$-caloric function $u$ that

$$\begin{aligned} u(\hat{x}) - u(\hat{y}) &= E\big[u(\hat{X}_T) - u(\hat{Y}_S)\big] \\ &= E\big[(u(\hat{X}_T) - u(\hat{Y}_S))\mathbb{1}_{\{\hat{X}_T \neq \hat{Y}_S\}}\big] \\ &\le \operatorname*{osc}_{\Theta^p(K_{\varsigma R})} u \; P(\hat{X}_T \neq \hat{Y}_S) \\ &\le \gamma \operatorname*{osc}_{\Theta^p(K_{\varsigma R})} u. \end{aligned}$$

This implies the oscillation inequality.

Another key tool for the proof of Theorem 1.2 is the following quantitative homogenization estimate: Let $F\colon \overline{\mathbb{K}}_1 \to \mathbb{R}$ be a continuous function of class $C^{2,3}$ on $\mathbb{K}_1$ such that

$$\frac{dF}{dt} + \frac{1}{2}\sum_{i,j=1}^{d} \mathfrak{A}_{ij}\frac{d^2F}{dx_i dx_j} = 0 \quad \text{on } \mathbb{K}_1. \tag{1.4}$$

The existence of $F$ is classical. We define $F_R\colon \mathbb{K}_R \to \mathbb{R}$ by

$$F_R(x,t) \triangleq F\left(\tfrac{x}{R}, \tfrac{t}{R^2}\right), \quad (x,t) \in \mathbb{K}_R.$$

For $u\colon \overline{Q}_R \to \mathbb{R}$, we define

$$(\mathcal{L}_\omega u)(y,s) \triangleq \sum_{i=1}^{2d} \omega(y,e_i)u(y+e_i,1+s) - u(y,s), \quad (y,s) \in Q_R.$$

For $\omega \in \Omega$ let $G_\omega\colon \overline{Q}_R \to \mathbb{R}$ be such that

$$\begin{cases} \mathcal{L}_\omega G_\omega = 0, & \text{on } Q_R, \\ G_\omega = F_R, & \text{on } \partial^p Q_R. \end{cases}$$

It is easy to see that $G_\omega$ exists in a unique manner: Indeed, first set $G_\omega = F_R$ on $\partial^p Q_R$ and then use $\mathcal{L}_\omega G_\omega = 0$ on $Q_R$ to define $G_\omega$ recursively. The following quantitative estimate is a parabolic version of [10, Theorem 1.4].

**Theorem 1.5.** *For all $\varepsilon \in (0,1)$ there exist $R_0 = R_0(\varepsilon) \ge \frac{1}{\varepsilon^2}, C_1 = C_1(F) > 0$, $C_2 = C_2(\varepsilon) > 0$, $C_3 = C_3(\varepsilon) > 0$ and $\delta > 0$ such that for all $R \ge R_0$ we have*

$$P\Big(\Big\{\omega \in \Omega\colon \sup_{Q_R} |F_R - G_\omega| \le \varepsilon\, C_1\Big\}\Big) \ge 1 - C_2 e^{-C_3 R^\delta}.$$

The main tool in the proof of Theorem 1.5 is a new parabolic Aleksandrov–Bakelman–Pucci maximum principle.

The remaining chapter is organized as follows: In Section 1.3 we prove the quantitative estimate (Theorem 1.5), in Section 1.4 we provided estimates on the exit probabilities from a cylinder through a part of the boundary, in Section 1.5 we prove the oscillation inequality (Theorem 1.4), in Section 1.6 we prove the PHI (Theorem 1.2) and finally in Section 1.7 we prove transience for $d \ge 3$.

## 1.3 Proof of the Quantitative Estimate: Theorem 1.5

### 1.3.1 A Parabolic Maximum Principle

In this section we prove a parabolic version of the Aleksandrov–Bakelman–Pucci maximum principle [12, Theorem 3.1]. We need further notation: For $k \in \mathbb{Z}_+$ set

$$\begin{aligned}
\partial^k O_R &\triangleq \big\{x \notin O_R \colon \exists_{y \in O_R}, \|x-y\|_\infty \le k\big\},\\
\partial^k Q_R &\triangleq \big(\partial^k O_R \times [\lfloor R^2 \rfloor + k]\big) \cup \big(O_R \times \{\lfloor R^2\rfloor, \dots, \lfloor R^2 \rfloor + k\}\big),\\
Q^k_R &\triangleq Q_R \cup \partial^k Q_R.
\end{aligned}$$

Fix a function $u \colon Q^k_R \to \mathbb{R}$ and define for $(y,s) \in Q_R$

$$\begin{aligned}
I_u(y,s) &\triangleq \big\{p \in \mathbb{R}^d \colon u(y,s) - u(x,t) \ge \langle p, y-x\rangle \ \forall\ (x,t) \in Q^k_R \text{ with } t > s\big\},\\
\Gamma_u &\triangleq \big\{(y,s) \in Q_R \colon I_u(y,s) \neq \emptyset\big\}.
\end{aligned}$$

Let $\alpha(n)$ be the coordinate that changes between $X_{n-1}$ and $X_n$ and define

$$T \triangleq \inf(n \in \mathbb{Z}_+ \colon \{\alpha(1), \dots, \alpha(n)\} = \{1, \dots, d\}), \qquad T^{(k)} \triangleq T \wedge k. \tag{1.5}$$

We are in the position to state the main result of this section.

**Theorem 1.6.** *There exists an $R_o > 0$ such that for all $R \ge R_o, 0 < k < R$ and all $\omega \in \mathsf{B}$ the following implication holds: If $u \le 0$ on $\partial^k Q_R$ and for all $z \in O_R$*

$$P^z_\omega(T > k) < e^{-(\log R)^3}, \tag{1.6}$$

*then*

$$\sup_{Q_R} u \le \mathfrak{c} R^{\frac{d}{d+1}} \Big( \sum_{(y,s) \in \Gamma_u} \big| E^y_\omega\big[u(X_{T^{(k)}}, s+1+T^{(k)})\big] - u(y, s+1)\big|^{d+1}\Big)^{\frac{1}{d+1}}. \tag{1.7}$$

*Proof.* We borrow ideas from the proofs of [12, Theorem 3.1] and [42, Theorem 2.2]. Define

$$M \triangleq \sup_{Q_R} u, \qquad \Theta \triangleq \big\{(y,s) \in \mathbb{R}^{d+1} \colon (2+\sqrt{d})R\|y\|_2 < s < \tfrac{M}{2}\big\}.$$

W.l.o.g. we assume that $M > 0$. Note that

$$\ell(\Theta) = \int_0^{\frac{M}{2}} \Big(\int_{\mathbb{R}^d} \mathbb{1}_{\{\|x\|_2 < \frac{s}{2R}\}} dx\Big) ds = \mathfrak{c} \int_0^{\frac{M}{2}} \frac{s^d ds}{R^d} = \frac{\mathfrak{c} M^{d+1}}{R^d},$$

where $\ell$ denotes the Lebesgue measure. Consequently, we have

$$M = \mathfrak{c} R^{\frac{d}{d+1}} \ell(\Theta)^{\frac{1}{d+1}}.$$

In other words, (1.7) follows once we show that

$$\ell(\Theta) \le \mathfrak{c} \sum_{(y,s) \in \Gamma_u} \big| E^y_\omega\big[u(X_{T^{(k)}}, s+1+T^{(k)})\big] - u(y,s+1)\big|^{d+1}. \tag{1.8}$$

The proofs of the following lemmata are postponed till the proof of Theorem 1.6 is complete.

**Lemma 1.3.** *We have*

$$\ell(\Theta) \leq \sum_{(y,s)\in\Gamma_u} (u(y,s) - u(y,s+1))\ell(I(y,s)).$$

*(Note that $u(y,s) - u(y,s+1) \geq 0$ whenever $(y,s) \in \Gamma_u$.)*

**Lemma 1.4.** *There exists an $R_o > 0$ such that whenever $R \geq R_o$ for all $(y,s) \in \Gamma_u$*

$$\ell(I(y,s)) \leq 4^d \big(E^y_\omega\big[u(X_{T^{(k)}}, s+1+T^{(k)})\big] - u(y,s)\big)^d_+.$$

Set

$$\Psi \triangleq \big\{(y,s) \in \Gamma_u \colon E^y_\omega\big[u(X_{T^{(k)}}, s+1+T^{(k)})\big] - u(y,s) > 0\big\}.$$

Using Lemmata 1.3 and 1.4 and the arithmetic-geometric mean inequality, we obtain for all $R \geq R_o$

$$\begin{aligned}
\ell(\Theta) &\leq \mathfrak{c} \sum_{(y,s)\in\Psi} (u(y,s) - u(y,s+1))\big(E^y_\omega\big[u(X_{T^{(k)}}, s+1+T^{(k)})\big] - u(y,s)\big)^d \\
&\leq \mathfrak{c} \sum_{(y,s)\in\Psi} \big(u(y,s) - u(y,s+1) + dE^y_\omega\big[u(X_{T^{(k)}}, s+1+T^{(k)})\big] - du(y,s)\big)^{d+1} \\
&\leq \mathfrak{c} \sum_{(y,s)\in\Psi} \big(E^y_\omega\big[u(X_{T^{(k)}}, s+1+T^{(k)})\big] - u(y,s+1)\big)^{d+1} \\
&\leq \mathfrak{c} \sum_{(y,s)\in\Gamma_u} \big|E^y_\omega\big[u(X_{T^{(k)}}, s+1+T^{(k)})\big] - u(y,s+1)\big|^{d+1}.
\end{aligned}$$

The claim of Theorem 1.6 follows. □

It remains to prove the Lemmata 1.3 and 1.4.

*Proof of Lemma 1.3.* We borrow arguments from the proof of [42, Theorem 2.2]. Set

$$\chi(y,s) \triangleq \big\{(p, q - \langle y, p\rangle) \colon p \in I_u(y,s) \text{ and } q \in [u(y,s+1), u(y,s)]\big\} \subset \mathbb{R}^{d+1}.$$

The key observation is the following inclusion:

$$\Theta \subseteq \chi(\Gamma_u) \triangleq \bigcup_{(y,s)\in\Gamma_u} \chi(y,s). \tag{1.9}$$

Let us accept (1.9) for a moment. Then, using that the map $(y,z) \mapsto (y, z - \langle \beta, y\rangle)$ preserves volume, because it has determinant one, we obtain

$$\ell(\Theta) \leq \ell(\chi(\Gamma_u)) \leq \sum_{(y,s)\in\Gamma_u} (u(y,s) - u(y,s+1))\ell(I_u(y,s)),$$

which is the claim.

It remains to prove (1.9). Let $(y,s) \in \Theta$ and define

$$\phi(x,t) \triangleq u(x,t) - \langle y, x\rangle - s, \quad (x,t) \in Q^k_R.$$

Let $(y_0, s_0) \in Q_R$ be such that $u(y_0, s_0) = M$. Recalling the definition of $\Theta$, we see that $\phi(y_0, s_0) > 0$ and that $\phi(x,t) < 0$ for all $(x,t) \in Q_R^k$ with $u(x,t) \leq 0$. Let

$$N_x \triangleq \max(t\colon (x,t) \in Q_R^k \text{ and } \phi(x,t) \geq 0), \quad \max(\emptyset) \triangleq -\infty.$$

Note that $s_0 \leq N_{y_0} \leq s_1 \triangleq \max(N_x \colon x \in O_R \cup \partial^k O_R) = \max(N_x \colon x \in O_R) \leq \lfloor R^2 \rfloor - 1$. Let $y_1$ be such that $s_1 = N_{y_1}$, and note that $(y_1, s_1) \in Q_R$. For all $(x,t) \in Q_R^k$ with $t > s_1$ we have $\phi(x,t) < 0$, which yields that $u(x,t) - \langle y, x\rangle < s \leq u(y_1, s_1) - \langle y, y_1\rangle$, because $\phi(y_1, s_1) \geq 0$. This implies that $y \in I_u(y_1, s_1)$. By definition of $s_1$, we have $\phi(y_1, s_1 + 1) < 0$, and hence $u(y_1, s_1 + 1) < \langle y, y_1 \rangle + s$. We conclude that $u(y_1, s_1 + 1) < \langle y, y_1\rangle + s \leq u(y_1, s_1)$, which finally implies $(y, s) \in \chi(y_1, s_1)$ and thus (1.9) holds. □

*Proof of Lemma 1.4.* We borrow the idea of the proof of [12, Lemma 3.4]. Fix $(y, s) \in Q_R$. For $i = 1, \dots, d$ define

$$u_i \triangleq \inf(n \in \mathbb{Z}_+ \colon \alpha(n) = i).$$

Furthermore, we define the following events:

$$A_i^{(+)} \triangleq \big\{X_{u_i} - X_{u_i - 1} = e_i, u_i \leq k\big\},$$
$$A_i^{(-)} \triangleq \big\{X_{u_i} - X_{u_i - 1} = -e_i, u_i \leq k\big\}.$$

Let $W$ be a random variable independent of the walk, which takes the values $\pm 1$ with probability $\frac{1}{2}$. Finally, define

$$B_i^{(+)} \triangleq A_i^{(+)} \cup \big(\{W = +1\} \cap \{u_i > k\}\big),$$
$$B_i^{(-)} \triangleq A_i^{(-)} \cup \big(\{W = -1\} \cap \{u_i > k\}\big).$$

We note that $B_i^{(+)}$ and $B_i^{(+)}$ are disjoint and that the union is $P_\omega^y$-full. Thus, due to symmetry, we have

$$P_\omega^y(B_i^{(+)}) = P_\omega^y(B_i^{(-)}) = \tfrac{1}{2}.$$

Because $\omega \in \mathsf{B}$, the walk $X$ is a $P_\omega^y$-martingale and $E_\omega^y[X_{T(k)}] = y$ follows from the optional stopping theorem. In summary, we obtain

$$\begin{aligned} E_\omega^y\big[X_{T(k)} | B_i^{(+)}\big] &= 2E_\omega^y\big[X_{T(k)} \mathbb{1}_{B_i^{(+)}}\big] \\ &= 2\big(E_\omega^y\big[X_{T(k)}\big] - E_\omega^y\big[X_{T(k)} \mathbb{1}_{B_i^{(-)}}\big]\big) \\ &= 2y - E_\omega^y\big[X_{T(k)} | B_i^{(-)}\big]. \end{aligned}$$

Hence, we have

$$\mathcal{O}_i \triangleq E_\omega^y\big[X_{T(k)} | B_i^{(+)}\big] - y = y - E_\omega^y\big[X_{T(k)} | B_i^{(-)}\big].$$

Take $\beta \in I_u(y,s)$. Using the definition of $I_u(y,s)$, we obtain

$$\begin{aligned}\langle \beta, \mathcal{O}_i \rangle &= \sum_{x \in O_R \cup \partial^k O_R} \langle \beta, x-y \rangle P_\omega^y(X_{T^{(k)}} = x | B_i^{(+)}) \\ &= \sum_{(x,t) \in Q_R \cup \partial^k Q_R} \langle \beta, x-y \rangle P_\omega^y(X_{T^{(k)}} = x, T^{(k)} = t-s-1 | B_i^{(+)}) \\ &\geq \sum_{(x,t) \in Q_R \cup \partial^k Q_R} (u(x,t) - u(y,s)) P_\omega^y(X_{T^{(k)}} = x, T^{(k)} = t-s-1 | B_i^{(+)}) \\ &= E_\omega^y\big[u(X_{T^{(k)}}, s+1+T^{(k)}) | B_i^{(+)}\big] - u(y,s).\end{aligned}$$

Similarly, we see that

$$\langle \beta, -\mathcal{O}_i \rangle \geq E_\omega^y\big[u(X_{T^{(k)}}, s+1+T^{(k)}) | B_i^{(-)}\big] - u(y,s).$$

Consequently, $\langle \beta, \mathcal{O}_i \rangle$ lies in an interval which length is bounded by $2L$, where

$$L \triangleq u(y,s) - E_\omega^y\big[u(X_{T^{(k)}}, s+1+T^{(k)})\big].$$

We conclude that

$$\ell(I_u(y,s)) \leq \ell\big(\big\{z \in \mathbb{R}^d : \forall_{i=1,\dots,d} \ \big\langle z, \mathcal{O}_i \big\rangle \in [0, 2L]\big\}\big).$$

Note that

$$\ell\big(\big\{z \in \mathbb{R}^d : \forall_{i=1,\dots,d} \ \big\langle z, \mathcal{O}_i \big\rangle \in [0, 2L]\big\}\big) = (2L)^d |\det\,(M)|,$$

where $M = \big(\mathcal{O}_1 \ \cdots \ \mathcal{O}_d\big)$. Due to Hadamard's determinant inequality, we have

$$|\det(M)| \leq \prod_{i=1}^d \big\|\mathcal{O}_i\big\|.$$

For large enough $R$, we deduce from [12, Claim 3.5] that

$$\big\|e_i - \mathcal{O}_i\big\| < \exp\big(-(\log R)^2\big).$$

Consequently, by the triangle inequality, we have

$$|\det(M)| \leq \big(1 + \exp\big(-(\log R)^2\big)\big)^d.$$

Now, the claim of the lemma follows. □

### 1.3.2 Proof of Theorem 1.5

We start with some general comments:

1. We fix $\varepsilon > 0, n_0 \in \mathbb{N}, K > 0$ and $\alpha \in (0, \frac{1}{2})$. Here, $\varepsilon$ is as in the statement of the theorem and $\alpha = \alpha(\varepsilon)$ is a free parameter, which we choose small enough in the end of the proof. First, we determine $K = K(d, P)$, then $\alpha = \alpha(\varepsilon)$, $n_0 = n_0(\varepsilon)$ and $R_0 = R_0(\varepsilon)$.

2. We denote by $\mathsf{c}$ any constant which only depends on the dimension $d$, the function

$F\colon \overline{\mathbb{K}}_1 \to \mathbb{R}$ and the environment measure $P$. The constant might change from line to line.

3. We simplify the notation and write $G$ instead of $G_\omega$. Moreover, we set

$$k = k(R) \triangleq \lfloor \sqrt{R} \rfloor.$$

The idea is to control

$$H \triangleq F_R - G$$

with Theorem 1.6. By definition of $F_R$ and $G$, we see that

$$\begin{cases} \mathcal{L}_\omega H = \mathcal{L}_\omega F_R, & \text{on } Q_R, \\ H = 0, & \text{on } \partial^p Q_R. \end{cases}$$

Because $H$ is only defined on $Q_R$ instead of $Q_R^k$ we cannot apply Theorem 1.6 directly. To overcome this problem, we consider an extension $h$ of $H$. Set $R^* \triangleq R + \sqrt{d}k$ and let $H'\colon \overline{K}_{R^*} \to \mathbb{R}$ be a solution to

$$\begin{cases} \mathcal{L}_\omega H' = (\mathcal{L}_\omega F_R)\mathbb{1}_{Q_R}, & \text{on } K_{R^*}, \\ H' = 0, & \text{on } \partial^p K_{R^*}. \end{cases}$$

**Lemma 1.5.** (i) $\max_{Q_R} |H - H'| \le \max_{\partial^p Q_R} |H'|$.

(ii) $\max_{\partial^k Q_R} |H'| \le \frac{\mathsf{c}}{\sqrt{R}}$.

*Proof.* (i). Fix $(y,s) \in Q_R$, set $\rho \triangleq \inf(t \in \mathbb{Z}_+\colon (X_t, s+t) \in \partial^p Q_R)$ and note that $\mathcal{L}_\omega(H - H') = 0$ on $Q_R$. Thus, we deduce from the optional stopping theorem that

$$\begin{aligned} |H(y,s) - H'(y,s)| &= \big| E^y_\omega \big[ H(X_\rho, s+\rho) - H'(X_\rho, s+\rho) \big] \big| \\ &= \big| E^y_\omega \big[ H'(X_\rho, s+\rho) \big] \big| \\ &\le \max_{\partial^p Q_R} |H'|. \end{aligned}$$

Thus, (i) follows.

(ii). By Taylor's theorem, we obtain for all $(y,s), (x,t) \in Q_R$

$$\begin{aligned} F_R(y,s) - F_R(x,t) = \tfrac{1}{R}\big\langle \nabla F\big(\tfrac{x}{R}, \tfrac{t}{R^2}\big), y - x \big\rangle &+ \tfrac{1}{2R^2}\big\langle y - x, \nabla^2 F\big(\tfrac{x}{R}, \tfrac{t}{R^2}\big)(y-x)\big\rangle \\ &+ \tfrac{1}{R^2}\tfrac{dF}{dt}\big(\tfrac{x}{R}, \tfrac{t}{R^2}\big)(s-t) \\ &+ \rho^{\mathrm{t}}_s |s-t|^2 + \rho^{\mathrm{s}}_y \|y - x\|_2^3, \end{aligned} \tag{1.10}$$

where $\rho^{\mathrm{s}}_y$ is bounded by $\frac{\mathsf{c}}{R^3}$ and $\rho^{\mathrm{t}}_s$ is bounded by $\frac{\mathsf{c}}{R^4}$. Thus, for all $(x,t) \in Q_R$

$$\big|(\mathcal{L}_\omega F_R)(x,t)\big| = \big| E^x_\omega\big[F_R(X_1, 1+t)\big] - F_R(x,t)\big| \le \tfrac{\mathsf{c}}{R^2}. \tag{1.11}$$

We set

$$\tau \triangleq \inf(t \in \mathbb{Z}_+\colon (X_t, s+t) \in \partial^p K_{R^*}), \qquad \rho \triangleq \inf(t \in \mathbb{Z}_+\colon X_t \in \partial B_{R^*}).$$

Because $(\|X_n\|_2^2 - n)_{n\in\mathbb{Z}_+}$ is a $P_\omega^x$-martingale, the optional stopping theorem yields that

$$\max_{x\in\partial^k O_R} E_\omega^x\big[\rho\big] = \max_{x\in\partial^k O_R} \big(E_\omega^x\big[\|X_\rho\|_2^2\big] - \|x\|_2^2\big) \leq \mathfrak{c}Rk. \tag{1.12}$$

Fix $(y,s)\in\partial^k Q_R$. The optional stopping theorem also yields that

$$\begin{aligned} H'(y,s) &= E_\omega^y\big[H'(X_\tau, s+\tau)\big] - E_\omega^y\Big[\sum_{t=0}^{\tau-1}\mathcal{L}_\omega H'(X_t, s+t)\Big] \\ &= -E_\omega^y\Big[\sum_{t=0}^{\tau-1}\mathcal{L}_\omega F_R(X_t,s+t)\mathbb{1}_{Q_R}(X_t,s+t)\Big]. \end{aligned} \tag{1.13}$$

If $s=\lfloor R^2\rfloor$, we have $H'(y,s)=0$ and if $s<\lfloor R^2\rfloor$, then $y\in\partial O_R$ and we deduce from (1.11), (1.12) and (1.13) that

$$\big|H'(y,s)\big| \leq \tfrac{\mathsf{c}}{R^2}E_\omega^y\big[\tau\big] \leq \tfrac{\mathsf{c}}{R^2}E_\omega^y\big[\rho\big] \leq \tfrac{\mathsf{c}Rk}{R^2} = \tfrac{\mathsf{c}}{\sqrt{R}}.$$

The proof is complete. □

Next, we add a quadratic penalty term to the function $H'$. Define

$$h(y,s) \triangleq H'(y,s) + \frac{\mathsf{c}'\varepsilon}{R^2}\|y\|_2^2, \quad (y,s)\in\overline{K}_{R^*}.$$

We will determine the constant $\mathsf{c}'=\mathsf{c}'(F)>0$ in Lemma 1.6 below.

To apply Theorem 1.6 to $h$, we have to control the upper contact set of $h$ and the $\omega$-Laplacian of $h$. In the next lemma we show that $\mathsf{c}'$ can be chosen such that only a few points are in the upper contact set. To formulate the lemma, we need more notation: Recall that we fixed a constant $n_0=n_0(\varepsilon)$. Set

$$(M_\omega^{(n_0)}(x))_{ij} \triangleq \frac{1}{n_0}E_\omega^x\big[(X_{n_0}^{(i)}-x^{(i)})(X_{n_0}^{(j)}-x^{(j)})\big], \quad 1\leq i,j\leq d,$$

where $X_{n_0}^{(k)}$ and $x^{(k)}$ denote the $k^{\text{th}}$ coordinate of $X_{n_0}$ and $x$. Moreover, set

$$A_{n_0}(x) \triangleq \big\{\omega\in\Omega\colon \|M_\omega^{(n_0)}(x)-\mathfrak{A}\|<\varepsilon\big\},$$

where $\|\cdot\|$ denotes the trace norm, i.e. $\|M\|\triangleq \operatorname{tr}(\sqrt{MM^*})$. Here, $\mathfrak{A}$ is the limiting covariance matrix as given by Theorem 1.1. Finally, set

$$J_{n_0}(R) \triangleq \big\{\hat{x}\in\overline{Q}_R\colon d(\hat{x},\partial^p Q_R)>n_0\big\}, \quad d\equiv \text{distance function.}$$

We are in the position to formulate the lemma announced above:

**Lemma 1.6.** *The constant $\mathsf{c}'$ can be chosen such that the following holds: Let $R>\frac{\sqrt{n_0}}{\varepsilon}\vee n_0$. If $(x,t)\in J_{n_0}(R)$ and $\omega\in A_{n_0}(x)$, then $(x,t)\notin\Gamma_h$.*

*Proof.* Take $R>\frac{\sqrt{n_0}}{\varepsilon}\vee n_0$, $(x,t)\in J_{n_0}(R)$ and $\omega\in A_{n_0}(x)$. Recalling (1.10) and using that the walk $X$ is a $P^x_\omega$-martingale, we obtain

$$\begin{aligned} E^x_\omega\big[F_R(X_{n_0}, t+n_0)\big] - F_R(x,t) \\ &= \tfrac{1}{2R^2} E^x_\omega\big[\langle X_{n_0}-x, \nabla^2 F\big(\tfrac{x}{R}, \tfrac{t}{R^2}\big)(X_{n_0}-x)\rangle\big] \\ &\quad + \tfrac{n_0}{R^2}\tfrac{dF}{dt}\big(\tfrac{x}{R}, \tfrac{t}{R^2}\big) + \rho^{\mathrm{t}}_{t+n_0} n_0^2 + E^x_\omega\big[\rho^{\mathrm{s}}_{X_{n_0}} \|X_{n_0}-x\|_2^3\big]. \end{aligned}$$

Because $\omega \in A_{n_0}(x)$, we have

$$\begin{aligned} E^x_\omega\big[\langle X_{n_0}-x, \nabla^2 F\big(\tfrac{x}{R}, \tfrac{t}{R^2}\big)(X_{n_0}-x)\rangle\big] &= n_0 \mathrm{tr}\big(\nabla^2 F\big(\tfrac{x}{R}, \tfrac{t}{R^2}\big) M^{(n_0)}_\omega(x)\big) \\ &\le \mathsf{c} n_0 \varepsilon + n_0 \mathrm{tr}\big(\mathfrak{A}\nabla^2 F\big(\tfrac{x}{R}, \tfrac{t}{R^2}\big)\big). \end{aligned}$$

Using that $\frac{dF}{dt} + \frac{1}{2}\mathrm{tr}(\mathfrak{A}\nabla^2 F) = 0$, we obtain

$$\begin{aligned} \big|E^x_\omega\big[F_R(X_{n_0}, t+n_0)\big] - F_R(x,t)\big| &\le \tfrac{n_0 \mathsf{c}\varepsilon}{2R^2} + \tfrac{n_0}{R^2}\big(\tfrac{1}{2}\mathrm{tr}\big(\mathfrak{A}\nabla^2 F\big(\tfrac{x}{R}, \tfrac{t}{R^2}\big)\big) + \tfrac{dF}{dt}\big(\tfrac{x}{R}, \tfrac{t}{R^2}\big)\big) \\ &\quad + \rho^{\mathrm{t}}_{t+n_0} n_0^2 + E^x_\omega\big[\rho^{\mathrm{s}}_{X_{n_0}} \|X_{n_0}-x\|_2^3\big] \\ &= \tfrac{\mathsf{c} n_0 \varepsilon}{2R^2} + \rho^{\mathrm{t}}_{t+n_0} n_0^2 + E^x_\omega\big[\rho^{\mathrm{s}}_{X_{n_0}} \|X_{n_0}-x\|_2^3\big] \\ &\le \tfrac{\mathsf{c} n_0 \varepsilon}{2R^2} + \tfrac{\mathsf{c} n_0^2}{R^4} + \tfrac{\mathsf{c}}{R^3}\, E^x_\omega\big[\|X_{n_0}-x\|_2^3\big]. \end{aligned}$$

We deduce from the Burkholder–Davis–Gundy inequality that $E^x_\omega\big[\|X_{n_0}-x\|_2^3\big] \le \mathfrak{c} n_0^{\frac{3}{2}}$. In summary, we have

$$\big|E^\omega_x\big[F_R(X_{n_0}, t+n_0)\big] - F_R(x,t)\big| \le \tfrac{n_0\varepsilon}{R^2}\big(\tfrac{\mathsf{c}}{2} \; + \mathsf{c}\big(\tfrac{n_0}{\varepsilon R^2} + \tfrac{\mathsf{c}\sqrt{n_0}}{\varepsilon R}\big)\big) \le \mathsf{c}\tfrac{n_0\varepsilon}{R^2} \triangleq \mathsf{c}' \tfrac{n_0\varepsilon}{2R^2}.$$

Because $\mathcal{L}_\omega(H' - F_R) = 0$ on $Q_R$, we have

$$E^x_\omega\big[H'(X_{n_0}, t+n_0)\big] - H'(x,t) = E^x_\omega\big[F_R(X_{n_0}, t+n_0)\big] - F_R(x,t).$$

We obtain

$$\begin{aligned} E^x_\omega\big[h(X_{n_0}, t+n_0)\big] - h(x,t) \\ &= E^x_\omega\big[F_R(X_{n_0}, t+n_0)\big] - F_R(x,t) + \mathsf{c}' \tfrac{\varepsilon}{R^2} E^x_\omega\big[\|X_{n_0}\|_2^2 - \|x\|_2^2\big] \\ &= E^x_\omega\big[F_R(X_{n_0}, t+n_0)\big] - F_R(x,t) + \mathsf{c}' \tfrac{n_0\varepsilon}{R^2} \\ &\ge -\mathsf{c}' \tfrac{n_0\varepsilon}{2R^2} + \mathsf{c}' \tfrac{n_0\varepsilon}{R^2} = \mathsf{c}' \tfrac{n_0\varepsilon}{2R^2} > 0. \end{aligned}$$

Using this inequality and the fact that martingales have constant expectation, we obtain for all $p \in \mathbb{R}^d$

$$E^x_\omega\big[h(X_{n_0}, t+n_0) + \langle p, x - X_{n_0}\rangle\big] - h(x,t) > 0.$$

Thus, for all $p \in \mathbb{R}^d$ there exists a $y$ in the $P^x_\omega$-support of $X_{n_0}$ such that

$$\langle p, x - y\rangle > h(x,t) - h(y, t+n_0).$$

We conclude that $(x,t) \notin \Gamma_h$. The proof is complete. □

Lemma 1.6 suggests that we should restrict our attention to environments which are in

$A_{n_0}(x)$ for many $x \in O_R$. Motivated by this observation, we define

$$A^{(1)} = A_R^{(1)}(\alpha) \triangleq \Big\{\omega \in \Omega \colon \frac{1}{|B_R|} \sum_{x \in B_R} \mathbb{1}_{\{\omega \in A_{n_0}(x)\}} > 1 - 2\alpha\Big\},$$

where $\alpha = \alpha(\varepsilon)$ is one of the free constants fixed in the beginning of the proof.

Next, we control the $\omega$-Laplacian of $h$:

$$(L_\omega h)(y,s) \triangleq E_\omega^y\big[h(X_{T^{(k)}}, s+1+T^{(k)})\big] - h(y, s+1), \quad (y,s) \in K_R,$$

where $T^{(k)} = T \wedge k$ is the stopping time defined in (1.5).

**Lemma 1.7.** *For all* $(x,s) \in K_R$

$$\big|(L_\omega h)(x,s)\big| \le \frac{\mathsf{c} E_\omega^x[T]}{R^2}.$$

*Proof.* Taylor's theorem yields that $|(\mathcal{L}_\omega F_R)(x,t)| \le \frac{\mathsf{c}}{R^2}$ for all $(x,t) \in K_R$. For $f(x) = \|x\|_2^2$, note that $(\mathcal{L}_\omega f)(x,t) = E_\omega^x\big[\|X_1\|_2^2\big] - \|x\|_2^2 = 1$. Consequently, we obtain that

$$|\mathcal{L}_\omega h| = |(\mathcal{L}_\omega F_R)\mathbb{1}_{Q_R} + \mathsf{c}'\tfrac{\varepsilon}{R^2}| \le \tfrac{\mathsf{c}}{R^2}.$$

We deduce from the optional stopping theorem that

$$\big|(L_\omega h)(x,s)\big| = \Big|E_\omega^x\Big[\sum_{t=0}^{T^{(k)}-1} (\mathcal{L}_\omega h)(X_t, s+1+t)\Big]\Big| \le E_\omega^x[T]\tfrac{\mathsf{c}}{R^2}.$$

This completes the proof. □

Lemma 1.7 shows that the $\omega$-Laplacian of $h$ can be controlled via $x \mapsto E_\omega^x[T]$. Motivated by this observation, we define

$$A_R^{(2)} \triangleq \Big\{\omega \in \Omega \colon \frac{1}{|B_R|} \sum_{x \in B_R} \big|E_\omega^x[T]\big|^{d+2} \le K\Big\},$$

where $K$ is one of the constants we fixed in the beginning.

As a last step before we apply Theorem 1.6, we introduce the following:

$$A_R^{(3)} \triangleq \Big\{\omega \in \Omega \colon P_\omega^x(T > k) < e^{-(\log R)^3} \text{ for all } x \in O_R\Big\},$$

which is in conjunction with the statement of Theorem 1.6.

We are in the position to complete the proof of Theorem 1.5. Take $\omega \in A^{(1)} \cap A^{(2)} \cap A^{(3)}$ and let $R$ be such that $R_o \vee \frac{n_0}{\alpha} \vee \frac{\sqrt{n_0}}{\varepsilon} \vee n_0 \le R$, where $R_o$ is as in Theorem 1.6. Note that

$$\ell(\mathbb{B}_R \backslash \mathbb{B}_{R-n_0}) = \mathfrak{c} \int_{R-n_0}^{R} r^{d-1} dr \le \mathfrak{c} n_0 R^{d-1}.$$

Because $\omega \in A_R^{(1)}$, we have for $s < \lceil R^2 \rceil - n_0$

$$\frac{1}{|B_R|} \sum_{x \in B_R} \mathbb{1}_{\{(x,s) \notin J_{n_0}(R) \text{ or } \omega \notin A_{n_0}(x)\}} \le \tfrac{\mathfrak{c} n_0}{R} + 2\alpha \le (\mathsf{c}+2)\alpha = \mathsf{c}\alpha. \tag{1.14}$$

Moreover, because $\omega \in A_R^{(2)}$, we deduce from Lemma 1.7 that

$$\frac{1}{|B_R|}\sum_{y\in B_R}\left|(L_\omega h)(y,s)\right|^{d+2} \le \frac{\mathsf{c}}{R^{2(d+2)}}\frac{1}{|B_R|}\sum_{y\in B_R}\left|E_\omega^y[T]\right|^{d+2} \le \frac{\mathsf{c}K}{R^{2(d+2)}}, \tag{1.15}$$

and

$$\frac{1}{|B_R|}\sum_{x\in B_R}\left|(L_\omega h)(y,s)\right|^{d+1} \le \frac{\mathsf{c}}{R^{2(d+1)}}\ \frac{1}{|B_R|}\sum_{y\in B_R}\left|E_\omega^y[T]\right|^{d+2} \le \frac{\mathsf{c}K}{R^{2(d+1)}}. \tag{1.16}$$

Furthermore, Lemma 1.5 yields that

$$\max_{\partial^k Q_R} h \le \max_{\partial^k Q_R} H' + \mathsf{c}'\varepsilon\tfrac{(R+k)^2}{R^2} \le \mathsf{c}\big(\tfrac{1}{\sqrt{R}}+\varepsilon\big). \tag{1.17}$$

Because $\omega \in A_R^{(3)}$, we can apply Theorem 1.6 and obtain that

$$\max_{Q_R} h - \max_{\partial^k Q_R} h \le \mathsf{c}R^{\frac{d}{d+1}}\Big(\sum_{(y,s)\in\Gamma_h}\left|(L_\omega h)(y,s)\right|^{d+1}\Big)^{\frac{1}{d+1}}. \tag{1.18}$$

Using Lemma 1.6, we obtain

$$\begin{aligned}(1.18) &= \mathsf{c}R^{\frac{2d}{d+1}}\Big(\frac{1}{|B_R|}\sum_{(y,s)\in\Gamma_h}\mathbb{1}_{\{(x,s)\notin J_{n_0}(R)\text{ or }\omega\notin A_{n_0}(x)\}}\left|(L_\omega h)(y,s)\right|^{d+1}\Big)^{\frac{1}{d+1}}\\ &\le \mathsf{c}R^{\frac{2d}{d+1}}\Big(\frac{1}{|B_R|}\sum_{(y,s)\in K_R}\mathbb{1}_{\{(x,s)\notin J_{n_0}(R)\text{ or }\omega\notin A_{n_0}(x)\}}\left|(L_\omega h)(y,s)\right|^{d+1}\Big)^{\frac{1}{d+1}}.\end{aligned}$$

Using Hölder's inequality, (1.14), (1.15) and (1.16), we further obtain that

$$\begin{aligned}(1.18) \le \mathsf{c}R^{\frac{2d}{d+1}}\Big(\sum_{s=0}^{R^2-n_0-1}&\Big[\frac{1}{|B_R|}\sum_{x\in B_R}\mathbb{1}_{\{(x,s)\notin J_{n_0}(R)\text{ or }\omega\notin A_{n_0}(x)\}}\Big]^{\frac{1}{d+2}}\\ &\Big[\frac{1}{|B_R|}\sum_{y\in B_R}\left|(L_\omega h)(y,s)\right|^{d+2}\Big]^{\frac{d+1}{d+2}}\\ &+\sum_{s=R^2-n_0}^{R^2-1}\frac{1}{|B_R|}\sum_{x\in B_R}\left|(L_\omega h)(y,s)\right|^{d+1}\Big)^{\frac{1}{d+1}}\\ \le \mathsf{c}R^{\frac{2d}{d+1}}&\Big(R^2\big[\mathsf{c}\alpha\big]^{\frac{1}{d+2}}\big[\mathsf{c}K\big]^{\frac{d+1}{d+2}}R^{-2(d+1)} + n_0\mathsf{c}KR^{-2(d+1)}\Big)^{\frac{1}{d+1}}\\ \le \mathsf{c}&\Big(\alpha^{\frac{1}{d+2}}K^{\frac{d+1}{d+2}}+\alpha K\Big)^{\frac{1}{d+1}}.\end{aligned}$$

Combining this bound with Lemma 1.5 and (1.17) shows that

$$\begin{aligned}\max_{Q_R} H &\leq \max_{Q_R} H' + \tfrac{\mathsf{c}}{\sqrt{R}}\\ &\leq \max_{Q_R} h + \tfrac{\mathsf{c}}{\sqrt{R}}\\ &\leq \mathsf{c}\big(\alpha^{\frac{1}{d+2}} K^{\frac{d+1}{d+2}} + \alpha K\big)^{\frac{1}{d+1}} + \mathsf{c}\big(\tfrac{1}{\sqrt{R}} + \varepsilon\big) + \tfrac{\mathsf{c}}{\sqrt{R}}\\ &= \mathsf{c}\big(\alpha^{\frac{1}{d+2}} K^{\frac{d+1}{d+2}} + \alpha K\big)^{\frac{1}{d+1}} + \mathsf{c}\varepsilon + \tfrac{\mathsf{c}}{\sqrt{R}}.\end{aligned}$$

Replacing the roles of $F_R$ with $G$ yields that

$$\max_{Q_R} \ |H| \leq \mathsf{c}\big(\alpha^{\frac{1}{d+2}} K^{\frac{d+1}{d+2}} + \alpha K\big)^{\frac{1}{d+1}} + \mathsf{c}\varepsilon + \tfrac{\mathsf{c}}{\sqrt{R}}.$$

To complete the proof we determine the constants. First, we choose $K$ according to the following lemma:

**Lemma 1.8.** ([10, Lemma 2.3]) *One can choose $K$ such that the following holds: There exists a constant $\delta$ such that*

$$P\big(A_R^{(2)}\big) > 1 - Ke^{-R^{\delta}}.$$

Next, we choose $\alpha = \alpha(\varepsilon)$ such that $(\alpha^{\frac{1}{d+2}} + \alpha)^{\frac{1}{d+1}} \leq \varepsilon$. Then,

$$\max_{Q_R} |H| \leq \mathsf{c}\varepsilon + \tfrac{\mathsf{c}}{\sqrt{R}}.$$

We choose $n_0 = n_0(\varepsilon)$ according to the following lemma:

**Lemma 1.9.** ([10, Lemma 2.1]) *There exists a $n_0 = n_0(\varepsilon)$ and constants $c = c(n_0) > 0$ and $C = C(n_0) > 0$ such that*

$$P\big(A_R^{(1)} \cap A_R^{(3)}\big) > 1 - Ce^{-cR^{\frac{1}{7}}}.$$

Now, we choose $R_0 = R_0(\varepsilon) \geq R_o \vee \frac{n_0}{\alpha} \vee \frac{1}{\varepsilon^2} \vee \frac{\sqrt{n_0}}{\varepsilon} \vee n_0$, where $R_o$ is as in Theorem 1.6. In summary, for all $R \geq R_0$ and $\omega \in A_R^{(1)} \cap A_R^{(2)} \cap A_R^{(3)}$ we have $\max_{Q_R} |H| \leq \mathsf{c}\varepsilon$, and $P\big(A_R^{(1)} \cap A_R^{(2)} \cap A_R^{(3)}\big) \geq 1 - C(\varepsilon)e^{-c(\varepsilon)R^{\delta}}$. The proof of Theorem 1.5 is complete. □

## 1.4 An Estimate for the Exit Measure

Take a Borel set $\mathbb{A} \subseteq \partial\mathbb{K}_1$ whose boundary has zero measure, i.e.

$$\operatorname{meas}\big(\{x \in \partial\mathbb{K}_1 \colon d(x,\mathbb{A}) = 0 = d(x, \partial^p\mathbb{K}_1\backslash\mathbb{A})\}\big) = 0, \quad d \equiv \text{distance function}, \tag{1.19}$$

and define for $(x,t) \in \mathbb{Z}^d \times \mathbb{Z}_+$

$$RA(x,t) \triangleq \big\{(y,s) \in \partial^p K_R(x,t) \colon \big(\tfrac{y-x}{\|y-x\|_2 \vee R}, \tfrac{s-t}{\lceil R^2 \rceil}\big) \in \mathbb{A}\big\}. \tag{1.20}$$

We also set $RA \triangleq RA(0)$. Furthermore, set

$$\begin{aligned}
\tau_s &\triangleq \inf(t \in \mathbb{R}_+ \colon (X_t, t+s) \notin \mathbb{K}_1), \quad s \in [0,1] \\
\rho_s &\triangleq \inf(t \in \mathbb{Z}_+ \colon (X_t, t+s) \notin K_R), \quad s \in [[\lceil R^2 \rceil]], \\
\chi(x,s) &\triangleq P^x_{\mathrm{BM}}((X_{\tau_s}, \tau_s + s) \in \mathbb{A}), \quad (x,s) \in \overline{\mathbb{K}}_1, \\
\chi_R(x,s) &\triangleq \chi\big(\tfrac{x}{R}, \tfrac{s}{R^2}\big), \quad (x,s) \in \overline{\mathbb{K}}_R, \\
\Phi_R(x,s) &\triangleq P^x_\omega((X_{\rho_s}, \rho_s + s) \in RA), \quad (x,s) \in \overline{K}_R.
\end{aligned} \tag{1.21}$$

Here, $P^x_{\mathrm{BM}}$ denotes the law of a Brownian motion with covariance matrix $\mathfrak{A}$ and starting value $x$.

**Corollary 1.1.** *For every $\varepsilon > 0$ and $\theta \in (0,1)$ there exist $R_o = R_o(\mathbb{A}, \varepsilon, \theta) > 0, \mathfrak{c}_1 = \mathfrak{c}_1(\mathbb{A}, \varepsilon, \theta), \mathfrak{c}_2 = \mathfrak{c}_2(\mathbb{A}, \varepsilon, \theta)$ and $\delta > 0$ such that for all $R \geq R_o$*

$$P\Big(\Big\{\omega \in \Omega \colon \sup_{K_{\theta R}} |\chi_R - \Phi_R| \leq \varepsilon\Big\}\Big) \geq 1 - \mathfrak{c}_1 e^{-\mathfrak{c}_2 R^\delta}.$$

*Proof. Step 1:* Fix a small number $\gamma > 0$ and define

$$\begin{aligned}
\mathbb{A}^+_\gamma &\triangleq \big\{x \in \partial^p \mathbb{K}_1 \colon d(x, \mathbb{A}) \leq \gamma\big\}, \\
\mathbb{A}^-_\gamma &\triangleq \big\{x \in \mathbb{A} \colon d(x, \partial^p \mathbb{K}_1 \backslash \mathbb{A}) \geq \gamma\big\}.
\end{aligned}$$

Note that

$$\mathbb{A}^-_{2\gamma} \subseteq \mathbb{A}^-_\gamma \subseteq \mathbb{A} \subseteq \mathbb{A}^+_\gamma \subseteq \mathbb{A}^+_{2\gamma}.$$

Let $f^{(1)}, f^{(2)} \colon \partial^p \mathbb{K}_1 \to [0,1]$ be sufficiently smooth functions such that

$$f^{(1)} = \begin{cases} 1, & \text{on } \mathbb{A}^-_\gamma, \\ 0, & \text{on } \partial^p \mathbb{K}_1 \backslash \mathbb{A}^-_{2\gamma}, \end{cases} \qquad f^{(2)} = \begin{cases} 1, & \text{on } \mathbb{A}^+_\gamma, \\ 0, & \text{on } \partial^p \mathbb{K}_1 \backslash \mathbb{A}^+_{2\gamma}. \end{cases}$$

For $k = 1,2$ let $J^{(k)} \colon \overline{\mathbb{K}}_1 \to \mathbb{R}$ be a solution to the boundary value problem

$$\begin{cases} \frac{1}{2} \sum_{i,j=1}^d \mathfrak{A}_{ij} \frac{d^2 J^{(k)}}{dx_i dx_j} + \frac{dJ^{(k)}}{dt} = 0, & \text{on } \mathbb{K}_1, \\ J^{(k)} = f^{(k)}, & \text{on } \partial^p \mathbb{K}_1. \end{cases}$$

The optional stopping theorem yields that

$$J^{(k)}(x,s) = E^x_{\mathrm{BM}}[f^{(k)}(X_{\tau_s}, \tau_s + s)], \quad (x,s) \in \overline{\mathbb{K}}_1, k = 1,2. \tag{1.22}$$

Next, we set

$$F^{(k)}_{R+1}(x,t) \triangleq J^{(k)}\big(\tfrac{x}{R+1}, \tfrac{t}{(R+1)^2}\big), \quad (x,t) \in \overline{Q}_{R+1}.$$

Note that

$$\bigcap_{\gamma > 0} \mathbb{A}^+_{2\gamma} = \{x \in \partial^p \mathbb{K}_1 \colon d(x, \mathbb{A}) = 0\},$$

which implies that $\bigcap_{\gamma>0} \mathbb{A}_{2\gamma}^{+} \backslash \mathbb{A} \subseteq \partial \mathbb{A}$. Thus, due to (1.19) and (1.22), we obtain that

$$\max_{(x,t)\in K_{\theta(R+1)}} (F_{R+1}^{(2)}(x,t) - \chi_{R+1}(x,t)) \\ \leq \max_{(x,t)\in \mathbb{K}_\theta} P_{\mathrm{BM}}^{x}((X_{\tau_t}, \tau_t + t) \in \mathbb{A}_{2\gamma}^{+} \backslash \mathbb{A}) \to 0 \text{ as } \gamma \searrow 0.$$

Next, note that

$$\bigcup_{\gamma>0} \mathbb{A}_{\gamma}^{-} = \mathbb{A} \cap \{x \in \partial^p \mathbb{K}_1 \colon d(x, \partial^p \mathbb{K}_1 \backslash \mathbb{A}) > 0\},$$

which implies that $\mathbb{A} \backslash \bigcup_{\gamma>0} \mathbb{A}_{\gamma}^{-} \subseteq \partial \mathbb{A}$. Due to (1.19) and (1.22), we obtain

$$\max_{(x,t)\in K_{\theta(R+1)}} (\chi_{R+1}(x,t) - F_{R+1}^{(1)}(x,t)) \\ \leq \max_{(x,t)\in \mathbb{K}_\theta} P_{\mathrm{BM}}^{x}((X_{\tau_t}, \tau_t + t) \in \mathbb{A} \backslash \mathbb{A}_{\gamma}^{-}) \to 0 \text{ as } \gamma \searrow 0.$$

Consequently, there exists a $\gamma = \gamma(\varepsilon, \theta) > 0$ such that the following holds:

$$F_{R+1}^{(2)} - \varepsilon \leq \chi_{R+1} \leq F_{R+1}^{(1)} + \varepsilon \text{ on } K_{\theta(R+1)}. \tag{1.23}$$

Take this $\gamma$. Note that the function $\chi$ is uniformly continuous on $\overline{\mathbb{K}}_\theta$, as it is continuous on $\mathbb{K}_1$. Thus, assuming that $R^o$ is large enough, we have

$$\max_{\mathbb{K}_{\theta R}} |\chi_R - \chi_{R+1}| \leq \varepsilon.$$

Now, it follows from (1.23) that

$$F_{R+1}^{(2)} - 2\varepsilon \leq \chi_R \leq F_{R+1}^{(1)} + 2\varepsilon \text{ on } K_{\theta R}. \tag{1.24}$$

*Step 2:* For $P$-a.a. $\omega \in \Omega$ and $k = 1, 2$ we define $G_{R+1}^{(k)} \colon K_R \to \mathbb{R}_+$ as solutions to the following boundary value problem:

$$\begin{cases} \mathcal{L}_\omega G_{R+1}^{(k)} = 0, & \text{on } Q_{R+1}, \\ G_{R+1}^{(k)} = F_{R+1}^{(k)}, & \text{on } \partial^p Q_{R+1}. \end{cases}$$

For $k = 1, 2$, let $C_1^{(k)} = C_1^{(k)}(J^{(k)}) = C_1^{(k)}(\mathbb{A}, \varepsilon, \theta) > 0$ be the constant from Theorem 1.5 and set

$$\widehat{\varepsilon} \triangleq \frac{\varepsilon}{C_1^{(1)} \vee C_2^{(2)}}.$$

Using Theorem 1.5 with $\widehat{\varepsilon}$ instead of $\varepsilon$ yields that there exists a set $G = G(\mathbb{A}, \varepsilon, \theta, R) \in \mathcal{F}$ such that, after eventually enlarging $R_o$, for all $\omega \in G$, all $R \geq R_o$ and $k = 1, 2$

$$\max_{Q_{R+1}} |F_{R+1}^{(k)} - G_{R+1}^{(k)}| \leq C^{(k)} \widehat{\varepsilon} \leq \varepsilon. \tag{1.25}$$

*Step 3:* In this step we show that

$$G_{R+1}^{(1)} - 2\varepsilon \leq \Phi_R \leq G_{R+1}^{(2)} + 2\varepsilon \text{ on } K_R. \tag{1.26}$$

For $(x,t) \in \partial K_R \backslash RA$, we obtain for sufficiently large $R_o$ that $F^{(1)}_{R+1}(x,t) < \varepsilon$. To see this, recall that $J^{(1)} = 0$ on $\partial^p \mathbb{K}_1 \backslash \mathbb{A}$ and note that for $(x,t) \in \partial K_R \backslash RA$

$$F^{(1)}_{R+1}(x,t) = \left| J^{(1)}\left(\tfrac{x}{R+1}, \tfrac{t}{(R+1)^2}\right) - J^{(1)}\left(\tfrac{x}{\|x\|_2 \vee R}, \tfrac{t}{\lceil R^2 \rceil}\right)\right|.$$

Because

$$\left\| \tfrac{x}{R+1} - \tfrac{x}{\|x\|_2 \vee R}\right\|_2 \vee \left| \tfrac{t}{(R+1)^2} - \tfrac{t}{\lceil R^2 \rceil}\right|^{\frac{1}{2}} \le \left(1 - \tfrac{R}{R+1}\right) \vee \left(1 - \tfrac{\lceil R^2 \rceil}{(R+1)^2}\right)^{\frac{1}{2}} \to 0 \text{ as } R \to \infty,$$

the uniform continuity of $J^{(1)}$ on $\overline{\mathbb{K}}_1$ yields the claim. In the same manner, eventually enlarging $R_o$ again, we obtain $1 - F^{(2)}_{R+1}(x,t) \le \varepsilon$ for $(x,t) \in RA$. In summary,

$$F^{(1)}_{R+1} - \varepsilon \le \mathbb{1}_{RA} \le F^{(2)}_{R+1} + \varepsilon \quad \text{ on } \partial K_R.$$

Together with (1.25), we conclude that on $\partial K_R$

$$G^{(1)}_{R+1} - 2\varepsilon \le \mathbb{1}_{RA} \le G^{(2)}_{R+1} + 2\varepsilon.$$

Using once again the optional stopping theorem yields (1.26).

*Step 4:* Due to (1.24) and (1.26), we obtain that on $K_{\theta R}$

$$G^{(1)}_{R+1} - F^{(1)}_{R+1} - 4\varepsilon \le \Psi - \chi_R \le G^{(2)}_{R+1} - F^{(2)}_{R+1} + 4\varepsilon.$$

Finally, with (1.25), we conclude that on $K_{\theta R}$

$$|\Psi - \chi_R| \le 4\varepsilon + |G^{(1)}_{R+1} - F^{(1)}_{R+1}| + |G^{(2)}_{R+1} - F^{(2)}_{R+1}| \le 6\varepsilon.$$

The proof is complete. □

## 1.5 Proof of the Oscillation Inequality: Theorem 1.4

### 1.5.1 An Oscillation Inequality on a Small Scale

The main result of this section is the following oscillation inequality on a small scale:

**Proposition 1.1.** *There exist constants $\alpha > 0, c \in \mathbb{N}$ such that for all $R \ge 1$ there is a constant $C \in (0,1)$ and a set $G \in \mathcal{F}$ with $P(G) \ge 1 - \mathfrak{c} R^{3d} e^{-R^\alpha}$ such that for all $\omega \in G, p \in \{o, e\}$ and every $\omega$-caloric function $u\colon \overline{K}_{(c+3)R} \to \mathbb{R}$ the following oscillation inequality holds:*

$$\operatorname*{osc}_{\Theta^p(K_R)} u \le C \operatorname*{osc}_{\Theta^p(K_{(c+3)R})} u. \tag{1.27}$$

*Proof.* To prove this result we need input from [10]: For $x, y \in \mathbb{Z}^d$ we write $x \xrightarrow{\omega} y$ in case

$$P^x_\omega(\exists_{n\in\mathbb{N}} \colon X_n = y) > 0.$$

We call a set $A \subseteq \mathbb{Z}^d$ to be *strongly connected* w.r.t. $\omega \in \Omega$ if $x \xrightarrow{\omega} y$ for every $x, y \in A$. Moreover, we call a set $A \subseteq \mathbb{Z}^d$ to be a *sink* w.r.t. $\omega \in \Omega$ if it is strongly connected w.r.t.

$\omega$ and for every $x \in A$ and $y \notin A$

$$P^x_\omega(\exists_{n\in\mathbb{N}}\colon X_n = y) = 0.$$

In other words, a sink is a strongly connected set from which the walk cannot escape. Due to [10, Proposition 1.13], for $P$-a.a. $\omega \in \Omega$ there exists a unique sink $\mathcal{C}_\omega$.

We now turn to the main proof of Proposition 1.1. Fix two parameters $c \in \mathbb{N}$ and $\xi > 0$ and a radius $R \geq 1$, and define

$$\begin{aligned}
\mathcal{E} = \mathcal{E}(R) &\triangleq \big\{\omega \in \Omega\colon \forall_{k=1,\dots,d}\ \nexists_{z\in B_{(c+3)R}}\ \omega(z, e_k) \in (0, \xi)\big\},\\
\mathcal{H} = \mathcal{H}(R) &\triangleq \big\{\omega \in \Omega\colon \forall_{z\in B_R}\ \nexists_{x\in\mathbb{Z}^d} \text{ such that } z \xrightarrow{\omega} x, x \notin \mathcal{C}_\omega, \|x - z\|_\infty = \lfloor R\rfloor\big\},\\
\mathcal{S} = \mathcal{S}(R) &\triangleq \big\{\omega \in \Omega\colon \forall_{x,y\in\mathcal{C}_\omega\cap B_{2R}}\ \mathrm{dist}_\omega(x, y) \leq cR\big\}.
\end{aligned}$$

Providing an intuition, we have the following:

- If $\omega \in \mathcal{E}$, the walk in $\omega$ is elliptic in $\mathcal{C}_\omega \cap B_{(c+3)R}$.
- If $\omega \in \mathcal{H}$, when starting in $B_R$ the worst case is that the walk in $\omega$ is in a hole of the sink $\mathcal{C}_\omega$ with radius $\lfloor R\rfloor$.
- If $\omega \in \mathcal{S}$, all points in $\mathcal{C}_\omega \cap B_{2R}$ can be reached by a walk in $\omega$ in at least $\lfloor cR\rfloor$ steps.

We set $G = G(R) \triangleq \mathcal{E}\cap\mathcal{S}\cap\mathcal{H}$ and take $\xi = \xi(R)$ small enough such that $P(\mathcal{E}^c) \leq R^{3d}e^{-R^\alpha}$. Due to [10, Proposition 3.1], there exists a constant $\alpha > 0$ (only depending on the dimension) such that $P(\mathcal{H}^c) \leq \mathfrak{c}R^d e^{-R^\alpha}$. Moreover, due to [10, Proposition 3.2], we can choose $c$ (depending only on $P$) in the definition of the set $\mathcal{S}$ (and the statement of the proposition) such that $P(\mathcal{S}^c) \leq \mathfrak{c}R^{3d}e^{-R^\alpha}$. In summary, we have

$$P(G) = 1 - P(G^c) \geq 1 - P(\mathcal{E}^c) - P(\mathcal{H}^c) - P(\mathcal{S}^c) \geq 1 - \mathfrak{c}R^{3d}e^{-R^\alpha}.$$

It is left to show that the oscillation inequality (1.27) holds $P$-a.e. on $G$. Let $\omega \in G\cap \mathsf{B}$ and fix $(x, t), (y, s) \in \Theta^p(K_R)$. Furthermore, let $(Z_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be independent walks in $\omega$ such that $Z_0 = x$ and $Y_0 = y$. With abuse of notation, we denote the underlying probability measure by $P_\omega$. Let $u$ be an $\omega$-caloric function on $\overline{K}_{(c+3)R}$. Denote

$$\begin{aligned}
\tau &\triangleq \inf(n \in \mathbb{Z}_+\colon (Z_n, n + t) \notin K_{(c+3)R}),\\
\rho &\triangleq \inf(n \in \mathbb{Z}_+\colon (Y_n, n + s) \notin K_{(c+3)R}).
\end{aligned}$$

Then,

$$\begin{aligned}
u(x, t) - u(y, s) &= E_\omega\big[u(Z_\tau, \tau + t) - u(Y_\rho, \rho + s)\big]\\
&= E_\omega\big[\big(u(Z_\tau, \tau + t) - u(Y_\rho, \rho + s)\big)\mathbb{1}_{\{(Z_\tau,\tau+t)\neq(Y_\rho,\rho+s)\}}\big]\\
&\leq \operatorname*{osc}_{\Theta^p(K_{(c+3)R})} u\ P_\omega((Z_\tau, \tau + t) \neq (Y_\rho, \rho + s)).
\end{aligned}$$

The oscillation inequality follows in case there exists a constant $C = C(R) > 0$ such that

$$P_\omega((Z_\tau, \tau + t) = (Y_\rho, \rho + s)) \geq C. \tag{1.28}$$

*Case 1:* $x, y \in \mathcal{C}_\omega$. Because $\omega \in \mathcal{E}\cap\mathcal{S}$, we can guide the space-time walks to meet at some point and afterwards to proceed together. Thus,

$$\xi^{2(c+3)^2R^2} \le P_\omega((Z_\tau, \tau+t) = (Y_\rho, \rho+s)).$$

*Case 2:* $x \notin \mathcal{C}_\omega$ *or* $y \notin \mathcal{C}_\omega$. In this case we first bring the walks into the sink. Because $\omega \in \mathcal{H}$, the worst case is that the initial points $x, y$ are in a hole of the sink of radius $\lfloor R \rfloor$. Furthermore, using $\omega \in \mathsf{B}$, the walk can step in direction of the sink with probability at least $\frac{1}{2d}$. Consequently, again guiding the walks and using that $\omega \in \mathcal{E} \cap \mathcal{S}$, we obtain that

$$(2d)^{-2(c+3)^2R^2}\xi^{2(c+3)^2R^2} \le P_\omega((Z_\tau, \tau+t) = (Y_\rho, \rho+s)).$$

Hence, (1.28) holds with $C \equiv (2d)^{-2(c+3)^2R^2}\xi^{2(c+3)^2R^2} > 0$. The proof is complete. □

The following is an application of Proposition 1.1:

**Corollary 1.2.** *There exist constants* $\alpha > 0, c \in \mathbb{N}$ *such that for all* $R \ge 1$ *there is a constant* $C = C(R) \in (0,1)$ *and a set* $G = G(R) \in \mathcal{F}$ *with* $P(G) \ge 1 - \mathfrak{c}R^{3d}e^{-R^\alpha}$ *such that for all* $\omega \in G, p \in \{o, e\}$

$$\max_{A \subseteq \partial^p K_{(c+3)R}} \operatorname*{osc}_{\Theta^p(K_R)} \mathfrak{p}_\omega(A) \le C,$$

*where*

$$\rho_t \triangleq \inf(n \in \mathbb{Z}_+ : (X_n, n+t) \notin K_{(c+3)R}), \qquad \mathfrak{p}_\omega^{(x,t)}(A) \triangleq P_\omega^x((X_{\rho_t}, \rho_t + t) \in A).$$

*Proof.* The Markov property of the walk in the environment $\omega$ yields that $\mathfrak{p}_\omega(A)$ is $\omega$-caloric in $K_{(c+3)R}$ and consequently, Proposition 1.1 implies the claim. □

### 1.5.2 Multi-Scale Structure

Let $R_o, M, K, N, \varepsilon_o > 0$ be parameters which we will determine later. The constant $M$ will be taken large (at least 10, say) and $N, K \in \mathbb{N}$.

Furthermore, let $\{\mathbb{A}_1, \dots, \mathbb{A}_N\}$ be a covering of $\partial\mathbb{K}_1$ intersecting only in their boundaries, which are supposed to have no measure, i.e.

$$\text{meas}\left(\{x \in \partial^p\mathbb{K}_1 : d(x, \mathbb{A}_i) = d(x, \partial^p\mathbb{K}_1 \backslash \mathbb{A}_i) = 0\}\right) = 0.$$

Moreover, we assume that

$$\forall_{i=1,\dots,N}\ \exists \hat{z}_i \in \partial^p\mathbb{K}_1 : \mathbb{A}_i \subset \mathbb{K}_{\frac{1}{4M^2}}(\hat{z}_i). \tag{1.29}$$

In the following we will denote space-time points by $\hat{x}, \hat{y}, \hat{z}$, etc. For $R \ge 1, j \in \{1, \dots, N\}$, $\hat{z} \in \mathbb{Z}^d \times \mathbb{Z}_+$ and $s \in \mathbb{R}_+$ we define

$$\rho_s^{\hat{z},R} \triangleq \inf(t \in \mathbb{Z}_+ : (X_t, t+s) \notin K_R(\hat{z})),$$
$$\tau_s^{\hat{z},R} \triangleq \inf(t \in \mathbb{R}_+ : (X_t, t+s) \notin \mathbb{K}_R(\hat{z})),$$

and

$$\mathfrak{p}_{\omega}^{(x,t),\hat{z},R}(j) \triangleq P_{\omega}^{x}((X_{\rho_t^{\hat{z},R}}, \rho_t^{\hat{z},R} + t) \in \ RA_j(\hat{z})),$$
$$\mathfrak{p}_{\mathrm{BM}}^{(x,t),\hat{z},R}(j) \triangleq P_{\mathrm{BM}}^{x}((X_{\tau_t^{\hat{z},R}}, \tau_t^{\hat{z},R} + t) \in R\mathbb{A}_j(\hat{z})),$$

where $R\mathbb{A}_j(x,t) \triangleq \big\{(y,s) \in \partial^p \mathbb{K}_R(x,t) \colon \big(\frac{y-x}{R}, \frac{s-t}{R^2}\big) \in \mathbb{A}_j\big\}$ and $RA_j(\hat{z})$ as in (1.20).

**Definition 1.1.** *Let $c \in \mathbb{N}$ and $C = C(R_o) \in (0,1)$ be as in Corollary 1.2.*

(i) *For $R \le R_o$ we say that the cylinder $K_R(\hat{z})$ is $\omega$-*good*, if*

$$\max_{p=o,e} \max \big( \|\mathfrak{p}_{\omega}^{\hat{x},\hat{z},(c+3)R_o} - \mathfrak{p}_{\omega}^{\hat{y},\hat{z},(c+3)R_o}\|_{tv} \colon \hat{x}, \hat{y} \in \Theta^p(K_R(\hat{z}))\big) \le C,$$

*where $\|\cdot\|_{tv}$ denotes the total variation distance.*

(ii) *For $R > R_o$ we say that the cylinder $K_R(\hat{z})$ is $\omega$-*good*, if for all $\hat{x} \in K_R(\hat{z})$*

$$\|\mathfrak{p}_{\omega}^{\hat{x},\hat{z},MR} - \mathfrak{p}_{\mathrm{BM}}^{\hat{x},\hat{z},MR}\|_{tv} < \varepsilon_o.$$

The following lemma shows that in case $R_o$ is large the probability for a cylinder to be good is high. The lemma follows from Corollaries 1.1 and 1.2.

**Lemma 1.10.** *There exist constants $R^* = R^*(\mathbb{A}_1, \dots, \mathbb{A}_N, \varepsilon_o, M) \ge 1$ and $\delta > 0$ such that whenever $R_o \ge R^*$*

$$P\big(\big\{\omega \in \Omega \colon K_R(\hat{z}) \text{ is } \omega\text{-good}\big\}\big) \ge 1 - e^{-R^{\delta}} \quad \text{for all } R \ge 1. \tag{1.30}$$

In the following, let $\delta$ be as in Lemma 1.10. Our next step is to set up a multi-scale structure. Define

$$R_k \triangleq R_o^{K^k} \text{ for } k \in \mathbb{Z}_+,$$

and take a constant

$$\nu < \frac{\delta}{K}.$$

**Definition 1.2.** (i) *A cylinder of radius $R_o^2$ is called $\omega$-*admissible*, if all sub-cylinders of radius $R_o$ are $\omega$-good.*

(ii) *For $k \in \mathbb{N}$ a cylinder of radius $R_k^2$ is called $\omega$-*admissible*, if*

- *every sub-cylinder of radius $> R_{k-1}$ is $\omega$-good.*
- *there are at most $R_k^{\nu}$ non-$\omega$-admissible sub-cylinders of radius $R_{k-1}^2$.*

**Lemma 1.11.** *There exists a constant $R^* = R^*(\mathbb{A}_1, \dots, \mathbb{A}_N, \varepsilon_o, M) \ge 1$ such that for $R_o \ge R^*$ the following holds: For all $(\hat{z}, k) \in \mathbb{Z}^d \times \mathbb{Z}_+ \times \mathbb{Z}_+$*

$$P\big(\big\{\omega \in \Omega \colon K_{R_k^2}(\hat{z}) \text{ is } \omega\text{-admissible}\big\}\big) \ge 1 - e^{-R_k^{\nu/2}}.$$

*Proof.* We use induction. For $k = 0$ the claim follows from Lemma 1.10 and a union bound. For the induction step, assume that the claim holds for $k \in \mathbb{Z}_+$. Denote

$$\begin{aligned} A &\triangleq \big\{\omega \in \Omega\colon \text{in } K_{R_k^2}(\hat{z}) \text{ there exists a sub-cylinder} \\ &\qquad\qquad \text{of radius} > R_{k-1} \text{ which is not } \omega\text{-good}\big\}, \\ B &\triangleq \big\{\omega \in \Omega\colon \text{in } K_{R_k^2}(\hat{z}) \text{ there are more than } R_k^\nu \\ &\qquad\qquad \text{non-}\omega\text{-admissible sub-cylinders of radius } R_{k-1}^2\big\}. \end{aligned}$$

Due to Lemma 1.10, each sub-cylinder with radius $> R_{k-1}$ is bad with probability less than $e^{-R_{k-1}^\delta} \leq e^{-R_k^\nu}$. Thus, due to a union bound, we obtain for $R_o$ large enough that

$$P(A) \leq \tfrac{1}{2}\, e^{-R_k^{\nu/2}}.$$

We denote by $\mathsf{A}(\hat{x}, R)$ the set of all $\omega \in \Omega$ such that $K_R(\hat{x})$ is $\omega$-admissible. To estimate $P(B)$, we partition the cylinder $K_{R_k^2}$ in $\rho \leq$ polynomial of $R_k$ subsets $\{U_1, \dots, U_\rho\}$ such that $\mathsf{A}(\hat{x}, R_{k-1}^2)$ and $\mathsf{A}(\hat{y}, R_{k-1}^2)$ are independent for all $\hat{x}, \hat{y} \in U_i, i = 1, \dots, \rho$. For $i = 1, \dots, \rho$ we have

$$Z_i \triangleq \sum_{\hat{x} \in U_i} \big(1 - \mathbb{1}_{\mathsf{A}(\hat{x}, R_{k-1}^2)}\big) \sim \operatorname{bin}(|U_i|, 1 - P(\mathsf{A}(0, R_{k-1}^2)) \leq_{\mathrm{st}} \operatorname{bin}(|U_i|, e^{-R_{k-1}^{\nu/2}}),$$

where $\leq_{\mathrm{st}}$ denotes the usual stochastic order. Note the following:

**Lemma 1.12.** *For $n \in \mathbb{N}$, let $S_1, \dots, S_n$ be i.i.d. Bernoulli random variables with parameter $p \in (0,1)$. Then, for all $k \in [n]$*

$$P(S_1 + \dots + S_n \geq k) \leq (np)^k.$$

*Proof.* We use induction over $k \in [n]$. For $k = 0$ the claim is obvious. Assume the claim holds for $0 \leq k < n$. Then,

$$\begin{aligned} P(S_1 + \dots + S_n \geq k+1) &= P(S_1 + \dots + S_n \geq k+1, \exists_{m \leq n}\colon S_m = 1) \\ &\leq \sum_{m=1}^n P(S_1 + \dots + S_n - S_m \geq k, S_m = 1) \\ &= \sum_{m=1}^n P(S_1 + \dots + S_{n-1} \geq k) P(S_m = 1) \\ &\leq \sum_{m=1}^n P(S_1 + \dots + S_n \geq k) p \\ &\leq \sum_{m=1}^n (np)^k p = (np)^{k+1}. \end{aligned}$$

The proof is complete. □

Using Lemma 1.12 and Chebyshev's inequality, we obtain that

$$P(B) \le \sum_{i=1}^{\rho} P(Z_i > \rho^{-1} R_k^{\nu}) \le \rho (2R_k)^{2(d+2)\rho^{-1} R_k^{\nu}} e^{-\rho^{-1} R_k^{\nu} R_{k-1}^{\nu/2}}$$
$$= \rho e^{\rho^{-1} R_k^{\nu} (\log(2R_k) 2(d+2) - R_{k-1}^{\nu/2})} \le \tfrac{1}{2} e^{-R_k^{\nu/2}},$$

provided $R_o$ is sufficiently large. We conclude that $P(A \cup B) \le e^{-R_k^{\nu/2}}$. The proof is complete. □

### 1.5.3 The Coupling

We use the notation from Section 1.5.2.

#### 1.5.3.1 Definition

In this section we define a coupling, which *success* will prove the oscillation inequality. We define the coupling via a (random) sequence

$$\{\hat{x}^{(m)}, \hat{y}^{(m)}, \hat{z}^{(m)}, R^{(m)}, Y^{(m)}, Z^{(m)} : m \in \mathbb{Z}_+\}.$$

The starting point is a so-called *basic coupling*: For $\hat{x} \in \mathbb{Z}^d \times \mathbb{Z}_+, R \ge 1$, $\hat{y} = (\hat{y}_1, \hat{y}_2), \hat{z} = (\hat{z}_1, \hat{z}_2) \in K_R(\hat{x})$, let $\mathfrak{q}_\omega^{(\hat{x},R,\hat{y},\hat{z})}$ be a Borel probability measure on the product space $(\mathbb{Z}^d \times \mathbb{Z}_+) \times (\mathbb{Z}^d \times \mathbb{Z}_+) \times D(\mathbb{Z}_+, \mathbb{Z}^d) \times D(\mathbb{Z}_+, \mathbb{Z}^d)$ such that the generic element $(\hat{Z}^1, \hat{Z}^2, X^1, X^2)$ is sampled as follows:

- If $R > R_o$, then $X^1$ and $X^2$ are two walks in $\omega$ starting at $\hat{y}_1$ and $\hat{z}_1$ respectively, such that the probability of $(X_n^1, \hat{y}_2 + n)_{n \in \mathbb{Z}_+}$ and $(X_n^2, \hat{z}_2 + n)_{n \in \mathbb{Z}_+}$ leaving $K_{MR}(\hat{x})$ in the same element of $\{RMA_1(\hat{x}), \dots, RMA_N(\hat{x})\}$ is maximized. Moreover, $\hat{Z}^1$ and $\hat{Z}^2$ are the points where $(X_n^1, \hat{y}_2 + n)_{n \in \mathbb{Z}_+}$ and $(X_n^2, \hat{z}_2 + n)_{n \in \mathbb{Z}_+}$ leave $K_{MR}(\hat{x})$.
- If $R \le R_o$, then $X^1$ and $X^2$ are two walks in $\omega$ starting at $\hat{y}_1$ and $\hat{z}_1$ respectively, such that the probability of $(X_n^1, \hat{y}_2 + n)_{n \in \mathbb{Z}_+}$ and $(X_n^2, \hat{z}_2 + n)_{n \in \mathbb{Z}_+}$ leaving $K_{(c+3)R_o}(\hat{x})$ in the same point is maximized. Moreover, $\hat{Z}^1$ and $\hat{Z}^2$ are the points where $(X_n^1, \hat{y}_2 + n)_{n \in \mathbb{Z}_+}$ and $(X_n^2, \hat{z}_2 + n)_{n \in \mathbb{Z}_+}$ leave $K_{(c+3)R_o}(\hat{x})$.

Before we turn to the main coupling, let us explain that on good cylinders there is a reasonable probability that the walks leave a cylinder in the same region or point.

**Lemma 1.13.** *Take $\hat{x} \in \mathbb{Z}^d \times \mathbb{Z}_+, R \ge 1$ and $\hat{y}, \hat{z} \in K_R(\hat{x})$ of the same parity and assume that $\omega \in \Omega$ is such that $K_R(\hat{x})$ is $\omega$-good.*

(i) *There exist two constant $\mathfrak{c}_1, \mathfrak{c}_2 > 0$ only depending on the dimension $d$ and the covariance matrix $\mathfrak{A}$ such that in case $R > R_o$*

$$\mathfrak{q}_\omega^{(\hat{x},R,\hat{y},\hat{z})}\big(\exists \hat{v} \in \partial^p K_{RM}(\hat{x}) \colon \hat{Z}^1, \hat{Z}^2 \in K_{RM^{-1}}(\hat{v})\big) > 1 - \mathfrak{c}_1 M^{-\mathfrak{c}_2} - 2\varepsilon_o.$$

(ii) *If $R \le R_o$, then*

$$\mathfrak{q}_\omega^{(\hat{x},R,\hat{y},\hat{z})}\big(\hat{Z}^1 = \hat{Z}^2\big) > 1 - C,$$

*where $C \in (0,1)$ is as in the definition of the good cylinder, see Corollary 1.2.*

*Proof.* (i). The proof is based on the relation of oscillation, total variation and couplings: In view of [95, Proposition 4.7, Remark 4.8] and of assumption (1.29), it suffices to show that

$$\|\mathfrak{p}_\omega^{\hat{y},\hat{x},MR} - \mathfrak{p}_\omega^{\hat{z},\hat{x},MR}\|_{tv} \le \mathfrak{c}_1 M^{-\mathfrak{c}_2} + 2\varepsilon_o.$$

Take $k \in \{1, \dots, N\}$. Because $K_R(\hat{x})$ is $\omega$-good, we have

$$\begin{aligned} |\mathfrak{p}_\omega^{\hat{y},\hat{x},MR}(k) - \mathfrak{p}_\omega^{\hat{z},\hat{x},MR}(k)| &\le |\mathfrak{p}_\omega^{\hat{y},\hat{x},MR}(k) - \mathfrak{p}_{\mathrm{BM}}^{\hat{y},\hat{x},MR}(k)| + |\mathfrak{p}_\omega^{\hat{z},\hat{x},MR}(k) - \mathfrak{p}_{\mathrm{BM}}^{\hat{z},\hat{x},MR}(k)| \\ &\quad + |\mathfrak{p}_{\mathrm{BM}}^{\hat{y},\hat{x},MR}(k) - \mathfrak{p}_{\mathrm{BM}}^{\hat{z},\hat{x},MR}(k)| \\ &\le 2\varepsilon_o + |\mathfrak{p}_{\mathrm{BM}}^{\hat{y},\hat{x},MR}(k) - \mathfrak{p}_{\mathrm{BM}}^{\hat{z},\hat{x},MR}(k)|. \end{aligned} \tag{1.31}$$

Furthermore, because $\hat{v} \mapsto u(\hat{v}) \triangleq \mathfrak{p}_{\mathrm{BM}}^{\hat{v},\hat{x},MR}$ solves the (backward) heat equation $\frac{d}{dt}u + \frac{1}{2}\mathrm{tr}(\mathfrak{A}\nabla^2 u) = 0$ on $\mathbb{K}_{MR}(\hat{x})$, [98, Theorem 6.28] yields the existence of two constants $\mathfrak{c}_1, \mathfrak{c}_2 > 0$ such that

$$|\mathfrak{p}_{\mathrm{BM}}^{\hat{y},\hat{x},MR}(k) - \mathfrak{p}_{\mathrm{BM}}^{\hat{z},\hat{x},MR}(k)| \le \mathfrak{c}_1 M^{-\mathfrak{c}_1}.$$

Together with (1.31), we conclude (i).

(ii). This follows from [95, Proposition 4.7, Remark 4.8] and the definition of a good cylinder. □

We can (and will) take $M$ and $\varepsilon_o$ such that

$$1 - \mathfrak{c}_1 M^{-\mathfrak{c}_2} - 2\varepsilon_o \ge \tfrac{2}{3}. \tag{1.32}$$

In other words, on good cylinders the coupling is *successful* (in some sense) with a reasonable probability.

From now on we fix $R > R_o, \omega \in \Omega$ and two points $\hat{y}, \hat{z} \in K_R$ of the same parity. The following are the initial values:

- $R^{(0)} \triangleq R$.
- $\hat{x}^{(0)} \triangleq (0, 0)$.
- sample $(\hat{y}^{(0)}, \hat{z}^{(0)}, Y^{(0)}, Z^{(0)})$ according to $\mathfrak{q}_\omega^{0,R,\hat{y},\hat{z}}$.

Now, we proceed inductively. Namely, once the $m^{\text{th}}$ element of the sequence is fixed, we generate the $(m+1)^{\text{th}}$ element as follows: Set $R^{(m)}$ and $\hat{x}^{(m)}$ according to the following rule:

- *Case 1*: $R^{(m-1)} > R_o$. If there exists a point $\hat{v}$ in the boundary of the cylinder $K_{MR^{(m-1)}}(\hat{x}^{(m-1)})$ such that $\hat{y}^{(m-1)}, \hat{z}^{(m-1)} \in K_{M^{-1}R^{(m-1)}}(\hat{v})$, set $R^{(m)} \equiv M^{-1}R^{(m-1)}$ and $\hat{x}^{(m)} \equiv \hat{v}$. Otherwise, take $R^{(m)} \equiv MR^{(m-1)}$ and $\hat{x}^{(m)} \equiv \hat{x}^{(m-1)}$.
- *Case 2*: $R^{(m-1)} \le R_o$. We set $R^{(m)} \equiv (c+3)R_o$ and $\hat{x}^{(m)} \equiv \hat{x}^{(m-1)}$.

Then, sample $(\hat{y}^{(m)}, \hat{z}^{(m)}, Y^{(m)}, Z^{(m)})$ according to $\mathfrak{q}_\omega^{(\hat{x}^{(m)},R^{(m)},\hat{y}^{(m-1)},\hat{z}^{(m-1)})}$. Finally, let $Y$ and $Z$ be the walks in $\omega$ that are obtained from $Y^{(m)}$ and $Z^{(m)}$ by pasting. To simplify our notation, we denote the probability measure underlying the coupling by $\mathbb{Q}$.

#### 1.5.3.2 A Technical Lemma

Fix $k \in \mathbb{Z}_+, \hat{x} \in \mathbb{Z}^d \times \mathbb{Z}_+$ and let $R_k < R \le R_{k+1}$. Further, define two stopping times:

$$\begin{aligned} T &\triangleq \inf\big(m \in \mathbb{N}\colon R^{(m)} \le R_k\big), \\ S &\triangleq \inf\big(m \in \mathbb{N}\colon R^{(m)} \ge R_{k+1}^2 \text{ or } \hat{x}^{(m)} \notin K_{R_{k+1}^2/2}(\hat{x})\big). \end{aligned}$$

**Remark 1.2.** *If $\omega \in \mathsf{A}(\hat{x}, R_{k+1}^2)$, then till $T \wedge S$ the coupling only sees $\omega$-good cylinders.*

**Lemma 1.14.** (i) *There exist constants $\theta, \mathsf{c} > 0$ such that if $\omega \in \mathsf{A}(\hat{x}, R_{k+1}^2)$, then*

$$\mathbb{Q}(S < T) \le \mathsf{c} R_k^{-\theta K}.$$

(ii) *If $R_{k+1}M^{-1} \le R$ there exist constants $\rho, \mathsf{c} > 0$ such that if $\omega \in \mathsf{A}(\hat{x}, R_{k+1}^2)$, then for all $\hat{z} \in K_{R_{k+1}^2}(\hat{x})$*

$$\mathbb{Q}(\hat{x}^{(T)} \in K_{R_k}(\hat{z})) \le \mathsf{c} R_k^{-\rho K}.$$

*Proof.* (i). Note that

$$\begin{aligned} \mathbb{Q}(S < T) &= \mathbb{Q}\Big(\Big\{\max_{m<T} R^{(m)} \ge R_{k+1}^2 \text{ or } \exists_{m<T}\colon\ \hat{x}^{(m)} \notin K_{R_{k+1}^2/2}(\hat{x})\Big\} \cap \Big\{S < T\Big\}\Big) \\ &\le \mathbb{Q}\Big(\max_{m<S\wedge T} R^{(m)} \ge R_{k+1}^2\Big) + \mathbb{Q}\Big(\exists_{m<S\wedge T}\colon\ \hat{x}^{(m)} \notin K_{R_{k+1}^2/2}(\hat{x})\Big). \end{aligned} \tag{1.33}$$

In view of Remark 1.2, [10, Lemma 4.10] yields that

$$\mathbb{Q}\Big(\max_{m<S\wedge T} R^{(m)} \ge R_{k+1}^2\Big) \le 2^{-\frac{K\log(R_k)}{\log(M)}} \equiv R_k^{-\theta' K} \text{ with } \theta' \triangleq \frac{\log(2)}{\log(M)}.$$

To control the second term in (1.33), we first consider the process

$$L_m \triangleq \frac{\log(R^{(m)}) - \log(R)}{\log(M)}, \quad m \le S \wedge T.$$

Note that $(L_m)_{m\le S\wedge T}$ has step size one and that it steps down with a probability larger than $\frac{2}{3}$, see Lemma 1.13, Remark 1.2 and (1.32). Consequently, $(L_m)_{m\le S\wedge T}$ is stochastically dominated by a biased random walk which steps down with probability $\frac{2}{3}$. This means that there exists a sequence of i.i.d. random variables $\xi_1, \xi_2 - \xi_1, \dots$ such that $\mathbb{Q}(\xi_1 = 1) = 1 - \mathbb{Q}(\xi_1 = -1) = \frac{1}{3}$ and $\mathbb{Q}$-a.s. on $\{m \le S \wedge T\}$

$$L_{m+1} - L_m \le \xi_{m+1} - \xi_m. \tag{1.34}$$

**Lemma 1.15.**

$$\mathbb{E}^{\mathbb{Q}}\big[S \wedge T\big] \le 3\Big\lceil \frac{\log(R) - \log(R_k)}{\log(M)} \Big\rceil.$$

*Proof.* We set

$$\tau_a \triangleq \inf(m \in \mathbb{N}\colon \xi_m \le -a), \quad a \in \mathbb{Z}_+,$$

It is well-known that $\mathbb{E}^{\mathbb{Q}}[\tau_a] = 3a$, which follows from the fact that $(\xi_m + \frac{m}{3})_{m\in\mathbb{Z}_+}$ is a martingale and the optional stopping theorem. Now, set

$$a \triangleq \Big\lceil \frac{\log(R) - \log(R_k)}{\log(M)} \Big\rceil,$$

and note that $\mathbb{Q}$-a.s. $S \wedge T = S \wedge T \wedge \tau_a \leq \tau_a$. The claim follows. $\square$

Next, set

$$B \triangleq \Big\{ \max_{m < S \wedge T} R^{(m)} < R_{k+1}^{3/2} \Big\}.$$

Using again [10, Lemma 4.10] yields that

$$\mathbb{Q}(B^c) \leq R_k^{-\theta' K/2}.$$

Denote $\hat{x}^{(m)} = (x^{(m)}, t^{(m)})$ and $\hat{x} = (x, t)$. Using Lemma 1.15 and Chebyshev's inequality, we obtain

$$\begin{aligned} \mathbb{Q}\big(\exists_{m < S \wedge T} : \|x^{(m)} - x\|_2 \geq R_{k+1}^2/2, B\big) &\leq \frac{2}{R_{k+1}^2} \mathbb{E}^{\mathbb{Q}}\Big[ \sum_{n < S \wedge T} \|x^{(n)} - x^{(n-1)}\|_2 \mathbb{1}_B \Big] \\ &\leq \frac{2}{R_{k+1}^2} \mathbb{E}^{\mathbb{Q}}\Big[ \sum_{n < S \wedge T} 2 M R^{(n-1)} \mathbb{1}_B \Big] \\ &\leq \frac{\mathsf{c}}{R_{k+1}^{1/2}} \mathbb{E}^{\mathbb{Q}}\big[ S \wedge T \big] \\ &\leq \frac{\mathsf{c}}{R_{k+1}^{1/2}} \frac{\log(R_{k+1})}{\log(M)} \\ &\leq \mathsf{c} R_k^{-K/8}. \end{aligned}$$

Similarly, we obtain that

$$\mathbb{Q}\big(\exists_{m < S \wedge T} \colon |t^{(m)} - t| \geq R_{k+1}^4/4, B\big) \leq \mathsf{c} R_k^{-K/8}.$$

In summary, we have

$$\mathbb{Q}\big(\exists_{m < S \wedge T} : \hat{x}^{(m)} \not\in K_{R_{k+1}^2/2}(\hat{x})\big) \leq \mathsf{c} R_k^{-\theta K},$$

for some suitable $\theta > 0$. We proved part (i).

(ii). Using (i) we see that

$$\begin{aligned} \mathbb{Q}(\hat{x}^{(T)} \in K_{R_k}(\hat{z})) &\leq \mathbb{Q}(S < T) + \mathbb{Q}(\hat{x}^{(T)} \in K_{R_k}(\hat{z}), T \leq S) \\ &\leq \mathsf{c} R_k^{-\theta K} + \mathbb{Q}(\hat{x}^{(T)} \in K_{R_k}(\hat{z}), T \leq S). \end{aligned}$$

It remains to control the second term.

Let $(\xi_k)_{k \in \mathbb{Z}_+}$ be as in the proof of part (i). For $m \in \mathbb{Z}_+$ we set

$$\mathcal{R}(m) \triangleq \Big\{ \xi_{m+1} - \xi_m = -1, \sum_{k=m+2}^{\infty} M^{\xi_k - \xi_{m+1}} < 1 \Big\}.$$

We obtain that on $\mathcal{R}(m) \cap \{m+2 \le T \le S\}$

$$\begin{aligned}\|x^{(T)} - x^{(m+1)}\|_2 &\le \sum_{k=m+2}^{T} \|x^{(k)} - x^{(k-1)}\|_2 \le \sum_{k=m+2}^{T} 2MR^{(k-1)} \\ &\le 2MR^{(m+1)} + 2MR^{(m+1)} \sum_{k=m+2}^{\infty} M^{\xi_k - \xi_{m+1}} \\ &\le 4MR^{(m+1)} = 4R^{(m)},\end{aligned}$$

and similarly, $|t^{(T)} - t^{(m+1)}|^{\frac{1}{2}} \le 2R^{(m)}$. Thus, on $\mathcal{R}(m) \cap \{m+2 \le T \le S, \hat{x}^{(T)} \in K_{R_k}(\hat{z})\}$ we have $\|\hat{z}_1 - x^{(m+1)}\|_2 \le 2R_k + 4R^{(m)} \le 6R^{(m)}$, and $|\hat{z}_1 - t^{(m+1)}|^{\frac{1}{2}} \le 3R^{(m)}$, which happens with probability bounded from above by a constant $p = p(\varepsilon_o, M) < 1$, because $(x^{(m+1)}, t^{(m+1)}) \in (x^{(m)}, t^{(m)}) + \partial^p K_{MR^{(m)}}$ and the definition of a good cylinder. Next, we need the following large deviation estimate:

**Lemma 1.16.** *There exist constants $\kappa, v > 0$ such that for all $n \in \mathbb{N}$*

$$\mathbb{Q}\Big(\frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{\mathcal{R}(k)} \le \kappa\Big) \le e^{-vn}.$$

*Proof.* We call $m \in \mathbb{N}$ a *renewal*, if

$$\begin{cases} \xi_m > \xi_n, & n > m, \\ \xi_m < \xi_n, & n < m. \end{cases}$$

For $k \in \mathbb{N}$, let $\tau_k$ be the $k^{\text{th}}$ renewal and note that $\xi_{\tau_i+2} - \xi_{\tau_k+1} \le -(i-k+1) = k-i-1$ for every $i \ge k$. Consequently, we see that

$$\begin{aligned}\sum_{i=\tau_k+2}^{\infty} M^{\xi_i - \xi_{\tau_k+1}} &= M^{-\xi_{\tau_k+1}} \sum_{i=k}^{\infty} \sum_{j=\tau_i+2}^{\tau_{i+1}+1} M^{\xi_j} \\ &\le M^{-\xi_{\tau_k+1}} \sum_{i=k}^{\infty} (\tau_{i+1} - \tau_i) M^{\xi_{\tau_i+2}} \le \sum_{i=k} (\tau_{i+1} - \tau_i) M^{k-i-1}.\end{aligned}$$

This shows that the event $\mathcal{R}(\tau_k)$ happens in case $(\tau_{i+1} - \tau_i) M^{k-i-1} < 2^{k-i-1}$ for all $i \ge k$. Now, the proof concludes identical to the proof of [10, Claim 4.12]. □

Let

$$Z_N \triangleq \sum_{k=1}^{N} \mathbb{1}_{\mathcal{R}(k)}, \qquad N \in \mathbb{N}.$$

Note that

$$T \ge \Big\lceil \frac{K \log(R_k)}{2\log(M)} \Big\rceil \equiv n + 2,$$

because $R^{(m)} \geq \frac{R}{M^m}$ and $R \geq R_{k+1}M^{-1}$. Now, we obtain that

$$\begin{aligned}\mathbb{Q}(\hat{x}^{(T)} \in K_{R_k}(\hat{z}), T \leq S) &\leq \mathbb{Q}(Z_n \leq \kappa n) + \mathbb{Q}(\hat{x}^{(T)} \in K_{R_k}(\hat{z}), T \leq S, Z_n > \kappa n) \\ &\leq R_k^{-\frac{v}{4\log(M)}K} + R_k^{\frac{\log(p)\kappa}{4\log(M)}K}.\end{aligned}$$

The claim of (ii) follows. □

#### 1.5.3.3 Success of the Coupling

Let $k \triangleq \max(n \in \mathbb{N} \colon R > R_n)$ and define the following sets:

$$A_1 \triangleq \bigcap_{i=0}^{k} \mathsf{A}(\hat{x}^{(T^{(i)})}, R_i^2), \qquad A_2 \triangleq \bigcap_{i=0}^{k} \big\{T^{(i)} \leq S^{(i)}\big\},$$

where $T^{(k+1)} \triangleq 0$ and

$$\begin{aligned}T^{(i)} &\triangleq \inf\big(m \geq T^{(i+1)} \colon R^{(m)} \leq R_i\big), \\ S^{(i)} &\triangleq \inf\big(m \geq T^{(i+1)} \colon R^{(m)} \geq R_{i+1}^2 \text{ or } \hat{x}^{(m)} \not\in K_{R_{i+1}^2/2}(\hat{x}^{(T^{(i+1)})})\big),\end{aligned}$$

and

$$\begin{aligned}A_3 &\triangleq \big\{(Z_\tau, \tau + \hat{z}_1) = (Y_\rho, \rho + \hat{y}_1)\big\}, \\ A_4 &\triangleq \big\{\{(Z_{n\wedge\tau}, n\wedge\tau + \hat{z}_1), (Y_{n\wedge\rho}, n\wedge\rho + \hat{y}_1) \colon n \in \mathbb{Z}_+\} \subset K_{(c+6)MR}\big\},\end{aligned}$$

where

$$\begin{aligned}\tau &\triangleq \hat{z}_1^{(T^{(0)})} + \inf\big(m \in \mathbb{Z}_+ \colon (Z_m^{(T^{(0)})}, m + \hat{z}_1^{(T^{(0)})}) \not\in K_{(c+3)R_o}(\hat{x}^{(T^{(0)})})\big), \\ \rho &\triangleq \hat{y}_1^{(T^{(0)})} + \inf\big(m \in \mathbb{Z}_+ \colon (Y_m^{(T^{(0)})}, m + \hat{y}_1^{(T^{(0)})}) \not\in K_{(c+3)R_o}(\hat{x}^{(T^{(0)})})\big).\end{aligned}$$

Moreover, set

$$A \triangleq \bigcap_{i=1}^{4} A_i.$$

We define $\mathsf{A}(0, R^2)$ to be the set of all environments $\omega \in \Omega$ for which every sub-cylinder of $K_{R^2}$ with radius $R_k^2$ is $\omega$-admissible and every sub-cylinder with radius $> R_k$ is $\omega$-good.

**Lemma 1.17.** *If $\delta < \rho K$ there exist two constants $R' \in \mathbb{N}$ and $\zeta > 0$ such that if $R_o \geq R'$ and $\omega \in \mathsf{A}(0, R^2)$, then*

$$\mathbb{Q}(A) \geq \zeta.$$

*Proof.* Using Lemma 1.14 and the definition of admissibility, we conclude the existence of a constant $\kappa > 0$ such that

$$\mathbb{Q}\Big(\mathsf{A}(\hat{x}^{(T^{(n)})}, R_n^2)^c \Big| \bigcap_{i=n}^{k+1} \mathsf{A}(\hat{x}^{(T^{(i)})}, R_i^2)\Big) \leq \mathsf{c}R_n^{-\kappa}, \quad n \in [k],$$

where with abuse of notation $\hat{x}^{(0)} \equiv 0$ and $R_{k+1} \equiv R$. Using the elementary inequality:

$$\prod_{i=1}^{n}(1-a_i) \geq 1 - \sum_{i=1}^{n} a_i, \quad a_i \in (0,1),$$

we obtain that

$$\mathbb{Q}\Big(\bigcap_{i=0}^{k} \mathsf{A}(\hat{x}^{(T^{(i)})}, R_i^2)\Big) \geq 1 - \sum_{i=0}^{k} \mathsf{c}R_i^{-\kappa} \geq 1 - \sum_{i=0}^{\infty} \mathsf{c}R_i^{-\kappa} = 1 - \sum_{i=0}^{\infty} R_o^{-\kappa K^i}.$$

Choosing $R_o$ large enough, we get

$$\mathbb{Q}\Big(\bigcap_{i=0}^{k} \mathsf{A}(\hat{x}^{T^{(i)}}, R_i^2)\Big) \geq 1 - \varepsilon,$$

for a fixed $\varepsilon \in (0,1)$. Using Lemma 1.14 (i), we also obtain that

$$\mathbb{Q}(A_2) \geq 1 - \mathsf{c}\sum_{i=0}^{\infty} R_i^{-\theta K} \geq 1 - \varepsilon,$$

provided $R_o$ is large enough. Let

$$\mathcal{R} \triangleq \Big\{\sum_{i=1}^{\infty} M^{\xi_i} < 1\Big\}.$$

We note that $\mathbb{Q}(\mathcal{R}) > 0$, see Lemma 1.16. Let $\hat{x} = (x,t)$ and note that on $A_1 \cap A_2 \cap \mathcal{R}$

$$\begin{aligned}\|x^{(T^{(0)})} - x\|_2 &\leq \|x^{(1)} - x\|_2 + \sum_{i=2}^{T^{(0)}} \|x^{(i)} - x^{(i-1)}\|_2 \leq 2MR + \sum_{i=2}^{T^{(0)}} 2MR^{(i-1)} \\ &\leq 2MR + 2MR\sum_{i=1}^{\infty} M^{\xi_i} \leq 4MR.\end{aligned}$$

Similarly, we see that $|t^{(T^{(0)})} - t| \leq 4M^2R^2$ on $A_1 \cap A_2 \cap \mathcal{R}$. Hence, $\mathbb{Q}(A_1 \cap A_2 \cap A_4) \geq \mathbb{Q}(A_1 \cap A_2 \cap \mathcal{R})$. Due to Lemma 1.13, we also have $\mathbb{Q}(A_3 | A_1 \cap A_2 \cap A_4) \geq C$. Finally, we conclude that $\mathbb{Q}(A) \geq C(\mathbb{Q}(\mathcal{R}) - 2\varepsilon)$. Taking $\varepsilon$ small enough completes the proof. □

### 1.5.4 Proof of Theorem 1.4

Let $(Z_n)_{n\in\mathbb{Z}_+}$ and $(Y_n)_{n\in\mathbb{Z}_+}$ be the coupled processes as defined in Section 1.5.3.1 and let $\tau$ and $\rho$ be as in Section 1.5.3.3. Take $\omega \in \mathsf{A}(0, R^2)$ and let $u\colon K_{(c+6)MR} \to \mathbb{R}$ be $\omega$-caloric. Now, we have

$$u(\hat{z}) - u(\hat{y}) \leq \mathbb{Q}((Z_\tau, \tau + \hat{z}_1) \neq (Y_\rho, \rho + \hat{y}_1)) \operatorname*{osc}_{\Theta^p(K_{(c+6)MR})} u \leq (1-\zeta) \operatorname*{osc}_{\Theta^p(K_{(c+6)MR})} u,$$

where $\zeta > 0$ is as in Lemma 1.17 and $p$ is the parity of $\hat{z}$ and $\hat{y}$. In view of Lemma 1.11, this proves Theorem 1.4. □

## 1.6 Proof of The Harnack Inequality: Theorem 1.2

### 1.6.1 Some Notation

In the following, fix a parity $p \in \{o, e\}$. Take $\nu \in (0,1)$ and $N \in \mathbb{N}$, and let $\{\mathbb{A}_1, \dots, \mathbb{A}_N\}$ be a covering of $\partial^p \mathbb{K}_1$ and let $\{\mathbb{C}_1, \dots, \mathbb{C}_N\} \subset \partial^p \mathbb{K}_1$. Further, we suppose that

$$\max_{i=1,\dots,N} \operatorname{diam}(\mathbb{A}_i) \leq \tfrac{\nu}{4}.$$

We assume that the boundary of each $\mathbb{A}_i$ and $\mathbb{C}_i$ has zero measure.

Let $\zeta > 1$ and $\gamma \in (0,1)$ be as in Theorem 1.4 and let $\mathcal{O}_{\hat{x},R}$ be the set of all $\omega \in \Omega$ such that the oscillation inequality

$$\operatorname*{osc}_{\Theta^p(K_R(\hat{x}))} u \leq \gamma \operatorname*{osc}_{\Theta^p(K_{\zeta R}(\hat{x}))} u$$

holds for all $\omega$-caloric functions $u$ on $\overline{K}_{\zeta R}(\hat{x})$.

For $i = 1, \dots, N$ define $\chi_{\hat{x},R,i}$ and $\Psi_{\hat{x},R,i}$ as in (1.21) with $\mathbb{A}$ replaced by $\mathbb{A}_i$ and $\overline{\mathbb{K}}_R$ replaced by $\overline{\mathbb{K}}_R(\hat{x})$. Moreover, set $\alpha \triangleq 2 - \varepsilon$ and

$$\mathcal{U}_{\hat{x},R} \triangleq \Big\{\omega \in \Omega \colon \forall_{\hat{y} \in K_R(\hat{x})} \forall_{i=1,\dots,N} \ \frac{|\Phi_{\hat{x},\alpha R,i}(\hat{y}) - \chi_{\hat{x},\alpha R,i}(\hat{y})|}{\chi_{\hat{x},\alpha R,i}(\hat{y})} \leq \varepsilon\Big\}.$$

The $\omega$-dependence in the above definition stems from $\Phi$.

For $i = 1, \dots, N$ define $\chi^*_{\hat{x},R,i}$ and $\Phi^*_{\hat{x},R,i}$ as in (1.21) with $\mathbb{A}$ replaced by $\mathbb{C}_i$ and $\overline{\mathbb{K}}_R$ replaced by $\overline{\mathbb{K}}_R(\hat{x})$. Fix a $\theta_1, \dots, \theta_N > 1$ and $\delta^* \in (0,1)$, and set

$$\mathcal{U}^*_{\hat{x},R} \triangleq \big\{\omega \in \Omega \colon \forall_{\hat{y} \in K_R(\hat{x})} \forall_{i=1,\dots,N} \ |\Phi^*_{\hat{x},\theta_i R,i}(\hat{y}) - \chi^*_{\hat{x},\theta_i R,i}(\hat{y})| \leq \delta^*\big\}.$$

Let $\kappa \in (0, \frac{1}{2d})$. Define the map $\widetilde{J} \colon \Omega \to \Omega, \widetilde{J}(\omega) \triangleq \widetilde{\omega}$ as follows: For $x \in \mathbb{Z}^d$ and $i = 1, \dots, 2d$ set

$$\widetilde{\omega}(x, e_i) \triangleq \begin{cases} 0, & \omega(x, e_i) < \kappa, \\ \omega(x, e_i) + \frac{M}{N}, & \omega(x, e_i) \geq \kappa, \end{cases}$$

where $N \triangleq \sum_{i=1}^{2d} \mathbb{1}_{\{\omega(x,e_i) \geq \kappa\}}$ and $M \triangleq \sum_{i=1}^{2d} \omega(x, e_i) \mathbb{1}_{\{\omega(x,e_i) < \kappa\}}$.

Next, take $\delta \in (\xi, \frac{1}{5})$, where $\xi \in (0, \frac{1}{5})$ is as in the statement of Theorem 1.2, and define

$$\begin{aligned} \mathcal{J}_R &\triangleq \Big\{\omega \in \Omega \colon \forall_{y \in B_{2R}} \ \nexists_{z \in \mathbb{Z}^d} \text{ such that } y \xrightarrow{\widetilde{\omega}} z, z \notin \mathcal{C}_{\widetilde{\omega}}, \|z - x\|_\infty = \lfloor R^\xi \rfloor\Big\}, \\ \mathcal{I}_R &\triangleq \Big\{\omega \in \Omega \colon \forall_{y \in B_{2R}} \text{ all self-avoiding paths in } \omega \text{ with length } \lfloor \mathfrak{o} R^{\xi/4} \rfloor \\ &\qquad\qquad \text{and starting value } y \text{ have visited } \mathcal{C}_{\widetilde{\omega}}\Big\}, \\ \mathcal{S}_R &\triangleq \Big\{\omega \in \Omega \colon \forall_{x,y \in \mathcal{C}_{\widetilde{\omega}} \cap B_{3R^\delta}} \ \operatorname{dist}_{\widetilde{\omega}}(x, y) \leq \mathsf{c} R^\delta\Big\}, \end{aligned}$$

where $\mathsf{c}, \mathfrak{o} > 0$ are constants determined in the following lemmata. The proof of the next lemma is given in Section 1.6.4 below.

**Lemma 1.18.** *If $\kappa$ is small enough, $\mathfrak{o} > 0$ can be chosen such that there are constants*

*$R' > 0, \mathfrak{c}_1, \mathfrak{c}_2 > 0$ and $\zeta > 0$ such that for all $R \geq R'$*

$$P(\mathcal{I}_R) \geq 1 - \mathfrak{c}_1 e^{-\mathfrak{c}_2 R^\zeta}.$$

Finally, define

$$\mathcal{Z}_R \triangleq \bigcap_{\hat{y} \in K_R} \bigcap_{r \in (R^\delta, R]} [\mathcal{O}_{\hat{y},r} \cap \mathcal{U}_{\hat{y},r} \cap \mathcal{U}^*_{\hat{y},r}] \bigcap \mathcal{J}_R \bigcap \mathcal{I}_R \bigcap \mathcal{S}_R.$$

**Lemma 1.19.** *If $\kappa$ is small enough, $\mathsf{c} > 0$ can be chosen such that there are constants $R' > 0, \mathfrak{c}_1, \mathfrak{c}_2 > 0$ and $\zeta > 0$ such that for all $R \geq R'$*

$$P(\mathcal{Z}_R) \geq 1 - \mathfrak{c}_1 e^{\mathfrak{c}_2 R^\zeta}.$$

*Proof.* If $\kappa$ is small enough the probability measure $P \circ \widetilde{J}^{-1}$ is balanced and genuinely $d$-dimensional. Thus, the claim follows from Theorem 1.4, Corollary 1.1, [10, Propositions 3.1 and 3.2] and Lemma 1.18 with a union bound. □

From now on, we will assume that $R \geq R'$ and that $\omega \in \mathcal{Z}_R \cap \mathsf{B}$. It might be that we enlarge $R'$ even further. Under these assumptions we will prove the parabolic Harnack inequality, which completes the proof of Theorem 1.2.

### 1.6.2 The Proof

Let $u$ be a non-negative $\omega$-caloric function satisfying the growth condition (1.2) in Theorem 1.2. For contradiction, assume that $\hat{x}^* \in \Theta^p(K_R^+)$ and $\hat{y}^* \in \Theta^p(K_R^-)$ satisfy

$$u(\hat{x}^*) \geq \frac{(1+3\varepsilon) H_{2-\varepsilon} u(\hat{y}^*)}{(1-\varepsilon)^2}, \qquad H \equiv H_{2-\varepsilon} = H_\alpha. \tag{1.35}$$

Furthermore, let $K^+_{(2-\varepsilon/2)R}$ be the discrete version of $\mathbb{B}_{(2-\varepsilon/2)R} \times (2R^2, (2-\varepsilon/2)^2 R^2)$ and let $K^{++}_{2R}$ be the discrete version of $\mathbb{B}_{2R} \times (1.5R^2, 4R^2)$.

The proof of Theorem 1.2 is based on the following three lemmata:

**Lemma 1.20.** *There exists a constant $M \geq 1$ such that every subcylinder of $K^{++}_{2R}$ with radius $R^\delta$ contains a point $\hat{z}$ of the same parity as $\hat{x}^*$ such that*

$$u(\hat{z}) \leq M u(\hat{y}^*).$$

*Moreover, there exists a subcylinder of $K^+_{(2-\varepsilon/2)R}$ with radius $R^\delta$ which contains a point $\hat{x}$ of the same parity as $\hat{x}^*$ such that*

$$u(\hat{x}) \geq M u(\hat{y}^*) 2^{\mathfrak{c} R^{\frac{1-\delta}{2}}}.$$

From now on let $\hat{x} = (x, t)$ be as in Lemma 1.20.

**Lemma 1.21.** *For every $\hat{z} \in \mathcal{C}_{\widetilde{\omega}} \times \mathbb{Z}_+ \cap K^+_{(2-\varepsilon/2)R}$ with the same parity as $\hat{x}^*$ it holds that*

$$u(\hat{z}) \leq M u(\hat{y}^*) \kappa^{-2R^{2\delta}}.$$

Noting that $(1-\delta)/2 > \frac{2}{5}$ and $2\delta < \frac{2}{5}$, we see that $\hat{x} \notin \mathcal{C}_{\widetilde{\omega}} \times \mathbb{Z}_+$. Let

$$T \triangleq \inf(n \in \mathbb{Z}_+ \colon (X_n, n+t) \notin K_{2R} \text{ or } X_n \in \mathcal{C}_{\widetilde{\omega}}). \tag{1.36}$$

**Lemma 1.22.** $P^x_\omega(X_T \notin \mathcal{C}_{\widetilde{\omega}}) \leq \mathfrak{w}^{-R^{2-\xi}}$.

Next, we put these pieces together. The optional stopping theorem and Lemmata 1.21 and 1.22 yield that

$$\begin{aligned} u(\hat{x}) &= E^x_\omega\big[u(X_T, T+t)\mathbb{1}_{\{X_T \in \mathcal{C}_{\widetilde{\omega}}\}}\big] + E^x_\omega\big[u(X_T, T+t)|X_T \notin \mathcal{C}_{\widetilde{\omega}}\big] P^x_\omega(X_T \notin \mathcal{C}_{\widetilde{\omega}}) \\ &\leq Mu(\hat{y}^*)\kappa^{-2R^{2\delta}} + E^x_\omega\big[u(X_T, T+t)|X_T \notin \mathcal{C}_{\widetilde{\omega}}\big]\mathfrak{w}^{-R^{2-\xi}}. \end{aligned}$$

Now, rearranging and using Lemma 1.20 shows that

$$E^x_\omega\big[u(X_T, T+t)|X_T \notin \mathcal{C}_{\widetilde{\omega}}\big] \geq Mu(y^*)\big(2^{\mathfrak{c}R^{\frac{1-\delta}{2}}} - \kappa^{-2R^{2\delta}}\big)\mathfrak{w}^{R^{2-\xi}}.$$

Because $2^{\mathfrak{c}R^{\frac{1-\delta}{2}}} - \kappa^{-2R^{2\delta}} > 1$ for large enough $R$, we obtained a contradiction to the growth assumption (1.2). Except for the proofs of Lemmata 1.20, 1.21 and 1.22, which are given in the next subsection, the proof of Theorem 1.2 is complete. □

### 1.6.3 Proof of Lemmata 1.18, 1.20, 1.21 and 1.22

### 1.6.4 Proof of Lemma 1.18

For $z \in \mathbb{Z}^d$ and $n \in \mathbb{N}$ we write

$$C_n(z) \triangleq [-n, n]^d + (2n+1)z.$$

Adapting terminology from [10], we call $C_n(z)$ to be *$\omega$-good*, if $C_{2n}(z)$ contains a unique sink and for every $x \in C_n(z)$ every self-avoiding path in $\omega$ of length $\geq n/10$ reaches the unique sink in $C_n(z)$, cf. [10, Lemma 3.6].

Fix a small $\varepsilon > 0$. Then, by [10, Lemma 3.6] there exists an $N \in \mathbb{N}$ such that

$$P\big(\big\{\omega \in \Omega \colon C_N(z) \text{ is } \omega\text{-good}\big\}\big) \geq 1 - \frac{\varepsilon}{2}.$$

Next, take the parameter $\kappa \in (0, \frac{1}{2d})$ in the definition of $\widetilde{J}$ small enough such that

$$P\big(\big\{\omega \in \Omega \colon \exists_{i=k,\dots,2d}\ \omega(0, e_k) < \kappa\big\}\big) \leq \frac{\varepsilon}{2|C_{2N}(0)|}.$$

Let $\mathcal{C}^z_\omega$ be a sink in $Q_{2N}(z)$ w.r.t. the environment $\omega$. In case there are several sinks, take one in an arbitrary manner. Now, we have

$$\begin{aligned} P\big(\big\{\omega \in \Omega \colon \mathcal{C}^z_\omega = \mathcal{C}^z_{\widetilde{\omega}}\big\}\big) &\geq P\big(\big\{\omega \in \Omega \colon \forall_{y \in C_{2N}(z)} \forall_{k=1,\dots,2d}\ \omega(y, e_k) \geq \kappa\big\}\big) \\ &\geq 1 - |C_{2N}(z)|\ P\big(\big\{\omega \in \Omega \colon \exists_{k=1,\dots,2d}\ \omega(0, e_k) < \kappa\big\}\big) \geq 1 - \frac{\varepsilon}{2}. \end{aligned}$$

We say that $C_{2N}(z)$ is *very $\omega$-good*, if it is $\omega$-good and $\mathcal{C}^z_\omega = \mathcal{C}^z_{\widetilde{\omega}}$. Now, define the $\{0,1\}$-valued random variables

$$G_z(\omega) \triangleq \mathbb{1}\{C_{2N}(z) \text{ is very } \omega\text{-good}\}, \quad z \in \mathbb{Z}^d,$$

and note that

$$P(G_z = 1) \geq 1 - \varepsilon.$$

Because the environment measure $P$ is an i.i.d. measure, the random variables $G_z$ and $G_y$ are independent whenever $\|x - z\|_\infty \geq 2$. Consequently, we can apply [100, Theorem 0.0] and conclude that in case we have chosen $\varepsilon$ small enough from the beginning, the family $(G_z)_{z\in\mathbb{Z}^d}$ stochastically dominates supercritical Bernoulli site percolation. With abuse of notation, this means that the percolation process and $(G_z)_{z\in\mathbb{Z}^d}$ can be realized on the same probability space such that a.s.

$$\mathbb{1}\{z \in \mathcal{D}\} \leq G_z, \quad z \in \mathbb{Z}^d,$$

where $\mathcal{D}$ is the (a.s. unique) infinite cluster of the supercritical percolation process. Thus, we note that a.s.

$$\bigcup_{z\in\mathcal{D}} \mathcal{C}^z_\omega \subseteq \mathcal{C}^z_{\widetilde{\omega}}.$$

Denote by $A_z$ the connected component of $z$ in $\mathbb{Z}^d\backslash\mathcal{D}$. In case $z \in \mathcal{D}$ we have $A_z = \emptyset$. Furthermore, set

$$K^z_\omega \triangleq \begin{cases} |A_z \cup \partial A_z|, & z \notin \mathcal{D}, \\ 1, & z \in \mathcal{D}. \end{cases}$$

Of course, the $\omega$-dependence stems from $\mathcal{D}$.

**Lemma 1.23.** *For a.a. $\omega$ every self-avoiding path in $\omega$ with length $|C_{2N}(0)|K^z_\omega + 1$ and starting value $z \in \mathbb{Z}^d$ must have visited $\mathcal{C}_{\widetilde{\omega}}$.*

*Proof.* Note that every self-avoiding path in $\omega$ with length $|C_{2N}(0)|K^z_\omega + 1$ and starting value $z$ has visited a point $y$ such that

$$y \notin \bigcup_{u\in A_z\cup\partial A_z} C_N(u).$$

Consequently, the path has crossed a cube $C_N(u)$ with $u \in \partial A_z$. Because $\partial A_z \subset \mathcal{D}$, for a.a. $\omega \in \Omega$ we have $G_u(\omega) = 1$ and the definition of very $\omega$-good implies that the path must have visited $\mathcal{C}_{\widetilde{\omega}}$. □

The following lemma follows from [57, Theorem 8, Remark 10].

**Lemma 1.24.** *There exists an $\alpha > 0$ such that for all $k \in \mathbb{N}$*

$$P(|A_z| \geq k) \leq \mathfrak{c}e^{-k^\alpha}.$$

Now, for $\mathfrak{o} \triangleq 2|C_{2N}(0)|$ and large enough $R$, Lemmata 1.23 and 1.24 yield that

$$\begin{aligned} P(\mathcal{I}_R) &\geq P(\mathcal{I}_R, \forall_{z\in B_{2R}}\, |A_z| < \lfloor R^{\xi/4}\rfloor) \\ &= P(\forall_{z\in B_{2R}}\, |A_z| < \lfloor R^{\xi/4}\rfloor) \\ &\geq 1 - \mathfrak{c}|B_{2R}|e^{-\lfloor R^{\xi/4}\rfloor^\alpha}. \end{aligned}$$

This bound completes the proof of Lemma 1.18. □

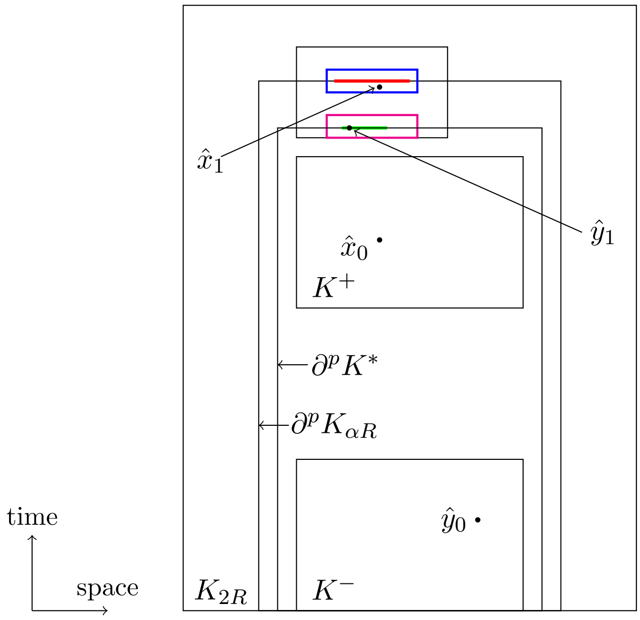


Figure 1.3: An illustration of the first step in the iteration procedure.

#### 1.6.4.1 Proof of Lemma 1.20

The proof is based on an iterative scheme in the spirit of an argument by Fabes and Stroock [48]. Let $(r_n)_{n\in\mathbb{Z}_+}$ be a sequence of radii defined as follows:

$$r_0 \triangleq R, \qquad r_1 \triangleq \frac{\varepsilon R}{8\alpha}, \qquad r_n \triangleq \frac{r_1}{n^2}.$$

Note that

$$\sum_{n=0}^{\infty} \alpha r_n = \alpha R + \frac{\varepsilon R}{8} \sum_{n=1}^{\infty} \frac{1}{n^2} \le \alpha R + \frac{\varepsilon R}{4} < \Big(2 - \frac{\varepsilon}{2}\Big) R,$$

and that

$$\sum_{n=0}^{\infty} \alpha^2 r_n^2 = \alpha^2 R^2 + \frac{\varepsilon^2 R^2}{64} \sum_{n=1}^{\infty} \frac{1}{n^4} < \Big(2 - \frac{\varepsilon}{2}\Big)^2 R^2.$$

Set $\mathfrak{X} \triangleq \max(n \in \mathbb{N}\colon r_n > R^\delta)$ and note that

$$\frac{r_1}{n^2} > R^\delta \quad \Leftrightarrow \quad \sqrt{\frac{\varepsilon}{8\alpha}} R^{\frac{1-\delta}{2}} > n \quad \Rightarrow \quad \mathfrak{X} \ge \Big\lfloor \sqrt{\frac{\varepsilon}{8\alpha}} R^{\frac{1-\delta}{2}} \Big\rfloor. \tag{1.37}$$

Next, we construct two sequences $(\hat{x}_n)_{n\in[\mathfrak{X}]}$ and $(\hat{y}_n)_{n\in[\mathfrak{X}]}$ of points in $K_{(2-\varepsilon/2)R}$ with the same parity as $\hat{x}^*$. As initial points we take $\hat{x}_0 \triangleq \hat{x}^*$ and $\hat{y}_0 \triangleq \hat{y}^*$.

Before we explain mathematically how $\hat{x}_{n+1}$ and $\hat{y}_{n+1}$ are chosen once $\hat{x}_n$ and $\hat{y}_n$ are known, we describe the idea in an informal manner, see also Figure 1.3. The initial step is to show the existence of a subset $\alpha R A_k$ (red in Figure 1.3) of $\partial^p K_{\alpha R}$ with the two properties that it can be reached by the space-time walk starting at $\hat{x}_0$ and that $\max_{\alpha R A_k} u$ and $\max_{\alpha R A_k} u / \min_{\alpha R A_k} u$ are reasonably large compared to $u(\hat{x}_0)$ and $u(\hat{x}_0)/u(\hat{y}_0)$, respectively. The oscillation inequality shows the existence of a cylinder $K^u$ (blue in Figure 1.3) containing $\alpha R A_k$ in which the ratio $\max_{K^u} u / \max_{\alpha R A_k} u$ is reasonably large. Using these properties, we obtain that

$$\max_{K^u} u \gg \max_{\alpha R A_k} u \gg u(\hat{x}_0),$$

where $b \gg a$ means that $b$ is in some sense larger than $a$. We now take $\hat{x}_1$ to be the point in $K^u$ (with the correct parity) where $u$ attains its maximum. The next step then is to chose $\hat{y}_1$ and to iterate. Before we comment on how $\hat{y}_1$ is chosen, let us stress that the sequence $(\hat{x}_n)_{n\in[\mathfrak{X}]}$ grows fast and the terminal point $\hat{x}_{\mathfrak{X}}$ will have the properties as described in the second part of Lemma 1.20. Because we want to iterate, the point $\hat{y}_1$ should be an element of a shifted version of $K^u$, say $K^l$ (magenta in Figure 1.3). Suppose that $\theta_k RC_k$ (green in Figure 1.3) is a subset of $K^l$ and part of the boundary of a cylinder $K^*$. We will chose $\theta_k$ and $C_k$ such that the space-time walk starting at $\hat{y}_0$ has a reasonable probability of exiting $K^*$ through $\theta_k RC_k$. Then, we take $\hat{y}_1$ to be the point in $\theta_k RC_k$ (with the correct parity) where $u$ attains its minimum. We proceed the iteration up to time $\mathfrak{X}$.

We now make this precise. The first step is based on the definition of $\mathcal{U}_{\hat{z}_n,r_n}$. Due the Harnack inequality for Brownian motion, we have

$$\chi_{\hat{z}_n,\alpha r_n,i}(\hat{x}_n) \le H\chi_{\hat{z}_n,\alpha r_n,i}(\hat{y}_n).$$

Using $\omega \in \mathcal{U}_{\hat{z}_n,r_n}$, we obtain

$$\frac{\Phi_{\hat{z}_n,\alpha r_n,i}(\hat{x}_n)}{\chi_{\hat{z}_n,\alpha r_n,i}(\hat{x}_n)} = 1 + \frac{\Phi_{\hat{z}_n,\alpha r_n,i}(\hat{x}_n) - \chi_{\hat{z}_n,\alpha r_n,i}(\hat{x}_n)}{\chi_{\hat{z}_n,\alpha r_n,i}(\hat{x}_n)} \le 1+\varepsilon.$$

Similarly, we see that $\Phi_{\hat{z}_n,\alpha r_n,i}(\hat{y}_n) \ge (1-\varepsilon)\chi_{\hat{z}_n,\alpha r_n,i}(\hat{y}_n)$. Therefore, we obtain

$$\begin{aligned}\Phi_{\hat{z}_n,\alpha r_n,i}(\hat{x}_n) &\le (1+\varepsilon)\chi_{\hat{z}_n,\alpha r_n,i}(\hat{x}_n)\\ &\le H(1+\varepsilon)\chi_{\hat{z}_n,\alpha r_n,i}(\hat{y}_n)\\ &\le \frac{(1+\varepsilon)H\Phi_{\hat{z}_n,\alpha r_n,i}(\hat{y}_n)}{1-\varepsilon}.\end{aligned} \tag{1.38}$$

In the following we use the short notation $A_{n,k} \triangleq \Theta^p(\alpha r_n A_k(\hat{z}_n))$.

**Lemma 1.25.** *There exists $k \in \{1,\dots,N\}$ such that*

$$\max_{A_{n,k}} u > \frac{\varepsilon u(\hat{x}_n)}{1+3\varepsilon}, \quad \max_{A_{n,k}} u > \frac{(1-\varepsilon)\min_{A_{n,k}} u}{(1+3\varepsilon)H}\frac{u(\hat{x}_n)}{u(\hat{y}_n)}.$$

*Proof.* We denote

$$\Theta \triangleq \Big\{k\in\{1,\dots,N\}\colon \max_{A_{n,k}} u > \frac{\varepsilon u(\hat{x}_n)}{1+3\varepsilon}\Big\}.$$

Note that

$$\sum_{k\notin\Theta} \max_{A_{k,n}} u\ \Phi_{z_n,\alpha r_n,k}(\hat{x}_n) \le \frac{\varepsilon}{1+3\varepsilon}\sum_{k\notin\Theta} u(\hat{x}_n)\Phi_{z_n,\alpha r_n,k}(\hat{x}_n) \le \frac{\varepsilon u(\hat{x}_n)}{1+3\varepsilon}. \tag{1.39}$$

This yields that

$$\begin{aligned}\sum_{k\in\Theta}\max_{A_{n,k}} u\ \Phi_{z_n,\alpha r_n,k}(\hat{x}_n) &= \sum_{k=1}^{N}\max_{A_{n,k}} u\ \Phi_{z_n,\alpha r_n,k}(\hat{x}_n) - \sum_{k\notin\Theta}\max_{A_{n,k}} u\ \Phi_{z_n,\alpha r_n,k}(\hat{x}_n)\\ &\ge \frac{1+2\varepsilon}{1+3\varepsilon}u(\hat{x}_n).\end{aligned}$$

Because the last term is positive, we conclude that $\Theta \neq \emptyset$.

For contradiction, assume that for all $k \in \Theta$

$$\max_{A_{n,k}} u \leq \frac{(1-\varepsilon)\min_{A_{n,k}} u}{(1+3\varepsilon)H} \frac{u(\hat{x}_n)}{u(\hat{y}_n)}. \tag{1.40}$$

Using the optional stopping theorem, (1.38), (1.39) and (1.40) yields that

$$\begin{aligned}
u(\hat{x}_n) &\leq \sum_{k\in\Theta} \max_{A_{n,k}} u \ \Phi_{z_n,\alpha r_n,k}(\hat{x}_n) + \sum_{k\notin\Theta} \max_{A_{n,k}} u \ \Phi_{z_n,\alpha r_n,k}(\hat{x}_n) \\
&\leq \sum_{k\in\Theta} \max_{A_{k,n}} u \ \Phi_{z_n,\alpha r_n,k}(\hat{x}_n) + \frac{\varepsilon u(\hat{x}_n)}{1+3\varepsilon} \\
&\leq \sum_{k\in\Theta} \frac{(1-\varepsilon)\min_{A_{n,k}} u}{(1+3\varepsilon)H} \frac{u(\hat{x}_n)}{u(\hat{y}_n)} \Phi_{z_n,\alpha r_n,k}(\hat{x}_n) + \frac{\varepsilon u(\hat{x}_n)}{1+3\varepsilon} \\
&\leq \frac{(1+\varepsilon)u(\hat{x}_n)}{(1+3\varepsilon)u(\hat{y}_n)} \sum_{k\in\Theta} \min_{A_{n,k}} u \ \Phi_{z_n,\alpha r_n,k}(\hat{y}_n) + \frac{\varepsilon u(\hat{x}_n)}{1+3\varepsilon} \\
&\leq \frac{(1+2\varepsilon)u(\hat{x}_n)}{1+3\varepsilon}.
\end{aligned}$$

This is a contradiction. The proof is complete. □

Let $k \in \{1, \dots, N\}$ be as in Lemma 1.25 and take $\hat{z}_{n+1} \in K_{2R}$ such that

$$A_{n,k} \subseteq K_{\alpha\nu r_n}(\hat{z}_{n+1}).$$

Due to Lemma 1.25, we have

$$\frac{(1-\varepsilon)u(\hat{x}_n)}{(1+3\varepsilon)Hu(\hat{y}_n)} \leq \frac{\max_{\Theta^p(K_{\alpha\nu r_n}(\hat{z}_{n+1}))} u}{\min_{\Theta^p(K_{\alpha\nu r_n}(\hat{z}_{n+1}))} u}. \tag{1.41}$$

Now, we explain how $\nu$ has to be chosen. Namely, take $\nu$ such that

$$\nu \cdot \alpha \cdot \zeta^{\frac{\log(\mathfrak{t})}{-\log(\gamma)}} \leq \inf_{i\in\mathbb{Z}_+} \left(\frac{r_{i+1}}{r_i}\right),$$

where $\mathfrak{t} > 1$ is a constant we determine later. With this choice of $\nu$ we can apply the oscillation inequality and obtain that

$$\operatorname*{osc}_{\Theta^p(K_{r_{n+1}}(\hat{z}_{n+1}))} u \geq \mathfrak{t} \operatorname*{osc}_{\Theta^p(K_{\alpha\nu r_n}(\hat{z}_{n+1}))} u. \tag{1.42}$$

Using (1.41) and (1.42), we further obtain that

$$\begin{aligned}\frac{\max_{\Theta^p(K_{r_{n+1}}(\hat{z}_{n+1}))} u}{\max_{A_{n,k}} u} &= \frac{\min_{\Theta^p(K_{r_{n+1}}(\hat{z}_{n+1}))} u}{\max_{A_{n,k}} u} + \frac{\operatorname{osc}_{\Theta^p(K_{r_{n+1}}(\hat{z}_{n+1}))} u}{\max_{A_{n,k}} u} \\ &\geq \frac{\mathfrak{t} \operatorname{osc}_{\Theta^p(K_{\alpha\nu r_n}(\hat{z}_{n+1}))} u}{\max_{\Theta^p(K_{\alpha\nu r_n}(\hat{z}_{n+1}))} u} \\ &= \mathfrak{t} \cdot \Big(1 - \frac{\min_{\Theta^p(K_{\alpha\nu r_n}(\hat{z}_{n+1}))} u}{\max_{\Theta^p(K_{\alpha\nu r_n}(\hat{z}_{n+1}))} u}\Big) \\ &\geq \mathfrak{t} \cdot \Big(1 - \frac{(1+3\varepsilon)Hu(\hat{y}_n)}{(1-\varepsilon)u(\hat{x}_n)}\Big).\end{aligned} \tag{1.43}$$

Let $\hat{x}_{n+1}$ be the point where $u$ attains its maximum on $\Theta^p(K_{r_{n+1}}(\hat{z}_{n+1}))$.

Next, we explain how $\hat{y}_{n+1}$ and $\mathfrak{t}$ are chosen. At this point we also explain how $\{\mathbb{C}_1, \ldots, \mathbb{C}_N\}, \theta_1, \ldots, \theta_N$ and $\delta^*$ are chosen. Take $\mathbb{C}_k$ and $\theta_k$ such that there is a cylinder $K_{\theta_k r_n}(\hat{u})$ with $\hat{y}_n \in K_{\theta_k r_n}(\hat{u})$ and $\theta_k r_n C_k(\hat{u}) \subset K_{r_{n+1}}(\hat{z}_{n+1}) - (0, 2r_{n+1}^2)$, see Figure 1.3. Here, $\theta_k r_n C_k(\hat{u})$ is defined in the same manner as for $\{\mathbb{A}_1, \ldots, \mathbb{A}_N\}$. Recalling that $\omega \in \mathcal{U}^*_{\hat{u},\theta_k r_n}$, we can take $\delta^*$ small enough such that there exists a uniform constant $\mathfrak{m} > 1$ such that $\Phi^*_{\hat{u},\theta_k r_n,k}(\hat{y}_n) \geq \mathfrak{m}^{-1}$. Then, take $\hat{y}_{n+1}$ to be the point in $\Theta^p(\theta_k r_n C_k(\hat{u}))$ where $u$ attains its minimum. The optional stopping theorem yields that

$$u(\hat{y}_n) \geq \frac{u(\hat{y}_{n+1})}{\mathfrak{m}}. \tag{1.44}$$

We now impose an assumption on $\mathfrak{t}$:

$$\mathfrak{t} \geq \frac{2\mathfrak{m}(1+3\varepsilon)}{\varepsilon^2}. \tag{1.45}$$

Using Lemma 1.25 and (1.43), we obtain that

$$\begin{aligned}u(\hat{x}_{n+1}) &\geq \mathfrak{t}\Big(1 - \frac{(1+3\varepsilon)Hu(\hat{y}_n)}{(1-\varepsilon)u(\hat{x}_n)}\Big) \max_{A_{n,k}} u \\ &\geq \frac{\mathfrak{t}\varepsilon}{1+3\varepsilon}\Big(1 - \frac{(1+3\varepsilon)Hu(\hat{y}_n)}{(1-\varepsilon)u(\hat{x}_n)}\Big) u(\hat{x}_n).\end{aligned} \tag{1.46}$$

**Lemma 1.26.** *For $n \in [\mathfrak{X} - 1]$ we have*

$$1 - \frac{(1+3\varepsilon)Hu(\hat{y}_{n+1})}{(1-\varepsilon)u(\hat{x}_{n+1})} \geq \varepsilon \quad \Big(\Leftrightarrow \quad u(\hat{x}_{n+1}) \geq \frac{(1+3\varepsilon)Hu(\hat{y}_{n+1})}{(1-\varepsilon)^2}\Big).$$

*Proof.* We use induction. For $n = 0$ the claim follows from (1.35). Suppose that the claim holds for $n \in [\mathfrak{X} - 2]$. Together with (1.44), the induction hypothesis yields that

$$\frac{(1+3\varepsilon)Hu(\hat{y}_{n+1})}{(1-\varepsilon)^2\mathfrak{m}} \leq \frac{(1+3\varepsilon)Hu(\hat{y}_n)}{(1-\varepsilon)^2} \leq u(\hat{x}_n).$$

Using this bound, (1.46) and the induction hypothesis again, we obtain that

$$u(\hat{x}_{n+1}) \geq \frac{\mathfrak{t}\varepsilon^2 u(\hat{x}_n)}{1+3\varepsilon} \geq \frac{\mathfrak{t}\varepsilon^2 Hu(\hat{y}_{n+1})}{(1-\varepsilon)^2\mathfrak{m}}.$$

The assumption (1.45) implies the claim. □

Now, (1.45), (1.46) and Lemma 1.26 yield that

$$u(\hat{x}_{n+1}) \geq \frac{\mathfrak{t}\varepsilon^2}{1+3\varepsilon} u(\hat{x}_n) \geq 2\mathfrak{m} u(\hat{x}_n).$$

Inductively, we see that

$$u(\hat{x}_{\mathfrak{X}}) \geq 2^{\mathfrak{X}} \mathfrak{m}^{\mathfrak{X}} u(\hat{x}_0)$$

and (1.37) completes the proof of the second claim in Lemma 1.20 with $M = \mathfrak{m}^{\mathfrak{X}}$. To see that the first claim holds, note that

$$u(\hat{y}_K) \leq \mathfrak{m}^{\mathfrak{X}} u(\hat{y}_0) = \mathfrak{m}^{\mathfrak{X}} u(\hat{y}^*).$$

Thus, the first claim follows from the argument we used to generate $(y_n)_{n \in [\mathfrak{X}]}$. The proof is complete. □

### 1.6.5 Proof of Lemma 1.21

By the first part of Lemma 1.20 there exists a point $\hat{y}$ of the same parity as $\hat{x}^*$ such that its space coordinate is in $B_{2R}$ and at most at distance $R^\delta$ from those of $\hat{z}$, the time coordinate of $\hat{y}$ is at least at distance $R^{2\delta}$ and at most at distance $2R^{2\delta}$ from those of $\hat{z}$, and $u(\hat{y}) \leq M u(\hat{y}^*)$. We now distinguish two cases.

First, if $\hat{y} \in \mathcal{C}_{\widetilde{\omega}}$ we use $\omega \in \mathcal{S}_R$ and the optional stopping theorem to obtain that $u(\hat{y}) \geq u(\hat{z}) \kappa^{2R^{2\delta}}$, provided $R$ is large enough. This yields the claim.

Second, if $\hat{y} \notin \mathcal{C}_{\widetilde{\omega}}$ we guide the walk into $\mathcal{C}_{\widetilde{\omega}}$. Because $\omega \in \mathcal{J}_R$, the worst case is that $\hat{y}$ is in a hole of $\mathcal{C}_{\widetilde{\omega}}$ of radius $\lfloor R^\xi \rfloor$. Because $\omega \in \mathsf{B}$, with probability at least $\frac{1}{2d}$ the walk in $\omega$ goes a step in direction of the boundary of the hole. Thus, with probability at least $(2d)^{-d\lfloor R^\xi \rfloor}$ the walk is in $\mathcal{C}_{\widetilde{\omega}}$. Recalling that $\xi < \delta$ and that $\kappa < \frac{1}{2d}$, the claim follows as before. □

### 1.6.6 Proof of Lemma 1.22

Recall that $\omega \in \mathcal{I}_R$. Thus, to be at time $T$ not in $\mathcal{C}_{\widetilde{\omega}}$, the walk may not leave the ball $B_{\mathfrak{o}\sqrt{d}R^{\xi/4}}(x) \equiv B_{\mathfrak{c}R^{\xi/4}}(x)$ before it leaves the cylinder $K_{2R}$, which is necessarily via its time boundary when $R$ is large enough. In other words, we have

$$\{X_T \notin \mathcal{C}_{\widetilde{\omega}}\} \subseteq \{S > \varepsilon R^2\},$$

where

$$S \triangleq \inf(n \in \mathbb{Z}_+ \colon X_n \notin B_{\mathfrak{c}R^{\xi/4}}(x)).$$

Set $o \triangleq \lfloor R^{2-\xi} \rfloor$. We show by induction that for $n = 1, \dots, o$

$$\sup \big( P^y_\omega(S > n\lfloor \varepsilon R^\xi \rfloor) \colon y \in B_{\mathfrak{c}R^{\xi/4}}(x) \big) \leq \mathfrak{w}^{-2n}. \tag{1.47}$$

For the induction base note that for all $y \in B_{\mathfrak{c}R^{\xi/4}}(x)$

$$P^y_\omega(S > \lfloor \varepsilon R^\xi \rfloor) \leq \frac{E^y_\omega[S]}{\lfloor \varepsilon R^\xi \rfloor} \leq \mathsf{c} R^{-\xi/2} < \mathfrak{w}^{-2},$$

in case $R$ is large enough. For the induction step assume that (1.47) holds for $n \in \{1, \dots, o-1\}$. The Markov property of the walk yields that for all $y \in B_{\mathfrak{c}R^{\xi/4}}(x)$

$$\begin{aligned} P^y_\omega(S > (n+1)\lfloor \varepsilon R^\xi \rfloor) &= P^y_\omega(S > (n+1)\lfloor \varepsilon R^\xi \rfloor, S > \lfloor \varepsilon R^\xi \rfloor) \\ &= E^y_\omega\big[P^{X(\lfloor \varepsilon R^\xi \rfloor)}_\omega (S > n\lfloor \varepsilon R^\xi \rfloor) \mathbb{1}_{\{S > \lfloor \varepsilon R^\xi \rfloor\}}\big] \\ &\le \mathfrak{w}^{-2n} P^y_\omega(S > \lfloor \varepsilon R^\xi \rfloor) \\ &\le \mathfrak{w}^{-2(n+1)}. \end{aligned}$$

Using (1.47) with $n = o$ yields that

$$P^x_\omega(S > \varepsilon R^2) \le P^x_\omega(S > o\lfloor \varepsilon R^\xi \rfloor) \le \mathfrak{w}^{-R^{2-\xi}}.$$

The lemma is proven. □

## 1.7 Proof of Theorem 1.3

The proof is similar to those of [143, Theorem 3.3.22]. The only differences are that instead of the EHI [143, Lemma 3.3.8] one has to use [10, Theorem 1.6] and that instead of [143, Eq. 3.3.23] one can use the martingale property of the walk and the optional stopping theorem, see also the proof of [57, Theorem 2 (i)].

We give some details: Let $R_0 \ge 1$ be a large constant and set $R_i \triangleq R_0^i$ and

$$B^i(z) \triangleq \big\{x \in \mathbb{Z}^d : \|x - z\|_\infty < R_i\big\}, \quad i \in \mathbb{Z}_+.$$

We shall also write $B^i \triangleq B^i(0)$. Set

$$\tau_i \triangleq \inf(n \in \mathbb{Z}_+ : X_n \notin B^i), \quad i \in \mathbb{Z}_+.$$

Due to [10, Theorem 1.6], provided $R_0$ is large enough, there exist constants $\gamma, \delta > 0$ and a set $G_i \in \mathcal{F}$ such that for every $\omega \in G_i$, every $z \in \partial B^i$ and every $x \in B^{i-1}$ it holds that

$$\max_{y \in B^{i-1}(z)} E^y_\omega\big[\#\text{ visits of } x \text{ before } \tau_{i+2}\big] \le \gamma \min_{y \in B^{i-1}(z)} E^y_\omega\big[\#\text{ visits of } x \text{ before } \tau_{i+2}\big],$$

and

$$P(G_i) \ge 1 - e^{-R^\delta_{i-1}}.$$

Let $(\theta^x)_{x \in \mathbb{Z}^d}$ be the canonical shifts on $\Omega$, i.e. $(\theta^x \omega)(y, e) = \omega(x + y, e)$. We obtain for every $\omega \in G_i$ and all $z \in \partial B^i$ that

$$\begin{aligned} &\sum_{x \in B^{i-1}} E^z_{\theta^x\omega}\big[\#\text{ visits of } 0 \text{ before } \tau_{i+1}\big] \\ &\quad\le \sum_{x \in B^{i-1}} \max_{y \in B^{i-1}(z)} E^y_\omega\big[\#\text{ visits of } x \text{ before } \tau_{i+2}\big] \\ &\quad\le \sum_{x \in B^{i-1}} \gamma \min_{y \in B^{i-1}(z)} E^y_\omega\big[\#\text{ visits of } x \text{ before } \tau_{i+2}\big] \end{aligned}$$

$$\leq \gamma E_\omega^z\big[\#\text{ visits of } B^{i-1} \text{ before } \tau_{i+2}\big]$$
$$\leq \gamma E_\omega^z\big[\tau_{i+2}\big]$$
$$\leq \gamma \mathfrak{c} R_{i+2}^2.$$

Using the shift invariance of $P$ and the Markov property of the walk yields that

$$\int E_\omega^0\big[\#\text{ visits of 0 in } (\tau_i, \tau_{i+1}]\big] P(d\omega)$$
$$= \frac{1}{|B^{i-1}|} \int \sum_{y \in B^{i-1}} E_{\theta^y\omega}^0\big[\#\text{ visits of 0 in } (\tau_i, \tau_{i+1}]\big] P(d\omega)$$
$$\leq \frac{1}{|B^{i-1}|} \int_{G_i} \sum_{y \in B^{i-1}} E_{\theta^y\omega}^0\big[E_{\theta^y\omega}^{X_{\tau_i}}\big[\#\text{ visits of 0 before } \tau_{i+1}\big]\big] P(d\omega) + \mathfrak{c} R_{i+1}^2 P(G_i^c)$$
$$\leq \mathsf{c}\big(R_i^{2-d} + R_{i+1}^2 e^{-R_i^\delta}\big).$$

Recalling that $d \geq 3$ and summing over $i$ shows that

$$\int E_\omega^0\big[\#\text{ visits of } 0\big] P(d\omega) < \infty,$$

which implies that the walk is transient for $P$-a.a. environments. $\square$

# 2 Lyapunov Criteria for the Feller–Dynkin Property of Martingale Problems

## 2.1 Introduction

It is a classical question for a Markov process whether its transition semigroup is a self-map on the space of bounded continuous functions and on the space of continuous functions vanishing at infinity, respectively. If the first property holds the Markov process is called $C_b$-Feller process and when the second property holds it is called Feller–Dynkin process. In the literature Feller–Dynkin processes are often also called Feller processes, see, for instance, [94, 99, 123]. Our terminology is borrowed from [124].

Knowing when a Markov process is Feller–Dynkin opens the door for many interesting results, such as existence theorems for evolution equations. To be more precise, let $(L, \mathcal{D}(L))$ be the generator of a Feller–Dynkin semigroup and $\mathbb{X} = (C_0, \|\cdot\|_\infty)$ is the Banach space of continuous functions vanishing at infinity. Then, for any non-linearity $h$ satisfying a Lipschitz condition the deterministic evolution equation

$$du(t) = \big(Lu(t) + h(t, u(t))\big)dt, \quad u(0) = f \in C_0,$$

has a mild solution in $\mathbb{X}$, see, for instance, [118, Section 6.1]. While such an existence result is of purely analytic nature, the connection of the semigroup and its generator to a stochastic process can be useful to verify its prerequisites. Moreover, the Feynman–Kac formula provides a stochastic representation of the solutions, see [99, Theorem 3.47] for a precise statement.

As Markov processes are usually defined by its infinitesimal description, it is interesting to find criteria for the Feller properties in terms of the generalized infinitesimal generator of the Markov process.

In this chapter we give such criteria for Markov processes defined via martingale problems (MPs). Our contributions are two-fold. First, we show that the Feller–Dynkin property can be described by a Lyapunov-type criterion in the spirit of the classical Lyapunov-type criteria for explosion, recurrence and transience, see, e.g. [80, 120]. More precisely, we prove a sufficient condition for the Feller–Dynkin property, see Theorem 2.1, and a condition to reject the Feller–Dynkin property, see Theorem 2.2. Under additional assumptions on the input data, we extend the sufficient condition for the Feller–Dynkin property to be necessary, see Theorem 2.3. The necessity is for instance useful when one studies coupled processes, i.e. processes whose infinitesimal description is built from the infinitesimal description of other processes. We illustrate this in our applications. Moreover, we provide a technical condition for a reduction or an enlargement of the input data of a MP, see Proposition 2.3. A reduction helps to check the additional assumption of our necessary and sufficient criterion, while an enlargement simplifies finding Lyapunov functions for our sufficient conditions. We apply our criteria to derive conditions for the Feller–Dynkin property of multidimensional diffusions with random switching. In particular, we derive a Khasminskii-type integral test for the Feller–Dynkin property.

Our second contribution is a systematic study of the Feller–Dynkin property of switching diffusions with state-independent switching. In other words, we consider a process $(Y, Z)$, where $Z$ is a continuous-time Feller–Dynkin Markov chain and $Y$ solves the stochastic differential equation (SDE)

$$dY_t = b(Y_t, Z_t)dt + \sigma(Y_t, Z_t)dW_t,$$

where $W$ is a Brownian motion. One may think of the process $Y$ as a diffusion in a random environment given by the Markov chain $Z$. The process $Y$ has a natural relation to processes with fixed environments, i.e. solutions to the SDEs

$$dY_t^k = b(Y_t^k, k)dt + \sigma(Y_t^k, k)dW_t, \tag{2.1}$$

where $k$ is in the state space of $Z$. Provided $(Y, Z)$ is a $C_b$-Feller process and the SDEs (2.1) satisfy weak existence and pathwise uniqueness, we show that $(Y, Z)$ is a Feller–Dynkin process if and only if the processes in the fixed environments are Feller–Dynkin processes. Furthermore, using a limit theorem for switching diffusions, see Theorem 2.5, we show that $(Y, Z)$ is a $C_b$-Feller process whenever it exists uniquely and the coefficients are continuous. We also explain that the uniqueness of $(Y, Z)$ is implied by weak existence and pathwise uniqueness of the diffusions in the fixed environments. For the one dimensional case we deduce an equivalent integral-test for the Feller–Dynkin property of $(Y, Z)$ and for multidimensional settings we give a Khasminskii-type integral test.

We now comment on related literature. To the best of our current knowledge, Lyapunov-type criteria for the Feller–Dynkin property are only used in specific case studies and a systematic study as given here does not appear in the literature. For continuous-time Markov chains, explicit conditions for the Feller–Dynkin property can be found in [97, 122]. In [97] also a Lyapunov-type condition appears. Infinitesimal conditions for the Feller–Dynkin property of diffusions are given in [4]. In the context of jump-diffusions, linear growth conditions for the Feller–Dynkin property were recently proven in [85, 86]. The proofs include a Lyapunov-type argument based on Gronwall's lemma. The $C_b$-Feller and the strong Feller property of switching diffusions are studied profoundly, see, for instance, [112, 131, 139, 142]. We think that our result for the Feller–Dynkin property is the first of its kind. In particular, the first which applies to switching diffusions with countable many states. Recall that the strong Feller property holds if the transition semigroup maps bounded functions to bounded continuous functions. It is clear that the strong Feller property implies the $C_b$-Feller property. We stress that neither the strong Feller property nor the Feller–Dynkin property implies the other. An easy example for a Feller–Dynkin process, which does not have the strong Feller property is the linear motion and an example for a strong Feller process, which does not have the Feller–Dynkin property is given in Example 2.6.

The chapter is structured as follows. In Section 2.2 we explain our setup. In particular, in Section 2.2.2 we recall the different concepts for the Feller properties of martingale problems. In Section 2.3 we discuss Lyapunov-type conditions for the Feller–Dynkin property in a general abstract setting and in Section 2.4 we discuss the case of switching diffusions. In Section 2.5 we give an existence theorem for switching diffusions and in Section 2.6 we comment on the role of initial laws.

## 2.2 The Feller Properties of Martingale Problems

### 2.2.1 The Setup

Let $S$ be a locally compact Hausdorff space with countable base (LCCB space), define $\Omega$ to be the space of all càdlàg functions $\mathbb{R}_+ \to S$ and let $X$ be the coordinate process on $\Omega$, i.e. the process defined by $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$ and $t \in \mathbb{R}_+$. We set $\mathcal{F} \triangleq \sigma(X_t, t \in \mathbb{R}_+)$ and $\mathcal{F}_t \triangleq \bigcap_{s>t} \mathcal{F}^o_s$, where $\mathcal{F}^o_t \triangleq \sigma(X_s, s \in [0,t])$. If not stated otherwise, all terms such as *local martingale, supermartingale,* etc. refer to $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ as the underlying filtration. We equip $\Omega$ with the Skorokhod topology (see [47, 70]). In this case, $\mathcal{F}$ is the Borel $\sigma$-field on $\Omega$, see [47, Proposition 3.7.1].

We use standard notation for function spaces, i.e. for example we denote by $M(S)$ the set of Borel functions $S \to \mathbb{R}$, by $B(S)$ the set of bounded Borel functions $S \to \mathbb{R}$, by $C(S)$ the set of continuous functions $S \to \mathbb{R}$ and by $C_0(S)$ the space of continuous functions $S \to \mathbb{R}$ which are vanishing at infinity, etc. We take the following four objects as input data for our abstract MP:

(i) A set $D \subseteq C(S)$ of test functions.

(ii) A map $\mathcal{L}\colon D \to M(S)$ satisfying

$$\int_0^t \big|\mathcal{L}f(X_s(\omega))\big| ds < \infty \tag{2.2}$$

for all $t \in \mathbb{R}_+, \omega \in \Omega$ and $f \in D$. We think of $\mathcal{L}$ as a candidate for an extended generator in the spirit of [123, Definition VII.1.8].

(iii) A set $\Sigma \in \mathcal{F}$, which can be seen as the state space for the paths.

(iv) A Borel probability measure $\eta$ on $S$, which we use as initial law.

**Definition 2.1.** *A probability measure $P$ on $(\Omega, \mathcal{F})$ is called a solution to the MP $(D, \mathcal{L}, \Sigma, \eta)$ if $P(\Sigma) = 1, P \circ X_0^{-1} = \eta$ and for all $f \in D$ the process*

$$f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds, \quad t \in \mathbb{R}_+, \tag{2.3}$$

*is a local $P$-martingale. When $\eta = \delta_x$ for some $x \in S$, we write $(D, \mathcal{L}, \Sigma, x)$ instead of $(D, \mathcal{L}, \Sigma, \delta_x)$. Here, $\delta_x$ denotes the Dirac measure on the point $x \in S$.*

**Example 2.1.** *The following MP corresponds to the classical MP of Stroock and Varadhan [137]. Let $S \triangleq \mathbb{R}^d, D \triangleq C^2_b(\mathbb{R}^d)$,*

$$\mathcal{L}f(x) \triangleq \langle \nabla f(x), b(x)\rangle + \tfrac{1}{2} \operatorname{trace} (\nabla^2 f(x)a(x)), \tag{2.4}$$

*where $\nabla$ denotes the gradient, $\nabla^2$ denotes the Hessian matrix and $b\colon \mathbb{R}^d \to \mathbb{R}^d$ and $a\colon \mathbb{R}^d \to \mathbb{S}^d$ are locally bounded Borel functions with $\mathbb{S}^d$ denoting the set of all real symmetric non-negative definite $d\times d$ matrices, and $\Sigma = C(\mathbb{R}_+, \mathbb{R}^d)$. We have $\Sigma \in \mathcal{F}$, because $\Sigma$ is a closed subset of $\Omega$, see [47, Problem 3.25].*

In the remaining of this article we impose the following assumption.

**Standing Assumption 2.1.** *For all $x \in S$ the MP $(D, \mathcal{L}, \Sigma, x)$ has a solution $P_x$.*

Conditions for the existence of solutions in diffusion settings can be found in [77, 123, 137]. For conditions in jump-diffusions setups we refer to [17, 88] and Chapter 5. Conditions for abstract MPs can be found in [47]. For switching diffusions with state-independent switching we provide a Skorokhod-type existence result in Section 2.5.

### 2.2.2 The Markov, the $C_b$-Feller and the Feller–Dynkin Property of Martingale Problems

The family $(P_x)_{x\in S}$ is called a *Markov family* or simply *Markov* if the map $x \mapsto P_x(A)$ is Borel for all $A \in \mathcal{F}$ and for all $x \in S, t \in \mathbb{R}_+$ and all $G \in \mathcal{F}$ we have $P_x$-a.s.

$$P_x\big(\theta_t^{-1}G|\mathcal{F}_t\big) = P_{X_t}(G), \tag{2.5}$$

where $\theta_t\omega(s) \triangleq \omega(t+s)$ denotes the shift operator. We call (2.5) the *Markov property.* The family $(P_x)_{x\in S}$ is called a *strong Markov family* or simply *strongly Markov* if $(P_x)_{x\in S}$ is Markov and for all $x \in S$, all stopping times $\xi$ and all $G \in \mathcal{F}$ we have $P_x$-a.s. on $\{\xi < \infty\}$

$$P_x\big(\theta_\xi^{-1}G|\mathcal{F}_\xi\big) = P_{X_\xi}(G). \tag{2.6}$$

The identity (2.6) is called the *strong Markov property.* As the following proposition shows, many families of solutions to MPs are strongly Markov. For reader's convenience we provide a sketch of the proof, which mimics the proof of [47, Theorem 4.4.2].

**Proposition 2.1.** *If $D$ is countable, $D \subseteq C_b(S), \mathcal{L}(D) \subseteq B_{\mathrm{loc}}(S)$, $(P_x)_{x\in S}$ is unique and $\Sigma \subseteq \theta_\xi^{-1}\Sigma$ for all bounded stopping times $\xi$, then $(P_x)_{x\in S}$ is strongly Markov.*

*Sketch of Proof.* Due to Proposition 2.9 in Section 2.6, the map $x \mapsto P_x(A)$ is Borel for all $A \in \mathcal{F}$ and, due to the argument used in the solution to [77, Problem 2.6.9] (see [77, p. 121]), it suffices to show the strong Markov property for all bounded stopping times. Let $\xi$ be a bounded stopping time, set $P \equiv P_x$ and fix $F \in \mathcal{F}_\xi$ with $P(F) > 0$. Using the argument from the proof of [77, Lemma 5.4.19] one checks that the probability measures

$$P_1 \triangleq \frac{E^P\big[\mathbb{1}_F P(\theta_\xi^{-1}\cdot|\mathcal{F}_\xi)\big]}{P(F)}, \quad P_2 \triangleq \frac{E^P\big[\mathbb{1}_F P_{X_\xi}\big]}{P(F)}$$

both solve the MP $(D, \mathcal{L}, \Sigma, \zeta)$, where $\zeta \triangleq P(F)^{-1}E^P[\mathbb{1}_F\mathbb{1}\{X_\xi \in \ \cdot\ \}]$. Due to Proposition 2.9 in Section 2.6, we have $P_1 = P_2$, which implies that

$$E^P\big[\mathbb{1}_F P(\theta_\xi^{-1}G|\mathcal{F}_\xi)\big] = E^P\big[\mathbb{1}_F P_{X_\xi}(G)\big], \quad G \in \mathcal{F}.$$

Since this identity holds trivially when $P(F) = 0$, for all $G \in \mathcal{F}$ we conclude that $P$-a.s. $P(\theta_\xi^{-1}G|\mathcal{F}_\xi) = P_{X_\xi}(G)$. In other words, the strong Markov property holds for all bounded stopping times. □

If $(P_x)_{x\in S}$ is not unique it might still be possible to pick a Markov family from the set of solutions. For instance, in the setting of Example 2.1, this is the case when $a$ and $b$ are bounded and continuous, see [137, Theorem 12.2.3]. Conditions for the selection of a Markov family in jump-diffusion cases can be found in [88].

In the case where $(P_x)_{x\in S}$ is Markov, we can define a semigroup $(T_t)_{t\geq 0}$ of positive contraction operators on $B(S)$ via

$$T_t f(x) \triangleq E_x\big[f(X_t)\big], \quad f \in B(S).$$

It is obvious that $T_t$ is a positive contraction, i.e. if $f(S) \subseteq [0,1]$ then also $T_t f(S) \subseteq [0,1]$, and the semigroup property follows easily from the Markov property (2.5).

If $(P_x)_{x\in S}$ is Markov and

$$T_t(C_b(S)) \subseteq C_b(S), \tag{2.7}$$

we call $(P_x)_{x\in S}$ a *$C_b$-Feller family* or simply *$C_b$-Feller*. The inclusion (2.7) is called the *$C_b$-Feller property*. Any $C_b$-Feller family is also strongly Markov, see the proof of [74, Theorem 17.17]. The $C_b$-Feller property of the family $(P_x)_{x\in S}$ has a natural relation to the continuity of $x \mapsto P_x$ for which many conditions are known, see, e.g. [70, Theorem IX.4.8] for conditions in a jump diffusion setting. Here, $x \mapsto P_x$ is said to be continuous if $P_{x_n} \to P_x$ weakly as $n \to \infty$ whenever $x_n \to x$ as $n \to \infty$. In the setup of Example 2.1, if $(P_x)_{x\in S}$ is unique, $(P_x)_{x\in S}$ is $C_b$-Feller whenever $b$ and $a$ are continuous. However, in the same setting, if $(P_x)_{x\in S}$ is not unique, it might not be possible to choose a $C_b$-Feller family from the set of solutions, even if the coefficients are continuous and bounded, see [137, Exercise 12.4.2].

We call $(P_x)_{x\in S}$ a *Feller–Dynkin family* or simply *Feller–Dynkin* if it is a $C_b$-Feller family and

$$T_t(C_0(S)) \subseteq C_0(S). \tag{2.8}$$

The inclusion (2.8) is called the *Feller–Dynkin property*. From a semigroup point of view, the definition of a Feller–Dynkin semigroup also includes strong continuity in zero, see, e.g. [124, Definition III.6.5]. In our case, when $(P_x)_{x\in S}$ is Feller–Dynkin, the semigroup $(T_t)_{t\geq 0}$ is strongly continuous in zero due to the right-continuous paths of $X$, the dominated convergence theorem and [124, Lemma III.6.7]. Let us also comment on the issue of uniqueness. If $(P_x)_{x\in S}$ is Feller–Dynkin and $(L, \mathcal{D}(L))$ is its generator, i.e.

$$Lf \triangleq \lim_{t \searrow 0} \frac{T_t f - f}{t} \tag{2.9}$$

for $f \in \mathcal{D}(L)$, where

$$\mathcal{D}(L) \triangleq \left\{ f \in C_0(S) \colon \exists g \in C_0(S) \text{ such that } \lim_{t\searrow 0} \left\| \frac{T_t f - f}{t} - g \right\|_\infty = 0 \right\}, \tag{2.10}$$

then $P_x$ is the unique solution to the MP $(D, L, \Sigma, x)$, where $D$ is any core for $L$, see [83, Theorem 4.10.3]. Consequently, conditions for the Feller–Dynkin property imply in some cases also uniqueness.

For an overview on different concepts of Feller properties from a semigroup point of view we refer to the first chapter in [17].

If $S = \mathbb{R}^d$ and $(P_x)_{x\in\mathbb{R}^d}$ is Feller–Dynkin with generator $(L, \mathcal{D}(L))$ such that $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(L)$, then $L$ is of the following form

$$Lf(x) = -\int e^{i\langle x,y\rangle} q(x,y) \hat{f}(y) dy, \quad f \in C_c^\infty(\mathbb{R}^d),$$

where $i$ is the imaginary number, $\hat{f}(y) \triangleq (2\pi)^{-d} \int e^{-i\langle y,x\rangle} f(x) dx$ denotes the Fourier transform of $f$ and

$$q(x,\xi) = q(x,0) - i\langle b(x),\xi\rangle + \tfrac{1}{2}\langle a(x)\xi,\xi\rangle + \int \big(1 - e^{i\langle y,\xi\rangle} + i\langle \xi, y\rangle \mathbb{1}\{\|y\| \le 1\}\big) K(x,dy)$$

for a Lévy triplet $(b(x), a(x), K(x,dy))$, see [17, Corollary 2.23]. The function $q$ is called the *symbol* of the family $(P_x)_{x\in\mathbb{R}^d}$. Starting with a candidate $q$ for a symbol corresponds to a MP with input data $\Sigma \triangleq \Omega, D \triangleq C_c^\infty(\mathbb{R}^d)$ and

$$\mathcal{L}f(x) \triangleq -\int e^{i\langle x,y\rangle} q(x,y)\hat{f}(y)dy, \quad f \in D.$$

We refer to the second and the third chapter of [17] for a survey on the approach via the symbol.

Most of the general conditions for the Feller–Dynkin property are formulated in terms of the semigroup $(T_t)_{t\ge 0}$ and therefore are often not easy to check, see, e.g., [17, Theorem 1.10] and the discussion below its proof. In the following section we give a criterion for the Feller–Dynkin property in terms of the existence of Lyapunov functions.

## 2.3 Lyapunov Criteria for the Feller–Dynkin Property

Lyapunov-type criteria often appear in the context of explosion, recurrence and transience of a Markov process, see, e.g. [80, 120]. In this section we present such criteria for the Feller–Dynkin property of $(P_x)_{x\in S}$. We start with a sufficient condition.

**Theorem 2.1.** *Fix $t \in \mathbb{R}_+$ and suppose that $T_t(C_0(S)) \subseteq C(S)$. Assume that for any compact set $K \subseteq S$ there exists a function $V : S \to \mathbb{R}_+$ with the following properties:*

(i) $V \in D \cap C_0(S)$.

(ii) $\underline{V} \triangleq \min_{x\in K} V(x) > 0$.

(iii) $\mathcal{L}V \le cV$ *for a constant* $c > 0$.

*Then, $T_t(C_0(S)) \subseteq C_0(S)$. The function $V$ is called a* Lyapunov function *for $K$.*

*Proof.* We first explain that it suffices to show that for all compact sets $K \subseteq S$ and all $\varepsilon > 0$ there exists a compact set $O \subseteq S$ such that $P_x(X_t \in K) < \varepsilon$ for all $x \notin O$. To see this, let $f \in C_0(S)$ and $\varepsilon > 0$. By the definition of $C_0(S)$, there exists a compact set $K \subseteq S$ such that $|f(x)| < \frac{\varepsilon}{2}$ for all $x \notin K$. By hypothesis, there exists a compact set $O \subseteq S$ such that

$$\sup_{y\in S} |f(y)|\ P_x(X_t \in K) < \tfrac{\varepsilon}{2}$$

for all $x \notin O$. Thus, for all $x \notin O$ we have

$$\begin{aligned} \big|E_x\big[f(X_t)\big]\big| &\le E_x\big[|f(X_t)|\big(\mathbb{1}\{X_t \in K\} + \mathbb{1}\{X_t \notin K\}\big)\big] \\ &\le \sup_{y\in S}|f(y)|\ P_x(X_t \in K) + \tfrac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

In other words, $T_t f \in C_0(S)$, i.e. the claim is proven.

Next, we verify that this condition holds under the hypothesis of the theorem. Fix $x \in S$ and a compact set $K \subseteq S$. Let $V$ be as described in the prerequisites of the theorem. The following lemma is an easy consequence of the integration by parts formula. For completeness, we give a proof after the proof of Theorem 2.1 is complete.

**Lemma 2.1.** *Assume that $f \in C(S)$ and $\mathcal{L}f \in M(S)$ are such that (2.2) holds and such that the process (2.3) is a local martingale and that $c\colon \mathbb{R}_+ \to \mathbb{R}$ is an absolutely continuous function with Lebesgue density $c'$. Then, the process*

$$f(X_t)c(t) - f(X_0)c(0) - \int_0^t \big(f(X_s)c'(s) + c(s)\mathcal{L}f(X_s)\big)ds, \quad t \in \mathbb{R}_+, \tag{2.11}$$

*is a local martingale.*

Since $V \in D$, the definition of the martingale problem and Lemma 2.1 imply that the process

$$Y_s \triangleq V(X_s)e^{-cs} - \int_0^s e^{-cr}\left(\mathcal{L}V(X_r) - cV(X_r)\right)dr, \quad s \in \mathbb{R}_+,$$

is a local $P_x$-martingale. Using (iii), we see that $Y_s \geq V(X_s)e^{-cs} \geq 0$ for all $s \in \mathbb{R}_+$. Thus, since non-negative local martingales are supermartingales due to Fatou's lemma, $Y$ is a $P_x$-supermartingale. Using Markov's inequality, we obtain that

$$\begin{aligned} P_x(X_t \in K) &\leq P_x(V(X_t) \geq \underline{V}) \\ &\leq \underline{V}^{-1}E_x\big[V(X_t)\big] \\ &\leq e^{ct}\underline{V}^{-1}E_x\big[Y_t\big] \\ &\leq e^{ct}\underline{V}^{-1}E_x\big[Y_0\big] \\ &= e^{ct}\underline{V}^{-1}V(x). \end{aligned}$$

Take an $\varepsilon > 0$. Since we assume that $V \in C_0(S)$, there exists a compact set $O \subseteq S$ such that

$$V(y) < e^{-ct}\underline{V}\,\varepsilon$$

for all $y \notin O$. We conclude that

$$P_x(X_t \in K) \leq e^{ct}\underline{V}^{-1}V(x) < \varepsilon$$

for all $x \notin O$. This finishes the proof. □

*Proof of Lemma 2.1:* Denote the local martingale (2.3) by $M$. Moreover, set

$$N_t \triangleq \int_0^t \mathcal{L}f(X_s)ds, \quad t \in \mathbb{R}_+.$$

As an absolutely continuous function, $c$ is of finite variation over finite intervals. Thus, integration by parts yields that

$$d\big(M_tc(t)\big) = c(t)dM_t + \big(f(X_t) - f(X_0)\big)c'(t)dt - d\big(N_tc(t)\big) + c(t)\mathcal{L}f(X_t)dt.$$

We conclude that the process (2.11) equals the local martingale $\int_0^\cdot c(s)dM_s$. □

Next, we give a condition for rejecting the Feller–Dynkin property.

**Theorem 2.2.** *Suppose that $S$ is not compact and that there exist compact sets $K, C \subset S$, a constant $\alpha > 0$ and a bounded function $U\colon S \to \mathbb{R}_+$ with the following properties:*

(i) $U \in D$.

(ii) $\max_{y\in K} U(y) > 0$.

(iii) $\inf_{y\in S\backslash C} U(y) > 0$.

(iv) $\mathcal{L}U \geq \alpha U$ *on* $S\backslash K$.

*Then,* $(P_x)_{x\in S}$ *cannot be Feller–Dynkin. The function $U$ is called a* Lyapunov function for the sets $K, C$.

*Proof.* For contradiction, assume that $(P_x)_{x\in S}$ is Feller–Dynkin. For a moment we fix $x \in S$. Let $(\mathcal{F}^x_t)_{t\geq 0}$ be the $P_x$-completion of $(\mathcal{F}_t)_{t\geq 0}$, i.e.

$$\mathcal{F}^x_t \triangleq \sigma\big(\mathcal{F}_t, \mathcal{N}_x\big) = \bigcap_{s>t} \sigma\big(\mathcal{F}^o_s, \mathcal{N}_x\big), \tag{2.12}$$

where

$$\mathcal{N}_x \triangleq \big\{F \subseteq \Omega\colon \exists G \in \mathcal{F} \text{ with } F \subseteq G, P_x(G) = 0\big\}.$$

We set

$$\tau \triangleq \inf(t \in \mathbb{R}_+\colon X_t \in K), \tag{2.13}$$

which is well-known to be an $(\mathcal{F}^x_t)_{t\geq 0}$-stopping time, see [74, Theorem 6.7].

*Step 1:* The proof of the following observation is given after the proof of Theorem 2.2 is complete.

**Proposition 2.2.** *Assume that $(P_x)_{x\in S}$ is Feller–Dynkin and denote its generator by $(L, \mathcal{D}(L))$ (see (2.9) and (2.10)). For any compact set $K \subseteq S$ and any $\alpha > 0$ there exists a function $V\colon S \to \mathbb{R}_+$ with the following properties:*

(i) $V \in \mathcal{D}(L)$.

(ii) $\min_{y\in K} V(y) > 0$.

(iii) $LV \leq \alpha V$.

Let $V$ be as in Proposition 2.2. Due to Dynkin's formula ([123, Proposition VII.1.6]) and Lemma 2.1 the process

$$Z_t \triangleq e^{-\alpha t} V(X_t) + \int_0^t e^{-\alpha s}\big(\alpha V(X_s) - LV(X_s)\big)ds, \quad t \in \mathbb{R}_+,$$

is a local $P_x$-martingale. As $Z$ is bounded (recall that $\mathcal{D}(L) \subseteq C_0(S)$ and that $Lf \in C_0(S)$ for all $f \in \mathcal{D}(L)$), it is even a true $P_x$-martingale. Consequently, for $s < t$ we have $P_x$-a.s.

$$\begin{aligned} E_x\big[e^{-\alpha t} V(X_t)|\mathcal{F}_s\big] &\leq E_x\big[Z_t|\mathcal{F}_s\big] - \int_0^s e^{-\alpha r}\big(\alpha V(X_r) - LV(X_r)\big)dr \\ &= Z_s - \int_0^s e^{-\alpha r}\big(\alpha V(X_r) - LV(X_r)\big)dr = e^{-\alpha s} V(X_s), \end{aligned} \tag{2.14}$$

which implies that the process $(e^{-\alpha t}V(X_t))_{t\geq 0}$ is a non-negative $P_x$-supermartingale, which has a terminal value due to the submartingale convergence theorem. In particular, due to [124, Lemma 67.10], $(e^{-\alpha t}V(X_t))_{t\geq 0}$ is also a non-negative bounded $P_x$-supermartingale for the filtration $(\mathcal{F}^x_t)_{t\geq 0}$. Recalling that $\tau$ as defined in (2.13) is an $(\mathcal{F}^x_t)_{t\geq 0}$-stopping time, we deduce from the optional stopping theorem that

$$\begin{aligned} V(x) &\geq E_x\big[e^{-\alpha\tau}V(X_\tau)\big] \\ &\geq E_x\big[e^{-\alpha\tau}V(X_\tau)\mathbb{1}\{\tau<\infty\}\big] \\ &\geq E_x\big[e^{-\alpha\tau}\big]\min_{y\in K} V(y). \end{aligned} \tag{2.15}$$

Here, we use the fact that $X_\tau \in K$ on $\{\tau < \infty\}$, which follows from the right-continuity of $(X_t)_{t\geq 0}$ because $K$ is closed.

*Step 2:* In the following all terms such as *local martingale, submartingale*, etc. refer to $(\mathcal{F}^x_t)_{t\geq 0}$ as the underlying filtration. Lemma 2.1 and [124, Lemma 67.10] imply that the stopped process

$$Y_t \triangleq e^{-\alpha(t\wedge\tau)}U(X_{t\wedge\tau}) + \int_0^{t\wedge\tau} e^{-\alpha s}\big(\alpha U(X_s) - \mathcal{L}U(X_s)\big)ds, \quad t\in\mathbb{R}_+,$$

is a local $P_x$-martingale. Due to property (iv) of the function $U$, we have

$$Y_t \leq e^{-\alpha(t\wedge\tau)}U(X_{t\wedge\tau}) \leq \text{const.}$$

for all $t \in \mathbb{R}_+$. We note that local martingales bounded from above are submartingales. To see this, let $(M_t)_{t\geq 0}$ be a local martingale bounded from above by a constant $c$. Then, the process $(c - M_t)_{t\geq 0}$ is a non-negative local martingale and hence a supermartingale by Fatou's lemma. This implies that $(M_t)_{t\geq 0}$ is submartingale. Therefore, the process $(Y_t)_{t\geq 0}$ is a $P_x$-submartingale and it follows similar to (2.14) that the stopped process $(e^{-\alpha(t\wedge\tau)}U(X_{t\wedge\tau}))_{t\geq 0}$ is a non-negative bounded $P_x$-submartingale, which has a terminal value $e^{-\alpha\tau}U(X_\tau)$ by the submartingale convergence theorem. As $U$ is bounded, we note that on $\{\tau = \infty\}$ up to a null set we have $e^{-\alpha\tau}U(X_\tau) = 0$. Another application of the optional stopping theorem yields that

$$\begin{aligned} U(x) &\leq E_x\big[e^{-\alpha\tau}U(X_\tau)\big] \\ &= E_x\big[e^{-\alpha\tau}U(X_\tau)\mathbb{1}\{\tau<\infty\}\big] \\ &\leq \max_{y\in K} U(y) E_x\big[e^{-\alpha\tau}\big]. \end{aligned} \tag{2.16}$$

*Step 3:* We deduce from (2.15) and (2.16) that for all $x \notin C$

$$\frac{\inf_{y\in S\backslash C} U(y)}{\max_{y\in K} U(y)} \leq E_x\big[e^{-\alpha\tau}\big] \leq \frac{V(x)}{\min_{y\in K} V(y)}.$$

As $V \in \mathcal{D}(L) \subseteq C_0(S)$, we find a compact set $G \subset S$ such that for all $x \notin G$

$$V(x) \leq \frac{1}{2}\frac{\inf_{y\in S\backslash C} U(y)\min_{y\in K} V(y)}{\max_{y\in K} U(y)} > 0,$$

which implies that for all $x \notin C \cup G \neq S$

$$0 < \frac{\inf_{y \in S \backslash C} U(y)}{\max_{y \in K} U(y)} \leq E_x\big[e^{-\alpha\tau}\big] \leq \frac{1}{2}\frac{\inf_{y \in S \backslash C} U(y)}{\max_{y \in K} U(y)}.$$

This is a contradiction and the proof of Theorem 2.2 is complete. □

*Proof of Proposition 2.2:* We construct $V$ via the $\alpha$-potential operator of $(T_t)_{t \geq 0}$, i.e. the operator $U_\alpha \colon C_0(S) \to C_0(S)$ defined by

$$U_\alpha f(x) \triangleq \int_0^\infty e^{-\alpha s} T_s f(x) ds, \quad f \in C_0(S), x \in S.$$

Due to Urysohn's lemma for locally compact spaces there exists a function $f \in C_0(S)$ with $0 \leq f \leq 1$ and $f \equiv 1$ on $K$. We set $V \triangleq U_\alpha f$. It is well-known that $V = U_\alpha f \in \mathcal{D}(L)$ and

$$(\alpha \mathbb{1} - L)V = (\alpha \mathbb{1} - L)U_\alpha f = f \geq 0, \tag{2.17}$$

see, e.g. [94, Proposition 6.12]. Thus, $V$ has the first and the third property. One way to see that $V$ also has the second property, is to recall that $T$ is strongly continuous in the origin. An alternative argument is the following: Since $U_\alpha$ is positivity preserving we have $V \geq 0$. For contradiction, assume that $\min_{y \in K} V(y) = 0$. Then, there exists an $x_0 \in K$ such that $V(x_0) = 0$ and we obtain

$$LV(x_0) = \lim_{t \searrow 0} \tfrac{1}{t}\big(T_t V(x_0) - V(x_0)\big) = \lim_{t \searrow 0} \tfrac{1}{t} E_{x_0}\big[V(X_t)\big] \geq 0.$$

Therefore, we conclude from (2.17) that

$$\alpha V(x_0) = f(x_0) + LV(x_0) = 1 + LV(x_0) \geq 1.$$

This is a contradiction and it follows that $V$ has also the second property. □

**Remark 2.1.** *The arguments from the proofs of Theorems 2.1 and 2.2 imply a version of [4, Proposition 3.1] beyond a diffusion setting. More precisely, when $(P_x)_{x \in S}$ is $C_b$-Feller, the following are equivalent:*

(i) $(P_x)_{x \in S}$ *is Feller–Dynkin.*

(ii) *For all compact sets $K \subset S$ and all constants $\alpha > 0$ the function $x \mapsto E_x\big[e^{-\alpha\tau}\big]$ vanishes at infinity, where $\tau$ is defined in* (2.13).

(iii) *For all compact sets $K \subset S$ and all constants $\alpha > 0$ the function $x \mapsto P_x(\tau \leq \alpha)$ vanishes at infinity, where $\tau$ is defined in* (2.13).

*The implication* (i) ⇒ (ii) *is shown in the proof of Theorem 2.2. The implication* (ii) ⇒ (iii) *follows from the inequality*

$$P_x\big(\tau \leq \alpha\big) \leq e^{\alpha^2} E_x\big[e^{-\alpha\tau} \mathbb{1}\{\tau \leq \alpha\}\big] \leq e^{\alpha^2} E_x\big[e^{-\alpha\tau}\big],$$

*and the final implication* (iii) ⇒ (i) *follows from the fact that*

$$P_x\big(X_\alpha \in K\big) \leq P_x\big(\tau \leq \alpha\big)$$

*and the argument in the proof of Theorem 2.1. A version of the equivalence of* (i) *and* (iii) *is also given in [56, Theorem 4.8].*

In some cases Theorem 2.1 and Proposition 2.2 can be combined to one sufficient and necessary Lyapunov-type condition for the Feller–Dynkin property:

**Example 2.2.** *Suppose that $S$ is a countable discrete space and let $Q = (q_{ij})_{i,j\in S}$ be a conservative $Q$-matrix, i.e. $q_{ij} \in \mathbb{R}_+$ for all $i \neq j$ and*

$$-q_{ii} = \sum_{j\neq i} q_{ij} < \infty.$$

*Set $\Sigma \triangleq \Omega, D \triangleq \{f \in C_0(S)\colon Qf \in C_0(S)\}$, and $\mathcal{L} \triangleq Q$, where $Qf$ is defined by*

$$Qf(i) = \sum_{j\in S} q_{ij} f(j). \tag{2.18}$$

*We stress that the r.h.s. of* (2.18) *converges absolutely whenever $f \in C_0(S)$. If $(P_x)_{x\in S}$ is Feller–Dynkin, the corresponding generator $(L, \mathcal{D}(L))$ is given by $(\mathcal{L}, D)$, see [122, Theorem 5]. Thus, when $(P_x)_{x\in S}$ is Markov (or, equivalently, $C_b$-Feller, because of the discrete topology), Theorem 2.1 and Proposition 2.2 imply that the following are equivalent:*

(i) *$(P_x)_{x\in S}$ is Feller–Dynkin.*

(ii) *For each $x \in S$ there exists a function $V\colon S \to \mathbb{R}_+$ such that $V \in D$, $V(x) > 0$, $QV \leq cV$ for a constant $c > 0$.*

*This observation is also contained in [97, Theorem 3.2].*

Under reasonable assumptions on the input data, we can deduce a related equivalence for more general martingale problems. To formulate it we need further terminology. By an extension of the input data $(D, \mathcal{L})$ we mean a pair $(D', \mathcal{L}')$ consisting of $D' \subseteq C(S)$ and $\mathcal{L}'\colon D' \to M(S)$ such that $D \subseteq D'$, $\mathcal{L}' = \mathcal{L}$ on $D$,

$$\int_0^t \big|\mathcal{L}' f(X_s(\omega))\big| ds < \infty$$

for all $t \in \mathbb{R}_+, \omega \in \Omega$ and $f \in D'$, and such that for all $x \in S$ the probability measure $P_x$ solves the MP $(D', \mathcal{L}', \Sigma, x)$.

**Theorem 2.3.** *Suppose that for all $f \in D \cap C_0(S)$ we have $\mathcal{L}f \in C_0(S)$ and that $(P_x)_{x\in S}$ is $C_b$-Feller. Then, the following are equivalent:*

(i) *$(P_x)_{x\in S}$ is Feller–Dynkin.*

(ii) *The input data $(D, \mathcal{L})$ can be extended such that for any compact set $K \subset S$ a Lyapunov function for $K$ in the sense of Theorem 2.1 exists.*

*Proof.* The implication (ii) $\Rightarrow$ (i) is due to Theorem 2.1. Assume that (i) holds, let $(L, \mathcal{D}(L))$ be the generator of $(P_x)_{x\in S}$ and set $D' \triangleq D \cup \mathcal{D}(L)$ and

$$\mathcal{L}' f \triangleq \begin{cases} \mathcal{L} f, & f \in D, \\ L f, & f \in \mathcal{D}(L). \end{cases}$$

Of course, we have to explain that $\mathcal{L}'$ is well-defined, i.e. that $Lf = \mathcal{L}f$ for all $f \in D \cap \mathcal{D}(L)$. Since $\mathcal{L}f \in C_0(S)$ for any $f \in D \cap \mathcal{D}(L)$ by assumption, the process

$$f(X_t) - f(x) - \int_0^t \mathcal{L}f(X_s)ds, \quad t \in \mathbb{R}_+,$$

is a $P_x$-martingale for all $x \in S$, because it is a bounded (on finite time intervals) local $P_x$-martingale. Consequently, [123, Proposition VII.1.7] implies $\mathcal{L}f = Lf$. Due to Dynkin's formula, $P_x$ solves also the MP $(D', \mathcal{L}', \Sigma, x)$ for all $x \in S$. In other words, $(D', \mathcal{L}')$ is an extension of $(D, \mathcal{L})$. Now, (ii) follows from Proposition 2.2. □

Let us comment on the prerequisites of the previous theorem. Even if the coefficients are continuous, in the case of Example 2.1 it is not always true that $\mathcal{L}f \in C_0(\mathbb{R}^d)$ whenever $f \in D \cap C_0(\mathbb{R}^d) = C^2_b(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$. However, if we could replace $D = C^2_b(\mathbb{R}^d)$ by $D = C^2_c(\mathbb{R}^d)$, then $\mathcal{L}f \in C_0(\mathbb{R}^d)$ holds for all $f \in D = D \cap C_0(\mathbb{R}^d)$ provided the coefficients are continuous. In other words, when we could reduce the input data, we would get an equivalent characterization of the Feller–Dynkin property from Theorem 2.3. Next, we explain that a reduction of the input data is often possible.

A sequence $(f_n)_{n\in\mathbb{N}} \subset M(S)$ is said to converge *locally bounded pointwise* to a function $f \in M(S)$ if

(i) $\sup_{n\in\mathbb{N}} \sup_{y\in K} |f_n(y)| < \infty$ for all compact sets $K \subseteq S$;

(ii) $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in S$.

Moreover, we say that $(f_n)_{n\in\mathbb{N}} \subset B(S)$ converges *bounded pointwise* to $f \in M(S)$ if $f_n \to f$ as $n \to \infty$ locally bounded pointwise and $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$.

For a set $A \subseteq C(S) \times M(S)$ we denote by $\mathrm{cl}(A)$ the set of all $(f, g) \in C(S) \times M(S)$ for which there exist sequences $(f_n, g_n)_{n\in\mathbb{N}} \subset A$ such that $f_n \to f$ as $n \to \infty$ bounded pointwise and $g_n \to g$ as $n \to \infty$ locally bounded pointwise. The following proposition can be viewed as an extension of [47, Proposition 4.3.1], which allows a local convergence in the second variable.

**Proposition 2.3.** *Let $D_1, D_2 \subseteq C(S), \mathcal{L}_1 \colon D_1 \to M(S)$ and $\mathcal{L}_2 \colon D_2 \to M(S)$ be such that*

$$\int_0^t \big(\big|\mathcal{L}_1 f(X_s(\omega))\big| + \big|\mathcal{L}_2 g(X_s(\omega))\big|\big)ds < \infty$$

*for all $t \in \mathbb{R}_+, \omega \in \Omega, f \in D_1$ and $g \in D_2$. Suppose that*

$$\{(f, \mathcal{L}_2 f) \colon f \in D_2\} \subseteq \mathrm{cl}(\{(f, \mathcal{L}_1 f) \colon f \in D_1\}). \tag{2.19}$$

*If $P$ is a solution to the MP $(D_1, \mathcal{L}_1, \Sigma, \eta)$, then $P$ is also a solution to the MP $(D_2, \mathcal{L}_2, \Sigma, \eta)$.*

*Proof.* There exists a sequence $(K_n)_{n\in\mathbb{N}} \subset S$ of compact sets such that $K_n \subset \mathrm{int}(K_{n+1})$ and $\bigcup_{n\in\mathbb{N}} K_n = S$. Now, define

$$\tau_n \triangleq \inf\big(t \in \mathbb{R}_+ \colon X_t \not\in \mathrm{int}(K_n) \text{ or } X_{t-} \not\in \mathrm{int}(K_n)\big), \quad n \in \mathbb{N}. \tag{2.20}$$

It is well-known that $\tau_n$ is a stopping time ([47, Proposition 2.1.5]) and it is easy to see that $\tau_n \nearrow \infty$ as $n \to \infty$. Take $f \in D_2$. Due to (2.19) there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset D_1$

such that $f_n \to f$ as $n \to \infty$ bounded pointwise and $\mathcal{L}_1 f_n \to \mathcal{L}_2 f$ as $n \to \infty$ locally bounded pointwise. For $i = 1, 2$ and $g \in D_i$ we set

$$M_t^{g,i} \triangleq g(X_t) - g(X_0) - \int_0^t \mathcal{L}_i g(X_s) ds, \quad t \in \mathbb{R}_+.$$

Since the class of local martingales is stable under stopping, the process $M_{\cdot \wedge \tau_m}^{f_n,1}$ is a local $P$-martingale. Furthermore,

$$\sup_{s \in [0,t]} \big| M_{s \wedge \tau_m}^{f_n,1} \big| \le 2 \sup_{k \in \mathbb{N}} \| f_k \|_\infty + t \sup_{k \in \mathbb{N}} \sup_{y \in K_m} |\mathcal{L}_1 f_k(y)| < \infty,$$

by the definition of (local) bounded pointwise convergence. Consequently, $M_{\cdot \wedge \tau_m}^{f_n,1}$ is a $P$-martingale by the dominated convergence theorem. Since

$$\sup_{s \in [0, t \wedge \tau_m)} |\mathcal{L}_1 f_n(X_{s-})| \le \sup_{k \in \mathbb{N}} \sup_{y \in K_m} |\mathcal{L}_1 f_k(y)| < \infty,$$

the dominated convergence theorem also yields that for any $t \in \mathbb{R}_+$ we have $\omega$-wise $M_{t \wedge \tau_m}^{f_n,1} \to M_{t \wedge \tau_m}^{f,2}$ as $n \to \infty$. Thus, for all $s < t$, applying the dominated convergence theorem a third time yields that $M_{t \wedge \tau_m}^{f,2}, M_{s \wedge \tau_m}^{f,2} \in L^1(P)$ and that for all $G \in \mathcal{F}_s$

$$E^P\big[M_{t \wedge \tau_m}^{f,2} \mathbb{1}_G\big] = \lim_{n \to \infty} E^P\big[M_{t \wedge \tau_m}^{f_n,1} \mathbb{1}_G\big] = \lim_{n \to \infty} E^P\big[M_{s \wedge \tau_m}^{f_n,1} \mathbb{1}_G\big] = E^P\big[M_{s \wedge \tau_m}^{f,2} \mathbb{1}_G\big].$$

In other words, the stopped process $M_{\cdot \wedge \tau_m}^{f,2}$ is a $P$-martingale. Since $\tau_m \nearrow \infty$ as $m \to \infty$, we conclude that $P$ solves the MP $(D_2, \mathcal{L}_2, \Sigma, x)$. □

**Example 2.3.** *In the setting of Example 2.1 we have*

$$\big\{(f, \mathcal{L}f) \colon f \in C_b^2(\mathbb{R}^d)\big\} \subseteq \operatorname{cl}\big(\big\{(f, \mathcal{L}f) \colon f \in C_c^2(\mathbb{R}^d)\big\}\big).$$

*To see this, let $g_n \in C_c^2(\mathbb{R}^d)$ be such that $0 \le g_n \le 1$ and $g_n \equiv 1$ on $\{x \in \mathbb{R}^d \colon \|x\| \le n\}$. For any $f \in C_b^2(\mathbb{R}^d)$ it is easy to verify that $f_n \triangleq f g_n \in C_c^2(\mathbb{R}^d)$, $f_n \to f$ as $n \to \infty$ bounded pointwise and $\mathcal{L} f_n \to \mathcal{L} f$ as $n \to \infty$ locally bounded pointwise. Consequently, a Borel probability measure on $\Omega$ solves the MP $(C_b^2(\mathbb{R}^d), \mathcal{L}, \Sigma, \eta)$ if and only if it solves the MP $(C_c^2(\mathbb{R}^d), \mathcal{L}, \Sigma, \eta)$. This fact is of course well-known, see, e.g. [77, Proposition 5.4.11]. In summary, if the family $(P_x)_{x \in \mathbb{R}^d}$ is unique and $b$ and $a$ are continuous, then $(P_x)_{x \in \mathbb{R}^d}$ is $C_b$-Feller (see [137, Corollary 11.1.5]) and Theorem 2.3 implies that the following are equivalent:*

(i) $(P_x)_{x \in \mathbb{R}^d}$ *is Feller–Dynkin.*

(ii) *The input data $(C_c^2(\mathbb{R}^d), \mathcal{L})$ can be extended such that for all compact sets $K \subset \mathbb{R}^d$ a Lyapunov function for $K$ (in the sense of Theorem 2.1) exists.*

The larger the set $D$, the easier it is to find a suitable Lyapunov function and to apply Theorems 2.1 and 2.2. Thus, when we have applications in mind, we would like to choose $D$ as large as possible. We stress that Proposition 2.3 also works in this direction, i.e. it gives a condition such that $D$ can be enlarged.

Proposition 2.3 can also be used to verify the prerequisites of Proposition 2.1 as the following example shows.

**Example 2.4.** *Suppose that*

$$A \triangleq \big\{(f, \mathcal{L}f) \colon f \in D\big\} \subseteq C_0(S) \times C_0(S).$$

*As $C_0(S)$ endowed with the uniform metric is a separable metric space, the space $A$ is a separable metric space endowed with the metric $d$ given by*

$$d((f_1, g_1), (f_2, g_2)) \triangleq \|f_1 - f_2\|_\infty + \|g_1 - g_2\|_\infty, \quad (f_1, g_1), (f_2, g_2) \in A.$$

*Consequently, we find a countable set $C \subseteq D$ such that for any $(f, g) \in A$ there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset C$ with*

$$d((f_n, \mathcal{L}f_n), (f, g)) \to 0$$

*as $n \to \infty$. Now, Proposition 2.3 implies that a Borel probability measure on $\Omega$ solves the MP $(D, \mathcal{L}, \Sigma, \eta)$ if and only if it solves the MP $(C, \mathcal{L}, \Sigma, \eta)$.*

## 2.4 The Feller–Dynkin Property of Switching Diffusions

In this section we derive Khasminskii-type integral tests for the Feller–Dynkin property of diffusions with random switching. Moreover, we give an equivalent characterization for the state-independent case and present equivalent integral-type conditions for the Feller–Dynkin property for one dimensional state-independent switching diffusions.

Before we start our program, we fix some notation. Let $S_d$ be a countable discrete space and let $S \triangleq \mathbb{R}^d \times S_d$ equipped with the product topology. Take the following coefficients:

(i) $b \colon S \to \mathbb{R}^d$ being Borel and locally bounded.

(ii) $a \colon S \to \mathbb{S}^d$ being Borel and locally bounded.

(iii) For each $x \in \mathbb{R}^d$, let $Q(x) = (q_{ij}(x))_{i,j\in S_d}$ be a conservative $Q$-matrix (see Example 2.2 for a definition), such that the map $x \mapsto Q(x)$ is Borel.

### 2.4.1 Conditions for the Feller–Dynkin Property

For $i, j \in S_d$, we set

$$\overline{q}_{ij} \triangleq \begin{cases} \sup_{x\in\mathbb{R}^d} q_{ij}(x), & i \neq j, \\ -\sum_{k\neq i} \overline{q}_{ik}, & i = j. \end{cases}$$

In this section, we impose the following standing assumption.

**Standing Assumption 2.2.** *For all $i \in S_d$ we have $|\overline{q}_{ii}| < \infty$ and*

$$\sup_{j\in S_d} \sup_{x\in\mathbb{R}^d} |q_{jj}(x) - \overline{q}_{jj}| < \infty. \tag{2.21}$$

Set $\overline{Q} \triangleq (\overline{q}_{ij})_{i,j\in S_d}$ and note that $\overline{Q}$ is a conservative $Q$-matrix. Furthermore, we set

$$C \triangleq \big\{f \in C_0(S_d) \colon \overline{Q}f \in C_0(S_d)\big\},$$
$$\Sigma_d \triangleq \big\{\omega \colon \mathbb{R}_+ \to S_d \colon t \mapsto \omega(t) \text{ is càdlàg}\big\}.$$

We also impose the following standing assumption.

**Standing Assumption 2.3.** *For all $i \in S_d$ the MP $(C, \overline{Q}, \Sigma_d, i)$ has a unique solution $P_i^d$ such that the family $(P_i^d)_{i\in S_d}$ is Feller–Dynkin. Here, the state space for the MP is assumed to be $S_d$.*

If $|S_d| < \infty$ this standing assumption holds. In the following remark we collect also some conditions when the previous standing assumption holds for the case $|S_d| = \infty$.

**Remark 2.2.** *(i) Conditions for the existence of $(P_i^d)_{i\in S_d}$ can be found in [2, Corollary 2.2.5, Theorem 2.2.27] and [20, Theorem 16]. If, in addition to one of these conditions, we have*

$$\forall \lambda > 0, k \in S_d, \quad \{y \in l_1 \colon y(\lambda \mathbb{1} - \overline{Q}) = 0\} = \{0\} \text{ and } \overline{q}_{\cdot k} \in C_0(S_d), \tag{2.22}$$

*then $(P_i^d)_{i\in S_d}$ is Feller–Dynkin, see [122, Theorem 8]. Here, $l_1$ denotes the space of all functions $f \colon S_d \to \mathbb{R}$ with $\sum_{i\in S_d} |f(i)| < \infty$.*

*(ii) If $\sup_{n\in S_d} |\overline{q}_{nn}| < \infty$, then $(P_i^d)_{i\in S_d}$ exists, see [2, Corollary 2.2.5, Proposition 2.2.9], and $\{y \in l_1 \colon y(\lambda \mathbb{1} - \overline{Q}) = 0\} = \{0\}$ holds for all $\lambda > 0$, see [122, pp. 273]. In this case, the second part of (2.22) is necessary and sufficient for $(P_i^d)_{i\in S_d}$ to be Feller–Dynkin, see [122, Theorem 9].*

*(iii) If $S_d = \{0, 1, 2, \dots\}$ and $\overline{q}_{ij} = 0$ for all $i \geq j + 2$, then [96, Proposition 2] yields that the following are equivalent:*

*(a)* $\{y \in l_1 \colon y(\lambda \mathbb{1} - \overline{Q}) = 0\} = \{0\}$.

*(b)* $\{y \in l_1^+ \colon y(\lambda \mathbb{1} - \overline{Q}) = 0\} = \{0\}$.

*Part (b) is necessary for $(P_i^d)_{i\in S_d}$ to be Feller–Dynkin, see [122, Theorem 7]. Here, $l_1^+$ denotes the set of all non-negative $f \in l_1$.*

For reader's convenience we recall our notation: $\Omega$ denotes the space of all càdlàg functions $\mathbb{R}_+ \to S$ equipped with the Skorokhod topology, $\mathcal{F}$ is the corresponding Borel $\sigma$-field, $(X_t)_{t\geq 0}$ is the coordinate process on $\Omega$ and $D \subseteq C(S)$ is a set of test functions.

We suppose that

$$\big\{f, fg, g \colon f \in C_b^2(\mathbb{R}^d), g \in C\big\} \subseteq D,$$

and set

$$\Sigma \triangleq \big\{(\omega^1, \omega^2) \in \Omega \colon \omega^1 \colon \mathbb{R}_+ \to \mathbb{R}^d \text{ is continuous}\big\}$$

and

$$\mathcal{L}f(x, i) \triangleq \mathcal{K}f(x, i) + \sum_{j\in S_d} q_{ij}(x) f(x, j), \quad (x, i) \in S, \tag{2.23}$$

where

$$\mathcal{K}f(x, i) \triangleq \langle \nabla_x f(x, i), b(x, i)\rangle + \tfrac{1}{2}\text{trace}\,(\nabla_x^2 f(x, i) a(x, i)), \quad (x, i) \in S.$$

In the proof of Lemma 2.11 below we explain that $\Sigma$ is closed, which yields $\Sigma \in \mathcal{F}$. Recalling the Standing Assumption in Section 2.2.1, we assume that for each $x \in S$ there exists a solution $P_x$ to the MP $(D, \mathcal{L}, \Sigma, x)$.

By our assumption that $(P_x^d)_{x\in S_d}$ is Feller–Dynkin, due to Proposition 2.2 (see also Example 2.2), for any compact subset of $S_d$ there exists a Lyapunov function (in the sense of Theorem 2.1) for $(P^d_x)_{x\in S_d}$. We will combine these Lyapunov functions with Lyapunov functions for the diffusion part, which we can define under each of the following two conditions.

**Condition 2.1.** *There exist two locally Hölder continuous functions* $a_d\colon [\frac{1}{2},\infty)\to(0,\infty)$ *and* $b_d\colon[\frac{1}{2},\infty)\to\mathbb{R}$ *such that*

$$\langle x, a(x,i)x\rangle \le a_d\left(\tfrac{\|x\|^2}{2}\right),$$
$$\operatorname{trace} a(x,i) + 2\langle x, b(x,i)\rangle \ge b_d\left(\tfrac{\|x\|^2}{2}\right)\langle x, a(x,i)x\rangle$$

*for all* $i\in S_d$ *and* $x\in\mathbb{R}^d\colon \|x\|\ge 1$. *Moreover, either*

$$p(r) \triangleq \int_1^r \exp\left(-\int_1^y b_d(z)dz\right)dy, \quad \lim_{r\to\infty} p(r) < \infty, \tag{2.24}$$

*or*

$$\lim_{r\to\infty} p(r) = \infty \text{ and } \int_1^\infty p'(y)\int_y^\infty \frac{dz}{a_d(z)p'(z)}dy = \infty. \tag{2.25}$$

*Furthermore, we have*

$$\sup_{j\in S_d}\sup_{\|x\|\le 1}\left(\|b(x,j)\| + \operatorname{trace} a(x,j)\right) < \infty. \tag{2.26}$$

**Condition 2.2.** *There exists a constant* $\beta>0$ *such that*

$$\|b(x,i)\| \le \beta(1+\|x\|), \quad \operatorname{trace} a(x,i) \le \beta(1+\|x\|^2),$$

*for all* $(x,i)\in S$.

**Proposition 2.4.** *If the family* $(P_x)_{x\in S}$ *is* $C_b$*-Feller and one of the Conditions 2.1 and 2.2 holds, then* $(P_x)_{x\in S}$ *is also Feller–Dynkin.*

*Proof.* We assume that Condition 2.1 holds. Fix an arbitrary compact set $K\subset S$. Since the projections $\pi_1\colon S\to\mathbb{R}^d$ and $\pi_2\colon S\to S_d$ are continuous for the product topology, the sets $\pi_1(K)$ and $\pi_2(K)$ are compact and $K\subseteq \pi_1(K)\times\pi_2(K)$.

Since we assume the family $(P^d_x)_{x\in S_d}$ to be Feller–Dynkin, Proposition 2.2 (see also Example 2.2) implies that there exists a function $\zeta\colon S_d\to\mathbb{R}_+$ such that $\zeta\in C, \zeta>0$ on $\pi_2(K)$ and $\overline{Q}\zeta\le c\zeta$ for a constant $c>0$. Applying the change of variable as explained in [4, Section 4.1] together with [4, Lemma 4.2], we obtain that there exists a twice continuously differentiable decreasing solution $u\colon[\frac{1}{2},\infty)\to(0,\infty)$ to the differential equation

$$\tfrac{1}{2}a_d b_d u' + \tfrac{1}{2}a_d u'' = u, \quad u\left(\tfrac{1}{2}\right) = 1, \tag{2.27}$$

which satisfies $\lim_{x\nearrow+\infty} u(x) = 0$. For the last property we require that either (2.24) or (2.25) holds. We find a twice continuously differentiable function $\phi\colon[0,\infty)\to(0,\infty)$ such that $\phi\ge 1$ on $[0,\frac{1}{2}]$ and $\phi=u$ on $(\frac{1}{2},\infty)$. Now, we define

$$V(x,i) \triangleq \phi\left(\tfrac{\|x\|^2}{2}\right)\zeta(i), \quad (x,i)\in S.$$

We see that $V \geq 0$, $V \in D$ and that $V > 0$ on $K$ and one readily checks that $V \in C_0(S)$. It remains to show that $\mathcal{L}V \leq \text{const.}\ V$. For all $i \in S_d$ and $x \in \mathbb{R}^d\colon \|x\| > 1$ we have

$$\begin{aligned}\mathcal{K}V(x,i) &= \zeta(i)\tfrac{1}{2}\Big(\langle x, a(x,i)x\rangle u''\left(\tfrac{\|x\|^2}{2}\right) + (\operatorname{trace} a(x,i) + 2\langle x, b(x,i)\rangle)\, u'\left(\tfrac{\|x\|^2}{2}\right)\Big)\\ &\leq \zeta(i)\tfrac{\langle x,a(x,i)x\rangle}{2}\left(u''\left(\tfrac{\|x\|^2}{2}\right) + b_d\left(\tfrac{\|x\|^2}{2}\right)u'\left(\tfrac{\|x\|^2}{2}\right)\right),\end{aligned}$$

where we used that $u$ is decreasing, i.e. that $u' \leq 0$. Due to (2.27), we have

$$u'' + b_d u' = \tfrac{2u}{a_d} \geq 0.$$

Thus, we obtain

$$\mathcal{K}V(x,i) \leq \zeta(i)\tfrac{1}{2}a_d\left(\tfrac{\|x\|^2}{2}\right)\left(u''\left(\tfrac{\|x\|^2}{2}\right) + b_d\left(\tfrac{\|x\|^2}{2}\right)u'\left(\tfrac{\|x\|^2}{2}\right)\right) = V(x,i) \tag{2.28}$$

for all $i \in S_d$ and $x \in \mathbb{R}^d\colon \|x\| > 1$. Due to (2.26), we find a constant $c^* \geq 1$ such that $\mathcal{K}V(x,i) \leq c^*\zeta(i) \leq c^*V(x,i)$ for all $i \in S_d$ and $x \in \mathbb{R}^d\colon \|x\| \leq 1$. In summary, using (2.21) and (2.28), we obtain

$$\begin{aligned}\mathcal{L}V(x,i) &\leq c^*V(x,i) + \Big(\sum_{j\neq i} q_{ij}(x)\zeta(j) + q_{ii}(x)\zeta(i)\Big)\phi\left(\tfrac{\|x\|^2}{2}\right)\\ &\leq c^*V(x,i) + \Big(\sum_{j\in S_d} \overline{q}_{ij}\zeta(j) + (q_{ii}(x) - \overline{q}_{ii})\zeta(i)\Big)\phi\left(\tfrac{\|x\|^2}{2}\right)\\ &\leq \Big(c^* + c + \sup_{j\in S_d}\sup_{y\in\mathbb{R}^d}|q_{jj}(y) - \overline{q}_{jj}|\Big)V(x,i) = \text{const.}\ V(x,i).\end{aligned}$$

Consequently, Theorem 2.1 implies the claim.

For the case where Condition 2.2 holds, we only have to replace $\phi(x)$ by $(1+2x)^{-1}$. The remaining argument stays unchanged. We omit the details. □

Conditions for the $C_b$-Feller property of $(P_x)_{x\in S}$ can be found in [112, 131, 139, 142]. We collect some of these in the following corollary, where we also assume that

$$D \equiv \big\{f\colon S \to \mathbb{R}\colon x \mapsto f(x,j) \in C_b^2(\mathbb{R}^d), i \mapsto f(y,i) \in B(S_d) \text{ for all } (y,j) \in S\big\}.$$

**Corollary 2.1.** *Suppose the following:*

(i) $S_d = \{0, 1, \ldots, N\}$ *for* $1 \leq N \leq \infty$*, where we mean* $S_d = \mathbb{N}_0$ *when* $N = \infty$.

(ii) *There exists a constant* $c_1 > 0$ *such that for all* $(x,i) \in S$ *we have* $q_{ij}(x) = 0$ *for all* $j \in S_d$ *with* $|j - i| > c_1$.

(iii) *There exits a constant* $c_2 > 0$ *such that for all* $i \in S_d$

$$\sup_{x\in\mathbb{R}_d}|q_{ii}(x)| \leq c_2(i+1).$$

(iv) *There exists a constant* $c_3 > 0$ *such that for all* $i \in S_d$ *and* $x, y \in \mathbb{R}^d$

$$\sum_{j\neq i}|q_{ij}(x) - q_{ij}(y)| \leq c_3\|x - y\|.$$

(v) *Condition 2.2 holds and there exists a constant* $c_4 > 0$ *and a root* $a^{\frac{1}{2}}$ *of* $a$ *such that for all* $i \in S_d$ *and* $x, y \in \mathbb{R}^d$

$$\|b(x,i) - b(x,i)\| + \|a^{\frac{1}{2}}(x,i) - a^{\frac{1}{2}}(y,i)\| \leq c_4\|x-y\|.$$

*Then, a Feller–Dynkin family* $(P_x)_{x\in S}$ *exists.*

*Proof.* The existence of a family $(P_x)_{x\in S}$ follows from [139, Theorem 2.1]. Furthermore, [139, Theorem 3.3] yields that $(P_x)_{x\in S}$ is $C_b$-Feller. Thus, Proposition 2.4 implies that $(P_x)_{x\in S}$ is Feller–Dynkin, too. □

**Remark 2.3.** (i) *Assumption (ii) in Corollary 2.1 can be replaced by a weaker condition of Lyapunov-type, see [139, Assumption 1.2].*

(ii) *In general, the conditions from Corollary 2.1 do not imply the strong Feller property of* $(P_x)_{x\in S}$. *For example, it is allowed to take the first coordinate as linear motion, which gives a process without the strong Feller property.*

*If, in addition to (i) – (v) in Corollary 2.1, we assume that there exists a constant* $c > 0$ *such that for all* $(x,i) \in S$ *and* $y \in \mathbb{R}^d$

$$\langle y, a(x,i)y\rangle \geq c\|y\|^2,$$

*then [131, Theorem 3.1] implies that* $(P_x)_{x\in S}$ *has the strong Feller property, too. In this case,* $(P_x)_{x\in S}$ *has the* $C_b$*-Feller, the strong Feller and the Feller–Dynkin property.*

The following example illustrates that our results include cases where $Q$ is unbounded.

**Example 2.5.** *Suppose that* $\overline{Q}$ *corresponds to a classical birth-death chain, i.e.* $S_d \triangleq \{0,1,2,\dots\}$ *and*

$$\overline{q}_{ij} \triangleq \begin{cases} \lambda_i, & j = i+1, i \geq 0, \\ \mu_i, & j = i-1, i \geq 1, \\ -(\lambda_i + \mu_i), & i = j, i \geq 0, \\ 0, & \textit{otherwise}, \end{cases}$$

*for strictly positive sequences* $(\lambda_n)_{n\in\mathbb{N}}$ *and* $(\rho_n)_{n\in\mathbb{N}}$ *and* $\mu_0 = 0$ *and* $\lambda_0 > 0$. *Set*

$$r \triangleq \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n\lambda_{n-1}} + \frac{\mu_n\mu_{n-1}}{\lambda_n\lambda_{n-1}\lambda_{n-2}} + \dots + \frac{\mu_n\cdots\mu_2}{\lambda_n\cdots\lambda_2\lambda_1} \right),$$
$$s \triangleq \sum_{n=1}^{\infty} \frac{1}{\mu_{n+1}} \left( 1 + \frac{\lambda_n}{\mu_n} + \frac{\lambda_n\lambda_{n-1}}{\mu_n\mu_{n-1}} + \dots + \frac{\lambda_n\lambda_{n-1}\cdots\lambda_2\lambda_1}{\mu_n\mu_{n-1}\cdots\mu_2\mu_1} \right).$$

*If* $r = s = \infty$ *it is well-known that a Feller–Dynkin family* $(P^d_i)_{i\in S_d}$ *exists, see [2, Theorems 3.2.2, 3.2.3] and Remark 2.2 (i) and (iii). In this case, if also one of the Conditions 2.1 and 2.2 holds, the family* $(P_x)_{x\in S}$ *is Feller–Dynkin whenever it is* $C_b$*-Feller. To be more concrete, if we choose*

$$\lambda_n \triangleq n^\alpha\lambda, \quad \mu_n \triangleq n^\alpha\mu, \quad \alpha \geq 0, \lambda, \mu > 0,$$

*then $s = r = \infty$ if and only if either $\alpha \le 1$ or [$\alpha \in (1,2]$ and $\lambda = \mu$]. In other words, we find coefficients $a, b$ and $Q$ which satisfy the conditions from Corollary 2.1 with an unbounded $Q$.*

### 2.4.2 Conditions *not* to be Feller–Dynkin

Next, we give conditions for rejecting the Feller–Dynkin property under the following standing assumption.

**Standing Assumption 2.4.** *$|S_d| < \infty$.*

Let $\Sigma$ and $\mathcal{L}$ be as in Section 2.4.1 and define

$$D \triangleq \big\{ f, fg, g \colon f \in C_b^2(\mathbb{R}^d), g \colon S_d \to \mathbb{R} \big\}.$$

**Proposition 2.5.** *Assume that there exist an $r > 0$ and two locally Hölder continuous functions $b_d \colon [r, \infty) \to \mathbb{R}$ and $a_d \colon [r, \infty) \to (0, \infty)$ such that for all $i \in S_d$ and $x \in \mathbb{R}^d \colon \|x\| \ge 2r$*

$$\langle x, a(x,i)x \rangle \ge a_d\left(\tfrac{\|x\|^2}{2}\right),$$
$$\operatorname{trace} a(x,i) + 2\langle x, b(x,i)\rangle \le b_d\left(\tfrac{\|x\|^2}{2}\right) \langle x, a(x,i)x\rangle,$$

*and*

$$p(t) \triangleq \int_{r+1}^{t} \exp\left( - \int_{r+1}^{y} b_d(z)dz \right) dy \to \infty \text{ as } t \to \infty,$$

*and*

$$\int_{r+1}^{\infty} p'(y) \int_{y}^{\infty} \frac{dz}{a_d(z)p'(z)} dy < \infty.$$

*Then $(P_x)_{x \in S}$ is not Feller–Dynkin.*

*Proof.* Applying the change of variable as explained in [4, Section 4.1] together with [4, Lemma 4.2], we obtain that there exists a twice continuously differentiable decreasing solution $u \colon [r, \infty) \to (0, \infty)$ to the differential equation

$$\tfrac{1}{2} a_d b_d u' + \tfrac{1}{2} a_d u'' = u, \quad u(r) = 1,$$

which satisfies $\lim_{x \nearrow +\infty} u(x) > 0$. We find a twice continuously differentiable function $\phi \colon [0, \infty) \to (0, \infty)$ such that $\phi \ge 1$ on $[0, r]$ and $\phi = u$ on $(r, \infty)$. It follows similarly to the proof of Proposition 2.4 that

$$U(x,i) \triangleq \phi\left(\tfrac{\|x\|^2}{2}\right), \quad (x,i) \in S,$$

has the properties from Theorem 2.2 for the compact sets $C \equiv K \triangleq \{x \in \mathbb{R}^d \colon \|x\| \le \sqrt{2r}\} \times S_d$, which implies the claim. $\square$

### 2.4.3 Equivalent Characterization for the State-Independent Case

In this section we study the state-independent case and characterize the Feller–Dynkin property via the Feller–Dynkin property of diffusions in fixed environments.

#### 2.4.3.1 The Setup

We impose the following:

**Standing Assumption 2.5.** *We have $S_d = \{1, \dots, N\}$ for $1 \leq N \leq \infty$, $Q(x) \equiv Q$ and there exists a continuous-time Markov chain with Q-matrix $Q$. For us a Markov chain is always non-explosive. We denote its unique law by $(P^\star_i)_{i \in S_d}$, where the subscript indicates the starting value. Furthermore, $(P^\star_i)_{i \in S_d}$ is Feller–Dynkin.*

From now on we fix a root $a^{\frac{1}{2}}$ of $a$. Let $\mathcal{L}$ and $\Sigma$ be as in Section 2.4.1 and set

$$D \triangleq \big\{ f, fg, g \colon f \in C^2_b(\mathbb{R}^d), g \in C \big\}, \quad C \triangleq \big\{ g \in C_0(S_d) \colon Qf \in C_0(S_d) \big\}. \tag{2.29}$$

Due to [122, Theorem 5], $(Q, C)$ is the generator of $(P^\star_i)_{i \in S_d}$ and consequently, for each $i \in S_d$ the probability measure $P^\star_i$ is the unique solution to the MP $(C, Q, \Sigma_d, i)$. It seems to be known that the family $(P_x)_{x \in S}$ has a one-to-one relation to a switching diffusion defined via an SDE, see, for instance, [8] for a partial result in this direction. However, we did not find a complete reference, such that we provide a statement and a proof.

**Lemma 2.2.** *Fix $y = (x, i) \in S$. A probability measure $P_y$ solves the MP $(D, \mathcal{L}, \Sigma, y)$ if and only if there exists a filtered probability space with right-continuous complete filtration $(\mathcal{G}_t)_{t \geq 0}$ which supports a Markov chain $Z$ for the filtration $(\mathcal{G}_t)_{t \geq 0}$ with Q-matrix $Q$ and initial value $Z_0 = i$ and a continuous, $(\mathcal{G}_t)_{t \geq 0}$-adapted process $Y$ with dynamics*

$$dY_t = b(Y_t, Z_t)dt + a^{\frac{1}{2}}(Y_t, Z_t)dW_t, \quad Y_0 = x, \tag{2.30}$$

*where $W$ is a Brownian motion for the filtration $(\mathcal{G}_t)_{t \geq 0}$ such that the law of $(Y, Z)$ is given by $P_y$ and the $\sigma$-fields $\sigma(W_t, t \in \mathbb{R}_+)$ and $\sigma(Z_t, t \in \mathbb{R}_+)$ are independent.*

*Proof.* The implication $\Leftarrow$ is a consequence of the integration by parts formula.

It remains to show the implication $\Rightarrow$. We consider the completion of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P_y)$ as underlying filtered probability space. Denote $X = (Y, Z)$, where $Y$ is $\mathbb{R}^d$-valued and $Z$ is $S_d$-valued. In view of [77, Remark 5.4.12], we can argue as in the proof of [77, Proposition 5.4.6] to conclude the existence of a Brownian motion $W$ (possibly defined on a standard extension of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P_y)$, see [77, Remark 3.4.1]) such that $Y$ satisfies the SDE (2.30). With abuse of notation, we denote the standard extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P_y)$ again by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P_y)$. Due to [65, Proposition 10.46] the martingale property is not affected by a standard extension. Thus, we deduce from Examples 2.2 and 2.4, Proposition 2.9 in Section 2.6 and [47, Theorem 4.4.2] that $Z$ is a Markov chain for the filtration $(\mathcal{F}_t)_{t \geq 0}$ with $Q$-matrix $Q$ and $Z_0 = i$. It remains to explain that the $\sigma$-fields $\sigma(W_t, t \in \mathbb{R}_+)$ and $\sigma(Z_t, t \in \mathbb{R}_+)$ are independent. We adapt an idea from [47, Theorem 4.10.1]. For all $f \in C$ the process

$$M^f_t \triangleq f(Z_t) - f(i) - \int_0^t Qf(Z_s)ds, \quad t \in \mathbb{R}_+,$$

is a $P_y$-martingale. For $g \in C_b^2(\mathbb{R}^d)$ with $\inf_{x\in\mathbb{R}^d} g(x) > 0$ set

$$K_t^g \triangleq g(W_t) \exp\Big( -\frac{1}{2}\int_0^t \frac{\Delta g(W_s)}{g(W_s)} ds\Big), \quad t \in \mathbb{R}_+,$$

where $\Delta$ denotes the Laplacian. Itô's formula yields that

$$dK_t^g = \exp\Big( -\frac{1}{2}\int_0^t \frac{\Delta g(W_s)}{g(W_s)} ds\Big) \langle \nabla g(W_t), dW_t\rangle,$$

which implies that also $K^g$ is a $P_y$-martingale, because it is a bounded (on finite time intervals) local $P_y$-martingale. As $Z$ has only finitely many jumps in a finite interval, $M^f$ is of finite variation on finite intervals and we have $P_y$-a.s.

$$[M^f, K^g]_t = 0 \text{ for all } t \in \mathbb{R}_+,$$

see [70, Proposition I.4.49]. Here, $[\cdot,\cdot]$ denotes the quadratic variation process. Consequently, integration by parts yields that $M^f K^g$ is a local $P_y$-martingale and a true $P_y$-martingale due to its boundedness on finite time intervals. Fix an arbitrary bounded stopping time $\psi$ and define

$$Q(G) \triangleq \frac{E_y\big[\mathbb{1}_G K^g_\psi\big]}{g(0)}, \quad G \in \mathcal{F}.$$

Due to the optional stopping theorem, for all bounded stopping times $\phi$ we have

$$E^Q\big[M^f_\phi\big] = \frac{E_y\big[M^f_{\phi\wedge\psi} K^g_{\phi\wedge\psi}\big]}{g(0)} = 0.$$

We conclude from [123, Proposition II.1.4] that $M^f$ is a $Q$-martingale. Consequently, in view of Example 2.2, we have

$$Q(\Gamma) = P_y(\Gamma),$$

where

$$\Gamma \triangleq \big\{Z_{t_1} \in F_1, \dots, Z_{t_n} \in F_n\big\}$$

for arbitrary $0 \le t_1 < \dots < t_n < \infty$ and $F_1, \dots, F_n \in \mathcal{B}(S_d)$. Suppose that $P_y(\Gamma) > 0$ and set

$$\widehat{Q}(G) \triangleq \frac{E_y\big[\mathbb{1}_G \mathbb{1}_\Gamma\big]}{P_y(\Gamma)}, \quad G \in \mathcal{F}.$$

We have

$$E^{\widehat{Q}}\big[K^g_\psi\big] = \frac{E_y\big[K^g_\psi \mathbb{1}_\Gamma\big]}{P_y(\Gamma)} = \frac{Q(\Gamma) g(0)}{P_y(\Gamma)} = g(0).$$

Thus, because $\psi$ was arbitrary, we deduce from [123, Proposition II.1.4] and [47, Proposition 4.3.3] that $W$ is a $\widehat{Q}$-Brownian motion and the uniqueness of the Wiener measure yields that

$$\widehat{Q}\big(W_{s_1} \in G_1, \dots, W_{s_k} \in G_k\big) = P_y\big(W_{s_1} \in G_1, \dots, W_{s_k} \in G_k\big)$$

for arbitrary $0 \le s_1 < \dots < s_k < \infty$ and $G_1, \dots, G_k \in \mathcal{B}(\mathbb{R}^d)$. Using the definition of $\widehat{Q}$, we conclude that

$$\begin{aligned}P_y\big(Z_{t_1} \in F_1, \dots, Z_{t_n} \in F_s, W_{s_1} \in G_1, \dots, W_{s_k} \in G_k\big)\\ = P_y\big(Z_{t_1} \in F_1, \dots, Z_{t_n} \in F_s\big) P_y\big(W_{s_1} \in G_1, \dots, W_{s_k} \in G_k\big),\end{aligned}$$

which implies the desired independence. □

**Remark 2.4.** *An inspection of the proof of Lemma 2.2 shows the following:*

(i) *If $Z$ is a Feller–Dynkin Markov chain and $W$ is a Brownian motion both with deterministic initial values and for the same filtration, then the $\sigma$-fields $\sigma(W_t, t \in \mathbb{R}_+)$ and $\sigma(Z_t, t \in \mathbb{R}_+)$ are independent.*

(ii) *As explained in Example 2.4, we find a countable set $C^\star \subseteq C$ such that for all $f \in C$ there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset C^\star$ such that*

$$\|f - f_n\|_\infty + \|Qf - Qf_n\|_\infty \to 0$$

*as $n \to \infty$. The set of solutions to the MP $(D, \mathcal{L}, \Sigma, y)$ remains unchanged if we redefine $D$ to be the countable set*

$$\big\{f, g^k_{ij}, g^k_i \colon 1 \le i, j \le d, k \in \mathbb{N}, f \in C^\star\big\}, \tag{2.31}$$

*where $g^k_i, g^k_{ij}$ are functions in $C^2_c(\mathbb{R}^d)$ such that $g^k_i(x) = x_i$ and $g^k_{ij}(x) = x_i x_j$ for all $x \in \mathbb{R}^d \colon \|x\| \le k$.*

We set

$$\Sigma_c \triangleq \big\{\omega \colon \mathbb{R}_+ \to \mathbb{R}^d \colon t \mapsto \omega(t) \text{ is continuous}\big\},$$

and

$$\mathcal{K}^i f(x) \triangleq \langle \nabla f(x), b(x,i)\rangle + \tfrac{1}{2}\text{trace}\,(\nabla^2 f(x) a(x,i)) \tag{2.32}$$

for $f \in C^2_b(\mathbb{R}^d)$ and $(x,i) \in S$. We equip $\Sigma_c$ with the local uniform topology. In this case the Borel $\sigma$-field is generated by the coordinate process on $\Sigma_c$, see [137, p. 30]. A map $F \colon \mathbb{R}^d \times \Sigma_c \to \Sigma_c$ is called universally adapted, if it is adapted to the filtration $(\bigcap_{\mu\in\mathcal{P}} \mathcal{G}^\mu_t)_{t\ge0}$, where $\mathcal{P}$ is the set of all Borel probability measures on $\mathbb{R}^d$ and $(\mathcal{G}^\mu_t)_{t\ge0}$ is the completion of the canonical filtration on $\mathbb{R}^d \times \Sigma_c$ w.r.t. the product measure $\mu \otimes \mathscr{W}$, where $\mathscr{W}$ is the Wiener measure, see [74, p. 346].

**Definition 2.2.** *A family $(P^i_x)_{x\in\mathbb{R}^d}$ of solutions to the MP $(C^2_b(\mathbb{R}^d), \mathcal{K}^i, \Sigma_c)$ is said to exist strongly, if a universally adapted Borel map $F^i \colon \mathbb{R}^d \times \Sigma_c \to \Sigma_c$ exists such that on every filtered probability space with right-continuous complete filtration $(\mathcal{G}_t)_{t\ge0}$, which supports a Brownian motion $W$ and an $\mathbb{R}^d$-valued $\mathcal{G}_0$-measurable random variable $\pi$, the process $F^i(\pi, W)$ solves the SDE*

$$dY^i_t = b(Y^i_t, i)dt + a^{\frac{1}{2}}(Y^i_t, i)dW_t, \quad Y^i_0 = \pi, \tag{2.33}$$

*and every solution $Y^i$ to (2.33) satisfies $Y^i = F^i(\pi, W)$ up to a null set. Here, the state space for the MP is $\mathbb{R}^d$.*

**Remark 2.5.** *We stress that our definition of strong existence includes a version of pathwise uniqueness and that the function $F^i$ in the previous definition is independent* *of the law of $\pi$. A generalization of the classical Yamada–Watanabe theorem shows that $(P^i_x)_{x\in\mathbb{R}^d}$ exists strongly if and only if the SDE* (2.33) *satisfies weak existence and pathwise uniqueness for all degenerated initial values, see [74, Theorem 18.14]. In the classical formulation of the Yamada–Watanabe theorem as given, for instance, in [77] the function $F^i$ depends on the law of $\pi$. This dependence was removed in [73].*

#### 2.4.3.2 Main Results

Next, we state the main results for this section. The proofs can be found in the following subsections.

**Condition 2.3.** *We have $q_{ii} \neq 0$ for all $i \in S_d$.*

**Condition 2.4.** *The family $(P_y)_{y\in S}$ is unique and $C_b$-Feller, and for all $(x, i) \in S$ the MP $(C^2_b(\mathbb{R}^d), \mathcal{K}^i, \Sigma_c, x)$, where $\mathcal{K}^i$ is given as in* (2.32)*, has a unique solution $P^i_x$. Furthermore, for all $i \in S_d$ the family $(P^i_x)_{x\in\mathbb{R}^d}$ is $C_b$-Feller and exists strongly.*

The following observation is the main result of this section.

**Theorem 2.4.** *Suppose that the Conditions 2.3 and 2.4 hold. The following are equivalent:*

(i) *The family $(P_y)_{y\in S}$ is Feller–Dynkin.*

(ii) *For all $i \in S_d$ the family $(P^i_x)_{x\in\mathbb{R}^d}$ is Feller–Dynkin.*

For the strong Feller property a related result is known, see [131, Theorem 3.2]. One implication in the previous theorem can be generalized as the following proposition shows.

**Proposition 2.6.** *Suppose that there exists an $i \in S_d$ such that for all $x \in \mathbb{R}^d$ the MP $(C^2_b(\mathbb{R}^d), \mathcal{K}^i, \Sigma_c, x)$ has a (unique) solution $P^i_x$ and that the family $(P^i_x)_{x\in\mathbb{R}^d}$ exists strongly and is $C_b$-Feller, but not Feller–Dynkin. Then, $(P_x)_{x\in S}$ is not Feller–Dynkin.*

The next two results provide conditions implying Condition 2.4.

**Proposition 2.7.** *Suppose that $b$ and $a$ are continuous and that $(P_y)_{y\in S}$ is unique, then $(P_y)_{y\in S}$ is $C_b$-Feller. In particular, $(P_y)_{y\in S}$ is strongly Markov.*

**Proposition 2.8.** *Suppose that Condition 2.3 holds and that for all $i \in S_d$ the family $(P^i_x)_{x\in\mathbb{R}^d}$ exists strongly, then a unique family $(P_y)_{y\in S}$ exists.*

An existence result without uniqueness is given in Section 2.5. We collect some consequences of the preceding results.

**Corollary 2.2.** *Suppose that $d = 1$, that Condition 2.3 holds and that for all $i \in S_d$ the map $x \mapsto b(x, i)$ is continuous and the map $x \mapsto a^{\frac{1}{2}}(x, i)$ is locally Hölder continuous with exponent larger or equal than $\frac{1}{2}$ and that $a^{\frac{1}{2}}(\cdot, i) \neq 0$. Furthermore, for all $i \in S_d$ suppose that*

$$\lim_{x\to\pm\infty} \int_0^x \exp\Big( -2\int_0^y \frac{b(z,i)}{a(z,i)}dz\Big) \int_0^y \frac{2\exp\big(2\int_0^u \frac{b(z,i)}{a(z,i)}dz\big)}{a(u,i)}dudy = \infty. \tag{2.34}$$

*Then, the family $(P_x)_{x\in S}$ exists uniquely and is $C_b$-Feller. Moreover, the following are equivalent:*

(i) $(P_x)_{x\in S}$ *is Feller–Dynkin.*

(ii) *For all* $i \in S_d$ *one of the conditions* (2.35) *and* (2.36) *holds and one of the conditions* (2.37) *and* (2.38) *holds:*

$$\int_0^\infty \exp\Big( -2\int_0^y \frac{b(z,i)}{a(z,i)}dz\Big)dy < \infty. \tag{2.35}$$

$$\begin{cases} \displaystyle\int_0^\infty \exp\Big( -2\int_0^y \frac{b(z,i)}{a(z,i)}dz\Big)dy = \infty, \\ \displaystyle\int_0^\infty \exp\Big( -2\int_0^y \frac{b(z,i)}{a(z,i)}dz\Big)\int_y^\infty \frac{\exp\big(2\int_0^u \frac{b(z,i)}{a(z,i)}dz\big)}{a(u,i)}dudy = \infty. \end{cases} \tag{2.36}$$

$$\int_{-\infty}^0 \exp\Big( 2\int_y^0 \frac{b(z,i)}{a(z,i)}dz\Big)dy < \infty. \tag{2.37}$$

$$\begin{cases} \displaystyle\int_{-\infty}^0 \exp\Big( 2\int_y^0 \frac{b(z,i)}{a(z,i)}dz\Big)dy = \infty, \\ \displaystyle\int_{-\infty}^0 \exp\Big( 2\int_y^0 \frac{b(z,i)}{a(z,i)}dz\Big)\int_{-\infty}^y \frac{\exp\big(-2\int_u^0 \frac{b(z,i)}{a(z,i)}dz\big)}{a(u,i)}dudy = \infty. \end{cases} \tag{2.38}$$

**Remark 2.6.** *If* $b \equiv 0$*, then the conditions in part (ii) of Corollary 2.2 are satisfies if and only if for all* $i \in S_d$ *the following hold:*

$$\int_0^\infty \frac{u}{a(u,i)}du = \int_{-\infty}^0 \frac{-u}{a(u,i)}du = \infty. \tag{2.39}$$

**Corollary 2.3.** *Assume that Condition 2.3 holds and that for all* $i \in S_d$ *the maps* $x \mapsto b(x,i)$ *and* $x \mapsto a^{\frac{1}{2}}(x,i)$ *are locally Lipschitz continuous and that for all* $(x,i) \in S$ *the MP* $(C_b^2(\mathbb{R}^d), \mathcal{K}^i, \Sigma_c, x)$ *has a solution. Furthermore, suppose that for each* $i \in S_d$ *there is an* $r_i > 0$ *and two locally Hölder continuous functions* $b_i\colon [r_i,\infty) \to \mathbb{R}$ *and* $a_i\colon [r_i,\infty) \to (0,\infty)$ *such that for all* $x \in \mathbb{R}^d\colon \|x\| \geq 2r_i$

$$\begin{aligned} \langle x, a(x,i)x\rangle &\leq a_i\Big(\tfrac{\|x\|^2}{2}\Big), \\ \operatorname{trace} a(x,i) + 2\langle x, b(x,i)\rangle &\geq b_i\Big(\tfrac{\|x\|^2}{2}\Big)\langle x, a(x,i)x\rangle, \end{aligned}$$

*and either*

$$p_i(r) \triangleq \int_1^{r_i} \exp\Big( -\int_1^y b_i(z)dz\Big)dy, \quad \lim_{r\to\infty} p_i(r) < \infty,$$

*or*

$$\lim_{r\to\infty} p(r) = \infty \text{ and } \int_1^\infty p_i'(y)\int_y^\infty \frac{dz}{a_i(z)p_i'(z)}dy = \infty.$$

*Then,* $(P_x)_{x\in S}$ *is Feller–Dynkin.*

Explicit conditions for the assumption that for all $(x,i) \in S$ the MP $(C_b^2(\mathbb{R}^d), \mathcal{K}^i, \Sigma_c, x)$

has a solution can, e.g. be found in [137, Chapter 10].

**Corollary 2.4.** *Assume that there exists an* $i \in S_d$ *such that the maps* $x \mapsto b(x,i)$ *and* $x \mapsto a^{\frac{1}{2}}(x,i)$ *are locally Lipschitz continuous and that for all* $x \in \mathbb{R}^d$ *the MP* $(C_b^2(\mathbb{R}^d), \mathcal{K}^i, \Sigma_c, x)$ *has a solution. Furthermore, suppose there is an* $r > 0$ *and two locally Hölder continuous functions* $b_d \colon [r, \infty) \to \mathbb{R}$ *and* $a_d \colon [r,\infty) \to (0,\infty)$ *such that for all* $x \in \mathbb{R}^d \colon \|x\| \geq 2r$

$$\langle x, a(x,i)x\rangle \geq a_d\left(\tfrac{\|x\|^2}{2}\right),$$
$$\operatorname{trace} a(x,i) + 2\langle x, b(x,i)\rangle \leq b_d\left(\tfrac{\|x\|^2}{2}\right)\langle x, a(x,i)x\rangle,$$

*and*

$$p(t) \triangleq \int_{r+1}^{t} \exp\left(-\int_{r+1}^{y} b_d(z)dz\right)dy \to \infty \text{ as } t \to \infty,$$

*and*

$$\int_{r+1}^{\infty} p'(y)\int_{y}^{\infty} \frac{dz}{a_d(z)p'(z)}dy < \infty.$$

*Then,* $(P_x)_{x\in S}$ *is not Feller–Dynkin.*

By [131, Theorem 3.2], the family $(P_x)_{x\in S}$ has the strong Feller property if it is $C_b$-Feller and for all $i \in S_d$ the families $(P^i_x)_{x\in\mathbb{R}^d}$ have the strong Feller property. Consequently, the strong Feller property and the Feller–Dynkin property are both inherited from the relative properties of processes in the fixed environments. We give a short example for a switching diffusion which has the strong Feller property, but not the Feller–Dynkin property.

**Example 2.6.** *Let* $d = 1, S_d = \{1,2\}, b \equiv 0$ *and*

$$a(x,i) \triangleq \begin{cases} 1 + x^4, & i = 1,\\ 1, & i = 2,\end{cases}$$

*for* $(x,i) \in S$. *Due to [77, Problem 5.5.27],* (2.34) *holds in the case* $b \equiv 0$. *Thus, we conclude from Corollary 2.2 that* $(P_y)_{y\in S}$ *exists uniquely and is* $C_b$*-Feller. Furthermore, due to [137, Corollary 10.1.4],* $(P^i_x)_{x\in\mathbb{R}}$ *has the strong Feller property for* $i = 1,2$. *Of course, the family* $(P^2_x)_{x\in\mathbb{R}}$ *consists of Wiener measures and is well-known to be strongly Feller. Therefore, [131, Theorem 3.2] implies that* $(P_y)_{y\in S}$ *has the strong Feller property, too. However, for* $i = 1$ *the condition* (2.39) *fails because*

$$\int_0^\infty \frac{x\ dx}{1+x^4} = \frac{\pi}{4} < \infty.$$

*The family* $(P_x)_{x\in S}$ *is not Feller–Dynkin due to Corollary 2.2, see Remark 2.6.*

#### 2.4.3.3 Proof of Proposition 2.6

Since $(P^i_x)_{x\in\mathbb{R}^d}$ is $C_b$-Feller, one can show as in the proof of Theorem 2.1 that if for any compact set $K \subset \mathbb{R}^d$ and any $t > 0$ it holds that

$$\limsup_{\|x\|\to\infty} P^i_x(X_t \in K) = 0,$$

then $(P^i_x)_{x\in\mathbb{R}^d}$ is Feller–Dynkin. Consequently, since we assume $(P^i_x)_{x\in\mathbb{R}^d}$ not to be Feller–Dynkin, there exists a sequence $(x_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ with $\|x_k\| \to \infty$ as $k \to \infty$, a compact set $K^o \subset \mathbb{R}^d$ and a $t^o > 0$ such that

$$\limsup_{k\to\infty} P^i_{x_k}(X_{t^o} \in K^o) > 0. \tag{2.40}$$

The set $G \triangleq K^o \times \{i\} \subset S$ is compact. If we show that

$$\limsup_{k\to\infty} P_{(x_k,i)}(X_{t^o} \in G) > 0, \tag{2.41}$$

then $(P_x)_{x\in S}$ cannot be Feller–Dynkin. To see this, assume for contradiction that $(P_x)_{x\in S}$ is Feller–Dynkin. Due to the locally compact version of Urysohn's lemma, there exists a function $f \in C_0(S)$ such that $0 \le f \le 1$ and $f \equiv 1$ on $G$. Consequently, we have

$$\begin{aligned} P_{(x_k,i)}(X_{t^o} \in G) &= E_{(x_k,i)}\big[f(X_{t^o})\mathbb{1}\{X_{t^o} \in G\}\big] \\ &\le E_{(x_k,i)}\big[f(X_{t^o})\big] \to 0 \text{ as } k \to \infty, \end{aligned}$$

because $(P_x)_{x\in S}$ is Feller–Dynkin. This, however, is a contradiction and we conclude that $(P_x)_{x\in S}$ cannot be Feller–Dynkin. In summary, it suffices to show (2.41).

For a càdlàg $S_d$-valued process $(Z_t)_{t\ge 0}$, we set

$$\tau(Z) \triangleq \inf\big(t \in \mathbb{R}_+ \colon Z_t \neq Z_0\big),$$

which is a stopping time for any filtration to which $Z$ is adapted, see [47, Proposition 2.1.5]. In the following let $Y, Z$ and $W$ be as in Lemma 2.2 for $y = (x, i)$. On $\{t \le \tau(Z)\}$ we have

$$Y_t = x + \int_0^t b(Y_s, i)ds + \int_0^t a^{\frac{1}{2}}(Y_s, i)dW_s,$$

which is the SDE corresponding to the MP $(C^2_b(\mathbb{R}^d), \mathcal{K}^i, \Sigma_c, x)$, see [77, Corollary 5.4.8]. We now need a local version of pathwise uniqueness. The proof of the following lemma is given after the proof of Proposition 2.6 is complete.

**Lemma 2.3.** *Suppose that the SDE*

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t \tag{2.42}$$

*satisfies weak existence and pathwise uniqueness (see [123, Section IX.1]). In other words, we assume that the martingale problem corresponding to the SDE* (2.42) *exists strongly, see Remark 2.5 and [77, Section 5.4]. Consider a filtered probability space with right-continuous complete filtration* $(\mathcal{G}_t)_{t\ge 0}$*, which supports a Brownian motion* $W$ *and an* $\mathbb{R}^d$*-valued* $\mathcal{G}_0$*-measurable random variable* $\psi$*. Take a* $(\mathcal{G}_t)_{t\ge 0}$*-stopping time* $\tau$ *and let* $Y$ *be the* *solution to* (2.42) *with initial value* $\psi$. *Then, all solutions to*

$$dO_t = \mu(O_t)\mathbb{1}_{\{t\leq\tau\}}dt + \sigma(O_t)\mathbb{1}_{\{t\leq\tau\}}dW_t, \quad O_0 = \psi,$$

*are indistinguishable from* $Y_{\cdot\wedge\tau}$.

Since we assume that $P^i_x$ exists strongly, Lemma 2.3 and the independence of the $\sigma$-fields $\sigma(W_t, t \in \mathbb{R}_+)$ and $\sigma(Z_t, t \in \mathbb{R}_+)$, see Lemma 2.2, imply that

$$\begin{aligned} P_{(x,i)}\big(X_{t^0} \in G\big) &\geq P\big(Y_{t^o\wedge\tau(Z)} \in K^o, Z_{t^o} = i, t^o < \tau(Z)\big) \\ &= P\big(F^i(x,W)_{t^o\wedge\tau(Z)} \in K^o, t^o < \tau(Z)\big) \\ &= P\big(F^i(x,W)_{t^o} \in K^o\big)P\big(t^o < \tau(Z)\big) \\ &= P^i_x\big(X_{t^o} \in K^o\big)P\big(t^o < \tau(Z)\big), \end{aligned}$$

where $F^i$ is as in Definition 2.2. It is well-known that $\tau(Z)$ is exponentially distributed with parameter $-q_{ii}$, see, e.g. [74, Lemma 10.18]. Therefore, we have

$$P_{(x,i)}\big(X_{t^o} \in G\big) \geq P^i_x\big(X_{t^o} \in K^o\big)e^{q_{ii}t^o}.$$

We conclude (2.41) from (2.40). This finishes the proof. □

*Proof of Lemma 2.3:* Due to localization, we can assume that $\tau$ is finite. Let $B$ be defined by

$$B_t \triangleq W_{t+\tau} - W_\tau, \quad t \in \mathbb{R}_+.$$

Due to [123, Proposition V.1.5] and Lévy's characterization (see, e.g. [77, Theorem 3.3.16]), the process $B$ is a $(\mathcal{G}_{t+\tau})_{t\geq0}$-Brownian motion and, due to the strong existence hypothesis, there exists a solution $U$ to the SDE

$$dU_t = \mu(U_t)dt + \sigma(U_t)dB_t, \quad U_0 = O_\tau.$$

Now, we set

$$V_t \triangleq \begin{cases} O_t, & t \leq \tau, \\ U_{t-\tau}, & t > \tau. \end{cases}$$

As $U_0 = O_\tau$, the process $V$ has continuous paths. We claim that $(U_{t-\tau}\mathbb{1}_{\{\tau<t\}})_{t\geq0}$ is progressively measurable. This implies that $V$ is adapted. Note that $t \mapsto U_{t-\tau}\mathbb{1}_{\{\tau<t\}}$ is left-continuous and that $s \mapsto U_{t-s}\mathbb{1}_{\{s<t\}}$ is right-continuous. Thus, by an approximation argument, it suffices to show that $(h_t)_{t\geq0} \triangleq (U_{t-\rho}\mathbb{1}_{\{\rho<t\}})_{t\geq0}$ is adapted for any stopping time $\rho$ which takes values in the countable set $2^{-n}\overline{\mathbb{N}}$ for some $n \in \mathbb{N}$ and satisfies $\rho \geq \tau$. Let $G \in \mathcal{B}(\mathbb{R}^d)$ and set $N_t \triangleq 2^{-n}\overline{\mathbb{N}} \cap [0,t)$. We have

$$\{h_t \in G\} = \Big(\bigcup_{k\in N_t} \big(\{U_{t-k} \in G\} \cap \{\rho = k\}\big)\Big) \cup \big(\{0 \in G\} \cap \{\rho \geq t\}\big) \in \mathcal{G}_t.$$

Here, we use that $\{U_{t-k} \in G\} \in \mathcal{G}_{t-k+\tau} \subseteq \mathcal{G}_{t-k+\rho}$ and the fact that $\mathcal{G}_{t-k+\rho}\cap\{\rho = k\} \subseteq \mathcal{G}_t$. Therefore, $(U_{t-\tau}\mathbb{1}_{\{\tau<t\}})_{t\geq0}$ is progressive. On $\{t \leq \tau\}$ we have

$$V_t = \psi + \int_0^t \mu(V_s)ds + \int_0^t \sigma(V_s)dW_s.$$

Classical rules for time-changed stochastic integrals (see, e.g. [123, Propositions V.1.4, V.1.5]) yield that on $\{t > \tau\}$

$$V_t = O_\tau + \int_0^{t-\tau} \mu(U_s)ds + \int_0^{t-\tau} \sigma(U_s)dB_s \tag{2.43}$$

$$= V_\tau + \int_\tau^t \mu(U_{s-\tau})ds + \int_\tau^t \sigma(U_{s-\tau})dW_s \tag{2.44}$$

$$= V_\tau + \int_\tau^t \mu(V_s)ds + \int_\tau^t \sigma(V_s)dW_s$$

$$= \psi + \int_0^t \mu(V_s)ds + \int_0^t \sigma(V_s)dW_s.$$

Consequently, $(V_t)_{t\geq 0}$ solves the SDE

$$dV_t = \mu(V_t)dt + \sigma(V_t)dW_t, \quad V_0 = \psi.$$

By the strong existence hypothesis, we conclude that a.s. $V_t = Y_t$ for all $t \in \mathbb{R}_+$. The definition of $V$ implies the claim. □

#### 2.4.3.4 Proof of Theorem 2.4

The implication (i) ⇒ (ii) follows from Proposition 2.6.

We prove the implication (ii) ⇒ (i) using an explicit construction of the family $(P_y)_{y\in S}$. Take a filtered probability space $(\Theta, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ satisfying the usual hypothesis of a right-continuous and complete filtration, which supports a Brownian motion $W$ for the filtration $(\mathcal{G}_t)_{t\geq 0}$ and an $S_d$-valued continuous-time Markov chain $Z$ for the filtration $(\mathcal{G}_t)_{t\geq 0}$ with $Q$-matrix $Q$ and $Z_0 = i$. Recalling Remark 2.4, we note that the $\sigma$-fields $\sigma(W_t, t \in \mathbb{R}_+)$ and $\sigma(Z_t, t \in \mathbb{R}_+)$ are independent. Define inductively

$$\tau_0 \triangleq 0, \quad \tau_n \triangleq \inf\big(t \geq \tau_{n-1} \colon Z_t \neq Z_{\tau_{n-1}}\big), \quad n \geq 1, \tag{2.45}$$

and

$$\sigma_0 \triangleq 0, \quad \sigma_n \triangleq \tau_n - \tau_{n-1} = \inf\big(t \in \mathbb{R}_+ \colon Z_{t+\tau_{n-1}} \neq Z_{\tau_{n-1}}\big), \quad n \geq 1.$$

Since no state of $Z$ is absorbing due to Condition 2.3, we have a.s. $\tau_n < \infty$ for all $n \in \mathbb{N}$. Furthermore, for all $n \in \mathbb{N}$ the random time $\tau_n$ is a $(\mathcal{G}_t)_{t\geq 0}$-stopping time and the random time $\sigma_n$ is a $(\mathcal{G}_{t+\tau_{n-1}})_{t\geq 0}$-stopping time, see [77, Proposition 1.1.12] and [74, Lemma 6.5, Theorem 6.7]. Due to [123, Proposition V.1.5] and Lévy's characterization, the process $W^n = (W_{t+\tau_n} - W_{\tau_n})_{t\geq 0}$ is a $(\mathcal{G}_{t+\tau_n})_{t\geq 0}$-Brownian motion and therefore independent of $\mathcal{G}_{\tau_n}$. For all $k \in S_d$ let $F^k \colon \mathbb{R}^d \times \Sigma_c \to \Sigma_c$ be as in Definition 2.2 and set $Y^{0,x} \triangleq F^i(x, W)$. By induction, define further

$$Y^{n,x} \triangleq \sum_{k=1}^N F^k(Y^{n-1,x}_{\sigma_n}, W^n)\mathbb{1}\{Z_{\tau_n} = k\}, \quad n \in \mathbb{N},$$

and set

$$Y^x_t \triangleq x\mathbb{1}\{t = 0\} + \sum_{n=0}^\infty Y^{n,x}_{t-\tau_n}\mathbb{1}\{\tau_n < t \leq \tau_{n+1}\}, \quad t \in \mathbb{R}_+.$$

The process $Y^x$ has continuous paths and similar arguments as used in the proof of

Lemma 2.3 show that $Y^x$ is adapted, too. Next, five technical lemmata follow.

**Lemma 2.4.** *The law of* $(Y^x, Z)$ *is given by* $P_{(x,i)}$.

*Proof.* The process $V \triangleq F^k(Y^{n-1,x}_{\sigma_n}, W^n)$ has the dynamics

$$dV_t = b(V_t, k)dt + a^{\frac{1}{2}}(V_t, k)dW^n_t, \quad V_0 = Y^{n-1,x}_{\sigma_n}.$$

Thus, due to classical rules for time-changed stochastic integrals, for $t \in [\tau_n, \tau_{n+1}]$ on $\{Z_{\tau_n} = k\}$ we have

$$\begin{aligned} Y^{n,x}_{t-\tau_n} &= F^k(Y^{n-1,x}_{\sigma_n}, W^n)_{t-\tau_n} \\ &= Y^{n-1,x}_{\sigma_n} + \int_0^{t-\tau_n} b(V_s, k)ds + \int_0^{t-\tau_n} a^{\frac{1}{2}}(V_s, k)dW^n_s \\ &= Y^{n-1,x}_{\sigma_n} + \int_{\tau_n}^t b(Y^{n,x}_{s-\tau_n}, k)ds + \int_{\tau_n}^t a^{\frac{1}{2}}(Y^{n,x}_{s-\tau_n}, k)dW_s \\ &= Y^{n-1,x}_{\sigma_n} + \int_{\tau_n}^t b(Y^x_s, k)ds + \int_{\tau_n}^t a^{\frac{1}{2}}(Y^x_s, k)dW_s \\ &= Y^{n-1,x}_{\sigma_n} + \int_{\tau_n}^t b(Y^x_s, Z_s)ds + \int_{\tau_n}^t a^{\frac{1}{2}}(Y^x_s, Z_s)dW_s. \end{aligned}$$

Iterating yields that for $t \in [\tau_n, \tau_{n+1}]$

$$Y^{n,x}_{t-\tau_n} = x + \int_0^t b(Y^x_s, Z_s)ds + \int_0^t a^{\frac{1}{2}}(Y^x_s, Z_s)dW_s.$$

Therefore, the process $(Y^x_t)_{t\geq 0}$ satisfies the SDE

$$dY^x_t = b(Y^x_t, Z_t)dt + a^{\frac{1}{2}}(Y^x_t, Z_t)dW_t, \quad Y^x_0 = x,$$

and, consequently, the uniqueness of $P_{(x,i)}$ and Lemma 2.2 imply that the law of $(Y^x, Z)$ coincides with $P_{(x,i)}$. □

**Lemma 2.5.** *For all Borel sets* $G \subseteq \Sigma_c$ *we have a.s.*

$$P(W^n \in G | \sigma(\mathcal{G}_{\tau_n}, \sigma_{n+1})) = P(W^n \in G).$$

*Proof.* Let $\mathscr{W}_z$ be the Wiener measure with starting value $z \in \mathbb{R}^d$ and $P^\star_k$ be the law of a Markov chain with $Q$-matrix $Q$ and starting value $k \in S_d$. Due to Remark 2.4, Proposition 2.9 in Section 2.6, and [47, Proposition 4.1.5, Theorem 4.4.2], the map $(z, k) \mapsto \mathscr{W}_z \otimes P^\star_k$ is Borel and the process $(W, Z)$ is a strong Markov process in the following sense: For all $F \in \mathcal{F}$ and all a.s. finite $(\mathcal{G}_t)_{t\geq 0}$-stopping times $\theta$ a.s.

$$P\big((W_{t+\theta}, Z_{t+\theta})_{t\geq 0} \in F | \mathcal{G}_\theta\big) = \big(\mathscr{W}_{W_\theta} \otimes P^\star_{Z_\theta}\big)(F).$$

Let $F \subseteq \Sigma_d$ be Borel. The strong Markov properties of $Z, W$ and $(W, Z)$ imply that a.s.

$$\begin{aligned} &P\big((W_{t+\tau_n})_{t\geq 0} \in G, (Z_{t+\tau_n})_{t\geq 0} \in F | \mathcal{G}_{\tau_n}\big) \\ &\qquad = \mathscr{W}_{W_{\tau_n}}(G)\ P^\star_{Z_{\tau_n}}(F) \\ &\qquad = P\big((W_{t+\tau_n})_{t\geq 0} \in G | \mathcal{G}_{\tau_n}\big) P\big((Z_{t+\tau_n})_{t\geq 0} \in F | \mathcal{G}_{\tau_n}\big). \end{aligned}$$

This implies that $\sigma(W^n_t, t \in \mathbb{R}_+)$ and $\sigma(\sigma_{n+1})$ are independent given $\mathcal{G}_{\tau_n}$. Thus, [74, Proposition 5.6] and the independence of $\sigma(W^n_t, t \in \mathbb{R}_+)$ and $\mathcal{G}_{\tau_n}$ yield that a.s.

$$P(W^n \in G|\sigma(\mathcal{G}_{\tau_n}, \sigma_{n+1})) = P(W^n \in G|\mathcal{G}_{\tau_n}) = P(W^n \in G),$$

which is the claim. □

**Lemma 2.6.** *For all $n \in \mathbb{N}_0$ we have $\|Y^{n,x}_{\sigma_{n+1}}\| \to \infty$ in probability as $\|x\| \to \infty$.*

*Proof.* We use induction. As the process $Y^{0,x}$ has law $P^i_x$ (by the uniqueness assumption) and $Y^{0,x}$ is independent of $\sigma_1 = \tau_1$, we can conclude the induction base from the hypothesis (ii) of Theorem 2.4. More precisely, we have for all $m \in \mathbb{N}$

$$P(\|Y^{0,x}_{\sigma_1}\| \le m) = \int_0^\infty P^i_x(\|X_s\| \le m) P(\sigma_1 \in ds) \to 0$$

as $\|x\| \to \infty$, see the proof of Proposition 2.6. Suppose now that the claim holds for $n \in \mathbb{N}_0$. Using the Lemmata 2.2 and 2.5 and [74, Theorem 5.4], we obtain

$$\begin{aligned}
&P(\|Y^{n+1,x}_{\sigma_{n+2}}\| \le m)\\
&\quad= \sum_{k=1}^N P(\|F^k(Y^{n,x}_{\sigma_{n+1}}, W^{n+1})_{\sigma_{n+2}}\| \le m, Z_{\tau_{n+1}} = k)\\
&\quad= \sum_{k=1}^N E\big[P(\|F^k(Y^{n,x}_{\sigma_{n+1}}, W^{n+1})_{\sigma_{n+2}}\| \le m|\sigma(\mathcal{G}_{\tau_{n+1}}, \sigma_{n+2}))\mathbb{1}\{Z_{\tau_{n+1}} = k\}\big]\\
&\quad= \sum_{k=1}^N \int P(\|F^k(Y^{n,x}_{\sigma_{n+1}(\omega)}(\omega), W^{n+1})_{\sigma_{n+2}(\omega)}\| \le m)\mathbb{1}\{Z_{\tau_{n+1}(\omega)}(\omega) = k\} P(d\omega)\\
&\quad= \sum_{k=1}^N \int P^k_{Y^{n,x}_{\sigma_{n+1}(\omega)}(\omega)}(\|X_{\sigma_{n+2}(\omega)}\| \le m)\mathbb{1}\{Z_{\tau_{n+1}(\omega)}(\omega) = k\} P(d\omega).
\end{aligned} \tag{2.46}$$

Take $(x_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ such that $\|x_k\| \to \infty$ as $k \to \infty$. A well-known characterization of convergence in probability is the following: A sequence $(Z^k)_{k\in\mathbb{N}}$ converges in probability to a random variable $Z$ if and only if each subsequence of $(Z^k)_{k\in\mathbb{N}}$ contains a further subsequence which converges almost surely to $Z$. Consequently, $(x_k)_{k\in\mathbb{N}}$ contains a subsequence $(x_{n'_k})_{k\in\mathbb{N}}$ such that

$$\left\|Y^{n,x_{n'_k}}_{\sigma_{n+1}}\right\| \to \infty$$

almost surely as $k \to \infty$. Due to the dominated convergence theorem, we deduce from (2.46) that

$$\left\|Y^{n+1,x_{n'_k}}_{\sigma_{n+2}}\right\| \to \infty$$

in probability as $k \to \infty$. Thus, applying again the subsequence criterion, we can extract a further subsequence such that the convergence holds almost surely. Finally, applying the subsequence criterion a third time (but this time the converse direction), we conclude the claim. □

**Lemma 2.7.** *For all $n \in \mathbb{N}_0, t > 0$ we have $\|Y^{n,x}_{t-\tau_n}\| \to \infty$ on $\{\tau_n < t\}$ in probability as $\|x\| \to \infty$.*

*Proof.* Since $\sigma(W^n_t, t \in \mathbb{R}_+)$ is independent of $\mathcal{G}_{\tau_n}$, we show as in the proof of Lemma 2.6 that

$$P(\|Y^{n,x}_{t-\tau_n}\| \leq m, \tau_n < t) \\ = \sum_{k=1}^{N} \int P^k_{Y^{n-1,x}_{\sigma_n(\omega)}(\omega)}(\|X_{t-\tau_n(\omega)}\| \leq m) \mathbb{1}\{\tau_n(\omega) < t\} \mathbb{1}\{Z_{\tau_n(\omega)}(\omega) = k\} P(d\omega).$$

Using Lemma 2.6 and the argument in its proof, we see that the claim follows. □

**Lemma 2.8.** *For all compact sets $K \subset \mathbb{R}^d$ and all $t, \varepsilon > 0$ there exists a compact set $K^* \subset \mathbb{R}^d$ such that*

$$P_{(x,i)}(X_t \in K \times S_d) < \varepsilon$$

*for all $x \notin K^*$.*

*Proof.* Let $f \in C_0(\mathbb{R}^d)$ be such that $0 \leq f \leq 1$ and $f \equiv 1$ on $K$. As before $f$ exists due to Urysohn's lemma for locally compact spaces. We have

$$\begin{aligned} E\big[f(Y^x_t)\big] &= \sum_{n=0}^{\infty} E\big[f(Y^x_t)\mathbb{1}\{\tau_n < t \leq \tau_{n+1}\}\big] \\ &= \sum_{n=0}^{\infty} E\big[f(Y^{n,x}_{t-\tau_n})\mathbb{1}\{\tau_n < t \leq \tau_{n+1}\}\big] \to 0 \end{aligned}$$

as $\|x\| \to \infty$, which follows from Lemma 2.7 and the dominated convergence theorem. Thus, the map $x \mapsto E\big[f(Y^x_t)\big]$ is an element of $C_0(\mathbb{R}^d)$. Finally, noting that

$$P_{(x,i)}(X_t \in K \times S_d) \leq E\big[f(Y^x_t)\big]$$

implies the claim. □

We are in the position to complete the proof. Fix $t, \varepsilon > 0$ and a compact set $K \subset S$. Recall that $\pi_1 \colon S \to \mathbb{R}^d$ and $\pi_2 \colon S \to S_d$ are the usual projections. As $(P^\star_i)_{i \in S_d}$ is Feller–Dynkin, there exists a compact set $K^* \subset S_d$ such that

$$P^\star_i\big(X_t \in \pi_2(K)\big) < \varepsilon$$

for all $i \notin K^*$. By Lemma 2.8, for each $i \in K^*$ we find a compact set $K^*_i \subset \mathbb{R}^d$ such that

$$P_{(x,i)}\big(X_t \in \pi_1(K) \times S_d\big) < \varepsilon$$

for all $x \notin K^*_i$. Define $\widehat{K} \triangleq \big(\bigcup_{i \in K^*} K^*_i\big) \times K^* \subset S$. Clearly, $\widehat{K}$ is compact. We claim that

$$P_{(x,i)}\big(X_t \in K\big) < \varepsilon$$

for all $(x, i) \notin \widehat{K}$. To see this, note that

$$\widehat{K}^c = \bigg(\Big(\bigcap_{i \in K^*} (K^*_i)^c\Big) \times K^*\bigg) \cup \Big(\mathbb{R}^d \times (K^*)^c\Big).$$

Now, if $(x,i) \in \mathbb{R}^d \times (K^*)^c$ we have

$$P_{(x,i)}\big(X_t \in K\big) \leq P_{(x,i)}\big(X_t \in \pi_1(K) \times \pi_2(K)\big) \leq P^\star_i\big(X_t \in \pi_2(K)\big) < \varepsilon.$$

If $(x,i) \in \big(\bigcap_{j \in K^*} (K^*_j)^c\big) \times K^*$ we have $x \notin K^*_i$ and hence

$$P_{(x,i)}\big(X_t \in K\big) \leq P_{(x,i)}\big(X_t \in \pi_1(K) \times S_d\big) < \varepsilon.$$

This proves the claim, which itself implies that $(P_x)_{x \in S}$ is Feller–Dynkin, see the proof of Theorem 2.1. □

#### 2.4.3.5 Proof of Proposition 2.7

It suffices to show that $x \mapsto P_x$ is continuous, i.e. that $x_n \to x$ implies $P_{x_n} \to P_x$ weakly as $n \to \infty$. In this case, because for all $x \in S$ and $t \in \mathbb{R}_+$ the map $\omega \mapsto \omega(t)$ is $P_x$-a.s. continuous (see [47, Proposition 3.5.2] and note that $P_x(\Delta X_t \neq 0) = 0$), the continuous mapping theorem implies that $(P_x)_{x \in S}$ has the $C_b$-Feller property. The continuity of $x \mapsto P_x$ follows from Theorem 2.5 below. □

**Theorem 2.5.** *For all $n \in \mathbb{N}$ let $b_n \colon S \to \mathbb{R}^d$ and $a_n \colon S \to \mathbb{S}^d$ be Borel functions such that for all $m \in \mathbb{R}_+$*

$$\sup_{n \in \mathbb{N}} \sup_{\|y\| \leq m} \big(\|b_n(y)\| + \|a_n(y)\|\big) < \infty, \tag{2.47}$$

*where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^{d+1}$. Assume that $b \colon S \to \mathbb{R}^d$ and $a \colon S \to \mathbb{S}^d$ are continuous functions and that for all $m \in \mathbb{R}_+$*

$$\sup_{\|y\| \leq m} \big(\|b(y) - b_n(y)\| + \|a(y) - a_n(y)\|\big) \to 0 \tag{2.48}$$

*as $n \to \infty$. Furthermore, let $(Q_n)_{n \in \mathbb{N}}$ be a sequence of $Q$-matrices on $S_d$ such that for all $n \in \mathbb{N}$ and $i \in S_d$ the MP $(C_n, Q_n, \Sigma_d, i)$, where*

$$C_n \triangleq \big\{f \in C_0(S_d) \colon Q_n f \in C_0(S_d)\big\},$$

*has a unique solution $P^n_i$ such that $(P^n_i)_{i \in S_d}$ is Feller–Dynkin. Let $C^\star \subseteq C$ be as in Remark 2.4. Suppose that for all $f \in C^\star$ there exists a sequence $(f_n)_{n \in \mathbb{N}}$ consisting of $f_n \in C_n$ such that*

$$\|f - f_n\|_\infty + \|Qf - Q_n f_n\|_\infty \to 0 \tag{2.49}$$

*as $n \to \infty$. Finally, take $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d$ and $(i_n)_{n \in \mathbb{N}} \subset S_d$ such that $x_n \to x \in \mathbb{R}^d$ and $i_n \to i \in S_d$ as $n \to \infty$. Set $\mathcal{L}$ as in (2.23), $\mathcal{L}_n$ as in (2.23) with $b$ replaced by $b_n$, $a$ replaced by $a_n$ and $Q$ replaced by $Q_n$, and $D$ as in (2.31). If $P^n$ is a solution to the MP $(D_n, \mathcal{L}_n, \Sigma, (x_n, i_n))$, where*

$$D_n \triangleq \big\{f, g \colon f \in C^2_c(\mathbb{R}^d), g \in C_n\big\},$$

*and for all $y \in S$ the MP $(D, \mathcal{L}, \Sigma, y)$ has a unique solution $P_y$, then $P^n \to P_{(x,i)}$ weakly as $n \to \infty$.*

*Proof.* We adapt the proof of [70, Theorem IX.3.39]. Let us start with a clarification of our terminology: When we say that a sequence of càdlàg processes is tight, we mean that its laws are tight or, equivalently, relatively compact by Prohorov's theorem (see [47, Theorem 3.2.2]). If we speak of an accumulation point of a sequence of processes, we refer to an accumulation point of the corresponding sequence of laws.

Because of the discrete topology of $S_d$ we can assume that $i_n \equiv i$. For all $n \in \mathbb{N}$ denote by $Y^n, Z^n$ and $W^n$ the processes from Lemma 2.2 corresponding to $P^n$. For $m \in \mathbb{R}_+$ we define

$$\tau_m \triangleq \inf\big(t \in \mathbb{R}_+ \colon \|X_t\| \geq m \text{ or } \|X_{t-}\| \geq m\big). \tag{2.50}$$

We note that $\tau_m$ is an $(\mathcal{F}^o_t)_{t\geq 0}$-stopping time, see [47, Proposition 2.1.5]. For $n \in \mathbb{N}$ and $m \in \mathbb{R}_+$ we set

$$\tau_{n,m} \triangleq \tau_m \circ (Y^n, Z^n).$$

Next, four technical lemmata follow.

**Lemma 2.9.** *For all $m \in \mathbb{R}_+$ the sequence $\{(Y^n_{\cdot\wedge\tau_{n,m}}, Z^n), n \in \mathbb{N}\}$ is tight.*

*Proof.* The Kato–Trotter theorem [74, Theorem 17.25] implies that $\{Z^n, n \in \mathbb{N}\}$ is tight in $\Sigma_d$ equipped with the Skorokhod topology. For all $n \in \mathbb{N}$ the process $Y^n_{\cdot\wedge\tau_{n,m}}$ has continuous paths. Below, we show that $\{Y^n_{\cdot\wedge\tau_{n,m}}, n \in \mathbb{N}\}$ is tight in $\Sigma_c$ equipped with the local uniform topology. In this case, [47, Problem 4.25] implies that $\{(Y^n_{t\wedge\tau_{n,m}})_{t\geq 0}, n \in \mathbb{N}\}$ is also tight in the space of càdlàg functions $\mathbb{R}_+ \to \mathbb{R}^d$ equipped with the Skorokhod topology, which we denote by $D(\mathbb{R}_+, \mathbb{R}^d)$. Due to [70, Corollary VI.3.33], this implies tightness for $\{(Y^n_{\cdot\wedge\tau_{n,m}}, Z^n), n \in \mathbb{N}\}$.

It remains to show that $\{Y^n_{\cdot\wedge\tau_{n,m}}, n \in \mathbb{N}\}$ is tight in $\Sigma_c$. Let $p > 2$ and recall the inequalities

$$\big(v+u\big)^p \leq 2^p\big(v^p + u^p\big), \quad v, u \geq 0, \qquad \left\| \int_0^t f(s)ds \right\| \leq \int_0^t \|f(s)\|ds. \tag{2.51}$$

Let $T \in \mathbb{R}_+$ and $s < t \leq T$. We write $x \preceq y$ whenever $x \leq$ const. $y$ where the constant only depends on $T, p, m$ and (2.47). We deduce from the triangle inequality, (2.51) and [77, Remark 3.3.30] (i.e. a multidimensional version of the Burkholder-Davis-Gundy inequality) that

$$\begin{aligned}
&E\big[\|Y^n_{t\wedge\tau_{n,m}} - Y^n_{s\wedge\tau_{n,m}}\|^p\big] \\
&\quad= E\bigg[\bigg\| \int_{s\wedge\tau_{n,m}}^{t\wedge\tau_{n,m}} b_n(Y^n_r, Z^n_r)dr + \int_{s\wedge\tau_{n,m}}^{t\wedge\tau_{n,m}} a_n^{\frac{1}{2}}(Y^n_r, Z^n_r)dW^n_r \bigg\|^p\bigg] \\
&\quad\leq 2^p E\bigg[\bigg\| \int_{s\wedge\tau_{n,m}}^{t\wedge\tau_{n,m}} b_n(Y^n_r, Z^n_r)dr \bigg\|^p\bigg] + 2^p E\bigg[\bigg\| \int_{s\wedge\tau_{n,m}}^{t\wedge\tau_{n,m}} a_n^{\frac{1}{2}}(Y^n_r, Z^n_r)dW^n_r \bigg\|^p\bigg] \\
&\quad\preceq E\bigg[\bigg( \int_{s\wedge\tau_{n,m}}^{t\wedge\tau_{n,m}} \|b_n(Y^n_r, Z^n_r)\|dr \bigg)^p\bigg] + E\bigg[\bigg( \int_{s\wedge\tau_{n,m}}^{t\wedge\tau_{n,m}} \|a_n(Y^n_r, Z^n_r)\|dr \bigg)^{\frac{p}{2}}\bigg] \\
&\quad\preceq \big(|t-s|^p + |t-s|^{\frac{p}{2}}\big) \\
&\quad\preceq |t-s|^{\frac{p}{2}}.
\end{aligned} \tag{2.52}$$

Furthermore, we have

$$\sup_{n\in\mathbb{N}} E\big[\|Y^n_0\|\big] = \sup_{n\in\mathbb{N}} \|x_n\| < \infty,$$

because convergent sequences are bounded. Consequently, [77, Problem 2.4.11, Remark 2.4.13] (i.e. Kolmogorov's tightness criterion) imply that $\{Y^n_{\cdot\wedge\tau_{n,m}}, n \in \mathbb{N}\}$ is tight in $\Sigma_c$. This completes the proof. □

The following lemma is a version of Lemma 2.3 for uniqueness in law instead of pathwise uniqueness.

**Lemma 2.10.** *Let $\rho$ be an $(\mathcal{F}^o_t)_{t\geq 0}$-stopping time and suppose that $P$ is a probability measure on $(\Omega, \mathcal{F})$ such that $P(X_0 = x) = P(\Sigma) = 1$ and*

$$M^f_{t\wedge\rho} = f(X_{t\wedge\rho}) - f(X_0) - \int_0^{t\wedge\rho} \mathcal{L}f(X_s)ds, \quad t \in \mathbb{R}_+, \tag{2.53}$$

*is a $P$-martingale for all $f \in D$. Then, $P = P_x$ on $\mathcal{F}^o_\rho$.*

*Proof.* The claim of this lemma is closely related to the concept of local uniqueness as introduced in [70] and it can be proven with the strategy from [70, Theorem III.2.40]. To each $G \in \mathcal{F}$ we can associate a (not necessarily unique) set $G' \in \mathcal{F}^o_\rho \otimes \mathcal{F}$ such that

$$G \cap \{\rho < \infty\} = \big\{\omega \in \Omega \colon \rho(\omega) < \infty, (\omega, \theta_{\rho(\omega)}\omega) \in G'\big\},$$

see [70, Lemma III.2.44]. Recalling Remark 2.4, we note that $y \mapsto P_y$ is Borel due to Proposition 2.9. Now, set

$$Q(G) \triangleq P(G \cap \{\rho = \infty\}) + \iint \mathbb{1}_{\{\rho(\omega)<\infty\}} \mathbb{1}_{G'}(\omega, \omega^*) P_{\omega(\rho(\omega))}(d\omega^*) P(d\omega).$$

Due to [70, Lemma III.2.47], $Q$ is a probability measure on $(\Omega, \mathcal{F})$. For $G \in \mathcal{F}^o_0$ we can choose $G' = G \times \Omega$. Consequently, we have $Q(X_0 = x) = P(X_0 = x) = 1$. Set

$$\Sigma^* \triangleq \big\{\omega \in \Omega \colon (\omega_{t\wedge\rho(\omega)})_{t\geq 0} \in \Sigma\big\} \supseteq \Sigma$$

and note that

$$\Sigma \cap \{\rho < \infty\} = \big\{\omega \in \Omega \colon \rho(\omega) < \infty, (\omega, \theta_{\rho(\omega)}\omega) \in \Sigma^* \times \Sigma\big\}.$$

Consequently, we have

$$\begin{aligned} Q(\Sigma) &= P(\Sigma \cap \{\rho = \infty\}) + \int \mathbb{1}_{\{\rho(\omega)<\infty\}} \mathbb{1}_{\Sigma^*}(\omega) P_{\omega(\rho(\omega))}(\Sigma) P(d\omega) \\ &= P(\Sigma \cap \{\rho = \infty\}) + P(\Sigma^* \cap \{\rho < \infty\}) \geq P(\Sigma) = 1. \end{aligned}$$

Fix a bounded $(\mathcal{F}^o_t)_{t\geq 0}$-stopping time $\psi$. For $\omega, \alpha \in \Omega$ and $t \in \mathbb{R}_+$ we set

$$z(\omega, \alpha)(t) \triangleq \begin{cases} \omega(t), & t < \rho(\omega), \\ \alpha(t - \rho(\omega)), & t \geq \rho(\omega), \end{cases}$$

and for a fixed $m \in \mathbb{N}$ we set

$$V(\omega, \alpha) \triangleq \begin{cases} \big((\psi \wedge \tau_m) \vee \rho - \rho\big)(z(\omega, \alpha)), & \alpha(0) = \omega(\rho(\omega)), \rho(\omega) < \infty, \\ 0, & \text{otherwise}. \end{cases}$$

Due to [40, Theorem IV.103] the map $V$ is $\mathcal{F}^o_\rho \otimes \mathcal{F}$-measurable and $V(\omega, \cdot)$ is an $(\mathcal{F}^o_t)_{t\geq 0}$-stopping time for all $\omega \in \Omega$. Furthermore, it is evident from the definition that

$$(\psi \wedge \tau_m)(\omega) \vee \rho(\omega) = \rho(\omega) + V(\omega, \theta_{\rho(\omega)}\omega)$$

for $\omega \in \Omega$. We take $f \in D$ and note that for $\omega \in \{\rho < \psi\}$

$$M^f_{V(\omega,\theta_{\rho(\omega)}\omega)}(\theta_{\rho(\omega)}\omega) = M^f_{\psi(\omega)\wedge\tau_m(\omega)-\rho(\omega)}(\theta_{\rho(\omega)}\omega) = M^f_{\psi(\omega)\wedge\tau_m(\omega)}(\omega) - M^f_{\rho(\omega)}(\omega).$$

Since $M^f_{\cdot\wedge\rho}$ is a $P$-martingale and $\psi$ is bounded, the optional stopping theorem yields that

$$E^Q\big[M^f_{\rho\wedge\psi\wedge\tau_m}\big] = E^P\big[M^f_{\rho\wedge\psi\wedge\tau_m}\big] = 0.$$

Therefore, we have

$$\begin{aligned} E^Q\big[M^f_{\psi\wedge\tau_m}\big] &= E^Q\big[M^f_{\psi\wedge\tau_m} - M^f_{\rho\wedge\psi\wedge\tau_m}\big] \\ &= E^Q\big[\big(M^f_{\psi\wedge\tau_m} - M^f_\rho\big)\mathbb{1}_{\{\rho<\psi\wedge\tau_m\}}\big] \\ &= E^Q\big[M^f_{V(\cdot,\theta_\rho)}(\theta_\rho)\mathbb{1}_{\{\rho<\psi\wedge\tau_m\}}\big] \\ &= \int E^{P_{\omega(\rho(\omega))}}\big[M^f_{V(\omega,\cdot)\wedge\tau_m}\big]\mathbb{1}_{\{\rho(\omega)<\psi(\omega)\wedge\tau_m(\omega)\}}P(d\omega) = 0, \end{aligned}$$

again due to the optional stopping theorem (recall that $V(\omega,\cdot)$ is bounded and that $M^f_{\cdot\wedge\tau_m}$ is a $P_y$-martingale for all $y \in S$). We conclude from [123, Proposition II.1.4] and the downwards theorem ([124, Theorem II.51.1]) that $M^f$ is a local $Q$-martingale, which implies that $Q$ solves the MP $(D, \mathcal{L}, \Sigma, x)$. The uniqueness assumption yields that $Q = P_x$. Since also for $G \in \mathcal{F}^o_\rho$ we can choose $G' = G \times \Omega$, we obtain that

$$P_x(G) = Q(G) = P(G).$$

This finishes the proof. □

**Lemma 2.11.** *For all $m \in \mathbb{N}$, all accumulation points of $\{(Y^n_{t\wedge\tau_{n,m}}, Z^n_t)_{t\geq 0}, n \in \mathbb{N}\}$ coincide with $P_{(x,i)}$ on $\mathcal{F}^o_{\tau_{m-1}}$.*

*Proof.* We recall some continuity properties of functions on $\Omega$. For $\omega \in \Omega$, define

$$\begin{aligned} J(\omega) &\triangleq \big\{t > 0\colon \omega(t) \neq \omega(t-)\big\}, \\ V(\omega) &\triangleq \big\{k > 0\colon \tau_k(\omega) < \tau_{k+}(\omega)\big\}, \\ V'(\omega) &\triangleq \big\{u > 0\colon \omega(\tau_u(\omega)) \neq \omega(\tau_u(\omega)-) \text{ and } \|\omega(\tau_u(\omega)-)\| = u\big\}, \end{aligned}$$

which are countable sets, see [70, Lemma VI.2.10]. The map $\omega \mapsto \omega(t)$ is continuous at $\omega$ whenever $t \notin J(\omega)$, see [47, Proposition 3.5.2], and the map $\omega \mapsto \tau_m(\omega)$ is continuous at $\omega$ whenever $m \notin V(\omega)$, see [47, Problem 13, p. 151] and [70, Proposition VI.2.11]. Furthermore, the map $\omega \mapsto \omega(\cdot \wedge \tau_m(\omega))$ is continuous at $\omega$ whenever $m \notin V(\omega) \cup V'(\omega)$, see [47, Problem 13, p. 151] and [70, Proposition VI.2.12].

Fix $f \in D$ and let $Q^m$ be an accumulation point of $\{(Y^n_{\cdot\wedge\tau_{n,m}}, Z^n), n \in \mathbb{N}\}$. Without loss of generality we assume that the law of $(Y^n_{\cdot\wedge\tau_{n,m}}, Z^n)$ converges weakly to $Q^m$ as $n \to \infty$. The set

$$F \triangleq \big\{t > 0\colon Q^m(t \in V \cup V') > 0\big\}$$

is countable, see the proof of [70, Proposition IX.1.17]. Thus, we find a $t_m \in [m-1, m]$ such that $t_m \notin F$. Set

$$U \triangleq \big\{ t \in \mathbb{R}_+ \colon Q^m\big(t \in J(X_{\cdot \wedge \tau_{t_m}})\big) = 0 \big\}.$$

By [47, Lemma 3.7.7], the complement of $U$ in $\mathbb{R}_+$ is countable. Thus, $U$ is dense in $\mathbb{R}_+$. Next, we explain that for all $z \in \mathbb{R}_+$ the map

$$\omega \mapsto I_{t \wedge \tau_z(\omega)}(\omega) \triangleq \int_0^{t \wedge \tau_z(\omega)} \mathcal{L} f(\omega(s)) ds$$

is continuous at all continuity points of $\omega \mapsto \tau_z(\omega)$. Let $(\omega_n)_{n \in \mathbb{N}} \subset \Omega$ and $\omega \in \Omega$ be such that $\omega_n \to \omega$ and $\tau_z(\omega_n) \to \tau_z(\omega)$ as $n \to \infty$. We deduce from [47, Proposition 3.5.2], the fact that $J(\omega)$ is countable, the dominated convergence theorem and the continuity of $x \mapsto \mathcal{L} f(x)$, which is due to the hypothesis that $b$ and $a$ are continuous, that

$$\big| I_{t \wedge \tau_z(\omega)}(\omega) - I_{t \wedge \tau_z(\omega)}(\omega_n) \big| \to 0$$

as $n \to \infty$. We obtain

$$\begin{aligned} &\big| I_{t \wedge \tau_z(\omega)}(\omega) - I_{t \wedge \tau_z(\omega_n)}(\omega_n) \big| \\ &\qquad \le \big| I_{t \wedge \tau_z(\omega)}(\omega) - I_{t \wedge \tau_z(\omega)}(\omega_n) \big| + \big| I_{t \wedge \tau_z(\omega)}(\omega_n) - I_{t \wedge \tau_z(\omega_n)}(\omega_n) \big| \to 0 \end{aligned}$$

as $n \to \infty$, where we use that $\tau_z(\omega_n) \to \tau_z(\omega)$ as $n \to \infty$. It follows that for each $t \in U$ there exists a $Q^m$-null set $N_t$ such that the map

$$\omega \mapsto M^f_{t \wedge \tau_{t_m}(\omega)}(\omega) = f(\omega(t \wedge \tau_{t_m}(\omega))) - f(\omega(0)) - \int_0^{t \wedge \tau_{t_m}(\omega)} \mathcal{L} f(\omega(s)) ds \tag{2.54}$$

is continuous at all $\omega \notin N_t$. For a moment we fix $t \in U$. Suppose that $f \in D$ is independent of the $\mathbb{R}^d$-coordinate (i.e. $f \in C^\star$) and let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions $f_n \in C_n$ such that (2.49) holds. Define $M^{f,n}$ as in (2.53) with $f$ replaced by $f_n$ and $\mathcal{L}$ replaced by $\mathcal{L}_n$. Furthermore, fix $\omega \notin N_t$ and let $(\omega_n)_{n \in \mathbb{N}} \subset \Omega$ be a sequence such that $\omega_n \to \omega$ as $n \to \infty$. Then, for any bounded continuous function $v \colon \Omega \to \mathbb{R}$ we have

$$\begin{aligned} &\big| M^f_{t \wedge \tau_{t_m}(\omega)}(\omega) v(\omega) - M^{f,n}_{t \wedge \tau_{t_m}(\omega_n)}(\omega_n) v(\omega_n) \big| \\ &\qquad \le \big| M^f_{t \wedge \tau_{t_m}(\omega)}(\omega) v(\omega) - M^f_{t \wedge \tau_{t_m}(\omega_n)}(\omega_n) v(\omega_n) \big| \\ &\qquad\quad + \|v\|_\infty \big| M^f_{t \wedge \tau_{t_m}(\omega_n)}(\omega_n) - M^{f,n}_{t \wedge \tau_{t_m}(\omega_n)}(\omega_n) \big| \to 0 \end{aligned} \tag{2.55}$$

as $n \to \infty$, where the first term converges to zero because of the continuity of (2.54) at $\omega$ and the second term converges to zero because

$$\big| M^f_{t \wedge \tau_{t_m}(\omega_n)}(\omega_n) - M^{f,n}_{t \wedge \tau_{t_m}(\omega_n)}(\omega_n) \big| \le 2 \|f - f_n\|_\infty + t \|Qf - Q_n f_n\|_\infty \to 0$$

as $n \to \infty$ by (2.49). Similarly, (2.55) holds if $f \in D$ depends only on the $\mathbb{R}^d$-coordinate provided $M^{f,n}$ is defined as in (2.53) with $\mathcal{L}$ replaced by $\mathcal{L}_n$. In this case, the second term in (2.55) converges to zero because

$$\begin{aligned}&\big|M^f_{t\wedge\tau_{t_m}(\omega_n)}(\omega_n) - M^{f,n}_{t\wedge\tau_{t_m}(\omega_n)}(\omega_n)\big|\\&\quad\leq \text{const. } t \sup_{\|y\|\leq m}\Big(\|b(y)-b_n(y)\| + \|a(y)-a_n(y)\|\Big)\to 0\end{aligned}$$

as $n\to\infty$, due to (2.48). We conclude from [74, Theorem 3.27] that for all $f\in D$ and $t\in U$

$$E^{P^{n,m}}\Big[M^{f,n}_{t\wedge\tau_{t_m}}v\Big]\to E^{Q^m}\Big[M^f_{t\wedge\tau_{t_m}}v\Big] \tag{2.56}$$

as $n\to\infty$, where $P^{n,m}$ denotes the law of $(Y^n_{\cdot\wedge\tau_{n,m}}, Z^n)$.

Fix $s<t$. As $U$ is dense in $\mathbb{R}_+$, we find a sequence $(z_n)_{n\in\mathbb{N}}\subset U$ such that $z_n\searrow t$ as $n\to\infty$ and a sequence $(u_n)_{n\in\mathbb{N}}\subset U$ such that $u_n\searrow s$ as $n\to\infty$. W.l.o.g. we can assume that $u_n\leq z_n$ for all $n\in\mathbb{N}$. Let $v\colon\Omega\to\mathbb{R}$ be continuous, bounded and $\mathcal{F}_s$-measurable. Using the dominated convergence theorem, the right-continuity of $X$ and (2.56), we obtain

$$E^{Q^m}\big[M^f_{t\wedge\tau_{t_m}}v\big] = \lim_{k\to\infty}E^{Q^m}\big[M^f_{z_k\wedge\tau_{t_m}}v\big] = \lim_{k\to\infty}\lim_{n\to\infty}E^{P^{n,m}}\big[M^{f,n}_{z_k\wedge\tau_{t_m}}v\big]. \tag{2.57}$$

The process $M^{f,n}_{\cdot\wedge\tau_{t_m}}$ is a $P^{n,m}$-martingale. To see this, note that

$$\tau_{t_m}\circ(Y^n_{s\wedge\tau_{n,m}}, Z^n_s)_{s\geq 0} = \tau_{n,t_m},$$

see [70, Lemma III.2.43], and recall that martingales are stable under stopping. Consequently, using again (2.56) and the dominated convergence theorem, we conclude from (2.57) that

$$E^{Q^m}\big[M^f_{t\wedge\tau_{t_m}}v\big] = \lim_{k\to\infty}\lim_{n\to\infty}E^{P^{n,m}}\big[M^{f,n}_{u_k\wedge\tau_{t_m}}v\big] = E^{Q^m}\big[M^f_{s\wedge\tau_{t_m}}v\big].$$

Recall that $s<t$ and $v$ were arbitrary.

We claim that this already implies that $M^f_{\cdot\wedge\tau_{t_m}}$ is a $Q^m$-martingale. Take $g\in C_b(S)$ and let $(m_k)_{k\in\mathbb{N}}\subset(0,\infty)$ be such that $m_k\searrow 0$ as $k\to\infty$. We set

$$g^k(q)\triangleq\frac{1}{m_k}\int_q^{q+m_k}g(X_r)dr,\quad k\in\mathbb{N}, q\in\mathbb{R}_+,$$

and note that $g^k(q)\colon\Omega\to\mathbb{R}$ is continuous, bounded and $\mathcal{F}^o_{q+m_k}$-measurable and that $g^k(q)\to g(X_q)$ as $k\to\infty$. Thus, using an approximation argument, we can deduce from the fact that $E^{Q^m}\big[M^f_{t\wedge\tau_{t_m}}v\big] = E^{Q^m}\big[M^f_{s\wedge\tau_{t_m}}v\big]$ holds for all $s<t$ and all continuous, bounded and $\mathcal{F}_s$-measurable $v$ that

$$E^{Q^m}\Big[M^f_{t\wedge\tau_{t_m}}\prod_{i=1}^l g_i(X_{q_i})\Big] = E^{Q^m}\Big[M^f_{s\wedge\tau_{t_m}}\prod_{i=1}^l g_i(X_{q_i})\Big],$$

for all $s<t$, $l\in\mathbb{N}$, $g_1,\dots,g_l\in C_b(S)$ and $q_1,\dots,q_l\in[0,s]$. Using a monotone class argument and the downwards theorem shows that $M^f_{\cdot\wedge\tau_{t_m}}$ is a $Q_m$-martingale.

Since $\omega\mapsto\omega(0)$ is continuous, we have $Q^m(X_0=(x,i))=1$ due to the continuous mapping theorem. Due to [47, Problem 4.25] the set $\Sigma=\Sigma_c\times\Sigma_d$ is a closed set in the product Skorokhod topology on $\Omega = D(\mathbb{R}_+, \mathbb{R}^d) \times \Sigma_d$, and [47, Proposition 3.5.3] implies that $\Sigma$ is closed in $\Omega$, too. Thus, by the Portmanteau theorem, we have $Q^m(\Sigma) = 1$. It follows from Lemma 2.10 that $Q^m$ coincides with $P_{(x,i)}$ on $\mathcal{F}^o_{\tau_{t_m}}$ and thus also on $\mathcal{F}^o_{\tau_{m-1}}$, because $t_m \geq m-1$ implies $\tau_{t_m} \geq \tau_{m-1}$. This completes the proof. □

**Lemma 2.12.** *The sequence $\{(Y^n, Z^n), n \in \mathbb{N}\}$ is tight.*

*Proof.* We use [47, Corollary 3.7.4]. Let us recall it as a fact:

**Fact 2.1.** *Let $(E, r)$ be a Polish space. A sequence $(\mu^n)_{n\in\mathbb{N}}$ of Borel probability measures on $D(\mathbb{R}_+, E)$ is tight if and only if the following hold:*

(a) *For all $t \in \mathbb{Q}_+$ and $\varepsilon > 0$ there exists a compact set $C(t, \varepsilon) \subseteq E$ such that*

$$\limsup_{n\to\infty} \mu^n(X_t \notin C(t,\varepsilon)) \leq \varepsilon.$$

(b) *For all $\varepsilon > 0$ and $t > 0$ there exists a $\delta > 0$ such that*

$$\limsup_{n\to\infty} \mu^n(w'(X, \delta, t) \geq \varepsilon) \leq \varepsilon,$$

*where*

$$w'(\alpha, \theta, t) \triangleq \inf_{\{t_i\}} \max_i \sup_{u,v\in[t_{i-1},t_i)} r(\alpha(u), \alpha(v)),$$

*with $\{t_i\}$ ranging over all partitions of the form $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n \leq t$ with $\min_{1\leq i<n}(t_i - t_{i-1}) \geq \theta$ and $n \geq 1$.*

As in the proof of the Lemma 2.11, let $P^{n,m}$ be the law of $(Y^n_{\cdot\wedge\tau_{n,m}}, Z^n)$ and $P^n$ be the law of $(Y^n, Z^n)$. We fix $t \in \mathbb{R}_+$. Due to [47, Problem 13, p. 151] and [59, Lemma 15.20], the set $\{\tau_{m-1} \leq t\}$ is closed. Moreover, $\{\tau_{m-1} \leq t\} \in \mathcal{F}^o_{\tau_{m-1}}$, because $\tau_{m-1}$ is an $(\mathcal{F}^o_t)_{t\geq 0}$-stopping time. We deduce from the Portmanteau theorem and Lemma 2.11 that

$$\limsup_{n\to\infty} \; P^{n,m}(\tau_{m-1} \leq t) \leq P_{(x,i)}(\tau_{m-1} \leq t). \tag{2.58}$$

Fix $\varepsilon > 0$. Since $P_{(x,i)}(\tau_{m-1} \leq t) \searrow 0$ as $m \to \infty$, we find an $m^o \in \mathbb{N}_{\geq 2}$ such that

$$P_{(x,i)}(\tau_{m^o-1} \leq t) \leq \tfrac{\varepsilon}{2}. \tag{2.59}$$

As $(P^{n,m^o-1})_{n\in\mathbb{N}}$ is tight due to Lemma 2.9, we deduce from Fact 2.1 that there exists a compact set $C(t,\varepsilon) \subseteq S$ such that

$$\limsup_{n\to\infty} P^{n,m^o-1}(X_t \notin C(t,\varepsilon)) \leq \tfrac{\varepsilon}{2}. \tag{2.60}$$

In view of [70, Lemma III.2.43] we obtain

$$\begin{aligned} P^n(X_t \notin C(t,\varepsilon)) &= P^n(X_t \notin C(t,\varepsilon), \tau_{m^o-1} > t) + P^n(X_t \notin C(t,\varepsilon), \tau_{m^o-1} \leq t) \\ &\leq P^{n,m^o-1}(X_t \notin C(t,\varepsilon)) + P^{n,m^o}(\tau_{m^o-1} \leq t). \end{aligned}$$

From this, (2.58), (2.59) and (2.60), we deduce that

$$\limsup_{n\to\infty} P^n(X_t \notin C(t,\varepsilon)) \leq \varepsilon.$$

This proves that the sequence $(P^n)_{n\in\mathbb{N}}$ satisfies (a) in Fact 2.1.

Next, we show that $(P^n)_{n\in\mathbb{N}}$ satisfies (b) in Fact 2.1. Let $\varepsilon, t$ and $m^o$ be as before. As $(P^{n,m^o-1})_{n\in\mathbb{N}}$ is tight due to Lemma 2.9 there exists a $\delta > 0$ such that

$$\limsup_{n\to\infty} P^{n,m^o-1}\big(w'((X_s)_{s\geq 0}, \delta, t) \geq \varepsilon\big) \leq \tfrac{\varepsilon}{2}. \tag{2.61}$$

Thus, similar as above, using (2.58), (2.59) and (2.61), we obtain

$$\begin{aligned}&\limsup_{n\to\infty} P^n(w'((X_s)_{s\geq 0}, \delta, t) \geq \varepsilon)\\&\qquad\leq \limsup_{n\to\infty} P^{n,m^o-1}(w'((X_s)_{s\geq 0}, \delta, t) \geq \varepsilon) + \limsup_{n\to\infty} P^{n,m^o}(\tau_{m^o-1} \leq t) \leq \varepsilon.\end{aligned}$$

In other words, $(P^n)_{n\in\mathbb{N}}$ satisfies also (b) in Fact 2.1 and the proof is complete. □

We are in the position to complete the proof of Theorem 2.5. To wit, in view of [14, Corollary to Theorem 5.1], because $\{(Y^n, Z^n), n \in \mathbb{N}\}$ is tight by the previous lemma, for $P^n \to P_{(x,i)}$ weakly as $n \to \infty$, it remains to show that any accumulation point $Q$ of $\{(Y^n, Z^n), n \in \mathbb{N}\}$ coincides with $P_{(x,i)}$. It follows as in the proof of Lemma 2.11 that the process $M^f$ is a local $Q$-martingale for all $f \in D$. Since $\omega \mapsto \omega(0)$ is continuous, we also have $Q(X_0 = (x, i)) = 1$ and, because $\Sigma$ is closed in $\Omega$, the Portmanteau theorem yields that $Q(\Sigma) = 1$. It follows that $Q$ solves the MP $(D, \mathcal{L}, \Sigma, (x, i))$. Due to the uniqueness assumption, $Q = P_{(x,i)}$ and the proof is complete. □

#### 2.4.3.6 Proof of Proposition 2.8

The existence is shown in the proof of Theorem 2.4. The uniqueness follows from a Yamada–Watanabe argument, which we only sketch. Fix $y = (x, i) \in S$ and suppose that $P_y$ and $Q_y$ solve the MP $(\mathcal{L}, D, \Sigma, y)$. Using similar arguments as in the proof of [66, Theorem 8.3], we obtain the following: We find a filtered probability space satisfying the usual hypothesis on which we can realize $P_y$ as the law of the process $(Y, Z)$, where $Z$ is a Markov chain with $Q$-matrix $Q$ and $Z_0 = i$ and

$$dY_t = b(Y_t, Z_t)dt + a^{\frac{1}{2}}(Y_t, Z_t)dW_t, \quad Y_0 = x,$$

where $W$ is a Brownian motion. On the same probability space, we can realize $Q_y$ as the law of $(V, Z)$, where

$$dV_t = b(V_t, Z_t)dt + a^{\frac{1}{2}}(V_t, Z_t)dW_t, \quad V_0 = x.$$

We stress that the driving system $(Z, W)$ coincides for $Y$ and $V$ Now, we claim that $Y_t = V_t$ for all $t \in \mathbb{R}_+$ up to a null set. This immediately implies $Q_y = P_y$. We prove this claim by induction. Let $(\tau_n)_{n\in\mathbb{N}}$ be the stopping times as defined in (2.45). We stress that a.s. $\tau_n \nearrow \infty$ as $n \to \infty$. On $\{t \leq \tau_1\}$ we have

$$\begin{aligned}Y_t &= x + \int_0^t b(Y_s, i)ds + \int_0^t a^{\frac{1}{2}}(Y_s, i)dW_s,\\V_t &= x + \int_0^t b(V_s, i)ds + \int_0^t a^{\frac{1}{2}}(V_s, i)dW_s.\end{aligned}$$

The strong existence hypothesis and Lemma 6.6 imply that $Y_t = V_t$ for all $t \leq \tau_1$ up to a null set. Suppose that $n \in \mathbb{N}$ is such that $Y_t = V_t$ for all $t \leq \tau_n$ up to a null set. Using classical rules for time-changed stochastic integrals, we obtain that on $\{t \leq \tau_{n+1} - \tau_n\} \cap \{Z_{\tau_n} = k\}$

$$\begin{aligned} Y_{t+\tau_n} &= Y_{\tau_n} + \int_{\tau_n}^{t+\tau_n} b(Y_s, k)ds + \int_{\tau_n}^{t+\tau_n} a^{\frac{1}{2}}(Y_s, k)dW_s \\ &= Y_{\tau_n} + \int_0^t b(Y_{s+\tau_n}, k)ds + \int_0^t a^{\frac{1}{2}}(Y_{s+\tau_n}, k)dW_s^n \end{aligned}$$

and

$$V_{t+\tau_n} = V_{\tau_n} + \int_0^t b(V_{s+\tau_n}, k)ds + \int_0^t a^{\frac{1}{2}}(V_{s+\tau_n}, k)dW_s^n,$$

where

$$W_t^n \triangleq W_{t+\tau_n} - W_{\tau_n}, \quad t \in \mathbb{R}_+.$$

We conclude again from the strong existence hypothesis and Lemma 6.6 that $Y_{t+\tau_n} = V_{t+\tau_n}$ for all $t \leq \tau_{n+1} - \tau_n$ up to a null set. Consequently, $Y_t = V_t$ for all $t \leq \tau_{n+1}$ up to a null set and our claim follows. □

#### 2.4.3.7 Proof of Corollary 2.2

Due to [77, Theorems 5.5.15, 5.5.29] and [137, Corollary 11.1.5], for all $i \in S_d$ the family $(P_x^i)_{x\in\mathbb{R}^d}$ exists uniquely and is $C_b$-Feller. Using the local Hölder condition on the diffusion coefficient, [123, Lemma IX.3.3, Proposition IX.3.2] and [74, Theorem 18.14] imply that $(P_x^i)_{x\in\mathbb{R}^d}$ exists strongly. Consequently, $(P_x)_{x\in S}$ exists uniquely due to Proposition 2.8. Now, $(P_x)_{x\in S}$ is strongly Markov and $C_b$-Feller due to Proposition 2.7 and the equivalence of (i) and (ii) follows from Theorem 2.4, Remark 2.1 and [120, Theorem 8.4.1]. □

#### 2.4.3.8 Proof of Corollary 2.3

Due to [77, Theorem 5.2.5], [74, Theorem 18.14] and [137, Corollary 11.1.5], for all $i \in S_d$ the family $(P_x^i)_{x\in\mathbb{R}^d}$ exists strongly and is $C_b$-Feller. Consequently, $(P_x)_{x\in S}$ exists uniquely due to Proposition 2.8. As in the proof of Proposition 2.4, we deduce from Theorem 2.1 that $(P_x^i)_{x\in\mathbb{R}^d}$ is Feller–Dynkin for all $i \in S_d$. Now, $(P_x)_{x\in S}$ is strongly Markov and $C_b$-Feller due to Proposition 2.7 and Feller–Dynkin due to Theorem 2.4. □

#### 2.4.3.9 Proof of Corollary 2.4

Due to [77, Theorem 5.2.5], [74, Theorem 18.14] and [137, Corollary 11.1.5], the family $(P_x^i)_{x\in\mathbb{R}^d}$ exists strongly and is $C_b$-Feller. Moreover, as in the proof of Proposition 2.5, we deduce from Theorem 2.2 that $(P_x^i)_{x\in\mathbb{R}^d}$ is not Feller–Dynkin. Finally, the claim follows from Proposition 2.6. □

## 2.5 An Existence Theorem for Switching Diffusions

In this section we give an existence theorem for switching diffusions with state-independent switching. We pose ourselves in the setting of Section 2.4.3.

**Theorem 2.6.** *Let $b\colon S \to \mathbb{R}^d$ and $a\colon S \to \mathbb{S}^d$ be continuous functions such that for all $m \in \mathbb{R}_+$*

$$\sup_{\|x\|\le m} \sup_{i\in S_d} \big(\|b(x,i)\| + \|a(x,i)\|\big) < \infty. \tag{2.62}$$

*Let $\mathcal{K}^i$ be given as in (2.32). Suppose that there exists two constants $c, \lambda > 0$, a function $v\colon \mathbb{R}_+ \to (0,\infty)$ and a twice continuously differentiable function $V\colon \mathbb{R}^d \to (0,\infty)$ such that $V(x) \ge v(\|x\|)$ for all $x \in \mathbb{R}^d\colon \|x\| \ge \lambda$, $\limsup_{n\to\infty} v(n) = \infty$ and*

$$\mathcal{K}^i V(x) \le cV(x),$$

*for all $(x,i) \in S$. Then, for any Borel probability measure $\eta$ on $S$ there exists a solution to the MP $(D,\mathcal{L},\Sigma,\eta)$.*

*Proof.* Due to Proposition 2.9 in Section 2.6, it suffices to show the claim for degenerated initial laws, i.e. we assume that $\eta(\{y\}) = 1$ for some $y \in S$.

*Step 1.* We first show the claim under the assumptions that $b$ and $a$ are continuous and bounded, i.e. $\|b(x,i)\| + \|a(x,i)\| \le c^*$ for all $(x,i) \in S$. Our initial step is a standard mollification argument. Let $\phi$ be the standard mollifier, i.e.

$$\phi(x) \triangleq \begin{cases} \theta \exp\big\{-(1-\|x\|^2)^{-1}\big\}, & \text{if } \|x\| < 1,\\ 0, & \text{otherwise,}\end{cases}$$

where $\theta > 0$ is a constant such that $\int \phi(x)dx = 1$. Let $\sigma$ be a root of $a$. For $(x,i) \in S$ we set

$$b_n(x,i) \triangleq n^d \int b(y,i)\phi(n(x-y))dy,$$
$$\sigma_n(x,i) \triangleq n^d \int \sigma(y,i)\phi(n(x-y))dy.$$

It is well-known that $x \mapsto b_n(x,i)$ and $x \mapsto \sigma_n(x,i)$ are smooth for all $i \in S_d$ and that $b_n \to b$ and $\sigma_n\sigma_n^* \to a$ as $n \to \infty$ uniformly on compact subsets of $S$. Furthermore, using that $\int \phi(x)dx = 1$, we obtain

$$\|b_n(x,i)\| \le n^d \int \|b(y,i)\|\phi(n(x-y))dy = \int \|b(x-n^{-1}z,i)\|\phi(z)dz \le c^*$$

and, in the same manner, $\|\sigma_n(x,i)\| \le c^*$ for all $(x,i) \in S$. Since smooth functions are locally Lipschitz continuous, we deduce from [117, Theorem 18.16], [74, Theorem 18.14] and Proposition 2.8 that for each $n \in \mathbb{N}$ there exists a solution $P^n$ to the MP $(D,\mathcal{L}_n,\Sigma,y)$, where $\mathcal{L}_n$ is defined as in (2.23) with $b$ replaced by $b_n$ and $a$ replaced by $a_n$. If we show that the sequence $(P^n)_{n\in\mathbb{N}}$ is tight and that any accumulation point of it solves the MP $(D,\mathcal{L},\Sigma,y)$ the claim of the theorem follows. That any accumulation point of $(P^n)_{n\in\mathbb{N}}$ solves the MP $(D,\mathcal{L},\Sigma,y)$ can be shown as in the proof of Theorem 2.5 and that $(P^n)_{n\in\mathbb{N}}$ is tight follows as in the proof of Lemma 2.9. Thus, the claim holds under the assumptions that $b$ and $a$ are continuous and bounded.

*Step 2.* We now deal with the general case. Let $\psi^n\colon \mathbb{R}^d \to [0,1]$ be a sequence of cutoff functions, i.e. non-negative smooth functions with compact support such that $\psi^n(x) = 1$

for $x \in \mathbb{R}^d$: $\|x\| \le n$. We set

$$b_n(x,i) \triangleq \psi^n(x)b(x,i), \qquad a_n(x,i) \triangleq \psi^n(x)a(x,i), \quad (x,i) \in S.$$

The functions $b_n$ and $a_n$ are continuous and bounded. Therefore, due to our first step, for each $n \in \mathbb{N}$ there exist a solution $P^n$ to the MP $(D, \mathcal{L}_n, \Sigma, y)$. We write $X = (X^1, X^2)$ and set

$$\tau_m \triangleq \inf\big(t \in \mathbb{R}_+ \colon \|X^1_t\| \ge m \text{ or } \|X^1_{t-}\| \ge m\big), \quad m \in \mathbb{R}_+.$$

Furthermore, we denote $P^{n,m} \triangleq P^n \circ (X^1_{\cdot\wedge\tau_m}, X^2)^{-1}$. It follows as in the proof of Lemma 2.9 that the sequence $(P^{n,m})_{n\in\mathbb{N}}$ is tight for every $m \in \mathbb{R}_+$. We note that for all $m \in \mathbb{R}_+$

$$\begin{aligned}\sup_{\|x\|\le m} &\big(\|b(x) - b_n(x)\| + \|a(x) - a_n(x)\|\big) \\ &\le 2 \sup_{\|x\|\le m}\big(\|b(x)\| + \|a(x)\|\big) \sup_{\|z\|\le m} |1 - \psi^n(z)| \to 0\end{aligned}$$

as $n \to \infty$. Thus, recalling the proof of Lemma 2.12 and Step 1 reveal that the existence of a solution to the MP $(D, \mathcal{L}, \Sigma, y)$ follows once we prove that for each $T > 0$ and $\varepsilon > 0$ we find an $m \in \mathbb{R}_+$ such that

$$\limsup_{n\to\infty} P^n(\tau_m \le T) \le \varepsilon. \tag{2.63}$$

Define $\mathcal{K}^{i,n}$ as $\mathcal{K}^i$ with $b$ and $a$ replaced by $b_n$ and $a_n$. We have

$$\mathcal{K}^{i,n}V(x) = \psi^n(x)\mathcal{K}^iV(x) \le c\psi^n(x)V(x) \le cV(x)$$

for all $(x,i) \in S$ and $n \in \mathbb{N}$. By Lemma 2.1 the process

$$U_t \triangleq e^{-c(t\wedge\tau_m)}V(X^1_{t\wedge\tau_m}) + \int_0^{t\wedge\tau_m} e^{-cs}\big(cV(X^1_s) - \mathcal{K}^{X^2_s,n}V(X^1_s)\big)ds, \quad t \in \mathbb{R}_+,$$

is a local $P^n$-martingale. Furthermore, because $U_t \ge e^{-c(t\wedge\tau_m)}V(X^1_{t\wedge\tau_m}) \ge 0$ for all $t \in \mathbb{R}_+$, the process $U$ is a non-negative $P^n$-supermartingale. We deduce that for all $m \ge \lambda \vee \|x\|$

$$\begin{aligned}P^n(\tau_m \le T)e^{-cT}v(m) &= E^n\Big[\mathbb{1}_{\{\tau_m\le T\}}e^{-cT}v(\|X^1_{\tau_m}\|)\Big] \\ &\le E^n\Big[\mathbb{1}_{\{\tau_m\le T\}}e^{-c(T\wedge\tau_m)}V\big(X^1_{T\wedge\tau_m}\big)\Big] \\ &\le E^n\Big[e^{-c(T\wedge\tau_m)}V\big(X^1_{T\wedge\tau_m}\big)\Big] \\ &\le E^n\Big[U_T\Big] \le V(x),\end{aligned}$$

where $y = (x,i)$. The assumption $\limsup_{m\to\infty} v(m) = \infty$ yields that we find an $m \ge \lambda$ such that (2.63) holds. This completes the proof. □

**Remark 2.7.** (i) *On one hand, the previous existence result does not require any uniqueness or strong existence assumption for the SDEs for the fixed environments. On the other hand, it does not provide a uniqueness statement.*

(ii) *Using* $V(x) = 1 + \|x\|^2$ *yields that the growth condition*

$$2\langle x, b(x,k)\rangle + \operatorname{trace} a(x,k) \le c\big(1 + \|x\|^2\big), \quad \textit{for all } (x,k) \in S,$$

*implies the existence of a solution to the MP* $(D, \mathcal{L}, \Sigma, \eta)$ *whenever the coefficients* $b$ *and* $a$ *are continuous and satisfy* (2.62).

## 2.6 The Role of Initial Laws

For the setting of Example 2.1 it is known that the existence of (unique) solutions for all degenerated initial laws implies the existence of (unique) solutions for all initial laws, see [73, Propositions 1 and 2]. The following proposition shows that these observations also hold in our setting. The proof is close to the diffusion case and we only sketch it.

**Proposition 2.9.** *Suppose that* $D$ *is countable, that* $D \subseteq C_b(S)$ *and that* $\mathcal{L}(D) \subseteq B_{\mathrm{loc}}(S)$. *Furthermore, let* $\eta$ *be a Borel probability measure on* $S$. *If for all* $y \in S$ *the MP* $(D, \mathcal{L}, \Sigma, y)$ *has a solution* $P_y$, *then also the MP* $(D, \mathcal{L}, \Sigma, \eta)$ *has a solution. Moreover, if the family* $(P_y)_{y \in S}$ *is unique, then* $y \mapsto P_y(A)$ *is Borel for all* $A \in \mathcal{F}$ *and* $\int P_y \eta(dy)$ *is the unique solution to the MP* $(D, \mathcal{L}, \Sigma, \eta)$.

*Sketch of Proof.* We assume that the MP $(D, \mathcal{L}, \Sigma, y)$ has a solution for all $y \in S$. Let $\eta$ be a Borel probability measure on $S$ and let $\mathcal{P}$ denote the set of all solutions to the MP $(D, \mathcal{L}, \Sigma, y)$ for all $y \in S$. We consider $\mathcal{P}$ as a subspace of the Polish space $\mathscr{P}$ of probability measures on $(\Omega, \mathcal{F})$ equipped with the topology of convergence in distribution. Let $(K_n)_{n \in \mathbb{N}} \subset S$ be a sequence of compact sets such that $K_n \subset \operatorname{int}(K_{n+1})$ and $\bigcup_{n \in \mathbb{N}} K_n = S$. For all $n \in \mathbb{N}$ define $\tau_n \triangleq \inf(t \in \mathbb{R}_+ \colon X_t \notin \operatorname{int}(K_n) \text{ or } X_{t-} \notin \operatorname{int}(K_n))$ and for $f \in D$ denote the process (2.3) by $(M^f_t)_{t \ge 0}$. As we assume that $\mathcal{L}(D) \subseteq B_{\mathrm{loc}}(S)$, a probability measure $P$ solves the MP $(D, \mathcal{L}, \Sigma, \eta)$ if and only if $P(\Sigma) = 1$, $P \circ X_0^{-1} = \eta$ and for all $f \in D$ and $n \in \mathbb{N}$ the stopped process $(M^f_{t \wedge \tau_n})_{t \ge 0}$ is a $P$-martingale for the filtration $(\mathcal{F}^o_t)_{t \ge 0}$. Since $D$ is assumed to be countable, the argument outlined in [137, Exercise 6.7.4] shows that $\mathcal{P}$ is a Borel subset of $\mathscr{P}$. Thus, $\mathcal{P}$ is a Borel space in the sense of [74, p. 456]. Let $\Phi \colon \mathcal{P} \to S$ be such that $\Phi(P)$ is the starting point associated to $P \in \mathcal{P}$. We note that $\Phi$ is continuous and that its graph $G \triangleq \big\{(P, \Phi(P)) \colon P \in \mathcal{P}\big\}$ is a Borel subset of $\mathcal{P} \times S$. We have $\bigcup_{P \in \mathcal{P}} \big\{s \in S \colon s = \Phi(P)\big\} = S$, by the assumption that there exist solutions for all degenerated initial laws. Using the section theorem [74, Theorem A.1.8] we see that there exists a Borel map $x \mapsto P_x$ and a $\eta$-null set $N \in \mathcal{B}(S)$ such that $(P_x, x) \in G$ for all $x \notin N$. By the definition of $G$, for all $x \notin N$ the probability measure $P_x$ solves the MP $(D, \mathcal{L}, \Sigma, x)$. It follows that the probability measure $\int P_x \eta(dx)$ solves the MP $(D, \mathcal{L}, \Sigma, \eta)$.

Assume now that $P_x$ is the unique solution to the MP $(D, \mathcal{L}, \Sigma, x)$ for all $x \in S$. Using Kuratovski's theorem as outlined in [137, Exercise 6.7.4] shows that $x \mapsto P_x$ is Borel. Let $P$ be a solution to the MP $(D, \mathcal{L}, \Sigma, \eta)$. Arguing as in the proof of [77, Lemma 5.4.19] shows that there exists a null set $N \in \mathcal{F}^o_0$ such that $P(\cdot | \mathcal{F}^o_0)(\omega)$ solves the MP $(D, \mathcal{L}, \Sigma, X_0(\omega))$ for all $\omega \notin N$. By the uniqueness assumption, this yields that $P$-a.s. $P_{X_0} = P(\cdot | \mathcal{F}^o_0)$. Using this observation together with the tower rule shows that $P = \int P_x \eta(dx)$. □

It is often the case that the input data of a martingale problem can be reduced such that the prerequisites of Proposition 2.9 are met, see Proposition 2.3 and Example 2.4.

# 3 Absolute Continuity of Laws of Semimartingales

## 3.1 Introduction

In the 1970s, probabilists studied conditions under which laws of semimartingales are (locally) absolutely continuous. The most general results were obtained by Jacod and Mémin [67] and Kabanov, Liptser and Shiryaev [71, 72] under a strong uniqueness assumption, called local uniqueness in the monograph by Jacod and Shiryaev [70].

In this chapter we provide equivalent statements for the (local) absolute continuity of semimartingales on random sets under the assumption that the dominated law is unique. While in Markovian settings local uniqueness is implied by uniqueness, it is surprising that this weaker condition suffices also beyond Markovian setups.

Our main tool is a generalized version of Girsanov's theorem for semimartingales, which relates two laws of semimartingales on random sets through a local martingale density. Key of the proof is to replace the classical Skorokhod space by a slightly larger path space whose topological properties allow the extension of relevant consistent families of probability measures.

Let us highlight related results from the literature. Under the so-called Engelbert–Schmidt conditions, a deterministic characterization of the (local) absolute continuity of one-dimensional Itô-diffusions was given by Cherny and Urusov [23]. In a similar setting, Mijativić and Urusov [107] proved equivalent conditions for the martingale property of stochastic exponentials. In both cases, the proofs are different from ours. We relate our main result to these observations and explain that the deterministic characterizations also follow from our main result, see Section 3.4.1. In an Itô jump-diffusion setting, Cheridito, Filipović and Yor [21] proved local absolute continuity if the dominated measure is unique and non-explosive. In Section 3.4.2, we explain the relation of their result to ours. In a multidimensional Itô diffusion setting, Ruf [127] proved equivalent conditions for the martingale property of stochastic exponentials using an extension argument similar to ours. This result can be deduced from ours.

In Section 3.5 we generalize Beneš's [7] linear growth condition for the martingale property of stochastic exponentials to continuous Itô-process drivers. This application does not require any uniqueness assumption.

Let us also comment on further related literature. An extension argument similar to ours was used by Ruf and Perkowski [119] to study Föllmer measures, and by Kardaras, Kreher and Nikeghbali [79] to study the influence of strict local martingales on pricing financial derivatives.

The chapter is structured as follows. In Section 3.2 we introduce our setting and present our main results. Criteria for absolute continuity of semimartingales are studied in Section 3.3 and in Section 3.4 we relate our results to those in [21, 107]. Finally, in Section 3.5 we derive criteria for the martingale property of stochastic exponentials.

## 3.2 A Generalized Girsanov Theorem

We start by introducing our probabilistic setup. We adjoint an isolated point $\Delta$ to $\mathbb{R}^d$ and write $\mathbb{R}^d_\Delta \triangleq \mathbb{R}^d \cup \{\Delta\}$. For a function $\alpha\colon \mathbb{R}_+ \to \mathbb{R}^d_\Delta$ we define

$$\tau_\Delta(\alpha) \triangleq \inf(t \in \mathbb{R}_+\colon \alpha(t) = \Delta).$$

Let $\Omega$ to be the set of all functions $\alpha\colon \mathbb{R}_+ \to \mathbb{R}^d_\Delta$ such that $\alpha$ is càdlàg on $[0, \tau_\Delta(\alpha))$ and $\alpha(t) = \Delta$ for all $t \geq \tau_\Delta(\alpha)$. Let $X_t(\alpha) = \alpha(t)$ be the coordinate process and define $\mathcal{F} \triangleq \sigma(X_t, t \in \mathbb{R}_+)$. Moreover, for each $t \in \mathbb{R}_+$ we define $\mathcal{F}^o_t \triangleq \sigma(X_s, s \in [0,t])$ and $\mathcal{F}_t \triangleq \bigcap_{s>t} \mathcal{F}^o_s$. We work with the right-continuous filtration $\mathbf{F} \triangleq (\mathcal{F}_t)_{t\geq 0}$. In general, if we use terms such as *local martingale, semimartingale, stopping time, predictable*, etc. we refer to $\mathbf{F}$ as the underlying filtration.

Note that for all $t \in \mathbb{R}_+$

$$\{\tau_\Delta \leq t\} = \{X_t = \Delta\} \in \mathcal{F}^o_t \subseteq \mathcal{F}_t,$$

which implies that $\tau_\Delta$ is a stopping time.

For a stopping time $\xi$ we set

$$\mathcal{F}_\xi \triangleq \{A \in \mathcal{F}\colon A \cap \{\xi \leq t\} \in \mathcal{F}_t \text{ for all } t \in \mathbb{R}_+\},$$

and

$$\mathcal{F}_{\xi-} \triangleq \sigma\left(\mathcal{F}^o_0, \{A \cap \{\xi > t\}\colon t \in \mathbb{R}_+, A \in \mathcal{F}_t\}\right).$$

We note that in the second definition the treatment of the initial $\sigma$-field is different from the classical definition, where $\mathcal{F}_0$ is used instead of $\mathcal{F}^o_0$. In our case, $\mathcal{F}_{\xi-}$ is countably generated, see [119, Lemma E.1], which is important for the extension argument in the proof of our first main result, Theorem 3.1 below.

The following facts for $\mathcal{F}_{\xi-}$ can be verified as in the classical case:

(a) $\mathcal{F}_{\xi-} \subseteq \mathcal{F}_\xi$.

(b) For two stopping times $\xi$ and $\rho$ and any $G \in \mathcal{F}_\xi$ we have $G \cap \{\rho > \xi\} \in \mathcal{F}_{\rho-}$ and for all $G \in \mathcal{F}$ we have $G \cap \{\rho = \infty\} \in \mathcal{F}_{\rho-}$.

(c) For an increasing sequence $(\rho_n)_{n\in\mathbb{N}}$ of stopping times with $\rho \triangleq \lim_{n\to\infty} \rho_n$ it holds that

$$\bigvee_{n\in\mathbb{N}} \mathcal{F}_{\rho_n-} = \mathcal{F}_{\rho-}.$$

For two stopping times $\xi$ and $\rho$ we define the stochastic interval

$$[\![\xi, \rho]\!] \triangleq \{(\omega, t) \in \Omega \times \mathbb{R}_+\colon \xi(\omega) \leq t \leq \rho(\omega)\}.$$

All other stochastic intervals $[\![\xi, \rho[\![, ]\!]\xi, \rho]\!]$ and $]\!]\xi, \rho[\![$ are defined in the same manner.

In the spirit of stochastic differential equations up to explosion, we now formulate a semimartingale problem up to explosion. We start by introducing the parameters:

(i) Let $(B, C, \nu)$ be a so-called *candidate triplet* consisting of

- a predictable $\mathbb{R}^d_\Delta$-valued process $B$.

– a predictable $(\mathbb{R} \cup \{\infty\})^{d\times d}$-valued process $C$, which admits a decomposition

$$C = \int_0^\cdot c_s dA_s,$$

where $c$ is a predictable $\mathbb{S}^d$-valued process and $A$ is a non-negative, increasing, predictable and right-continuous process starting in the origin. Here, $\mathbb{S}^d$ denotes the set of all symmetric non-negative definite real $d \times d$ matrices. The entries of the integral are set to be $\infty$ whenever they diverge.

– a predictable random measure $\nu$ on $\mathbb{R}_+ \times \mathbb{R}^d$.

(ii) Let $\eta$ be a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, which we call *initial law.*

(iii) Let $\rho$ be a stopping time, which we call *lifetime.*

We fix a truncation function $h$ and suppose that all terms such as *semimartingale characteristics* refer to this truncation function.

The idea of the semimartingale problem formulated below is to find a probability measure on $(\Omega, \mathcal{F})$ such that the coordinate process $X$ is a semimartingale with characteristics $(B, C, \nu)$ up to the lifetime $\rho$ and with initial law $\eta$.

**Definition 3.1.** *We call a probability measure $P$ on $(\Omega, \mathcal{F})$ a* solution to the semimartingale problem (SMP) *associated with $(\rho; \eta; B, C, \nu)$, if there exists an increasing sequence $(\rho_n)_{n\in\mathbb{N}}$ of stopping times and a sequence of $P$-semimartingales $(X^n)_{n\in\mathbb{N}}$ such that $\rho_n \nearrow \rho$ as $n \to \infty$ and for all $n \in \mathbb{N}$ the following holds:*

(i) *The stopped process $X^{\rho_n} \triangleq (X_{t\wedge\rho_n})_{t\geq 0}$ is $P$-indistinguishable from $X^n$.*

(ii) *The $P$-characteristics of the $P$-semimartingale $X^n$ are $P$-indistinguishable from the stopped triplet $(B^{\rho_n}, C^{\rho_n}, \nu^{\rho_n})$, where*

$$\nu^{\rho_n}(\omega, dt \times dx) \triangleq \mathbb{1}_{[\![0,\rho_n]\!]\times\mathbb{R}^d}(\omega, t, x)\nu(\omega, dt \times dx).$$

(iii) $P \circ X_0^{-1} = \eta$.

*The sequence $(\rho_n)_{n\in\mathbb{N}}$ is called* $\rho$-localization sequence *and the sequence $(X^n)_{n\in\mathbb{N}}$ is called* fundamental sequence. *If $P(\rho = \infty) = 1$, we say that $P$ is* conservative.

In a conservative setting the semimartingale problem was first introduced by Jacod [64]. In this section we impose the following standing assumption.

**Standing Assumption 3.1.** *The underlying probability measure $P$ is a solution to the SMP $(\rho; \eta; B, C, \nu)$ with $\rho$-localization sequence $(\rho_n)_{n\in\mathbb{N}}$ and fundamental sequence $(X^n)_{n\in\mathbb{N}}$, and $Z$ is a non-negative local $P$-martingale such that $E^P[Z_0] = 1$ and $(\sigma_n)_{n\in\mathbb{N}}$ is an increasing sequence of stopping times such that $Z^{\sigma_n}$ is a uniformly integrable $P$-martingale. Furthermore, $P$-a.s. $\sigma_n < \sigma \triangleq \lim_{n\to\infty} \sigma_n$ and, w.l.o.g., $\rho_n \vee \sigma_n \leq n$ for all $n \in \mathbb{N}$.*

Of course, since we assume that $\sigma_n \leq n$, the stopped process $Z^{\sigma_n}$ is a uniformly integrable $P$-martingale whenever it is a $P$-martingale.

Let us further comment on this standing assumption. Our aim is to relate $P$ and $Z$ to another solution of an SMP. We start with a local relation and define a sequence $(Q_n)_{n\in\mathbb{N}}$ of probability measures via

$$Q_n \triangleq Z_{\sigma_n} \cdot P, \tag{3.1}$$

which means $Q_n(G) = E^P\left[Z_{\sigma_n}\mathbb{1}_G\right]$ for all $G \in \mathcal{F}$. Each $Q_n$ solves an SMP by Girsanov's theorem. The next step is to extend this sequence and to show that the extension also solves an SMP. We observe that the sequence $(Q_n)_{n\in\mathbb{N}}$ is consistent and consequently classical extension arguments yield that we find a probability measure $Q$ such that $Q = Q_n$ on $\mathcal{F}_{\sigma_n-}$ for all $n \in \mathbb{N}$. Next, we want to conclude that $Q$ solves an SMP. For this aim, however, the identity $Q = Q_n$ on $\mathcal{F}_{\sigma_n-}$ is not sufficient. At this point, the last part of our standing assumption comes into play. In the proof of Theorem 3.1 below we show that for any $G \in \mathcal{F}_{\sigma_n}$

$$Q(G \cap \{\sigma_n < \sigma\}) = E^P\left[Z_{\sigma_n}\mathbb{1}_{G\cap\{\sigma_n<\sigma\}}\right].$$

Using our assumption that $P$-a.s. $\sigma_n < \sigma$, this identity implies that $Q_n = Q$ on $\mathcal{F}_{\sigma_n}$, which allows us to conclude that $Q$ solves an SMP.

In the following two remarks we comment on choices for $(\sigma_n)_{n\in\mathbb{N}}$ and explain how to construct $Z$ from a non-negative local $P$-martingale, which is only defined on a random set.

**Remark 3.1.** *An example for the sequence $(\sigma_n)_{n\in\mathbb{N}}$ in Standing Assumption 3.1 is*

$$\sigma_n \triangleq \inf(t \in \mathbb{R}_+ \colon Z_t > n) \wedge n.$$

*To see this, it suffices to note that $Z_{t\wedge\sigma_n} \le n + Z_{\sigma_n}$. Since $Z_{\sigma_n}$ is $P$-integrable by Fatou's lemma, $Z^{\sigma_n}$ is a uniformly integrable $P$-martingale by the dominated convergence theorem. Furthermore, in this case $\{\sigma = \infty\}$ and $\{\sigma_n < \sigma\}$ are $P$-full sets. More generally, $\sigma_n$ can be chosen as $\gamma_n \wedge n$, where $(\gamma_n)_{n\in\mathbb{N}}$ is a $P$-localizing sequence for $Z$.*

**Remark 3.2.** *Let $(\xi_n)_{n\in\mathbb{N}}$ be an increasing sequence of stopping times. We say that a process $\widehat{Z}$ is a non-negative local $P$-martingale on the random set $\bigcup_{n\in\mathbb{N}}[\![0,\xi_n]\!]$, if the stopped process $\widehat{Z}^{\xi_n}$ is a non-negative local $P$-martingale. It is always possible to extend the process to a globally defined non-negative local $P$-martingale by setting*

$$Z \triangleq \begin{cases} \widehat{Z}, & \text{on } \bigcup_{n\in\mathbb{N}}[\![0,\xi_n]\!], \\ \liminf_{n\to\infty} \widehat{Z}_{\xi_n}, & \text{otherwise.} \end{cases} \tag{3.2}$$

*By Fatou's lemma, the extension $Z$ is a $P$-supermartingale. Using the Doob–Meyer decomposition theorem for supermartingales, it can be shown that $Z$ is even a local $P$-martingale, see [65, Lemma 12.43].*

So far we have explained that we want to relate $P$ and $Z$ to a solution of an SMP. Our next step is to formally introduce the parameters of the SMP to which we want to connect $P$ and $Z$.

For $n \in \mathbb{N}$ denote by $X^{c,n}$ the continuous local $P$-martingale part of $X^n$ and by $Z^c$ the continuous local $P$-martingale part of $Z$. Both are unique up to $P$-indistinguishability. The predictable quadratic covariation process (w.r.t. $P$) is denoted by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. We set

$\sigma_0 \triangleq \rho_0 \triangleq 0$. For all $k \in \mathbb{N}$ let $\beta^k$ be an $\mathbb{R}^d$-valued predictable process such that up to $P$-evanescence

$$\langle\!\langle Z^c, X^{c,k}\rangle\!\rangle_{\sigma_k\wedge\rho_k} - \langle\!\langle Z^c, X^{c,k}\rangle\!\rangle_{\sigma_{k-1}\wedge\rho_{k-1}} = \int_0^\cdot \mathbb{1}_{\{\sigma_{k-1}\wedge\rho_{k-1} < s \leq \sigma_k\wedge\rho_k\}} Z_{s-} c_s \beta^k_s dA_s,$$

and $Y^k$ be a non-negative $\widetilde{\mathcal{P}} \triangleq \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable function such that $M^P_{\mu^k}$-a.e.

$$Z_- Y^k \mathbb{1}_{]\!]\sigma_{k-1}\wedge\rho_{k-1},\sigma_k\wedge\rho_k]\!]\times\mathbb{R}^d} = M^P_{\mu^k}\big(Z|\widetilde{\mathcal{P}}\big)\mathbb{1}_{]\!]\sigma_{k-1}\wedge\rho_{k-1},\sigma_k\wedge\rho_k]\!]\times\mathbb{R}^d},$$

where $M^P_{\mu^k}(\,\cdot\,|\widetilde{\mathcal{P}})$ denotes the conditional expectation w.r.t. the Doléans measure

$$M^P_{\mu^k}(d\omega \times dt \times dx) \triangleq \mu^k(\omega, dt \times dx) P(d\omega)$$

conditioned on $\widetilde{\mathcal{P}}$, see [70, Section III.3.c)] for more details. Here, $\mu^k$ is the random measure of jumps associated to $X^k$. We set

$$\beta \triangleq \sum_{k=1}^\infty \beta^k \mathbb{1}_{]\!]\sigma_{k-1}\wedge\rho_{k-1},\sigma_k\wedge\rho_k]\!]},$$
$$Y \triangleq \sum_{k=1}^\infty Y^k \mathbb{1}_{]\!]\sigma_{k-1}\wedge\rho_{k-1},\sigma_k\wedge\rho_k]\!]\times\mathbb{R}^d}.$$

We stress that the sequences $(\beta^k)_{k\in\mathbb{N}}$ and $(Y^k)_{k\in\mathbb{N}}$ are consistent in the sense that for all $k \leq n$ we have $\beta^k = \beta^n$ on $[\![0, \sigma_k \wedge \rho_k]\!]$ up to $P$-evanescence and $Y^k = Y^n$ on $[\![0, \sigma_k \wedge \rho_k]\!] \times \mathbb{R}^d$ up to a $M^P_{\mu^n}$-null set. This observation is implied by the following two facts: If $U$ and $V$ are two $\mathbb{R}^d$-valued semimartingales and $\xi$ is a stopping time such that $U = V$ on $[\![0, \xi]\!]$ up to evanescence, then $V^c = U^c$ on $[\![0, \xi]\!]$ up to evanescence, and

$$M_{\mu^U}\big(\,\cdot\,|\widetilde{\mathcal{P}}\big) = M_{\mu^V}\big(\,\cdot\,|\widetilde{\mathcal{P}}\big)$$

on $[\![0, \xi]\!] \times \mathbb{R}^d$ up to a $M_{\mu^U}$-null set. The first claim follows from the uniqueness of the continuous local martingale part, and the second claim follows from the fact that the set $[\![0, \xi]\!] \times \mathbb{R}^d$ is $\widetilde{\mathcal{P}}$-measurable and the definition of the conditional expectation w.r.t. the Doléans measures.

Finally, let $(B', C, \nu')$ be a candidate triplet, such that on $\bigcup_{n\in\mathbb{N}}[\![0, \sigma_n \wedge \rho_n]\!]$ up to $P$-evanescence

$$\begin{aligned} B' &= B + \int_0^\cdot c_s \beta_s dA_s + h(x)(Y - 1) \star \nu, \\ \nu' &= Y \cdot \nu, \end{aligned} \tag{3.3}$$

where

$$h(x)(Y - 1) \star \nu \triangleq \int_0^\cdot \int h(x)(Y(s, x) - 1)\nu(ds \times dx)$$

and

$$(Y \cdot \nu)(dt \times dx) \triangleq Y(t, x)\nu(dt \times dx).$$

The consistency of the sequences $(\beta^k)_{k\in\mathbb{N}}$ and $(Y^k)_{k\in\mathbb{N}}$ yields that the integrals in (3.3) are well-defined, see the proof of [70, Theorem III.3.24] for details.

Let us shortly comment on the intuition behind the modified candidate triplet $(B', C, \nu')$. The idea is to consider the probability measure $Q_n$ as defined in (3.1). Then, by Girsanov's theorem, the stopped process $X^n_{\cdot\wedge\sigma_n}$ is a $Q_n$-semimartingale whose characteristics are $Q_n$-indistinguishable from the stopped modified triplet

$$(B'_{\cdot\wedge\sigma_n\wedge\rho_n}, C_{\cdot\wedge\sigma_n\wedge\rho_n}, \mathbb{1}_{[\![0,\sigma_n\wedge\rho_n]\!]}\cdot\nu').$$

Thus, if an extension of $Q_n$ solves an SMP with lifetime $\sigma\wedge\rho$, the corresponding candidate triplet should be $(B', C, \nu')$.

For a second probability measure $Q$ on $(\Omega, \mathcal{F})$, we write $Q \ll_{\text{loc}} P$ if $Q \ll P$ on $\mathcal{F}_t$ for all $t \in \mathbb{R}_+$. Moreover, we set

$$\zeta \triangleq \sigma\wedge\rho, \qquad \zeta_n \triangleq \sigma_n \wedge \rho_n, \qquad n\in\mathbb{N}.$$

We are now in the position to state our first main result.

**Theorem 3.1.** *There exists a solution $Q$ to the SMP $(\zeta; \eta'; B', C, \nu')$, where*

$$\eta'(G) \triangleq E^P\left[Z_0 \mathbb{1}_{\{X_0\in G\}}\right]$$

*for $G \in \mathcal{B}(\mathbb{R}^d)$, with $\zeta$-localizing sequence $(\zeta_n)_{n\in\mathbb{N}}$ and such that*

$$Q = Z_{\sigma_n}\cdot P \text{ on } \mathcal{F}_{\sigma_n} \text{ for all } n\in\mathbb{N}. \tag{3.4}$$

*Moreover, the following hold:*

(a) *For all stopping times $\xi$ we have*

$$Q = Z_\xi \cdot P \text{ on } \mathcal{F}_\xi \cap \{\sigma > \xi\}. \tag{3.5}$$

(b) *The following are equivalent:*

(b.i) *$Q$-a.s. $\sigma = \infty$.*

(b.ii) *The process $Z$ is a $P$-martingale and $P$-a.s. $Z = 0$ on $[\![\sigma, \infty[\![$.*

*If these statements hold true, then $Q \ll_{\text{loc}} P$ with $\frac{dQ}{dP}|_{\mathcal{F}_t} = Z_t$ for all $t \in \mathbb{R}_+$.*

(c) *The following are equivalent:*

(c.i) *There exists an increasing sequence $(\gamma_n)_{n\in\mathbb{N}}$ of stopping times such that $\gamma_n \nearrow \sigma$ as $n\to\infty$, $Z^{\gamma_n}$ is a uniformly integrable $P$-martingale and*

$$\lim_{n\to\infty} Q(\gamma_n = \sigma = \infty) = 1.$$

(c.ii) *The process $Z$ is a uniformly integrable $P$-martingale with $P$-a.s. $Z = 0$ on $[\![\sigma, \infty[\![$.*

*If these statements hold true, then $Q \ll P$ with $\frac{dQ}{dP} = \lim_{t\to\infty} Z_t \triangleq Z_\infty$.*

(d) *Suppose that at least one of the following conditions holds:*

(d.i) *$Q$-a.s. $\rho_n < \sigma$ for all $n \in \mathbb{N}$.*

(d.ii) *$P$-a.s. $\rho_n < \sigma$ and $E^P\big[Z_{\rho_n}\big] = 1$ for all $n \in \mathbb{N}$.*

*Then, $Q$ solves the SMP $(\rho; \eta'; B', C, \nu')$ with $\rho$-localizing sequence $(\rho_n)_{n\in\mathbb{N}}$. Moreover,* (d.ii) *implies* (d.i).

We stress that the terminal random variable $Z_\infty$ is well-defined due to the supermartingale convergence theorem.

Part (a) of this theorem is a Girsanov-type formula, part (b) gives a criterion for the local absolute continuity of $Q$ and $P$ and part (c) gives a criterion for the global absolute continuity. In part (d) we give conditions such that $Q$ solves an SMP with lifetime $\rho$. In this case, our observations from (b) and (c) give criteria for the (local) absolute continuity of solutions of two SMPs with the same lifetime. In (d) we present a condition which only depends on $Q$ and a condition which only depends on $P$. The latter is important for applications because it allows us to check properties of $P$ to conclude that $Q$ solves an SMP with lifetime $\rho$. The condition $E^P\big[Z_{\rho_n}\big] = 1$ means that the stopped process $Z^{\rho_n}$ is a uniformly integrable $P$-martingale.

**Remark 3.3.** *If $P$-a.s. $Z = 0$ on $[\![\sigma, \infty[\![$, then* (b.i) *and* (b.ii) *in Theorem 3.1 are equivalent to $Q \ll_{\text{loc}} P$ with $\frac{dQ}{dP}|_{\mathcal{F}_t} = Z_t$ for all $t \in \mathbb{R}_+$, and* (c.i) *and* (c.ii) *in Theorem 3.1 are equivalent to $Q \ll P$ with $\frac{dQ}{dP} = Z_\infty$.*

We would like to choose $(\sigma_n)_{n\in\mathbb{N}}$ such that $P$-a.s. $Z = 0$ on $[\![\sigma, \infty[\![$. Of course, this is the case if $P$-a.s. $\sigma = \infty$, which is true when $\sigma_n$ is chosen as proposed in Remark 3.1. In particular, it is interesting to note that when $P$-a.s. $\sigma = \infty$, then $(\sigma_n)_{n\in\mathbb{N}}$ is a $P$-localization sequence for the local $P$-martingale $Z$.

Let us mention another natural choice for $(\sigma_n)_{n\in\mathbb{N}}$. Suppose that $U$ is a continuous local martingale on $[\![0, \rho[\![$ and set $\langle\!\langle U, U\rangle\!\rangle_t = \langle\!\langle U, U\rangle\!\rangle_{\rho-}$ for all $t \geq \rho$. Let $\widehat{Z}$ be the stochastic exponential of $U$ on $[\![0, \rho[\![$, i.e. on $[\![0, \rho[\![$

$$\widehat{Z} \triangleq \exp\left(U - \tfrac{1}{2}\langle\!\langle U, U\rangle\!\rangle\right).$$

Take $Z$ to be the extension of $\widehat{Z}$ as introduced in Remark 3.2 and set

$$\sigma_n \triangleq \inf(t \in \mathbb{R}_+ \colon \langle\!\langle U, U\rangle\!\rangle_t \geq n) \wedge n.$$

Then, $P$-a.s. $Z = 0$ on $[\![\sigma, \infty[\![$ follows from the strong law of large numbers for continuous local martingales, see [123, Exercise V.1.16]. In view of Theorem 3.1, this choice of $(\sigma_n)_{n\in\mathbb{N}}$ shows that $Q \ll_{\text{loc}} P$ if and only if $Q$-a.s. $\langle\!\langle U, U\rangle\!\rangle_t < \infty$ for all $t \in \mathbb{R}_+$, and that $Q \ll P$ if and only if $Q$-a.s. $\langle\!\langle U, U\rangle\!\rangle_\infty < \infty$. This observation is in the spirit of classical results for the local absolute continuity of globally defined semimartingales. We comment on this in Section 3.3 below, where we also define a version of $\sigma_n$ in the presence of jumps.

*Proof of Theorem 3.1:* We construct $Q$ using the extension theorem of Parthasarathy. We recall a definition due to Föllmer [51]. Let $\mathbb{T} \subseteq \mathbb{R}_+$ be an index set and $(\Omega^*, \mathcal{F}^*_t)_{t\in\mathbb{T}}$ be a sequence of measurable spaces.

**Definition 3.2.** *We call $(\Omega^*, \mathcal{F}^*_t)_{t\in\mathbb{T}}$ a standard system, if*

(i) *$\mathcal{F}^*_t \subseteq \mathcal{F}^*_s$ for $t, s \in \mathbb{T}$ with $t < s$,*

(ii) *for each $t \in \mathbb{T}$ the space $(\Omega^*, \mathcal{F}^*_t)$ is a standard Borel space, i.e. $\mathcal{F}^*_t$ is $\sigma$-isomorphic to the Borel $\sigma$-field of a Polish space,*

(iii) *for each increasing sequence* $(t_n)_{n\in\mathbb{N}}$ *of elements in* $\mathbb{T}$ *and any decreasing sequence* $(A_n)_{n\in\mathbb{N}}$*, where* $A_n$ *is an atom in* $\mathcal{F}^*_{t_n}$*, we have* $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$.

It is shown in the Appendix of [51] that $(\Omega, \mathcal{F}_{\sigma_n -})_{n\in\mathbb{N}}$ is a standard system. Here, the choice of the underlying measurable space is crucial, because $(\Omega, \mathcal{F}^o_t)_{t\geq 0}$ is a standard system, too. Furthermore, it is important to define $\mathcal{F}_{\sigma_n -}$ with $\mathcal{F}^o_0$ instead of $\mathcal{F}_0$, because the proof of Definition 3.2 (ii) requires $\mathcal{F}_{\sigma_n -}$ to be countably generated. For $n \in \mathbb{N}$ define the probability measure

$$Q_n \triangleq Z_{\sigma_n} \cdot P$$

on $(\Omega, \mathcal{F})$. We deduce from Parthasarathy's extension theorem, see [116, Theorem V.4.2], together with [119, Theorem D.4, Lemma E.1], where it is again important that $\mathcal{F}_{\sigma -}$ is countably generated, that there exists a probability measure $Q$ on $(\Omega, \mathcal{F})$ such that

$$Q = Q_n \text{ on } \mathcal{F}_{\sigma_n -} \text{ for all } n \in \mathbb{N}. \tag{3.6}$$

Next, we show (a).

(a) Let $\xi$ be a stopping time and $A \in \mathcal{F}_\xi$, then $A \cap \{\sigma_n > \xi\} \in \mathcal{F}_\xi \cap \mathcal{F}_{\sigma_n -}$ and hence

$$\begin{aligned} Q(A \cap \{\sigma > \xi\}) &= \lim_{n\to\infty} Q(A \cap \{\sigma_n > \xi\}) \\ &= \lim_{n\to\infty} E^P\big[Z_{\sigma_n} \mathbb{1}_{A\cap\{\sigma_n > \xi\}}\big] \\ &= \lim_{n\to\infty} E^P\big[Z_{\xi} \mathbb{1}_{A\cap\{\sigma_n > \xi\}}\big] \\ &= E^P\big[Z_{\xi} \mathbb{1}_{A\cap\{\sigma > \xi\}}\big], \end{aligned}$$

due to the monotone convergence theorem and the optional stopping theorem.

This observation allows us to deduce (3.4) from (3.6). For all $G \in \mathcal{F}_{\sigma_n}$

$$\begin{aligned} Q(G \cap \{\sigma > \sigma_n\}) &= E^P\big[Z_{\sigma_n} \mathbb{1}_{G\cap\{\sigma > \sigma_n\}}\big] \\ &= E^P\big[Z_{\sigma_n} \mathbb{1}_{G}\big] \\ &= Q_n(G), \end{aligned}$$

where we also use our assumption that $P$-a.s. $\sigma_n < \sigma$. In particular, $Q(\sigma > \sigma_n) = Q_n(\Omega) = 1$. Thus, we have shown that

$$Q(G) = Q(G \cap \{\sigma > \sigma_n\}) = Q_n(G),$$

i.e. in other words

$$Q = Q_n \text{ on } \mathcal{F}_{\sigma_n} \text{ for all } n \in \mathbb{N}.$$

Next, we show that $Q$ solves the SMP $(\zeta; \eta; B', C, \nu')$. Since $Q_n \ll P$ with density process

$$Z_{\sigma_n \wedge t} = \frac{dQ_n}{dP}\bigg|_{\mathcal{F}_t},$$

we deduce from Girsanov's theorem for semimartingales, see [70, Theorem III.3.24], that the stopped process $X^n_{\cdot\wedge\zeta_n}$ is a $Q_n$-semimartingale whose characteristics are $Q_n$-indistinguishable from $(B'_{\cdot\wedge\zeta_n}, C_{\cdot\wedge\zeta_n}, \mathbb{1}_{[\![0,\zeta_n]\!]} \cdot \nu')$. Here, we use the consistency of the sequences $(\beta^k)_{k\in\mathbb{N}}$ and $(Y^k)_{k\in\mathbb{N}}$.

Let us now transfer this observation from $Q_n$ to the extension $Q$. We can consider $X^n_{\cdot\wedge\zeta_n}$ as a semimartingale on the filtered probability space $(\Omega, \mathcal{F}_{\zeta_n}, (\mathcal{F}_{t\wedge\zeta_n})_{t\geq 0}, Q_n)$, see [65, Section 10.1]. The identity $Q = Q_n$ on $\mathcal{F}_{\zeta_n} \subseteq \mathcal{F}_{\sigma_n}$ implies that $X^n_{\cdot\wedge\zeta_n}$ is an $Q$-semimartingale whose characteristics are $Q$-indistinguishable from $(B'_{\cdot\wedge\zeta_n}, C_{\cdot\wedge\zeta_n}, \mathbb{1}_{[\![0,\zeta_n]\!]} \cdot \nu')$. We conclude that $Q$ solves the SMP $(\zeta; \eta'; B', C, \nu')$ with $\zeta$-localizing sequence $(\zeta_n)_{n\in\mathbb{N}}$.

We proceed with the proofs of (b) – (d).

(b) Suppose that (b.i) holds. Then, due to (a), we obtain

$$1 = Q(\sigma > t) = E^P\big[Z_t \mathbb{1}_{\{\sigma > t\}}\big].$$

Since $Z$ is a $P$-supermartingale by Fatou's lemma, we have

$$E^P\big[Z_t\big] \leq E^P\big[Z_0\big] = 1.$$

We conclude that

$$0 \leq E^P\big[Z_t \mathbb{1}_{\{t\geq\sigma\}}\big] = E^P\big[Z_t\big] - 1 \leq 0,$$

which implies that $P$-a.s. $Z_t = 0$ on $\{t \geq \sigma\}$. This yields that for all $G \in \mathcal{F}_t$

$$Q(G) = E^P\big[Z_t \mathbb{1}_G\big],$$

and $Q \ll_{\text{loc}} P$ with $Z_t = \frac{dQ}{dP}|_{\mathcal{F}_t}$ follows immediately. In particular, $Z$ is a $P$-martingale. In other words, we have shown that (b.i) $\Rightarrow$ (b.ii) and that (b.i) implies $Q \ll_{\text{loc}} P$ with $Z_t = \frac{dQ}{dP}|_{\mathcal{F}_t}$.

It remains to prove the implication (b.ii) $\Rightarrow$ (b.i). If (b.ii) holds, (a) implies that for all $t \in \mathbb{R}_+$

$$Q(\sigma > t) = E^P\big[Z_t \mathbb{1}_{\{\sigma>t\}}\big] = E^P\big[Z_t\big] = E^P\big[Z_0\big] = 1.$$

It follows that $Q(\sigma = \infty) = 1$, i.e. that (b.i) holds.

(c) To see the implication (c.ii) $\Rightarrow$ (c.i), we set $\gamma_n \triangleq \sigma$ for all $n \in \mathbb{N}$. Then, the implication (c.ii) $\Rightarrow$ (b.ii) $\Leftrightarrow$ (b.i) yields that this sequence has all properties as claimed in (c.i).

Let us assume that (c.i) holds. Since (c.i) $\Rightarrow$ (b.i) $\Leftrightarrow$ (b.ii), it suffices to prove that $Z$ is a uniformly integrable $P$-martingale. In fact, since $Z$ is a $P$-supermartingale, it suffices to show that $E^P[Z_\infty] \geq 1$. Let $A \in \mathcal{F}_{\gamma_n} \cap \mathcal{F}_{\sigma_m} = \mathcal{F}_{\gamma_n\wedge\sigma_m}$. Then,

$$Q(A) = E^P\big[Z_{\sigma_m}\mathbb{1}_A\big] = E^P\big[Z_{\gamma_n}\mathbb{1}_A\big],$$

where we use (3.4) and the optional stopping theorem. By a monotone class argument, we have

$$Q = Z_{\gamma_n} \cdot P \text{ on } \mathcal{F}_{\gamma_n-},$$

where we use that $\gamma_n \leq \sigma$. Note that $\{\gamma_n = \sigma = \infty\} \in \mathcal{F}_{\gamma_n -}$. Thus, we obtain

$$\begin{aligned} 1 = \lim_{n\to\infty} Q(\gamma_n = \sigma = \infty) &= \lim_{n\to\infty} E^P\big[Z_{\gamma_n} \mathbb{1}_{\{\gamma_n = \sigma = \infty\}}\big] \\ &= \lim_{n\to\infty} E^P\big[Z_\infty \mathbb{1}_{\{\gamma_n = \sigma = \infty\}}\big] \\ &\leq E^P\big[Z_\infty\big]. \end{aligned}$$

This proves (c.i) ⇒ (c.ii).

Finally, if (c.ii) holds, then (b) implies that $Q \ll_{\text{loc}} P$ with $\frac{dQ}{dP}|_{\mathcal{F}_t} = Z_t$. Hence, $Q \ll P$ with $\frac{dQ}{dP} = Z_\infty$ follows immediately from [70, Proposition III.3.5] and the uniform integrability of $Z$.

(d) We first show that (d.ii) ⇒ (d.i). If $P$-a.s. $\rho_n < \sigma$ and $E^P\big[Z_{\rho_n}\big] = 1$, then (a) yields

$$Q(\rho_n < \sigma) = E^P\big[Z_{\rho_n} \mathbb{1}_{\{\rho_n < \sigma\}}\big] = E^P\big[Z_{\rho_n}\big] = 1. \tag{3.7}$$

Thus, (d.ii) ⇒ (d.i).

Suppose that (d.i) holds. We define the sequence of stopping times (see, e.g. [59, Theorem III.3.9] for the fact that the following are stopping times)

$$\gamma_m \triangleq \begin{cases} \sigma_m, & \text{on } \{\rho_n \geq \sigma_m\}, \\ \infty, & \text{otherwise}, \end{cases}$$

and note that $Q$-a.s. $\gamma_m \nearrow \infty$ as $m \to \infty$ and that $\rho_n \wedge \gamma_m = \rho_n \wedge \sigma_m$. As above, it follows from Girsanov's theorem that for all $m \in \mathbb{N}$ the process $X^n_{\cdot \wedge \rho_n \wedge \gamma_m} = X^n_{\cdot \wedge \rho_n \wedge \sigma_m}$ is a $Q$-semimartingale whose characteristics are $Q$-indistinguishable from the triplet $(B'_{\cdot \wedge \rho_n \wedge \gamma_m}, C_{\cdot \wedge \rho_n \wedge \gamma_m}, \mathbb{1}_{[\![0,\rho_n \wedge \gamma_m]\!]} \cdot \nu')$. Recalling that the class of semimartingales is stable under localization, see [70, Proposition I.4.25], we conclude that the process $X^n_{\cdot \wedge \rho_n}$ is a $Q$-semimartingale whose characteristics are $Q$-indistinguishable from the triplet $(B'_{\cdot \wedge \rho_n}, C_{\cdot \wedge \rho_n}, \mathbb{1}_{[\![0,\rho_n]\!]} \cdot \nu')$. In other words, we have shown that $Q$ solves the SMP $(\rho; \eta'; B', C, \nu')$ with $\rho$-localizing sequence $(\rho_n)_{n\in\mathbb{N}}$.

The proof is complete. □

In the next section, we discuss consequences of Theorem 3.1.

## 3.3 Absolute Continuity of Semimartingales

In this section we study absolute continuity of semimartingales. Systematic approaches in conservative settings were given by Kabanov, Liptser and Shiryaev [71, 72], Jacod and Mémin [67] and Jacod [65] under a strong uniqueness assumption, called local uniqueness in the monograph [70]. As we show below, the local uniqueness assumption can be replaced by a usual uniqueness assumption. This is well-known to be true in Markovian settings and surprising to hold in all generality.

Let $\beta \colon \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ be predictable and $U \colon \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ be $\widetilde{\mathcal{P}}$-measurable.

Furthermore, let $B, C$ and $\nu$ are given as in Section 3.2. We further set

$$\begin{aligned} B' &\triangleq B + \int_0^{\cdot} c_s \beta_s dA_s + h(x)(U-1) \star \nu, \\ \nu' &\triangleq U \cdot \nu, \end{aligned} \tag{3.8}$$

where $B'$ is set to be $\Delta$ whenever one of the integrals diverges. The first standing assumption in this section is the following.

**Standing Assumption 3.2.** *Let $P$ be a solution to the SMP $(\rho; \eta; B, C, \nu)$, with $\rho$-localizing sequence $(\rho_n)_{n\in\mathbb{N}}$ and fundamental sequence $(X^n)_{n\in\mathbb{N}}$, and let $Q^*$ be a solution to the SMP $(\rho; \eta; B', C, \nu')$ with $\rho$-localizing sequence $(\rho_n)_{n\in\mathbb{N}}$. W.l.o.g. $\rho_n \leq n$ for all $n \in \mathbb{N}$.*

For all $t \in \mathbb{R}_+$ we define

$$\widehat{U}_t \triangleq \int U(t,x)\nu(\{t\} \times dx), \qquad a_t \triangleq \nu(\{t\} \times \mathbb{R}^d).$$

**Standing Assumption 3.3.** *For all $t \in \mathbb{R}_+$ we have identically $a_t \leq 1$ and $\widehat{U}_t \leq 1$.*

We define the $[0,\infty]$-valued predictable process

$$H \triangleq \int_0^{\cdot \wedge \rho} \langle \beta_s, c_s \beta_s \rangle dA_s + \left(1 - \sqrt{U}\right)^2 \star \nu_{\cdot \wedge \rho} + \sum_{s \leq \cdot \wedge \rho} \left(\sqrt{1-a_s} - \sqrt{1-\widehat{U}_s}\right)^2, \tag{3.9}$$

where the process is defined to be $\infty$ whenever one of the terms diverges, and set

$$\sigma_n \triangleq \inf(t \in \mathbb{R}_+ \colon H_t \geq n) \wedge n, \qquad \sigma \triangleq \lim_{n\to\infty} \sigma_n. \tag{3.10}$$

The process $H$ is increasing, but not in the sense of [70], because it may fail to be right-continuous, i.e. on $\{\sigma < \infty\}$ it can happen that $H_\sigma < \infty$ and $H_{\sigma+} = \infty$. Here, we stress that increasing functions have left and right limits. Nevertheless, for all $n \in \mathbb{N}$ the random time $\sigma_n$ is a stopping time. This follows from the fact that

$$\{\sigma_n \leq t\} = \big(\{H_t \geq n\} \cap \{H_{t+} < \infty\}\big) \cup \{H_{t+} = \infty\} \cup \{n \leq t\} \in \mathcal{F}_t, \quad t \in \mathbb{R}_+,$$

due to the right-continuity of the filtration. Consequently, also $\sigma$ is a stopping time.

In the following standing assumption we suppose that $P$-a.a. paths of $H$ are right-continuous and do not jump to $\infty$. We comment on this in Remark 3.5 below.

**Standing Assumption 3.4.** *Up to $P$-evanescence, on $\bigcup_{n\in\mathbb{N}} [\![0, \sigma_n \wedge \rho_n]\!]$*

$$a = 1 \ \Rightarrow \ \widehat{U} = 1. \tag{3.11}$$

*Furthermore, one of the following holds:*

(a) *$P$-a.s. $H_{\sigma-} = \infty$ on $\{\sigma < \infty\}$, and $P$-a.s. $\rho_n < \sigma$ for all $n \in \mathbb{N}$.*

(b) *$P$-a.s. $H_{\sigma-} = \infty$ on $\{\sigma < \infty\}$, and for all solutions $\widehat{Q}$ to the SMP $(\sigma\wedge\rho; \eta; B', C, \nu')$ with $\sigma \wedge \rho$-localizing sequence $(\sigma_n \wedge \rho_n)_{n\in\mathbb{N}}$ we have $\widehat{Q}$-a.s. $\rho_n < \sigma$ for all $n \in \mathbb{N}$.*

To get an intuition for the condition (3.11), suppose that $P$ and $Q^*$ are laws of $\mathbb{R}^d$-valued semimartingales with independent increments. In this case, the triplets $(B, C, \nu)$

and $(B', C, \nu')$ are deterministic and we have

$$
\begin{aligned}
P(\Delta X_t \in dx) &= \mathbb{1}_{\mathbb{R}^d \backslash \{0\}} \nu(\{t\} \times dx) + (1 - a_t)\, \delta_0(dx), \\
Q^*(\Delta X_t \in dx) &= \mathbb{1}_{\mathbb{R}^d \backslash \{0\}} U(t,x) \nu(\{t\} \times dx) + \big(1 - \widehat{U}_t\big) \delta_0(dx),
\end{aligned}
$$

where $\delta$ denotes the Dirac measure, see [70, Theorem II.4.15]. If $a_t = 1$, then $Q^*(\Delta X_t \in dx) \ll P(\Delta X_t \in dx)$ can only be true when $\widehat{U}_t = 1$. The absolute continuity $Q^*(\Delta X_t \in dx) \ll P(\Delta X_t \in dx)$ is implied by $Q^* \ll_{\text{loc}} P$ and therefore (3.11) is very natural.

We stress that $P$-a.s. $H_{\sigma-} = \infty$ on $\{\sigma < \infty\}$ implies that $P$-a.s. $\sigma_n < \sigma$ for all $n \in \mathbb{N}$. If $P$-a.s. $\rho_n < \sigma$ for all $n \in \mathbb{N}$, then $P$-a.s.

$$
\sigma = \begin{cases} \rho, & \text{if } H_\rho = \infty \text{ and } \rho < \infty, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.12}
$$

**Remark 3.4.** *If the process $H$ is left-continuous the condition $P$-a.s. $H_{\sigma-} = \infty$ on $\{\sigma < \infty\}$ is implied by $P$-a.s. $H_\sigma = \infty$ on $\{\sigma < \infty\}$, which itself is implied by $P$-a.s. $\rho_n < \sigma$ for all $n \in \mathbb{N}$, see* (3.12)*.*

Regardless whether (a) or (b) in Standing Assumption 3.4 holds, we have $P$-a.s.

$$
\begin{aligned}
H_{\sigma_n} &\le n + \Delta H_{\sigma_n} \\
&\le n + 2\Big(a_{\sigma_n} + \widehat{U}_{\sigma_n} + 1 - a_{\sigma_n} + 1 - \widehat{U}_{\sigma_n}\Big) \\
&= n + 4.
\end{aligned} \tag{3.13}
$$

Next, we define a non-negative local martingale $Z$ which relates $P$ and $Q^*$. We find a non-negative local $P$-martingale on $\bigcup_{n \in \mathbb{N}} [\![0, \sigma_n \wedge \rho_n]\!]$ which coincides with the stochastic exponential of

$$
\int_0^\cdot \langle \beta_s, dX^{n,c}_s \rangle + \Big(U - 1 + \frac{\widehat{U} - a}{1 - a}\Big) \star \big(\mu^n - \nu^{\rho_n}\big) \tag{3.14}
$$

on the random set $[\![0, \sigma_n \wedge \rho_n]\!]$, see [70, Proposition II.1.16]. Here, we use the convention that $\frac{0}{0} \equiv 0$. The second stochastic integral denotes the discontinuous local $P$-martingale whose jump process is $P$-indistinguishable from

$$
(U(\cdot, \Delta X^n) - 1) \mathbb{1}_{\{\Delta X^n \neq 0\}} - \Big(\frac{\widehat{U} - a}{1 - a}\Big) \mathbb{1}_{\{\Delta X^n = 0\}}, \tag{3.15}
$$

see [70, Section II.1] for more details. The non-negativity follows from the fact that (3.11) implies that (3.15) is, up to $P$-evanescence, greater or equal than $-1$ on $\bigcup_{n \in \mathbb{N}} [\![0, \sigma_n \wedge \rho_n]\!]$, see [70, Theorem I.4.61]. As pointed out in Remark 3.2, we can extend this non-negative local $P$-martingale on $\bigcup_{n \in \mathbb{N}} [\![0, \sigma_n \wedge \rho_n]\!]$ to a global one, which we denote $Z$ in the following.

By [65, Theorem 8.25] and similar arguments as used in the proof of [65, Lemma 12.44], the bound (3.13) implies that the stopped process $Z^{\sigma_n}$ is a uniformly integrable $P$-martingale and, since $P$-a.s. $H_{\sigma-} = \infty$ on $\{\sigma < \infty\}$, [65, Theorem 8.10] yields that $P$-a.s. $Z = 0$ on $[\![\sigma, \infty[\![$.

**Remark 3.5.** *If $H$ is allowed to jump to $\infty$, it may happen that $Z$ is positive on $[\![\sigma, \infty[\![$ with positive $P$-probability. In this case, (b) and (c) in Theorem 3.1 do not provide a statement on (local) absolute continuity. Of course, we could modify $Z$ to be zero on*

*$[\![\sigma, \infty[\![$, but the modification might only be a supermartingale. Let us discuss an explicit example. Consider a $[-\infty, \infty]$-valued diffusion*

$$dY_t = \mu(Y_t)dt + a(Y_t)dW_t,$$

*where $W$ is a one-dimensional Brownian motion. If the coefficients $\mu$ and $a$ satisfy the Engelbert–Schmidt conditions, see [77] or Standing Assumption 3.6 below, then $Y$ exists up to an explosion time $\theta$. In this case, for any Borel function $f\colon \mathbb{R} \to \mathbb{R}_+$ the integral process*

$$K \triangleq \int_0^{\cdot\wedge\theta} f(Y_s)ds$$

*is in the spirit of $H$. Let $D$ be the set of all $x \in \mathbb{R}$ for which there is no $\varepsilon > 0$ such that*

$$\int_{x-\varepsilon}^{x+\varepsilon} \frac{f(y)}{a^2(y)}dy < \infty,$$

*and denote*

$$\eta_D \triangleq \theta \wedge \inf(t \in \mathbb{R}_+ \colon Y_t \in D).$$

*By [105, Theorem 2.6], we have a.s.*

$$K_t \begin{cases} < \infty, & t \in [0, \eta_D), \\ = \infty, & t \in (\eta_D, \theta]. \end{cases}$$

*This characterization follows from the occupation times formula, which states that a.s. on $\{t < \theta\}$*

$$\int_0^t f(Y_s)ds = \int_0^t \frac{f(Y_s)}{a^2(Y_s)} d\langle\!\langle Y, Y\rangle\!\rangle_s = \int_{-\infty}^{\infty} \frac{f(y)}{a^2(y)} L_t^y(Y)dy,$$

*where $L$ denotes the local time, see [46, Equation (4.4)]. On the set $\{\eta_D < \theta\}$ it might happen with positive probability that $K_{\eta_D} < \infty$, see [105, Sections 2.4, 2.5] for more details. In this case, $K$ jumps to infinity and the extension $Z$ is positive on $[\![\eta_D, \infty[\![$ with positive probability. Deterministic conditions for this case can be found in [105]. Finally, we stress that a.a. paths of $K$ do not jump to infinity if $D = \emptyset$.*

**Standing Assumption 3.5.** *Standing Assumption 3.4 holds with* (a) *replaced by*

(a)' *$P$-a.s. $H_{\sigma-} = \infty$ on $\{\sigma < \infty\}$, and $P$-a.s. $\rho_n < \sigma$ and $E^P\big[Z_{\rho_n}\big] = 1$ for all $n \in \mathbb{N}$.*

The additional moment assumption has a local character. In fact, in many cases it follows easily from classical moment conditions such as Novikov's condition. For the readers convenience we collect two conditions:

**Proposition 3.1.** *Let $n \in \mathbb{N}$. Assume that at least one of the following conditions holds:*

(i) *The random variable $H_{\rho_n}$ is bounded up to a $P$-null set.*

(ii) *Set*

$$H^* \triangleq \frac{1}{2}\int_0^{\cdot} \langle \beta_s, c_s\beta_s\rangle dA_s + \sum_{s\leq \cdot}\Big(\big(1-\widehat{U}_s\big)\log\Big(\frac{1-\widehat{U}_s}{1-a_s}\Big) + \widehat{U}_s - a_s\Big)$$
$$+ \big(U\log(U) - U + 1\big)\star\nu,$$

*where we use the conventions that* $0/0 \equiv 0, \log(0) \equiv -\infty$ *and* $0\times(-\infty)\equiv 0$. *It holds that* $E^P\big[\exp(H^*_{\rho_n})\big] < \infty$.

*Then,* $E^P\big[Z_{\rho_n}\big] = 1$.

*Proof.* The identity $E^P[Z_{\rho_n}] = 1$ is implied by (i) due to similar arguments as used in the proof of [65, Lemma 12.44] together with [65, Theorem 8.25]. Furthermore, $E^P[Z_{\rho_n}] = 1$ is implied by (ii) due to [65, Corollary 8.44]. □

Next, we state the main result of this section.

**Corollary 3.1.** *Assume that all solutions to the SMP* $(\rho;\eta;B',C,\nu')$ *with* $\rho$*-localizing sequence* $(\rho_n)_{n\in\mathbb{N}}$ *coincide on the* $\sigma$*-field* $\mathcal{F}_{\sigma-}$. *Then, for all stopping times* $\xi$ *we have*

$$Q^* = Z_\xi\cdot P \text{ on } \mathcal{F}_\xi\cap\{\sigma>\xi\}. \tag{3.16}$$

*Moreover, we have the following:*

(a) *The following are equivalent:*
  - (a.i) $Q^*$*-a.s.* $H_t<\infty$ *for all* $t\in\mathbb{R}_+$.
  - (a.ii) *The process* $Z$ *is a* $P$*-martingale.*
  - (a.iii) $Q^* \ll_{\text{loc}} P$ *with* $\frac{dQ^*}{dP}|_{\mathcal{F}_t} = Z_t$.

(b) *The following are equivalent:*
  - (b.i) $Q^*$*-a.s.* $H_\rho<\infty$.
  - (b.ii) *The process* $Z$ *is a uniformly integrable* $P$*-martingale.*
  - (b.iii) $Q^*\ll P$ *with* $\frac{dQ^*}{dP} = Z_\infty$.

*Proof.* First, note that Standing Assumption 3.1 holds. Let $Q$ be as in Theorem 3.1. We have, up to $P$-evanescence, for all $n\in\mathbb{N}$

$$\langle\!\langle Z^c, X^{c,n}\rangle\!\rangle_{\cdot\wedge\sigma_n\wedge\rho_n} = \int_0^{\cdot\wedge\sigma_n\wedge\rho_n} Z_{s-}c_s\beta_s dA_s,$$

and, in view of (3.15), $M^P_{\mu^n}$-a.e. on $[\![0,\sigma_n\wedge\rho_n]\!]\times\mathbb{R}^d$

$$Z = Z_- + \Delta Z = Z_-\big(1+U(\cdot,\Delta X^n)-1\big) = Z_-U(\cdot,\Delta X^n).$$

This implies that $M^P_{\mu^n}$-a.e.

$$\mathbb{1}_{[\![0,\sigma_n\wedge\rho_n]\!]\times\mathbb{R}^d} M^P_{\mu^n}\big(Z\big|\widetilde{\mathcal{P}}\big) = \mathbb{1}_{[\![0,\sigma_n\wedge\rho_n]\!]\times\mathbb{R}^d} Z_-U.$$

Therefore, $\beta$ and $U$ play the same role as in Section 3.2. Consequently, due to Standing Assumption 3.5, Theorem 3.1 implies that $Q$ solves the SMP $(\rho;\eta;B',C,\nu')$ with

$\rho$-localizing sequence $(\rho_n)_{n\in\mathbb{N}}$. Furthermore, by hypothesis, $Q$ coincides with $Q^*$ on $\mathcal{F}_{\sigma-}$. Hence, the formula (3.16) immediately follows from Theorem 3.1 (a).

Since for all $G \in \mathcal{F}$ we have $G \cap \{\sigma = \infty\} \in \mathcal{F}_{\sigma-}$, the equivalence (a.i) $\Leftrightarrow$ (a.ii) follows from Theorem 3.1 (b). If (a.i) holds, then $Q = Q^*$ on $\mathcal{F}$. We explain this with more details. Since $Q = Q^*$ on $\mathcal{F}_{\sigma-}$ and $\{\sigma = \infty\} \in \mathcal{F}_{\sigma-}$, we have $Q$-a.s. $\sigma = \infty$. Now, for all $G \in \mathcal{F}$, we have $G \cap \{\sigma = \infty\} \in \mathcal{F}_{\sigma-}$ and therefore

$$Q(G) = Q(G \cap \{\sigma = \infty\}) = Q^*(G \cap \{\sigma = \infty\}) = Q^*(G).$$

Consequently, (a.i) $\Rightarrow$ (a.iii) follows from Theorem 3.1 (b), too. Since the implication (a.iii) $\Rightarrow$ (a.ii) is trivial, this completes the proof of (a).

Set

$$\gamma_n \triangleq \inf\left(t \in \mathbb{R}_+ \colon H_t \geq n\right),$$

and note that (b.i) implies that

$$\lim_{n\to\infty} Q(\gamma_n = \sigma = \infty) = \lim_{n\to\infty} Q^*(\gamma_n = \sigma = \infty) = 1.$$

Moreover, note that $P$-a.s. $H_{\gamma_n} \leq n + 4$ for all $n \in \mathbb{N}$. Thus, by [65, Theorem 8.25] and similar arguments as used in the proof of [65, Lemma 12.44], the stopped process $Z^{\gamma_n}$ is a uniformly integrable $P$-martingale. Thus, the implication (b.i) $\Rightarrow$ (b.ii) follows from Theorem 3.1 (c).

If (b.ii) holds, then (a.ii) and thus also (a.i) holds and we have $Q = Q^*$ on $\mathcal{F}$. Hence, the implication (b.ii) $\Rightarrow$ (b.iii) is due to Theorem 3.1 (c).

Finally, the implication (b.iii) $\Rightarrow$ (b.i) follows from [65, Theorem 8.21, Lemma 12.44] and the proof is complete. □

Corollary 3.1 can be seen as a version of [65, Theorem 12.55] and [70, Theorem III.5.34] with a simple uniqueness assumption replacing the local uniqueness assumption.

**Remark 3.6.** *Recalling the equalities* (3.7) *and* (3.12)*, if Standing Assumption 3.5 holds, then (a.i) in Corollary 3.1 is equivalent to $Q^*$-a.s. $H_\rho < \infty$ on $\{\rho < \infty\}$. In this case, the difference between local absolute continuity and absolute continuity is captured by the behavior of $H_\rho$ on the set $\{\rho = \infty\}$.*

**Remark 3.7.** *In many cases, for instance due to parametric constraints, all solutions to a SMP are supported on a path space $\Omega^o \subseteq \Omega$. In particular, this is the case when $\rho = \infty$ with $\Omega^o$ being the classical Skorokhod space, i.e. the space of all càdlàg functions $\mathbb{R}_+ \to \mathbb{R}^d$. In such a situation, uniqueness on $\mathcal{F}_{\sigma-}$ is equivalent to uniqueness on the trace $\sigma$-field $\mathcal{F}_{\sigma-} \cap \Omega^o$. If in addition $\rho = \tau_\Delta$ on $\Omega^o$, then*

$$\mathcal{F}_{\rho-} \cap \Omega^o = \mathcal{F}_{\tau_\Delta -} \cap \Omega^o = \mathcal{F} \cap \Omega^o. \tag{3.17}$$

*Here, we use the identity $\mathcal{F}_{\tau_\Delta -} = \mathcal{F}$, which holds because for all $G \in \mathcal{B}(\mathbb{R}^d_\Delta)$*

$$\{X_t \in G\} = \begin{cases} \{\tau_\Delta \leq t\} \cup \left(\{X_t \in G\} \cap \{\tau_\Delta > t\}\right), & \Delta \in G, \\ \{X_t \in G\} \cap \{\tau_\Delta > t\}, & \Delta \notin G. \end{cases}$$

*Returning to the identity* (3.17)*, we see that uniqueness on $\mathcal{F}_{\rho-} \cap \Omega^o$ implies uniqueness on $\mathcal{F}$ and in particular uniqueness on the sub $\sigma$-field $\mathcal{F}_{\sigma-}$.*

The following corollary is a localization of [65, Theorem 12.49] without local uniqueness.

**Corollary 3.2.** *Suppose that $P$-a.s. $\rho = \infty$, that $Q^*$ is the only solution of the SMP $(\rho; \eta; B', C, \nu')$ with $\rho$-localizing sequence $(\rho_n)_{n\in\mathbb{N}}$ and that* (a)' *in Standing Assumption 3.5 holds. Then, the following are equivalent:*

(i) *$Q^*$-a.s. $\rho = \infty$.*

(ii) *$Q^* \ll_{\text{loc}} P$ with $\frac{dQ^*}{dP}|_{\mathcal{F}_t} = Z_t$.*

*If these statements hold true, then also the following are equivalent:*

(iii) *$P \ll_{\text{loc}} Q^*$*

(iv) *$P$-a.s. $\mathbb{1}_{\{U=0\}} \star \nu_\infty = 0$ and $\widehat{U} = 1 \Rightarrow a = 1$.*

*Proof.* In view of Remark 3.6, Corollary 3.1 yields that (i) implies (ii). Now, if (ii) holds, then $Q^*(\rho < t) = 0$ for all $t \in \mathbb{R}_+$, because we assume $P(\rho < t) = 0$. Consequently, we conclude that (i) holds.

If (i) and (ii) hold, the equivalence of (iii) and (iv) follows from [65, Theorem 12.48]. □

Another consequence of Corollary 3.1 is the following: If $Q^*$ is unique and $H$ is finite and deterministic, then $Q^* \ll_{\text{loc}} P$ with $\frac{dQ^*}{dP}|_{\mathcal{F}_t} = Z_t$. This observation can be proven directly with the same strategy as used in the proof of Theorem 3.1: Indeed, if $H$ is finite and deterministic, the local $P$-martingale $Z$ has a deterministic $P$-localizing sequence, namely $(\sigma_n)_{n\in\mathbb{N}}$. Consequently, $Z$ is a true $P$-martingale. Now, because $(\Omega, \mathcal{F}^o_n)_{n\in\mathbb{N}}$ is a standard system, see [51] for more details, Parthasaraty's extension theorem yields that $Q^*$ can be constructed from $P$ as an extension of the consistent sequence $(Z_n \cdot P)_{n\in\mathbb{N}}$. By construction, $Q^* \ll_{\text{loc}} P$ with $\frac{dQ^*}{dP}|_{\mathcal{F}_t} = Z_t$.

In the following section we comment on related literature.

## 3.4 Comments on the Literature

In this section we relate our results to the literature. In Section 3.4.1, we show that Corollary 3.1 is in line with the main results of Cherny and Urusov [23] and Mijatović and Urusov [107]. In Section 3.4.2, we relate Corollary 3.1 to the main result of Cheridito, Filipović and Yor [21].

### 3.4.1 Absolute Continuity of One-Dimensional Diffusions

The (local) absolute continuity of laws of one-dimensional diffusions was intensively studied by Cherny and Urusov [23], who gave deterministic equivalent conditions under the Engelbert–Schmidt conditions. In the same setting, deterministic equivalent conditions for the martingale property of stochastic exponentials were given by Mijatović and Urusov [107]. Both approaches are based on so-called separation times and are quite different from ours. As we will illustrate in this section, the results can be deduced from Corollary 3.1.

We start with a formal introduction to the setup. In the following, $\nu$ will always be the zero measure and we will remove it from all notations.

Let $b, \beta \colon \mathbb{R} \to \mathbb{R}$ and $c \colon \mathbb{R} \to \mathbb{R}_+$ be Borel functions. We extend these functions to $\mathbb{R}_\Delta$ by setting them to zero outside $\mathbb{R}$. Furthermore, we set

$$\begin{aligned} B &\triangleq \int_0^{\cdot} b(X_s)ds, \\ B' &\triangleq \int_0^{\cdot} \left(b(X_s) + (\beta c)(X_s)\right) ds, \\ C &\triangleq \int_0^{\cdot} c(X_s)ds, \end{aligned}$$

where $B$ and $B'$ are set to be $\Delta$ and $C$ is set to be $\infty$ whenever the integrals diverge, and define the stopping times

$$\rho_n \triangleq \inf(t \in \mathbb{R}_+ \colon \|X_t\| > n) \wedge n, \quad \rho \triangleq \lim_{n\to\infty} \rho_n, \tag{3.18}$$

where $\|\Delta\| \triangleq \infty$.

**Standing Assumption 3.6.** *The Engelbert–Schmidt conditions hold for the pairs $(b, c)$ and $(b + \beta c, c)$, i.e.*

$$c > 0, \qquad \frac{1 + |b| + |b + \beta c|}{c} \in L^1_{\text{loc}}(\mathbb{R}).$$

In this case, for all $x \in \mathbb{R}$ the SMP $(\rho; \delta_x; B, C)$ has a solution $P$ and the SMP $(\rho; \delta_x; B', C)$ has a solution $Q^*$ both with $\rho$-localizing sequence $(\rho_n)_{n\in\mathbb{N}}$. Let $\Omega^o$ be the set of all $\omega \in \Omega$ which are continuous functions $\mathbb{R}_+ \to \mathbb{R}_\Delta$ when $\mathbb{R}_\Delta$ is equipped with the one-point compactification topology. The solutions to each of these SMPs with $\rho$-localizing sequence $(\rho_n)_{n\in\mathbb{N}}$ are supported on the set $\Omega^o$ and coincide on $\mathcal{F}$, see Remark 3.7. In particular, we have $P$-a.s.

$$\rho_n < \rho. \tag{3.19}$$

In other words, the probability measures $P$ and $Q^*$ and the sequence $(\rho_n)_{n\in\mathbb{N}}$ are as in Standing Assumption 3.2 and the uniqueness assumption of Corollary 3.1 holds. Proofs for these claims can be found in [77, Section 5.5] or [46].

**Standing Assumption 3.7.** *We have*

$$\beta^2 \in L^1_{\text{loc}}(\mathbb{R}). \tag{3.20}$$

Standing Assumption 3.7 is also imposed in [107], but not in [23], where it is shown to be necessary for $Q^* \ll_{\text{loc}} P$.

We define the $[0, \infty]$-valued process

$$H \triangleq \int_0^{\cdot\wedge\rho} (\beta^2 c)(X_s)ds$$

and set $\sigma_n$ and $\sigma$ as in (3.10). Note that $H$ is left-continuous due to the monotone convergence theorem. The condition (3.20) implies that

$$\int_0^t (\beta^2 c)(X_s)ds < \infty, \quad P\text{-a.s. for all } t < \rho, \tag{3.21}$$

see [105, Theorem 2.6] and Remark 3.5. Moreover, (3.19) and (3.21) imply $P$-a.s.

$$\rho_n < \rho \leq \sigma. \tag{3.22}$$

In other words, recalling Remark 3.4, we conclude that Standing Assumption 3.4 (a) holds. We define a non-negative local $P$-martingale $Z$ as in Section 3.3.

In this setting, (a)' in Standing Assumption 3.5 holds, too. If the function $\beta^2 c$ is locally bounded, $E^P\big[Z_{\rho_n}\big] = 1$ follows immediately from Novikov's condition, see also Proposition 3.1. However, under the weaker assumption that $\beta^2 \in L^1_{\text{loc}}(\mathbb{R})$ the verification becomes more challenging. We refer to [23, Lemma 5.30] for a proof.

Consequently, the following result follows from Corollary 3.1 and Remark 3.6.

**Corollary 3.3.** (a) *The following are equivalent:*

(a.i) $Q^*$*-a.s.* $H_\rho = \int_0^\rho (\beta^2 c)(X_s)ds < \infty$ *on* $\{\rho < \infty\}$.

(a.ii) $Z$ *is a* $P$*-martingale.*

(a.iii) $Q^* \ll_{\text{loc}} P$ *with* $\frac{dQ^*}{dP}|_{\mathcal{F}_t} = Z_t$.

(b) *The following are equivalent:*

(b.i) $Q^*$*-a.s.* $H_\rho = \int_0^\rho (\beta^2 c)(X_s)ds < \infty$.

(b.ii) $Z$ *is a uniformly integrable* $P$*-martingale.*

(b.iii) $Q^* \ll P$ *with* $\frac{dQ^*}{dP} = Z_\infty$.

The equivalences (a.i) $\Leftrightarrow$ (a.ii) and (b.i) $\Leftrightarrow$ (b.ii) in Corollary 3.3 are also implied by [127, Theorem 3.3].

Next, we explain that this corollary is in line with the deterministic conditions for the (local) absolute continuity as given in [23] and for the (uniformly integrable) $P$-martingale property of $Z$ as given in [107]. We start with notation:

$$\begin{aligned} p(x) &\triangleq \exp\left(-\int_0^x \frac{2(b(y) + (\beta c)(y))}{c(y)} dy\right), \ x \in \mathbb{R}, \\ s(x) &\triangleq \int_0^x p(y)dy, \ x \in \mathbb{R}, \\ s(+\infty) &\triangleq \lim_{x \nearrow +\infty} s(x), \\ s(-\infty) &\triangleq \lim_{x \searrow -\infty} s(x). \end{aligned} \tag{3.23}$$

Furthermore, for $z \in \{-\infty, \infty\}$ and a Borel function $f\colon \mathbb{R} \to \mathbb{R}_+$ we write $f \in L^1_{\text{loc}}(z)$ if there is an $x \in \mathbb{R}$ such that $\int_{x \wedge z}^{x \vee z} f(y)dy < \infty$. We define the following conditions:

$$s(+\infty) = \infty, \tag{3.24}$$

$$s(+\infty) < \infty \quad \text{and} \quad \frac{s(+\infty) - s}{pc} \not\in L^1_{\text{loc}}(\infty), \tag{3.25}$$

$$s(+\infty) < \infty \quad \text{and} \quad \frac{(s(+\infty) - s)\beta^2}{p} \in L^1_{\text{loc}}(-\infty), \tag{3.26}$$

and similarly

$$s(-\infty) = -\infty, \tag{3.27}$$

$$s(-\infty) > -\infty \quad \text{and} \quad \frac{s - s(-\infty)}{pc} \notin L^1_{\text{loc}}(-\infty), \tag{3.28}$$

$$s(-\infty) > -\infty \quad \text{and} \quad \frac{(s - s(-\infty))\beta^2}{p} \in L^1_{\text{loc}}(-\infty). \tag{3.29}$$

Let us relate these conditions to (a.i) in Corollary 3.3. Define

$$\rho_+ \triangleq \lim_{n\to+\infty} \inf(t \in \mathbb{R}_+ \colon X_t > n),$$
$$\rho_- \triangleq \lim_{n\to+\infty} \inf(t \in \mathbb{R}_+ \colon X_t < -n),$$

and note that $Q$-a.s.

$$\{\rho < \infty\} = \{\rho_+ < \infty\} \cup \{\rho_- < \infty\}.$$

We discuss the finiteness of $H_\rho$ separately on the two sets on the right hand side. By Feller's test for explosion, see [105, Propositions 2.4, 2.5, 2.12], $\{\rho_+ < \infty\}$ is $Q^*$-null if and only if either (3.24) or (3.25) holds.

If $Q^*(\rho_+ < \infty) > 0$, then $H_\rho$ is $Q^*$-a.s. finite on $\{\rho_+ < \infty\}$ if and only if (3.26) holds, see [105, Theorem 2.11].

Similar arguments yield that $H_\rho$ is $Q^*$-a.s. finite on $\{\rho_- < \infty\}$ if and only if one of the conditions (3.27), (3.28) or (3.29) holds. Finally, we recover the following version of [107, Theorem 2.1] and [23, Corollary 5.2] from Corollary 3.3.

**Corollary 3.4.** (a.i), (a.ii) *and* (a.iii) *from Corollary 3.3 are equivalent to the following:*

(a.iv) *One of the conditions* (3.24), (3.25) *or* (3.26) *holds and one of the conditions* (3.27), (3.28) *or* (3.29) *holds.*

Let us now explain when $H_\rho$ is $Q^*$-a.s. finite. We distinguish four cases:

1. If $s(+\infty) = \infty$ and $s(-\infty) = \infty$, then $Q^*$-a.s. $H_\rho < \infty$ if and only if Lebesgue almost everywhere $\beta = 0$, see [105, Theorem 2.10].

2. If $s(+\infty) < \infty$ and $s(-\infty) = \infty$, then $Q^*$-a.s. $H_\rho < \infty$ if and only if the second part in (3.26) holds, see [105, Proposition 2.4, Theorem 2.11].

3. If $s(+\infty) = \infty$ and $s(-\infty) < \infty$, then $Q^*$-a.s. $H_\rho < \infty$ if and only if the second part in (3.29) holds, see [105, Proposition 2.4, Theorem 2.11].

4. If $s(+\infty) < \infty$ and $s(-\infty) < \infty$, then $Q^*$-a.s. $H_\rho < \infty$ if and only if the second parts in (3.26) and (3.29) hold, see [105, Proposition 2.4, Theorem 2.11].

Corollary 3.3 implies the following version of [107, Theorem 2.3] and [23, Corollary 5.1].

**Corollary 3.5.** (b.i), (b.ii) *and* (b.iii) *from Corollary 3.3 are equivalent to the following:*

(b.iv) *One of the following conditions holds:*

(1) *Lebesgue almost everywhere* $\beta = 0$.

(2) (3.26) *and* (3.27) *hold.*

(3) (3.24) *and* (3.29) *hold.*

(4) (3.26) *and* (3.29) *hold.*

**Remark 3.8.** *Let us comment on the first case, which is closely related to the recurrence of $P$. It is well-known that $P$ is recurrent if and only if $s(+\infty) = s(-\infty) = \infty$, see [77, Proposition 5.5.22] or [120, Theorem 5.1.1]. Here, we call $P$ recurrent if*

$$P(X_t = y \text{ for some } t \in \mathbb{R}_+) = 1 \text{ for all } y \in \mathbb{R}.$$

*In particular, $P$ is conservative. For recurrent diffusions, we have an ergodic theorem, i.e. $P$-a.s.*

$$\frac{\int_0^t f(X_s)ds}{\int_0^t g(X_s)ds} \xrightarrow{t\to\infty} \frac{\int f(y)m(dy)}{\int g(y)m(dy)},$$

*where $m$ is the speed measure of $X$ and $f, g\colon \mathbb{R} \to \mathbb{R}_+$ are Borel functions such that $\int f(y)m(dy) < \infty$ and $\int g(y)m(dy) > 0$. For a proof see [74, Theorem 20.14]. If $Q^* \ll P$, then the $P$-a.s. convergence transfers to $Q^*$ and it follows that the speed measures of $X$ coincide under $P$ and $Q^*$. This implies that $\beta = 0$ Lebesgue almost everywhere.*

For explicit examples of all possible situations, we refer to [23]. In summary, we have seen that Corollary 3.1 is in line with the results proven in [23, 107].

### 3.4.2 Absolute Continuity of Itô-Jump-Diffusions

In this section, we compare Corollary 3.1 to the main result of Cheridito, Filipović and Yor [21]. The proof in [21] heavily relies on the concept of local uniqueness, which is in Markovian setups implied by the existence of unique solutions for all deterministic initial values, see [70, Theorem III.2.40].

Next, we recall a version of the setup of [21]. We stress that [21] includes a killing rate, which is not included in our case. Moreover, the underlying filtered spaces are different, such that the uniqueness assumptions are not identical, but very similar.

Let $b, \beta\colon \mathbb{R}^d \to \mathbb{R}^d$ and $c\colon \mathbb{R}^d \to \mathbb{S}^d$ be Borel functions and let $K$ be a Borel transition kernel from $\mathbb{R}^d$ into $\mathbb{R}^d$. Furthermore, let $U\colon \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty)$ be Borel. We extend $b, \beta, c, K$ and $x \mapsto U(x, y)$ for all $y \in \mathbb{R}^d$ to $\mathbb{R}^d_\Delta$ by setting them zero outside $\mathbb{R}^d$. More precisely, we mean here the zero vector, the zero matrix, etc. In [21] the following local boundedness assumptions are imposed:

The functions

$$b, b + c\beta, c, \int \left(1 \wedge \|y\|^2\right) K(\cdot, dy) \text{ and } \int \left(1 + \|y\|^2\right) U(\cdot, y) K(\cdot, dy)$$

are locally bounded on $\mathbb{R}^d$.

We set

$$\begin{aligned} B &\triangleq \int_0^\cdot b(X_s)ds, \\ B' &\triangleq \int_0^\cdot \left(b(X_s) + c(X_s)\beta(X_s)\right) ds, \\ C &\triangleq \int_0^\cdot c(X_s)ds, \end{aligned}$$

where $B$ and $B'$ are set to be $\Delta$ and each entry of $C$ is set to be $\infty$ whenever the integrals diverge, and

$$\nu(dt \times dx) \triangleq K(X_t, dx)dt,$$
$$\nu'(dt \times dx) \triangleq U(X_t, x)K(X_t, dx)dt.$$

Let $\rho_n$ and $\rho$ be as in (3.18) and let $\eta$ be a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

In the following, $P$ is a solution to the SMP $(\rho; \eta; B, C, \nu)$ and $Q^*$ is a solution to the SMP $(\rho; \eta; B', C, \nu')$ both with $\rho$-localizing sequence $(\rho_n)_{n\in\mathbb{N}}$.

Define

$$H^* \triangleq \frac{1}{2}\int_0^{\cdot\wedge\rho} \langle \beta(X_t), c(X_t)\beta(X_t)\rangle dt + \int_0^{\cdot\wedge\rho}\int \big(U(X_t, x)\log(U(X_t, x)) - U(X_t, x) + 1\big)K(X_t, dx)dt,$$

see also Proposition 3.1. The main result in [21] can be rephrased as follows:

If $Q^*$ is the only solution to the SMP $(\rho; \eta; B', C, \nu')$ with $\rho$-localizing sequence $(\rho_n)_{n\in\mathbb{N}}$ and

$$E^P\left[\exp\left(H^*_{\rho_n}\right)\right] < \infty \text{ for all } n \in \mathbb{N}, \tag{3.30}$$

then a formula like (3.16) holds for all $(\mathcal{F}^o_t)_{t\geq 0}$-stopping times $\xi$, and $Q^* \ll_{\text{loc}} P$ holds if $Q^*$ is conservative.

The condition (3.30) is a Novikov-type condition, which ensures that $Z^{\rho_n}$ is a uniformly integrable $P$-martingale, where $Z$ is defined as in Section 3.3, see also Standing Assumption 3.1. In particular, it implies that $E^P\big[Z_{\rho_n}\big] = 1$, see Proposition 3.1 and Standing Assumption 3.5.

Next, we compare this statement to Corollary 3.1. Let $H$ be defined as in Section 3.3, i.e. in this case

$$H = \int_0^{\cdot\wedge\rho} \langle \beta(X_t), c(X_t)\beta(X_t)\rangle dt + \int_0^{\cdot\wedge\rho}\int \Big(1 - \sqrt{U(X_t, x)}\Big)^2 K(X_t, dx)dt. \tag{3.31}$$

We note that $H \leq 2H^*$, which follows from the elementary inequality

$$\big(1 - \sqrt{x}\big)^2 \leq x\log(x) - x + 1 \text{ for all } x > 0. \tag{3.32}$$

Thus, (3.30) implies that $P$-a.s. $H_{\rho_n} < \infty$, which yields that $P$-a.s. $\rho \leq \sigma$. In this setting, it can be shown that $P$-a.s. $\rho_n < \rho$, see [21, Lemma 3.1] and the paragraph below its proof. Thus, $P$-a.s. $\rho_n < \sigma$ and, because $H$ is left-continuous (due to the monotone convergence theorem), Remark 3.4 implies that (a)' in Standing Assumption 3.5 holds. Consequently, Corollary 3.1 implies that $Q^* \ll_{\text{loc}} P$ is true when $Q^*$ is conservative. Furthermore, the formula (3.16) holds for all stopping times $\xi$. In this regard, our result is different from the main result in [21], which applies for stopping times of the canonical filtration $(\mathcal{F}^o_t)_{t\geq 0}$.

## 3.5 Martingale Property of Stochastic Exponentials

In this section, we generalize Beneš' [7] classical linear growth condition for the martingale property of exponential Brownian martingales. This application does not require any uniqueness assumption. Let us shortly explain the idea. If a local martingale has a localizing sequence, which is also a localizing sequence for a modified SMP, then the local martingale is a true martingale. In the following, we will formulate conditions which imply the existence of such a localizing sequence for any solution of the modified SMP. Thus, no uniqueness assumption is required.

We recall the result of Beneš [7]: Assume that $W$ is a $d$-dimensional Brownian motion and that $\mu$ is an $\mathbb{R}^d$-valued predictable process on the Wiener space. Then, the stochastic exponential

$$\exp\left(\int_0^\cdot \langle \mu_s(W), dW_s\rangle - \frac{1}{2}\int_0^\cdot \|\mu_s(W)\|^2 ds\right)$$

is a martingale if $\mu$ is at most of linear growth. We refer to [77, Corollary 3.5.16] for a precise statement.

In the following we generalize this result to cases where $W$ is a continuous Itô process. Of course, it is possible to allow additionally jumps. However, we think that focusing on the less technical continuous setup suffices to explain the main idea. For similar conditions in an Itô jump-diffusion setup we refer to [82].

Since $\nu$ will always be the zero measure we remove it from all notations. Let $b$ and $\beta$ be $\mathbb{R}^d$-valued predictable processes and $c$ be a predictable process with values in $\mathbb{S}^d$. We set

$$\begin{aligned} B &\triangleq \int_0^\cdot b_s ds, \\ B' &\triangleq \int_0^\cdot (b_s + c_s\beta_s)\, ds, \\ C &\triangleq \int_0^\cdot c_s ds, \end{aligned}$$

where $B$ and $B'$ are set to be $\Delta$ and each entry of $C$ is set to be $\infty$ whenever the integrals diverge. Let $\rho_n$ and $\rho$ be as in (3.18) and let $\eta$ be a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

**Standing Assumption 3.8.** *Let $P$ be a solution to the SMP $(\rho;\eta;B,C)$ with $\rho$-localizing sequence $(\rho_n)_{n\in\mathbb{N}}$. Let $\sigma$ be as in (3.10) and define $Z$ as in Section 3.3. Furthermore, $P$-a.s. $\rho_n < \sigma$ and $E^P\big[Z_{\rho_n}\big] = 1$ for all $n \in \mathbb{N}$.*

**Corollary 3.6.** *Suppose there exists a Borel function $\gamma \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R}_+)$ such that for all continuous functions $\omega\colon \mathbb{R}_+ \to \mathbb{R}^d$ and all $t \in \mathbb{R}_+$*

$$\begin{aligned} \|b_t(\omega) + \beta_t(\omega)c_t(\omega)\|^2 &\leq \gamma(t)\Big(1 + \sup_{s\in[0,t]} \|\omega(s)\|^2\Big), \\ \operatorname{trace} c_t(\omega) &\leq \gamma(t)\Big(1 + \sup_{s\in[0,t]} \|\omega(s)\|^2\Big). \end{aligned}$$

*Then, $Z$ is a $P$-martingale.*

*Proof.* The strategy is to apply Theorem 3.1: It suffices to show that all solutions $Q$ to the SMP $(\rho;\eta;B',C)$ with $\rho$-localizing sequence $(\rho_n)_{n\in\mathbb{N}}$ satisfy $Q(\rho = \infty) = 1$. We fix

$\varepsilon > 0$. Since $\eta(\{x \in \mathbb{R}^d \colon \|x\| \geq m\}) \to 0$ as $m \to \infty$, there exists an $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\eta(\{x \in \mathbb{R}^d \colon \|x\| \geq N\}) \leq \varepsilon.$$

We also fix $T \in \mathbb{N}$. It is not difficult to see that, due to our linear growth conditions, we find a constant $k = k(T, N)$ such that for all $t \in [0, T]$

$$E^Q\Big[\sup_{s\in[0,t\wedge\rho_n]} \|X_s\|^2 \mathbb{1}_{\{\|X_0\|<N\}}\Big] \leq k\Big(1 + E^Q\Big[\int_0^t \gamma(s) \sup_{r\in[0,s\wedge\rho_n]} \|X_r\|^2 ds \mathbb{1}_{\{\|X_0\|<N\}}\Big]\Big).$$

Now, we deduce from Gronwall's lemma that for all $t \in [0, T]$

$$E^Q\Big[\sup_{s\in[0,t\wedge\rho_n]} \|X_s\|^2 \mathbb{1}_{\{\|X_0\|<N\}}\Big] \leq \text{const. independent of } n.$$

Using Chebyshev's inequality, we obtain that for all $t \in [0, T]$

$$\begin{aligned} Q(\rho_n \leq t) &= Q(\rho_n \leq t, \|X_0\| < N) + Q(\rho_n \leq t, \|X_0\| \geq N) \\ &\leq Q\Big(\sup_{s\in[0,t\wedge\rho_n]} \|X_s\| \geq n, \|X_0\| < N\Big) + \varepsilon \\ &\leq \frac{\text{const. independent of } n}{n^2} + \varepsilon \to \varepsilon \end{aligned}$$

as $n \to \infty$. We conclude that $Q(\rho = \infty) = 1$, which completes the proof. □

Under the Engelbert–Schmidt conditions we have already seen equivalent conditions for the martingale property of $Z$, see Section 3.4.1 or [107]. The linear growth condition presented in Corollary 3.6 is not necessary. However, it applies in multidimensional setups, in non-Markovian cases and does not require any uniqueness assumption. Furthermore, it is typically easy to verify.

# 4 Absolute Continuity and Singularity of Multidimensional Diffusions

## 4.1 Introduction

Consider two laws $P$ and $Q$ of multidimensional possibly explosive diffusions with common diffusion coefficient $\mathfrak{a}$ and drift coefficients $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{a}\mathfrak{c}$, respectively. We are interested in finding analytic conditions for the absolute continuity $P \ll Q$ and the singularity $P \perp Q$. Such conditions are of interest in many branches of probability theory. In mathematical finance, for instance, mutual absolute continuity is of importance in the study of the absence of arbitrage, see [22, 38].

For one-dimensional diffusions precise integral tests were proven in [23] under the Engelbert–Schmidt conditions. For multidimensional diffusions the situation is less well-understood and only a few analytic conditions are known, see [6] for an integral test for Fuchsian diffusions. Here, a diffusion is called Fuchsian if the coefficients $\mathfrak{a}$ and $\mathfrak{b}$ are locally Hölder continuous, $\mathfrak{a}$ is uniformly elliptic and $\sup_{x \in \mathbb{R}^d} \|\mathfrak{b}(x)\|(1 + \|x\|) < \infty$.

The starting point for our research is the following probabilistic characterization of absolute continuity and singularity: Let $X$ be the coordinate process and set

$$A_t \triangleq \int_0^{t\wedge\theta} \langle \mathfrak{c}(X_s), \mathfrak{a}(X_s)\mathfrak{c}(X_s)\rangle ds, \quad t \in \mathbb{R}_+,$$

where $\theta$ is the explosion time. It has been proven in [6] and in Chapter 3 that $P \ll Q$ is equivalent to $P(A_\theta < \infty) = 1$ and that $P \perp Q$ is equivalent to $P(A_\theta = \infty) = 1$. In other words, $P \ll Q$ and $P \perp Q$ are characterized by $P$-a.s. divergence and convergence of the perpetual integral $A_\theta$. Again, these properties are well-understood for one-dimensional diffusions, see [33, 81, 105, 110], and it seems that less work has been done for the multidimensional case, see [45, 140] for results concerning Bessel processes, [32] for some conditions in radial cases, and [87] for results on divergence in case $X$ is a conservative Feller process possibly with jumps.

In [16, 33, 81, 87] the perpetual integral $A_\theta$ was related to the hitting time of a time-changed process. We pick up this idea and prove the following: Let $\mathfrak{f}\colon \mathbb{R}^d \to (0, \infty)$ be a Borel function which is locally bounded away from zero and infinity. Under the assumptions that the diffusion $P$ exists and that $\mathfrak{b}$ and $\mathfrak{a}$ are locally bounded, we show existence of a diffusion $P^\circ$ with diffusion coefficient $\mathfrak{f}^{-1}\mathfrak{a}$ and drift coefficient $\mathfrak{f}^{-1}\mathfrak{b}$ such that the law of the perpetual integral

$$T_\theta \triangleq \int_0^\theta \mathfrak{f}(X_s)ds$$

under $P$ coincides with the law of the explosion time $\theta$ under $P^\circ$. Furthermore, we show that $P^\circ$ is unique whenever $P$ is unique.

Returning to our initial problem, we note that in case $\mathfrak{f} = \langle \mathfrak{c}, \mathfrak{a}\mathfrak{c}\rangle$ the absolute continuity

$P \ll Q$ is equivalent to $P^\circ(\theta < \infty) = 1$ and the singularity $P \perp Q$ is equivalent to $P^\circ(\theta = \infty) = 1$. This observation is very useful, because the literature contains many conditions for explosion and non-explosion of multidimensional diffusions, see [104, 120, 137]. For illustration, we formulate a Khasminskii-type integral test for absolute continuity and singularity.

The result can also be applied in the converse direction: In case we have criteria for absolute continuity and singularity, these can be used to deduce explosion criteria for time-changed diffusions. To illustrate this, we derive an integral test for almost sure explosion and non-explosion of time-changed Brownian motion, using results on singularity of Fuchsian diffusions proven in [6].

The absolute continuity $P \ll Q$ is intrinsically connected to the uniform integrable (UI) $Q$-martingale property of a certain stochastic exponential (see Eq. 4.2 below), which has been studied for one-dimensional diffusions in [107]. Independent of the dimension, for the conservative case it is known that the loss of the martingale property has a one-to-one relation to the explosion of an auxiliary diffusion, see, e.g. [21, 134]. This turned out to be wrong in the non-conservative setting of [107]. Our result shows that for the UI martingale property the statement is true irrespective whether the diffusions are conservative or non-conservative.

As a third application, we use the relation of the UI martingale property and absolute continuity to study a problem in mathematical finance: More precisely, for certain exponential diffusion models with infinite time horizon we derive analytic criteria for the existence of an equivalent local martingale measure (ELMM), and explosion criteria for the existence of an equivalent martingale measure (EMM).

Let us end the introduction with comments on related literature. To the best of our knowledge, the relation of absolute continuity/singularity and explosion of a time-changed process has not been reported before. We think that our new integral tests for absolute continuity/singularity and explosion/non-explosion illustrate that working out this connection is fruitful. The integral tests in [23, 107] for absolute continuity, singularity and the UI martingale property in one-dimensional frameworks follow from our result and Feller's test for explosion under additional assumptions on the coefficients. The existence of E(L)MMs for one- and multidimensional diffusion models with finite time horizon has, e.g. been studied in [27, 106, 134] and will also be studied in Chapter 6 below. Certain one-dimensional diffusion models with infinite time horizon have been studied in [106]. In this chapter we focus on multidimensional models. Beginning with [138], existence and uniqueness results for time-changed Markov processes have a long history, see, e.g. [17, 137] for more information. In most of the classical work, the function $\mathfrak{f}$ is assumed to be uniformly bounded away from zero, which implies that time-changes of conservative diffusions are themselves conservative. More general positive continuous $\mathfrak{f}$ are considered in the recent article [87] in combination with linear growth conditions for non-explosion. The novelty of our existence and uniqueness result is that we work without additional assumptions for non-explosion. This is crucial for the question of absolute continuity and singularity. Moreover, we work under sort of minimal assumptions on the original diffusion $P$ by assuming only existence and locally bounded coefficients.

The chapter is structured as follows: In Section 4.2 we present our main results, in Section 4.3 we discuss applications and in Section 4.4 we prove our main theorem. In Section 4.6 we recall the relation of martingale problems and weak solutions of stochastic differential equations, and in Section 4.7 we collect some existence and uniqueness results for solutions of martingale problems.

## 4.2 Main Results

Let $\mathbb{R}^d_\Delta \triangleq \mathbb{R}^d \cup \{\Delta\}$ be the one-point compactification of $\mathbb{R}^d$ and let $\Omega$ be the space of all continuous functions $\mathbb{R}_+ \to \mathbb{R}^d_\Delta$ which are absorbed in $\Delta$. Define $X$ to be the coordinate process on $\Omega$, i.e. $X_t(\omega) = \omega(t)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}_+$, and define $\mathcal{F} \triangleq \sigma(X_t, t \in \mathbb{R}_+), \mathcal{F}^o_t \triangleq \sigma(X_s, s \in [0,t])$ and $\mathcal{F}_t \triangleq \mathcal{F}^o_{t+} \triangleq \bigcap_{s>t} \mathcal{F}^o_s$ for all $t \in \mathbb{R}_+$. Except stated otherwise, all terms such as *martingale, stopping time* etc. correspond to $\mathbf{F} \triangleq (\mathcal{F}_t)_{t\geq 0}$ as underlying filtration. Let $\mathbb{S}^d$ be the space of symmetric non-negative definite real-valued $d \times d$ matrices.

For $n \in \mathbb{N}$ we set

$$\theta_n \triangleq \inf(t \in \mathbb{R}_+ \colon \|X_t\| \geq n), \quad \theta \triangleq \inf(t \in \mathbb{R}_+ \colon X_t = \Delta) = \lim_{m\to\infty} \theta_m.$$

It is well-known that $\theta_n$ and $\theta$ are stopping times. We fix two Borel functions $\mathfrak{b} \colon \mathbb{R}^d \to \mathbb{R}^d$ and $\mathfrak{a} \colon \mathbb{R}^d \to \mathbb{S}^d$ and assume the following:

(S1) $\mathfrak{b}$ and $\mathfrak{a}$ are locally bounded.

Here, $S$ is an acronym for *standing*, which indicates that the assumption is in force for the remainder of the section. For reader's convenience and because in some cases not all standing assumptions are needed, we indicate in every result which standing assumptions are used.

The following definition of a *martingale problem* is taken from [120], where it is called *generalized martingale problem* due to the possibility of explosion. For simplicity we drop the term *generalized*.

**Definition 4.1.** *We say that a probability measure $P$ on $(\Omega, \mathcal{F})$ solves the* martingale problem *MP* $(\mathfrak{a}, \mathfrak{b}, x_0)$, *where $x_0 \in \mathbb{R}^d$, if $P(X_0 = x_0) = 1$ and for all $n \in \mathbb{N}$ and $f \in C^2(\mathbb{R}^d)$ the process*

$$f(X_{\cdot\wedge\theta_n}) - f(x_0) - \int_0^{\cdot\wedge\theta_n} \big(\langle \nabla f(X_s), \mathfrak{b}(X_s)\rangle + \tfrac{1}{2}\operatorname{tr}(\nabla^2 f(X_s)\mathfrak{a}(X_s))\big)ds$$

*is a $P$-martingale. A solution $P$ is called* conservative *(or* non-explosive*), if $P(\theta = \infty) = 1$, and* almost surely explosive*, if $P(\theta < \infty) = 1$.*

It is well-known that martingale problems have a one-to-one relation to weak solutions of stochastic differential equations, see, e.g. [47, Section 5.3] or [77, Section 5.5.B]. For reader's convenience, we recall this connection in Section 4.6.

The following theorem is the key observation in this article. It shows that perpetual integrals are distributed as the explosion time of a time-changed diffusion.

**Theorem 4.1.** *Assume* (S1), *let $\mathfrak{f} \colon \mathbb{R}^d \to (0,\infty)$ be Borel and locally bounded away from zero and infinity and let $x_0 \in \mathbb{R}^d$. There exists a measurable map $V \colon \Omega \to \Omega$ such that for every solution $P_{x_0}$ to the MP $(\mathfrak{a}, \mathfrak{b}, x_0)$ the following hold:*

(i) $P^\circ_{x_0} \triangleq P_{x_0} \circ V^{-1}$ *solves the MP* $(\mathfrak{f}^{-1}\mathfrak{a}, \mathfrak{f}^{-1}\mathfrak{b}, x_0)$.

(ii) *For all Borel sets $A \subseteq [0, \infty]$*

$$P_{x_0}\Big(\int_0^\theta \mathfrak{f}(X_s)ds \in A\Big) = P^\circ_{x_0}(\theta \in A). \tag{4.1}$$

*Moreover, if $P^\circ_{x_0}$ is the unique solution to the MP $(\mathfrak{f}^{-1}\mathfrak{a}, \mathfrak{f}^{-1}\mathfrak{b}, x_0)$, then $P_{x_0}$ is the unique solution to the MP $(\mathfrak{a}, \mathfrak{b}, x_0)$.*

**Remark 4.1.** *By symmetry, Theorem 4.1 yields that existence and uniqueness hold simultaneously for the MPs $(\mathfrak{a}, \mathfrak{b}, x_0)$ and $(\mathfrak{f}^{-1}\mathfrak{a}, \mathfrak{f}^{-1}\mathfrak{b}, x_0)$, i.e. one of these problems has a solution precisely in case the other has a solution and this solution is unique precisely if the other problem has a unique solution.*

The proof of Theorem 4.1 is given in Section 4.4 below. Let us shortly explain the main idea: We first define a right-continuous measurable process $Y$ via a *random time-change*, i.e. we set

$$Y_t \triangleq \begin{cases} X_{L_t}, & t < T_\theta, \\ \Delta, & t \geq T_\theta, \end{cases}$$

with

$$T_t \triangleq \int_0^{t\wedge\theta} \mathfrak{f}(X_s)ds, \quad L_t \triangleq \inf(s \in \mathbb{R}_+ \colon T_s > t), \quad t \in \mathbb{R}_+.$$

The technical core in the proof of Theorem 4.1 is to show that $Y$ has almost surely continuous paths. This observation allows us to define the map $V$ as a modification of $Y$. Noting that $\theta(Y) = T_\theta$ explains (ii). To understand (i), consider the simplified case where $\mathfrak{f}$ is uniformly bounded away from zero and $P_{x_0}$ is conservative. Then, $P_{x_0}$-a.s.

$$T_\theta = \int_0^\infty \mathfrak{f}(X_s)ds \geq \inf_{x\in\mathbb{R}^d} \mathfrak{f}(x) \int_0^\infty ds = \infty,$$

and $P_{x_0}$-a.s. $Y = X_L$ has $\mathbb{R}^d$-valued continuous paths. Due to a change of variable, for every $f \in C^2(\mathbb{R}^d)$ we obtain that $P_{x_0}$-a.s.

$$\int_0^\cdot \big(\langle \nabla f(Y_s), \mathfrak{f}^{-1}(Y_s)\mathfrak{b}(Y_s)\rangle + \tfrac{1}{2}\operatorname{tr}(\nabla^2 f(Y_s)\mathfrak{f}^{-1}(Y_s)\mathfrak{a}(Y_s))\big)ds$$
$$= \int_0^{L_\cdot} \big(\langle \nabla f(X_s), \mathfrak{b}(X_s)\rangle + \tfrac{1}{2}\operatorname{tr}(\nabla^2 f(X_s)\mathfrak{a}(X_s))\big)ds.$$

This observation allows us to deduce (i) from the optional stopping theorem. The assumption that $\mathfrak{f}$ is uniformly bounded away from zero simplifies the argument substantially. As we explain in Remark 4.2 below, a uniform boundedness assumption is typically too strong for our purpose.

While we are mainly interested in (4.1), the existence and uniqueness parts of Theorem 4.1 are also useful, because they lead to localizations of known existence and uniqueness theorems. For example, if (S1) holds it follows from Theorem 4.1 and an existence result from [126] that existence of a solution to MP $(\mathfrak{a}, \mathfrak{b}, x_0)$ is implied by the weak ellipticity condition $\det(\mathfrak{a})^{-1} \in L^1_{\text{loc}}(\mathbb{R}^d)$. To see this, note first that there exists a continuous function $\mathfrak{f} \colon \mathbb{R}^d \to (0, \infty)$ such that $\mathfrak{f}\mathfrak{b}$ and $\mathfrak{f}\mathfrak{a}$ are bounded. Then, [126, Theorem 2] yields that the MP $(\mathfrak{f}\mathfrak{a}, \mathfrak{f}\mathfrak{b}, x_0)$ has a (conservative) solution. Finally, Theorem 4.1 implies that also the MP $(\mathfrak{a}, \mathfrak{b}, x_0)$ has a (not necessarily conservative) solution. We give more details on this in Section 4.7.

Fix a third Borel function $\mathfrak{c} \colon \mathbb{R}^d \to \mathbb{R}^d$ and assume the following:

(S2) $\mathfrak{a}\mathfrak{c}$ is locally bounded.

Before we turn to our main application, we report a simple observation which we believe to be of interest: Absolute continuity and singularity are invariant under time-changes. We outline an application of this observation in Section 4.3.3.1 below.

**Corollary 4.1.** *Assume that* (S1) *and* (S2) *hold and let* $\mathfrak{f}\colon \mathbb{R}^d \to (0,\infty)$ *be a Borel function which is locally bounded away from zero and infinity and take* $x_0 \in \mathbb{R}^d$*. Further, let* $P_{x_0}$ *be the unique solution to the MP* $(\mathfrak{a}, \mathfrak{b}, x_0)$*, let* $Q_{x_0}$ *be the unique solution to the MP* $(\mathfrak{a}, \mathfrak{b} + \mathfrak{a}\mathfrak{c}, x_0)$*, let* $P^\circ_{x_0}$ *be the unique solution to the MP* $(\mathfrak{f}^{-1}\mathfrak{a}, \mathfrak{f}^{-1}\mathfrak{b}, x_0)$ *and let* $Q^\circ_{x_0}$ *be the unique solution to the MP* $(\mathfrak{f}^{-1}\mathfrak{a}, \mathfrak{f}^{-1}(\mathfrak{b} + \mathfrak{a}\mathfrak{c}), x_0)$*. The following hold:*

(i) $P_{x_0} \ll Q_{x_0}$ *if and only if* $P^\circ_{x_0} \ll Q^\circ_{x_0}$.

(ii) $P_{x_0} \perp Q_{x_0}$ *if and only if* $P^\circ_{x_0} \perp Q^\circ_{x_0}$.

*Proof.* Let $V$ be as in Theorem 4.1, let $A \in \mathcal{F}$ be such that $Q^\circ_{x_0}(A) = 0$ and set $B \triangleq \{V \in A\} \in \mathcal{F}$. Then, Theorem 4.1 yields that $Q_{x_0}(B) = Q^\circ_{x_0}(A) = 0$. Thus, $P_{x_0} \ll Q_{x_0}$ implies $P^\circ_{x_0}(A) = P_{x_0}(B) = 0$ and consequently, $P^\circ_{x_0} \ll Q^\circ_{x_0}$. The converse implication in (i) follows by symmetry. Part (ii) can be shown in the same manner. □

From now on, we also assume the following:

(S3) For every $x \in \mathbb{R}^d$ there exists a unique solution $P_x$ to the MP $(\mathfrak{a}, \mathfrak{b}, x)$, and for a fixed $x_0 \in \mathbb{R}^d$ there exists a solution $Q_{x_0}$ to the MP $(\mathfrak{a}, \mathfrak{b} + \mathfrak{a}\mathfrak{c}, x_0)$.

Analytic conditions for (S3) are given in Proposition 4.6 below.

Next, we introduce a non-negative local $Q_{x_0}$-martingale which relates $Q_{x_0}$ and $P_{x_0}$. For this, we assume the following:

(S4) $\langle \mathfrak{c}, \mathfrak{a}\mathfrak{c}\rangle$ is locally bounded.

If (S1) – (S4) hold, $Q_{x_0}$ from (S3) is unique by Proposition 4.7 below. We set

$$\overline{X}_{\cdot\wedge\theta_n} \triangleq X_{\cdot\wedge\theta_n} - X_0 - \int_0^{\cdot\wedge\theta_n} \big(\mathfrak{b}(X_s) + \mathfrak{a}(X_s)\mathfrak{c}(X_s)\big)ds.$$

By definition of the martingale problem, $\overline{X}_{\cdot\wedge\theta_n}$ is a continuous $Q_{x_0}$-martingale with quadratic variation process

$$[\overline{X}_{\cdot\wedge\theta_n}, \overline{X}_{\cdot\wedge\theta_n}] = \int_0^{\cdot\wedge\theta_n} \mathfrak{a}(X_s)ds.$$

By assumption (S4), the integral process $\overline{Y}_{\cdot\wedge\theta_n} \triangleq \int_0^{\cdot\wedge\theta_n}\langle \mathfrak{c}(X_s), d\overline{X}_s\rangle$ is well-defined as a continuous $Q_{x_0}$-martingale with quadratic variation process

$$[\overline{Y}_{\cdot\wedge\theta_n}, \overline{Y}_{\cdot\wedge\theta_n}] = \int_0^{\cdot\wedge\theta_n} \langle \mathfrak{c}(X_s), \mathfrak{a}(X_s)\mathfrak{c}(X_s)\rangle ds.$$

**Lemma 4.1.** *Assume that* (S1) – (S4) *hold. Then, the process*

$$Z_t \triangleq \begin{cases} \exp\big(-\int_0^t \langle \mathfrak{c}(X_s), d\overline{X}_s\rangle - \frac{1}{2}\int_0^t \langle \mathfrak{c}(X_s), \mathfrak{a}(X_s)\mathfrak{c}(X_s)\rangle ds\big), & t < \theta,\\ \liminf_{n\to\infty} Z_{\theta_n}, & t \geq \theta, \end{cases} \tag{4.2}$$

*is a non-negative local* $Q_{x_0}$*-martingale and* $Q_{x_0}$*-a.s. the terminal value* $Z_\infty \triangleq \lim_{t\to\infty} Z_t$ *exists and is finite.*

*Proof.* It follows similar to the proof of [65, Lemma 12.43] that $Z$ is a non-negative local $Q_{x_0}$-martingale. Thus, $Z$ is a non-negative $Q_{x_0}$-supermartingale by Fatou's lemma, and the existence of a finite terminal value follows from the supermartingale convergence theorem. □

As in the introduction, we set

$$A_\theta \triangleq \int_0^\theta \langle \mathfrak{c}(X_s), \mathfrak{a}(X_s)\mathfrak{c}(X_s)\rangle ds.$$

The next proposition is a version of [6, Theorem 1] for possibly non-conservative martingale problems. Closely related results are also given by Corollary 3.1, [32, Corollary 5.1] and [127, Theorem 3.3]. The setting in Chapter 3 and [127] is not completely identical to those in this chapter, because the path space in Chapter 3 and [127] allows also discontinuous explosion. This freedom is crucial for the extension arguments used there. In Section 4.5 we explain how part (i) of the next proposition can be deduced from Corollary 3.1. The proof of (ii) is identical to those of [6, Theorem 1] and given in Section 4.5.2.

**Proposition 4.1.** *Assume that* (S1) – (S4) *hold.*

(i) *The following are equivalent:*
  (a) $P_{x_0} \ll Q_{x_0}$ *with* $\frac{dP_{x_0}}{dQ_{x_0}} = Z_\infty$.
  (b) *The local* $Q_{x_0}$*-martingale* $Z$ *as defined in* (4.2) *is a uniformly integrable (UI)* $Q_{x_0}$*-martingale.*
  (c) $P_{x_0}(A_\theta < \infty) = 1$.

(ii) *The following are equivalent:*
  (a) $P_{x_0} \perp Q_{x_0}$.
  (b) $P_{x_0}(A_\theta = \infty) = 1$.

From now on we also assume the following:

(S5) $\langle \mathfrak{c}, \mathfrak{a}\mathfrak{c}\rangle$ is locally bounded away from zero.

**Remark 4.2.** *In case* $\langle \mathfrak{c}, \mathfrak{a}\mathfrak{c}\rangle$ *is* uniformly *bounded away from zero and* $P_{x_0}$ *is conservative, we have* $P_{x_0}(A_\theta = \infty) = 1$ *and Proposition 4.1 shows that* $P_{x_0} \perp Q_{x_0}$. *This observation explains that a uniform boundedness assumption is too strong for a characterization of absolute continuity.*

Due to Theorem 4.1, there exists a unique solution $P^\circ_{x_0}$ to the time-changed MP $(\langle \mathfrak{c}, \mathfrak{a}\mathfrak{c}\rangle^{-1}\mathfrak{a}, \langle \mathfrak{c}, \mathfrak{a}\mathfrak{c}\rangle^{-1}\mathfrak{b}, x_0)$ and $P_{x_0}(A_\theta < \infty) = P^\circ_{x_0}(\theta < \infty)$. In view of Proposition 4.1, this observation allows us to relate absolute continuity and singularity of $Q_{x_0}$ and $P_{x_0}$ to almost sure (non-)explosion properties of $P^\circ_{x_0}$:

**Corollary 4.2.** *Assume that* (S1) – (S5) *hold.*

(i) (i.a) – (i.c) *in Proposition 4.1 are equivalent to* $P^\circ_{x_0}(\theta < \infty) = 1$.

(ii) (ii.a) *and* (ii.b) *in Proposition 4.1 are equivalent to* $P^\circ_{x_0}(\theta = \infty) = 1$.

Applications of Corollary 4.2 are discussed in Section 4.3 below. Next, we comment on the difference between finite and infinite time horizons. As we explain in Section 4.5, the next proposition, which can be seen as a local version of Proposition 4.1 (i), follows from Corollary 3.1, see also [32, Corollary 5.1] and [127, Theorem 3.3] for closely related statements.

**Proposition 4.2.** *Suppose that* (S1) – (S4) *hold. The following are equivalent:*

(i) $P_{x_0} \ll_{\text{loc}} Q_{x_0}$*, i.e.* $P_{x_0} \ll Q_{x_0}$ *on* $\mathcal{F}_t$ *for all* $t \in \mathbb{R}_+$*.*

(ii) *The local* $Q_{x_0}$*-martingale* $Z$ *as defined in* (4.2) *is a* $Q_{x_0}$*-martingale.*

(iii) $P_{x_0}$*-a.s.* $A_\theta < \infty$ *on* $\{\theta < \infty\}$*.*

As the next corollary shows, in case $Q_{x_0}$ is conservative local absolute continuity of $P_{x_0}$ and $Q_{x_0}$ and the martingale property of $Z$ are equivalent to non-explosion of $P_{x_0}$. This observation is well-known, see, e.g. [137, Exercise 10.3.2] or [21, 107, 134].

**Corollary 4.3.** *Suppose that* (S1) – (S4) *hold and that* $Q_{x_0}$ *is conservative. The following are equivalent:*

(i) $P_{x_0} \ll_{\text{loc}} Q_{x_0}$*.*

(ii) *The local* $Q_{x_0}$*-martingale* $Z$ *is a* $Q_{x_0}$*-martingale.*

(iii) $P_{x_0}$ *is conservative.*

*Proof.* The implications (iii) $\Rightarrow$ (i) $\Leftrightarrow$ (ii) follow directly from Proposition 4.2. If (i) holds, then $P_{x_0}(\theta \le t) = 0$ for all $t \in \mathbb{R}_+$, because $Q_{x_0}$ is conservative. This shows that (i) implies (iii). □

In view of Proposition 4.2 and Corollary 4.3, we observe an interesting difference between the characterizations of *local* absolute continuous and *global* absolute continuity, and between the *martingale property* and the *UI martingale property* of the local martingale $Z$: The local absolute continuity $P_{x_0} \ll_{\text{loc}} Q_{x_0}$ and the $Q_{x_0}$-martingale property of $Z$ are related to *non-explosion* of $P_{x_0}$, while the absolute continuity $P_{x_0} \ll Q_{x_0}$ and the UI $Q_{x_0}$-martingale property of $Z$ are related to almost sure *explosion* of $P^{\circ}_{x_0}$. Moreover, Corollary 4.2 shows that absolute continuity and singularity can be related to the explosion of *one* auxiliary diffusion. This is not necessarily true for local absolute continuity and the martingale property, see Proposition 4.2 and [107, Remark (ii), p. 10].

Let us also comment on the role played by the initial value.

**Lemma 4.2.** *Assume that* $\mathfrak{b}$ *is locally bounded, that* $\mathfrak{a}$ *is locally Hölder continuous and that* $\langle \xi, \mathfrak{a}(x)\xi \rangle > 0$ *for all* $x \in \mathbb{R}^d$ *and* $\xi \in \mathbb{R}^d \backslash \{0\}$*. Then, for every* $x_0 \in \mathbb{R}^d$ *there exists a unique solution* $P_{x_0}$ *to the MP* $(\mathfrak{a}, \mathfrak{b}, x_0)$ *and the following hold:*

(i) $P_{x_0}(\theta = \infty) = 1$ *holds for all* $x_0 \in \mathbb{R}^d$ *if it holds for some* $x_0 \in \mathbb{R}^d$*.*

(ii) $P_{x_0}(\theta < \infty) = 1$ *holds for all* $x_0 \in \mathbb{R}^d$ *if it holds for some* $x_0 \in \mathbb{R}^d$*.*

*Proof.* The existence and uniqueness of $P_x$ follows from Proposition 4.6 below. The maximum principle (see [13, Lemma 1.4]) implies that non-negative harmonic functions vanish at all points whenever they vanish at one point. Since $x \mapsto P_x(\theta < \infty)$ and $x \mapsto 1 - P_x(\theta < \infty)$ are harmonic (see [13, Lemma 1.2]), the claim follows. □

For the conservative case the following observation is implied by [6, Corollary 1].

**Corollary 4.4.** *Suppose that* $\mathfrak{b}$ *and* $\mathfrak{a}\mathfrak{c}$ *are locally bounded, that* $\mathfrak{a}$ *satisfies the assumptions from Lemma 4.2 and that* $\langle \mathfrak{c}, \mathfrak{a}\mathfrak{c}\rangle$ *is strictly positive and locally Hölder continuous. Then, for every* $x_0 \in \mathbb{R}^d$ *there exist unique solutions* $P_{x_0}$ *and* $Q_{x_0}$ *to the MPs* $(\mathfrak{a}, \mathfrak{b}, x_0)$ *and* $(\mathfrak{a}, \mathfrak{b} + \mathfrak{a}\mathfrak{c}, x_0)$*, respectively. Moreover, the following hold:*

(i) $P_{x_0} \ll Q_{x_0}$ *holds for all* $x_0 \in \mathbb{R}^d$ *if it holds for some* $x_0 \in \mathbb{R}^d$*.*

(ii) $P_{x_0} \perp Q_{x_0}$ *holds for all* $x_0 \in \mathbb{R}^d$ *if it holds for some* $x_0 \in \mathbb{R}^d$*.*

In the following section we present three applications of our results. First, we derive a Khasminskii-type integral test for absolute continuity and singularity, second, we derive a Feller-type integral test for explosion of a multidimensional time-changed Brownian motion, and, third, we outline applications to mathematical finance.

## 4.3 Three Applications

### 4.3.1 A Khasminskii-Test for Absolute Continuity/Singularity

For $d = 1$ almost sure explosion and non-explosion can be characterized via analytic integral tests, see, e.g. [77, Theorem 5.5.29, Proposition 5.5.32]. In combination with Corollary 4.2 these characterizations lead to [23, Corollaries 5.1, 5.3] and [107, Theorem 2.3] under additional assumptions on the coefficients. Most importantly, the time-change argument requires $\mathfrak{c} \neq 0$, which is not needed in [23, 107]. In return, Corollary 4.2 can be applied independent of the dimension. Moreover, the characterization of absolute continuity and singularity via almost sure explosion and non-explosion is useful, because even for multidimensional diffusions many analytic conditions for almost sure (non-)explosion are known, see, e.g. [104, 120, 137]. In the following we use some of these conditions to formulate a Khasminskii-type integral test for $P_{x_0} \ll Q_{x_0}$ and $P_{x_0} \perp Q_{x_0}$.

**Condition 4.1.** *There exist continuous functions* $B\colon [\tfrac{1}{2}, \infty) \to \mathbb{R}$ *and* $A\colon [\tfrac{1}{2}, \infty) \to (0, \infty)$ *such that for all* $x \in \mathbb{R}^d\colon \|x\| \geq 1$

$$A\Big(\frac{\|x\|^2}{2}\Big) \leq \frac{\langle x, \mathfrak{a}(x)x\rangle}{\langle \mathfrak{c}(x), \mathfrak{a}(x)\mathfrak{c}(x)\rangle},$$
$$\langle x, \mathfrak{a}(x)x\rangle B\Big(\frac{\|x\|^2}{2}\Big) \leq \operatorname{tr}(\mathfrak{a}(x)) + 2\langle x, \mathfrak{b}(x)\rangle,$$

*and*

$$\int_{\frac{1}{2}}^{\infty} \frac{1}{C(z)} \int_{\frac{1}{2}}^{z} \frac{C(u)du}{A(u)} dz < \infty,$$

*where*

$$C(z) \triangleq \exp\Big(\int_{\frac{1}{2}}^{z} B(u)du\Big).$$

*Moreover,*

$$\inf_{\|\theta\|=1} \inf_{\|x\|\leq R} \langle \theta, \mathfrak{a}(x)\theta\rangle > 0$$

*for all* $R > 0$*.*

**Condition 4.2.** *There exists an* $R > 0$ *and continuous functions* $B\colon [R, \infty) \to \mathbb{R}$ *and* $A\colon [R, \infty) \to (0, \infty)$ *such that for all* $x \in \mathbb{R}^d\colon \|x\| \geq \sqrt{2R}$

$$A\Big(\frac{\|x\|^2}{2}\Big) \geq \frac{\langle x, \mathfrak{a}(x)x\rangle}{\langle \mathfrak{c}(x), \mathfrak{a}(x)\mathfrak{c}(x)\rangle},$$

$$\langle x, \mathfrak{a}(x)x\rangle B\Big(\frac{\|x\|^2}{2}\Big) \geq \operatorname{tr}(\mathfrak{a}(x)) + 2\langle x, \mathfrak{b}(x)\rangle,$$

*and*

$$\int_R^\infty \frac{1}{C(z)} \int_R^z \frac{C(u)du}{A(u)} dz = \infty,$$

*where*

$$C(z) \triangleq \exp\Big(\int_R^z B(u)du\Big).$$

**Corollary 4.5.** *Assume that* (S1) – (S5) *from Section 4.2 hold.*

(i) *Suppose that Condition 4.1 holds. Then,* $P_{x_0} \ll Q_{x_0}$ *with* $\frac{dP_{x_0}}{dQ_{x_0}} = Z_\infty$*. In particular,* $Z$ *as defined in* (4.2) *is a uniformly integrable* $Q_{x_0}$*-martingale.*

(ii) *Suppose that Condition 4.2 holds. Then,* $P_{x_0} \perp Q_{x_0}$ *and* $Z$ *as defined in* (4.2) *is no uniformly integrable* $Q_{x_0}$*-martingale.*

*Proof.* Due to [137, Theorem 10.2.4], Condition 4.1 implies that $P^\circ_{x_0}(\theta < \infty) = 1$. In case Condition 4.2 holds, [137, Theorem 10.2.3] yields that $P^\circ_{x_0}(\theta = \infty) = 1$. Now, all claims follow from Corollary 4.2. □

### 4.3.2 An Explosion-Test for Time-Changed Brownian Motion

Let $\mathfrak{g}\colon \mathbb{R}^d \to (0, \infty)$ be a Borel function which is locally bounded away from zero and infinity. Due to Theorem 4.1, for every $x_0 \in \mathbb{R}^d$ there exists a unique solution $P_{x_0}$ to the MP $(\mathfrak{g}\,\mathrm{Id}, 0, x_0)$. Providing an intuition, the probability measure $P_{x_0}$ is the (unique) law of the $\mathbb{R}^d_\Delta$-valued time-changed Brownian motion

$$Y_t \triangleq \begin{cases} x_0 + W_{L_t}, & t < T_\infty, \\ \Delta, & t \geq T_\infty, \end{cases}$$

where $W$ is a $d$-dimensional standard Brownian motion and

$$T_t \triangleq \int_0^t \frac{ds}{\mathfrak{g}(x_0 + W_s)}, \quad L_t \triangleq \inf(s \in \mathbb{R}_+\colon T_s > t), \quad t \in \mathbb{R}_+.$$

Since Brownian motion is recurrent for $d = 1, 2$, Theorem 4.1 directly implies that $P_{x_0}$ is non-explosive, see [108, Theorem 3.27] or [123, Proposition X.3.11]. We are interested in explosion properties of $P_{x_0}$ for the transient regime of Brownian motion.

For the remainder of this section let $d \geq 3$ and denote by $\mathscr{W}_{x_0}$ the $d$-dimensional Wiener measure with initial value $x_0$.

By the standard linear growth condition for non-explosion, we have $P_{x_0}(\theta = \infty) = 1$ in case

$$\mathfrak{g}(x) \leq \mathsf{C}(1 + \|x\|)^2, \quad x \in \mathbb{R}^d, \mathsf{C} > 0.$$

Using the Green kernel of Brownian motion, we also obtain a condition for almost sure explosion. More precisely, [108, Theorems 3.32, 3.33] yield that

$$E^{\mathscr{W}_{x_0}}\Big[\int_0^\infty \frac{ds}{\mathfrak{g}(X_s)}\Big] = \mathsf{C}_d \int_{\mathbb{R}^d} \frac{\|x-x_0\|^{2-d}dx}{\mathfrak{g}(x)},$$

for a dimension-dependent constant $\mathsf{C}_d > 0$. Together with Theorem 4.1 this observation implies the following:

**Corollary 4.6.** *If $\int_{\mathbb{R}^d} \mathfrak{g}^{-1}(x)\|x-x_0\|^{2-d}dx < \infty$, then $E_{x_0}[\theta] < \infty$.*

We now ask whether for certain choices of $\mathfrak{g}$ the convergence criterion in Corollary 4.6 is necessary for almost sure explosion. In other words, we ask whether in some cases $E_{x_0}[\theta] = \infty$ implies $P_{x_0}(\theta = \infty) > 0$, which is of course in general not true. The following corollary shows that in case $\mathfrak{g}$ is locally Hölder continuous and at least of quadratic growth, $E_{x_0}[\theta] = \infty$ even implies $P_{x_0}(\theta = \infty) = 1$.

**Corollary 4.7.** *Suppose that $\mathfrak{g}$ is locally Hölder continuous and*

$$\mathfrak{g}(x) \geq \mathsf{C}(1+\|x\|)^2, \qquad x \in \mathbb{R}^d, \mathsf{C} > 0. \tag{4.3}$$

*If $\int_{\mathbb{R}^d} \mathfrak{g}^{-1}(x)\|x\|^{2-d}dx = \infty$, then $P_{x_0}(\theta = \infty) = 1$.*

*Proof.* We define $\mathfrak{a} \triangleq \mathrm{Id}$ and $\mathfrak{c} \triangleq \mathfrak{g}^{-\frac{1}{2}}e_1$, where $e_1$ is the first unit vector. Let $Q_{x_0}$ be the unique solution to the MP $(\mathfrak{a}, \mathfrak{c}, x_0)$, see Proposition 4.6 below. As $\mathfrak{a}$ and $\mathfrak{c}$ are bounded, the solution $Q_{x_0}$ is conservative. Note that $\langle \mathfrak{c}, \mathfrak{a}\mathfrak{c}\rangle = \mathfrak{g}^{-1}$ is a strictly positive continuous function. Corollary 4.2 yields that $P_{x_0}(\theta = \infty) = 1$ if and only if $\mathscr{W}_{x_0} \perp Q_{x_0}$. It follows from [6, Corollary 4][1] that

$$\mathscr{W}_{x_0} \perp Q_{x_0} \Leftrightarrow \int_{\mathbb{R}^d} \frac{\|x\|^{2-d}dx}{\mathfrak{g}(x)} = \infty.$$

This completes the proof. □

The growth condition (4.3) and $\int_{\mathbb{R}^d} \mathfrak{g}^{-1}(x)\|x\|^{2-d}dx = \infty$ do not exclude each other: In case (4.3) holds, we have

$$\int_{\mathbb{R}^d} \frac{\|x\|^{2-d}dx}{\mathfrak{g}(x)} \leq \mathsf{C}\int_{\mathbb{R}^d} \frac{\|x\|^{2-d}dx}{1+\|x\|^2} = \mathsf{C}_d \int_0^\infty \frac{rdr}{1+r^2} = \infty.$$

The following proposition explains that in general the growth condition (4.3) is sharp.

**Proposition 4.3.** *Let $\rho\colon \mathbb{R}_+ \to [1,\infty)$ be an increasing function with $\rho(0) = 1$ and $\rho(x) \to \infty$ as $x \to \infty$. There exists a function $\mathfrak{g}$ such that the following hold:*

(i) $\mathfrak{g}(x) \geq \frac{1+\|x\|^2}{\rho(\|x\|)}$ *for all $x \in \mathbb{R}^d$.*

(ii) $P_0(\theta < \infty) = 1$.

(iii) $\int_{\mathbb{R}^d} \mathfrak{g}^{-1}(x)\|x\|^{2-d}dx = \infty$.

[1] The statement of [6, Corollary 4] contains a small typo: $|b(x)|$ has to be replaced by $|b(x)|^2$, see [6, Eq. 1.2].

*Proof.* We adapt the proof of [6, Theorem 3]. Let $e_1$ be the first unit vector, set $x_1 \triangleq e_1$ and define inductively

$$R_n \triangleq 3^{-n}\|x_n\|, \qquad x_{n+1} \in \big\{te_1 \colon t > 4\|x_n\|, \rho(\tfrac{t}{2}) > 4^{d(n+1)}\big\}. \tag{4.4}$$

Set $B_R(x) \triangleq \{y \in \mathbb{R}^d \colon \|x - y\| < R\}$ and note that the balls $(B_{R_n}(x_n))_{n\in\mathbb{N}}$ are disjoint, because

$$\|x_{n+1}\| - \|x_n\| > \frac{3\|x_{n+1}\|}{4} = \frac{3^{n+2}R_{n+1}}{4} > \frac{9}{8}\big(R_{n+1} + R_n\big),$$

where we use (4.4) and in particular that $3R_{n+1} > 4R_n$. Define

$$\mathfrak{g}(x) \triangleq \begin{cases} \frac{1+\|x\|^2}{\rho(\|x_n\|-R_n)}, & x \in B_{R_n}(x_n) \text{ for some } n \in \mathbb{N}, \\ 2 + \|x\|^4, & x \notin \bigcup_{n\in\mathbb{N}} B_{R_n}(x_n) \triangleq G. \end{cases}$$

It is clear that $\mathfrak{g}$ is Borel and locally bounded away from zero and infinity. If $x \in B_{R_n}(x_n)$ we have $\|x_n\| - R_n \le \|x\|$ and

$$\frac{\mathfrak{g}(x)\rho(\|x\|)}{1 + \|x\|^2} = \frac{\rho(\|x\|)}{\rho(\|x_n\| - R_n)} \ge 1,$$

because $\rho$ is increasing. If $x \notin G = \bigcup_{n\in\mathbb{N}} B_{R_n}(x_n)$ we have

$$\frac{\mathfrak{g}(x)\rho(\|x\|)}{1 + \|x\|^2} \ge \rho(\|x\|) \ge 1.$$

In other words, (i) holds.

Next, we show (ii). Due to [108, Corollary 3.19] we have

$$\sum_{n=1}^{\infty} \mathscr{W}_0(X \text{ hits } B_{R_n}(x_n)) = \sum_{n=1}^{\infty} \Big(\frac{R_n}{\|x_n\|}\Big)^{d-2} = \sum_{n=1}^{\infty} 3^{-n(d-2)} < \infty. \tag{4.5}$$

Thus, the Borel–Cantelli lemma yields that $\mathscr{W}_0$-a.a. paths of $X$ hit only finitely many elements of $(B_{R_n}(x_n))_{n\in\mathbb{N}}$. Recalling that Brownian motion is transient for $d \ge 3$, i.e. that $\mathscr{W}_0$-a.a. paths of $X$ leave bounded domains forever in finite time, we conclude that $\mathscr{W}_0$-a.s.

$$\int_0^\infty \frac{\mathbb{1}_G(X_s)ds}{\mathfrak{g}(X_s)} < \infty.$$

Note that

$$E^{\mathscr{W}_0}\Big[\int_0^\infty \frac{ds}{2 + \|X_s\|^4}\Big] = \int_{\mathbb{R}^d} \frac{\|x\|^{2-d}dx}{2 + \|x\|^4} = d\omega_d \int_0^\infty \frac{rdr}{2 + r^4} < \infty,$$

where $\omega_d$ is the volume of the unit ball in $\mathbb{R}^d$. We conclude that $\mathscr{W}_0$-a.s.

$$\int_0^\infty \frac{ds}{\mathfrak{g}(X_s)} < \infty.$$

Thus, Theorem 4.1 yields that $P_0(\theta < \infty) = 1$, i.e. (ii) holds.

It is left to verify (iii). Using (4.4), the fact that $f(x) = \|x\|^{2-d}$ is harmonic on $\mathbb{R}^d\backslash\{0\}$

and the mean-value property of harmonic functions, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^d} \frac{\|x\|^{2-d}dx}{\mathfrak{g}(x)} &\geq \sum_{n=1}^{\infty} \int_{B_{R_n}(x_n)} \frac{\|x\|^{2-d}dx}{\mathfrak{g}(x)} \\
&= \sum_{n=1}^{\infty} \rho(\|x_n\| - R_n) \int_{B_{R_n}(x_n)} \frac{\|x\|^{2-d}dx}{1+\|x\|^2} \\
&\geq \omega_d \sum_{n=1}^{\infty} \rho(\|x_n\| - R_n) \frac{\|x_n\|^{2-d}R_n^d}{1+(\|x_n\|+R_n)^2} \\
&\geq \omega_d \sum_{n=1}^{\infty} \frac{\rho((1-3^{-n})\|x_n\|)}{1+(1+3^{-n})^2} \Big( \frac{R_n}{\|x_n\|}\Big)^d \\
&\geq \frac{\omega_d}{5} \sum_{n=1}^{\infty} \rho\Big(\frac{\|x_n\|}{2}\Big) 3^{-dn} \\
&\geq \frac{\omega_d}{5} \sum_{n=1}^{\infty} \Big(\frac{4}{3}\Big)^{dn} = \infty.
\end{aligned}
$$

This implies (iii) and the proof is complete. □

In case $\mathfrak{g}$ is a radial function, the growth condition on $\mathfrak{g}$ is not needed:

**Corollary 4.8.** *Suppose that $\mathfrak{g}(x) = \mathfrak{s}(\|x\|)$ for a Borel function $\mathfrak{s}\colon \mathbb{R}_+ \to (0,\infty)$ which is locally bounded away from zero and infinity. The following hold:*

(i) *If $\int_{\|x_0\|}^{\infty} r\mathfrak{s}^{-1}(r)dr < \infty$, then $P_{x_0}(\theta < \infty) = 1$.*

(ii) *If $\int_{\|x_0\|}^{\infty} r\mathfrak{s}^{-1}(r)dr = \infty$, then $P_{x_0}(\theta = \infty) = 1$.*

*Proof.* Due to [45, Theorem 2] and [140, Corollary 3], for every locally bounded Borel function $\mathfrak{z}\colon \mathbb{R}_+ \to \mathbb{R}_+$ the following are equivalent:

(a) $\mathscr{W}_{x_0}(\int_0^{\infty} \mathfrak{z}(\|X_s\|)ds < \infty) > 0$.

(b) $\mathscr{W}_{x_0}(\int_0^{\infty} \mathfrak{z}(\|X_s\|)ds < \infty) = 1$.

(c) $\int_{\|x_0\|}^{\infty} z\mathfrak{z}(z)dz < \infty$.

The claims now follow directly from Theorem 4.1. □

Finally, we give a precise integrability condition, which is, in contrast to the conditions above, not completely analytic, because it involves probability via transient sets of Brownian motion. Let $\mathcal{T}$ be the collection of all sets $G \in \mathcal{B}(\mathbb{R}^d)$ such that $G^c$ is transient, i.e.

$$\mathscr{W}_x(X_s \in G \text{ for all } s \in \mathbb{R}_+) > 0 \text{ for some } x \in \mathbb{R}^d.$$

We stress that any Borel subset of $\mathbb{R}^d$ whose complement is bounded belongs to $\mathcal{T}$. However, there are examples of sets in $\mathcal{T}$ with unbounded complement, see the comment below the following corollary. For more comments on $\mathcal{T}$ we refer to [5, pp. 470 – 471].

**Corollary 4.9.** *Suppose that $\mathfrak{g}$ is locally Hölder continuous. The following are equivalent:*

(i) *$P_{x_0}(\theta = \infty) = 1$.*

(ii) $\int_G \mathfrak{g}^{-1}(x)\|x\|^{2-d}dx = \infty$ *for all* $G \in \mathcal{T}$.

*Proof.* According to [5, Proposition 3.1, Theorem 3.5], part (ii) is equivalent to

(iii) $\mathscr{W}_x(\int_0^\infty \mathfrak{g}^{-1}(X_s)ds = \infty) = 1$ for Lebesgue a.a. $x \in \mathbb{R}^d$.

The claim now follows from Theorem 4.1 and Lemma 4.2. □

Note that Proposition 4.3 is in line with Corollary 4.9: Let $G$ be as in the proof of Proposition 4.3. Then, (4.5) yields that

$$\mathscr{W}_0(X_s \in G \text{ for some } s \in \mathbb{R}_+) \leq \sum_{n=1}^\infty 3^{-n} = \frac{1}{2},$$

which implies $G^c \in \mathcal{T}$. Thus, (ii) in Corollary 4.9 is violated in case of Proposition 4.3, because $\mathfrak{g}(x) = 2 + \|x\|^4$ on $G^c$ and therefore

$$\int_{G^c} \frac{\|x\|^{2-d}dx}{\mathfrak{g}(x)} \leq \mathsf{C}_d \int_0^\infty \frac{rdr}{2+r^4} < \infty.$$

### 4.3.3 On the Absence of Arbitrage in Diffusion Markets

In this section we outline applications to mathematical finance. We start by introducing a stochastic model for a financial market. Let $\mathfrak{a}\colon \mathbb{R}^d \to \mathbb{S}^d$ and $\mathfrak{b}\colon \mathbb{R}^d \to \mathbb{R}^d$ be locally bounded Borel functions and fix an initial value $x_0 \in \mathbb{R}^d$. We assume that the MP $(\mathfrak{a}, \mathfrak{a}\mathfrak{b}, x_0)$ has a unique conservative solution $P_{x_0}$. In the following we consider $(\Omega, \mathcal{F}, \mathbf{F}, P_{x_0})$ as underlying filtered probability space. Providing an intuition, possibly on an extension of $(\Omega, \mathcal{F}, \mathbf{F}, P_{x_0})$, there exists a $d$-dimensional standard Brownian motion $B = (B^1, \dots, B^d)$ such that

$$dX_t = \mathfrak{a}^{\frac{1}{2}}(X_t)(dB_t + \mathfrak{a}^{\frac{1}{2}}(X_t)\mathfrak{b}(X_t)dt), \quad X_0 = x_0, \tag{4.6}$$

see Proposition 4.5 and Remark 4.5 below. For each $i = 1, \dots, d$ we define $S^i$ to be the stochastic exponential of $X^i$, i.e. the unique solution to the stochastic differential equation $dS^i_t = S^i_t dX^i_t$ with initial value $S_0 = 1$. We think of $S = (S^1, \dots, S^d)$ as discounted price process in a financial market with $d$ risky assets.

It is an important question in mathematical finance whether there are certain arbitrage opportunities in the market. A probabilistic characterization of the existence is given by so-called *fundamental theorems of asset pricing*, which state that the absence of certain arbitrage opportunities is equivalent to the existence of a so-called *equivalent (local) martingale measure (E(L)MM)*, i.e. a probability measure which is on one hand equivalent (i.e. mutually absolutely continuous) to the real-world measure $P_{x_0}$ and on the other hand turns the discounted asset price process $S$ into a (local) UI martingale, see, e.g. [22, Corollary 5.2] and [38, Corollary 9.1.2]. Let us assume that the MP $(\mathfrak{a}, 0, x_0)$ has a unique solution $Q_{x_0}$. Since any equivalent change of measure preserves the quadratic variation, and continuous local martingales have necessarily zero drift, $Q_{x_0}$ is the only candidate for an ELMM. Clearly, because $P_{x_0}$ is assumed to be conservative, $Q_{x_0}$ is conservative whenever $P_{x_0}$ and $Q_{x_0}$ are equivalent. Consequently, we have the following:

**Proposition 4.4.** *The following are equivalent:*

(i) *There exists an ELMM.*

(ii) $Q_{x_0}$ *is an ELMM.*

(iii) $Q_{x_0}$ *is equivalent to* $P_{x_0}$.

The plan for the next two subsections is the following: In Section 4.3.3.1 we consider diagonal diffusion coefficients and deduce analytic conditions for the existence of an ELMM from Corollary 4.1, Proposition 4.4 and results from [6]. In Section 4.3.3.2 we use Corollary 4.2 to formulate explosion criteria for an ELMM to be an EMM.

#### 4.3.3.1 On the Existence of Equivalent Local Martingale Measures

We assume that $\mathfrak{a} = \mathfrak{g}\,\mathrm{Id}$, where $\mathfrak{g}\colon \mathbb{R}^d \to (0,\infty)$ is locally bounded away from zero and infinity. In this case, (4.6) means that

$$dX_t^i = \sqrt{\mathfrak{g}}(X_t)(dB_t^i + \sqrt{\mathfrak{g}}(X_t)\mathfrak{b}^i(X_t)dt), \quad X_0^i = x_0^i, \quad i = 1,\dots,d.$$

Due to Proposition 4.6 below, there exists a unique solution $P^\circ_{x_0}$ to the MP $(\mathrm{Id}, \mathfrak{b}, x_0)$. Corollary 4.1 and Proposition 4.4 yield the following:

**Corollary 4.10.** (i) – (iii) *from Proposition 4.4 hold if and only if* $P^\circ_{x_0}$ *is equivalent to the* $d$*-dimensional Wiener measure* $\mathscr{W}_{x_0}$ *with initial value* $x_0$.

For a more general continuous setting with finite time horizon, a comparable characterization of the absence of arbitrage is given in [103]. More precisely, it is shown in [103, Proposition 2.3] that the existence of an ELMM is determined by the equivalence of a probability measure to the Wiener measure. Next, we use Corollary 4.10 to obtain analytic criteria for the existence of an ELMM. We start with a comment on the recurrent regime of Brownian motion. The Lebesgue measure is denoted by $\ell$.

**Corollary 4.11.** *If* $d = 1, 2$*, then an ELMM exists if and only if* $\ell$*-a.e.* $\mathfrak{b} = 0$.

*Proof.* In case $\ell(\mathfrak{b} \neq 0) = 0$ we have $P^\circ_{x_0} = \mathscr{W}_{x_0}$ and Theorem 4.1 yields that the real-world measure $P_{x_0}$ is an ELMM. To see that $P^\circ_{x_0} = \mathscr{W}_{x_0}$, recall the uniqueness of $P^\circ_{x_0}$ and note that for any Borel function $f\colon \mathbb{R}^d \to \mathbb{R}_+$ with $\ell(f \neq 0) = 0$

$$E^{\mathscr{W}_{x_0}}\Big[\int_0^\infty f(X_s)ds\Big] = \int_0^\infty E^{\mathscr{W}_{x_0}}\big[f(X_s)\big]ds = 0,$$

because $X_s$ is normally distributed under $\mathscr{W}_{x_0}$.

If $\ell(\mathfrak{b} \neq 0) > 0$, then [123, Proposition X.3.11] yields $\mathscr{W}_{x_0}$-a.s. $\int_0^\infty \|\mathfrak{b}(X_s)\|^2 ds = \infty$, and Proposition 4.1 and Corollary 4.10 show that no ELMM exists. □

This observation is different for our exponential model and the (one-dimensional) diffusion model studied in [106] for which an ELMM might exist for non-trivial cases, see [106, Theorem 3.5].

We now consider the transient regime of Brownian motion. Corollary 4.10 and [6, Corollary 4] imply the following:

**Corollary 4.12.** *Suppose that* $d \geq 3$*, that* $\mathfrak{b}$ *is locally Hölder continuous and that* $\sup_{x\in\mathbb{R}^d} \|\mathfrak{b}(x)\|(1 + \|x\|) < \infty$. *Then,* (i) – (iii) *from Proposition 4.4 hold if and only if* $\int_{\mathbb{R}^d} \|\mathfrak{b}(x)\|^2 \|x\|^{2-d} dx < \infty$.

Of course, the assumption $\sup_{x\in\mathbb{R}^d} \|\mathfrak{b}(x)\|(1 + \|x\|) < \infty$ implies that $\mathfrak{b}$ is bounded. While $P^\circ_{x_0}$ and $\mathcal{W}_{x_0}$ are locally equivalent whenever $\mathfrak{b}$ is bounded, this is not necessarily true for global equivalence. Indeed, the strong law of large numbers shows that the laws of Brownian motion with and without non-trivial linear drift are singular. This easy example also hints why we require $\|\mathfrak{b}(x)\| \to 0$ as $\|x\| \to \infty$.

#### 4.3.3.2 On the Existence of Equivalent Martingale Measures

Assume that $\mathfrak{b} = 0$ and that $\mathfrak{a}$ is continuous and maps into the set of strictly positive definite $d \times d$ matrices. In this case, the real-world measure $P_{x_0}$ is already an ELMM and we ask for a condition when $P_{x_0}$ is even an EMM. Note that $S^i$ is related to $Z$ as given in (4.2) when $-\mathfrak{c}$ is defined to be the $i$-th unit vector $e_i$ and consequently, that we are in the setting of Section 4.2. In particular, (S1) – (S5) from Section 4.2 hold by Proposition 4.6 below and the assumptions on $\mathfrak{a}$. Consequently, Corollary 4.2 implies the following:

**Corollary 4.13.** *$S^i$ is a UI $Q_{x_0}$-martingale if and only if $Q^i_{x_0}(\theta < \infty) = 1$, where $Q^i_{x_0}$ is the unique solution to the MP $(\mathfrak{a}^{-1}_{ii}\mathfrak{a}, \mathfrak{a}^{-1}_{ii}\mathfrak{a}e_i, x_0)$.*

Applying this corollary for all $i = 1, \dots, d$, we obtain an equivalent explosion condition for $Q_{x_0}$ to be an EMM. Based on results from [104, 120, 137] one can also formulate analytic conditions.

We stress that the results for finite and infinite time horizons are quite different. For example, in case $d = 1$ the probability measure $Q^1_{x_0}$ solves the MP $(1, 1, x_0)$, which corresponds to Brownian motion with linear drift. Thus, $Q^1_{x_0}$ is conservative and consequently, $S = S^1$ is no UI $Q_{x_0}$-martingale, while it is a $Q_{x_0}$-martingale if and only if $\int_0^\infty \frac{dx}{\mathfrak{a}(x)} = \infty$, see [27, Proposition 5.2].

## 4.4 Proof of Theorem 4.1

In this section we prove Theorem 4.1, i.e. we prove the following:

**Theorem.** *Assume* (S1)*, let $\mathfrak{f}\colon \mathbb{R}^d \to (0, \infty)$ be Borel and locally bounded away from zero and infinity and let $x_0 \in \mathbb{R}^d$. There exists a measurable map $V\colon \Omega \to \Omega$ such that for every solution $P_{x_0}$ to the MP $(\mathfrak{a}, \mathfrak{b}, x_0)$ the following hold:*

(i) *$P^\circ_{x_0} \triangleq P_{x_0} \circ V^{-1}$ solves the MP $(\mathfrak{f}^{-1}\mathfrak{a}, \mathfrak{f}^{-1}\mathfrak{b}, x_0)$.*

(ii) *For all Borel sets $A \subseteq [0, \infty]$*

$$P_{x_0}\Big(\int_0^\theta \mathfrak{f}(X_s)ds \in A\Big) = P^\circ_{x_0}(\theta \in A).$$

*Moreover, if $P^\circ_{x_0}$ is the unique solution to the MP $(\mathfrak{f}^{-1}\mathfrak{a}, \mathfrak{f}^{-1}\mathfrak{b}, x_0)$, then $P_{x_0}$ is the unique solution to the MP $(\mathfrak{a}, \mathfrak{b}, x_0)$.*

Let $x_0 \in \mathbb{R}^d$ and let $P_{x_0}$ be a solution to the MP $(\mathfrak{a}, \mathfrak{b}, x_0)$. To simplify the notation, we denote $P \equiv P_{x_0}$. We first define a right-continuous measurable process $Y$ via a random time-change. For $t \in \mathbb{R}_+$ we set

$$T_t \triangleq \int_0^{t\wedge\theta} \mathfrak{f}(X_s)ds, \qquad L_t \triangleq \inf(s \in \mathbb{R}_+ \colon T_s > t).$$

The functions $T, L\colon \mathbb{R}_+ \to [0,\infty]$ are increasing. As $\mathfrak{f}$ is locally bounded, we have $T_{\theta_n \wedge n} < \infty$ for all $n \in \mathbb{N}$. Using this and the strict positivity of $\mathfrak{f}$, we see that $T$ is finite, absolutely continuous and strictly increasing on $[0,\theta)$. Moreover, because $\lim_{t \nearrow \theta} T_t = T_\theta$ by the monotone convergence theorem, $T$ is everywhere continuous. We also note that $L$ is finite, continuous and strictly increasing on $[0, T_\theta)$, everywhere right-continuous, and that $L_{T_s} = s$ for $s < \theta$ and $T_{L_t} = t$ for $t < T_\theta$, see [123, pp. 7 – 9]. In particular, we have

$$\lim_{t \nearrow T_\theta} L_t = \lim_{t \nearrow \theta} L_{T_t} = \theta.$$

For $t \in \mathbb{R}_+$ we define

$$Y_t \triangleq \begin{cases} X_{L_t}, & t < T_\theta, \\ \Delta, & t \geq T_\theta. \end{cases}$$

It is easy to see that $Y$ is an $\mathbb{R}^d_\Delta$-valued right-continuous measurable process. Since $\{t < T_\theta\} = \{L_t < \theta\}$, we have $Y_t \in \mathbb{R}^d$ for every $t < T_\theta$ and consequently, $T_\theta \leq \theta(Y)$. Noting that $\theta(Y) \leq T_\theta$ by definition, we obtain that

$$T_\theta = \theta(Y) = \inf(t \in \mathbb{R}_+ \colon Y_t = \Delta). \tag{4.7}$$

The next step is to show that $Y$ has almost surely continuous paths, i.e. that $P$-a.s. $Y_{T_\theta -} = \Delta$ on $\{T_\theta < \infty\}$. This observation allows us to modify $Y$ on a null set in order to define $V$ as in Theorem 4.1.

**Discussion.** *On $\{T_\theta < \infty, \theta < \infty\}$ we simply have $Y_{T_\theta -} = X_\theta = \Delta$, but on $\{T_\theta < \infty, \theta = \infty\}$ it is necessary to understand the behavior of $X_t$ as $t \to \infty$. We stress that $\theta = \infty$ does not exclude $T_\theta < \infty$ in a pathwise sense. To see this, consider the following simple example:*

$$\mathfrak{f}(x) = \mathbb{1}_{(-\infty,0)}(x) + \sum_{k=1}^{\infty} a_k \mathbb{1}_{[k-1,k)}(x), \qquad x \in \mathbb{R}, 0 < a_k \leq 1.$$

*Clearly, $\mathfrak{f}$ is locally bounded away from zero and infinity and for $\omega(t) = t$ the integral*

$$\int_0^\infty \mathfrak{f}(X_s(\omega)) ds = \sum_{k=1}^{\infty} a_k$$

*converges or diverges depending on whether $(a_k)_{k \in \mathbb{N}}$ is summable or not. To understand why $P$-a.s. $Y_{T_\theta -} = \Delta$ on $\{T_\theta < \infty\}$ holds, note that problems with the limit of $X_t$ as $t \to \infty$ occur for paths which either stay in a bounded subset of $\mathbb{R}^d$ or have a recurrent behavior, where we think for instance of a one-dimensional Brownian path. These cases are excluded by considering the set $\{T_\theta < \infty\}$, because for some compact set $U \subset \mathbb{R}^d$ the positive value $\inf_{x \in U} \mathfrak{f}(x)$ will contribute to $T_\theta$ for an infinite time.*

For every $n, m \in \mathbb{N}$ we define

$$\begin{aligned} \sigma^m_1 &\triangleq 0, & \tau^m_1 &\triangleq \inf(t \in \mathbb{R}_+ \colon \|X_t\| \geq m+1), \\ \sigma^m_{n+1} &\triangleq \inf(t > \tau^m_n \colon \|X_t\| \leq m), & \tau^m_{n+1} &\triangleq \inf(t > \sigma^m_{n+1} \colon \|X_t\| \geq m+1). \end{aligned}$$

Set

$$\mathcal{O} \triangleq \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \{\tau^m_n < \infty,\ \sigma^m_{n+1} = \infty\},$$

and note that $\mathcal{O} \subseteq \{Y_{T_{\theta-}} = \Delta\} = \{X_{\theta-} = \Delta\}$. The proof of the following lemma borrows ideas from [62, Lemma IV.2.1].

**Lemma 4.3.** *$P$-a.s. $\mathcal{O}^c \subseteq \{T_\theta = \infty\}$.*

*Proof.* We obtain

$$\begin{aligned}
\mathcal{O}^c &= \bigcup_{m=1}^\infty \bigcap_{n=1}^\infty \big(\{\tau_n^m = \infty\} \cup \{\sigma_{n+1}^m < \infty\}\big) \\
&= \bigcup_{m=1}^\infty \bigcap_{n=1}^\infty \big(\{\tau_n^m = \infty, \sigma_n^m < \infty\} \cup \{\sigma_n^m = \infty\} \cup \{\sigma_{n+1}^m < \infty\}\big) \\
&\subseteq \bigcup_{m=1}^\infty \Big(\Big(\bigcup_{k=1}^\infty \{\tau_k^m = \infty, \sigma_k^m < \infty\}\Big) \cup \Big(\bigcap_{n=1}^\infty \big(\{\sigma_n^m = \infty\} \cup \{\sigma_{n+1}^m < \infty\}\big)\Big)\Big) \\
&= \bigcup_{m=1}^\infty \Big(\Big(\bigcup_{k=1}^\infty \{\tau_k^m = \infty, \sigma_k^m < \infty\}\Big) \cup \Big(\bigcap_{n=1}^\infty \{\sigma_n^m < \infty\}\Big)\Big) \\
&\subseteq \Big(\bigcup_{m=1}^\infty \bigcup_{k=1}^\infty \{\tau_k^m = \infty, \sigma_k^m < \infty\}\Big) \cup \Big(\bigcup_{i=1}^\infty \bigcap_{n=1}^\infty \{\sigma_n^i < \infty\}\Big) \\
&\triangleq \mathcal{O}_1 \cup \mathcal{O}_2.
\end{aligned}$$

Take $\omega \in \mathcal{O}_1$. Then, there exist $m = m(\omega), n = n(\omega) \in \mathbb{N}$ such that $\sigma_n^m(\omega) < \infty$ and $\|X_t(\omega)\| \leq m + 1$ for all $t \geq \sigma_n^m(\omega)$. Consequently, $\theta(\omega) = \infty$ and

$$T_{\theta(\omega)}(\omega) = \int_0^\infty \mathfrak{f}(X_s(\omega))ds \geq \int_{\sigma_n(\omega)}^\infty \mathfrak{f}(X_s(\omega))ds \geq \inf_{\|y\| \leq m+1} \mathfrak{f}(y) \int_{\sigma_n(\omega)}^\infty ds = \infty.$$

This implies $\mathcal{O}_1 \subseteq \{T_\theta = \infty\}$.

Set

$$\Theta \triangleq \bigcup_{m=1}^\infty \Big\{\sigma_n^m < \infty \text{ for all } n \in \mathbb{N} \text{ and } \sum_{k=1}^\infty \big(\tau_k^m - \sigma_k^m\big) = \infty\Big\}.$$

Take $\omega \in \Theta$ and let $m = m(\omega) \in \mathbb{N}$ be as in the definition of $\Theta$. Then,

$$T_{\theta(\omega)}(\omega) \geq \sum_{k=1}^\infty \int_{\sigma_k^m(\omega)}^{\tau_k^m(\omega)} \mathfrak{f}(X_s(\omega))ds \geq \inf_{\|y\| \leq m+1} \mathfrak{f}(y) \sum_{k=1}^\infty (\tau_k^m(\omega) - \sigma_k^m(\omega)) = \infty.$$

This implies that $\Theta \subseteq \{T_\theta = \infty\}$.

Next, we show that $P$-a.s. $\mathcal{O}_2 = \Theta$, which then implies that $P$-a.s. $\mathcal{O}^c \subseteq \{T_\theta = \infty\}$ and thereby completes the proof. We fix $m, n \in \mathbb{N}$. Clearly, we have on $\{\sigma_n^m < \infty\}$

$$\tau_n^m - \sigma_n^m = \inf(t \in \mathbb{R}_+ \colon \|X_{t+\sigma_n^m}\| \geq m + 1) \triangleq \gamma.$$

We set

$$K_t \triangleq \|X_t\|^2 - \|X_0\|^2 - \int_0^t \big(2\langle X_s, \mathfrak{b}(X_s)\rangle + \operatorname{tr}(\mathfrak{a}(X_s))\big)ds, \quad t < \theta,$$

and on $\{\sigma_n^m < \infty\}$ we define

$$M \triangleq K_{\cdot\wedge\gamma+\sigma_n^m} - K_{\sigma_n^m}, \qquad I \triangleq \int_{\sigma_n^m}^{\cdot\wedge\gamma+\sigma_n^m} \big(2\langle X_s, \mathfrak{b}(X_s)\rangle + \operatorname{tr}(\mathfrak{a}(X_s))\big) ds.$$

Using that for every $t \in \mathbb{R}_+$ on $\{\sigma_n^m < \infty\}$

$$\{\|X_{t\wedge\gamma+\sigma_n^m}\| \geq m+1\} \subseteq \{|M_t| \geq \tfrac{1}{2}\} \cup \{|I_t| \geq \tfrac{1}{2}\},$$

we obtain that

$$\gamma \geq \inf(t \in \mathbb{R}_+ \colon |M_t| \geq \tfrac{1}{2}) \wedge \inf(t \in \mathbb{R}_+ \colon |I_t| \geq \tfrac{1}{2}) \text{ on } \{\sigma_n^m < \infty\}. \tag{4.8}$$

Since

$$|I_t| \leq \Big( \sup_{\|y\| \leq m+1} \big|2\langle y, \mathfrak{b}(y)\rangle + \operatorname{tr}(\mathfrak{a}(y))\big| \vee 1\Big) t \triangleq \alpha t \text{ on } \{\sigma_n^m < \infty\},$$

we obtain that

$$\inf(t \in \mathbb{R}_+ \colon |I_t| \geq \tfrac{1}{2}) \geq \frac{1}{2\alpha} \text{ on } \{\sigma_n^m < \infty\}. \tag{4.9}$$

For every $t \in \mathbb{R}_+$ we have $t \wedge \gamma + \sigma_n^m < \theta$ on $\{\sigma_n^m < \infty\}$. Consequently,

$$(t \wedge \gamma + \sigma_n^m) \wedge \theta_k \wedge k \nearrow t \wedge \gamma + \sigma_n^m \text{ as } k \to \infty \text{ on } \{\sigma_n^m < \infty\}. \tag{4.10}$$

Applying the definition of the martingale problem with $f(x) = \|x\|^2$ yields that for every $k \in \mathbb{N}$ the process $K_{\cdot\wedge\theta_k\wedge k}$ is a $P$-martingale. Note that for every $t \in \mathbb{R}_+$

$$\sup_{k\in\mathbb{N}} \big|K_{(t\wedge\gamma+\sigma_n^m)\wedge\theta_k\wedge k} - K_{\sigma_n^m\wedge\theta_k\wedge k}\big| \mathbb{1}_{\{\sigma_n^m<\infty\}} \leq 2(m+1)^2 + \alpha t. \tag{4.11}$$

It is well-known that $\sigma_n^m$ and $\tau_n^m$ are $(\mathcal{F}_t^o)_{t\geq 0}$-stopping times, see [47, Proposition 2.1.5]. We note that $t \wedge \gamma + \sigma_n^m$, which is set to be $\infty$ in case $\sigma_n^m = \infty$, is an $(\mathcal{F}_t^o)_{t\geq 0}$-stopping time, too. To see this, note that for all $s \in \mathbb{R}_+$

$$\begin{aligned}\{t \wedge \gamma + \sigma_n^m \leq s\} = \{t + \sigma_n^m \leq s, \sigma_n^m < \infty, t + \sigma_n^m \leq \tau_n^m\} \\ \cup \{\tau_n^m \leq s, \sigma_n^m < \infty, \tau_n^m \leq t + \sigma_n^m\} \in \mathcal{F}_s^o,\end{aligned}$$

which holds due to the following facts: For any $(\mathcal{F}_t^o)_{t\geq 0}$-stopping times $\rho$ and $\tau$ it holds that $\mathcal{F}_\rho^o \cap \{\rho \leq s\} \subseteq \mathcal{F}_s^o, \{\rho \leq \tau\} \in \mathcal{F}_\rho^o \cap \mathcal{F}_\tau^o$, and $\mathcal{F}_\rho^o \subseteq \mathcal{F}_\tau^o$ whenever $\rho \leq \tau$.

Let $s < t$ and take $A \in \mathcal{F}_{s+\sigma_n^m}^o$ and $G \in \mathcal{F}_{\sigma_n^m}^o$. Recalling (4.10) and (4.11), the dominated convergence and the optional stopping theorem yield that

$$\begin{aligned}
&E^P\big[M_t \mathbb{1}_A \mathbb{1}_G \mathbb{1}_{\{\sigma_n^m<\infty\}}\big] \\
&\quad = \lim_{k\to\infty} E^P\big[\big(K_{(t\wedge\gamma+\sigma_n^m)\wedge\theta_k\wedge k} - K_{\sigma_n^m\wedge\theta_k\wedge k}\big)\mathbb{1}_A \mathbb{1}_G \mathbb{1}_{\{\sigma_n^m<\infty\}}\big] \\
&\quad = \lim_{k\to\infty} E^P\big[\big(E^P\big[K_{(t\wedge\gamma+\sigma_n^m)\wedge\theta_k\wedge k} | \mathcal{F}_{s+\sigma_n^m}\big] - K_{\sigma_n^m\wedge\theta_k\wedge k}\big)\mathbb{1}_A \mathbb{1}_G \mathbb{1}_{\{\sigma_n^m<\infty\}}\big] \\
&\quad = \lim_{k\to\infty} E^P\big[\big(K_{(s\wedge\gamma+\sigma_n^m)\wedge\theta_k\wedge k} - K_{\sigma_n^m\wedge\theta_k\wedge k}\big)\mathbb{1}_A \mathbb{1}_G \mathbb{1}_{\{\sigma_n^m<\infty\}}\big] \\
&\quad = E^P\big[M_s \mathbb{1}_A \mathbb{1}_G \mathbb{1}_{\{\sigma_n^m<\infty\}}\big].
\end{aligned}$$

We conclude the existence of a $P$-null set $N(s,t,A)$ such that

$$E^P\big[(M_t - M_s)\mathbb{1}_A\mathbb{1}_{\{\sigma^m_n<\infty\}}|\mathcal{F}^o_{\sigma^m_n}\big](\omega) = 0$$

for all $\omega \notin N(s,t,A)$. Since $\mathcal{F}^o_{s+\sigma^m_n} = \sigma(X_{t\wedge(s+\sigma^m_n)}, t \in \mathbb{Q}_+)$, see [132, Theorem I.6], there exists a countable system $\mathcal{C}_s$ of generators of $\mathcal{F}^o_{s+\sigma^m_n}$. Set

$$N \triangleq \bigcup_{t\in\mathbb{Q}_+}\ \bigcup_{\mathbb{Q}_+\ni s<t}\ \bigcup_{A\in\mathcal{C}_s} N(s,t,A),$$

which is a $P$-null set. Now, we conclude that for all $\omega \notin N\cup\{\sigma^m_n = \infty\}$ the process $M$ is a continuous $P(\cdot|\mathcal{F}^o_{\sigma^m_n})(\omega)$-martingale for the shifted filtration $(\mathcal{F}^o_{t+\sigma^m_n})_{t\geq 0}$ and, by the backwards martingale convergence theorem, also for its right-continuous version $\mathbf{F}_{\sigma^m_n} \triangleq (\mathcal{F}_{t+\sigma^m_n})_{t\geq 0}$, see also [74, Lemma 6.2].

Fix $\omega \notin N\cup\{\sigma^m_n = \infty\}$. It follows similar to the proof of [123, Proposition VIII.3.3] that $P(\cdot|\mathcal{F}^o_{\sigma^m_n})(\omega)$-a.s.

$$[M,M] = 4\int_{\sigma^m_n}^{\cdot\wedge\gamma+\sigma^m_n} \langle X_s, \mathfrak{a}(X_s)X_s\rangle ds.$$

The Dambis, Dubins–Schwarz theorem (see, e.g. [74, Theorem 16.4]) yields that on a standard extension of the filtered probability space $(\Omega,\mathcal{F},\mathbf{F}_{\sigma^m_n}, P(\cdot|\mathcal{F}^o_{\sigma^m_n})(\omega))$, which we ignore in our notation for simplicity, there exists a one-dimensional Brownian motion $B$ such that $P(\cdot|\mathcal{F}^o_{\sigma^m_n})(\omega)$-a.s. $M = B_{[M,M]}$. As $P(\cdot|\mathcal{F}^o_{\sigma^m_n})(\omega)$-a.s.

$$4\int_{\sigma^m_n}^{t\wedge\gamma+\sigma^m_n} \langle X_s, \mathfrak{a}(X_s)X_s\rangle ds \leq 4\Big(\sup_{\|y\|\leq m+1}\langle y, \mathfrak{a}(y)y\rangle \vee 1\Big)t \triangleq \beta t, \quad t\in\mathbb{R}_+,$$

we have $P(\cdot|\mathcal{F}^o_{\sigma^m_n})(\omega)$-a.s.

$$\inf(t\in\mathbb{R}_+ : |B_{[M,M]_t}| \geq \tfrac{1}{2}) \geq \frac{\inf(t\in\mathbb{R}_+ : |B_t| \geq \frac{1}{2})}{\beta} \triangleq \frac{\tau}{\beta}. \tag{4.12}$$

In summary, (4.8), (4.9) and (4.12) imply that

$$E^P\big[e^{-(\tau^m_n - \sigma^m_n)}|\mathcal{F}^o_{\sigma^m_n}\big](\omega) \leq E\big[e^{-\frac{\tau}{\beta}\wedge\frac{1}{2\alpha}}\big] \triangleq \mathsf{C}.$$

We note that the law of $\tau$ under $P(\cdot|\mathcal{F}^o_{\sigma^m_n})(\omega)$ only depends on the Wiener measure, which means that $\mathsf{C}$ is a constant independent of $n, m$ and $\omega$. Note also that $\mathsf{C} < 1$. Now, we obtain for all $n\in\mathbb{Z}_+$

$$\begin{aligned} E^P\Big[\prod_{k=1}^{n+1}\mathbb{1}_{\{\sigma^m_k<\infty\}}e^{-(\tau^m_k-\sigma^m_k)}\Big] &= E^P\Big[\mathbb{1}_{\{\sigma^m_{n+1}<\infty\}}E^P\big[e^{-(\tau^m_{n+1}-\sigma^m_{n+1})}\big|\mathcal{F}^o_{\sigma^m_{n+1}}\big]\prod_{k=1}^{n}\mathbb{1}_{\{\sigma^m_k<\infty\}}e^{-(\tau^m_k-\sigma^m_k)}\Big] \\ &\leq \mathsf{C}E^P\Big[\prod_{k=1}^{n}\mathbb{1}_{\{\sigma^m_k<\infty\}}e^{-(\tau^m_k-\sigma^m_k)}\Big]. \end{aligned}$$

By induction, we conclude

$$E^P\Big[\prod_{k=1}^n \mathbb{1}_{\{\sigma_k^m<\infty\}} e^{-(\tau_k^m-\sigma_k^m)}\Big] \le \mathsf{C}^n, \quad n \in \mathbb{N}.$$

Letting $n \to \infty$ and using the dominated convergence theorem yields that

$$E^P\Big[\prod_{k=1}^\infty \mathbb{1}_{\{\sigma_k^m<\infty\}} e^{-(\tau_k^m-\sigma_k^m)}\Big] = 0.$$

This implies that $P$-a.s. for all $m \in \mathbb{N}$

$$e^{-\sum_{i=1}^\infty(\tau_i^m-\sigma_i^m)} \prod_{k=1}^\infty \mathbb{1}_{\{\sigma_k^m<\infty\}} = \prod_{k=1}^\infty \mathbb{1}_{\{\sigma_k^m<\infty\}} e^{-(\tau_k^m-\sigma_k^m)} = 0.$$

We conclude that $P$-a.s. $\mathcal{O}_2 = \Theta$. The proof is complete. □

**Remark 4.3.** *In case the MP $(\mathfrak{a}, \mathfrak{b}, x)$ has a unique solution $P_x$ for all $x \in \mathbb{R}^d$ and $x \mapsto P_x$ is continuous, the proof of $P$-a.s. $\mathcal{O}_2 = \Theta$ in Lemma 4.3 simplifies substantially: We equip $\Omega$ with the usual local uniform topology, which renders it into a Polish space, see [120, pp. 33 – 34] for details. As $\omega \mapsto \tau_1^m(\omega)$ is lower semi-continuous (see [120, Exercise 2.2.1]), the map $\omega \mapsto e^{-\tau_1^m(\omega)}$ is upper semi-continuous. Thus, [1, Theorem 15.5] yields that also $x \mapsto E_x[e^{-\tau_1^m}]$ is upper semi-continuous. Consequently, because on compact sets upper semi-continuous functions attain a maximum value, $\mathsf{C} \triangleq \sup_{\|x\|\le m} E_x[e^{-\tau_1^m}] < 1$. Now, using the strong Markov property, which is implied by uniqueness of $(P_x)_{x\in\mathbb{R}^d}$, we obtain*

$$\begin{aligned}
&E_{x_0}\Big[\prod_{k=1}^{n+1} \mathbb{1}_{\{\sigma_k^m<\infty\}} e^{-(\tau_k^m-\sigma_k^m)}\Big] \\
&\qquad = E_{x_0}\Big[\mathbb{1}_{\{\sigma_{n+1}^m<\infty\}} E_{x_0}\big[e^{-(\tau_{n+1}^m-\sigma_{n+1}^m)}\big|\mathcal{F}^o_{\sigma_{n+1}^m}\big] \prod_{k=1}^n \mathbb{1}_{\{\sigma_k^m<\infty\}} e^{-(\tau_k^m-\sigma_k^m)}\Big] \\
&\qquad = E_{x_0}\Big[\mathbb{1}_{\{\sigma_{n+1}^m<\infty\}} E_{X_{\sigma_{n+1}^m}}\big[e^{-\tau_1^m}\big] \prod_{k=1}^n \mathbb{1}_{\{\sigma_k^m<\infty\}} e^{-(\tau_k^m-\sigma_k^m)}\Big] \\
&\qquad \le \mathsf{C} E_{x_0}\Big[\prod_{k=1}^n \mathbb{1}_{\{\sigma_k^m<\infty\}} e^{-(\tau_k^m-\sigma_k^m)}\Big] \le \mathsf{C}^{n+1} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

*The previous proof of Lemma 4.3 requires no uniqueness assumption on $P$ and no continuity assumptions on $\mathfrak{b}$ and/or $\mathfrak{a}$. Latter are often imposed to obtain continuity of $x \mapsto P_x$, see, for instance, [120, 137].*

We now set

$$V \triangleq \begin{cases} Y, & \text{on } \{T_\theta = \infty\} \cup (\{T_\theta < \infty\} \cap \mathcal{O}), \\ x_0, & \text{on } \{T_\theta < \infty\} \cap \mathcal{O}^c. \end{cases}$$

Clearly, $V$ is a measurable map from $\Omega$ into $\Omega$. Furthermore, Lemma 4.3 implies $P$-a.s. $V = Y$. For $n \in \mathbb{N}$ set $\gamma_n \triangleq T_{\theta_n \wedge n}$ and note that $L_{\gamma_n} = \theta_n \wedge n$. It follows from [123, Proposition V.1.4] that for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$

$$L_{t\wedge\gamma_n} = \int_0^{L_{t\wedge\gamma_n}} \mathfrak{f}^{-1}(X_s)dT_s = \int_0^{t\wedge\gamma_n} \mathfrak{f}^{-1}(X_{L_s})dT_{L_s} = \int_0^{t\wedge\gamma_n} \mathfrak{f}^{-1}(X_{L_s})ds.$$

In other words, we have for all $n \in \mathbb{N}$

$$\mathbb{1}_{\{t\leq\gamma_n\}}dL_t = \mathbb{1}_{\{t\leq\gamma_n\}}\mathfrak{f}^{-1}(X_{L_t})dt. \tag{4.13}$$

Using (4.13) and again [123, Proposition V.1.4], we obtain for every locally bounded Borel function $\mathfrak{g}\colon \mathbb{R}^d \to \mathbb{R}$ that for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$

$$\int_0^{t\wedge\gamma_n} \frac{\mathfrak{g}(Y_s)ds}{\mathfrak{f}(Y_s)} = \int_0^{t\wedge\gamma_n} \frac{\mathfrak{g}(X_{L_s})ds}{\mathfrak{f}(X_{L_s})} = \int_0^{t\wedge\gamma_n} \mathfrak{g}(X_{L_s})dL_s = \int_0^{L_{t\wedge\gamma_n}} \mathfrak{g}(X_s)ds. \tag{4.14}$$

Note that $L_t$ is an $\mathbf{F}$-stopping time and define the time-changed filtration $\mathbf{G} = (\mathcal{G}_t)_{t\geq 0} \triangleq (\mathcal{F}_{L_t})_{t\geq 0}$. Since $(L_t)_{t\geq 0}$ is right-continuous, also $\mathbf{G}$ is right-continuous, and, because $\theta_n \wedge n$ is an $\mathbf{F}$-stopping time, [65, Lemma 10.5] implies that $\gamma_n = T_{\theta_n\wedge n}$ is a $\mathbf{G}$-stopping time and that $t \mapsto L_{t\wedge\gamma_n}$ is an increasing sequence of $\mathbf{F}$-stopping times. We set

$$\mathfrak{K}f \triangleq \langle \nabla f, \mathfrak{b}\rangle + \tfrac{1}{2}\operatorname{tr}(\nabla^2 f \mathfrak{a}), \quad f \in C^2(\mathbb{R}^d).$$

By the definition of the MP $(\mathfrak{a}, \mathfrak{b}, x_0)$, the process

$$f(X_{\cdot\wedge\theta_n}) - f(x_0) - \int_0^{\cdot\wedge\theta_n} \mathfrak{K}f(X_s)ds$$

is a $P$-martingale. Recall further that $L_{t\wedge\gamma_n} \leq \theta_n \wedge n$. Using (4.14) and the optional stopping theorem, for $s < t, n \in \mathbb{N}$ and $f \in C^2(\mathbb{R}^d)$ we obtain that $P$-a.s.

$$\begin{aligned}
E^P\Big[f(Y_{t\wedge\gamma_n}) &- f(x_0) - \int_0^{t\wedge\gamma_n} \frac{\mathfrak{K}f(Y_r)dr}{\mathfrak{f}(V_r)}\Big|\mathcal{G}_s\Big] \\
&= E^P\Big[f(X_{L_{t\wedge\gamma_n}\wedge\theta_n\wedge n}) - f(x_0) - \int_0^{L_{t\wedge\gamma_n}\wedge\theta_n\wedge n} \mathfrak{K}f(X_r)dr\Big|\mathcal{F}_{L_s}\Big] \\
&= f(X_{L_{t\wedge\gamma_n}\wedge\theta_n\wedge n\wedge L_s}) - f(x_0) - \int_0^{L_{t\wedge\gamma_n}\wedge\theta_n\wedge n\wedge L_s} \mathfrak{K}f(X_r)dr \\
&= f(X_{L_{s\wedge\gamma_n}}) - f(x_0) - \int_0^{L_{s\wedge\gamma_n}} \mathfrak{K}f(X_r)dr \\
&= f(Y_{s\wedge\gamma_n}) - f(x_0) - \int_0^{s\wedge\gamma_n} \frac{\mathfrak{K}f(Y_r)dr}{\mathfrak{f}(Y_r)}.
\end{aligned}$$

This yields that

$$f(V_{\cdot\wedge\gamma_n}) - f(x_0) - \int_0^{\cdot\wedge\gamma_n} \frac{\mathfrak{K}f(V_r)dr}{\mathfrak{f}(V_r)}$$

is a $P$-martingale for the $P$-augmentation of $\mathbf{G}$, which we denote by $\mathbf{G}^P$. Note that $\theta_n(V)$ is a $\mathbf{G}^P$-stopping time. Recalling that $P$-a.s. $\gamma_n \nearrow T_\theta = \theta(V)$ and $t \wedge \theta_n(V) < \theta(V)$, the dominated convergence theorem yields for all $s < t, n \in \mathbb{N}$ and $f \in C^2(\mathbb{R}^d)$ that $P$-a.s.

$$\begin{aligned}
E^P\Big[f(V_{t\wedge\theta_n(V)}) - f(x_0) - \int_0^{t\wedge\theta_n(V)} \frac{\mathfrak{K}f(V_r)dr}{\mathfrak{f}(V_r)}\big|\mathcal{G}^P_s\Big]
&= \lim_{m\to\infty} E^P\Big[f(Y_{t\wedge\theta_n(V)\wedge\gamma_m}) - f(x_0) - \int_0^{t\wedge\theta_n(V)\wedge\gamma_m} \frac{\mathfrak{K}f(V_r)dr}{\mathfrak{f}(V_r)}\big|\mathcal{G}^P_s\Big] \\
&= \lim_{m\to\infty} \Big(f(V_{s\wedge\theta_n(V)\wedge\gamma_m}) - f(x_0) - \int_0^{s\wedge\theta_n(V)\wedge\gamma_m} \frac{\mathfrak{K}f(V_r)dr}{\mathfrak{f}(V_r)}\Big) \\
&= f(V_{s\wedge\theta_n(V)}) - f(x_0) - \int_0^{s\wedge\theta_n(V)} \frac{\mathfrak{K}f(V_r)dr}{\mathfrak{f}(V_r)}.
\end{aligned}$$

Using the tower rule, we conclude that

$$f(V_{\cdot\wedge\theta_n(V)}) - f(x_0) - \int_0^{\cdot\wedge\theta_n(V)} \frac{\mathfrak{K}f(V_r)dr}{\mathfrak{f}(V_r)}$$

is a $P$-martingale for the filtration generated by $V$. Consequently, the push-forward $P \circ V^{-1}$ solves the MP $(\mathfrak{f}^{-1}\mathfrak{a}, \mathfrak{f}^{-1}\mathfrak{b}, x_0)$, which is part (i) of Theorem 4.1. Recalling (4.7) shows the formula (4.1), i.e. part (ii) of Theorem 4.1.

To prove the uniqueness claim in Theorem 4.1, we introduce a right-continuous measurable process $U$. We define

$$S_t \triangleq \int_0^{t\wedge\theta} \mathfrak{f}^{-1}(X_s)ds, \qquad A_t \triangleq \inf(s \in \mathbb{R}_+ : S_s > t), \qquad t \in \mathbb{R}_+,$$

and

$$U_t \triangleq \begin{cases} X_{A_t}, & t < S_\theta, \\ \Delta, & t \geq S_\theta. \end{cases}$$

Using (4.7) and (4.13), we obtain $P$-a.s. for all $t \in \mathbb{R}_+$

$$S_t \circ V = \int_0^{t\wedge T_\theta} \mathfrak{f}^{-1}(Y_s)ds = \lim_{n\to\infty} \int_0^{t\wedge\gamma_n\wedge n} \mathfrak{f}^{-1}(X_{L_s})ds = \lim_{n\to\infty} L_{t\wedge\gamma_n\wedge n} = L_{(t\wedge T_\theta)-}.$$

In particular, $P$-a.s. $S_\theta \circ V = \theta$. We deduce $P$-a.s. $A_t \circ V = T_t$ for all $t < \theta$, which implies $P$-a.s. $X_{A_t} \circ V = X_{L_{T_t}} = X_t$ for all $t < S_\theta \circ V = \theta$. We conclude that $P$-a.s. $U \circ V = X$.

To prove the last claim in Theorem 4.1, suppose that $P \circ V^{-1}$ is the unique solution to the MP $(\mathfrak{f}^{-1}\mathfrak{a}, \mathfrak{f}^{-1}\ \mathfrak{b}, x_0)$. For $n \in \mathbb{N}$ let $0 \leq t_1 < t_2 < \cdots < t_n < \infty$ and $G_1, \dots, G_n \in \mathcal{B}(\mathbb{R}^d_\Delta)$. Suppose that $Q$ is a second solution to the MP $(\mathfrak{a}, \mathfrak{b}, x_0)$. Then, the push-forwards $P \circ V^{-1}$ and $Q \circ V^{-1}$ both solve the MP $(\mathfrak{f}^{-1}\mathfrak{a}, \mathfrak{f}^{-1}\mathfrak{b}, x_0)$ and we deduce from the uniqueness assumption that

$$\begin{aligned}
P(X_{t_1} \in G_1, \dots, X_{t_n} \in G_n) &= P \circ V^{-1}(U_{t_1} \in G_1, \dots, U_{t_n} \in G_n) \\
&= Q \circ V^{-1}(U_{t_1} \in G_1, \dots, U_{t_n} \in G_n) \\
&= Q(X_{t_1} \in G_1, \dots, X_{t_n} \in G_n).
\end{aligned}$$

By a monotone class argument, $P = Q$. The proof is complete. □

## 4.5 Proof of Propositions 4.1 and 4.2

### 4.5.1 Proof of Propositions 4.1 (i) and 4.2

In this section we explain that Propositions 4.1 (i) and 4.2 can be deduced from Corollary 3.1. The difference between the setting in Chapter 3 and the current setting is that the underlying path space in Chapter 3 is slightly bigger and allows explosion in a discontinuous manner. We now introduce the continuous version of the path space from Chapter 3. Let $\Sigma$ be the space of right-continuous functions $\omega\colon \mathbb{R}_+ \to \mathbb{R}^d_\Delta$ which are continuous on $[0, \theta(\omega))$ and $\omega(t) = \Delta$ for $t \geq \theta(\omega) = \inf(t \in \mathbb{R}_+ \colon \omega(t) = \Delta)$. Let $Y$ be the coordinate process on $\Sigma$ and define $\mathcal{A} \triangleq \sigma(Y_t, t \in \mathbb{R}_+), \mathcal{A}^o_t \triangleq \sigma(Y_s, s \in [0,t])$ and $\mathcal{A}_t \triangleq \bigcap_{s>t} \mathcal{A}^o_s$ for $t \in \mathbb{R}_+$. The MP $(\mathfrak{a}, \mathfrak{b}, x_0)$ on $(\Sigma, \mathcal{A}, \mathbf{A} = (\mathcal{A}_t)_{t\geq 0})$ is defined in the same manner as on the filtered space $(\Omega, \mathcal{F}, \mathbf{F})$ with the additional requirement that a solution $P$ has to satisfy $P$-a.s. $\theta_n(Y) < \theta(Y)$ on $\{\theta(Y) < \infty\}$ for all $n \in \mathbb{N}$.

Equivalently, one could define the martingale problem as follows: Using the convention that all functions $f$ on $\mathbb{R}^d$ are extended to $\mathbb{R}^d_\Delta$ by setting $f(\Delta) \equiv 0$, we say that $P$ solves the MP $(\mathfrak{a}, \mathfrak{b}, x_0)$ on $(\Sigma, \mathcal{A}, \mathbf{A})$ if $P(X_0 = x_0) = 1$ and for all $f \in C^2_c(\mathbb{R}^d)$ the process

$$f(Y_{\cdot\wedge\theta_n(Y)}) - f(x_0) - \int_0^{\cdot\wedge\theta_n(Y)} \big(\langle \nabla f(Y_s), \mathfrak{b}(Y_s)\rangle + \tfrac{1}{2}\operatorname{tr}(\nabla^2 f(Y_s)\mathfrak{a}(Y_s))\big) ds$$

is a $P$-martingale. In this case, it always holds that $P$-a.s. $\theta_n(Y) < \theta(Y)$ on $\{\theta(Y) < \infty\}$ for all $n \in \mathbb{N}$ and consequently, the definitions are equivalent. This follows from an argument in the proof of [21, Lemma 3.1]: Let $f_k \in C^2_c(\mathbb{R}^d)$ be such that $0 \leq f_k \leq 1$ and $f_k(x) = 1$ whenever $\|x\| \leq k$. The process $f_k(Y_{\cdot\wedge\theta_n(Y)}) - f_k(x_0)$ is a $P$-martingale for all $k > n$, because $\nabla f_k(Y_s)$ and $\nabla^2 f_k(Y_s)$ vanish for all $s < \theta_n(Y)$. Thus, by dominated convergence,

$$0 = \lim_{k\to\infty} E^P\big[f_k(Y_{t\wedge\theta_n(Y)}) - f_k(x_0)\big] = P(t \wedge \theta_n(Y) < \theta(Y)) - 1, \quad t \in \mathbb{R}_+.$$

This shows that $P$-a.s. $\theta_n(Y) < \theta(Y)$ on $\{\theta(Y) < \infty\}$ for all $n \in \mathbb{N}$.

A third equivalent definition is the following: A probability measure $P$ on $(\Sigma, \mathcal{A})$ solves the MP $(\mathfrak{a}, \mathfrak{b}, x_0)$ if $P(Y_0 = x_0) = 1$ and for all $n \in \mathbb{N}$ the stopped process $Y_{\cdot\wedge\theta_n(Y)}$ is an $\mathbb{R}^d$-valued continuous semimartingale with semimartingale characteristics $(B^n, C^n, 0)$, where

$$B^n = \int_0^{\cdot\wedge\theta_n(Y)} \mathfrak{b}(Y_s) ds, \qquad C^n = \int_0^{\cdot\wedge\theta_n(Y)} \mathfrak{a}(Y_s) ds,$$

see [65, Theorem 13.55] for more details. This relates the martingale problem to the semimartingale problem defined in Definition 3.1.

**Lemma 4.4.** *The path space $\Omega$ is a measurable subset of the path space $\Sigma$, i.e. $\Omega \in \mathcal{A}$.*

*Proof.* Let $d$ be a metric on $\mathbb{R}^d_\Delta$ (which induces the topology, of course) and set $Z_t \triangleq \limsup_{s\to t-, s\in\mathbb{Q}_+} d(Y_s, \Delta)$ for $t > 0$. Due to [40, Theorem IV.17], the process $(Z_t)_{t>0}$ is progressively measurable for the filtration $\mathbf{A}$. Moreover, [47, Lemma 2.2.8] yields that

$$\Omega = \{\theta(Y) \notin (0, \infty)\} \cup \{Z_{\theta(Y)} = 0, \theta(Y) \in (0,\infty)\}.$$

Hence, the claim follows. □

We define the following well-posedness condition:

(C) $\mathfrak{a}$ and $\mathfrak{b}$ are locally bounded and for all $x \in \mathbb{R}^d$ the MP $(\mathfrak{a}, \mathfrak{b}, x)$ on $(\Omega, \mathcal{F}, \mathbf{F})$ has a unique solution $P_x$.

The following lemma relates the martingale problems on the path spaces $\Sigma$ and $\Omega$.

**Lemma 4.5.** *Suppose that* (C) *holds. Then,* $P_x$*, extended to* $(\Sigma, \mathcal{A})$*, see Lemma 4.4, is also the unique solution to the MP* $(\mathfrak{a}, \mathfrak{b}, x)$ *on* $(\Sigma, \mathcal{A}, \mathbf{A})$*.*

*Proof.* On an intuitive level, any solution to the MP $(\mathfrak{a}, \mathfrak{b}, x)$ on $(\Sigma, \mathcal{A}, \mathbf{A})$ should coincide locally with $P_x$ and consequently, explosion should happen in the same manner for both problems. We now make this intuition precise.

The following local uniqueness property of well-posed martingale problems can be proven similar to Lemma 2.10.

**Lemma 4.6.** *Suppose that* (C) *holds, let* $\tau$ *be an* $(\mathcal{F}^o)_{t\geq 0}$*-stopping time and let* $R$ *be a probability measure on* $(\Omega, \mathcal{F})$ *with* $R(X_0 = x) = 1$ *and with the property that for all* $n \in \mathbb{N}$ *and* $f \in C^2(\mathbb{R}^d)$ *the process*

$$f(X_{\cdot\wedge\tau\wedge\theta_n}) - f(x) - \int_0^{\cdot\wedge\tau\wedge\theta_n} \big(\langle \nabla f(X_s), \mathfrak{b}(X_s)\rangle + \tfrac{1}{2}\operatorname{tr}(\nabla^2 f(X_s)\mathfrak{a}(X_s))\big) ds$$

*is an* $R$*-martingale. Then,* $R = P_x$ *on* $\mathcal{F}^o_\tau$*.*

Note that $\theta_n$ and $\theta_n(Y)$ are stopping times for the filtrations $(\mathcal{F}^o_t)_{t\geq 0}$ and $(\mathcal{A}^o_t)_{t\geq 0}$, respectively. In the first case this is a well-known fact ([123, Proposition I.4.5]), because the coordinate process on $\Omega$ has continuous paths. On $\Sigma$ the coordinate process is not continuous and the classical result does not apply, but its proof can be adapted easily:

$$\{\theta_n(Y) \leq t\} = \Big\{ \inf_{q\in\mathbb{Q}\cap[0,t]} d(Y_q, B_n^c) = 0, Y_t \neq \Delta\Big\} \cup \{Y_t = \Delta\} \in \mathcal{A}^o_t, \qquad t \in \mathbb{R}_+,$$

where $d$ is a metric on $\mathbb{R}^d_\Delta$ and $B_n^c \triangleq \{x \in \mathbb{R}^d \colon \|x\| \geq n\} \cup \{\Delta\}$.

Let $Q$ be a solution to the MP $(\mathfrak{a}, \mathfrak{b}, x)$ on $(\Sigma, \mathcal{A}, \mathbf{A})$. By definition of the martingale problem, the push-forward $R \equiv Q \circ Y^{-1}_{\cdot\wedge\theta_n(Y)}$ satisfies the assumptions in Lemma 4.6 for $\tau \equiv \theta_n$. Thus, Lemma 4.6 implies that $R = P_x$ on $\mathcal{F}^o_{\theta_n}$. Provided $P_x$ is extended to $(\Sigma, \mathcal{A})$, this implies that $Q = P_x$ on $\mathcal{A}^o_{\theta_n(Y)}$. Now, because $\bigvee_{n\in\mathbb{N}} \mathcal{A}^o_{\theta_n(Y)-} = \mathcal{A}^o_{\theta(Y)-}$, a monotone class argument shows that $Q = P_x$ on $\mathcal{A}^o_{\theta(Y)-}$. Finally, note that $\mathcal{A}^o_{\theta(Y)-} = \mathcal{A}$, which follows from the observation

$$\{Y_t \in G\} = \begin{cases} \{\theta(Y) \leq t\} \cup (\{Y_t \in G\} \cap \{\theta(Y) > t\}), & \Delta \in G, \\ \{Y_t \in G\} \cap \{\theta(Y) > t\}, & \Delta \notin G, \end{cases}$$

for all $t \in \mathbb{R}_+$ and $G \in \mathcal{B}(\mathbb{R}^d_\Delta)$. The proof is complete. □

We are in the position to deduce Propositions 4.1 (i) and 4.2 from Corollary 3.1. Note that Standing Assumption 3.2, which is only concerned with the existence of $Q_{x_0}$, is implied by (S3), and that Standing Assumption 3.3 and the first part of Standing Assumption 3.4 are not needed in our continuous setting. Note that (S4) implies Standing Assumption 3.5. Thus, Propositions 4.1 (i) and 4.2 follow from Corollary 3.1, because its uniqueness assumption is implied by Lemma 4.5. □

**Remark 4.4.** *Using Lemma 4.6, Propositions 4.1 (i) and 4.2 could have been proven directly without Corollary 3.1. We think it is interesting to relate the setting of this chapter to those of Chapter 3.*

### 4.5.2 Proof of Proposition 4.1 (ii)

The following proof is identical to those of [6, Theorem 1 (i)] for the conservative setting. Let $Z$ be as in (4.2). Due to Novikov's condition, (S4) implies that the stopped process $Z_{\cdot\wedge\theta_n\wedge n}$ is a UI $Q_{x_0}$-martingale. Now, using Girsanov's theorem and Lemma 4.6 implies that $E^{Q_{x_0}}[Z_{\theta_n\wedge n}\mathbb{1}_G] = P_{x_0}(G)$ for all $G \in \mathcal{F}^o_{\theta_n\wedge n}$. In other words, $Q_{x_0} \sim P_{x_0}$ on $\mathcal{F}^o_{\theta_n\wedge n}$ with

$$\frac{dP_{x_0}}{dQ_{x_0}}\bigg|_{\mathcal{F}^o_{\theta_n\wedge n}} = Z_{\theta_n\wedge n}.$$

Due to [6, Lemma 1], the following are equivalent:

(a) $Q_{x_0} \perp P_{x_0}$

(b) $Q_{x_0}$-a.s. $\lim_{n\to\infty} Z_{\theta_n\wedge n} = 0$.

It follows from [6, Lemma 2] that (b) is equivalent to $Q_{x_0}(A_\theta = \infty) = 1$. The claim of Proposition 4.1 (ii) follows now by symmetry. □

## 4.6 Martingale Problems and Stochastic Differential Equations

In this section we recall the relation of martingale problems and weak solutions of stochastic differential equations. The following definition can be viewed as a multidimensional version of [77, Definition 5.5.1].

**Definition 4.2.** *Let* $\mathfrak{b}\colon \mathbb{R}^d \to \mathbb{R}^d$ *and* $\mathfrak{s}\colon \mathbb{R}^d \to \mathbb{R}^{d\times r}$ *be locally bounded Borel functions and denote the adjoint of* $\mathfrak{s}(x)$ *by* $\mathfrak{s}^*(x)$*. We call a triplet* $((\Sigma, \mathcal{A}, \mathbf{A}, P), Y, W)$ *a* weak solution to the SDE $(\mathfrak{s}, \mathfrak{b}, x_0)$*, if the following hold:*

(i) *The triplet* $(\Sigma, \mathcal{A}, P)$ *is a complete probability space,* $\mathbf{A}$ *is an augmented filtration on* $(\Sigma, \mathcal{A}, P)$*, and* $Y = (Y_t)_{t\geq 0}$ *and* $W = (W_t)_{t\geq 0}$ *are measurable processes on* $(\Sigma, \mathcal{A})$*.*

(ii) *The process* $Y$ *is* $\mathbb{R}^d_\Delta$*-valued, continuous and* $\mathbf{A}$*-adapted, and* $W$ *is an* $r$*-dimensional standard* $\mathbf{A}$*-Brownian motion.*

(iii) *For every* $n \in \mathbb{N}$ *the process* $Y$ *satisfies* $P$*-a.s.*

$$Y_{\cdot\wedge\theta_n(Y)} = x_0 + \int_0^{\cdot\wedge\theta_n(Y)} \mathfrak{b}(Y_s)ds + \int_0^{\cdot\wedge\theta_n(Y)} \mathfrak{s}(Y_s)dW_s,$$

*where the integrals are well-defined due the local boundedness of* $\mathfrak{b}$ *and* $\mathfrak{s}$*. Moreover, we stipulate that* $Y_t = Y_{\theta(Y)}$ *for all* $t \geq \theta(Y)$*.*

The following is a version of [47, Corollary 5.3.4] or [77, Corollary 5.4.8] for possibly explosive MPs and SDEs, see [62, Theorem IV.6.1] for a statement in a non-conservative setting with continuous coefficients. The proof is identical to the non-explosive case and omitted.

**Proposition 4.5.** *Suppose that* $\mathfrak{b}\colon \mathbb{R}^d \to \mathbb{R}^d$ *and* $\mathfrak{s}\colon \mathbb{R}^d \to \mathbb{R}^{d\times r}$ *are locally bounded Borel functions.*

(i) *If* $((\Sigma, \mathcal{A}, \mathbf{A}, P), Y, W)$ *is a weak solution to the SDE* $(\mathfrak{s}, \mathfrak{b}, x_0)$*, then the push-forward* $P \circ Y^{-1}$ *solves the MP* $(\mathfrak{s}\mathfrak{s}^*, \mathfrak{b}, x_0)$*.*

(ii) *If* $Q$ *solves the MP* $(\mathfrak{s}\mathfrak{s}^*, \mathfrak{b}, x_0)$*, then there exists a weak solution* $((\Sigma, \mathcal{A}, \mathbf{A}, P), Y, W)$ *to the SDE* $(\mathfrak{s}, \mathfrak{b}, x_0)$ *and* $Q = P \circ Y^{-1}$*.*

**Remark 4.5.** *In case one starts with a coefficient* $\mathfrak{a}\colon \mathbb{R}^d \to \mathbb{S}^d$ *it is always possible to find a decomposition* $\mathfrak{a} = \mathfrak{s}\mathfrak{s} = \mathfrak{s}\mathfrak{s}^*$*, where* $\mathfrak{s}\colon \mathbb{R}^d \to \mathbb{S}^d$ *is Borel, locally bounded or continuous whenever* $\mathfrak{a}$ *is Borel, locally bounded or continuous, respectively. Let us explain this in more detail: It is well-known that for a matrix* $A \in \mathbb{S}^d$ *there exists a unique matrix* $A^{\frac{1}{2}} \in \mathbb{S}^d$ *such that* $A = A^{\frac{1}{2}} A^{\frac{1}{2}}$*. Moreover, the map* $S\colon \mathbb{S}^d \to \mathbb{S}^d$ *defined by* $S(A) = A^{\frac{1}{2}}$ *is continuous (for the matrix-norm topology on* $\mathbb{S}^d$*). In fact,* $S$ *is even Hölder continuous with exponent* $\frac{1}{2}$*, which follows from the Powers–Størmer inequality ([121, Lemma 4.1]):*

$$\|A^{\frac{1}{2}} - B^{\frac{1}{2}}\|_2 \le \sqrt{\|A - B\|_1} \le d^{\frac{1}{4}} \sqrt{\|A - B\|_2}, \qquad A, B \in \mathbb{S}^d,$$

*where* $\|\cdot\|_1$ *denotes the trace norm and* $\|\cdot\|_2$ *denotes the Hilbert–Schmidt norm. The function* $\mathfrak{s} \equiv S(\mathfrak{a})$ *has the claimed properties. Although continuity transfers from* $\mathfrak{a}$ *to its root* $\mathfrak{s} = S(\mathfrak{a})$*, the same is not necessarily true for higher regularities, see [54, Section 6.1] or [137, Section 5.2] for comments.*

## 4.7 A Few Existence and Uniqueness Results

In this section we collect some existence and uniqueness results for martingale problems. We assume that $\mathfrak{b}\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\mathfrak{a}\colon \mathbb{R}^d \to \mathbb{S}^d$ are locally bounded Borel functions and we formulate the following conditions:

(A1) $\mathfrak{b}$ and $\mathfrak{a}$ are continuous.

(A2) $\{x \in \mathbb{R}^d\colon \int_{B_r(x)} \frac{dy}{\det(\mathfrak{a}(y))} = \infty \text{ for all } r > 0\} \subseteq \{x \in \mathbb{R}^d\colon \mathfrak{b}(x) = 0, \mathfrak{a}(x) = 0\}$, where $B_r(x) = \{y \in \mathbb{R}^d\colon \|x - y\| < r\}$ denotes the open ball with center $x$ and radius $r$.

(A3) $\mathfrak{a}$ is continuous and $\langle \xi, \mathfrak{a}(x)\xi \rangle > 0$ for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d \backslash \{0\}$.

(A4) $\mathfrak{b}$ is locally Lipschitz continuous and $\mathfrak{a}$ has a decomposition $\mathfrak{a} = \mathfrak{s}\mathfrak{s}^*$, where $\mathfrak{s}\colon \mathbb{R}^d \to \mathbb{R}^{d\times r}$ is locally Lipschitz continuous.

We use this opportunity and illustrate that Theorem 4.1 can be used to prove conditions for existence and uniqueness of martingale problems.

**Proposition 4.6.** *Let* $x_0 \in \mathbb{R}^d$*. If* (A1) *or* (A2) *holds, then there exists a solution to the MP* $(\mathfrak{a}, \mathfrak{b}, x_0)$*. If* (A3) *or* (A4) *holds, then there exists a unique solution to the MP* $(\mathfrak{a}, \mathfrak{b}, x_0)$*.*

*Proof.* The following strategy is borrowed from the proof of [62, Theorem IV.2.3]. Let $\mathfrak{f}\colon \mathbb{R}^d \to (0, \infty)$ be a continuous function such that $\mathfrak{f}\mathfrak{a}$ and $\mathfrak{f}\mathfrak{b}$ are bounded. Such a function can be constructed as follows: Set

$$\mathfrak{g} \triangleq \sum_{k=1}^{\infty} a_k^{-1} \mathbb{1}_{[k-1,k)}, \text{ where } a_k \triangleq \sup_{\|x\|\le k} \|\mathfrak{b}(x)\| \vee \sup_{\|x\|\le k} \|\mathfrak{a}(x)\| \ \vee 1,$$

and let $\mathfrak{z}\colon \mathbb{R}_+ \to (0, \infty)$ be a continuous function $\mathfrak{z} \le \mathfrak{g}$. Then, $\mathfrak{f}(x) \triangleq \mathfrak{z}(\|x\|)$ has the claimed properties. In case one of (A1) – (A3) holds, the MP $(\mathfrak{f}\mathfrak{a}, \mathfrak{f}\mathfrak{b}, x_0)$ has a (conservative) solution and in case (A3) holds the solution is even unique. With these observations at hand, Theorem 4.1 implies that existence holds for the MP $(\mathfrak{a}, \mathfrak{b}, x_0)$ under either of (A1) – (A4) and that uniqueness holds under (A3). That uniqueness also holds under (A4) is well-known, see [62, Theorem IV.3.1]. Finally, we provide references for the existence and uniqueness statements concerning the MP $(\mathfrak{f}\mathfrak{a}, \mathfrak{f}\mathfrak{b}, x_0)$: For existence under (A1) see [137, Theorem 6.1.7], and for existence and uniqueness under (A3) see [137, Theorem 7.2.1]. Recalling Proposition 4.5 and Remark 4.5, existence under (A2) is implied by [126, Theorem 2]. □

**Remark 4.6.** *Existence under* (A1) *is also implied by [62, Theorem IV.2.3] and existence and uniqueness under* (A3) *is implied by [120, Theorem 1.13.1]. Under* (A2)*, existence of a solution with not necessarily continuous paths (more precisely with paths in* $\Sigma$ *as defined in Section 4.5) is implied by [90, Theorem 4.4].*

Finally, we recall that Girsanov's theorem is helpful in the study of uniqueness, see also [77, Proposition 5.3.10] and [74, Proposition 18.12].

**Proposition 4.7.** *Let* $\mathfrak{b}, \mathfrak{c} \colon \mathbb{R}^d \to \mathbb{R}^d$ *and* $\mathfrak{a} \colon \mathbb{R}^d \to \mathbb{S}^d$ *be Borel functions such that* $\mathfrak{a}, \mathfrak{b}, \mathfrak{a}\mathfrak{c}$ *and* $\langle \mathfrak{c}, \mathfrak{a}\mathfrak{c} \rangle$ *are locally bounded. Assume that for all* $x \in \mathbb{R}^d$ *the MP* $(\mathfrak{a}, \mathfrak{b}, x)$ *has a unique solution* $P_x$*. Then, for every* $x \in \mathbb{R}^d$ *the MP* $(\mathfrak{a}, \mathfrak{b} + \mathfrak{a}\mathfrak{c}, x)$ *has at most one solution.*

*Proof.* Lemma 4.6, Proposition 4.5 and [74, Proposition 18.12] yield that all solutions to the MP $(\mathfrak{a}, \mathfrak{b} + \mathfrak{a}\mathfrak{c}, x)$ coincide on $\mathcal{F}^o_{\theta_n}$ for all $n \in \mathbb{N}$. By a monotone class argument, this implies the claim. □

# 5 Existence of Semimartingales with Continuous Characteristics

## 5.1 Introduction

Existence theorems for solutions to stochastic equations are of fundamental interest in many areas of probability theory. In the context of *weak solutions* to stochastic differential equations (SDEs) important contributions were made by Skorokhod and by Stroock and Varadhan. Skorokhod (see [135]) showed that SDEs with continuous coefficients of linear growth have weak solutions. Stroock and Varadhan (see [137]) introduced the concept of the *martingale problem*, which is nowadays one of the most important tools for studying existence, uniqueness and limit theorems for stochastic processes. In many of the classical monographs on stochastic analysis (e.g. [77, 123]) Skorokhod's existence theorem is proven by the martingale problem argument of Stroock and Varadhan. The main idea is to construct an approximation sequence of probability measures on a path space, to show its tightness and finally to use the martingale problem method to verify that any of its accumulation points is the law of a weak solution.

In case of SDEs with Wiener noise and coefficients of linear growth, tightness can be verified via Kolmogorov's tightness criterion. Gatarek and Goldys [55] proposed a more direct argument based on the compactness of a fractional operator and the factorization method of Da Prato, Kwapien and Zabczyk [35]. This method was used by Hofmanová and Seidler [61] to replace the linear growth assumption in Skorokhod's theorem by a Lyapunov-type condition.

Skorokhod's original theorem is not restricted to path continuous settings. For general semimartingales Jacod and Mémin [68] proved conditions for tightness in terms of the so-called *semimartingale characteristics*. These criteria were used by Jacod and Mémin [69] to prove continuity and uniform boundedness conditions for the existence of weak solutions to SDEs driven by general semimartingales.

Refinements of the tightness criteria from [68] are proved in the monograph [70] of Jacod and Shiryaev. The conditions are used to prove a Skorokhod-type existence result for semimartingales. More precisely, Jacod and Shiryaev consider a candidate for semimartingale characteristics on the Skorokhod space and formulate continuity and uniform boundedness conditions which imply the existence of a probability measure for which the coordinate process is a semimartingle with the candidate as semimartingale characteristics.

In this chapter we generalize the existence result of Jacod and Shiryaev for the quasi-left continuous case by replacing the uniform boundedness assumption by local boundedness assumptions together with a Lyapunov-type or a linear growth condition. The linear growth condition takes the whole history of the paths into consideration. We prove the result as follows: First, we construct an approximation sequence with the help of the existence result of Jacod and Shiryaev. Second, we show tightness by a localization of a criterion from [70] together with a Lyapunov-type or a Gronwall-type argument. In this step we also adapt arguments used by Liptser and Shiryaev [101]. Finally, we use arguments based on the martingale problem for semimartingales to verify that any accumulation point of our approximation sequence is the law of a semimartingale with the correct semimartingale characteristics.

Let us shortly comment on continuative problems. The weak convergence argument heavily relies on the continuous mapping theorem, which is applicable when the coefficients have a continuity property. It is only natural to ask what can be said for discontinuous coefficients. We do not touch this topic in this thesis and refer the curious reader to the recent articles [63, 88] where interesting progress in this direction is made.

The chapter is structured as follows. In Section 5.2.1 we explain the mathematical setting. In Section 5.2.2 we state our main results. In particular, we discuss its assumptions. Finally, we comment on the method based on the extension of local solutions and on a possible expansion of our results via Girsanov-type arguments. In Section 5.2.3 we apply our results in a jump-diffusion setting. The proofs are given in Section 5.3.

## 5.2 Formulation of the Main Results

### 5.2.1 The Mathematical Setting

Let $\Omega$ be the Skorokhod space of càdlàg functions $\mathbb{R}_+ \to \mathbb{R}^d$ equipped with the Skorokhod topology (see [70] for details). We denote the coordinate process on $\Omega$ by $X$, i.e. $X_t(\omega) = \omega(t)$ for $t \in \mathbb{R}_+$ and $\omega \in \Omega$. Let $\mathcal{F} \triangleq \sigma(X_t, t \in \mathbb{R}_+)$ and $\mathcal{F}_t \triangleq \bigcap_{s>t} \mathcal{F}^o_s$, where $\mathcal{F}^o_s \triangleq \sigma(X_t, t \in [0, s])$. Except stated otherwise, when we use terms such as *adapted, predictable, etc.* we refer to the right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$.

Throughout the chapter we fix a continuous truncation function $h\colon \mathbb{R}^d \to \mathbb{R}^d$, i.e. a bounded continuous function which equals the identity around the origin.

A càdlàg $\mathbb{R}^d$-valued adapted process $Y$ is called a semimartingale if it admits a decomposition $Y = Y_0 + M + V$, where $M$ is a càdlàg local martingale starting at the origin and $V$ is a càdlàg adapted process of finite variation starting at the origin. Here, we adapt the terminology from [70] and call a process $V$ of finite variation if for all $\omega \in \Omega$ the map $t \mapsto V_t(\omega)$ is locally of finite variation. To a semimartingale $Y$ we associate a quadruple $(b, c, K; A)$ consisting of an $\mathbb{R}^d$-valued predictable process $b$, a predictable process $c$ taking values in the set $\mathbb{S}^d$ of symmetric non-negative definite $d \times d$ matrices, a predictable kernel $K$ from $\Omega \times \mathbb{R}_+$ into $\mathbb{R}^d$ and a predictable increasing càdlàg process $A$, see [70, Definition II.2.6, Proposition II.2.9, II.2.12 – II.2.14] for precise definitions and properties. When $(B, C, \nu)$ are the semimartingale characteristics of $Y$ (see [70, Definition II.2.6]), then

$$\frac{dB_t}{dA_t} = b_t, \qquad \frac{dC_t}{dA_t} = c_t, \qquad \frac{\nu(dt, dx)}{dA_t} = K_t(dx),$$

i.e. in other words $(b, c, K)$ are the densities of $(B, C, \nu)$ w.r.t. the reference measure $dA_t$. Thus, we call the quadruple $(b, c, K; A)$ the *local characteristics of* $Y$. Providing an intuition, $b$ represents the drift and depends on the truncation function $h$, $c$ encodes the continuous local martingale component and $K$ reflects the jump structure. In addition, for $i, j = 1, \dots, d$ we define by

$$\widetilde{c}^{ij} \triangleq c^{ij} + \int h^i(x)h^j(x)K(dx) - \Delta A \int h^i(x)K(dx) \int h^j(x)K(dx)$$

a *modified second characteristic*, see [70, Proposition II.2.17].

Let us shortly comment on the role played by the initial law. For SDEs with Wiener noise Kallenberg [73] proved that weak solutions exist for all initial laws if and only if weak solutions exist for all degenerated initial laws. Although the result is fairly old, it seems not to be commonly known. We now state a version for a general semimartingale setting. The proof is similar as in the diffusion case (see Proposition 2.9 in Section 2.6) and can be found in Section 5.4 below.

**Proposition 5.1.** *Assume that for all $z \in \mathbb{R}^d$ there exists a probability measure $P_z$ on $(\Omega, \mathcal{F})$ such that the coordinate process is a $P_z$-semimartingale with local characteristics $(b, c, K; A)$ and initial law $\delta_z$. Then, for any Borel probability measure $\eta$ on $\mathbb{R}^d$ there exists a probability measure $P_\eta$ on $(\Omega, \mathcal{F})$ such that the coordinate process is a $P_\eta$-semimartingale with local characteristics $(b, c, K; A)$ and initial law $\eta$.*

From now on we fix a deterministic continuous increasing function $A\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $A_0 = 0$ and a Borel probability measure $\eta$ on $\mathbb{R}^d$. Next, we define a so-called *candidate triplet* $(b, c, K)$ on $(\Omega, \mathcal{F})$. Let us shortly clarify some notations: For $x, y \in \mathbb{R}^d$ we write $\|x\|$ for the Euclidean norm, $\langle x, y\rangle$ for the Euclidean scalar product, and for $M \in \mathbb{S}^d$ we write $\|M\| \triangleq \operatorname{trace} M$.

**Definition 5.1.** *We call $(b, c, K)$ a candidate triplet, if it consists of the following:*

(i) *A predictable $\mathbb{R}^d$-valued process $b$ such that $\int_0^t \|b_s(\omega)\| dA_s < \infty$ for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$.*

(ii) *A predictable $\mathbb{S}^d$-valued process $c$ such that $\int_0^t \|c_s(\omega)\| dA_s < \infty$ for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$.*

(iii) *A predictable kernel $(\omega, s) \mapsto K_s(\omega; dx)$ from $\Omega \times \mathbb{R}_+$ into $\mathbb{R}^d$ such that for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$ we have $K_t(\omega; \{0\}) = 0$ and $\int_0^t \int (1 \wedge \|x\|^2) K_s(\omega; dx) dA_s < \infty$.*

In the following we fix also a candidate triplet $(b, c, K)$. The goal is to find a probability measure $P$ on $(\Omega, \mathcal{F})$ such that the coordinate process $X$ is a $P$-semimartingale with local characteristics $(b, c, K; A)$ and initial law $\eta$.

### 5.2.2 Existence Conditions for Semimartingales

Let $C_2(\mathbb{R}^d)$ be the set of all continuous bounded function $\mathbb{R}^d \to \mathbb{R}$ which vanish around the origin. Moreover, let $C_1(\mathbb{R}^d)$ be a subclass of the non-negative functions in $C_2(\mathbb{R}^d)$ which contains all functions $g(x) = (a\|x\| - 1)^+ \wedge 1$ for $a \in \mathbb{Q}$ and is convergence determining for the weak convergence induced by $C_2(\mathbb{R}^d)$ (see [70, p. 395] for more details).

For $f \in C^2(\mathbb{R}^d, \mathbb{R})$ and $a > 0$ we set

$$\widetilde{c}^{ij,a} \triangleq c^{ij} + \int_{\|x\| \le a} x^i x^j K(dx), \quad b^a \triangleq b - \int \big(h(x) - x \mathbb{1}\{\|x\| \le a\}\big) K(dx)$$

and for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$ and $x \in \mathbb{R}^d$ we set

$$(\mathcal{K}^d_a f)(\omega; t, x) \triangleq f(\omega(t-) + x) - f(\omega(t-)) - \sum_{k=1}^d \partial_k f(\omega(t-)) x^k,$$

$$(\mathcal{K}^l_a f)(\omega; t) \triangleq \sum_{k=1}^d \partial_k f(\omega(t-)) b^{k,a}_t(\omega) + \frac{1}{2} \sum_{k,j=1}^d \partial^2_{kj} f(\omega(t-)) c^{kj}_t(\omega),$$

and

$$(\mathcal{L}_a f)(\omega;t) \triangleq (\mathcal{K}^l_a f)(\omega;t) + \int_{\|x\|\le a} (\mathcal{K}^d_a f)(\omega;t,x) K_t(\omega;dx),$$

provided the last term is well-defined. Taylor's theorem yields that for all $(t,\omega) \in \mathbb{R}_+ \times \Omega$ there exists a constant $c = c(f,a,t,\omega)$ such that

$$\int_0^t \int_{\|x\|\le a} \big|(\mathcal{K}^d_a f)(\omega;s,x)\big| K_s(\omega;dx) dA_s \le c \int_0^t \int_{\|x\|\le a} \|x\|^2 K_s(\omega;dx) dA_s < \infty. \quad (5.1)$$

For $m > 0$ we define

$$\Theta_m \triangleq \Big\{(t,\omega) \in [0,m] \times \Omega \colon \sup_{s\in[0,t]} \|\omega(s-)\| \le m\Big\}.$$

**Remark 5.1.** *For $(t,\omega) \in (0,\infty) \times \Omega$ it holds that*

$$\sup_{s\in[0,t]} \|\omega(s-)\| = \sup_{s\in[0,t)} \|\omega(s-)\| = \sup_{s\in[0,t)} \|\omega(s)\|.$$

*To see this, note that for every $s \in [0,t)$ there exists a decreasing sequence $(s_n)_{n\in\mathbb{N}} \subset (s,t)$ such that*

$$\omega(s) = \lim_{n\to\infty} \omega(s_n-),$$

*which follows from the right-continuity of $\omega$ and the fact that càdlàg functions only have countably many discontinuities in any compact interval.*

**Condition 5.1.** (i) Local majoration property of $(b,c,K)$: *For all $m > 0$ it holds that*

$$\sup_{(t,\omega)\in\Theta_m} \Big(\|b_t(\omega)\| + \|c_t(\omega)\| + \int \big(1 \wedge \|x\|^2\big) K_t(\omega;dx)\Big) < \infty.$$

(ii) Skorokhod continuity property of $(b,c,K)$: *For all $\alpha \in \Omega$ each of the maps*

$$\omega \mapsto b_t(\omega), \widetilde{c}_t(\omega), \int g(x) K_t(\omega;dx), \quad g \in C_1(\mathbb{R}^d),$$

*is continuous at $\alpha$ for $dA_t$-a.a. $t \in \mathbb{R}_+$.*

(iii) Local uniform continuity property of $(b,c,K)$: *For all $t \in \mathbb{R}_+, g \in C_1(\mathbb{R}^d), i,j = 1,\dots,d$ and all Skorokhod compact sets $K \subset \Omega$ each $k \in \{b^i_t, \widetilde{c}^{ij}_t, \int g(x)K_t(dx)\}$ is uniformly continuous on $K$ equipped with the local uniform topology, i.e. for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for all $\omega, \alpha \in K$*

$$\sup_{s\in[0,t]} \|\omega(s) - \alpha(s)\| < \delta \quad \Rightarrow \quad |k(\omega) - k(\alpha)| < \varepsilon.$$

**Condition 5.2.** Big jump property of $K$: *For every $t \in \mathbb{R}_+$ there exists an $a > 0$ such that*

$$\sup_{s\in[0,t]} \sup_{\omega\in\Omega} K_s(\omega; \{x \in \mathbb{R}^d \colon \|x\| > a\}) < \infty, \quad (5.2)$$

*and*

$$\lim_{a\nearrow\infty}\sup_{\omega\in\Omega} K_t(\omega;\{x\in\mathbb{R}^d\colon \|x\|>a\})=0 \text{ for all } t\in\mathbb{R}_+, \tag{5.3}$$

**Condition 5.3.** Lyapunov condition I: *There exists a $\theta\in\mathbb{R}_+$ such that for all $a\in(\theta,\infty)$ there exist Borel functions $V_a\colon\mathbb{R}^d\to(0,\infty)$, $\gamma_a\colon\mathbb{R}_+\to\mathbb{R}_+$ and $\beta_a\colon\mathbb{R}_+\to\mathbb{R}_+$ with the following properties:*

(a) $V_a\in C^2(\mathbb{R}^d)$.

(b) $\int_0^t\gamma_a(s)dA_s<\infty$ *for all* $t\in\mathbb{R}_+$.

(c) $\beta_a$ *is increasing and* $\lim_{n\to\infty}\beta_a(n)=\infty$.

(d) *For all* $(t,\omega)\in\mathbb{R}_+\times\Omega$ *we have* $V_a(\omega(t))\geq\beta_a(\|\omega(t)\|)$ *and*

$$\int_0^t \mathbb{1}\{\gamma_a(s)V_a(\omega(s-))<(\mathcal{L}_aV)(\omega;s)\}dA_s=0.$$

**Condition 5.4.** Linear growth condition I: *There exists a $\theta\in\mathbb{R}_+$ such that for all $a\in(\theta,\infty)$ there exists a Borel function $\gamma_a\colon\mathbb{R}_+\to\mathbb{R}_+$ such that $\int_0^t\gamma_a(s)dA_s<\infty$ for all $t\in\mathbb{R}_+$ and for all $\omega\in\Omega$ and for $dA_t$-a.a. $t\in\mathbb{R}_+$*

$$\|b_t^a(\omega)\|^2+\|\widetilde{c}_t^a(\omega)\|\leq\gamma_a(t)\Big(1+\sup_{s\in[0,t]}\|\omega(s-)\|^2\Big). \tag{5.4}$$

The first main result of this chapter is the following:

**Theorem 5.1.** *Assume that the Conditions 5.1 and 5.2 hold and that one of the Conditions 5.3 and 5.4 holds. Then, there exists a probability measure $P$ on $(\Omega,\mathcal{F})$ such that the coordinate process $X$ is a $P$-semimartingale with local characteristics $(b,c,K;A)$ and initial law $\eta$.*

The theorem can be viewed as a generalization of [70, Theorem IX.2.31] in the sense that the uniform boundedness assumptions has been replaced by local boundedness assumptions and a Lyapunov-type condition or a linear growth condition. Recall that the function $A$ is assumed to be deterministic and continuous. The continuity of $A$ is not assumed in [70, Theorem IX.2.31]. It implies that any semimartingale with local characteristics $(b,c,K;A)$ is quasi-left continuous, see [70, Proposition II.2.9]. Theorem 5.1 is proven in Section 5.3 below.

We need the big jump condition on $K$ (Condition 5.2) to obtain the existence of our approximation sequence and to show its tightness. In fact, [70, Theorem VI.4.18] explains that a condition of this type is necessary for tightness of our approximation sequence. The big jump condition on $K$ can be replaced by a local big jump condition when the big jumps are also taken into consideration in the Lyapunov and the linear growth condition. To state this modification, we introduce some additional notation: For $f\in C^2(\mathbb{R}^d,\mathbb{R}),(t,\omega)\in$

$\mathbb{R}_+ \times \Omega$ and $x \in \mathbb{R}^d$ we set

$$(\mathcal{K}^d f)(\omega; t, x) \triangleq f(\omega(t-) + x) - f(\omega(t-)) - \sum_{k=1}^d \partial_k f(\omega(t-)) h^k(x),$$
$$(\mathcal{K}^l f)(\omega; t) \triangleq \sum_{k=1}^d \partial_k f(\omega(t-)) b_t^k(\omega) + \frac{1}{2} \sum_{k,j=1}^d \partial^2_{kj} f(\omega(t-)) c_t^{kj}(\omega),$$

and

$$(\mathcal{L} f)(\omega; t) \triangleq (\mathcal{K}^l f)(\omega; t) + \int (\mathcal{K}^d f)(\omega; t, x) K_t(\omega; dx),$$

provided the last term is well-defined. Furthermore, for $m > 0$ and $t \in [0, m]$ we set

$$\Theta^t_m \triangleq \big\{\omega \in \Omega \colon (t, \omega) \in \Theta_m\big\}.$$

**Condition 5.5.** Local big jump property of $K$: *For all $m > 0$ and $t \in [0, m]$*

$$\lim_{a \nearrow \infty} \sup_{\omega \in \Theta^t_m} K_t(\omega; \{x \in \mathbb{R}^d \colon \|x\| > a\}) = 0.$$

**Condition 5.6.** Lyapunov condition II: *There exist Borel functions $V \colon \mathbb{R}^d \to (0, \infty)$, $\gamma \colon \mathbb{R}_+ \to \mathbb{R}_+$ and $\beta \colon \mathbb{R}_+ \to \mathbb{R}_+$ with the following properties:*

(a) *$V \in C^2(\mathbb{R}^d)$.*

(b) *$\int_0^t \gamma(s) dA_s < \infty$ for all $t \in \mathbb{R}_+$.*

(c) *$\beta$ is increasing and $\lim_{n \to \infty} \beta(n) = \infty$.*

(d) *For all $(t, \omega) \in \mathbb{R}_+ \times \Omega$ we have $V(\omega(t)) \geq \beta(\|\omega(t)\|)$,*

$$\int_0^t \int \big|(\mathcal{K}^d V)(\omega, s, x)\big| K_s(\omega; dx) dA_s < \infty, \tag{5.5}$$

*and*

$$\int_0^t \mathbb{1}\{\gamma(s) V(\omega(s-)) < (\mathcal{L} V)(\omega; s)\} dA_s = 0.$$

**Condition 5.7.** Linear growth condition II: *There exits a Borel function $\gamma \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^t \gamma(s) dA_s < \infty$ for all $t \in \mathbb{R}_+$ and for all $\omega \in \Omega$ and for $dA_t$-a.a. $t \in \mathbb{R}_+$*

$$\|\widetilde{b}_t(\omega)\|^2 + \|\widetilde{c}_t(\omega)\| + \int \|h'(x)\|^2 K_t(\omega; dx) \leq \gamma(t) \Big(1 + \sup_{s \in [0, t]} \|\omega(s-)\|^2\Big), \tag{5.6}$$

*where $h'(x) \triangleq x - h(x)$ and*

$$\widetilde{b}_t(\omega) \triangleq b_t(\omega) + \int h'(x) K_t(\omega; dx).$$

Our second main result is the following:

**Theorem 5.2.** *Suppose that the Conditions 5.1 and 5.5 hold and that one of the Conditions 5.6 and 5.7 holds. Then, there exists a probability measure $P$ on $(\Omega, \mathcal{F})$ such that the coordinate process $X$ is a $P$-semimartingale with local characteristics $(b, c, K; A)$ and initial law $\eta$.*

Theorem 5.2 is also proven in Section 5.3 below.

**Remark 5.2.** *In Condition 5.1* (i) *one can replace $\Theta_m$ by*

$$\Theta^*_m \triangleq \Big\{(t, \omega) \in [0, m] \times \Omega \colon \sup_{s \in [0,t]} \|\omega(s)\| \leq m\Big\} \subset \Theta_m$$

*and in Condition 5.5 one can replace $\Theta^t_m$ by $\{\omega \in \Omega \colon (t, \omega) \in \Theta^*_m\} \subset \Theta^t_m$. Furthermore, in* (5.4) *and* (5.6) *one can replace $\sup_{s \in [0,t]} \|\omega(s-)\|$ by $\sup_{s \in [0,t]} \|\omega(s)\|$. This follows from part (d) of [70, Lemma III.2.43], which states that for a predictable process $H$ and all $t > 0$ and $\omega, \alpha \in \Omega$*

$$\omega(s) = \alpha(s) \text{ for all } s < t \quad \Rightarrow \quad H_t(\omega) = H_t(\alpha).$$

Due to this observation, we expect part (i) of Condition 5.1 to be close to optimal for a local boundedness condition. We give some examples for functions having the Skorokhod continuity property and the local uniform continuity property:

**Example 5.1.** *Let $g \colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ be a Borel function such that $x \mapsto g(t, x)$ is continuous for all $t \in \mathbb{R}_+$. Furthermore, fix $t > 0$.*

*(a) The map $\omega \mapsto g(t, \omega(t-))$ is continuous at each $\alpha \in \Omega$ such that $t \notin J(\alpha) \triangleq \{s > 0 \colon \alpha(s) \neq \alpha(s-)\}$, see [70, VI.2.3]. Recalling that $A$ is deterministic and continuous and that any càdlàg function has at most countably many discontinuities, we see that the set $J(\alpha)$ is a $dA_t$-null set and, consequently, that the Skorokhod continuity property holds. Furthermore, the local uniform continuity property holds. To see this, note that for each compact set $K \subset \Omega$ there exists a compact set $K_t \subset \mathbb{R}^d$ such that $\omega(s) \in K_t$ for all $\omega \in K$ and $s \in [0, t]$, see [47, Problem 16, p. 152]. Using that continuous functions on compact sets are uniformly continuous, for each $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that*

$$x, y \in K_t \colon \|x - y\| < \delta \quad \Rightarrow \quad |g(t, x) - g(t, y)| < \varepsilon.$$

*Now, if $\omega, \alpha \in K$ are such that $\sup_{s \in [0,t]} \|\omega(s) - \alpha(s)\| < \delta$ we have $\omega(t-), \alpha(t-) \in K_t$, because $K_t$ is closed, and $\|\omega(t-) - \alpha(t-)\| \leq \sup_{s \in [0,t]} \|\omega(s) - \alpha(s)\| < \delta$. Consequently, we have*

$$|g(t, \omega(t-)) - g(t, \alpha(t-))| < \varepsilon.$$

*This shows that the local uniform continuity property holds.*

*(b) If $g$ is continuous, the map $\omega \mapsto \int_0^t g(s, \omega(s-)) dA_s$ is continuous. This follows from the fact that $\omega \mapsto g(s, \omega(s-))$ is continuous at each $\alpha \in \Omega$ such that $s \notin J(\alpha)$, the dominated convergence theorem and the fact that $J(\alpha)$ is a $dA_t$-null set. Furthermore, the map $\omega \mapsto \int_0^t g(s, \omega(s-)) dA_s$ has the local uniform continuity property. To see this, let $K \subset \Omega$ and $K_t \subset \mathbb{R}^d$ be as in part (a) and fix $\varepsilon > 0$. Without loss of*

*generality we assume that* $A_t > 0$. *Since* $g$ *is uniformly continuous on* $[0,t] \times K_t$ *we find a* $\delta = \delta(\varepsilon) > 0$ *such that*

$$x, y \in K_t\colon \|x - y\| < \delta \quad \Rightarrow \quad |g(s,x) - g(s,y)| < \frac{\varepsilon}{2A_t}$$

*for all* $s \in [0,t]$. *Now, for all* $\omega, \alpha \in K$ *such that* $\sup_{s\in[0,t]} \|\omega(s) - \alpha(s)\| < \delta$ *we have*

$$\Big| \int_0^t g(s,\omega(s-))dA_s - \int_0^t g(r,\alpha(r-))dA_r \Big| \\ \leq \int_0^t \Big| g(s,\omega(s-)) - g(s,\alpha(s-)) \Big| dA_s < \varepsilon,$$

*which gives the local uniform continuity property.*

(c) *If* $g$ *is continuous, the map* $\omega \mapsto \sup_{s\in[0,t]} g(s,\omega(s-))$ *is continuous at each* $\alpha \in \Omega$ *such that* $t \notin J(\alpha)$. *This can be seen with the arguments used in the proof of Lemma 5.2 below. Furthermore, the local uniform continuity property holds, which follows with the argument from part (b) and the inequality*

$$\Big| \sup_{s\in[0,t]} g(s,\omega(s-)) - \sup_{r\in[0,t]} g(r,\alpha(r-)) \Big| \leq \sup_{s\in[0,t]} \Big| g(s,\omega(s-)) - g(s,\alpha(s-)) \Big|.$$

We now comment on the big jump property and the local big jump property.

**Example 5.2.** (a) *If* $K_t(\omega; dx) = F(dx)$ *for a Lévy measure* $F$, *then the big jump property of* $K$ *(Condition 5.2) holds, because*

$$F(\{x \in \mathbb{R}^d\colon \|x\| > a\}) \to 0 \text{ with } a \nearrow \infty.$$

*However, Condition 5.7 can fail, because* $\|h'\|$ *might not be* $F$*-integrable, i.e.* $F$ *corresponds to a Lévy process with infinite mean.*

(b) *When we consider a one-dimensional SDE of the type*

$$dX_t = g_t(X)dL_t,$$

*where* $g$ *is predictable and* $L$ *is a Lévy process, then* $\Delta X_t = g_t(X)\Delta L_t$ *and, consequently, we consider*

$$K_t(G) = \int \mathbb{1}_{G\setminus\{0\}}(g_t(X)y)F(dy), \quad G \in \mathcal{B}(\mathbb{R}),$$

*where* $F$ *is the Lévy measure corresponding to* $L$. *In this case, we obtain*

$$K_t(\{x \in \mathbb{R}\colon |x| > a\}) = F(\{y \in \mathbb{R}\colon |y||g_t(X)| > a\}).$$

*If for* $m \in \mathbb{N}$ *there is a constant* $c_m > 0$ *such that* $\sup_{(t,\omega)\in[0,m]\times\Omega} |g_t(\omega)| \leq c_m$, *then we have*

$$\sup_{t\in[0,m]} \sup_{\omega\in\Omega} F\big(\{y \in \mathbb{R}\colon |y||g_t(X(\omega))| > a\}\big) \leq F\big(\big\{y \in \mathbb{R}\colon |y| > \tfrac{a}{c_m}\big\}\big) \to 0$$

*with $a \nearrow \infty$. However, if $g$ is unbounded, the global big jump property of $K$ (Condition 5.2) might fail, while the local big jump property of $K$ (Condition 5.5) and Condition 5.7 might hold.*

(c) *For a jump-diffusion setting we discuss the local big jump property in Section 5.2.3 below.*

Next, we provide examples to understand the Lyapunov-type conditions.

**Example 5.3.** (a) *For $V(x) \triangleq 1 + \|x\|^2$ the Lyapunov-type Conditions 5.3 and 5.6 correspond to a linear growth condition. For example, if there exists a Borel function $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $t \in \mathbb{R}_+$ we have $\int_0^t \gamma(s) dA_s < \infty$ and*

$$\begin{aligned}\int_{\|x\|\leq a} \big(\|X_{t-} + x\|^2 - \|X_{t-}\|^2 - 2\langle X_{t-}, x\rangle\big) K_t(dx) \\ + 2\langle X_{t-}, b^a_t\rangle + \|c_t\| \leq \gamma(t)\big(1 + \|X_{t-}\|^2\big),\end{aligned} \tag{5.7}$$

*then Condition 5.3 is satisfied. This linear growth condition is different from Condition 5.4. On one hand, the growth condition (5.7) allows an interplay of the coefficients. For example, if $d = 1$ and $b_t \equiv -X^3_{t-}, c_t \equiv 2X^4_{t-}, K \equiv 0$, then*

$$2\langle X_{t-}, b_t\rangle + \|c_t\| = -2X^4_{t-} + 2X^4_{t-} = 0 \leq 1 + X^2_{t-},$$

*although $|b_t|$ and $|c_t|$ are not of linear growth. On the other hand, Condition 5.4 takes the whole history of the paths into consideration.*

(b) *Let us consider the case $d = 1$ where $b \equiv K \equiv 0$, i.e. we are looking for a probability measure $P$ on $(\Omega, \mathcal{F})$ such that the coordinate process is a one-dimensional continuous local $P$-martingale with quadratic variation process $\int_0^\cdot c_s dA_s$. Suppose there exists an $a > 1$ and a constant $\zeta < \infty$ such that for all $(t, \omega) \in \mathbb{R}_+ \times \Omega\colon |\omega(t-)| < a$ we have $c_t(\omega) \leq \zeta$. Then, the Lyapunov-type Conditions 5.3 and 5.6 hold with $\gamma(t) \triangleq \frac{a^2\zeta}{\log(a^2)}$ and $V(x) \triangleq \log(a^2 + |x|^2)$. To see this, note that*

$$\begin{aligned}\gamma(t)V(X_t) - (\mathcal{L}V)(t) &= \frac{a^2\zeta}{\log(a^2)} V(X_t) + \Big(\frac{|X_{t-}|^2 - a^2}{(a^2 + |X_{t-}|^2)^2}\Big) c_t \\ &\geq a^2\big(\zeta - c_t \mathbb{1}\{|X_{t-}| < a\}\big) \geq 0.\end{aligned}$$

*In particular, the Conditions 5.3 and 5.6 hold when $c_s(\omega) = \overline{c}(\omega(s-))\iota_s(\omega)$ for a locally bounded function $\overline{c}\colon \mathbb{R} \to \mathbb{R}_+$ and a bounded process $\iota$. This observation can be seen as a generalization of the well-known result that one-dimensional SDEs of the type*

$$dX_t = \sqrt{\overline{c}(X_t)}\; dW_t$$

*have non-exploding weak solutions whenever the coefficient $\overline{c}\colon \mathbb{R} \to \mathbb{R}_+$ is continuous.*

**Remark 5.3.** *As already indicated in Example 5.2, if we have*

$$K_t(\omega; G) = \int \mathbb{1}_{G\setminus\{0\}}(v(t, \omega, y)) F(dy), \quad G \in \mathcal{B}(\mathbb{R}^d),$$

*where $v$ is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable and $F$ is a Lévy measure on $\mathbb{R}^d$, then $(b, c, K)$ corresponds to an SDE driven by Lévy noise, see [70, Theorem III.2.26]. Here, $\mathcal{P}$ denotes the predictable $\sigma$-field. In this case, Condition 5.7 is in the spirit of the linear growth conditions from [65, Theorems 14.23, 14.95] and [70, Theorem III.2.32], which are stated together with local Lipschitz conditions. In particular, Condition 5.7 holds under the following linear growth condition:* There exist two Borel functions $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$ and $\theta\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ such that for all $(t, \omega, y) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^d$ we have $\int_0^t (\gamma(s) + \int |\theta(s, x)|^2 F(dx))dA_s < \infty$ and

$$\|b_t(\omega)\|^2 + \|\widetilde{c}_t(\omega)\| \le \gamma(t)\Big(1 + \sup_{s\in[0,t]} \|\omega(s)\|^2\Big),$$

$$\|h'(v(t, \omega, y))\| \le \Big(\theta(t, y) \wedge |\theta(t, y)|^2\Big)\Big(1 + \sup_{s\in[0,t]} \|\omega(s)\|^2\Big)^{\frac{1}{2}}.$$

*Local Lipschitz conditions imply the existence of a local solution. We do not work with a local solution, but construct a solution by approximation. The local Lipschitz conditions also imply uniqueness, which is a property not provided by the approximation argument. Uniform boundedness and continuity conditions for the existence of weak solutions to SDEs driven by semimartingales were proven by Jacod and Mémin [69] and Lebedev [92]. Lebedev [93] also proved Lyapunov-type conditions.*

As already indicated in the previous remark, Lyapunov-type and linear growth conditions for the existence of weak solutions to SDEs are sometimes combined with conditions implying the existence of a local solution. Next, we explain the method used by Stroock and Varadhan [137] to construct a global solution from a local solution and discuss some differences between arguments based on extension and approximation.

The following proposition is a version of Tulcea's extension theorem, which follows from [137, Theorem 1.1.9] in the same manner as its continuous analogous [137, Theorem 1.3.5] does.

**Proposition 5.2.** *Let $(\tau_n)_{n\in\mathbb{N}}$ be an increasing sequence of $(\mathcal{F}^o_t)_{t\ge 0}$-stopping times and let $(P^n)_{n\in\mathbb{N}}$ be a sequence of probability measures on $(\Omega, \mathcal{F})$ such that $P^n = P^{n+1}$ on $\mathcal{F}^o_{\tau_n}$ for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} P^n(\tau_n \le t) = 0$ for all $t \in \mathbb{R}_+$, then there exists a unique probability measure $P$ on $(\Omega, \mathcal{F})$ such that $P = P_n$ on $\mathcal{F}^o_{\tau_n}$ for all $n \in \mathbb{N}$.*

Supposing that $(P^n)_{n\in\mathbb{N}}$ is a local solution, the consistency assumption shows that the extension, provided it exists, is a global solution.

Stroock and Varadhan [137] construct a consistent sequence as in Proposition 5.2 under a uniqueness condition. In general semimartingale cases, the consistency holds when the sequence $(P^n)_{n\in\mathbb{N}}$ has a local uniqueness property as define in [70, Definition III.2.37]. Local uniqueness is a strong concept of uniqueness, which in particular implies (global) uniqueness. In Markovian settings, such as the diffusion setting of Stroock and Varadhan, local uniqueness is implied by the existence of (globally) unique solutions for all degenerated initial laws, see [70, Theorem III.2.40].[1] In more general cases, however, local uniqueness is considered to be difficult to show, see the comment in the beginning of [70, Section III.2d.2]. In our opinion, using local uniqueness is a natural approach to verify the consistency hypothesis. The approximation argument requires no uniqueness condition. However, it also provides no uniqueness statement.

A version of the convergence criterion $\lim_{n\to\infty} P^n(\tau_n \le t) = 0$ from Proposition 5.2 is also verified in the tightness argument as presented in Section 5.3.2 below. This is a

[1]The assumed kernel property in [70, Theorem III.2.40] is typically implied by the uniqueness assumption.

similarity between the extension and the approximation argument and illustrates that both are soul mates in the point that they prevent a loss of mass.

In some cases it is possible to construct a consistent sequence as in Proposition 5.2 without a uniqueness assumption. An example for such a case arises from a local change of measure. Suppose that $Q$ is a probability measure and that $Z$ is a non-negative normalized local $Q$-martingale with localizing sequence $(\tau_n)_{n\in\mathbb{N}}$. We define a sequence $(P^n)_{n\in\mathbb{N}}$ by $P^n(G) = E^Q[Z_{\tau_n}\mathbb{1}_G]$ for all $G \in \mathcal{F}$. The consistency follows from the martingale property of $Z_{\cdot\wedge\tau_n}$ via the optional stopping theorem. Consequently, the existence of an extension $P$ of $(P^n)_{n\in\mathbb{N}}$ follows from Proposition 5.2 if

$$1 = \lim_{n\to\infty} P^n(\tau_n > t) = \lim_{n\to\infty} E^Q[Z_{\tau_n}\mathbb{1}\{\tau_n > t\}] = E^Q[Z_t], \quad t \in \mathbb{R}_+,$$

which is equivalent to the $Q$-martingale property of $Z$.

Of course, in case one knows a priori that $Z$ is a $Q$-martingale one could simply apply Proposition 5.2 to the deterministic sequence $\tau_n = n$. The previous discussion explains that the $Q$-martingale property of $Z$ is naturally connected to Proposition 5.2.

The extension $P$ is locally absolutely continuous with respect to $Q$, because for all $G \in \mathcal{F}^o_t$ we have $G \cap \{\tau_n > t\} \in \mathcal{F}^o_{\tau_n}$ and thus

$$Q(G) = 0 \quad \Rightarrow \quad P(G) = \lim_{n\to\infty} P(G \cap \{\tau_n > t\}) = \lim_{n\to\infty} P^n(G \cap \{\tau_n > t\}) = 0.$$

Consequently, if $X$ is a $Q$-semimartingale, it is also a $P$-semimartingale due to [70, Theorem III.3.13]. This argument does not require any form of uniqueness. However, it requires that there exists a probability measure $Q$ for which the coordinate process is a semimartingale. Furthermore, the structure of the local characteristics under $P$ is determined by $Q$ and $Z$ via Girsanov's theorem (see [70, Theorem III.3.24]). Nevertheless, we think that this method provides a possibility to relax the assumptions in the Theorems 5.1 and 5.2. Namely, one can apply one of our main results to obtain the probability measure $Q$ and then deduce the existence of a probability measure $P$ corresponding to local characteristics which need not to satisfy the continuity conditions formulated in Condition 5.1.

### 5.2.3 Application: Existence Conditions for Jump-Diffusions

In this subsection we discuss the classical jump-diffusion case as an important example. Let $\overline{b}\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\overline{c}\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{S}^d$ be Borel functions. Furthermore, let $\overline{K}_t(x, dy)$ be a Borel transition kernel from $\mathbb{R}_+ \times \mathbb{R}^d$ into $\mathbb{R}^d$. Set for all $t \in \mathbb{R}_+$

$$b_t \triangleq \overline{b}(t, X_{t-}), \quad c_t \triangleq \overline{c}(t, X_{t-}), \quad K_t(dx) \triangleq \overline{K}_t(X_{t-}, dx).$$

We assume that for all $t \in \mathbb{R}_+$ and $g \in C_1(\mathbb{R}^d)$ the maps

$$x \mapsto \overline{b}(t, x), \overline{c}^{ij}(t, x) + \int h^i(y)h^j(y)\overline{K}_t(x, dy), \int g(y)\overline{K}_t(x, dy) \tag{5.8}$$

are continuous. Then, the Skorokhod continuity property and the local uniform continuity property hold, see part (a) of Example 5.1. Furthermore, we assume that the maps

$$(t, x) \mapsto \overline{b}(t, x), \overline{c}(t, x), \int \big(1 \wedge \|y\|^2\big)\overline{K}_t(x, dy)$$

are locally bounded. Then, the local majoration property holds.

The assumption that the maps (5.8) are continuous implies the local big jump condition (Condition 5.5):

**Lemma 5.1.** *Condition 5.5 holds.*

*Proof.* This follows from [44, Proposition 5.33], [113, Theorems 4.5.6, 4.5.7] and [129, Theorem 4.4]. □

Due to Lemma 5.1 the following corollary follows from Theorems 5.1 and 5.2.

**Corollary 5.1.** *In addition to the assumptions above, suppose that one of the following two conditions holds:*

(i) *For all $t \in \mathbb{R}_+$ there exists an $a > 0$ such that*

$$\sup_{s\in[0,t]} \sup_{x\in\mathbb{R}^d} \overline{K}_s(x, \{y \in \mathbb{R}^d : \|y\| > a\}) < \infty,$$

*for all $t \in \mathbb{R}_+$*

$$\lim_{a\nearrow\infty} \sup_{x\in\mathbb{R}^d} \overline{K}_t(x, \{y \in \mathbb{R}^d : \|y\| > a\}) = 0$$

*and there exists a Borel function $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$ we have $\int_0^t \gamma(s)dA_s < \infty$ and*

$$\|\overline{b}(t,x)\|^2 + \|\overline{c}(t,x)\| + \int \big(1 \wedge \|y\|^2\big)\overline{K}_t(x,dy) \le \gamma(t)\big(1 + \|x\|^2\big).$$

(ii) *There exits a Borel function $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$ we have $\int_0^t \gamma(s)dA_s < \infty$ and*

$$\|\widetilde{b}(t,x)\|^2 + \|\widetilde{c}(t,x)\| + \int \|h'(y)\|^2\overline{K}_t(x,dy) \le \gamma(t)\big(1 + \|x\|^2\big),$$

*where $h'(y) \triangleq y - h(y)$ and*

$$\widetilde{b}(t,x) \triangleq \overline{b}(t,x) + \int h'(y)\overline{K}_t(x,dy),$$
$$\widetilde{c}^{ij}(t,x) \triangleq \overline{c}^{ij}(t,x) + \int h^i(y)h^j(y)\overline{K}_t(x,dy).$$

*Then, there exists a probability measure $P$ on $(\Omega, \mathcal{F})$ such that the coordinate process $X$ is a $P$-semimartingale with local characteristics $(b, c, K; A)$ and initial law $\eta$.*

Theorems 5.1 and 5.2 also give Lyapunov-type existence criteria. We leave the statements to the reader. Corollary 5.1 can be viewed as a generalization of [70, Corollary IX.2.33] and a time-inhomogeneous version of [44, Theorem 5.36].

## 5.3 Proof of Theorem 5.1 and Theorem 5.2

In view of Proposition 5.1 it suffices to show the claim for all degenerated initial laws, i.e. we assume that $\eta = \delta_z$, where $z \in \mathbb{R}^d$ is chosen arbitrary. Here $\delta$ denotes the Dirac measure. The proof is split into three steps: First, we construct a sequence of probability measures, see Section 5.3.1. Second, we show that the sequence is tight, see Section 5.3.2. This step requires different arguments under the assumptions of Theorem 5.1 and Theorem 5.2. Third, we use a martingale problem argument to identify any accumulation point of the sequence as a probability measure under which the coordinate process is a semimartingale with local characteristics $(b, c, K; A)$ and initial law $\delta_z$, see Section 5.3.3.

In general, we assume that the Conditions 5.1 and 5.5 hold. In case we impose additional assumptions in one of the following sections we indicate these in the beginning.

### 5.3.1 The Approximation Sequence $(P^n)_{n\in\mathbb{N}}$

Let $\phi^n\colon \mathbb{R} \to [0,1]$ be a sequence of cutoff functions, i.e. $\phi^n \in C_c^\infty(\mathbb{R})$ with $\phi^n(x) = 1$ for $x \in [-n, n]$ and $\phi^n(x) = 0$ for $|x| \geq n+1$. We define $X^*_t \triangleq \sup_{s\in[0,t]} \|X_{s-}\|$ for $t \in \mathbb{R}_+$ and note that $X^*$ is a predictable process, because it is left continuous and adapted. Set

$$\begin{aligned} b^n_t &\triangleq \phi^n(X^*_t)\mathbb{1}\{t \leq n+1\} b_t, \\ c^n_t &\triangleq \phi^n(X^*_t)\mathbb{1}\{t \leq n+1\} c_t, \\ K^n_t(dy) &\triangleq \phi^n(X^*_t)\mathbb{1}\{t \leq n+1\} K_t(dy). \end{aligned}$$

It is clear that $(b^n, c^n, K^n)$ is a candidate triplet. Fix $n \in \mathbb{N}$. Our goal is to apply [70, Theorem IX.2.31] to conclude that there exists a probability measure $P^n$ such that the coordinate process is a $P^n$-semimartingale with local characteristics $(b^n, c^n, K^n; A)$ and initial law $\delta_z$. We proceed by checking the prerequisites of [70, Theorem IX.2.31].

By the local majoration property of the candidate triplet $(b, c, K)$ (Condition 5.1 (i)) the modified triplet $(b^n, c^n, K^n)$ has the following global majoration property:

$$\begin{aligned} &\sup_{t\in\mathbb{R}_+}\sup_{\omega\in\Omega}\Big(\|b^n_t(\omega)\| + \|c^n_t(\omega)\| + \int \big(1 \wedge \|x\|^2\big) K^n_t(\omega; dx)\Big) \\ &\qquad\leq \sup_{(t,\omega)\in\Theta_{n+1}} \Big(\|b_t(\omega)\| + \|c_t(\omega)\| + \int \big(1 \wedge \|x\|^2\big) K_t(\omega; dx)\Big) < \infty. \end{aligned}$$

Furthermore, the triplet $(b^n, c^n, K^n)$ has the following modified Skorokhod continuity property: For all $t \in \mathbb{R}_+$ and $g \in C_1(\mathbb{R}^d)$ the maps

$$\omega \mapsto \int_0^t b^n_s(\omega) dA_s, \int_0^t \widetilde{c}^n_s(\omega) dA_s, \int_0^t \int g(x) K^n_s(\omega; dx) dA_s$$

are continuous for the Skorokhod topology. To see this, we first note the following:

**Lemma 5.2.** *The map $\omega \mapsto \phi^n(X^*_t(\omega))$ is continuous at $\alpha \in \Omega$ for all $t \notin J(\alpha) = \{s > 0\colon \alpha(s) \neq \alpha(s-)\}$.*

*Proof.* Let $(\alpha_n)_{n\in\mathbb{N}} \subset \Omega$ such that $\alpha_n \to \alpha$ as $n \to \infty$. By [70, Theorem VI.1.14] there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of strictly increasing continuous functions $\mathbb{R}_+ \to \mathbb{R}_+$ such that $\lambda_n(0) = 0, \lambda_n(t) \nearrow \infty$ as $t \to \infty$ and for all $N \in \mathbb{N}$

$$\sup_{s\in\mathbb{R}_+} |\lambda_n(s) - s| + \sup_{s\in[0,N]} \|\alpha_n(\lambda_n(s)) - \alpha(s)\| \to 0 \tag{5.9}$$

as $n \to \infty$. Now, we have

$$\left| X_t^*(\alpha_n) - X_{\lambda_n^{-1}(t)}^*(\alpha) \right| \leq \sup_{s \in [0, \lambda_n^{-1}(t)]} \|\alpha_n(\lambda_n(s)) - \alpha(s)\| \to 0$$

as $n \to \infty$ by (5.9). In case $t \notin J(\alpha)$, (5.9) also yields that

$$\left| X_{\lambda_n^{-1}(t)}^*(\alpha) - X_t^*(\alpha) \right| \to 0 \text{ as } n \to \infty.$$

Thus, $\omega \mapsto X_t^*(\omega)$ is continuous at $\alpha$ for all $t \notin J(\alpha)$. As $\phi^n$ is continuous, this implies the claim. □

Since càdlàg functions have at most countably many discontinuities, for each $\alpha \in \Omega$ the set $J(\alpha)$ is at most countable. Thus, because the function $t \mapsto A_t$ is assumed to be continuous, the set $J(\alpha)$ is a $dA_t$-null set. Now, the modified Skorokhod continuity property of $(b^n, c^n, K^n)$ follows from the Skorokhod continuity property of $(b, c, K)$ (Condition 5.1 (ii)) and the dominated convergence theorem.

Finally, we also note that the modified triplet $(b^n, c^n, K^n)$ has the following modified local uniform continuity property:

**Lemma 5.3.** *For all $t \in \mathbb{R}_+, g \in C_1(\mathbb{R}^d), i, j = 1, \dots, d$ and all compact sets $K \subset \Omega$ any $k \in \{\omega \mapsto b_t^{n,i}(\omega), \widetilde{c}_t^{n,ij}(\omega), \int g(x) K_t^n(\omega; dx)\}$ has the uniform continuity property that for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for all $\omega, \alpha \in K$*

$$\sup_{s \in [0,t]} \|\omega(s) - \alpha(s)\| < \delta \quad \Rightarrow \quad |k(\omega) - k(\alpha)| < \varepsilon.$$

*Proof.* By the local uniform continuity property of $(b, c, K)$ (Condition 5.1 (iii)) it suffices to consider $k(\omega) = \phi^n(X_t^*(\omega))g(\omega)$, where $g$ already has the uniform continuity property and $|g|$ is bounded by a constant $\|g\|_\infty > 0$. We fix $\varepsilon > 0$. There exists a $\delta^* = \delta^*(\varepsilon) > 0$ such that for all $\omega, \alpha \in K$

$$\sup_{s \in [0,t]} \|\omega(s) - \alpha(s)\| < \delta^* \quad \Rightarrow \quad |g(\omega) - g(\alpha)| < \tfrac{\varepsilon}{2}.$$

Since smooth functions with compact support are Lipschitz continuous, there exists a constant $L > 0$ such that

$$|\phi^n(X_t^*(\omega)) - \phi^n(X_t^*(\alpha))| \leq L \sup_{s \in [0,t]} \|\omega(s) - \alpha(s)\|.$$

Now, choose $\delta \triangleq \min(\delta^*, \varepsilon(2L\|g\|_\infty)^{-1})$. Then, for $\omega, \alpha \in K$: $\sup_{s \in [0,t]} \|\omega(s) - \alpha(s)\| < \delta$

$$|k(\omega) - k(\alpha)| \leq \|g\|_\infty |\phi^n(X_t^*(\omega)) - \phi^n(X_t^*(\alpha))| + |g(\omega) - g(\alpha)| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

We conclude that $k$ has the uniform continuity property. □

Finally, we note that for all $t \in \mathbb{R}_+$

$$\begin{aligned} &\lim_{a \nearrow \infty} \sup_{\omega \in \Omega} K_t^n(\omega; \{x \in \mathbb{R}^d \colon \|x\| > a\}) \\ &\qquad \leq \lim_{a \nearrow \infty} \sup_{\omega \in \Theta_{n+1}^{t \wedge (n+1)}} K_{t \wedge (n+1)}(\omega; \{x \in \mathbb{R}^d \colon \|x\| > a\}) = 0, \end{aligned}$$

by the local big jump property of $K$ (Condition 5.5). In summary, we conclude that the prerequisites of [70, Theorem IX.2.31] are fulfilled. Consequently, there exists a probability measure $P^n$ such that the coordinate process $X$ is a $P^n$-semimartingale with local characteristics $(b^n, c^n, K^n; A)$ and initial law $\delta_z$.

### 5.3.2 Tightness of $(P^n)_{n\in\mathbb{N}}$

For $m > 0$ we define the stopping time

$$\rho_m \triangleq \inf\big(t \in \mathbb{R}_+ \colon \|X_t\| > m\big) \wedge m.$$

For $m > 0$ and $n \in \mathbb{N}$ we define $P^{n,m}$ to be the law of the stopped process $X_{\cdot\wedge\rho_m}$ under $P^n$. Our strategy is first to show tightness for $(P^{n,m})_{n\in\mathbb{N}}$ and then to deduce the tightness of $(P^n)_{n\in\mathbb{N}}$ with the help of the Lyapunov and linear growth conditions.

#### 5.3.2.1 Tightness of $(P^{n,m})_{n\in\mathbb{N}}$.

Let $(b^{n,m}, c^{n,m}, K^{n,m}; A)$ be the local characteristics of $X_{\cdot\wedge\rho_m}$ under $P_n$. Due to [75, Lemma 2.3], we have

$$b^{n,m} = \mathbb{1}_{[\![0,\rho_m]\!]} b^n, \quad c^{n,m} = \mathbb{1}_{[\![0,\rho_m]\!]} c^n, \quad K^{n,m}(dx) = \mathbb{1}_{[\![0,\rho_m]\!]} K^n(dx),$$

where $[\![0,\rho_m]\!] \triangleq \{(t,\omega) \in \mathbb{R}_+ \times \Omega \colon 0 \le t \le \rho_m(\omega)\}$. The tightness of $(P^{n,m})_{n\in\mathbb{N}}$ follows from [70, Theorem VI.5.10] once we show the following four conditions:

(i) The sequence $(P^{n,m} \circ X_0^{-1})_{n\in\mathbb{N}}$ is tight.

(ii) For all $t, \varepsilon > 0$ we have

$$\lim_{a\nearrow\infty} \limsup_{n\to\infty} P^{n,m}\Big(\int_0^t K_s^{n,m}(\{x \in \mathbb{R}^d \colon \|x\| > a\}) dA_s > \varepsilon\Big) = 0.$$

(iii) The sequence $(P^{n,m} \circ (\int_0^\cdot b_s^{n,m} dA_s)^{-1})_{n\in\mathbb{N}}$ is tight.

(iv) For all $p \in \mathbb{N}$ there exists a deterministic increasing process $G^p$ such that

$$G^p - \int_0^\cdot \Big(\sum_{i=1}^d c_t^{n,m,ii} + \int \Big(\sum_{i=1}^d |h^i(x)|^2 + (p\|x\| - 1)^+ \wedge 1\Big) K_t^{n,m}(dx)\Big) dA_t$$

is an increasing process for all $n \in \mathbb{N}$.

As $P^{n,m} \circ X_0^{-1} = \delta_z$ for all $n, m \in \mathbb{N}$, (i) is trivially satisfied. Due to [136, Fact 2.9, Theorem 2.17] the map

$$[0,m] \ni t \mapsto \sup_{\omega\in\Theta_m^t} K_t(\omega; \{x \in \mathbb{R}^d \colon \|x\| > a\})$$

is universally measurable (see [136, Definition 2.8]). Thus, the integral

$$\int_0^m \sup_{\omega\in\Theta_m^t} K_t(\omega; \{x \in \mathbb{R}^d \colon \|x\| > a\}) dA_t$$

is well-defined. Moreover, we have for all $a \geq 1$ and $t \in [0, m]$

$$\sup_{\omega \in \Theta_m^t} K_t(\omega; \{x \in \mathbb{R}^d \colon \|x\| > a\}) \leq \sup_{(s,\omega) \in \Theta_m} \int \big(1 \wedge \|x\|^2\big) K_s(\omega; dx) < \infty,$$

by local majoration property (Condition 5.1 (i)). Consequently, we deduce from Chebyshev's inequality, the local big jump property of $K$ (Condition 5.2) and the dominated convergence theorem that for all $t, \varepsilon > 0$

$$\begin{aligned}\limsup_{n \to \infty} P^{n,m}\Big(\int_0^t K_s^{n,m}(\{x \in \mathbb{R}^d \colon \|x\| > a\}) dA_s > \varepsilon\Big) \\ \leq \frac{1}{\varepsilon} \int_0^m \sup_{\omega \in \Theta_m^s} K_s(\omega; \{x \in \mathbb{R}^d \colon \|x\| > a\}) dA_s \to 0 \text{ with } a \nearrow \infty.\end{aligned}$$

We conclude that (ii) holds. We set

$$\gamma^i \triangleq \sup_{(s,\omega) \in \Theta_m} |b_s^i(\omega)|, \quad i = 1, \dots, d.$$

The local majoration property (Condition 5.1 (i)) implies that $\gamma^i < \infty$ for all $i = 1, \dots, d$. Denote by $\operatorname{Var}(\cdot)$ the variation process. It is easy to see that the process

$$\sum_{i=1}^d \gamma^i A - \sum_{i=1}^d \operatorname{Var}\Big(\int_0^\cdot b_s^{n,m,i} dA_s\Big) = \sum_{i=1}^d \int_0^\cdot \big(\gamma^i - |b_s^{n,m,i}|\big) dA_s$$

is increasing. Thus, we deduce from [70, Propositions VI.3.35, VI.3.36] that (iii) holds. Similarly, the local majoration property implies that (iv) holds. We conclude from [70, Theorem VI.5.10] that $(P^{n,m})_{n \in \mathbb{N}}$ is tight.

#### 5.3.2.2 Non-Explosion implies Tightness.

We recall [59, Theorem 15.47]: A sequence $(Q^n)_{n \in \mathbb{N}}$ of probability measures on $(\Omega, \mathcal{F})$ is tight if and only if for every $N \in \mathbb{N}$ and $\varepsilon, \delta > 0$ there exist $K, M > 0$ such that

$$\begin{aligned}\limsup_{n \to \infty} Q^n\Big(\sup_{t \in [0,N]} \|X_t\| \geq K\Big) \leq \varepsilon, \\ \limsup_{n \to \infty} Q^n\Big(w'(M, X, N) \geq \delta\Big) \leq \varepsilon,\end{aligned}$$

where $w'$ is the modulus of continuity defined on p. 438 in [59]. We only need the following property of $w'$: For a random time $\tau$ we have

$$w'(M, X, N) = w'(M, X_{\cdot \wedge \tau}, N)$$

on $\{N \leq \tau\}$. Fix $N \in \mathbb{N}$ and $\varepsilon, \delta > 0$. As $(P^{n,m})_{n \in \mathbb{N}}$ is tight, there exist $K, M > 0$, which depend on $m$, such that

$$\begin{aligned}\limsup_{n \to \infty} P^n\Big(\sup_{t \in [0,N]} \|X_{t \wedge \rho_m}\| \geq K\Big) \leq \frac{\varepsilon}{2}, \\ \limsup_{n \to \infty} P^n\Big(w'(M, X_{\cdot \wedge \rho_m}, N) \geq \delta\Big) \leq \frac{\varepsilon}{2}.\end{aligned} \tag{5.10}$$

Now, we have

$$P^n\Big(\sup_{t\in[0,N]}\|X_t\|\geq K\Big)\leq P^n\Big(\sup_{t\in[0,N]}\|X_{t\wedge\rho_m}\|\geq K\Big)+P^n\Big(N>\rho_m\Big),$$
$$P^n\Big(w'(M,X,N)\geq\delta\Big)\leq P^n\Big(w'(M,X_{\cdot\wedge\rho_m},N)\geq\delta\Big)+P^n\Big(N>\rho_m\Big).$$

Thus, using (5.10), $(P^n)_{n\in\mathbb{N}}$ is tight if we can chose $m>0$ such that

$$\limsup_{n\to\infty}P^n\Big(N>\rho_m\Big)\leq\frac{\varepsilon}{2}.$$

Of course, we would first determine $m>0$ and afterwards $K,M>0$.

From this point on the strategies for the conditions from the Theorems 5.1 and 5.2 distinguish. To prove Theorem 5.1 we separate the big jumps, which is a step we do not require in the proof of Theorem 5.2.

#### 5.3.2.3 Separation of the Big Jumps

In this section we use ideas from the proof of [101, Theorem 6.4.1]. We fix a constant $a\in(0,\infty]$ which we determine later and $m>\max(N,2)$. Set

$$Y^a\triangleq\sum_{s\leq\cdot}\Delta X_s\mathbb{1}\{\|\Delta X_s\|>a\},\quad X^a\triangleq X-Y^a.$$

Since $X$ has càdlàg paths, $\mathbb{1}\{\|\Delta X_s\|>a\}=1$ only for finitely many $s\in[0,t]$. Thus, $Y^a$ is well-defined. Note that for two non-negative random variables $U$ and $V$ we have

$$P(U+V\geq 2\varepsilon)\leq P(U\geq\varepsilon)+P(V\geq\varepsilon).$$

Hence, we obtain

$$P^n\Big(N>\rho_m\Big)\leq P^n\Big(\sup_{s\in[0,N\wedge\rho_m]}\|Y^a_s\|\geq\frac{m}{2}\Big)+P^n\Big(\sup_{s\in[0,N\wedge\rho_m]}\|X^a_s\|\geq\frac{m}{2}\Big).$$

Clearly, $\sup_{s\in[0,N\wedge\rho_m]}\|Y^a_s\|$ can only be larger than one in case that at least one jump with norm strictly larger than $a$ happens before time $N\wedge\rho_m$, i.e.

$$\Big\{\sup_{s\in[0,N\wedge\rho_m]}\|Y^a_s\|\geq 1\Big\}\subseteq\Big\{\sum_{s\in[0,N\wedge\rho_m]}\mathbb{1}\{\|\Delta X_s\|>a\}\geq 1\Big\}.$$

Thus, we deduce from Lenglart's domination property, see [70, Lemma I.3.30], and Chebyshev's inequality that for all $\varepsilon>0$

$$\begin{aligned}P^n\Big(\sup_{s\in[0,N\wedge\rho_m]}\|Y^a_s\|\geq 1\Big)&\leq P^n\Big(\sum_{s\in[0,N\wedge\rho_m]}\mathbb{1}\{\|\Delta X_s\|>a\}\geq 1\Big)\\&\leq\frac{\varepsilon}{7}+P^n\Big(\int_0^{N\wedge\rho_m}K^n_s(\{x\in\mathbb{R}^d\colon\|x\|>a\})dA_s\geq\frac{\varepsilon}{7}\Big)\\&\leq\frac{\varepsilon}{7}+\frac{7}{\varepsilon}E^n\Big[\int_0^{N\wedge\rho_m}K^n_s(\omega;\{x\in\mathbb{R}^d\colon\|x\|>a\})dA_s\Big].\end{aligned}$$

As in Section 5.3.2.1, it follows from [136, Fact 2.9, Theorem 2.17] that the integral

$$\int_0^N \sup_{\omega\in\Omega} K_s(\omega; \{x \in \mathbb{R}^d \colon \|x\| > a\}) dA_s$$

is well-defined. Hence, we obtain

$$P^n\Big(\sup_{s\in[0,N\wedge\rho_m]} \|Y^a_s\| \geq 1\Big) \leq \frac{\varepsilon}{7} + \frac{7}{\varepsilon}\int_0^N \sup_{\omega\in\Omega} K_s(\omega; \{x \in \mathbb{R}^d \colon \|x\| > a\}) dA_s.$$

In case Condition 5.2 is assumed (i.e. in the case of Theorem 5.1), the dominated convergence theorem yields that

$$\lim_{a\to\infty} \int_0^N \sup_{\omega\in\Omega} K_s(\omega; \{x \in \mathbb{R}^d \colon \|x\| > a\}) dA_s = 0,$$

and consequently, there exists an $a \in (\theta, \infty)$ independent of $n$ and $m$ such that

$$P^n\Big(\sup_{s\in[0,N\wedge\rho_m]} \|Y^a_s\| \geq \frac{m}{2}\Big) \leq P^n\Big(\sup_{s\in[0,N\wedge\rho_m]} \|Y^a_s\| \geq 1\Big) \leq \frac{\varepsilon}{6}. \tag{5.11}$$

In case Condition 5.2 is not assumed to hold (i.e. in the case of Theorem 5.2) we choose $a \equiv \infty$. As $\|Y^\infty\| = 0$, in this case we clearly have

$$P^n\Big(\sup_{s\in[0,N\wedge\rho_m]} \|Y^a_s\| \geq \frac{m}{2}\Big) = P^n\Big(\sup_{s\in[0,N\wedge\rho_m]} \|Y^\infty_s\| \geq \frac{m}{2}\Big) = 0.$$

These choices for $a$ stay fix from now on. Set

$$\zeta_m \triangleq \inf\big(t \in \mathbb{R}_+ \colon \|Y^a_{t\wedge\rho_m}\| > 1\big).$$

We note that

$$\begin{aligned} P^n\Big(\sup_{s\in[0,N\wedge\rho_m]} \|X^a_s\| \geq \frac{m}{2}\Big) &\leq P^n\Big(\sup_{s\in[0,N\wedge\rho_m\wedge\zeta_m]} \|X^a_s\| \geq \frac{m}{2}\Big) + P^n\Big(N > \zeta_m\Big) \\ &\leq P^n\Big(\sup_{s\in[0,N\wedge\rho_m\wedge\zeta_m]} \|X^a_s\| \geq \frac{m}{2}\Big) + \frac{\varepsilon}{6}. \end{aligned}$$

Consequently, it suffices to choose $m$ such that

$$\limsup_{n\to\infty} P^n\Big(\sup_{s\in[0,N\wedge\rho_m\wedge\zeta_m]} \|X^a_s\| \geq \frac{m}{2}\Big) \leq \frac{\varepsilon}{6}.$$

#### 5.3.2.4 Non-Explosion under the Lyapunov Conditions

In this section we assume that either the Conditions 5.2 and 5.3 hold or that Condition 5.6 holds.

In case $a < \infty$ we deduce from [70, Theorem II.2.21, Proposition II.2.24] that the process $X^a$ is a $P^n$-semimartingale with local characteristics $(b^{n,a}, c^n, K^{n,a}; A)$ corresponding to the truncation function $x\mathbb{1}\{\|x\| \leq a\}$, where

$$b^{n,a}_t \triangleq \phi^n(X^*_t)\mathbb{1}\{t \leq n+1\} b^a_t, \quad K^{n,a}_t(dx) \triangleq \mathbb{1}\{\|x\| \leq a\} K^n_t(dx), \quad t \in \mathbb{R}_+.$$

From now on we assume that Condition 5.6 holds. In case Conditions 5.2 and 5.3 hold it suffices to replace $\gamma, V, \beta, \mathcal{L}$ and $X$ in the following argument by $\gamma_a, V_a, \beta_a, \mathcal{L}_a$ and $X^a$.

As it is possible to replace $\beta$ with its left-continuous regularization $\beta^-(x) \equiv \beta(x-)$, we can and will assume that $\beta$ is left-continuous.

Set

$$Z \triangleq e^{-\int_0^\cdot \gamma(s)dA_s} V(X)$$

and

$$Y \triangleq Z + \int_0^\cdot e^{-\int_0^s \gamma(u)dA_u} \big(\gamma(s)V(X_{s-}) - (\mathcal{L}V)(s)\phi^n(X_s^*)\mathbb{1}\{s \leq n+1\}\big)dA_s.$$

Since we assume (5.5) (see (5.1) for the case where Condition 5.3 holds), we can deduce from Ito's formula (see, e.g. [70, Theorem I.4.57]) and [70, Lemma I.3.10, Proposition II.1.28] that $Y$ is a local $P^n$-martingale. For all $(t, \omega) \in \mathbb{R}_+ \times \Omega$ we have

$$\int_0^t \mathbb{1}\{\gamma(s)V(\omega(s-)) < (\mathcal{L}V)(\omega; s)\phi^n(X_s^*(\omega))\mathbb{1}\{s \leq n+1\}\}dA_s = 0,$$

by Condition 5.6. Thus, $Y \geq Z \geq 0$, which implies that $Y$ is a non-negative local $P^n$-martingale and hence a $P^n$-supermartingale by Fatou's lemma. As $\beta$ is increasing with $\beta(m) \nearrow \infty$ as $m \to \infty$, we find an $m > \max(N, 2)$ such that

$$\beta(k) \geq e^{\int_0^N \gamma(s)dA_s} \frac{6V(z)}{\varepsilon}$$

for all $k \geq \frac{m}{2}$.

Note that for every bounded set $G \subset \mathbb{R}_+$ it holds that

$$\sup \beta(G) = \beta(\sup G).$$

To see this, recall that $\beta$ is increasing and left-continuous and note that if $\sup G \notin G$, then there exists an increasing sequence $(g_n)_{n\in\mathbb{N}} \subset G$ such that $\lim_{n\to\infty} g_n = \sup G$.

Using that for all $t \in [0, N]$

$$Y_t \geq Z_t \geq e^{-\int_0^N \gamma(s)dA_s} V(X_t) \geq e^{-\int_0^N \gamma(s)dA_s} \beta(\|X_t\|),$$

we deduce from the supermartingale inequality (see, e.g. [77, Theorem 1.3.8 (ii)]) that

$$\begin{aligned} P^n\Big(\sup_{s\in[0,N]} \|X_s\| \geq \frac{m}{2}\Big) &\leq P^n\Big(\sup_{s\in[0,N]} \beta(\|X_s\|) \geq e^{\int_0^N \gamma(s)dA_s} \frac{6V(z)}{\varepsilon}\Big) \\ &\leq P^n\Big(\sup_{s\in[0,N]} Y_s \geq \frac{6V(z)}{\varepsilon}\Big) \leq \frac{\varepsilon V(z)}{6V(z)} = \frac{\varepsilon}{6}. \end{aligned}$$

We conclude that $(P^n)_{n\in\mathbb{N}}$ is tight.

#### 5.3.2.5 Non-Explosion under Conditions 5.2 and 5.4

In this section we assume that the Conditions 5.2 and 5.4 hold. We use an argument based on Gronwall's lemma.

Fix $T > N$ and set

$$M^a \triangleq X^a - \int_0^{\cdot} b_s^{n,a} dA_s - X_0.$$

Due to [70, Theorem II.2.21, Proposition II.2.24] the process $M^a$ is a square-integrable local $P^n$-martingale with predictable quadratic variation process

$$\langle\!\langle M^a, M^a \rangle\!\rangle = \int_0^{\cdot} \widetilde{c}_s^{n,a} dA_s,$$

where

$$\widetilde{c}_t^{n,a} \triangleq \phi^n(X_t^*)\mathbb{1}\{t \leq n+1\}\widetilde{c}_t^a, \quad t \in \mathbb{R}_+.$$

Thus, using Doob's inequality (see, e.g. [70, Theorem I.1.43]), we obtain

$$\begin{aligned} E^{P^n}\Big[&\sup_{s\in[0,N\wedge\rho_m\wedge\zeta_m]} \|M_s^a\|^2\Big] \\ &\leq 4E^{P^n}\Big[\int_0^{N\wedge\rho_m\wedge\zeta_m} \|\widetilde{c}_s^{n,a}\| dA_s\Big] \\ &\leq 4\int_0^T \gamma_a(s)dA_s + 4\int_0^N \gamma_a(s)E^{P^n}\Big[\sup_{t\in[0,s\wedge\rho_m\wedge\zeta_m]} \|X_{t-}\|^2\Big]dA_s. \end{aligned} \tag{5.12}$$

Hölder's inequality yields that

$$\begin{aligned} \sup_{t\in[0,N\wedge\rho_m\wedge\zeta_m]}&\Big\|\int_0^t b_s^{n,a} dA_s\Big\|^2 \\ &\leq A_T \int_0^{N\wedge\rho_m\wedge\zeta_m} \|b_s^{n,a}\|^2 dA_s \\ &\leq A_T\int_0^T \gamma_a(s)dA_s + A_T\int_0^N \gamma_a(s) \sup_{t\in[0,s\wedge\rho_m\wedge\zeta_m]} \|X_{t-}\|^2 dA_s. \end{aligned} \tag{5.13}$$

By the definition of $\zeta_m$, we deduce from the inequality $(a_1+a_2)^2 \leq 2(|a_1|^2+|a_2|^2)$ that

$$\begin{aligned} \sup_{t\in[0,s\wedge\rho_m\wedge\zeta_m]} \|X_{t-}\|^2 &\leq 2\Big(\sup_{t\in[0,s\wedge\rho_m\wedge\zeta_m]} \|Y_{t-}^a\|^2 + \sup_{t\in[0,s\wedge\rho_m\wedge\zeta_m]} \|X_{t-}^a\|^2\Big) \\ &\leq 2\Big(1 + \sup_{t\in[0,s\wedge\rho_m\wedge\zeta_m]} \|X_{t-}^a\|^2\Big). \end{aligned}$$

Using the inequality $(a_1+a_2+a_3)^2 \leq 3(|a_1|^2+|a_2|^2+|a_3|^2)$, we conclude that there exist a constant $c^* > 0$ and a $dA_t$-integrable Borel function $\iota\colon [0,T] \to \mathbb{R}_+$, which only depend on $z, T$ and $\gamma_a$, such that

$$E^{P^n}\Big[\sup_{s\in[0,N\wedge\rho_m\wedge\zeta_m]} \|X_s^a\|^2\Big] \leq c^* + \int_0^N \iota(s)E^{P^n}\Big[\sup_{t\in[0,s\wedge\rho_m\wedge\zeta_m]} \|X_{t-}^a\|^2\Big]dA_s.$$

Applying the Gronwall-type lemma [101, Theorem 2.4.3] we obtain

$$E^{P^n}\Big[\sup_{s\in[0,N\wedge\rho_m\wedge\zeta_m]} \|X_s^a\|^2\Big] \leq c^* e^{\int_0^N \iota(s)dA_s}.$$

Chebyshev's inequality yields that

$$\limsup_{n\to\infty} P^n\Big(\sup_{s\in[0,N\wedge\rho_m\wedge\zeta_m]} \|X^a_s\| \geq \frac{m}{2}\Big) \leq \frac{4c^* e^{\int_0^N \iota(s)dA_s}}{m^2}.$$

Consequently, we find $m > \max(N, 2)$ such that

$$\limsup_{n\to\infty} P^n\Big(\sup_{s\in[0,N\wedge\rho_m\wedge\zeta_m]} \|X^a_s\| \geq \frac{m}{2}\Big) \leq \frac{\varepsilon}{6}$$

and therefore we conclude that $(P^n)_{n\in\mathbb{N}}$ is tight.

#### 5.3.2.6 Non-Explosion under Condition 5.7

In this section we assume that Condition 5.7 holds. The argument is almost identical to the one given in Section 5.3.2.5. The only difference is that we have an additional big jump term. Namely, we have

$$X = X_0 + M + N + \int_0^\cdot b^n_s dA_s + \int_0^\cdot \int h'(x) K^n_s(dx) dA_s,$$

where

$$M \triangleq X - \int_0^\cdot b^n_s dA_s - \sum_{s\leq\cdot} h'(\Delta X_s) - X_0,$$

$$N \triangleq \sum_{s\leq\cdot} h'(\Delta X_s)\mathbb{1}\{\Delta X_s \neq 0\} - \int_0^\cdot \int h'(x) K^n_s(dx) dA_s.$$

Here, $h$ is the truncation function we fixed from the beginning and $h'(x) = x - h(x)$. We note that $\int_0^\cdot \int h'(x)K^n_s(dx)dA_s$ is well-defined due to Condition 5.7. Moreover, [70, Proposition II.1.28, Theorem II.1.33] imply that $N$ is a square integrable local $P^n$-martingale with predictable quadratic variation process

$$\langle\langle N^i, N^i\rangle\rangle = \int_0^\cdot \int |(h')^i(x)|^2 K^n_s(dx) dA_s, \quad i = 1, \dots, d.$$

We deduce from Doob's inequality that

$$E^{P^n}\Big[\sup_{s\in[0,N\wedge\rho_m]} \|N_s\|^2\Big] \leq 4\int_0^T \gamma(s)dA_s + 4\int_0^N \gamma(s) E^{P^n}\Big[\sup_{t\in[0,s\wedge\rho_m]} \|X_{t-}\|^2\Big] dA_s.$$

Using estimates similar to (5.12) and (5.13) and the Gronwall-type lemma [101, Theorem 2.4.3] yields that

$$E^{P^n}\Big[\sup_{s\in[0,N\wedge\rho_m]} \|X_s\|^2\Big] \leq c^* e^{\int_0^N \iota(s)dA_s}$$

for a constant $c^* > 0$ independent of $n$ and $m$ and a non-negative Borel function $\iota$ independent of $n$ and $m$ such that $\int_0^N \iota(s)dA_s < \infty$. Chebyshev's inequality completes the proof of the tightness of $(P^n)_{n\in\mathbb{N}}$.

### 5.3.3 Martingale Problem Argument

In this section we show that for every accumulation point of $(P^n)_{n\in\mathbb{N}}$ the coordinate process is a semimartingale with local characteristics $(b, c, K; A)$ and initial law $\delta_z$.

Let $P$ be an accumulation point of $(P^n)_{n\in\mathbb{N}}$. Without loss of generality, we assume that $P^n \to P$ weakly as $n \to \infty$. Since $\omega \mapsto \omega(0)$ is continuous, we clearly have $P \circ X_0^{-1} = \delta_z$. Set

$$\tau_m \triangleq \inf\big(t \in \mathbb{R}_+ \colon \|X_{t-}\| \geq m \text{ or } \|X_t\| \geq m\big), \quad m > 0,$$

and for $\alpha \in \Omega$ set

$$\begin{aligned} V(\alpha) &\triangleq \big\{m > 0 \colon \tau_m(\alpha) < \tau_{m+}(\alpha)\big\}, \\ V'(\alpha) &\triangleq \big\{m > 0 \colon \Delta\alpha(\tau_m(\alpha)) \neq 0, \|\alpha(\tau_m(\alpha)-)\| = m\big\}. \end{aligned}$$

Finally, we define

$$U \triangleq \Big\{m > 0 \colon P\big(\{\omega \in \Omega \colon m \in V(\omega) \cup V'(\omega)\}\big) = 0\Big\}.$$

Fix $m \in U$ and denote by $P_{n,m}$ the law of $X_{\cdot\wedge\tau_m}$ under $P^n$ and by $P_m$ the law of $X_{\cdot\wedge\tau_m}$ under $P$. Due to [70, Proposition VI.2.12] and the definition of $U$, the map $\omega \mapsto X_{\cdot\wedge\tau_m(\omega)}(\omega)$ is $P$-a.s. continuous. Thus, due to the continuous mapping theorem, we have $P_{n,m} \to P_m$ weakly as $n \to \infty$.

Due to [75, Lemma 2.3], the stopped coordinate process $X_{\cdot\wedge\tau_m}$ is a $P^n$-semimartingale with local characteristics $(\mathbb{1}_{[\![0,\tau_m]\!]}b^n, \mathbb{1}_{[\![0,\tau_m]\!]}c^n, \mathbb{1}_{[\![0,\tau_m]\!]}K^n; A)$.

Next, we use [70, Theorem IX.2.11] to conclude that the stopped coordinate process $X_{\cdot\wedge\tau_m}$ is a $P$-semimartingale with local characteristics $(\mathbb{1}_{[\![0,\tau_m]\!]}b, \mathbb{1}_{[\![0,\tau_m]\!]}c, \mathbb{1}_{[\![0,\tau_m]\!]}K; A)$. For reader's convenience we recall the prerequisites of [70, Theorem IX.2.11]:

(i) For all $t \in \mathbb{R}_+$ and $g \in C_1(\mathbb{R}^d)$ the maps

$$\omega \mapsto \int_0^{t\wedge\tau_m(\omega)} b_s(\omega) dA_s, \int_0^{t\wedge\tau_m(\omega)} \widetilde{c}_s(\omega) dA_s, \int_0^{t\wedge\tau_m(\omega)} \int g(x) K_s(\omega; dx) dA_s$$

are $P$-a.s. continuous.

(ii) For all $t \in \mathbb{R}_+$ and $g \in C_1(\mathbb{R}^d)$

$$\sup_{\omega\in\Omega} \Big(\Big\| \int_0^{t\wedge\tau_m(\omega)} \widetilde{c}_s(\omega) dA_s\Big\| + \Big\| \int_0^{t\wedge\tau_m(\omega)} \int g(x) K_s(\omega; dx) dA_s\Big\|\Big) < \infty.$$

(iii) For all $(k, k^n) \in \{(b, b^n), (\widetilde{c}, \widetilde{c}^n), (\int g(x)K(dx), \int g(x)K^n(dx)) \colon g \in C_1(\mathbb{R}^d)\}, t \in \mathbb{R}_+$ and $\varepsilon > 0$ it holds that

$$P^n\Big(\Big\| \int_0^{t\wedge\tau_m} (k_s - k_s^n) dA_s\Big\| > \varepsilon\Big) \to 0 \text{ with } n \to \infty.$$

Due to the local majoration property (Condition 5.1 (i)), the Skorokhod continuity property (Condition 5.1 (ii)) and the fact that the map $\omega \mapsto \tau_m(\omega)$ is $P$-a.s. continuous, because $m \in U$ and [70, Proposition VI.2.11], part (i) holds due to [70, IX.3.42].

Part (ii) follows from the local majoration property (Condition 5.1 (i)), because for each $g \in C_1(\mathbb{R}^d)$ we find a constant $c^* > 0$ such that $g(x) \leq c^*(1 \wedge \|x\|^2)$ for all $x \in \mathbb{R}^d$.

It remains to explain that (iii) holds. Let $(k, k^n)$ be one of the pairs $(b, b^n), (\widetilde{c}, \widetilde{c}^n)$ or $(\int g(x)K(dx), \int g(x)K^n(dx))$, where $g \in C_1(\mathbb{R}^d)$. Chebyshev's inequality yields that for all $t \in \mathbb{R}_+$ and $\varepsilon > 0$

$$\begin{aligned} P^n\Big(\Big\| \int_0^{t\wedge\tau_m} (k_s - k_s^n) dA_s\Big\| > \varepsilon\Big) &\leq \frac{1}{\varepsilon} E^{P^n}\Big[\Big\| \int_0^{t\wedge\tau_m} (k_s - k_s^n) dA_s\Big\|\Big] \\ &\leq \frac{1}{\varepsilon} E^{P^n}\Big[\int_0^{t\wedge\tau_m} \|k_s\|(1 - \phi^n(X_s^*)) dA_s\Big] \\ &\leq \frac{A_t \sup_{(s,\omega)\in\Theta_{m\vee t}} \|k_s(\omega)\|}{\varepsilon} \sup_{|x|\leq m} (1 - \phi^n(x)) \to 0 \end{aligned}$$

with $n \to \infty$. We conclude that (iii) holds.

In summary, we deduce from [70, Theorem IX.2.11] and [75, Lemma 2.3] that the stopped coordinate process $X_{\cdot\wedge\tau_m}$ is a $P_m$-semimartingale with local characteristics $(\mathbb{1}_{[\![0,\tau_m]\!]} b, \mathbb{1}_{[\![0,\tau_m]\!]} c, \mathbb{1}_{[\![0,\tau_m]\!]} K; A)$.

Next, we explain that this implies that the stopped coordinate process $X_{\cdot\wedge\tau_m}$ is also a $P$-semimartingale with local characteristics $(\mathbb{1}_{[\![0,\tau_m]\!]} b, \mathbb{1}_{[\![0,\tau_m]\!]} c, \mathbb{1}_{[\![0,\tau_m]\!]} K; A)$. Due to [70, Theorem II.2.42] the following are equivalent:

(i) The stopped coordinate process $X_{\cdot\wedge\tau_m}$ is a $P$-semimartingale with local characteristics $(\mathbb{1}_{[\![0,\tau_m]\!]} b, \mathbb{1}_{[\![0,\tau_m]\!]} c, \mathbb{1}_{[\![0,\tau_m]\!]} K; A)$.

(ii) For all bounded $f \in C^2(\mathbb{R}^d)$ the process

$$M^f \triangleq f(X_{\cdot\wedge\tau_m}) - f(X_0) - \int_0^{\cdot\wedge\tau_m} (\mathcal{L}f)(s) dA_s \tag{5.14}$$

is a local $P$-martingale.

Fix a bounded $f \in C^2(\mathbb{R}^d)$ and let $M^f$ be as in (5.14). The local majoration property (Condition 5.1 (i)) yields that $M^f$ is bounded on finite time intervals and therefore a martingale whenever it is a local martingale. As $X_{\cdot\wedge\tau_m}$ is a $P_m$-semimartingale with local characteristics $(\mathbb{1}_{[\![0,\tau_m]\!]} b, \mathbb{1}_{[\![0,\tau_m]\!]} c, \mathbb{1}_{[\![0,\tau_m]\!]} K; A)$, [70, Theorem II.2.42] implies that the process $M^f$ is a $P_m$-martingale. Let $\rho$ be a bounded $(\mathcal{F}_t^o)_{t\geq 0}$-stopping time. Due to [70, Lemma III.2.43] we have $M_\rho^f \circ X_{\cdot\wedge\tau_m} = M_\rho^f$. Thus, the optional stopping theorem yields that

$$E^P\big[M_\rho^f\big] = E^{P_m}\big[M_\rho^f\big] = 0. \tag{5.15}$$

Since predictable processes are $(\mathcal{F}_{t-})_{t\geq 0}$-adapted, see [70, Proposition I.2.4], and $\mathcal{F}_{t-} \subseteq \mathcal{F}_t^o$ for $t > 0$, see [70, p. 159], we conclude that $M^f$ is $(\mathcal{F}_t^o)_{t\geq 0}$-adapted. Hence, (5.15) and [123, Proposition II.1.4] yield that $M^f$ is a $P$-martingale for the filtration $(\mathcal{F}_t^o)_{t\geq 0}$. Finally, the backward martingale convergence theorem yields that $M^f$ is a $P$-martingale for the right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$, too. We conclude that the stopped coordinate process $X_{\cdot\wedge\tau_m}$ is a $P$-semimartingale with local characteristics $(\mathbb{1}_{[\![0,\tau_m]\!]} b, \mathbb{1}_{[\![0,\tau_m]\!]} c, \mathbb{1}_{[\![0,\tau_m]\!]} K; A)$.

Recall that $m \in U$ was arbitrary. As in the proof of [70, Proposition IX.1.17] we see that the complement of $U$ is at most countable. Consequently, we find a sequence $(m_k)_{k\in\mathbb{N}} \subset U$ such that $m_k \nearrow \infty$ as $k \to \infty$. In particular, we have $\tau_{m_k} \nearrow \infty$ as $k \to \infty$. It follows now from [70, Theorem II.2.42] that the coordinate process is a $P$-semimartingale with local characteristics $(b, c, K; A)$. The proof of the Theorems 5.1 and 5.2 is complete. □

## 5.4 Proof of Proposition 5.1

We first introduce a martingale problem for semimartingales. Let $\mathscr{C}^+(\mathbb{R}^d)$ be a countable sequence of test functions as defined in [70, II.2.20]. In particular, any function in $\mathscr{C}^+(\mathbb{R}^d)$ is bounded and vanishes around the origin. We set

$$X(h) \triangleq X - \sum_{s\leq \cdot}(\Delta X_s - h(\Delta X_s))\mathbb{1}\{\Delta X_s \neq 0\},$$
$$M(h) \triangleq X(h) - \int_0^\cdot b_s dA_s - X_0,$$

where $h$ is a truncation function. Let $\mathfrak{X}$ be the set of the following processes:

(i) $M^i(h)$ for $i = 1, \dots, d$.

(ii) $M^i(h)M^j(h) - \int_0^\cdot \tilde{c}_s^{ij} dA_s$ for $i, j = 1, \dots, d$.

(iii) $\sum_{s\leq\cdot} g(\Delta X_s) - \int_0^\cdot \int g(x)K_s(dx)dA_s$ for $g \in \mathscr{C}^+(\mathbb{R}^d)$.

For $n \in \mathbb{N}$ and a càdlàg process $Y$ we set

$$\tau_n^Y \triangleq \inf\big(t \in \mathbb{R}_+ \colon |Y_{t-}| \geq n \text{ or } |Y_t| \geq n\big).$$

Moreover, we define

$$\tau_n^i \triangleq \tau_n^Y \text{ with } Y = M^i(h),$$
$$\tau_n^{ij} \triangleq \tau_n^Y \text{ with } Y = M^i(h)M^j(h) - \int_0^\cdot \tilde{c}_s^{ij} dA_s,$$
$$\tau_n^g \triangleq \tau_n^Y \text{ with } Y = \sum_{s\leq\cdot} g(\Delta X_s) - \int_0^\cdot \int g(x)K_s(dx)dA_s.$$

Let $\mathfrak{X}_{\text{loc}}$ be the set of the following processes:

(i) $M^i(h)_{\cdot\wedge\tau_n^i}$ for $i = 1, \dots, d$ and $n \in \mathbb{N}$.

(ii) $\big(M^i(h)M^j(h) - \int_0^\cdot \tilde{c}_s^{ij} dA_s\big)_{\cdot\wedge\tau_n^i\wedge\tau_n^j\wedge\tau_n^{ij}}$ for $i, j = 1, \dots, d$ and $n \in \mathbb{N}$.

(iii) $\big(\sum_{s\leq\cdot} g(\Delta X_s) - \int_0^\cdot \int g(x)K_s(dx)dA_s\big)_{\cdot\wedge\tau_n^g}$ for $g \in \mathscr{C}^+(\mathbb{R}^d)$ and $n \in \mathbb{N}$.

We stress that the set $\mathfrak{X}_{\text{loc}}$ is countable.

Due to [70, Theorem II.2.21], $X$ is a $P$-semimartingale with local characterisics $(b, c, K; A)$ and initial law $\eta$ if and only if $P \circ X_0^{-1} = \eta$ and all processes in $\mathfrak{X}$ (or, equivalently, all processes in $\mathfrak{X}_{\text{loc}}$) are local $P$-martingales.

For a bounded function $f \colon \mathbb{R}^d \to \mathbb{R}^n$ we set $\|f\|_\infty \triangleq \sup_{x\in\mathbb{R}^d} \|f(x)\|$. We note that for any $g \in \mathscr{C}^+(\mathbb{R}^d)$

$$|\Delta M^i(h)| + \Big|\Delta\Big(\sum_{s\leq\cdot} g(\Delta X_s) - \int_0^\cdot \int g(x)K_s(dx)dA_s\Big)\Big| + \Big|\Delta\Big(\int_0^\cdot \tilde{c}_s^{ij} dA_s\Big)\Big|$$
$$\leq 2\|h^i\|_\infty + 2\|g\|_\infty + \|h^i h^j\|_\infty + \|h^i\|_\infty \|h^j\|_\infty,$$

see [70, II.2.11, Proposition II.2.17]. Furthermore, we note that for all $t \le \tau_n^i \wedge \tau_n^j$

$$\begin{aligned}\big|\Delta\big(M^i(h)M^j(h)\big)_t\big| &= \big|\Delta M^i(h)_t \Delta M^j(h)_t + M^i(h)_{t-}\Delta M^j(h)_t + M^j(h)_{t-}\Delta M^i(h)_t\big| \\ &\le 4\|h^i\|_\infty \|h^j\|_\infty + 2n\big(\|h^j\|_\infty + \|h^i\|_\infty\big).\end{aligned}$$

Hence, because for all $t \in \mathbb{R}_+$ we have

$$\big|Y_{t\wedge\tau_n^Y}\big| \le n + \big|\Delta Y_{t\wedge \tau_n^Y}\big|,$$

we conclude that all processes in $\mathfrak{X}_{\mathrm{loc}}$ are bounded and therefore martingales whenever they are local martingales. Furthermore, because predictable processes are $(\mathcal{F}_{t-})_{t\ge 0}$-adapted, see [70, Proposition I.2.4], and $\mathcal{F}_{t-} \subseteq \mathcal{F}_t^o$ for $t > 0$, see [70, p. 159], all processes in $\mathfrak{X}$ are $(\mathcal{F}_t^o)_{t\ge 0}$-adapted. As, due to [47, Proposition 2.1.5], the random time $\tau_n^Y$ is an $(\mathcal{F}_t^o)_{t\ge0}$-stopping time whenever $Y$ is $(\mathcal{F}_t^o)_{t\ge0}$-adapted, all processes in $\mathfrak{X}_{\mathrm{loc}}$ are $(\mathcal{F}_t)_{t\ge0}$-martingales if and only if they are $(\mathcal{F}_t^o)_{t\ge0}$-martingales. Here, the implication $\Rightarrow$ follows from the tower rule and the implication $\Leftarrow$ follows from the backward martingale convergence theorem.

In summary, we proved the following:

**Lemma 5.4.** *For a probability measure $P$ on $(\Omega, \mathcal{F})$ the coordinate process $X$ is a $P$-semimartingale with local characteristics $(b, c, K; A)$ and initial law $\eta$ if and only if $P \circ X_0^{-1} = \eta$ and all processes in $\mathfrak{X}_{\mathrm{loc}}$ are $P$-martingales for the filtration $(\mathcal{F}_t^o)_{t\ge0}$.*

With this observation at hand the claim of Proposition 5.1 follows from the arguments explained in the proof of Proposition 2.9. □

# 6 Existence and Non-Existence of Arbitrage in Continuous Financial Markets

## 6.1 Introduction

The absence of arbitrage is of fundamental interest in many areas of financial mathematics. In this chapter we give a systematic discussion for a financial market with one risky asset modeled via its discounted price process $P = (P_t)_{t\in[0,T]}$, which we assume to be either the stochastic exponential of an Itô process, i.e. to have dynamics

$$dP_t = P_t(b_t dt + \sigma_t dW_t), \tag{6.1}$$

or to be a positive diffusion with Markov switching, i.e. to have dynamics

$$dP_t = b(P_t, \xi_t)dt + \sigma(P_t, \xi_t)dW_t, \tag{6.2}$$

where $\xi = (\xi_t)_{t\in[0,T]}$ is a continuous-time Markov chain and $W = (W_t)_{t\in[0,T]}$ is a Brownian motion.

For semimartingale markets the classical concepts of no arbitrage are the notions of *no free lunch with vanishing risk (NFLVR)* as defined in [36, 37] and *no feasible free lunch with vanishing risk (NFFLVR)* as defined in [133]. The difference between (NFLVR) and (NFFLVR) is captured by the concept of a *financial bubble* in the sense of [25]. For our market it is well-known that (NFLVR) is equivalent to the existence of an *equivalent local martingale measure (ELMM)*, see [37], and that (NFFLVR) is equivalent to the existence of an *equivalent martingale measure (EMM)*, see [22, 133, 141]. The no arbitrage condition used in the stochastic portfolio theory of Fernholz [50] is *no relative arbitrage (NRA)*. For complete markets it is shown in [49] that (NRA) is equivalent to the existence of a *strict martingale density (SMD)*. A weaker concept is *no unbounded profit with bounded risk (NUPBR)*, which is known to be equivalent to the existence of a *strict local martingale density (SLMD)*, see [24]. (NUPBR) is considered to be the minimal notion needed for portfolio optimization, see [76].

The first results in this chapter are integral tests for the existence and non-existence of SMDs, ELMMs and EMMs. For (6.1) the tests are formulated in terms of Markovian upper and lower bounds for the volatility coefficient $\sigma$ and for (6.2) the tests depend on $x \mapsto \sigma(x, j)$ with $j$ in the state space of the Markov chain $\xi$. The main novelty of our results is that they apply in the presence of multiple sources of risk. Beside the Markov switching framework, this is for instance the case in diffusion models with a change point, which represents a change of the economical situation caused for instance by a sudden adjustment in the interest rates or a default of a major financial institution. In general, the question whether (NFLVR) and/or (NFFLVR) hold for a model with a change point is difficult, see [53] for some results in this direction. Our integral tests provide explicit criteria, which are easy to verify. For many applications of the Markov switching model (6.2) it is important to know how the change to an ELMM affects the dynamics of the Markov chain $\xi$. As a second step, we study this question form a general perspective for independent sources of risk modeled via martingale problems. In particular, we show that the *minimal local martingale measure (MLMM)*, see [60], preserves the independence and the laws of the sources of risk. To our knowledge, this property has not been reported in the literature. As a third contribution, we prove integral tests for the martingale property of certain stochastic exponentials driven by Itô processes or switching diffusions. These characterizations are our key tools to study the absence of arbitrage.

We comment on related literature. For continuous semimartingale models the absence of arbitrage has been studied in [27, 39, 103, 106]. Our results can be seen as generalizations of some results in [27, 39, 106] to an Itô process or Markov switching framework. For a model comparable to (6.1), it has been proven in [103] that the existence of an ELMM is determined by the equivalence of a probability measure to the Wiener measure. The structure of this characterization is very different from our results. In Section 6.3.3 we comment in more detail on the results in [27, 39, 103, 106]. The martingale property of stochastic exponentials is under frequent investigation. At this point we mention the articles [15, 21, 27, 75]. The arguments in [27] are based on Lyapunov functions and contradictions to verify the martingale property of certain stochastic exponentials in a multi-dimensional diffusion setting. We transfer these techniques to a general Itô process setting. In [21, 75] the martingale property of a stochastic exponential is related to explosion via a method based on the concept of *local uniqueness* as defined in [70]. This technique traces back to [67, 71, 72]. We use a similar argument for the Markov switching setting. The main difficulties are the proofs of explosion criteria and local uniqueness. Both approaches have a close relation to [15], where a tightness criterion for the martingale property of non-negative local martingales has been proven. For an explanation of the connection between Lyapunov functions, explosion and tightness see the discussion below Proposition 5.2.

Let us also comment on consecutive problems and extensions of our results: In case the discounted price process $P$ is a positive Itô process of the type

$$dP_t = b_t dt + \sigma_t dW_t,$$

our results on the martingale property of stochastic exponentials can be used to obtain characterizations for no arbitrage with a similar structure as for the model (6.2). Moreover, in case $P$ is the stochastic exponential of a diffusion with Markovian switching, i.e.

$$\begin{aligned} dP_t &= P_t dS_t, \\ dS_t &= b(S_t, \xi_t)dt + \sigma(S_t, \xi_t)dW_t, \end{aligned}$$

our martingale criteria yield conditions for no arbitrage with a similar structure as for (6.1). It is also interesting to ask about multidimensional models. In this case, results in the spirit of [27] can be proven by similar arguments as used in this chapter. However, the conditions are rather complicated to formulate. Therefore, we restrict ourselves to the one-dimensional case.

The chapter is structured as follows. In Section 6.2 we give conditions for the martingale and strict local martingale property of certain stochastic exponentials. In Section 6.3.1 we study the model (6.1) and in Section 6.3.2 we study the model (6.2). In Section 6.4 we show that the MLMM preserves independence and laws for sources of risk and we explain how the MLMM can be modified to affect the law of an additional source of risk. The proofs are collected in the remaining sections.

## 6.2 Martingale Property of Stochastic Exponentials

Fix a finite time horizon $0 < T < \infty$ and let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a complete filtered probability space with right-continuous and complete filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$. Moreover, fix a state space $I \triangleq (l, r)$ with $-\infty \le l < r \le +\infty$.

In the following two subsections we provide conditions for the martingale and strict local martingale property of certain stochastic exponentials.

### 6.2.1 The General Case

Assume that $S = (S_t)_{t\in[0,T]}$ is an $I$-valued Itô process with deterministic initial value $S_0 \in I$ and dynamics

$$dS_t = b_t dt + \sigma_t dW_t,$$

where $W = (W_t)_{t\in[0,T]}$ is a one-dimensional Brownian motion and $b = (b_t)_{t\in[0,T]}$ and $\sigma = (\sigma_t)_{t\in[0,T]}$ are real-valued progressively measurable processes. It is implicit that $b$ and $\sigma$ are such that the integrals are well-defined, i.e. a.s.

$$\int_0^T \big(|b_s| + \sigma_s^2\big) ds < \infty.$$

We assume that $\ell \otimes \mathbb{P}$-a.e. $\sigma \neq 0$, which latter will correspond to the assumption that we consider an asset price process with a non-vanishing volatility. Here, $\ell$ denotes the Lebesgue measure.

Let $c = (c_t)_{t\in[0,T]}$ be a real-valued progressively measurable process such that a.s.

$$\int_0^T c_s^2 ds < \infty,$$

and let $N = (N_t)_{t\in[0,T]}$ be a local martingale such that a.s. $\Delta N \ge -1$ and $[N, W] = 0$. We ask for conditions under which the non-negative local martingale

$$Z \triangleq \mathcal{E}\Big(N + \int_0^\cdot c_s dW_s\Big), \tag{6.3}$$

is a true or a strict local martingale. Here, $\mathcal{E}$ denotes the stochastic exponential. The structure of $Z$ is very important in mathematical finance, because $Z$ is the prototype of a strict local martingale density, see Lemma 6.2 below.

Let $\underline{a}, \overline{a}\colon I \to (0,\infty), \underline{u}, \overline{u}\colon I \to \mathbb{R}$ and $\zeta\colon [0,T] \to \mathbb{R}_+$ be Borel functions such that

$$\frac{1}{\overline{a}} + \frac{1}{\underline{a}} + |\overline{u}| + |\underline{u}| \in L^1_{\text{loc}}(I), \qquad \zeta \in L^1([0,T]).$$

In case $(f, g)$ is one of the pairs $(\underline{u}, \underline{a}), (\underline{u}, \overline{a}), \dots$ we set

$$v(f,g)(x) \triangleq \int_{x_0}^x \exp\Big(-\int_{x_0}^y 2f(z)dz\Big) \int_{x_0}^y \frac{2\exp\big(\int_{x_0}^u 2f(z)dz\big)}{g(u)} du dy, \quad x \in I, \tag{6.4}$$

where $x_0 \in I$ is fixed. Let $l_n \searrow l, r_n \nearrow r$ be sequences such that $l < l_{n+1} < l_n < r_n < r_{n+1} < r$.

The first main result of this section is the following:

**Theorem 6.1.** *Assume the following:*

(M1) *The sequence*

$$\tau_n \triangleq \inf(t \in [0,T] \colon S_t \notin (l_n, r_n)), \quad n \in \mathbb{N},$$

*is a localizing sequence for $Z$, i.e. $Z_{\cdot \wedge \tau_n}$ is a martingale for every $n \in \mathbb{N}$. We use the convention that* $\inf(\emptyset) \triangleq \infty$.

(M2) *For $\ell \otimes \mathbb{P}$-a.a.* $(t,\omega) \in [0,T] \times \Omega$

$$\begin{aligned} \sigma_t^2(\omega) &\leq \zeta(t)\overline{a}(S_t(\omega)), \\ \underline{u}(S_t(\omega))\sigma_t^2(\omega) &\leq b_t(\omega) + c_t(\omega)\sigma_t(\omega), \\ \overline{u}(S_t(\omega))\sigma_t^2(\omega) &\geq b_t(\omega) + c_t(\omega)\sigma_t(\omega). \end{aligned}$$

(M3) $\lim_{x \nearrow r} v(\overline{u}, \overline{a})(x) = \lim_{x \searrow l} v(\underline{u}, \overline{a})(x) = \infty$.

*Then, $Z$ is a martingale.*

The proof of this theorem is given in Section 6.5.

**Remark 6.1.** (M3) *is independent of the choice of $x_0$, see [77, Problem 5.5.28].*

Next, we provide a counterpart to Theorem 6.1. Let $\mathscr{H}$ be the set of all Borel functions $h \colon \mathbb{R}_+ \to \mathbb{R}_+$ which are starting at zero, are strictly increasing and satisfy

$$\int_0^\varepsilon \frac{dz}{h^2(z)} = \infty \text{ for all } \varepsilon > 0,$$

and let $\mathscr{K}$ be the set of all Borel functions $\kappa \colon \mathbb{R}_+ \to \mathbb{R}_+$, which are starting at zero, are strictly increasing and concave and satisfy

$$\int_0^\varepsilon \frac{dz}{\kappa(z)} = \infty \text{ for all } \varepsilon > 0.$$

In case $(f,g)$ is one of the pairs $(\underline{u}, \underline{a}), (\underline{u}, \overline{a}), \dots$ we say that $(f,g)$ satisfies the *Yamada–Watanabe (YW) conditions*, if for every $n \in \mathbb{N}$ there exist $h_n \in \mathscr{H}$ and $\kappa_n \in \mathscr{K}$ such that and for all $x, y \in [l_n, r_n]$

$$\begin{aligned} |g^{\frac{1}{2}}(x) - g^{\frac{1}{2}}(y)| &\leq h_n(|x-y|), \\ |g(x)f(x) - g(y)f(y)| &\leq \kappa_n(|x-y|). \end{aligned}$$

The second main result of this section is the following:

**Theorem 6.2.** *Assume one of the following conditions:*

(SL1) *The pair $(\underline{u}, \underline{a})$ satisfies the YW conditions, for $\ell \otimes \mathbb{P}$-a.a.* $(t,\omega) \in [0,T] \times \Omega$

$$\begin{aligned} \underline{a}(S_t(\omega)) &\leq \sigma_t^2(\omega), \\ \underline{u}(S_t(\omega))\sigma_t^2(\omega) &\leq b_t(\omega) + c_t(\omega)\sigma_t(\omega), \end{aligned} \tag{6.5}$$

*and* $\lim_{x \nearrow r} v(\underline{u}, \underline{a})(x) < \infty$.

(SL2) *The pair* $(\overline{u}, \underline{a})$ *satisfies the YW conditions, for* $\ell \otimes \mathbb{P}$*-a.a.* $(t, \omega) \in [0, T] \times \Omega$

$$\underline{a}(S_t(\omega)) \leq \sigma_t^2(\omega),$$
$$\overline{u}(S_t(\omega))\sigma_t^2(\omega) \geq b_t(\omega) + c_t(\omega)\sigma_t(\omega),$$

*and* $\lim_{x \searrow l} v(\overline{u}, \underline{a})(x) < \infty$.

*Then,* $Z$ *is a strict local martingale.*

The proof of this theorem is given in Section 6.5. In Section 6.2.3 below we comment on the assumptions of Theorems 6.1 and 6.2 and related literature.

### 6.2.2 Markov Switching Case

In this section we consider a special case of the setting from Section 6.2.1 and assume that $S$ is a switching diffusion. Before we introduce the setting in detail, we clarify terminology: A process is called a *Feller–Markov chain* if it is a Markov chain which is a Feller process in the sense that the corresponding transition semigroup is a self-map on the space of continuous functions vanishing at infinity. For conditions implying that a Markov chain is Feller–Markov we refer to [2]. It is also important to stress that whenever we have fixed a filtration and a Markov chain, we presume that the Markov chain is Markovian for the given filtration. All non-explained terminology for Markov chains can be found in [114].

We assume that $S = (S_t)_{t\in[0,T]}$ is an $I$-valued Itô process with deterministic initial value $S_0 \in I$ and dynamics

$$dS_t = b(S_t, \xi_t)dt + \sigma(S_t, \xi_t)dW_t, \tag{6.6}$$

where $W = (W_t)_{t\in[0,T]}$ is a one-dimensional Brownian motion, $\xi = (\xi_t)_{t\in[0,T]}$ is a continuous-time irreducible Feller–Markov chain with state space $J \triangleq \{1, \dots, N\}, 1 \leq N \leq \infty$, and deterministic initial value $j_0 \in J$, and $b\colon I \times J \to \mathbb{R}$ and $\sigma\colon I \times J \to \mathbb{R}\backslash\{0\}$ are Borel functions such that

$$\frac{1 + |b(\cdot, j)|}{\sigma^2(\cdot, j)} \in L^1_{\text{loc}}(I) \text{ for all } j \in J. \tag{6.7}$$

It is implicit that the integrals in (6.6) are well-defined. We allow $N = \infty$ in which case $J = \mathbb{N}$. A process of the type (6.6) is called a *switching diffusion* and the elements of $J$ are called *regimes.*

Let $c\colon I \times J \to \mathbb{R}$ be a Borel function such that

$$\frac{c(\cdot, j)}{\sigma(\cdot, j)} \in L^2_{\text{loc}}(I) \text{ for all } j \in J. \tag{6.8}$$

**Lemma 6.1.** *Almost surely* $\int_0^T c^2(S_s, \xi_s)ds < \infty$.

*Proof.* Set $F \triangleq \{\xi_s\colon s \in [0, T]\}$, $m \triangleq \min_{s\in[0,T]} S_s$ and $M \triangleq \max_{s\in[0,T]} S_s$. Using that $\xi$ only makes finitely many jumps in the finite time interval $[0, T]$, the occupation times formula for continuous semimartingales and (6.8), we obtain a.s.

$$\begin{aligned}\int_0^T c^2(S_s,\xi_s)ds &= \int_0^T \Big(\frac{c(S_s,\xi_s)}{\sigma(S_s,\xi_s)}\Big)^2 d[S,S]_s\\ &\le \sum_{j\in F}\int_0^T \Big(\frac{c(S_s,j)}{\sigma(S_s,j)}\Big)^2 d[S,S]_s\\ &= \sum_{j\in F}\int_m^M \Big(\frac{c(x,j)}{\sigma(x,j)}\Big)^2 2L_T^S(x)dx\\ &\le \max_{y\in[m,M]} 2L_T^S(y)\sum_{j\in F}\int_m^M \Big(\frac{c(x,j)}{\sigma(x,j)}\Big)^2 dx<\infty,\end{aligned}$$

where $L^S$ denotes the local time of $S$. The lemma is proven. □

We are interested in the martingale property of the non-negative local martingale

$$Z \triangleq \mathcal{E}\Big(\int_0^\cdot c(S_s,\xi_s)dW_s\Big).$$

This definition coincides with (6.13) for the choices $c = c(S,\xi)$ and $N = 0$.

Before we state the main result of this section, we fix some notation. Since $L^2_{\text{loc}}(I) \subset L^1_{\text{loc}}(I)$, (6.7) and (6.8) imply that

$$\frac{|b(\cdot,j)+c(\cdot,j)\sigma(\cdot,j)|}{\sigma^2(\cdot,j)} \in L^1_{\text{loc}}(I) \text{ for all } j\in J.$$

Thus, we can set

$$v(x,j) \triangleq \int_{x_0}^x \exp\Big(-\int_{x_0}^y \frac{2(b+c\sigma)(z,j)}{\sigma^2(z,j)}dz\Big)\int_{x_0}^y \frac{2\exp\big(\int_{x_0}^s \frac{2(b+c\sigma)(z,j)}{\sigma^2(z,j)}dz\big)}{\sigma^2(s,j)}dsdy$$

for $(x,j)\in I\times J$ and a fixed $x_0\in I$.

We say that $\sigma$ *satisfies the Engelbert–Schmidt (ES) conditions for* $j \in J$ if one of the following two conditions holds:

(ES1) For every compact set $K\subset I$ there are Borel functions $f\colon K\to[0,\infty]$ and $h\colon \mathbb{R}\to[0,\infty]$ and a constant $c>0$ such that the following properties are satisfied:
  (i) $\frac{f}{\sigma^2(\cdot,j)} \in L^1(K)$.
  (ii) For every neighborhood $U$ of the origin

$$\int_U \frac{dy}{h(y)} = \infty.$$

  (iii) For all $x, x+y\in K, y\in(-c,c)$

$$|\sigma(x+y,j)-\sigma(x,j)|^2 \le f(x)h(y).$$

(ES2) For every compact set $K\subset I$ there are Borel functions $g\colon K\to\mathbb{R}$ and $h\colon\mathbb{R}\to[0,\infty]$ and a constant $c>0$ such that the following properties are satisfied:

(i) $g$ is increasing.

(ii) For every neighborhood $U$ of the origin

$$\int_U \frac{dy}{h(y)} = \infty.$$

(iii) For all $x, x+y \in K, y \in (-c,c)\backslash\{0\}$

$$|\sigma(x+y,j) - \sigma(x,j)|^2 \le h(y)\frac{|g(x+y)-g(x)|}{|y|}.$$

(iv) $\inf_{x\in K} \sigma(x,j) > 0$.

The following theorem gives an almost complete answer to the question when $Z$ is a true or strict local martingale. A proof is given in Section 6.6.

**Theorem 6.3.** (i) *Suppose that $c$ is bounded on compact subsets of $I \times J$, that $\sigma$ satisfies the ES conditions for all $j \in J$ and that*

$$\lim_{x\nearrow r} v(x,j) = \lim_{x\searrow l} v(x,j) = \infty \text{ for all } j \in J. \tag{6.9}$$

*Then, $Z$ is a martingale.*

(ii) *Assume that there exists a $j \in J$ such that $\sigma$ satisfies the ES conditions for $j$ and*

$$\lim_{x\nearrow r} v(x,j) < \infty \text{ or } \lim_{x\searrow l} v(x,j) < \infty. \tag{6.10}$$

*Then, $Z$ is a strict local martingale.*

An immediate consequence of Theorem 6.3 is the following:

**Corollary 6.1.** *Suppose that $c$ is bounded on compact subsets of $I \times J$ and that $\sigma$ satisfies the ES conditions for all $j \in J$. Then, $Z$ is a martingale if and only if* (6.9) *holds.*

### 6.2.3 Comments on Related Literature

The martingale property of non-negative local martingales is under frequent investigation. We mention a few related works: A general semimartingale setting has been considered in [32, 65, 70] and a diffusion and/or jump-diffusion setting has been studied in [21, 75, 102, 107, 127, 134].

To the best of our knowledge, for a general Itô process or Markov switching setting Theorems 6.1, 6.2 and 6.3 are the first results which provide integral tests for the martingale property of certain stochastic exponentials.

For the diffusion case

$$dS_t = b(S_t)dt + \sigma(S_t)dW_t,$$

a complete characterization of the martingale property of the non-negative local martingale

$$Z = \mathcal{E}\Big(\int_0^\cdot c(S_s)dW_s\Big)$$

has been proven in [107] under local integrability conditions. We stress that in [107] the diffusion $S$ is allowed to explode, which is a feature not included in our framework.

Provided $S$ is non-explosive, the main theorem of [107] shows that $Z$ is a martingale if and only if

$$\lim_{x \nearrow r} v(u, \sigma^2) = \lim_{x \searrow l} v(u, \sigma^2) = \infty,$$

where $u \triangleq \frac{b+c\sigma}{\sigma^2}$ and $v$ is defined as in (6.4). The same condition is implied by either Theorems 6.1 and 6.2, or Corollary 6.1. For the strict local martingale property we require that $\sigma$ satisfies the ES conditions, which are not imposed in [107].

The key idea underlying Theorems 6.1, 6.2 and 6.3 is a local change of measure combined with either a Lyapunov-type argument (in case of Theorem 6.1), a comparison with one-dimensional diffusions (in case of Theorem 6.2) or a local uniqueness property (in case of Theorem 6.3).

The idea of using a local change of measure is not new. It has for instance been used in [21, 27, 32, 127, 134]. The Lyapunov and comparison arguments were inspired by [27], where a multi-dimensional diffusions has been studied. To use the ideas in our general setting, we prove a new Lyapunov condition for Itô processes and we transport the comparison arguments from a multi-dimensional diffusion setting to a one-dimensional Itô process framework, see Section 6.5 below. The idea of relating local uniqueness to the martingale property of a stochastic exponential traces back to [67, 71, 72]. More recently, the method was used in [21, 27, 75, 134]. Although the terminology suggests the converse, local uniqueness is a strong version of uniqueness in law. In the proof of Theorem 6.3 we deduce local uniqueness from pathwise uniqueness by a Yamada–Watanabe-type argument.

## 6.3 On the Absence and Existence of Arbitrage

Let $0 < T < \infty$ be a finite time horizon and let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a complete filtered probability space with right-continuous and complete filtration $\mathbf{F} = (\mathcal{F}_t)_{t \in [0,T]}$. We consider a financial market consisting of one risky asset with discounted price process $P = (P_t)_{t \in [0,T]}$, which is assumed to be a positive continuous semimartingale with deterministic initial value.

Recall the following classical terminology: A probability measure $\mathbb{Q}$ is called an *equivalent (local) martingale measure (E(L)MM)* if $\mathbb{Q} \sim \mathbb{P}$ and $P$ is a (local) $\mathbb{Q}$-martingale. A strictly positive local $\mathbb{P}$-martingale $Z = (Z_t)_{t \in [0,T]}$ with $Z_0 = 1$ is called a *strict (local) martingale density (S(L)MD)* if $ZP$ is a (local) $\mathbb{P}$-martingale.

In the following we study existence and non-existence of SMDs, ELMMs and EMMs in case $P$ is either the stochastic exponential of an Itô process or a positive switching diffusion. In case $P$ is a positive Itô process or the stochastic exponential of a real-valued switching diffusion similar results can be deduced from the martingale criteria in Section 6.2.

### 6.3.1 Stochastic Exponential Model

Suppose that $P$ is the stochastic exponential of the real-valued Itô process $S = (S_t)_{t \in [0,T]}$ with deterministic initial value $S_0 \in \mathbb{R}$ and dynamics

$$dS_t = b_t dt + \sigma_t dW_t, \tag{6.11}$$

where $W = (W_t)_{t \in [0,T]}$ is a one-dimensional Brownian motion and $b = (b_t)_{t \in [0,T]}$ and $\sigma = (\sigma_t)_{t \in [0,T]}$ are real-valued progressively measurable processes such that the stochastic integrals in (6.11) are well-defined. We assume that $\ell \otimes \mathbb{P}$-a.e. $\sigma \neq 0$, which corresponds to the assumption that $P$ has a non-vanishing volatility.

#### 6.3.1.1 Absence of Arbitrage

In the following we study when a SMD, ELMM or EMM exists. As a minimal condition we assume that (NUPBR) holds. This is equivalent to the existence of a *market price of risk* $\theta = (\theta_t)_{t\in[0,T]}$, i.e. a real-valued progressively measurable process such that a.s.

$$\int_0^T \theta_s^2 ds < \infty$$

and

$$\ell \otimes \mathbb{P}\text{-a.e. } b - \theta\sigma = 0. \tag{6.12}$$

We define the continuous local martingale

$$Z \triangleq \mathcal{E}\Big( - \int_0^{\cdot} \theta_s dW_s \Big). \tag{6.13}$$

Integration by parts and (6.12) yield that

$$dZ_t P_t = Z_t P_t(\sigma_t - \theta_t) dW_t, \tag{6.14}$$

which shows that $ZP$ is a local martingale or, equivalently, that $Z$ is a SLMD. We observe the following:

(O1) If $ZP$ is a martingale, then $Z$ is a SMD by definition.

(O2) If $Z$ is a martingale, we can define a probability measure $\mathbb{Q}$ by the Radon–Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq Z_T$ and $\mathbb{Q}$ is an ELMM by (6.14) and [70, Proposition III.3.8].

(O3) If $ZP$ and $Z$ are martingales, then $\mathbb{Q}$ as defined in (O2) is an EMM by [70, Proposition III.3.8].

In summary, to prove the existence of a SMD, ELMM and EMM we have to identify conditions for the martingale property of $ZP$ and $Z$. The following is the main result of this section:

**Theorem 6.4.** *Suppose the following:*

(L1) *The sequence*

$$\tau_n \triangleq \inf(t \in [0,T] \colon |S_t| \geq n), \quad n \in \mathbb{N}, \tag{6.15}$$

*is a localizing sequence for $Z$.*

(L2) *There are Borel functions $\overline{a} \colon \mathbb{R} \to (0,\infty)$ and $\zeta \colon [0,T] \to \mathbb{R}_+$ such that*

$$\frac{1}{\overline{a}} \in L^1_{\text{loc}}(\mathbb{R}), \quad \zeta \in L^1([0,T]),$$

*and $\sigma_t^2(\omega) \leq \zeta(t)\overline{a}(S_t(\omega))$ for $\ell \otimes \mathbb{P}$-a.a. $(t,\omega) \in [0,T] \times \Omega$.*

*Then, $Z$ is a martingale, $\mathbb{Q}$ defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq Z_T$ is an ELMM and*

$$B = W + \int_0^\cdot \theta_t dt \tag{6.16}$$

*is a $\mathbb{Q}$-Brownian motion such that*

$$S = S_0 + \int_0^\cdot \sigma_t dB_t.$$

*If in addition*

$$\int_1^\infty \frac{dz}{\overline{a}(z)} = \infty, \tag{6.17}$$

*then $\mathbb{Q}$ is an EMM and $Z$ is a SMD.*

*Proof.* We apply Theorem 6.1 with $I \triangleq \mathbb{R}, l_n \triangleq -n, r_n \triangleq n$ and $c \triangleq -\theta$. Note that (L1) equals (M1). Furthermore, set $\underline{u}(x) \equiv \overline{u}(x) \triangleq 0$. Then, (L2) implies (M2), because (6.12) implies $\ell \otimes \mathbb{P}$-a.e. $b + c\sigma = 0$. Finally, note that

$$\int_{x_0}^{\pm\infty} \exp\Big( -2 \int_{x_0}^x \underline{u}(y)dy\Big) dx = \int_{x_0}^{\pm\infty} \exp\Big( -2 \int_{x_0}^x \overline{u}(y)dy\Big) dx = \pm\infty,$$

which shows that (M3) holds due to [77, Problem 5.5.27]. We conclude that $Z$ is a martingale and that $\mathbb{Q}$ is an ELMM by (O2).

Next, we assume that (6.17) holds. We apply Theorem 6.1 with $I \triangleq \mathbb{R}, l_n \triangleq -n, r_n \triangleq n$ and $c \triangleq \sigma - \theta$ to show that the local martingale

$$Z' \triangleq \frac{ZP}{P_0} = \mathcal{E}\Big( \int_0^\cdot (\sigma_s - \theta_s) dW_s\Big)$$

is a martingale. In this case, $\mathbb{Q}$ is an EMM and $Z$ is a SMD by (O1) and (O3). By (L1), the set $\{Z_{T\wedge\gamma\wedge\tau_n} \colon \gamma \text{ stopping time}\}$ is uniformly integrable (see [70, Proposition I.1.47]). Thus,

$$\begin{aligned}&\sup_\gamma \mathbb{E}^\mathbb{P}\big[Z'_{\gamma\wedge\tau_n} \mathbb{1}_{\{Z'_{\gamma\wedge\tau_n} \geq K\}}\big] \\ &\qquad \leq e^{|S_0|+n} \sup_\gamma \mathbb{E}^\mathbb{P}\big[Z_{\gamma\wedge\tau_n} \mathbb{1}_{\{Z_{\gamma\wedge\tau_n} \geq e^{-|S_0|-n}K\}}\big] \to 0 \text{ as } K \to \infty,\end{aligned}$$

where the $\sup_\gamma$ is meant to be the supremum over all stopping times $\gamma \leq T$. Due to [70, Proposition I.1.47], we conclude that (M1) holds for $Z'$. Note that (6.12) implies that $\ell \otimes \mathbb{P}$-a.e. $b + c\sigma = \sigma^2$. Thus, we set $\underline{u}(x) \equiv \overline{u}(x) \triangleq 1$ and note that (L2) implies (M2) for $Z'$. Using Fubini's theorem and (6.17), we obtain that

$$\begin{aligned}\lim_{x \nearrow \infty} v(1, \overline{a})(x) &= 2 \int_{x_0}^\infty e^{-2y} \int_{x_0}^y \frac{e^{2u}}{\overline{a}(u)} du dy \\ &= 2 \int_{x_0}^\infty \frac{e^{2u}}{\overline{a}(u)} \int_u^\infty e^{-2y} dy du \\ &= \int_{x_0}^\infty \frac{du}{\overline{a}(u)} = \infty.\end{aligned}$$

As

$$\int_{x_0}^{-\infty} \exp\Big( -2 \int_{x_0}^{x} dy\Big) dx = -\infty,$$

[77, Problem 5.5.27] yields that $\lim_{x\searrow -\infty} v(1,\overline{a})(x) = \infty$. Hence, (M3) holds for $Z'$. We conclude that $Z'$ is a martingale and the proof is complete. □

In our setting there might exist several ELMMs and it is an important question which ELMM should be chosen for applications. The ELMM from Theorem 6.4 is the *minimal local martingale measure (MLMM)* as defined in [60].[1] For financial interpretations of the MLMM we refer to [60] and for a general overview on possible applications we refer to [52]. In Theorem 6.9 below we discover a new property of the MLMM: The MLMM preserves independence and laws of sources of risk.

In the following paragraph we relate the assumptions (L1) and (L2) to so-called *weakly equivalent local martingale measures (WELMM)* as introduced in [78]. We explain the connection from a general point of view under the assumptions that $\mathcal{F} = \mathcal{F}_T$ and that (NUPBR) holds. With slight abuse of notation, let $Z = (Z_t)_{t\in[0,T]}$ be a SLMD with localizing sequence $(\tau_n)_{n\in\mathbb{N}}$. For every $n \in \mathbb{N}$ we can define a probability measure $\mathbb{Q}^n$ by the Radon–Nikodym derivative $\frac{d\mathbb{Q}^n}{d\mathbb{P}} \triangleq Z_{T\wedge\tau_n}$. It is easy to see that $\mathbb{Q}^n$ is an ELMM for the stopped process $P_{\cdot\wedge\tau_n}$. In other words, for every $n \in \mathbb{N}$ the notion (NFLVR) holds for all admissible strategies which invest riskless after $\tau_n$. Roughly speaking, this observation suggests that (NFLVR) holds in case we can take the limit $n \to \infty$. As explained in Section 2.4.2 of [78], Alaoglu's theorem yields that $(\mathbb{Q}^n)_{n\in\mathbb{N}}$ has an accumulation point $\mathsf{Q}$ for the weak* topology on the dual of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, which is a finitely additive probability such that $\mathsf{Q}(A) = 0$ for all $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$, see the Appendix of [34]. We use the sans-serif typeface to highlight that $\mathsf{Q}$ is not necessarily a probability measure, because it may fail to be countably additive. Note that $\mathsf{Q} = \mathbb{Q}^n$ on $\mathcal{F}_{\tau_n}$ for every $n \in \mathbb{N}$. Using this fact, it follows that for all $A \in \mathcal{F}$ with $\mathsf{Q}(A) = 0$ we also have $\mathbb{P}(A) = 0$, which shows that $\mathsf{Q}$ and $\mathbb{P}$ have the same null-sets. Indeed, if $A \in \mathcal{F} = \mathcal{F}_T$ is such that $\mathsf{Q}(A) = 0$, we have $A \cap \{\tau_n > T\} \in \mathcal{F}_{\tau_n}$ and consequently

$$\mathbb{Q}^n(A \cap \{\tau_n > T\}) = \mathsf{Q}(A \cap \{\tau_n > T\}) = 0$$

for all $n \in \mathbb{N}$. This implies $\mathbb{P}(A \cap \{\tau_n > T\}) = 0$ and, because $\mathbb{P}$-a.s. $\tau_n \nearrow \infty$ as $n \to \infty$, we conclude that $\mathbb{P}(A) = 0$. Following [78], we call $\mathsf{Q}$ a WELMM. The main difference between WELMMs and ELMMs, and therefore between (NUPBR) and (NFLVR), is that a WELMM is not necessarily a measure.

The idea of condition (L1) is to identify a WELMM, which, as explained above, is a natural candidate for an ELMM. Assuming that $(\tau_n)_{n\in\mathbb{N}}$ is given by (6.15) means controlling the MPR via the size of the asset. This assumption is reasonable from a modeling perspective, because, as explained by Lyasoff [103, p. 488], "excessively large expected instantaneous net returns from risky securities entail excessively large demands for money (to invest in such securities), which, in turn, means higher and higher interest rates, which, in turn, means lower and lower market price of risk". In the diffusion settings of Mijatović and Urusov [106], (L1) is equivalent to the local integrability condition [106, Eq. 3.2] on the MPR, see [107, Lemma 6.3].

Condition (L2) takes care on the countable additivity of the candidate WELMM, which

[1] In [60] the MLMM has been called *minimal martingale measure*. Since we distinguish between ELMMs and EMMs we adjust the terminology.

corresponds to problems arising when $n \to \infty$. Indeed, $\mathsf{Q}$ is countably additive if and only if

$$\limsup_{n\to\infty} \mathsf{Q}(\tau_n > T) = \limsup_{n\to\infty} \mathbb{Q}^n(\tau_n > T) = 1, \tag{6.18}$$

which is also the condition we check in the proof of Theorem 6.1. If $\mathsf{Q}$ is countably additive, then (6.18) follows from the monotone convergence theorem and the fact that $\mathbb{P}$-a.s. $\tau_n \nearrow \infty$ as $n \to \infty$. Conversely, assume that (6.18) holds. Let $(E_k)_{k\in\mathbb{N}} \subset \mathcal{F}$ be a decreasing sequence with $\bigcap_{k\in\mathbb{N}} E_k = \emptyset$. Then, because $E_k \in \mathcal{F} = \mathcal{F}_T$, we have $E_k \cap \{\tau_n > T\} \in \mathcal{F}_{\tau_n}$, which yields that

$$\begin{aligned}\limsup_{k\to\infty} \mathsf{Q}(E_k) &\leq \mathsf{Q}(\tau_n \leq T) + \limsup_{k\to\infty} \mathsf{Q}(E_k \cap \{\tau_n > T\}) \\ &= \mathsf{Q}(\tau_n \leq T) + \limsup_{k\to\infty} \mathbb{Q}^n(E_k \cap \{\tau_n > T\}) \\ &= \mathsf{Q}(\tau_n \leq T) \to 0 \text{ with } n \to \infty.\end{aligned}$$

Thus, $\mathsf{Q}$ is continuous at zero, which implies that it is countably additive.

#### 6.3.1.2 Existence of a Financial Bubble

In Theorem 6.4 we gave conditions for the existence of an ELMM. In this section, we derive a counterpart to (6.17), which implies the existence of a financial bubble in the sense of [25].

As we explain next, the question when a SMD exists is strongly connected to the question when a non-negative local martingale is a strict local martingale. We recall the following:

**Lemma 6.2.** *If $Z$ is a SLMD, then there exists a market price of risk $\theta = (\theta_t)_{t\in[0,T]}$ and a local martingale $N = (N_t)_{t\in[0,T]}$ such that a.s. $\Delta N > -1, [N, W] = 0$ and*

$$Z = \mathcal{E}\Big(N - \int_0^\cdot \theta_s dW_s\Big). \tag{6.19}$$

*Proof.* See [130, Theorem 1]. □

In case $Z$ is a SMD, (6.19) holds and

$$ZP = P_0 \mathcal{E}\Big(N + \int_0^\cdot (\sigma_s - \theta_s) dW_s\Big) \tag{6.20}$$

is a martingale by definition. If this is not the case, we have a contradiction and no SMD exists.

The following is the main result of this section:

**Theorem 6.5.** *Suppose there exists a Borel function $\underline{a}\colon \mathbb{R} \to (0, \infty)$ such that $(1, \underline{a})$ satisfies the YW conditions (see Section 6.2.1 for this terminology), $\underline{a}(S_t(\omega)) \leq \sigma_t^2(\omega)$ for $\ell \otimes \mathbb{P}$-a.a. $(t, \omega) \in [0, T] \times \Omega$ and*

$$\int_1^\infty \frac{dz}{\underline{a}(z)} < \infty. \tag{6.21}$$

*Then, no SMD exists.*

*Proof.* We use Theorem 6.2 with $I \triangleq \mathbb{R}$ and $\underline{u} \triangleq 1$ to show that $ZP$ as defined in (6.20) is a strict local martingale. Since $\theta$ is a MPR, $\ell \otimes \mathbb{P}$-a.e. $b + (\sigma - \theta)\sigma = \sigma^2 = \underline{u}(S)\sigma^2$. Furthermore, Fubini's theorem and (6.21) yield that

$$\lim_{x \nearrow \infty} v(1, \underline{a})(x) = \int_{x_0}^{\infty} \frac{dz}{\underline{a}(z)} < \infty.$$

Thus, the conditions from part (ii) of Theorem 6.2 hold and we conclude that $ZP$ is a strict local martingale. Consequently, as explained above, no SMD exists. □

The conditions (6.17) and (6.21) provide a test for the MLMM to be an EMM or not. In a diffusion setting the conditions boil down to a single sufficient and necessary condition, which is also given in [27, Proposition 5.2].

#### 6.3.1.3 Example: Diffusion Models with a Change Point

Fontana et al. [53] study (NUPBR) and (NFLVR) for a model with a change point. The authors are interested in the influence of filtrations, which represent different levels of information. Under a weak form of the $\mathcal{H}'$-hypothesis the model can be included into our framework. More precisely, in this case $S$ is of the form

$$dS_t = \mu_t dt + \big(\sigma^{(1)}(t, S_t)\mathbb{1}_{\{t \le \tau\}} + \sigma^{(2)}(t, S_t)\mathbb{1}_{\{t > \tau\}}\big) dW_t,$$

where $\tau$ is a random time. The coefficient $\sigma^{(i)}$ is assumed to be positive, continuous and Lipschitz continuous in the second variable uniformly in the first, see [53, Condition I]. Theorem 6.4 provides local conditions for (NFLVR). For instance in the special cases described in [53, Section 3.3], Theorem 6.4 yields that (NFLVR) always holds, because

$$\mu_t = \mu^{(1)}(t, S_t)\mathbb{1}_{\{t \le \tau\}} + \mu^{(2)}(t, S_t)\mathbb{1}_{\{t > \tau\}}, \tag{6.22}$$

where $\mu^{(i)}$ is locally bounded. This extends the observation from [53] that (NUPBR) holds in these cases. Furthermore, if in addition to (6.22) for $i = 1, 2$

$$\big(\sigma^{(i)}(t, x)\big)^2 \le \text{const. } x, \quad (t, x) \in [0, T] \times [1, \infty),$$

then even (NFFLVR) holds.

### 6.3.2 Diffusion Model with Markov switching

In this section, we assume that $P$ is a positive continuous semimartingale with deterministic initial value $P_0 \in (0, \infty)$ and dynamics

$$dP_t = b(P_t, \xi_t)dt + \sigma(P_t, \xi_t)dW_t,$$

where $W = (W_t)_{t \in [0,T]}$ is a one-dimensional Brownian motion, $\xi = (\xi_t)_{t \in [0,T]}$ is a continuous-time irreducible Feller–Markov chain with state space $J \triangleq \{1, \dots, N\}$, $1 \le N \le \infty$, and deterministic initial value $j_0 \in J$, and $b\colon (0, \infty) \times J \to \mathbb{R}$ and $\sigma\colon (0, \infty) \times J \to \mathbb{R}\backslash\{0\}$ are Borel functions such that

$$\frac{1 + |b(\cdot, j)|}{\sigma^2(\cdot, j)} \in L^1_{\text{loc}}((0, \infty)) \text{ for all } j \in J.$$

We can interpret $N$ as the number of all possible states of the business cycle. The assumption of irreducibilily means that we exclude all states of the business cycle which are not attainable from the initial state. We assume $\xi$ to be a Feller process for technical reasons. In case $N < \infty$ any Markov chain with values in $J$ is a Feller process, because all real-valued functions on $J$ are continuous and vanishing at infinity. Due to Remark 2.4, the sources of risk $\xi$ and $W$ are independent. The remark even shows that it is not possible to model $\xi$ and $W$ as Markov processes for a superordinate filtration without their independence. This observation gives a novel interpretation for the independence assumption, which is typically interpreted as the price process being influenced by the business cycle and an additional independent source of risk represented by the driving Brownian motion.

#### 6.3.2.1 Absence and Existence of Arbitrage

We impose the following two assumptions: The coefficient $b$ is bounded on compact subsets of $(0,\infty) \times J$, $\sigma^2$ is bounded away from zero on compact subsets of $(0,\infty) \times J$ and $\sigma$ satisfies the ES conditions for all $j \in J$, see Section 6.2.2 for this terminology.

We define

$$\theta(x,j) \triangleq \frac{b(x,j)}{\sigma(x,j)},$$

which is a Borel map bounded on compact subsets of $(0,\infty)\times J$. The process $\theta_t \triangleq \theta(P_t, \xi_t)$ is a MPR. We define the continuous local martingale $Z$ as in (6.13). Note that the observations (O1) – (O3) in Section 6.3.1 also hold in this setting. We call the E(L)MM $\mathbb{Q}$ with Radon–Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$ the *minimal (local) martingale measure (M(L)MM)*. The following theorem provides conditions for the existence of the M(L)MM and for $Z$ to be a SMD.

**Theorem 6.6.** (i) *Assume that*

$$\int_0^1 \frac{z}{\sigma^2(z,j)} dz = \infty \text{ for all } j \in J. \tag{6.23}$$

*Then, $Z$ is a martingale and the probability measure $\mathbb{Q}$ defined by the Radon–Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq Z_T$ is an ELMM. Moreover, $B$ as defined in* (6.16) *is a $\mathbb{Q}$-Brownian motion such that*

$$P = P_0 + \int_0^{\cdot} \sigma(P_t, \xi_t) dB_t.$$

*If in addition*

$$\int_1^\infty \frac{z}{\sigma^2(z,j)} dz = \infty \text{ for all } j \in J, \tag{6.24}$$

*then $\mathbb{Q}$ is an EMM.*

(ii) *If* (6.24) *holds, then $Z$ is a SMD.*

*Proof.* The claim follows similar to the proof of Theorem 6.4 when Theorem 6.3 is used instead of Theorem 6.1. □

Theorem 6.3 suggests that the conditions in Theorem 6.6 are sufficient and necessary. The following theorem makes this precise.

**Theorem 6.7.** (i) *If there exists a* $j \in J$ *such that*

$$\int_0^1 \frac{z}{\sigma^2(z,j)} dz < \infty, \tag{6.25}$$

*then* $Z$ *is a strict local martingale and the MLMM does not exist.*

(ii) *If there exists a* $j \in J$ *such that*

$$\int_1^\infty \frac{z}{\sigma^2(z,j)} dz < \infty, \tag{6.26}$$

*then* $Z$ *is no SMD. In particular, the MMM does not exist.*

*Proof.* The claim follows similar to the proof of Theorem 6.5 when Theorem 6.3 is used instead of Theorem 6.2. □

In summary, we have the following:

**Corollary 6.2.** (a) *The MLMM exists if and only if* (6.23) *holds.*

(b) *The MMM exists if and only if* (6.23) *and* (6.24) *hold.*

(c) $Z$ *is a SMD if and only if* (6.24) *holds.*

With $N = 1$ we recover [106, Corollary 3.4, Theorems 3.6 and 3.11]. Corollary 6.2 means that the M(L)MM exists if and only if the M(L)MM exists for all markets with fixed regimes. We will see in the next section that in case one of the frozen markets allows arbitrage, it is not possible to find a risk-neutral market in which the business cycle has Markovian dynamics.

#### 6.3.2.2 Non-Existence of Structure-preserving ELMMs and EMMs

Let $\mathcal{L}_{sp}$ the set of all ELMMs $\mathbb{Q}$ such that $\xi$ is an irreducible Feller–Markov chain on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{Q})$ and let $\mathcal{M}_{sp}$ be the set of all EMMs in $\mathcal{L}_{sp}$. The main result of this section is the following:

**Theorem 6.8.** (i) *Suppose there exists a* $j \in J$ *such that* (6.25) *holds and* $\sigma$ *satisfies the ES conditions for* $j$*. Then,* $\mathcal{L}_{sp} = \emptyset$*.*

(ii) *Suppose there exists a* $j \in J$ *such that* (6.26) *holds and* $\sigma$ *satisfies the ES conditions for* $j$*. Then,* $\mathcal{M}_{sp} = \emptyset$*.*

*Proof.* The result follows from the contradiction argument used in the proof of Theorem 6.2, where Theorem 6.15 has to be used instead of Theorem 6.14. □

In Section 6.4 we show that an equivalent change to the MLMM does not affect the Markov chain $\xi$. Thus, Theorem 6.8 generalizes Theorem 6.7.

#### 6.3.2.3 Example: Markov Switching CEV Model

We consider a version of the CEV model (see [26]) with Markov switching. Take $\beta\colon J \to (0,\infty)$ and assume that

$$\sigma(x,j) = x^{\beta(j)}, \quad (x,j) \in (0,\infty)\times J.$$

Furthermore, assume that $b\colon (0,\infty)\times J \to \mathbb{R}$ is locally bounded such that

$$\int_1^\infty \int_1^y \frac{\exp(-\int_s^y \frac{2b(z,j)}{z^{2\beta(j)}}dz)}{s^{2\beta(j)}} dsdy = \int_0^1 \int_y^1 \frac{\exp(-\int_s^y \frac{2b(z,j)}{z^{2\beta(j)}}dz)}{s^{2\beta(j)}} dsdy = \infty$$

for all $j \in J$. Then, the discounted asset price process $P$ exists due to Theorem 6.15 below. Let $Z$ be defined as in (6.13) with $\theta_t = \frac{b(S_t,\xi_t)}{\sigma(S_t,\xi_t)}$. Corollary 6.2 shows the following:

(a) The MLMM exists if and only if $\beta(j) \geq 1$ for all $j \in J$.

(b) The MMM exists if and only if $\beta(j) = 1$ for all $j \in J$.

(c) $Z$ is a SMD if and only if $\beta(j) \leq 1$ for all $j \in J$.

### 6.3.3 Comments on Related Literature

For continuous semimartingale markets the existence and non-existence of SMDs, ELMMs and EMMs has been studied in [27, 39, 103, 106]. We comment on these works in more detail.

In [39, 106] a one-dimensional diffusion framework has been considered. We discuss the results from [106] and refer to [106, Remark 3.2] for comments on the relation between [39] and [106]. In [106] it is assumed that the price process $P = (P_t)_{t\in[0,T]}$ is a $[0,\infty)$-valued diffusion such that

$$dP_t = b(P_t)dt + \sigma(P_t)dW_t, \quad P_0 \in (0,\infty),$$

where $b\colon (0,\infty)\to\mathbb{R}$ and $\sigma\colon (0,\infty)\to\mathbb{R}\backslash\{0\}$ are Borel functions satisfying

$$\frac{1+|b|}{\sigma^2} \in L^1_{\text{loc}}((0,\infty)),$$

see also [77, Definition 5.5.20]. In the following we assume that $P$ cannot explode to zero. In [106] the notions (NFLVR) and (NFFLVR) are also studied in case $P$ can explode to zero and for the infinite time horizon. For the non-explosive case the results from [106] are as follows:

(a) (NFLVR) $\Leftrightarrow$ $\frac{b}{\sigma^2} \in L^2_{\text{loc}}((0,\infty))$ and $\int_0^1 \frac{x}{\sigma^2(x)}dx = \infty$, see [106, Corollary 3.4].

(b) (NFFLVR) $\Leftrightarrow$ $\frac{b}{\sigma^2} \in L^2_{\text{loc}}((0,\infty))$ and $\int_0^1 \frac{x}{\sigma^2(x)}dx = \int_1^\infty \frac{x}{\sigma^2(x)}dx = \infty$, see [106, Theorem 3.6].

(c) If $\frac{b}{\sigma^2} \in L^2_{\text{loc}}((0,\infty))$, then (NRA) $\Leftrightarrow$ $\int_1^\infty \frac{x}{\sigma^2(x)}dx = \infty$, see [106, Theorem 3.11].

Applying Corollary 6.2 with $N = 1$ shows versions of (a) − (c) under slightly more restrictive regularity assumptions on $b$ and $\sigma$. The novelty of Corollary 6.2 or more generally Theorems 6.6 and 6.7 is their scope.

A multi-dimensional diffusion setting has been studied in [27]. We explain the one-dimensional version: Assume that the price process $P = (P_t)_{t\in[0,T]}$ is the stochastic exponential of

$$dS_t = b(S_t)dt + \sigma(S_t)dW_t,$$

where $b, \sigma\colon \mathbb{R} \to \mathbb{R}$ are locally bounded Borel functions such that $\sigma^2$ is locally bounded away from zero. In this setting, [27, Propositions 5.1] shows that (NFLVR) always holds and [27, Proposition 5.2] implies that (NFFLVR) $\Leftrightarrow$ (NRA) $\Leftrightarrow \int_1^\infty \frac{dx}{\sigma^2(x)} = \infty$. Under slightly different regularity assumptions on $b$ and $\sigma$, the same observation follows from Theorems 6.4 and 6.5. The novelty of Theorems 6.4 and 6.5 is that no diffusion structure is needed. In particular, the coefficients $b$ and $\sigma$ are allowed to depend on the path of $S$ or several sources of risk. In [27] the main interest lies in the multi-dimensional setting. We stress that it is possible to extend our results to a multi-dimensional framework. The type of condition will be similar as in [27].

In [103] the price process $P = (P_t)_{t\in[0,T]}$ is assumed to be the stochastic exponential of

$$dS_t = -\alpha(t, S, X)\theta_t dt + \alpha(t, S, X)dW_t,$$

where $X = (X_t)_{t\in[0,T]}$ is a continuous process, $\alpha$ and $\theta$ are suitable processes such that the integrals are well-defined and $\ell \otimes \mathbb{P}$-a.e. $\alpha \neq 0$. The process $X$ is called *information process*. This setting is closely related to those from Section 6.3.1. Let $\mathscr{W}$ be the Wiener measure and let $\nu$ be the law of $W - \int_0^\cdot \theta_s ds$. The main result from [103] is the following: If a.s. $\int_0^T \theta_s^2 ds < \infty$, then (NFLVR) $\Leftrightarrow \mathscr{W} \sim \nu$, see [103, Proposition 2.3]. This result is very different from ours, which are intended to give easy to verify conditions for a large class of models.

## 6.4 Modifying Minimal Local Martingale Measures

In Section 6.3.2.1 we proved conditions for the existence of the minimal (local) martingale measure in a Markov switching framework. We ask the following consecutive questions:

1. Does the MLMM change the dynamics of the Markov chain?
2. Is it possible to modify the MLMM such that the dynamics of the Markov chain are changed in a tractable manner?

In this section we answer these questions from a general perspective under an independence assumption, which holds in our Markov switching framework.

### 6.4.1 Martingale Problems

To characterize additional sources of risk in our financial market, we introduce a martingale problem.

Let $J$ be a Polish space, define $D(\mathbb{R}_+, J)$ to be the space of all càdlàg functions $\mathbb{R}_+ \to J$ and $\mathcal{D}$ to be the $\sigma$-field generated by the coordinate process $X = (X_t)_{t\geq 0}$, i.e. $X_t(\omega) = \omega(t)$ for $\omega \in D(\mathbb{R}_+, J)$ and $t \in \mathbb{R}_+$. We equip $D(\mathbb{R}_+, J)$ with the Skorokhod topology, which renders it into a Polish space. It is well-known that $\mathcal{D}$ is the Borel $\sigma$-field on $D(\mathbb{R}_+, J)$. We refer to [47, 70] for more details. Let $\mathbf{D}^o \triangleq (\mathcal{D}^o_t)_{t\geq 0}$ be the filtration induced by $X$, i.e. $\mathcal{D}^o_t \triangleq \sigma(X_s, s \in [0, t])$, and let $\mathbf{D} \triangleq (\mathcal{D}_t)_{t\geq 0}$ be its right-continuous version, i.e. $\mathcal{D}_t \triangleq \bigcap_{s>t} \mathcal{D}^o_s$ for all $t \in \mathbb{R}_+$.

Let $(B_n)_{n\in\mathbb{N}}$ be an increasing sequence of nonempty open sets in $J$ such that $\bigcup_{n\in\mathbb{N}} B_n = J$ and define

$$\rho_n(\omega) \triangleq \inf\big(t\in\mathbb{R}_+ : \omega(t)\notin B_n \text{ or } \omega(t-)\notin B_n\big), \quad \omega\in D(\mathbb{R}_+,J), n\in\mathbb{N}. \tag{6.27}$$

Due to [47, Proposition 2.1.5], $\rho_n$ is a $\mathbf{D}^o$-stopping time and, due to [47, Problem 4.27], $\rho_n \nearrow \infty$ as $n\to\infty$. We will use the sequence $(\rho_n)_{n\in\mathbb{N}}$ as a localizing sequence for test martingales of our martingale problem. We fix this sequence, because for some arguments we need a common localizing sequence consisting of $\mathbf{D}^o$-stopping times.

The input data for our martingale problem is the following:

(i) A set $A\subseteq C(J,\mathbb{R})$, where $C(J,\mathbb{R})$ denotes the space of continuous functions $J\to\mathbb{R}$.

(ii) A map $L\colon A\to\mathcal{PM}$ such that for all $f\in A, t\in\mathbb{R}_+$ and $\omega\in D(\mathbb{R}_+,J)$

$$\int_0^t \big|Lf(\omega,s)\big|ds<\infty,$$

where $\mathcal{PM}$ denotes the space of all $\mathbf{D}$-progressively measurable processes.

(iii) An initial value $j_0\in J$.

(iv) A time horizon $0<T\le\infty$.

We use the convention that in case $T=\infty$ the interval $[0,T]$ is identified with $\mathbb{R}_+$.

**Definition 6.1.** (i) *Let $(\Omega^o,\mathcal{F}^o,\mathbf{F}^o,\mathbb{P}^o)$ be a filtered probability space with right-continuous filtration $\mathbf{F}^o=(\mathcal{F}^o_t)_{t\in[0,T]}$, supporting a càdlàg, adapted, $J$-valued process $\xi=(\xi_t)_{t\in[0,T]}$. We say that $\xi$ is a* solution process to the martingale problem $(A,L,j_0,T)$, *if for all $f\in A$ and $n\in\mathbb{N}$ the process*

$$M^{f,n}\triangleq f(\xi_{\cdot\wedge\rho_n(\xi)})-f(\xi_0)-\int_0^{\cdot\wedge\rho_n(\xi)} Lf(\xi,s)ds \tag{6.28}$$

*is a martingale, $\mathbb{P}^o(\xi_0=j_0)=1$ and for all $t\in[0,T]$ there exists a constant $C=C(f,n,t)>0$ such that a.s. $\sup_{s\in[0,t]}|M^{f,n}_s|\le C$.*

(ii) *We say that the* martingale problem has a solution *if there exists a filtered probability space which supports a solution process.*

(iii) *We say that the martingale problem satisfies* uniqueness *if the laws (seen as Borel probability measures on $D(\mathbb{R}_+,J)$) of any two solution processes, possibly defined on different filtered probability spaces, coincide.*

(iv) *If for all $j_0\in J$ the martingale problem $(A,L,j_0,T)$ has a solution and satisfied uniqueness, we call the martingale problem $(A,L,T)$* well-posed.

Martingale problems were introduced by Stroock and Varadhan [137] in a diffusion setting. Martingale problems for semimartingales were studied in [65] and Markovian martingale problems with a Polish state space were studied in [47]. Our definition is unifying in the sense that it deals with non-Markovian processes and a Polish state space. Most of the conditions for existence and uniqueness given in [47, 65, 137] also apply to our setting.

**Example 6.1** (Martingale problem for Markov chains)**.** *Suppose that* $J = \{1, \dots, N\}$ *with* $1 \le N \le \infty$. *We equip* $J$ *with the discrete topology. Let* $\xi = (\xi_t)_{t\ge 0}$ *be a Feller–Markov chain with initial value* $j_0 \in J$ *and* $Q$*-matrix* $Q$. *Due to [122, Theorem 5], the generator* $(\mathcal{L}, D(\mathcal{L}))$ *of* $\xi$ *is given by* $\mathcal{L} = Q$ *and* $D(\mathcal{L}) = \{f \in C_0(J)\colon Qf \in C_0(J)\}$, *where* $C_0(J)$ *denotes the space of all continuous functions* $J \to \mathbb{R}$ *which are vanishing at infinity. Due to Dynkin's formula (see [123, Proposition VII.1.6]) the process* $\xi$ *solves the martingale problem* $(D(\mathcal{L}), \mathcal{L}, j_0, \infty)$ *and, due to [99, Theorem 3.33], the martingale problem satisfies uniqueness.*

*Conversely, in case* $\xi$ *is a solution process to the martingale problem* $(\mathcal{L}, D(\mathcal{L}), j_0, \infty)$, *where* $(\mathcal{L}, D(\mathcal{L}))$ *given as above is the generator of a Feller process,* $\xi$ *is a Feller–Markov chain with* $Q$*-matrix* $Q$, *see [47, Theorem 3.4.2] and [99, Theorem 3.33].*

### 6.4.2 How to modify the MLMM

Fix a finite time horizon $0 < T < \infty$ and let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a complete filtered probability space with right-continuous and complete filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$, which supports a solution process $\xi = (\xi_t)_{t\in[0,T]}$ to the martingale problem $(A, L, j_0, T)$. Moreover, assume that the martingale problem $(A, L, j_0, T)$ satisfies uniqueness. Let $W = (W_t)_{t\in[0,T]}$ be a one-dimensional Brownian motion such that $\sigma(W_t, t \in [0,T])$ and $\sigma(\xi_t, t \in [0,T])$ are independent. We think of $W$ and $\xi$ as two independent sources of risk influencing the market. The independence assumption is satisfied when $\xi$ is a Feller–Markov chain, see Remark 2.4.

In the following theorem we find a new property of the MLMM. To wit, we show that the MLMM preserves the independence of the sources of risk and their laws. As the M(L)MM is often used for pricing, this observation is important for analytical and numerical computations. We prove the following theorem in Section 6.7.

**Theorem 6.9.** *Let* $c = (c_t)_{t\in[0,T]}$ *be a real-valued progressively measurable process such that a.s.*

$$\int_0^T c_s^2 ds < \infty$$

*and define*

$$Z \triangleq \mathcal{E}\Big( \int_0^\cdot c_s dW_s \Big), \quad B \triangleq W - \int_0^\cdot c_s ds.$$

*Suppose further that* $Z$ *is a martingale and that the martingale problem* $(A, L, j_0, T)$ *satisfies uniqueness. Define* $\mathbb{Q}$ *by the Radon–Nikodym derivative* $\frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq Z_T$. *Then,* $\sigma(B_t, t \in [0,T])$ *and* $\sigma(\xi_t, t \in [0,T])$ *are* $\mathbb{Q}$*-independent,* $B$ *is a* $\mathbb{Q}$*-Brownian motion and* $\xi$ *is a solution process to the martingale problem* $(A, L, j_0, T)$ *on* $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{Q})$.

Let us outline an important consequence of Theorem 6.9: If the MLMM exists, then its density is of the same type as $Z$ in Theorem 6.9 and it follows that the joint law of the sources of risk remains unchanged by an equivalent change to the MLMM. In particular, in the setting of Section 6.2.2 this means that $\xi$ stays a Markov chain after a change to the MLMM.

We ask further whether it is possible to modify the MLMM such that the law of $\xi$ can be affected in a tractable manner. An answer to this question is provided by the next theorem. A proof can be found in Section 6.8.

**Theorem 6.10.** *Let $f \in A$ be strictly positive and suppose that the process*

$$Z \triangleq \frac{f(\xi)}{f(j_0)} \exp\Big( - \int_0^{\cdot} \frac{Lf(\xi,s)}{f(\xi_s)} ds\Big) \tag{6.29}$$

*is a martingale. Set*

$$A^* \triangleq \big\{g \in C(J, \mathbb{R}) \colon fg \in A\big\},$$

*and*

$$L^* g \triangleq \frac{L(fg) - gLf}{f}.$$

*Suppose that for every $g \in A^*$ and $n \in \mathbb{N}$ there exists a constant $C = C(g,n) > 0$ such that a.s.*

$$\sup_{t \in [0,T]} \Big| g(\xi_{t \wedge \rho_n(\xi)}) - g(\xi_0) - \int_0^{t \wedge \rho_n(\xi)} L^* g(\xi, s) ds \Big| \leq C.$$

*Define the probability measure $\mathbb{Q}$ by the Radon–Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq Z_T$. Then, $\sigma(\xi_t, t \in [0,T])$ and $\sigma(W_t, t \in [0,T])$ are $\mathbb{Q}$-independent, $W$ is a $\mathbb{Q}$-Brownian motion and $\xi$ is a solution process for the martingale problem $(A^*, L^*, j_0, T)$ on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{Q})$.*

**Remark 6.2.** (i) *For all $\omega \in D(\mathbb{R}_+, J)$ and $g \in A^*$*

$$\int_0^T \Big( \Big| \frac{Lf(\omega, s)}{f(\omega(s))} \Big| + \big| L^* g(\omega, s) \big| \Big) ds < \infty,$$

*because $f$ and $g$ are continuous and the set $\{\omega(t) \colon t \in [0,T]\} \subseteq J$ is relatively compact, see [47, Problem 16, p. 152]. Consequently, $Z$ and the martingale problem $(A^*, L^*, j_0, T)$ are well-defined.*

(ii) *In view of [47, Corollary 2.3.3], the process* (6.29) *is always a local martingale by the definition of the martingale problem.*

We explain an application of Theorem 6.10: Suppose that the MLMM exists. Then, using the change of measure described in Theorem 6.10, the MLMM can be changed further such that the law of $\xi$ gets affected as described in the theorem, while the local martingale property of the price process is preserved. We stress that in this manner the MLMM induces a family of ELMMs, which is often infinite. In a Markov switching framework with $N < \infty$ the following proposition explains how the $Q$-matrix of the driving Feller–Markov chain can be changed.

**Proposition 6.1.** *Suppose that $J = \{1, \dots, N\}$ with $N < \infty$ and*

$$Lf(\omega, s) = Qf(\omega(s)), \quad \omega \in D(\mathbb{R}_+, J), s \in \mathbb{R}_+,$$

*for a $Q$-matrix $Q = (q_{ij})_{i,j \in J}$ and $f \in A \triangleq \mathbb{R}^N$. Let $f \in (0,\infty)^N$ and $A^*, L^*$ as in Theorem 6.10. Then, $A^* = \mathbb{R}^N$ and*

$$L^* f(\omega, s) = Q^* f(\omega(s)), \quad f \in \mathbb{R}^N, \omega \in D(\mathbb{R}_+, J), s \in \mathbb{R}_+,$$

*for $Q^* = (q^*_{ij})_{i,j \in J}$ with*

$$q^*_{ij} \triangleq \begin{cases} q_{ij} \frac{f(j)}{f(i)}, & i \neq j, \\ - \sum_{k \neq i} q_{ik} \frac{f(k)}{f(i)}, & i = j. \end{cases}$$

*Proof.* See [115, Proposition 5.1]. □

A useful criterion for the martingale property of (6.29) is given by Theorem 6.11 below. We consider it as an extension of results from [21, 65, 137]. In the following $X = (X_t)_{t\geq 0}$ denotes the coordinate process on $D(\mathbb{R}_+, J)$.

**Definition 6.2.** *A set $\widetilde{A} \subseteq A$ is called a* determining set *for the martingale problem $(A, L, \infty)$ if for all $j_0 \in J$ a Borel probability measure $\mu$ on $D(\mathbb{R}_+, J)$ is the law of a solution process to the martingale problem $(A, L, j_0, \infty)$ if and only if for all $f \in \widetilde{A}$ and $n \in \mathbb{N}$ the process*

$$f(X_{\cdot\wedge\rho_n}) - f(X_0) - \int_0^{\cdot\wedge\rho_n} Lf(X, s)ds$$

*is a $\mu$-martingale and $\mu(X_0 = j_0) = 1$.*

For the martingale problem associated to a Feller–Markov chain it is always possible to find a countable determining set, see Example 2.4.

A proof for the following theorem can be found in Section 6.9.

**Theorem 6.11.** *Let $f, A^*$ and $L^*$ be as in Theorem 6.10. Moreover, assume there exists a countable determining set for the martingale problem $(A^*, L^*, \infty)$ and that*

$$L^*g(\xi, t) = Kg(\xi_t), \quad g \in A^*, t \in \mathbb{R}_+,$$

*where $K$ maps $A^*$ into the space of Borel functions $J \to \mathbb{R}$. Finally, assume that the martingale problem $(A^*, L^*, \infty)$ is well-posed and that $(\rho_n(\xi))_{n\in\mathbb{N}}$ is a localizing sequence for the local martingale* (6.29)*, see Remark 6.2. Then, the process* (6.29) *is a martingale.*

Roughly speaking, this theorem shows that in Markovian settings we can modify the law of $\xi$ whenever the martingale problem $(A^*, L^*, \infty)$ is well-posed.

**Remark 6.3.** *The existence of a solution to the martingale problem $(A^*, L^*, j_0, T)$ is often necessary for the martingale property of $Z$, see Theorem 6.10.*

## 6.5 Proof of Theorems 6.1 and 6.2

The following section is divided into three parts. In the first part we prove Lyapunov-type conditions for non-explosion of Itô processes, in the second part we prove non-existence conditions for Itô processes and in the third part we deduce Theorems 6.1 and 6.2.

### 6.5.1 Criteria for Non-Explosion

In this section we pose ourselves into a version of the setting from Section 6.2.1. Let $I = (l, r)$ be as in Section 6.2.1 and $(\Omega, \mathcal{F})$ be a measurable space which supports three real-valued processes $S = (S_t)_{t\in[0,T]}, b = (b_t)_{t\in[0,T]}$ and $\sigma = (\sigma_t)_{t\in[0,T]}$. For every $n \in \mathbb{N}$ we fix a probability measure $\mathbb{Q}^n$ and a right-continuous $\mathbb{Q}^n$-complete filtration $\mathbf{F}^n = (\mathcal{F}^n_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F})$ such that $S, b$ and $\sigma$ are $\mathbf{F}^n$-progressively measurable. We set $\tau_n$ as in Theorem 6.1, i.e.

$$\tau_n = \inf(t \in [0, T]\colon S_t \notin (l_n, r_n)),$$

where $l_n \searrow l, r_n \nearrow r$ are sequences such that $l < l_{n+1} < l_n < r_n < r_{n+1} < r$. Moreover, suppose that $\mathbb{Q}^n$-a.s.

$$dS_{t\wedge\tau_n} = b_t \mathbb{1}_{\{t\leq\tau_n\}}dt + \sigma_t \mathbb{1}_{\{t\leq\tau_n\}}dW^n_t, \quad S_0 \in I,$$

where $W^n = (W^n_t)_{t\in[0,T]}$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathbf{F}^n, \mathbb{Q}^n)$. It is implicit that the integrals are well-defined. We also assume that

$$\ell \otimes \mathbb{Q}^n\text{-a.e. } \sigma \neq 0 \text{ for all } n \in \mathbb{N} \tag{6.30}$$

and we fix a Borel function $\zeta\colon [0,T] \to \mathbb{R}_+$ such that $\zeta \in L^1([0,T])$.

#### 6.5.1.1 A Lyapunov Criterion

In this section we give a Lyapunov-type condition for

$$\limsup_{n\to\infty} \mathbb{Q}^n(\tau_n = \infty) = 1. \tag{6.31}$$

For $f \in C^1(I,\mathbb{R})$ with locally absolutely continuous derivative, it is well-known that there exists a $\ell$-null set $N^f \subset I$ such that $f$ has a second derivative $f''$ on $I\backslash N^f$. In this case, we set

$$\mathcal{L}f \triangleq f'(S)\mathbb{1}_I(S)b + \tfrac{1}{2}f''(S)\mathbb{1}_{I\backslash N^f}(S)\sigma^2.$$

**Theorem 6.12.** *Let $V\colon I \to (0,\infty)$ be differentiable with locally absolutely continuous derivative such that*

$$\limsup_{n\to\infty} V(l_n) \wedge V(r_n) = \infty. \tag{6.32}$$

*Suppose there exists a $\ell$-null set $N \subset I$ such that*

$$\begin{aligned}\mathcal{L}V(t)(\omega)\mathbb{1}_{I\backslash N}(S_t(\omega)) \leq \zeta(t)V(S_t(\omega))\mathbb{1}_{I\backslash N}(S_t(\omega))\\ \text{for } \ell\otimes\mathbb{Q}^n\text{-a.a. } (t,\omega)\in[0,T]\times\Omega, \quad n\in\mathbb{N}.\end{aligned} \tag{6.33}$$

*Then,* (6.31) *holds.*

*Proof.* Let $L^S$ be the local time of the continuous $\mathbb{Q}^n$-semimartingale $S_{\cdot\wedge\tau_n}$. The occupation times formula yields that $\mathbb{Q}^n$-a.s.

$$\int_0^{\tau_n\wedge T} \mathbb{1}_N(S_s)\sigma^2_s ds = 2\int_{-\infty}^{\infty} \mathbb{1}_N(x)L^S_T(x)dx = 0,$$

which implies that $\mathbb{Q}^n$-a.s $\ell(\{t \in [0,\tau_n \wedge T]\colon S_t \in N\}) = 0$. We will use this fact in the following without further reference.

Set

$$U^n \triangleq \exp\Big(-\int_0^{\cdot\wedge\tau_n} \zeta(s)ds\Big)V(S_{\cdot\wedge\tau_n}).$$

Using a generalized version of Itô's formula (see [125, Lemma IV.45.9]), we obtain that the process

$$U^n + \int_0^{\cdot\wedge\tau_n} \exp\Big(-\int_0^s \zeta(z)dz\Big)\big(\zeta(s)V(S_s) - \mathcal{L}V(s)\big)ds$$

is a local $\mathbb{Q}^n$-martingale. We deduce from (6.33) and the fact that non-negative local martingales are supermartingales, that $\mathbb{Q}^n$-a.s.

$$U^n \leq \mathbb{Q}^n\text{-supermartingale starting at } U_0 = V(S_0).$$

W.l.o.g. we assume that $S_0 \in (l_1, r_1)$. Note that for all $n \in \mathbb{N}$ we have $\mathbb{Q}^n$-a.s. $S_{\tau_n} \in \{l_n, r_n\}$ on $\{\tau_n \leq T\}$. We conclude that for all $n \in \mathbb{N}$

$$\begin{aligned}\mathbb{Q}^n(\tau_n \leq T) \exp\Big( - \int_0^T \zeta(s)ds\Big)(V(l_n) \wedge V(r_n)) &\leq \mathbb{E}^{\mathbb{Q}^n}\big[U^n_{\tau_n} \mathbb{1}_{\{\tau_n \leq T\}}\big] \\ &\leq \mathbb{E}^{\mathbb{Q}^n}\big[U^n_T\big] \leq V(S_0).\end{aligned}$$

By (6.32) there exists a sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that $V(l_{n_k}) \wedge V(r_{n_k}) > 0$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} V(l_{n_k}) \wedge V(r_{n_k}) = \infty$. We deduce from

$$0 \leq \mathbb{Q}^{n_k}(\tau_{n_k} \leq T) \leq V(S_0) \exp\Big(\int_0^T \zeta(s)ds\Big) \frac{1}{V(l_{n_k}) \wedge V(r_{n_k})}$$

that

$$\lim_{k\to\infty} \mathbb{Q}^{n_k}(\tau_{n_k} \leq T) = 0.$$

Since $\{\tau_n \leq T\}^c = \{\tau_n = \infty\}$, we obtain

$$1 = \lim_{k\to\infty} \mathbb{Q}^{n_k}(\tau_{n_k} = \infty) \leq \limsup_{n\to\infty} \mathbb{Q}^n(\tau_n = \infty) \leq 1,$$

which implies (6.31). The proof is complete. □

#### 6.5.1.2 An Integral Test

Let $\overline{a}\colon I \to (0,\infty)$ and $\underline{u}, \overline{u}\colon I \to \mathbb{R}$ be Borel functions such that

$$\frac{1}{\overline{a}} + |\underline{u}| + |\overline{u}| \in L^1_{\text{loc}}(I).$$

Recall from Section 6.2 that in case $(f, g)$ is one of the pairs $(\underline{u}, \overline{a}), (\overline{u}, \overline{a})$ we set

$$v(f,g)(x) = \int_{x_0}^x \exp\Big( - \int_{x_0}^y 2f(z)dz\Big) \int_{x_0}^y \frac{2\exp(\int_{x_0}^u 2f(z)dz)}{g(u)} dudy, \quad x \in I, \tag{6.34}$$

for a fixed $x_0 \in I$. The main result of this section is the following:

**Theorem 6.13.** *Suppose that*

$$\lim_{x\nearrow r} v\left(\overline{u}, \overline{a}\right)(x) = \lim_{x\searrow l} v\left(\underline{u}, \overline{a}\right)(x) = \infty. \tag{6.35}$$

*Moreover, for all $n \in \mathbb{N}$ assume that for $\ell \otimes \mathbb{Q}^n$-a.a. $(t,\omega) \in [0,T] \times \Omega\colon S_t(\omega) \in I$*

$$\begin{aligned}\sigma_t^2(\omega) &\leq \zeta(t)\overline{a}(S_t(\omega)), \\ b_t(\omega) &\leq \sigma_t^2(\omega)\overline{u}(S_t(\omega)), \\ b_t(\omega) &\geq \sigma_t^2(\omega)\underline{u}(S_t(\omega)).\end{aligned} \tag{6.36}$$

*Then,* (6.31) *holds.*

*Proof.* Due to [77, Lemma 5.5.26], there are differentiable functions $U_1\colon [x_0, r) \to [1, \infty)$ and $U_2\colon (l, x_0] \to [1, \infty)$ with locally absolutely continuous derivatives and a $\ell$-null set $N' \subset I$ such that $U_1$ is increasing, $U_2$ is decreasing, $U_1(x_0) = U_2(x_0) = 1, U_1'(x_0) = U_2'(x_0) = 0$, for all $x \in [x_0, r)\backslash N'$ and for all $y \in (l, x_0]\backslash N'$

$$\overline{a}(x)\left(\tfrac{1}{2}U_1''(x) + \overline{u}U_1'(x)\right) = U_1(x) \quad \text{and} \quad \overline{a}(y)\left(\tfrac{1}{2}U_2''(y) + \underline{u}U_2'(y)\right) = U_2(y),$$

$1 + v(\overline{u}, \overline{a}) \le U_1$ and $1 + v(\underline{u}, \overline{a}) \le U_2$. We define

$$V \triangleq \begin{cases} U_1, & \text{on } [x_0, r), \\ U_2, & \text{on } (l, x_0], \end{cases}$$

which is a differentiable function with locally absolutely continuous derivative. In particular, $V' \ge 0$ on $[x_0, r)$, $V' \le 0$ on $(l, x_0]$, $\frac{1}{2}V'' + \underline{u}V' \ge 0$ on $(l, x_0]\backslash N'$ and $\frac{1}{2}V'' + \overline{u}V' \ge 0$ on $[x_0, r)\backslash N'$. Furthermore,

$$\lim_{x \nearrow r} V(x) = \lim_{x \searrow l} V(x) = \infty,$$

due to the assumption (6.35). Let $\widetilde{N}$ be the set of all $(t, \omega) \in [0, T] \times \Omega$ such that $S_t(\omega) \in I$ and (6.36) holds. For all $(t, \omega) \in \widetilde{N}\colon S_t(\omega) \in [x_0, r)\backslash N'$

$$\begin{aligned} \mathcal{L}V(t)(\omega) &= \tfrac{1}{2}\sigma_t^2(\omega)V''(S_t(\omega)) + b_t(\omega)V'(S_t(\omega)) \\ &\le \sigma_t^2(\omega)\left(\tfrac{1}{2}V''(S_t(\omega)) + \overline{u}(S_t(\omega))V'(S_t(\omega))\right) \\ &\le \zeta(t)\overline{a}(S_t(\omega))\left(\tfrac{1}{2}V''(S_t(\omega)) + \overline{u}(S_t(\omega))V'(S_t(\omega))\right) = \zeta(t)V(S_t(\omega)). \end{aligned}$$

In the same manner we see that for all $(t, \omega) \in \widetilde{N}\colon S_t(\omega) \in (l, x_0]\backslash N'$

$$\mathcal{L}V(t)(\omega) \le \zeta(t)V(S_t(\omega)).$$

We conclude that (6.33) holds for $N = N'$. The claim follows from Theorem 6.12. □

### 6.5.2 Criteria for Non-Existence

In this section we give a converse to Theorem 6.13. As in Section 6.2, let $I = (l, r)$ with $-\infty \le l < r \le +\infty$ and let $\underline{a}\colon I \to (0, \infty)$ and $\underline{u}, \overline{u}\colon I \to \mathbb{R}$ be Borel functions such that

$$\frac{1}{\underline{a}} + |\underline{u}| + |\overline{u}| \in L^1_{\text{loc}}(I).$$

If $(f, g)$ is one of the pairs $(\underline{u}, \underline{a}), (\overline{u}, \underline{a})$, we set $v(f, g)$ as in (6.34).

Let $0 < T < \infty$, $(\Omega, \mathcal{F})$ be a measurable space with right-continuous filtration $\mathbf{F} = (\mathcal{F}_t)_{t \in [0,T]}$ and $s_0 \in I$. Suppose that $(\Omega, \mathcal{F}, \mathbf{F})$ supports three progressively measurable processes $S = (S_t)_{t \in [0,T]}, b = (b_t)_{t \in [0,T]}$ and $\sigma = (\sigma_t)_{t \in [0,T]}$. We define $\mathcal{I}$ be the set of all pairs $(\mathbb{Q}, B)$ consisting of a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ and an $(\mathbf{F}, \mathbb{Q})$-Brownian motion $B = (B_t)_{t \in [0,T]}$ with the properties that $S$ is $\mathbb{Q}$-a.s. $I$-valued and

$$dS_t = b_t dt + \sigma_t dB_t, \quad S_0 = s_0,$$

where it is implicit that the integrals are well-defined.

**Theorem 6.14.** (i) *Suppose that the pair* $(\underline{u}, \underline{a})$ *satisfies the YW conditions (see Section 6.2.1 for this terminology) and*

$$\lim_{x \nearrow r} v\left(\underline{u}, \underline{a}\right)(x) < \infty.$$

*Then, there exists no pair* $(\mathbb{Q}, B) \in \mathcal{I}$ *such that for* $\ell \otimes \mathbb{Q}$*-a.a.* $(t, \omega) \in [0, T] \times \Omega$

$$\begin{aligned} \underline{a}(S_t(\omega)) &\leq \sigma_t^2(\omega), \\ \underline{u}(S_t(\omega))\sigma_t^2(\omega) &\leq b_t(\omega). \end{aligned} \tag{6.37}$$

(ii) *Suppose that the pair* $(\overline{u}, \underline{a})$ *satisfies the YW conditions and*

$$\lim_{x \searrow l} v\left(\overline{u}, \underline{a}\right)(x) < \infty.$$

*Then, there exists no pair* $(\mathbb{Q}, B) \in \mathcal{I}$ *such that for* $\ell \otimes \mathbb{Q}$*-a.a.* $(t, \omega) \in [0, T] \times \Omega$

$$\begin{aligned} \underline{a}(S_t(\omega)) &\leq \sigma_t^2(\omega), \\ \overline{u}(S_t(\omega))\sigma_t^2(\omega) &\geq b_t(\omega). \end{aligned} \tag{6.38}$$

*Proof.* **(i).** We use a comparison and contradiction argument as in the proof of [27, Theorem 4.1]. For contradiction, assume that $(\mathbb{Q}, B) \in \mathcal{I}$ is such that (6.37) holds. W.l.o.g. we assume that $\mathbf{F}$ is $\mathbb{Q}$-complete. In the following we work on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{Q})$. As $\underline{a}$ is positive and continuous and a.s

$$\ell(\{t \in [0, T] \colon \underline{a}(S_t) > \sigma_t^2\}) = 0, \qquad \int_0^T \sigma_s^2 ds < \infty,$$

the function

$$[0, T] \ni t \mapsto \int_0^t \frac{\sigma_s^2}{\underline{a}(S_s)} ds$$

is a.s. finite, continuous and strictly increasing, which implies that the same holds for the function

$$\phi_t \triangleq \inf \Big( s \in [0, T] \colon \int_0^s \frac{\sigma_r^2}{\underline{a}(S_r)} dr \geq t \Big), \quad t \in [0, T],$$

see [123, pp. 179 – 180]. Furthermore, we have a.s. $\phi_t \leq t$ for all $t \in [0, T]$. We redefine $\phi_t$ to be zero on the null sets where the previously mentioned properties fail. Since $\mathbf{F}$ is complete, this modification of $(\phi_t)_{t \in [0,T]}$ is an increasing and continuous sequence of finite stopping times.

Next, we set $\mathbf{F}_\phi \triangleq (\mathcal{F}_{\phi_t})_{t \in [0,T]}$. The following lemma follows from [123, Propositions V.1.4, V.1.5].

**Lemma 6.3.** *Suppose that* $(H_t)_{t \in [0,T]}$ *is progressively measurable. Then, the time-changed process* $(H_{\phi_t})_{t \in [0,T]}$ *is* $\mathbf{F}_\phi$*-progressively measurable and a.s.*

$$\int_0^t H_{\phi_s} ds = \int_0^{\phi_t} \frac{H_s \sigma_s^2}{\underline{a}(S_s)} ds, \quad t \in [0, T],$$

*provided the integrals are well-defined. Moreover, the process* $B_\phi = (B_{\phi_t})_{t \in [0,T]}$ *is a continuous local* $\mathbf{F}_\phi$*-martingale with a.s.* $[B_\phi, B_\phi] = \phi$*, and if a.s.* $\int_0^T H_s^2 ds < \infty$ *then also* *a.s.* $\int_0^T H^2_{\phi_s} d\phi_s < \infty$ *and a.s.*

$$\int_0^t H_{\phi_s} dB_{\phi_s} = \int_0^{\phi_t} H_s dB_s, \quad t \in [0,T].$$

We deduce from Lemma 6.3 that a.s.

$$\begin{aligned}\ell\big(\{t \in [0,T] \colon \underline{a}(S_{\phi_t}) > \sigma^2_{\phi_t} \text{ or } \underline{u}(S_{\phi_t})\sigma^2_{\phi_t} > b_{\phi_t}\}\big) \\ = \int_0^{\phi_T} \frac{\mathbb{1}_{\{\underline{a}(S_s) > \sigma^2_s\} \cup \{\underline{u}(S_s)\sigma^2_s > b_s\}} \sigma^2_s}{\underline{a}(S_s)} ds = 0.\end{aligned}$$

We will use this observation in the following without further reference.
Applying Lemma 6.3 with

$$H_t \triangleq \frac{\underline{a}(S_t)}{\sigma^2_t} \mathbb{1}_{\{\sigma^2_t > 0\}}, \quad t \in [0,T],$$

yields that a.s.

$$d\phi_t = \frac{\underline{a}(S_{\phi_t})}{\sigma^2_{\phi_t}} dt. \tag{6.39}$$

Using again Lemma 6.3, we obtain that a.s. for all $t \in [0,T]$

$$\begin{aligned}S_{\phi_t} &= S_{\phi_0} + \int_0^{\phi_t} b_s ds + \int_0^{\phi_t} \sigma_s dB_s \\ &= s_0 + \int_0^t \frac{b_{\phi_s}\underline{a}(S_{\phi_s})}{\sigma^2_{\phi_s}} ds + \int_0^t \sigma_{\phi_s} dB_{\phi_s} \\ &= s_0 + \int_0^t \frac{b_{\phi_s}\underline{a}(S_{\phi_s})}{\sigma^2_{\phi_s}} ds + \int_0^t \underline{a}^{\frac{1}{2}}(S_{\phi_s}) dB'_s,\end{aligned}$$

where

$$B' \triangleq \int_0^{\cdot} \frac{\sigma_{\phi_s} dB_{\phi_s}}{\underline{a}^{\frac{1}{2}}(S_{\phi_s})}.$$

Due to Lemma 6.3 and (6.39), we obtain that a.s. for all $t \in [0,T]$

$$\begin{aligned}[B',B']_t &= \int_0^t \frac{\sigma^2_{\phi_s}}{\underline{a}(S_{\phi_s})} d[B_\phi, B_\phi]_s \\ &= \int_0^t \frac{\sigma^2_{\phi_s}}{\underline{a}(S_{\phi_s})} d\phi_s \\ &= \int_0^t \frac{\sigma^2_{\phi_s}}{\underline{a}(S_{\phi_s})} \frac{\underline{a}(S_{\phi_s})}{\sigma^2_{\phi_s}} ds = t.\end{aligned}$$

Consequently, $B'$ is a continuous local $\mathbf{F}_\phi$-martingale with a.s. $[B',B']_t = t$ for $t \in [0,T]$, i.e. an $\mathbf{F}_\phi$-Brownian motion due to Lévy's characterization. We summarize that

$$dS_{\phi_t} = \underline{a}(S_{\phi_t}) \frac{b_{\phi_t}}{\sigma^2_{\phi_t}} dt + \underline{a}^{\frac{1}{2}}(S_{\phi_t}) dB'_t, \quad S_{\phi_0} = s_0.$$

Using a standard extension of $(\Omega, \mathcal{F}, \mathbf{F}_\phi, \mathbb{Q})$ we can extend $(B'_t)_{t\in[0,T]}$ to a Brownian motion $(B'_t)_{t\geq 0}$, see, e.g. the proof of [123, Theorem V.1.7].

We will use the following terminology: When we say that $(V_t)_{t\geq 0}$ is a continuous $[l,r]$-valued process we mean that all its paths are continuous in the $[l,r]$-topology *and* absorbed in $\{l,r\}$, i.e. that $V_t = V_{\tau(V)}$ for all $t \geq \tau(V) \triangleq \inf(t \in \mathbb{R}_+ \colon V_t \notin I)$. This convention is in line with [77, Definition 5.5.20].

**Definition 6.3.** *Let $\mu\colon I \to \mathbb{R}$ and $v\colon I \to \mathbb{R}$ be Borel functions. We say that an SDE*

$$dV_t = \mu(V_t)dt + v(Y_t)dB^*_t, \tag{6.40}$$

*where $(B^*_t)_{t\geq 0}$ is a one-dimensional Brownian motion, satisfies* strong existence and uniqueness up to explosion, *if on any complete probability space $(\Omega^o, \mathcal{F}^o, \mathbb{P}^o)$ with complete right-continuous filtration $\mathbf{F}^o = (\mathcal{F}^o_t)_{t\geq 0}$, which supports a Brownian motion $(B^*_t)_{t\geq 0}$ and an $I$-valued $\mathcal{F}^o_0$-measurable random variable $\psi$, there exists a up to indistinguishability unique adapted continuous $[l,r]$-valued process $(V_t)_{t\geq 0}$ such that a.s.*

$$V_{t\wedge\theta_n} = \psi + \int_0^{t\wedge\theta_n} \mu(V_s)ds + \int_0^{t\wedge\theta_n} v(V_s)dB^*_s, \quad t \geq 0, n \in \mathbb{N},$$

*where*

$$\theta_n \triangleq \inf(t \in \mathbb{R}_+ \colon V_t \notin (l_n, r_n)), \quad n \in \mathbb{N}.$$

*It is implicit that the integrals are well-defined. The process $(V_t)_{t\geq 0}$ is called the* solution process to (6.40) with driver $(B^*_t)_{t\geq 0}$.

Due to [46, Remark 4.50 (2), Theorem 4.53], the SDE

$$dV_t = \underline{a}(V_t)\underline{u}(V_t)dt + \underline{a}^{\frac{1}{2}}(V_t)dB^*_t \tag{6.41}$$

satisfies strong existence and uniqueness up to explosion.

Consequently, there exists a solution process $(Y_t)_{t\geq 0}$ to (6.41) with driver $(B'_t)_{t\geq 0}$. The following lemma is proven after the proof of Theorem 6.14 is complete.

**Lemma 6.4.** *Almost surely $Y_t \leq S_{\phi_t}$ for all $t \leq T \wedge \tau(Y)$.*

As $(Y_t)_{t\geq 0}$ is regular due to [105, Proposition 2.2] and $\lim_{x\nearrow r} v\left(\underline{u}, \underline{a}\right)(x) < \infty$, we deduce from [105, Proposition 2.12] and [19, Theorem 1.1] that $(Y_t)_{t\in[0,T]}$ reaches $r$ with positive probability. Consequently, due to Lemma 6.4, $(S_t)_{t\in[0,T]}$ reaches $r$ with positive probability. This is a contradiction.

**(ii).** For contradiction, assume that $(\mathbb{Q}, B) \in \mathcal{I}$ is such that (6.38) holds. By the same arguments as in part (i), there exists a process $(Y_t)_{t\geq 0}$ such that

$$dY_t = \underline{a}(Y_t)\overline{u}(Y_t)dt + \underline{a}^{\frac{1}{2}}(Y_t)dB'_t, \quad Y_0 = s_0,$$

and a.s. $S_{\phi_t} \leq Y_t$ for all $t \leq T \wedge \tau(Y)$. Since $\lim_{x\searrow l} v\left(\overline{u}, \underline{a}\right)(x) < \infty$, the process $(Y_t)_{t\in[0,T]}$ reaches $l$ with positive probability and again the pathwise ordering gives a contradiction. □

*Proof of Lemma 6.4:* There are functions $h_n \in \mathscr{H}$ and $\kappa_n \in \mathscr{K}$ such that for all $x, y \in$

$[l_n, r_n]$

$$|\underline{a}^{\frac{1}{2}}(x) - \underline{a}^{\frac{1}{2}}(y)| \leq h_n(|x-y|), \quad |\underline{a}(x)\underline{u}(x) - \underline{a}(y)\underline{u}(y)| \leq \kappa_n(|x-y|).$$

We set

$$\rho_n \triangleq \inf(t \in [0,T] \colon S_{\phi_t} \notin (l_n, r_n) \text{ or } Y_t \notin (l_n, r_n)).$$

Note that for all $t \in (0,T]$ we have

$$\int_0^{t\wedge\rho_n} \frac{d[Y - S_\phi, Y - S_\phi]_s}{h_n^2(|Y_s - S_{\phi_s}|)} = \int_0^{t\wedge\rho_n} \frac{\big(\underline{a}^{\frac{1}{2}}(Y_s) - \underline{a}^{\frac{1}{2}}(S_{\phi_s})\big)^2}{h_n^2(|Y_s - S_{\phi_s}|)} ds \leq \int_0^t ds = t.$$

Thus, [123, Lemma IX.3.3] implies that the local time of $Y_{\cdot\wedge\rho_n} - S_{\phi_{\cdot\wedge\rho_n}}$ in the origin is a.s. zero. We deduce from Tanaka's formula that a.s.

$$(Y_{t\wedge\rho_n} - S_{\phi_{t\wedge\rho_n}})^+ = \int_0^{t\wedge\rho_n} \mathbb{1}_{\{Y_s - S_{\phi_s} > 0\}} d(Y_s - S_{\phi_s}), \quad t \in [0,T].$$

Taking expectation, using the martingale property of the Brownian part of $Y_{\cdot\wedge\rho_n} - S_{\phi_{\cdot\wedge\rho_n}}$ and Jensen's inequality yields that for all $t \in [0,T]$

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\big[(Y_{t\wedge\rho_n} - S_{\phi_{t\wedge\rho_n}})^+\big] &= \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{t\wedge\rho_n} \mathbb{1}_{\{Y_s - S_{\phi_s} > 0\}} \Big(\underline{a}(Y_s)\underline{u}(Y_s) - \underline{a}(S_{\phi_s})\frac{b_{\phi_s}}{\sigma^2_{\phi_s}}\Big) ds\Big] \\
&\leq \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{t\wedge\rho_n} \mathbb{1}_{\{Y_s - S_{\phi_s} > 0\}} \big|\underline{a}(Y_s)\underline{u}(Y_s) - \underline{a}(S_{\phi_s})\underline{u}(S_{\phi_s})\big| ds\Big] \\
&\leq \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{t\wedge\rho_n} \mathbb{1}_{\{Y_s - S_{\phi_s} > 0\}} \kappa_n(|Y_s - S_{\phi_s}|) ds\Big] \\
&\leq \int_0^t \mathbb{E}^{\mathbb{Q}}\big[\kappa_n((Y_{s\wedge\rho_n} - S_{\phi_{s\wedge\rho_n}})^+)\big] ds \\
&\leq \int_0^t \kappa_n\big(\mathbb{E}^{\mathbb{Q}}\big[(Y_{s\wedge\rho_n} - S_{\phi_{s\wedge\rho_n}})^+\big]\big) ds.
\end{aligned}$$

Finally, Bihari's lemma (see [27, Lemma E.2]) yields that for all $t \in [0,T]$

$$\mathbb{E}^{\mathbb{Q}}\big[(Y_{t\wedge\rho_n} - S_{\phi_{t\wedge\rho_n}})^+\big] = 0.$$

Consequently, due to the continuous paths of $Y$ and $S_\phi$, the claim follows. □

### 6.5.3 Proof of Theorem 6.1

As non-negative local martingales are supermartingales, $Z$ is a martingale if and only if $\mathbb{E}^{\mathbb{P}}[Z_T] = 1$. By (M1), we can define $\mathbb{Q}^n$ by the Radon–Nikodym derivative $\frac{d\mathbb{Q}^n}{d\mathbb{P}} = Z_{T\wedge\tau_n}$. We note that the assumption $\ell \otimes \mathbb{P}$-a.e. $\sigma \neq 0$ implies (6.30). Due to Girsanov's theorem, there exists a $\mathbb{Q}^n$-Brownian motion $B^n = (B^n_t)_{t\in[0,T]}$ such that

$$dS_{t\wedge\tau_n} = (b_t + c_t\sigma_t)\mathbb{1}_{\{t\leq\tau_n\}} dt + \sigma_t \mathbb{1}_{\{t\leq\tau_n\}} dB^n_t.$$

The monotone convergence theorem yields that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[Z_T\big] &= \limsup_{n\to\infty} \mathbb{E}^{\mathbb{P}}\big[Z_T \mathbb{1}_{\{\tau_n=\infty\}}\big] \\ &= \limsup_{n\to\infty} \mathbb{Q}^n(\tau_n = \infty).\end{aligned}$$

In view of (M2) and (M3), Theorem 6.13 yields that

$$\limsup_{n\to\infty} \mathbb{Q}^n(\tau_n = \infty) = 1.$$

Thus, $\mathbb{E}^{\mathbb{P}}[Z_T] = 1$ and the proof is complete. □

### 6.5.4 Proof of Theorem 6.2

For contradiction, assume that $(Z_t)_{t\in[0,T]}$ is a martingale. Define a probability measure $\mathbb{Q}$ by the Radon–Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq Z_T$. By Girsanov's theorem, there exists a $\mathbb{Q}$-Brownian motion $B = (B_t)_{t\in[0,T]}$ such that

$$dS_t = (b_t + c_t\sigma_t)dt + \sigma_t dB_t.$$

Consequently, in case (SL1) holds we obtain a contradiction to part (i) of Theorem 6.14 and in case (SL2) holds we obtain a contradiction to part (ii) of Theorem 6.14. The proof is complete. □

## 6.6 Proof of Theorem 6.3

The section is split into two parts: First, we prove existence, non-existence and local uniqueness for switching diffusions and second, we deduce Theorem 6.3.

### 6.6.1 Existence and Non-Existence Criteria

As in Section 6.2.2, let $I = (l, r)$ with $-\infty \le l < r \le +\infty$ and $J = \{1, \dots, N\}$ with $1 \le N \le \infty$. Moreover, let $u\colon I \times J \to \mathbb{R}$ and $\sigma\colon I \times J \to \mathbb{R}\backslash\{0\}$ be Borel functions such that

$$\frac{1 + u(\cdot, j)}{\sigma^2(\cdot, j)} \in L^1_{\text{loc}}(I) \text{ for all } j \in J. \tag{6.42}$$

We fix $x_0 \in I$ and set

$$v(x, j) \triangleq \int_{x_0}^x \exp\left(-\int_{x_0}^y \frac{2u(z,j)}{\sigma^2(z,j)}dz\right) \int_{x_0}^y \frac{2\exp\left(\int_{x_0}^s \frac{2u(z,j)}{\sigma^2(z,j)}dz\right)}{\sigma^2(s,j)} ds dy$$

for $(x, j) \in I \times J$. Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a filtered complete probability space with a right-continuous and complete filtration $\mathbf{F} = (\mathcal{F}_t)_{t\ge 0}$, which supports a Brownian motion $W = (W_t)_{t\ge 0}$, a $J$-valued irreducible continuous-time Feller–Markov chain $\xi = (\xi_t)_{t\ge 0}$ and an $I$-valued $\mathcal{F}_0$-measurable random variable $\phi$. The main result of this section is the following:

**Theorem 6.15.** (i) *Suppose that $\sigma$ satisfies the ES conditions for all $j \in J$ (see Section 6.2.2 for this terminology) and that*

$$\lim_{x \searrow l} v(x, j) = \lim_{x \nearrow r} v(x, j) = \infty \text{ for all } j \in J. \tag{6.43}$$

*Then, there exists an adapted $I$-valued continuous process $(Y_t)_{t \geq 0}$ such that*

$$Y = \phi + \int_0^{\cdot} u(Y_s, \xi_s) ds + \int_0^{\cdot} \sigma(Y_s, \xi_s) dW_s, \tag{6.44}$$

*where it is implicit that the integrals are well-defined.*

(ii) *Assume there exists a $j \in J$ such that $\sigma$ satisfies the ES conditions for $j$ and*

$$\lim_{x \searrow l} v(x, j) < \infty \text{ or } \lim_{x \nearrow r} v(x, j) < \infty.$$

*Let $0 < T \leq \infty$ be a time horizon. There exists no adapted $I$-valued continuous process $Y = (Y_t)_{t \in [0,T]}$ such that (6.44) holds.*

*Proof.* The case $N = 1$ concerns classical diffusions for which all claims are known, see [19, 46, 77] for details. We prove the claim under the assumption $N \geq 2$.

**(i).** Due to [46, Remark 4.50 (2), Theorem 4.53] and Feller's test for explosion ([77, Theorem 5.5.29]), for every $j \in J$ the SDE

$$dX_t^j = u(X_t^j, j) dt + \sigma(X_t^j, j) dW_t' \tag{6.45}$$

satisfies weak existence and pathwise uniqueness for all deterministic initial values. Thus, the process $Y$ can be constructed as in the proof of Theorem 2.4.

**(ii).** For contradiction, assume that $Y$ satisfies (6.44). Let $j \in J$ be such that $\lim_{x \searrow l} v(x, j) < \infty$ or $\lim_{x \nearrow r} v(x, j) < \infty$. We define

$$\delta \triangleq \inf(t \in \mathbb{R}_+ \colon \xi_t = j), \quad \zeta \triangleq \inf(t \geq \delta \colon \xi_t \neq j).$$

Due to the strong Markov property of $\xi$ and [74, Lemma 10.18], for all $G \in \mathcal{B}(\mathbb{R}_+)$ it holds that

$$\mathbb{P}(\zeta - \delta \in G, \delta < \infty) = -\mathbb{P}(\delta < \infty) \int_G q_{jj} e^{q_{jj} x} dx, \tag{6.46}$$

where $q_{jj} < 0$ is the $j$-th diagonal element of the $Q$-matrix of $\xi$.

Recall our convention that we call a process $V = (V_t)_{t \geq 0}$ to be continuous and $[l, r]$-valued in case all paths are continuous in the $[l, r]$-topology *and* absorbed in $\{l, r\}$, i.e. that $V_t = V_{\tau(V)}$ for all $t \geq \tau(V) \triangleq \inf(t \in \mathbb{R}_+ \colon V_t \notin I)$.

It follows from [46, Remark 4.50 (2), Theorem 4.53] that the SDE (6.45) satisfies strong existence and uniqueness up to explosion in the sense of Definition 6.3.

Consequently, there exists a continuous $[l, r]$-valued process $X = (X_t)_{t \geq 0}$ such that

$$dX_t = u(X_t, j) dt + \sigma(X_t, j) dW_t^\delta, \quad X_0 = Y_{\delta \wedge T}, \tag{6.47}$$

where $W^\delta \triangleq W_{\cdot + \delta \wedge T} - W_{\delta \wedge T}$ is a Brownian motion for the filtration $\mathbf{F}^\delta \triangleq (\mathcal{F}_{t + \delta \wedge T})_{t \geq 0}$. We prove the following lemma after the proof of (ii) is complete.

**Lemma 6.5.** *Almost surely* $Y_{t+\delta} = X_t$ *for all* $0 \le t \le \zeta - \delta$ *on* $\{\zeta \le T\}$.

Since on $\{\tau(X) < \infty\}$ we have $X_{\tau(X)} \notin I$, Lemma 6.5 implies that

$$\mathbb{P}(\tau(X) \le \zeta - \delta, \zeta \le T) = 0. \tag{6.48}$$

The following lemma is an extension of Lemma 2.3 for possibly explosive SDEs. Its proof is given after the proof of (ii) is complete.

**Lemma 6.6.** *Suppose that the SDE* (6.40) *satisfies strong existence and uniqueness up to explosion. Let* $\psi$ *be an* $I$*-valued* $\mathcal{F}_0$*-measurable random variable and let* $(V_t)_{t\ge0}$ *be the solution process to* (6.40) *with driver* $W$ *and initial value* $\psi$ *and let* $\tau$ *be a stopping time. Then, all adapted* $I$*-valued continuous processes* $(U_t)_{t\ge0}$ *with*

$$dU_t = \mu(U_t)\mathbb{1}_{\{t\le\tau\}}dt + v(U_t)\mathbb{1}_{\{t\le\tau\}}dW_t, \quad U_0 = \psi,$$

*are indistinguishable from* $(V_{t\wedge\tau})_{t\ge0}$.

Let $l_n \searrow l, r_n \nearrow r$ be sequences such that $l < l_{n+1} < l_n < r_n < r_{n+1} < r$ and set for a function $\alpha\colon \mathbb{R}_+ \to [l, r]$

$$\tau_n(\alpha) \triangleq \inf(t \in \mathbb{R}_+ \colon \alpha(t) \notin (l_n, r_n)).$$

We conclude from Lemma 6.6 and Galmarino's test ([70, Lemma III.2.43]) that for all $n \in \mathbb{N}$ the SDE

$$dX^j_t = u(X^j_t, j)\mathbb{1}_{\{t\le\tau_n(X^j)\}}dt + \sigma(X^j_t, j)\mathbb{1}_{\{t\le\tau_n(X^j)\}}dW_t, \tag{6.49}$$

satisfies weak existence and pathwise uniqueness in the usual sense, see [77, Definitions 5.3.1, 5.3.2]. Thus, due to [74, Theorem 18.14], there exists a Borel function $F^n\colon \mathbb{R} \times C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, I)$ such that whenever $X^j$ solves (6.49) with driver $W = (W_t)_{t\ge0}$ and (possibly stochastic) initial value $X^j_0$, then a.s. $X^j = F^n(X^j_0, W)$.

Lemma 6.6 and Galmarino's test yield that a.s.

$$\tau_n(X) = \tau_n(F^n(Y_{\delta\wedge T}, W^\delta)). \tag{6.50}$$

As strong existence and uniqueness up to explosion holds for the SDE (6.45), for a.a. $\omega \in \Omega$ there exists an $\mathbf{F}^\delta$-adapted continuous $[l, r]$-valued process $Y^\omega = (Y^\omega_t)_{t\ge0}$ such that

$$dY^\omega_t = u(Y^\omega_t, j)dt + \sigma(Y^\omega_t, j)dW^\delta_t, \quad Y^\omega_0 = Y_{\delta(\omega)\wedge T}(\omega) \in I.$$

We stress that the initial value $Y_{\delta(\omega)\wedge T}(\omega)$ is deterministic. Lemma 6.6 and Galmarino's test yield that a.s.

$$\tau_n(Y^\omega) = \tau_n(F^n(Y_{\delta(\omega)\wedge T}(\omega), W^\delta)). \tag{6.51}$$

We prove the following lemma after the proof of (ii) is complete.

**Lemma 6.7.** *For all* $G \in \mathcal{B}(\mathbb{R}_+)$ *we have a.s. on* $\{\delta \le T\}$

$$\mathbb{P}(\zeta - \delta \in G | \mathcal{F}_{\delta\wedge T}, \sigma(W^\delta_t, t \in \mathbb{R}_+)) = -\int_G q_{jj} e^{q_{jj}x} dx.$$

Using (6.48), the monotone convergence theorem and then (6.50), we obtain that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \mathbb{P}(\tau_n(X) \le \zeta - \delta, \zeta \le T) \\ &= \lim_{n\to\infty} \mathbb{P}(\tau_n(F^n(Y_{\delta\wedge T}, W^\delta)) \le \zeta - \delta, \zeta - \delta + \delta \le T), \end{aligned}$$

using [74, Theorem 5.4] and Lemma 6.7 we further obtain that

$$\begin{aligned} &= \lim_{n\to\infty} \int_0^T \mathbb{P}(\tau_n(F^n(Y_{\delta\wedge T}, W^\delta)) \le s, s + \delta \le T)(-q_{jj})e^{q_{jj}s} ds \\ &= \lim_{n\to\infty} \int_0^T \mathbb{E}^{\mathbb{P}}\big[\mathbb{P}(\tau_n(F^n(Y_{\delta\wedge T}, W^\delta)) \le s | \mathcal{F}_{\delta\wedge T})\mathbb{1}_{\{s+\delta\le T\}}\big](-q_{jj})e^{q_{jj}s} ds, \end{aligned}$$

which, due to [74, Theorem 5.4] and the independence of $W^\delta$ and $\mathcal{F}_{\delta\wedge T}$, equals

$$= \lim_{n\to\infty} \int_0^T \int_\Omega \mathbb{P}(\tau_n(F^n(Y_{\delta(\omega)\wedge T}(\omega), W^\delta)) \le s)\mathbb{1}_{\{s+\delta(\omega)\le T\}}\mathbb{P}(d\omega)(-q_{jj})e^{q_{jj}s} ds,$$

and finally, with (6.51) and the monotone convergence theorem, we obtain

$$\begin{aligned} &= \lim_{n\to\infty} \int_0^T \int_\Omega \mathbb{P}(\tau_n(Y^\omega) \le s)\mathbb{1}_{\{s+\delta(\omega)\le T\}}\mathbb{P}(d\omega)(-q_{jj})e^{q_{jj}s} ds \\ &= \int_0^T \int_\Omega \mathbb{P}(\tau(Y^\omega) \le s)\mathbb{1}_{\{s+\delta(\omega)\le T\}}\mathbb{P}(d\omega)(-q_{jj})e^{q_{jj}s} ds. \end{aligned}$$

Due to Feller's test for explosion ([77, Theorem 5.5.29]), $Y^\omega$ reaches $l$ or $r$ in finite time with positive probability. In fact, because $Y^\omega$ is regular due to [105, Proposition 2.2], [19, Theorem 1.1] implies that $Y^\omega$ even reaches $l$ or $r$ arbitrarily fast with positive probability, i.e. $\mathbb{P}(\tau(Y^\omega) \le \varepsilon) > 0$ for all $\varepsilon > 0$. Consequently, the identity

$$\int_0^T \int_\Omega \mathbb{P}(\tau(Y^\omega) \le s)\mathbb{1}_{\{s+\delta(\omega)\le T\}}\mathbb{P}(d\omega)(-q_{jj})e^{q_{jj}s} ds = 0$$

implies that for $\ell$-a.a. $s \in (0, T)$ we have $\mathbb{P}(\delta \le T - s) = 0$. However, because $\xi$ is irreducible, we have $\mathbb{P}(\xi_t = j) > 0$ for all $t > 0$. This is a contradiction and the proof of (ii) is complete. □

*Proof of Lemma 6.5:* Define $\iota \triangleq \zeta \wedge T - \delta \wedge T$. Note that for all $t \in \mathbb{R}_+$

$$\{\iota \le t\} = \{\zeta \le t + \delta \wedge T\} \in \mathcal{F}_{t+\delta\wedge T},$$

which shows that $\iota$ is an $\mathbf{F}^\delta$-stopping time. Moreover, we have for all $s, t \in \mathbb{R}_+$

$$\begin{aligned} \{s \wedge \iota + \delta \wedge T \le t\} = \big(\{s + \delta \wedge T \le t\} \cap \overbrace{\{s + \delta \wedge T \le \zeta \wedge T\}}^{\in \mathcal{F}_{s+\delta\wedge T}}\big) \\ \cup \big(\{\zeta \wedge T \le t\} \cap \underbrace{\{s + \delta \wedge T > \zeta \wedge T\}}_{\in \mathcal{F}_{\zeta\wedge T}}\big) \in \mathcal{F}_t. \end{aligned}$$

Thus, the random time $s \wedge \iota + \delta \wedge T$ is an $\mathbf{F}$-stopping time. We deduce from classical rules for time-changed stochastic integrals that a.s. for all $t \in \mathbb{R}_+$

$$
\begin{aligned}
Y_{t\wedge\iota+\delta\wedge T} &= \phi + \int_0^{t\wedge\iota+\delta\wedge T} u(Y_s,\xi_s)ds + \int_0^{t\wedge\iota+\delta\wedge T} \sigma(Y_s,\xi_s)dW_s \\
&= Y_{\delta\wedge T} + \int_0^t u(Y_{s\wedge\iota+\delta\wedge T}, j)\mathbb{1}_{\{s\le\iota\}}ds + \int_0^t \sigma(Y_{s\wedge\iota+\delta\wedge T}, j)\mathbb{1}_{\{s\le\iota\}}dW^\delta_s.
\end{aligned}
$$

Since the SDE (6.45) satisfies strong existence and uniqueness up to explosion, Lemma 6.6 implies that a.s. $Y_{t\wedge\iota+\delta\wedge T} = X_{t\wedge\iota}$ for all $t \in \mathbb{R}_+$. On $\{\zeta \le T\} \subseteq \{\delta \le T\}$ we have $\iota = \zeta - \delta$ and the claim follows. □

*Proof of Lemma 6.6:* Due to localization, we can assume that $\tau$ is finite. By [123, Proposition V.1.5] and Lévy's characterization, the process

$$\widehat{W}_t \triangleq W_{t+\tau} - W_\tau, \quad t \in \mathbb{R}_+,$$

is an $(\mathcal{F}_{t+\tau})_{t\ge0}$-Brownian motion. Due to the strong existence and uniqueness hypothesis, there exists a solution process $O = (O_t)_{t\ge0}$ to the SDE

$$dO_t = \mu(O_t)dt + v(O_t)d\widehat{W}_t, \quad O_0 = U_\tau.$$

We set

$$Z_t \triangleq \begin{cases} U_t, & t \le \tau, \\ O_{t-\tau}, & t > \tau. \end{cases}$$

The process $Z$ has continuous paths and similar arguments as used in the proof of Lemma 2.3 show that it is $\mathbf{F}$-adapted. Let

$$\theta^Z_n \triangleq \inf(t \in \mathbb{R}_+ \colon Z_t \notin (l_n, r_n)).$$

On $\{\tau \ge t \wedge \theta^Z_n\}$ we have a.s.

$$Z_{t\wedge\theta^Z_n} = \psi + \int_0^{t\wedge\theta^Z_n} \mu(Z_s)ds + \int_0^{t\wedge\theta^Z_n} v(Z_s)dW_s.$$

Next, we discuss what happens on the set $\{\tau < t \wedge \theta^Z_n\}$. Set

$$\theta^O_n \triangleq \inf(t \in \mathbb{R}_+ \colon O_t \notin (l_n, r_n)).$$

On $\{\tau < \theta^Z_n\}$ we have a.s. $\theta^Z_n = \theta^O_n + \tau$. Moreover, note that

$$t \wedge (\theta^O_n + \tau) - \tau = \begin{cases} \theta^O_n, & \text{if } \theta^O_n + \tau \le t, \\ t - \tau, & \text{if } t \le \theta^O_n + \tau. \end{cases}$$

Thus, $t \wedge (\theta^O_n + \tau) - \tau \le \theta^O_n$. Classical rules for time-changed stochastic integrals yield that on $\{\tau < t \wedge \theta_n^Z\}$ a.s.

$$\begin{aligned}
Z_{t\wedge\theta_n^Z} &= Z_\tau + \int_0^{t\wedge\theta_n^Z-\tau} \mu(O_s)ds + \int_0^{t\wedge\theta_n^Z-\tau} v(O_s)d\widehat{W}_s \\
&= Z_\tau + \int_\tau^{t\wedge\theta_n^Z} \mu(O_{s-\tau})ds + \int_\tau^{t\wedge\theta_n^Z} v(O_{s-\tau})dW_s \\
&= \psi + \int_0^{t\wedge\theta_n^Z} \mu(Z_s)ds + \int_0^{t\wedge\theta_n^Z} v(Z_s)dW_s.
\end{aligned}$$

We conclude that $Z$ is a solution process of the SDE (6.40) with driver $W$ and initial value $\psi$. By the strong existence and uniqueness hypothesis, we conclude that a.s. $Z = V$. The definition of $Z$ implies the claim. □

*Proof of Lemma 6.7:* Denote the Wiener measure with initial value $x \in \mathbb{R}$ by $\mathscr{W}_x$ and by $\mu_j$ the law of a Feller–Markov chain with the same $Q$-matrix as $\xi$ and initial value $j \in J$. Let $\mathcal{C}$ be the $\sigma$-field on $C(\mathbb{R}_+, \mathbb{R})$ generated by the coordinate process. It follows as in the proof of Lemma 2.5 that $(j, x) \mapsto (\mu_j \otimes \mathscr{W}_x)(F)$ is Borel for every $F \in \mathcal{D} \otimes \mathcal{C}$ and that the process $(\xi, W)$ is a strong Markov process in the following sense: For all $F \in \mathcal{D} \otimes \mathcal{C}$ and every stopping time $\theta$ we have a.s. on $\{\theta < \infty\}$

$$\mathbb{P}((\xi_{\cdot+\theta}, W_{\cdot+\theta}) \in F|\mathcal{F}_\theta) = (\mu_{\xi_\theta} \otimes \mathscr{W}_{W_\theta})(F).$$

For all $A \in \mathcal{D}$ and $F \in \mathcal{C}$ the strong Markov properties of $\xi, W$ and $(\xi, W)$ imply that a.s.

$$\begin{aligned}
&\mathbb{P}(\xi_{\cdot+\delta\wedge T} \in A, W_{\cdot+\delta\wedge T} \in F|\mathcal{F}_{\delta\wedge T}) \\
&\qquad = \mu_{\xi_{\delta\wedge T}}(A)\ \mathscr{W}_{W_{\delta\wedge T}}(F) \\
&\qquad = \mathbb{P}(\xi_{\cdot+\delta\wedge T} \in A|\mathcal{F}_{\delta\wedge T})\mathbb{P}(W_{\cdot+\delta\wedge T} \in F|\mathcal{F}_{\delta\wedge T}).
\end{aligned}$$

This implies that $\sigma(\zeta - \delta)$ and $\sigma(W_t^\delta, t \in \mathbb{R}_+)$ are independent given $\mathcal{F}_{\delta\wedge T}$. Now, [74, Proposition 5.6] yields that a.s.

$$\mathbb{P}(\zeta - \delta \in G|\mathcal{F}_{\delta\wedge T}, \sigma(W_t^\delta, t \in \mathbb{R}_+)) = \mathbb{P}(\zeta - \delta \in G|\mathcal{F}_{\delta\wedge T}).$$

By the strong Markov property of $\xi$ and (6.46), we have for $F \in \mathcal{F}_\delta$

$$\mathbb{P}(\zeta - \delta \in G, \delta < \infty, F) = -\int_G q_{jj}e^{q_{jj}x}dx\ \mathbb{P}(\delta < \infty, F).$$

The proof is complete. □

### 6.6.2 Local Uniqueness

For the space of continuous functions from $\mathbb{R}_+$ into $I$ or $\mathbb{R}$, we denote by $\mathcal{C}$ the $\sigma$-field generated by the coordinate process. Moreover, we denote by $\mathbf{C}^o \triangleq (\mathcal{C}_t^o)_{t\geq 0}$ the filtration generated by the corresponding coordinate process and by $\mathbf{C} \triangleq (\mathcal{C}_t)_{t\geq 0}$ its right-continuous version. The image space will be clear from the context. Let

$$\rho\colon C(\mathbb{R}_+, I) \times D(\mathbb{R}_+, J) \to [0, \infty]$$

be a $\mathbf{C}^o \otimes \mathbf{D}^o$-stopping time. An example for $\rho$ is

$$\tau(\alpha, \omega) \triangleq \inf(t \in \mathbb{R}_+ \colon \alpha(t) \notin U \text{ or } \omega(t) \notin V),$$

where $U \subseteq I$ and $V \subseteq J$ are open:

**Lemma 6.8.** *$\tau$ is a $\mathbf{C}^o \otimes \mathbf{D}^o$-stopping time.*

*Proof.* See [123, Proposition I.4.5] and [47, Proposition 2.1.5]. □

Let $u \colon I \times J \to \mathbb{R}$ and $\sigma \colon I \times J \to \mathbb{R}\backslash\{0\}$ be Borel functions such that (6.42) holds, $\sigma$ satisfies (6.43) and the ES conditions for all $j \in J$ (see Section 6.2.2 for this terminology). In other words, we ask that the conditions from part (i) of Theorem 6.15 hold.

For $i = 1, 2$, let $(\Omega^i, \mathcal{F}^i, \mathbf{F}^i, \mathbb{P}^i)$ be a filtered probability space with right-continuous complete filtration $\mathbf{F}^i = (\mathcal{F}^i_t)_{t \geq 0}$. Let $W^i = (W^i_t)_{t\geq 0}$ be a one-dimensional Brownian motion, $\xi^i = (\xi^i_t)_{t \geq 0}$ be a $J$-valued irreducible Feller–Markov chain with $Q$-matrix $Q$ and $\xi^i_0 = j_0 \in J$, and let $X^i = (X^i_t)_{t\geq 0}$ be an adapted continuous $I$-valued process such that

$$dX^i_{t\wedge\rho(X^i,\xi^i)} = u(X^i_t, \xi^i_t)\mathbb{1}_{\{t \leq \rho(X^i,\xi^i)\}}dt + \sigma(X^i_t, \xi^i_t)\mathbb{1}_{\{t\leq\rho(X^i,\xi^i)\}}dW^i_t, \quad X^i_0 = y_0 \in I.$$

It is implicit that the stochastic integrals are well-defined. We stress that $\xi^1$ and $\xi^2$ have the same law, because they have the same $Q$-matrix, see Example 6.1.

The main observation of this section is the following:

**Theorem 6.16.** $\mathbb{P}^1 \circ (X^1_{\cdot\wedge\rho(X^1,\xi^1)}, \xi^1)^{-1} = \mathbb{P}^2 \circ (X^2_{\cdot\wedge\rho(X^2,\xi^2)}, \xi^2)^{-1}$.

*Proof.* We follow the Yamada–Watanabe-type idea used in [66]. Define

$$\Omega^* \triangleq C(\mathbb{R}_+, I) \times C(\mathbb{R}_+, I) \times D(\mathbb{R}_+, J) \times C(\mathbb{R}_+, \mathbb{R}),$$
$$\mathcal{F}^* \triangleq \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{D} \otimes \mathcal{C},$$

and for $i = 1, 2$

$$Y^i \colon \Omega^* \to C(\mathbb{R}_+, I), \quad Y^i(\omega^1, \omega^2, \omega^3, \omega^4) = \omega^i,$$
$$Z^1 \colon \Omega^* \to D(\mathbb{R}_+, J), \quad Z^1(\omega^1, \omega^2, \omega^3, \omega^4) = \omega^3,$$
$$Z^2 \colon \Omega^* \to C(\mathbb{R}_+, \mathbb{R}), \quad Z^2(\omega^1, \omega^2, \omega^3, \omega^4) = \omega^4.$$

Denote the Wiener measure by $\mathscr{W}$ and denote the unique law of $\xi^i$ by $\mu$. Due to Remark 2.4, we have

$$\mathbb{P}^i \circ (\xi^i, W^i)^{-1} = \mu \otimes \mathscr{W}.$$

When the space of continuous functions is equipped with the local uniform topology it is a Polish spaces and the corresponding Borel $\sigma$-fields is generated by the coordinate process. Thus, there exist regular conditional probabilities

$$Q^i \colon D(\mathbb{R}_+, J) \times C(\mathbb{R}_+, \mathbb{R}) \times \mathcal{C} \to [0, 1]$$

such that

$$\mathbb{P}^i(X^i \in d\omega^1, \xi^i \in d\omega^2, W^i \in d\omega^3) = Q^i(\omega^2, \omega^3, d\omega^1)\mu(d\omega^2)\mathscr{W}(d\omega^3).$$

We define a probability measure $\mathbb{Q}$ on $(\Omega^*, \mathcal{F}^*)$ by

$$\mathbb{Q}(d\omega^1 \times d\omega^2 \times d\omega^3 \times d\omega^4) \triangleq Q^1(\omega^3, \omega^4, d\omega^1) Q^2(\omega^3, \omega^4, d\omega^2) \mu(d\omega^3) \mathscr{W}(d\omega^4).$$

With abuse of notation, denote the $\mathbb{Q}$-completion of $\mathcal{F}^*$ again by $\mathcal{F}^*$ and denote by $\mathcal{F}^*_t$ the $\mathbb{Q}$-completion of

$$\bigcap_{s>t} (\mathcal{C}_s \otimes \mathcal{C}_s \otimes \mathcal{D}_s \otimes \mathcal{C}_s), \quad t \in \mathbb{R}_+.$$

From now on we consider $(\Omega^*, \mathcal{F}^*, \mathbf{F}^* = (\mathcal{F}^*_t)_{t\geq 0}, \mathbb{Q})$ as underlying filtered probability space. In view of [66, Propositions 4.6, 5.6], for all $A \in \mathcal{C}_t$ the map $\omega \mapsto Q^i(\omega, A)$ is measurable w.r.t. the $\mu \otimes \mathscr{W}$-completion of $\bigcap_{s>t}(\mathcal{D}^o_s \otimes \mathcal{C}^o_s)$. In other words, [65, Hypothesis 10.43] is satisfied and we deduce from [66, Lemmata 2.7, 2.9], [65, Proposition 10.46] and Lévy's characterization that $Z^1$ is a Markov chain with $Q$-matrix $Q$, $Z^2$ is a Brownian motion and

$$dY^i_{t\wedge\rho(Y^i,Z^1)} = u(Y^i_t, Z^1_t) \mathbb{1}_{\{t\leq\rho(Y^i,Z^1)\}} dt + \sigma(Y^i_t, Z^1_t) \mathbb{1}_{\{t\leq\rho(Y^i,Z^1)\}} dZ^2_t, \quad Y^i_0 = y_0.$$

The proof of the following lemma is given after the proof of Theorem 6.16 is complete.

**Lemma 6.9.** *Almost surely* $Y^1_{\cdot\wedge\rho(Y^1,Z^1)\wedge\rho(Y^2,Z^1)} = Y^2_{\cdot\wedge\rho(Y^1,Z^1)\wedge\rho(Y^2,Z^1)}$.

Due to Galmarino's test, this implies a.s. $\rho(Y^1, Z^1) = \rho(Y^2, Z^1)$. Thus, a.s. $Y^1_{\cdot\wedge\rho(Y^1,Z^1)} = Y^2_{\cdot\wedge\rho(Y^2,Z^1)}$ and the claim follows from the definition of $\mathbb{Q}$. □

*Proof of Lemma 6.9:* Due to localization, we can assume that $\rho(Y^1, Z^1) \wedge \rho(Y^2, Z^1)$ is finite. Recall the following fact (see [123, Proposition III.3.5]): If $(Z_t)_{t\geq 0}$ is a Feller–Markov chain for the right-continuous filtration $\mathbf{G} = (\mathcal{G}_t)_{t\geq 0}$ and $\gamma$ is a finite $\mathbf{G}$-stopping time, then $(Z_{t+\gamma})_{t\geq 0}$ is a Feller–Markov chain for a filtration $(\mathcal{G}_{t+\gamma})_{t\geq 0}$ and both chains have the same $Q$-matrix. Due to Theorem 6.10 (i), for $i = 1, 2$ there exists a process $(O^i_t)_{t\geq 0}$ defined by

$$dO^i_t = u(O^i_t, Z^1_{t+\rho(Y^1,Z^1)\wedge\rho(Y^2,Z^1)}) dt + \sigma(O^i_t, Z^1_{t+\rho(Y^1,Z^1)\wedge\rho(Y^2,Z^1)}) dW^\rho_t,$$

where

$$W^\rho_t \triangleq Z^2_{t+\rho(Y^1,Z^1)\wedge\rho(Y^2,Z^1)} - Z^2_{\rho(Y^1,Z^1)\wedge\rho(Y^2,Z^1)}, \quad t \in \mathbb{R}_+,$$

with initial value $O^i_0 = Y^i_{\rho(Y^1,Z^1)\wedge\rho(Y^2,Z^1)}$. Now, set

$$V^i_t \triangleq \begin{cases} Y^i_t, & t \leq \rho(Y^1, Z^1) \wedge \rho(Y^2, Z^1), \\ O^i_{t-\rho(Y^1,Z^1)\wedge\rho(Y^2,Z^1)}, & t > \rho(Y^1, Z^1) \wedge \rho(Y^2, Z^1). \end{cases}$$

As in the proof of Lemma 6.6, we deduce from classical rules for time-changed stochastic integrals that

$$dV^i_t = u(V^i_t, Z^1_t) dt + \sigma(V^i_t, Z^1_t) dZ^2_t, \quad V^i_0 = y_0, \tag{6.52}$$

i.e. that $V^1$ and $V^2$ are global solutions. The proof of Proposition 2.8 shows a version of pathwise uniqueness for the global equation (6.52). Thus, the claim follows. □

### 6.6.3 Proof of Theorem 6.3

*(i).* Recall that $J = \{1, \dots, N\}$ with $1 \le N \le \infty$. For $n \in \mathbb{N}$ define

$$\tau_n \triangleq \inf(t \in [0,T] \colon S_t \notin (l_n, r_n) \text{ or } \xi_t > n \wedge N).$$

As $c$ is assumed to be bounded on compact subsets of $I \times J$, Novikov's condition implies that $(\tau_n)_{n\in\mathbb{N}}$ is a localizing sequence for $Z$. We define $\mathbb{Q}^n$ by the Radon–Nikodym derivative $\frac{d\mathbb{Q}^n}{d\mathbb{P}} \triangleq Z_{T\wedge\tau_n}$. By Girsanov's theorem,

$$B^n \triangleq W - \int_0^{\cdot\wedge\tau_n} c(S_s, \xi_s) ds$$

is a $\mathbb{Q}^n$-Brownian motion such that

$$dS_{t\wedge\tau_n} = (b(S_t,\xi_t) + c(S_t,\xi_t)\sigma(S_t,\xi_t))\mathbb{1}_{\{t\le\tau_n\}}dt + \sigma(S_t,\xi_t)\mathbb{1}_{\{t\le\tau_n\}}dB^n_t.$$

We deduce from Remark 2.4, Example 6.1 and Theorem 6.9 that under $\mathbb{Q}^n$ the process $\xi$ remains a Feller–Markov chain with unchanged $Q$-matrix. W.l.o.g. we extend $W, \xi$ and $\mathbf{F}$ to the infinite time interval $\mathbb{R}_+$. Applying Theorem 6.15 with $u \triangleq b + c\sigma$ yields that on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ there exists an adapted continuous $I$-valued process $X = (X_t)_{t\ge0}$ such that

$$dX_t = (b(X_t,\xi_t) + c(X_t,\xi_t)\sigma(X_t,\xi_t))dt + \sigma(X_t,\xi_t)dW_t, \quad X_0 = S_0.$$

We set

$$\rho_n \triangleq \inf(t \in [0,T] \colon X_t \notin (l_n, r_n) \text{ or } \xi_t > n \wedge N).$$

It follows from Lemma 6.8 and Theorem 6.16 that

$$\mathbb{P} \circ (X_{\cdot\wedge\rho_n}, \xi)^{-1} = \mathbb{Q}^n \circ (S_{\cdot\wedge\tau_n}, \xi)^{-1}.$$

Consequently, using Galmarino's test, we obtain that

$$\lim_{n\to\infty} \mathbb{Q}^n(\tau_n = \infty) = \lim_{n\to\infty} \mathbb{P}(\rho_n = \infty) = 1.$$

Now, it follows as in the proof of Theorem 6.1 that $Z$ is a martingale.

*(ii).* This result follows similar as Theorem 6.2, where Theorem 6.15 has to be used instead of Theorem 6.14. We omit the details. □

## 6.7 Proof of Theorem 6.9

*Step 1.* Let $g \in A$ and set

$$M^g_t \triangleq g(\xi_t) - g(\xi_0) - \int_0^t Lg(\xi, s)ds, \quad t \in [0,T]. \tag{6.53}$$

Due to the definition of the martingale problem $(A, L, T)$, the process $M^g$ is a local martingale with localizing sequence $(\rho_n(\xi))_{n\in\mathbb{N}}$. Thus, the quadratic variation process $[M^g, W]$ is well-defined. Our first step is to show that a.s. $[M^g, W] = 0$. We explain that $WM^g$ is a local martingale for the completed right-continuous version of the natural filtration of $\xi$ and $W$. Let $0 \le s < t \le T$, $G \in \sigma(W_r, r \in [0,s]) \triangleq \mathcal{W}_s$ and $F \in \sigma(\xi_r, r \in$

$[0, s]) \triangleq \mathcal{E}_s$. The independence assumption yields that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[W_t M^g_{t\wedge\rho_m(\xi)}\mathbb{1}_{G\cap F}\big] &= \mathbb{E}^{\mathbb{P}}\big[W_t\mathbb{1}_G\big]\mathbb{E}^{\mathbb{P}}\big[M^g_{t\wedge\rho_m(\xi)}\mathbb{1}_F\big] \\ &= \mathbb{E}^{\mathbb{P}}\big[W_s\mathbb{1}_G\big]\mathbb{E}^{\mathbb{P}}\big[M^g_{s\wedge\rho_m(\xi)}\mathbb{1}_F\big] \\ &= \mathbb{E}^{\mathbb{P}}\big[W_s M^g_{s\wedge\rho_m(\xi)}\mathbb{1}_{G\cap F}\big].\end{aligned}$$

By a monotone class argument, we have

$$\mathbb{E}^{\mathbb{P}}\big[W_t M^g_{t\wedge\rho_m(\xi)}\mathbb{1}_B\big] = \mathbb{E}^{\mathbb{P}}\big[W_s M^g_{s\wedge\rho_m(\xi)}\mathbb{1}_B\big]$$

for all $B \in \mathcal{W}_s \vee \mathcal{E}_s$. Due to the downwards theorem ([124, Theorem II.51.1]), the process $WM^g_{\cdot\wedge\rho_m(\xi)}$ is a martingale for the completed right-continuous version $\mathbf{G} \triangleq (\mathcal{G}_t)_{t\in[0,T]}$ of $(\mathcal{W}_t \vee \mathcal{E}_t)_{t\in[0,T]}$. Consequently, because $\rho_m(\xi) \nearrow \infty$ as $m \to \infty$, $WM^g$ is a local $\mathbf{G}$-martingale. By the tower rule, also $W$ and $M^g$ are local $\mathbf{G}$-martingales. Integration by parts implies that

$$[W, M^g] = WM^g - \int_0^\cdot W_s dM^g_s - \int_0^\cdot M^g_{s-} dW_s,$$

where the stochastic integrals are defined as local $\mathbf{G}$-martingales. Here, we use that $[W, M^g]$ can be defined independently of the filtration. Thus, the process $[W, M^g]$ is a continuous local $\mathbf{G}$-martingale of finite variation and hence a.s. $[W, M^g] = 0$.

*Step 2.* In this step we identify the laws of $B$ and $\xi$ under $\mathbb{Q}$. Clearly, $B$ is a $\mathbb{Q}$-Brownian motion due to Girsanov's theorem. Next, we show that on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{Q})$ the process $\xi$ is a solution process for the martingale problem $(A, L, T)$. By Step 1 and Girsanov's theorem, the process

$$M^g - \int_0^t \frac{d[Z, M^g]_s}{Z_s} = M^g - \int_0^\cdot \theta_s d\,[W, M^g]_s = M^g$$

is a local $\mathbb{Q}$-martingale. The equivalence $\mathbb{Q} \sim \mathbb{P}$ implies that $\mathbb{Q}(\xi_0 = e_0) = 1$ and that $M^g_{\cdot\wedge\rho_n(\xi)}$ is $\mathbb{Q}$-a.s. bounded. Thus, the claim follows.

*Step 3.* We prove $\mathbb{Q}$-independence of $B$ and $\xi$ by the argument used in the proof of Lemma 2.2, which is based on an idea from [47, Theorem 4.10.1]. We define $C^2_b(\mathbb{R})$ to be the set of all bounded twice continuously differentiable functions $\mathbb{R} \to \mathbb{R}$ with bounded first and second derivative. Suppose that $f \in C^2_b(\mathbb{R})$ with $\inf_{x\in\mathbb{R}} f(x) > 0$ and define

$$K^f_t \triangleq f(B_t)\exp\Big(-\frac{1}{2}\int_0^t \frac{f''(B_s)}{f(B_s)}ds\Big), \quad t \in [0, T].$$

By Itô's formula, we have

$$\begin{aligned}dK^f_t &= \exp\Big(-\frac{1}{2}\int_0^t \frac{f''(B_s)}{f(B_s)}ds\Big)\big(df(B_t) - \tfrac{1}{2}f''(B_t)dt\big) \\ &= \exp\Big(-\frac{1}{2}\int_0^t \frac{f''(B_s)}{f(B_s)}ds\Big)f'(B_t)dB_t.\end{aligned}$$

Thus, $K^f$ is a $\mathbb{Q}$-martingale, as it is a bounded local $\mathbb{Q}$-martingale. Recall that the quadratic variation process is not affected by an equivalent change of measure. By Step 1,

$\mathbb{Q}$-a.s. $[B, M^g] = 0$. Due to integration by parts, we obtain that

$$\begin{aligned} dK^f_t M^g_t &= K^f_t dM^g_t + M^g_{t-} dK^f_t + d[K^f, M^g]_t \\ &= K^f_t dM^g_t + M^g_{t-} dK^f_t, \end{aligned}$$

which implies that $K^f M^g_{\cdot\wedge\rho_m(\xi)}$ is a $\mathbb{Q}$-martingale, as it is a bounded local $\mathbb{Q}$-martingale.
Let $\zeta$ be a stopping time such that $\zeta \le T$ and set

$$\widetilde{\mathbb{Q}}(G) \triangleq \frac{\mathbb{E}^{\mathbb{Q}}\big[\mathbb{1}_G K^f_\zeta\big]}{\mathbb{E}^{\mathbb{Q}}\big[K^f_\zeta\big]}, \quad G \in \mathcal{F}.$$

Using that $K^f M^g_{\cdot\wedge\rho_m(\xi)}, K^f$ and $M^f_{\cdot\wedge\rho_m(\xi)}$ are $\mathbb{Q}$-martingales (see also Step 2), the optional stopping theorem implies that for all stopping times $\psi \le T$

$$\mathbb{E}^{\widetilde{\mathbb{Q}}}\big[M^g_{\psi\wedge\rho_m(\xi)}\big] = \frac{\mathbb{E}^{\mathbb{Q}}\big[M^g_{\psi\wedge\rho_m(\xi)} K^f_\zeta\big]}{\mathbb{E}^{\mathbb{Q}}\big[K^f_\zeta\big]} = 0.$$

Consequently, by [123, Proposition II.1.4], $M^g_{\cdot\wedge\rho_m(\xi)}$ is a $\widetilde{\mathbb{Q}}$-martingale. Since $\widetilde{\mathbb{Q}} \sim \mathbb{Q}$, this implies that on $(\Omega, \mathcal{F}, \mathbf{F}, \widetilde{\mathbb{Q}})$ the process $\xi$ is a solution process for the martingale problem $(A, L, T)$. The uniqueness assumption for the martingale problem $(A, L, j_0, T)$ implies that

$$\widetilde{\mathbb{Q}}(\Gamma) = \mathbb{Q}(\Gamma) \tag{6.54}$$

for all

$$\Gamma \triangleq \big\{\xi_{t_1} \in G_1, \dots, \xi_{t_n} \in G_n\big\},$$

where $G_1, \dots, G_n \in \mathcal{B}(J)$ and $0 \le t_1 < \dots < t_n \le T$. We fix $\Gamma$ such that $\mathbb{Q}(\Gamma) > 0$ and define

$$\widehat{\mathbb{Q}}(F) \triangleq \frac{\mathbb{E}^{\mathbb{Q}}\big[\mathbb{1}_F \mathbb{1}_\Gamma\big]}{\mathbb{Q}(\Gamma)}, \quad F \in \mathcal{F}.$$

Using the definition of $\widetilde{\mathbb{Q}}$, (6.54), the fact that $K^f$ is a $\mathbb{Q}$-martingale and the optional stopping theorem, we obtain

$$\mathbb{E}^{\widehat{\mathbb{Q}}}\big[K^f_\zeta\big] = \frac{\mathbb{E}^{\mathbb{Q}}\big[K^f_\zeta \mathbb{1}_\Gamma\big]}{\mathbb{Q}(\Gamma)} = \frac{\widetilde{\mathbb{Q}}(\Gamma)\mathbb{E}^{\mathbb{Q}}\big[K^f_\zeta\big]}{\mathbb{Q}(\Gamma)} = \mathbb{E}^{\mathbb{Q}}\big[K^f_\zeta\big] = f(0).$$

As $\zeta$ was arbitrary, we conclude that $K^f$ is a $\widehat{\mathbb{Q}}$-martingale. Furthermore, $\widehat{\mathbb{Q}}(B_0 = 0) = 1$ follows from the fact that $B$ is a $\mathbb{Q}$-Brownian motion. Finally, due to [47, Proposition 4.3.3], the process $B$ is a $\widehat{\mathbb{Q}}$-Brownian motion. We conclude that

$$\widehat{\mathbb{Q}}\big(B_{s_1} \in F_1, \dots, B_{s_k} \in F_k\big) = \mathbb{Q}\big(B_{s_1} \in F_1, \dots, B_{s_k} \in F_k\big),$$

for all $F_1, \dots, F_k \in \mathcal{B}(\mathbb{R})$ and $0 \le s_1 < \dots < s_k \le T$. By the definition of $\widehat{\mathbb{Q}}$, we have proven that

$$\mathbb{Q}\big(B_{s_1} \in F_1, \ldots, B_{s_k} \in F_k, \xi_{t_1} \in G_1, \ldots, \xi_{t_n} \in G_n\big)$$
$$= \mathbb{Q}\big(B_{s_1} \in F_1, \ldots, B_{s_k} \in F_k\big)\mathbb{Q}\big(\xi_{t_1} \in G_1, \ldots, \xi_{t_n} \in G_n\big),$$

which implies that the $\sigma$-fields $\sigma(\xi_t, t \in [0,T])$ and $\sigma(B_t, t \in [0,T])$ are $\mathbb{Q}$-independent. The proof is complete. □

## 6.8 Proof of Theorem 6.10

As $\sigma(\xi_t, t \in [0,T])$ and $\sigma(W_t, t \in [0,T])$ are assumed to be $\mathbb{P}$-independent, it follows as in the proof of Theorem 6.9 that a.s. $[Z, W] = 0$. Thus, Girsanov's theorem implies that $W$ is a $\mathbb{Q}$-Brownian motion.

Take $0 \le s_1 < \cdots < s_m \le T, 0 \le t_1 < \cdots < t_n \le T, (G_k)_{k \le m} \subset \mathcal{B}(J)$ and $(F_k)_{k \le n} \subset \mathcal{B}(\mathbb{R})$, and set

$$\Gamma_1 \triangleq \big\{\xi_{s_1} \in G_1, \ldots, \xi_{s_m} \in G_m\big\},$$
$$\Gamma_2 \triangleq \big\{W_{t_1} \in F_1, \ldots, W_{t_n} \in F_n\big\}.$$

The $\mathbb{P}$-independence of $\sigma(\xi_t, t \in [0,T])$ and $\sigma(W_t, t \in [0,T])$ and the uniqueness of the Wiener measure yield that

$$\begin{aligned}\mathbb{Q}(\Gamma_1 \cap \Gamma_2) &= \mathbb{E}^{\mathbb{P}}\big[Z_T \mathbb{1}_{\Gamma_1 \cap \Gamma_2}\big] \\ &= \mathbb{E}^{\mathbb{P}}\big[Z_T \mathbb{1}_{\Gamma_1}\big]\mathbb{P}(\Gamma_2) \\ &= \mathbb{Q}(\Gamma_1)\mathbb{Q}(\Gamma_2).\end{aligned}$$

We conclude that $\sigma(\xi_t, t \in [0,T])$ and $\sigma(W_t, t \in [0,T])$ are $\mathbb{Q}$-independent.

For $g \in A^*$ we set

$$\begin{aligned}M^g_t &\triangleq g(\xi_t) - g(\xi_0) - \int_0^t L^* g(\xi, s) ds, \quad t \in [0,T], \\ K^f_t &\triangleq f(\xi_t) - f(\xi_0) - \int_0^t L f(\xi, s) ds, \quad t \in [0,T], \\ K^{fg}_t &\triangleq f(\xi_t)g(\xi_t) - f(\xi_0)g(\xi_0) - \int_0^t L(fg)(\xi, s) ds, \quad t \in [0,T].\end{aligned}$$

The processes $K^f$ and $K^{fg}$ are local $\mathbb{P}$-martingales. We set

$$V_t \triangleq \frac{1}{f(\xi_0)} \exp\Big( - \int_0^t \frac{Lf(\xi, s)}{f(\xi_s)} ds\Big), \quad t \in [0,T].$$

Integration by parts implies that

$$dZ_t = V_t\Big(df(\xi_t) - f(\xi_t)\frac{L(\xi, t)}{f(\xi_t)} dt\Big) = V_t dK^f_t.$$

Using again integration by parts and the identity $L^*g = \frac{1}{f}(L(fg) - gLf)$ yields

$$\begin{aligned} dZ_tM_t^g &= Z_{t-}dM_t^g + M_{t-}^g dZ_t + d[Z, M^g]_t \\ &= V_t\Big(f(\xi_{t-})dM_t^g + M_{t-}^g dK_t^f + d[f(\xi), g(\xi)]_t\Big) \\ &= V_t\Big(f(\xi_{t-})dg(\xi_t) - f(\xi_{t-})L^*g(\xi, t)dt + g(\xi_{t-})df(\xi_t) \\ &\qquad - g(\xi_{t-})Lf(\xi, t)dt - \Big(g(\xi_0) + \int_0^t L^*g(\xi, s)ds\Big)dK_t^f + d[f(\xi), g(\xi)]_t\Big) \\ &= V_t\Big(d\big((fg)(\xi_t)\big) - L(fg)(\xi, t)dt - \Big(g(\xi_0) + \int_0^t L^*g(\xi, s)ds\Big)dK_t^f\Big) \\ &= V_t\Big(dK_t^{fg} - \Big(g(\xi_0) + \int_0^t L^*g(\xi, s)ds\Big)dK_t^f\Big). \end{aligned}$$

We conclude that $ZM^g$ is a local $\mathbb{P}$-martingale and it follows from [70, Proposition III.3.8] that $M^g$ is a local $\mathbb{Q}$-martingale. Due to the equivalence $\mathbb{Q} \sim \mathbb{P}$, we conclude that on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{Q})$ the process $\xi$ is a solution process to the martingale problem $(A^*, L^*, j_0, T)$. The proof is complete. □

## 6.9 Proof of Theorem 6.11

Let $(X_t)_{t\geq 0}$ be the coordinate process on $D(\mathbb{R}_+, J)$ and denote

$$M_t^f \triangleq \frac{f(X_t)}{f(j_0)} \exp\Big( - \int_0^t \frac{Lf(X, s)}{f(X_s)} ds\Big), \quad t \in [0, T].$$

Define by $\mu \triangleq \mathbb{P} \circ \xi^{-1}$ a Borel probability measure on $D(\mathbb{R}_+, J)$. We have to show that

$$\mathbb{E}^\mu\big[M_T^f\big] = 1.$$

It follows from [65, Lemma 2.9] that $M^f$ is a local $\mu$-martingale with localizing sequence $(\rho_n)_{n\in\mathbb{N}}$. For all $n \in \mathbb{N}$, define a Borel probability measure $\mu_n$ on $D(\mathbb{R}_+, J)$ via the Radon–Nikodym derivative

$$\frac{d\mu_n}{d\mu} = M^f_{T\wedge\rho_n}.$$

The following lemma can be proven similar to Lemma 2.10 and Proposition 2.9.[2]

**Lemma 6.10.** *Let $\mu^*$ be the unique law of a solution process to the martingale problem $(A^*, L^*, j_0, \infty)$. For all $n \in \mathbb{N}$ we have $\mu_n = \mu^*$ on $\mathcal{D}^o_{T\wedge\rho_n}$.*

Recalling that $\{\rho_n > T\} \in \mathcal{D}^o_{T\wedge\rho_n}$, Lemma 6.10 implies that

$$\mathbb{E}^\mu\big[M_T^f\big] = \lim_{n\to\infty} \mathbb{E}^\mu\big[M^f_{T\wedge\rho_n}\mathbb{1}_{\{\rho_n > T\}}\big] = \lim_{n\to\infty} \mu^*(\rho_n > T) = 1.$$

This completes the proof. □

[2] The existence of a countable determining set is needed to follow the proof of Proposition 2.9.

# Bibliography


[1] Aliprantis, C. D. and Border, K. (2013). *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer Berlin.

[2] Anderson, W. J. (2012). *Continuous-Time Markov Chains: An Applications-Oriented Approach*. Springer New York.

[3] Andres, S., Deuschel, J.-D., and Slowik, M. (2016). Harnack inequalities on weighted graphs and some applications to the random conductance model. *Probability Theory and Related Fields*, 164(3):931–977.

[4] Azencott, R. (1974). Behavior of diffusion semi-groups at infinity. *Bulletin de la Société Mathématique de France*, 102:193–240.

[5] Batty, C. J. K. (1992). Asymptotic stability of Schrödinger semigroups: path integral methods. *Mathematische Annalen*, 292(3):457–492.

[6] Ben-Ari, I. and Pinsky, R. (2005). Absolute continuity/singularity and relative entropy properties for probability measures induced by diffusions on infinite time intervals. *Stochastic Processes and their Applications*, 115(2):179–206.

[7] Beneš, V. E. (1971). Existence of optimal stochastic control laws. *SIAM Journal on Control*, 9(3):446–472.

[8] Bensoussan, A. and Lions, P. (1981). Optimal control of random evolutions. *Stochastics*, 5(3):169–190.

[9] Berger, N., Biskup, M., Hoffman, C. E., and Kozma, G. (2008). Anomalous heat-kernel decay for random walk among bounded random conductances. *Annales de l'IHP Probabilités et statistiques*, 44(2):374–392.

[10] Berger, N., Cohen, M., Deuschel, J.-D., and Guo, X. (2018). An elliptic Harnack inequality for random walk in balanced environments. *arXiv:1807.03531v1*.

[11] Berger, N. and Criens, D. (2020). A parabolic Harnack principle for balanced difference equations in random environments. *in preparation*.

[12] Berger, N. and Deuschel, J.-D. (2014). A quenched invariance principle for non-elliptic random walk in i.i.d. balanced random environment. *Probability Theory and Related Fields*, 158(1):91–126.

[13] Bhattacharya, R. N. and Ramasubramanian, S. (1982). Recurrence and ergodicity of diffusions. *Journal of Multivariate Analysis*, 12(1):95–122.

[14] Billingsley, P. (1999). *Convergence of Probability Measures*. Wiley.

[15] Blanchet, J. and Ruf, J. (2016). A weak convergence criterion for constructing changes of measure. *Stochastic Models*, 32(2):233–252.

[16] Borodin, A. N. and Salminen, P. (2006). Some exponential integral functionals of BM($\mu$) and BES(3). *Journal of Mathematical Sciences*, 133(3):1231–1248.

[17] Böttcher, B., Schilling, R., and Wang, J. (2014). *Lévy Matters III: Lévy-Type Processes: Construction, Approximation and Sample Path Properties.* Springer International Publishing.

[18] Boukhadra, O., Kumagai, T., and Mathieu, P. (2015). Harnack inequalities and local central limit theorem for the polynomial lower tail random conductance model. *Journal of the Mathematical Society of Japan*, 67(4):1413–1448.

[19] Bruggeman, C. and Ruf, J. (2016). A one-dimensional diffusion hits points fast. *Electronic Communications in Probability*, 21(22):1–7.

[20] Chen, M. (1986). Coupling for jump processes. *Acta Mathematica Sinica*, 2(2):123–136.

[21] Cheridito, P., Filipovic, D., and Yor, M. (2005). Equivalent and absolutely continuous measure changes for jump-diffusion processes. *The Annals of Applied Probability*, 15(3):1713–1732.

[22] Cherny, A. S. (2007). General arbitrage pricing model: I – probability approach. In Donati-Martin, C., Émery, M., Rouault, A., and Stricker, C., editors, *Séminaire de Probabilités XL*, pages 415–445. Springer Berlin Heidelberg.

[23] Cherny, A. S. and Urusov, M. (2006). On the absolute continuity and singularity of measures on filtered spaces: Separating times. In *From Stochastic Calculus to Mathematical Finance: The Shiryaev Festschrift*, pages 125–168. Springer Berlin Heidelberg.

[24] Choulli, T. and Stricker, C. (1996). Deux applications de la décomposition de Galtchouk–Kunita–Watanabe. *Séminaire de probabilités de Strasbourg*, 30:12–23.

[25] Cox, A. and Hobson, D. (2005). Local martingales, bubbles and option prices. *Finance and Stochastics*, 9(4):477–492.

[26] Cox, J. C. (1996). The constant elasticity of variance option pricing model. *The Journal of Portfolio Management*, 23(5):15–17.

[27] Criens, D. (2018). Deterministic criteria for the absence and existence of arbitrage in multi-dimensional diffusion markets. *International Journal of Theoretical and Applied Finance*, 21(01):1850002.

[28] Criens, D. (2020a). Lyapunov criteria for the Feller–Dynkin property of martingale problems. *Stochastic Processes and their Applications*, 130(5):2693–2736.

[29] Criens, D. (2020b). No arbitrage in continuous financial markets. *Mathematics and Financial Economics*, 14(3):461–506.

[30] Criens, D. (2020c). On absolute continuity and singularity of multidimensional diffusions. *arXiv:2005.02276.*

[31] Criens, D. (2020d). On the existence of semimartingales with continuous characteristics. *Stochastics*, 92(5):785–813.

[32] Criens, D. and Glau, K. (2018). Absolute continuity of semimartingales. *Electronic Journal of Probability*, 23(125):1–28.

[33] Cui, Z. (2014). A new proof of an Engelbert—Schmidt type zero–one law for time-homogeneous diffusions. *Statistics & Probability Letters*, 89:118–123.

[34] Cvitanić, J., Schachermayer, W., and Wang, H. (2001). Utility maximization in incomplete markets with random endowment. *Finance and Stochastics*, 5(2):259–272.

[35] Da Prato, G., Kwapien, S., and Zabczyk, J. (1988). Regularity of solutions of linear stochastic equations in Hilbert spaces. *Stochastics*, 23(1):1–23.

[36] Delbaen, F. and Schachermayer, W. (1994). A general version of the fundamental theorem of asset pricing. *Mathematische Annalen*, 300:463–520.

[37] Delbaen, F. and Schachermayer, W. (1998). The fundamental theorem of asset pricing for unbounded stochastic processes. *Mathematische Annalen*, 312:215–250.

[38] Delbaen, F. and Schachermayer, W. (2006). *The Mathematics of Arbitrage*. Springer Berlin Heidelberg.

[39] Delbaen, F. and Shirakawa, H. (2002). No arbitrage condition for positive diffusion price processes. *Asia-Pacific Financial Markets*, 9(3):159–168.

[40] Dellacherie, C. and Meyer, P.-A. (1978). *Probabilities and Potential*. North-Holland.

[41] Deuschel, J.-D. and Guo, X. (2019). Quenched local central limit theorem for random walks in a time-dependent balanced random environment. *arXiv:1710.05508v2*.

[42] Deuschel, J.-D., Guo, X., and Ramírez, A. (2018). Quenched invariance principle for random walk in time-dependent balanced random environment. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 54(1):363–384.

[43] Drewitz, A. and Ramírez, A. F. (2014). Selected topics in random walks in random environment. In Ramírez, A. F., Ben Arous, G., Ferrari, P. A., Newman, C. M., Sidoravicius, V., and Vares, M. E., editors, *Topics in Percolative and Disordered Systems*, pages 23–83. Springer New York.

[44] Eberlein, E. and Kallsen, J. (2019). *Mathematical Finance*. Springer International Publishing.

[45] Engelbert, H.-J. and Schmidt, W. (1987). On the behaviour of certain Bessel functional. an application to a class of stochastic differential equations. *Mathematische Nachrichten*, 131(1):219–234.

[46] Engelbert, H.-J. and Schmidt, W. (1991). Strong Markov continuous local martingales and solutions of one-dimensional stochastic differential equations (part III). *Mathematische Nachrichten*, 151(1):149–197.

[47] Ethier, S. N. and Kurtz, T. G. (2005). *Markov Processes: Characterization and Convergence*. Wiley.

[48] Fabes, E. B. and Stroock, D. W. (1989). A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash. In *Analysis and Continuum Mechanics: A Collection of Papers Dedicated to J. Serrin on His Sixtieth Birthday*, pages 459–470. Springer Berlin Heidelberg.

[49] Fernholz, D. and Karatzas, I. (2010). On optimal arbitrage. *The Annals of Applied Probability*, 20(4):1179–1204.

[50] Fernholz, R. E. (2002). *Stochastic Portfolio Theory.* Springer New York.

[51] Föllmer, H. (1972). The exit measure of a supermartingale. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 21(2):154–166.

[52] Föllmer, H. and Schweizer, M. (2010). Minimal martingale measure. In Cont, R., editor, *Encyclopedia of Quantitative Finance*, pages 1200–1204. Wiley.

[53] Fontana, C., Grbac, Z., Jeanblanc, M., and Li, Q. (2014). Information, no-arbitrage and completeness for asset price models with a change point. *Stochastic Processes and their Applications*, 124(9):3009–3030.

[54] Friedman, A. (1975). *Stochastic Differential Equations and Applications: Volume 1.* Academic Press.

[55] Gatarek, D. and Goldys, B. (1994). On weak solutions of stochastic equations in Hilbert spaces. *Stochastics and Stochastic Reports*, 46(1-2):41–51.

[56] Gradinaru, M. and Haugomat, T. (2017). Locally Feller processes and martingale local problems. *arXiv:1706.04880v2.*

[57] Guo, X. and Zeitouni, O. (2012). Quenched invariance principle for random walks in balanced random environment. *Probability Theory and Related Fields*, 152(1):207–230.

[58] Hambly, B. and Barlow, M. (2009). Parabolic Harnack inequality and local limit theorem for percolation clusters. *Electronic Journal of Probability*, 14(1):1–26.

[59] He, S., Wang, J., and Yan, J. (1992). *Semimartingale Theory and Stochastic Calculus.* Science Press.

[60] Hofmann, N., Platen, E., and Schweizer, M. (1992). Option pricing under incompleteness and stochastic volatility. *Mathematical Finance*, 2(3):153–187.

[61] Hofmanová, M. and Seidler, J. (2013). On weak solutions of stochastic differential equations II. *Stochastic Analysis and Applications*, 31(4):663–670.

[62] Ikeda, N. and Watanabe, S. (1989). *Stochastic Differential Equations and Diffusion Processes.* Elsevier Science.

[63] Imkeller, P. and Willrich, N. (2016). Solutions of martingale problems for Lévy-type operators with discontinuous coefficients and related SDEs. *Stochastic Processes and their Applications*, 126(3):703–734.

[64] Jacod, J. (1977). A general theorem of representation for martingales. *Proceedings of the Symposia in Pure Mathematics*, 31:37–53.

[65] Jacod, J. (1979). *Calcul stochastique et problèmes de martingales.* Springer Berlin.

[66] Jacod, J. (1980). Weak and strong solutions of stochastic differential equations. *Stochastics*, 3(1-4):171–191.

[67] Jacod, J. and Mémin, J. (1976). Caractéristiques locales et conditions de continuité absolue pour les semi-martingales. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 35(1):1–37.

[68] Jacod, J. and Mémin, J. (1979). Un nouveau critère de compacité relative pour une suite de processus. *Publications mathématiques et informatique de Rennes*, (1):1–27.

[69] Jacod, J. and Mémin, J. (1981). Existence of weak solutions for stochastic differential equations with driving semimartingales. *Stochastics*, 4(4):317–337.

[70] Jacod, J. and Shiryaev, A. N. (2003). *Limit Theorems for Stochastic Processes*. Springer Berlin Heidelberg, 2nd edition.

[71] Kabanov, Y. M., Liptser, R. S., and Shiryaev, A. N. (1978). Absolute continuity and singularity of locally absolutely continuous probability distributions. I. *Matematicheskii Sbornik*, 107(149)(3(11)):631–680.

[72] Kabanov, Y. M., Liptser, R. S., and Shiryaev, A. N. (1979). Absolute continuity and singularity of locally absolutely continuous probability distributions. II. *Matematicheskii Sbornik*, 108(150)(1):32–61.

[73] Kallenberg, O. (1996). On the existence of universal functional solutions to classical SDE's. *The Annals of Probability*, 24(1):196–205.

[74] Kallenberg, O. (1997). *Foundations of Modern Probability*. Springer New York.

[75] Kallsen, J. and Muhle-Karbe, J. (2010). Exponentially affine martingales, affine measure changes and exponential moments of affine processes. *Stochastic Processes and their Applications*, 120(2):163–181.

[76] Karatzas, I. and Kardaras, C. (2007). The numéraire portfolio in semimartingale financial models. *Finance and Stochastics*, 11(4):447–493.

[77] Karatzas, I. and Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus*. Springer New York, 2nd edition.

[78] Kardaras, C. (2010). Finitely additive probabilities and the fundamental theorem of asset pricing. In Chiarella, C. and Novikov, A., editors, *Contemporary Quantitative Finance: Essays in Honour of Eckhard Platen*, pages 19–34. Springer Berlin Heidelberg.

[79] Kardaras, C., Kreher, D., and Nikeghbali, A. (2015). Strict local martingales and bubbles. *The Annals of Applied Probability*, 25(4):1827–1867.

[80] Khasminskii, R. and Milstein, G. (2011). *Stochastic Stability of Differential Equations*. Springer Berlin Heidelberg.

[81] Khoshnevisan, D., Salminen, P., and Yor, M. (2006). A note on a.s. finiteness of perpetual integral functionals of difusions. *Electronic Communications in Probability*, 11(11):108–117.

[82] Klebaner, F. and Liptser, R. (2014). When a stochastic exponential Is a true Martingale. Extension of the Beneš method. *Theory of Probability & Its Applications*, 58(1):38–62.

[83] Kolokoltsov, V. N. (2011). *Markov Processes, Semigroups, and Generators.* De Gruyter.

[84] Krylov, N. V. and Safonov, M. V. (1981). A certain property of solutions of parabolic equations with measurable coefficients. *Mathematics of the USSR-Izvestiya*, 16(1):151–164.

[85] Kühn, F. (2018a). On martingale problems and Feller processes. *Electronic Journal of Probability*, 23(13):1–18.

[86] Kühn, F. (2018b). Solutions of Lévy-driven SDEs with unbounded coefficients as Feller processes. *Proceedings of the American Mathematical Society*, 146:3591–3604.

[87] Kühn, F. (2019). Perpetual integrals via random time changes. *Bernoulli*, 25(3):1755–1769.

[88] Kühn, F. (2020). Existence of (Markovian) solutions to martingale problems associated with Lévy-type operators. *Electronic Journal of Probability*, 25(16):1–26.

[89] Kuo, H.-J. and Trudinger, N. S. (1998). Evolving monotone difference operators on general space-time meshes. *Duke Mathematical Journal*, 91(3):587 – 607.

[90] Kurenok, V. P. and Lepeyev, A. N. (2008). On multi-dimensional SDEs with locally integrable coefficients. *The Rocky Mountain Journal of Mathematics*, 38(1):139–174.

[91] Lawler, G. F. (1991). Estimates for differences and harnack inequality for difference operators coming from random walks with symmetric, spatially inhomogeneous, increments. *Proceedings of the London Mathematical Society*, s3-63(3):552–568.

[92] Lebedev, V. A. (1983). On the existence of weak solutions for stochastic differential equations with driving martingales and random measures. *Stochastics*, 9(1-2):37–76.

[93] Lebedev, V. A. (1984). On non-explosion for the solution of a stochastic differential equation. *Stochastics*, 11(3-4):301–314.

[94] LeGall, J.-F. (2016). *Brownian Motion, Martingales, and Stochastic Calculus.* Springer International Publishing.

[95] Levin, D. A., Peres, Y., and Wilmer, E. L. (2009). *Markov Chains and Mixing Times.* American Mathematical Society.

[96] Li, Y. (2006). Dual and Feller–Reuter–Riley transition functions. *Journal of Mathematical Analysis and Applications*, 313(2):461–474.

[97] Li, Y. and Li, J. (2009). Criteria for Feller transition functions. *Journal of Mathematical Analysis and Applications*, 359(2):653–665.

[98] Lieberman, G. M. (1996). *Second Order Parabolic Differential Equations.* World Scientific.

[99] Liggett, T. M. (2010). *Continuous Time Markov Processes: An Introduction.* American Mathematical Society.

[100] Liggett, T. M., Schonmann, R. H., and Stacey, A. M. (1997). Domination by product measures. *The Annals of Probability*, 25(1):71–95.

[101] Liptser, R. S. and Shiryaev, A. N. (1989). *Theory of Martingales.* Kluwer Academic Publishers.

[102] Liptser, R. S. and Shiryaev, A. N. (2001). *Statistics of Random Processes: I. General Theory.* Springer Berlin Heidelberg.

[103] Lyasoff, A. (2014). The two fundamental theorems of asset pricing for a class of continuous-time financial markets. *Mathematical Finance*, 24(3):485–504.

[104] McKean, H. P. J. (1969). *Stochastic Integrals.* Academic Press.

[105] Mijatović, A. and Urusov, M. (2012a). Convergence of integral functionals of one-dimensional diffusions. *Electronic Communications in Probability*, 17(61):1–13.

[106] Mijatović, A. and Urusov, M. (2012b). Deterministic criteria for the absence of arbitrage in one-dimensional diffusion models. *Finance and Stochastics*, 16(2):225–247.

[107] Mijatović, A. and Urusov, M. (2012c). On the martingale property of certain local martingales. *Probability Theory and Related Fields*, 152:1–30.

[108] Mörters, P. and Peres, Y. (2010). *Brownian Motion.* Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press.

[109] Moser, J. (1964). A Harnack inequality for parabolic differential equations. *Communications on Pure and Applied Mathematics*, 17(1):101–134.

[110] Musiela, M. (1986). On Kac functionals of one-dimensional diffusions. *Stochastic Processes and their Applications*, 22(1):79–88.

[111] Mustapha, S. (2006). Gaussian estimates for spatially inhomogeneous random walks on $Z^d$. *The Annals of Probability*, 34(1):264 – 283.

[112] Nguyen, D. and Yin, G. (2016). Modeling and analysis of switching diffusion systems: Past-dependent switching with a countable state space. *SIAM Journal on Control and Optimization*, 54(5):2450–2477.

[113] Niels, J. (2001). *Pseudo Differential Operators And Markov Processes, Volume I: Fourier Analysis And Semigroups.* World Scientific Publishing Company.

[114] Norris, J. R. (1997). *Markov Chains.* Cambridge University Press.

[115] Palmowski, Z. and Rolski, T. (2002). A technique for exponential change of measure for Markov processes. *Bernoulli*, 8(6):767–785.

[116] Parthasarathy, K. R. (1967). *Probability Measures on Metric Spaces.* Academic Press.

[117] Partzsch, L. and Schilling, R. L. (2014). *Brownian Motion: An Introduction to Stochastic Processes.* De Gruyter.

[118] Pazy, A. (2012). *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer New York.

[119] Perkowski, N. and Ruf, J. (2015). Supermartingales as Radon–Nikodym densities and related measure extensions. *The Annals of Probability*, 43(6):3133–3176.

[120] Pinsky, R. (1995). *Positive Harmonic Functions and Diffusion.* Cambridge University Press.

[121] Powers, R. T. and Størmer, E. (1970). Free states of the canonical anticommutation relations. *Communications in Mathematical Physics*, 16(1):1–33.

[122] Reuter, G. E. H. and Riley, P. W. (1972). The Feller property for Markov semigroups on a countable state space. *Journal of the London Mathematical Society*, s2-5(2):267–275.

[123] Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion.* Springer Berlin Heidelberg, 3rd edition.

[124] Rogers, L. C. G. and Williams, D. (2000a). *Diffusions, Markov Processes, and Martingales: Volume 1, Foundations.* Cambridge University Press.

[125] Rogers, L. C. G. and Williams, D. (2000b). *Diffusions, Markov Processes and Martingales: Volume 2, Itô Calculus.* Cambridge University Press.

[126] Rozkosz, A. and Słomiński, L. (1991). On existence and stability of weak solutions of multidimensional stochastic differential equations with measurable coefficients. *Stochastic Processes and their Applications*, 37(2):187–197.

[127] Ruf, J. (2015). The martingale property in the context of stochastic differential equations. *Electronic Communications in Probability*, 20(34):1–10.

[128] Safonov, M. V. (1983). Harnack's inequality for elliptic equations and the Hölder property of their solutions. *Journal of Soviet Mathematics*, 21(5):851–863.

[129] Schilling, R. L. (1998). Conservativeness and extensions of Feller semigroups. *Positivity*, 2(3):239–256.

[130] Schweizer, M. (1995). On the minimal martingale measure and the Föllmer–Schweizer decomposition. *Stochastic Analysis and Applications*, 13(5):573–599.

[131] Shao, J. (2015). Strong solutions and strong Feller properties for regime-switching diffusion processes in an infinite state space. *SIAM Journal on Control and Optimization*, 53(4):2462–2479.

[132] Shiryaev, A. N. (1978). *Optimal Stopping Rules.* Springer New York.

[133] Sin, C. (1996). *Strictly Local Martingales and Hedge Ratios on Stochatic Volatility Models.* PhD thesis.

[134] Sin, C. (1998). Complications with stochastic volatility models. *Advances in Applied Probability*, 30(1):256–268.

[135] Skorokhod, A. V. (2014). *Studies in the Theory of Random Processes.* Dover Books on Mathematics. Dover Publications.

[136] Stinchcombe, M. B. and White, H. (1992). Some measurability results for extrema of random functions over random sets. *The Review of Economic Studies*, 59(3):495–514.

[137] Stroock, D. W. and Varadhan, S. R. S. (1979). *Multidimensional Diffussion Processes*. Springer Berlin Heidelberg.

[138] Volkonskii, V. A. (1958). Random substitution of time in strong markov processes. *Theory of Probability & Its Applications*, 3(3):310–326.

[139] Xi, F. and Zhu, C. (2017). On Feller and strong Feller properties and exponential ergodicity of regime-switching jump diffusion processes with countable regimes. *SIAM Journal on Control and Optimization*, 55(3):1789–1818.

[140] Xue, X.-X. (1990). A zero-one law for integral functionals of the bessel process. In Azéma, J., Yor, M., and Meyer, P. A., editors, *Séminaire de Probabilités XXIV 1988/89*, pages 137–153. Springer Berlin Heidelberg.

[141] Yan, J.-A. (1998). A new look at the fundamental theorem of asset pricing. *Journal of the Korean Mathematical Society*, (33):659–673.

[142] Yin, G. and Zhu, C. (2009). *Hybrid Switching Diffusions: Properties and Applications*. Springer New York.

[143] Zeitouni, O. (2004). Part ii: Random walks in random environment. In Picard, J., editor, *Lectures on Probability Theory and Statistics: Ecole d'Eté de Probabilités de Saint-Flour XXXI - 2001*, pages 190–312. Springer Berlin Heidelberg.