Fakultät für Mathematik
Technische Universität München

Technische Universität München
Professur für Analysis und ihre Anwendungen

Programmable Quantum Processors

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Vollständiger Abdruck der von der Fakultät für Mathematik der Technischen Universität München zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigten Dissertation.

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   Universität Wien
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Die Dissertation wurde am 18.1.2021 bei der Technischen Universität München eingereicht und durch die Fakultät für Mathematik am 15.2.2021 angenommen.

Department of Mathematics
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Programmable Quantum Processors

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Full imprint of the dissertation approved by the Department of Mathematics of the Technical University of Munich to obtain the academic degree of

Doctor of Natural Sciences (Dr. rer. nat.).

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The dissertation was submitted to the Technical University of Munich on 18.1.2021 and was accepted by the Department of Mathematics on 15.2.2021.

Zusammenfassung

Programmierbare Quantenprozessoren stellen ein fundamentales Modell für Quantencomputing dar. Als solches implementieren sie Quantenkanäle mittels eines Quantenzustands des Programmregisters. Diese Dissertation erforscht zunächst die Implementierung einer Klasse von Quantenkanälen mit Symmetrieeigenschaften hinsichtlich der Größe des Programmregisters und des Präzisionsgrades. Darüber hinaus wird das Modell auf die Programmierbarkeit unendlich-dimensionaler Quantenkanäle erweitert, wobei insbesondere Gaußsche Quantenkanäle betrachtet werden.

Abstract

Programmable quantum processors represent a fundamental model for quantum computing. As such, they implement quantum channels using a quantum state of the program register. This dissertation first explores the implementation of a class of quantum channels with symmetry properties regarding the program size and the precision. Furthermore, the model is extended to the programmability of infinite-dimensional quantum channels, considering Gaussian quantum channels in particular.

Acknowledgements

I want to thank many supportive people for accompanying my journey in the last years leading to this dissertation. Many thanks go to Prof. Simone Warzel for the opportunity to pursue my doctorate, always taking time for any query, and for sending me to Prof. Andreas Winter at some point: I learnt so much from him in the last few months. It is beyond words how much I appreciate this rewarding experience. It is amazing how naturally and unintended this collaboration evolved. I would like to thank Prof. Christina Kuttler for being chairwoman of my thesis committee and being an enormously supporting, caring and encouraging professor. Furthermore, I thank the members of my thesis committee Prof. Simone Warzel, Prof. Norbert Schuch, and Prof. Andreas Winter. My gratitude also goes to my co-authors Prof. Andreas Winter, Dr. Andreas Bluhm, Dr. Burak Şahinoğlu, Eugene Tang and Prof. Robert König.
Furthermore, I thank my colleagues (of which some even became friends) both from the mathematics department and the Max Planck Institute of Quantum Optics – too many to list in person here – for enriching discussions, interesting meetings and for the joyful time we spent together. I am also grateful for many great colleagues I got to know during workshops and summer schools. Thank you for the amazing memories. Special thanks go to the participants and organizers of the Lindau Nobel Laureates meeting for the various interesting conversations, life lessons, insights and the creation of a very special atmosphere: the Lindau spirit. Moreover, I want to thank the Kavli Institute for Theoretical Physics, University of Santa Barbara (California), for my research stay. Many thanks go to Prof. Stefan Wolf and his group at USI Lugano for hosting my visit and for many inspiring conversations, the books, and his support.
I want to express my gratitude to all participants of the teaching mathematics course offered by *TUM ProLehre* and especially the organizers. Thank you for creating such a nice environment for loads of good, fruitful and inspiring conversations about teaching mathematics in all its aspects and various other topics throughout this year.
Being a member of the IMPRS-QST graduate school is another aspect I am grateful for. I want to thank all members especially from the first group for the nice discussions and the great time during various events, summer schools, workshops, seminars and extra courses. Special thanks go to the members of my Thesis Advisory Board for interesting conversations and Sonya Gzyl for always being available for all sorts of questions and open for any idea.
In particular, I want to thank Isabella Wiegand from the ISAM office for her enormous support. This was beyond one could wish for and I am very grateful for her advice and care. I also thank Prof. Folkmar Bornemann, Prof. Matthias Scherer

and PD. Dr. Nada Sissouno for their support during a very crucial period of my doctorate.
Moreover, I want to express my gratitude to the MCQST office, especially Barbara Tautz and Tatjana Wilk for their great support and open ears for any kind of matter. I am amazed by their encouragement. This made so much possible. Thank you for really implementing what the MCQST stands for and filling emty words with real action. Furthermore, I thank Barbara Tautz for organizing the mentoring program.
Many thanks go to Andreas Bluhm, Matthias Caro, Stefan Huber, and Simone Warzel for proof-reading (parts of) this thesis. I thank Silvia Schulz and Mariola Nicpon-Statsch for their support in various administrative aspects.
It is particularly important to me to thank all the students I have had the pleasure of teaching mathematics to in various groups over the past few years. I am very grateful for the interactive classes offline and online. They made it possible for me to live my passion for teaching.
I especially want to thank my friends inside and outside of academia who were there for me and supported me and went through the ups and downs with me, for amazing trips exploring the world, having fun on various adventures, for laughing and dancing.
It is my pleasure to furthermore thank every single person not mentioned here so far who accompanies my journey in the one or the other way: without them I would not be where I am today.
My deepest gratitude goes to my family: my parents for their support in so many ways, my mother in particular for also being a close friend supporting and encouraging me, especially in the challenging times of this doctorate; and my sister, for our adventures, for always structuring my ideas and the fun we have together.

List of Publications

Chapter 5 and Chapter 7 of this dissertation are based on the following articles:

Articles as principal author

1) Martina Gschwendtner, Andreas Bluhm, and Andreas Winter
Programmability of covariant quantum channels
Quantum 5:488, 2021 (24 pages)
(see Article [1] in the bibliography)

2) Martina Gschwendtner and Andreas Winter
Infinite-Dimensional Programmable Quantum Processors
PRX Quantum 2:030308, 2021 (38 pages)
(see Article [2] in the bibliography)

The author of this dissertation is the principal author of Articles 1) and 2).

Further article, not included in this dissertation

3) Martina Gschwendtner, Robert König, Burak Şahinoğlu, Eugene Tang
Quantum Error-detection at Low Energies
J. High Energ. Phys. 2019: 21, 2019 (79 pages)
(see Article [3] in the bibliography)

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1 The idea of programmable quantum processors

A quantum computer that implements any kind of quantum operation controlled by the user through program states has been a dream since the early days of quantum information theory. This is inspired by the functioning of classical computers which we can program to operate on a given input by providing additional data through the program states. Analogously, the aim of years of research was a quantum machine that takes any quantum state from an input register and outputs the state resulting from the action of a quantum channel on the input state. This channel is specified and at the same time varied by a program state of a finite-dimensional program register. Such a machine is highly relevant and convenient because it can be programmed and we can use the same machine for many operations.
However, in 1997, Nielsen and Chuang proved their *No-Programming Theorem* stating that it is not possible to universally and exactly implement all unitary operations with a finite-dimensional program register [4]. This result opened the space for research on suitable relaxations of this model in the following years. One option is to study special classes of channels, aiming for exact programmability instead of universality (see Ref. [5]). Another option of relaxing the model is seeking for an approximate implementation, which yields a tradeoff between the precision of the implementation and the size of the program register [6–13]. Recently, Kubicki *et al.* [14] and Yang *et al.* [15] presented new upper and lower bounds on the dimension of the program register. The former applied tools from Banach space theory while the latter used information-theoretic methods which provide the optimal scaling of the program dimension and the precision of implementation depending on the input dimension.
In this thesis, we seek to explore two main possibilities to study programmability in a setting not covered by the *No-Programming Theorem*: the first main contribution is considering the programmability of a class of channels with symmetries and, by generalizing the concept to infinite dimension, we secondly contribute to the field of continuous-variable systems.
Concerning the first possibility, we consider symmetry properties in the form of group-covariant quantum channels. This class of channels is present in and relevant for channel discrimination, capacities and communication tasks, for instance (see Ref. [16] and the references therein). So, if we know that a system has a certain symmetry, exact programmability can be considered because ultimately we are leaving the setting of the *No-Programming Theorem* which holds for universal programmability. In particular, we study a programmable quantum processor that implements all quantum channels which are covariant with respect to two unitary

representations of a compact Lie group.
Having encountered symmetries in finite dimension, we reach the second main contribution of this thesis. As continuous-variable systems are highly relevant in quantum information theory and gain more and more attention in quantum computing, we seek to generalize the concept of programmable quantum processors to infinite dimension. For physical reasons, we assume energy constraints on the input and output states and in particular, we first investigate the implementation of energy-limited gauge-covariant Gaussian channels of a single Bosonic mode because these are interesting special channels of physical relevance. Secondly, we consider the more general class of energy-limited Gaussian unitary channels of any finite number of Bosonic modes.

1.1 Overview of methods

In the next chapter, we present all relevant definitions and results from quantum information theory and representation theory. We start by introducing fundamental concepts from quantum information theory such as states and measurements in Subsection 2.1.1. As certain characteristics of quantum mechanics, such as entanglement for instance, rely on composite systems, we continue with presenting those in Subsection 2.1.2. Aiming for programmable quantum processors that implement quantum channels, we study operations on quantum states in Subsection 2.1.3. A central topic in quantum information theory is how to quantify information. This is usually covered by the entropy which we define in Section 2.2, followed by the Holevo information serving as a quantification of the information contained in an ensemble of states. Furthermore, we present fundamental definitions and results from representation theory in Section 2.3 as studying symmetries in physics is related to the action of groups.
In Chapter 3, we combine all these tools from quantum information theory and representation theory to study symmetries in quantum information theory including examples.
We mathematically define a programmable quantum processor in Chapter 4 and take a closer look at the *No-Programming Theorem* and its proof. In Section 4.3, we introduce the standard teleportation protocol as well as port-based teleportation, teleportation simulation and the relation to programmable quantum processors.
As continuous-variable systems play a major role in quantum mechanics, we introduce the properties of infinite-dimensional systems in Section 6.1. Starting with the general setting, we continue with the quantification of information in this broader context. In particular, we introduce Bosonic Gaussian systems, where we again consider symmetries in the form of gauge-covariant Bosonic Gaussian quantum channels to then study general Gaussian unitary channels. This is motivated by the aim of developing a programmable quantum processor that implements all gauge-covariant Gaussian channels and Gaussian unitary channels, respectively, assuming input and output energy constraints. With these fundamental concepts

and methods, we provide the tools required to investigate the aforementioned two main contributions of this thesis expressed in the two publications [1, 2].

1.2 Overview of results

The two publications as principal author [1, 2] investigate programmable quantum processors and especially the required dimension of the program register.

Chapter 5 is based on the article

> Martina Gschwendtner, Andreas Bluhm, and Andreas Winter
> Programmability of covariant quantum channels
> *Quantum* 5:488, 2021 (24 pages)
> (see Article [1] in the bibliography)

The author of this thesis is the principal author of the above publication.

Summary article 1 [1]. We consider a programmable quantum processor that uses a state of the program register to apply a group-covariant quantum channel to an arbitrary input state. Due to this symmetry, the No-Programming Theorem does not apply and therefore, we consider exact programmability. To obtain the dimension of the program register, we first consider a method which is based on the storage of extreme points if the commutant of the tensor representation of the group is abelian. It turns out that the program dimension equals the number of irreducible representations occurring in the direct sum decomposition of the tensor representation. We then look at a different construction which relies on teleportation. Concatenating this with a compression map yields a program dimension of at most the sum of the dimensions of the blocks occurring in the structure of the Choi-Jamiołkowski state corresponding to the covariant channels.
After studying exact programmability, we explore the possibility of approximate versions thereof. The upper bounds on the dimension of the program register, based on ϵ-nets, lack the precision of the exact result but are more generally applicable. The lower bounds are based on information-theoretic tools using entropies and reveal that the construction for the exact case is optimal.

Chapter 7 is based on the article

Martina Gschwendtner and Andreas Winter
Infinite-Dimensional Programmable Quantum Processors
PRX Quantum 2:030308, 2021 (38 pages)
(see Article [2] in the bibliography)

The author of this thesis is the principal author of the above publication.

Summary article 2 [2]. We generalize the model of programmable quantum processors to infinite-dimensional systems motivated by the importance of continuous-variable systems for quantum computing. We assume energy constraints on the input and output and aim to implement energy-limited quantum channels which map energy-bounded states to energy-bounded states.
We establish a connection between infinite-dimensional and finite-dimensional programmable quantum processors: one of an infinite-dimensional unitary-universal programmable quantum processor to a finite-dimensional one assuming the existence of the latter. The other one assuming an infinite-dimensional unitary-universal programmable quantum processor and constructing a finite-dimensional one. Due to these two constructions, we can import existing upper and lower bounds on the program register dimension from finite dimension.
We prove a central lemma stating that if a processor approximately implements a unitary, then the processor can be modified such that it both reuses the program state and implements the same unitary several times. This is followed by information-theoretic relations between the ensembles involved. Thereafter, we focus on the implementation of gauge-covariant Gaussian channels for one mode, using a concatenation property of this particular class of channels. Having provided upper and lower bounds on the program register dimension using ϵ-net constructions and information-theoretic tools based on entropies, respectively, we turn to the implementation of the more general multimode Gaussian unitary channels.
We conclude with upper and lower bounds on the program register dimension of an infinite-dimensional approximate programmable quantum processor implementing multimode Gaussian unitary channels.

2 Fundamentals of quantum information and representation theory

2.1 Quantum information theory

As a statistical theory, quantum mechanics predicts measurement outcomes probabilistically, i.e., through repetition of an experiment, we can obtain the relative frequencies of the outcomes. This process consists of two parts: the preparation of the system in a particular state and the subsequent measurement of the prepared quantum system. In the first step, we prepare the system in a certain state $\rho \in \mathcal{D}(\mathcal{H})$ which is independent of the preparation procedure. In the second part, an observable quantity, which is independent of the measurement procedure, is measured.

As long as we operate in finite dimension (until Chapter 6), we consider a d-dimensional Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ where $\langle\cdot,\cdot\rangle$ denotes the inner product, which we choose linear in the second variable. Let $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ denote the associated norm.
We use *Schatten p-norms* which are defined as

$$\|A\|_p = \operatorname{tr}\left((A^*A)^{p/2}\right)^{1/p} \qquad \forall\ A \in \mathcal{B}(\mathcal{H}),$$

where $\mathcal{B}(\mathcal{H})$ is the set of *bounded linear operators* $A : \mathcal{H} \to \mathcal{H}$. Considering the canonical basis of $\mathcal{H} \cong \mathbb{C}^d$, A^* is the *adjoint* and A^T is the *transpose*. Special cases of Schatten p-norms are the *operator norm*

$$\|A\| = \max\{\|Ax\| : x \in \mathcal{H}, \|x\| \leq 1\},$$

which is the Schatten ∞-norm and the *trace norm*

$$\|A\|_1 = \operatorname{tr}\sqrt{A^*A} \qquad \forall\ A \in \mathcal{B}(\mathcal{H}),$$

which corresponds to the Schatten 1-norm. We call an operator *positive semidefinite* ($A \geq 0$) if and only if it is self-adjoint with non-negative eigenvalues. The d-dimensional identity operator is denoted by $\mathbb{1}_d$ and the identity channel by id.
We use the bra-ket-notation where a vector $\varphi \in \mathcal{H}$ is denoted by $|\varphi\rangle$ and a dual vector as $\langle\psi| \in \mathcal{H}^*$. Let ψ, $\varphi \in \mathcal{H}$, then we write $|\psi\rangle\langle\varphi|$ for the corresponding rank-one operator $\mathcal{H} \to \mathcal{H}$ which acts on a vector $|\chi\rangle$ as $|\psi\rangle\langle\varphi|\chi\rangle$.

2.1.1 States and measurements

In the following, we present the concept of states and measurements mathematically. For further background, we refer to the textbooks [17–20].

Definition 2.1.1 (Quantum State)**.** *A d-dimensional* quantum state *is described by a density matrix $\rho \in \mathcal{B}(\mathcal{H})$ which obeys the properties*

(i) $\rho \geq 0$ *(positive semidefinite) and*

(ii) $\mathrm{tr}(\rho) = 1$ *(unit trace).*

In the two-dimensional case, i.e., $d = 2$, a state is referred to as a *qubit*, otherwise, it is called a *qudit*.
The set of all density operators is

$$\mathcal{D}(\mathcal{H}) = \{\rho \in \mathcal{B}(\mathcal{H}) \mid \rho \geq 0, \mathrm{tr}(\rho) = 1\}.$$

This is the set of all positive semidefinite complex $d \times d$-matrices $\mathcal{M}_d$ satisfying $\mathrm{tr}(\rho) = 1$ because $\mathcal{B}(\mathcal{H}) \cong \mathcal{M}_d$. The set is convex, i.e., $\lambda\rho_1 + (1-\lambda)\rho_2 \in \mathcal{D}(\mathcal{H})$ for all $\rho_1, \rho_2 \in \mathcal{D}(\mathcal{H})$ and $0 \leq \lambda \leq 1$, and compact. The extreme points of this set can, by definition, not be written as non-trivial convex-combination. These states form a special class of states called *pure states*. They are exactly the rank-one projections $P = |\psi\rangle\langle\psi|$ for a unit vector $|\psi\rangle \in \mathcal{H}$. The set of all pure states is denoted by $\mathcal{D}_P(\mathcal{H})$. All other states are called *mixed states*.

Example 2.1.2. *The special state $\frac{\mathbb{1}_d}{d} \in \mathcal{D}(\mathcal{H})$ is called* maximally mixed*.*

Any mixed state can be expressed as convex combination of pure states via the *spectral decomposition* [18, Theorem 1.65]

$$\rho = \sum_{i=1}^{d} \lambda_i |\psi_i\rangle\langle\psi_i|$$

with some $\lambda_i \in [0,1]$ for all $i \in \{1, \ldots, d\}$ and $\sum_i \lambda_i = 1$, and some orthonormal basis (ONB) $\{|\psi_i\rangle\}_{i=1}^d$ of $\mathcal{H}$.

Having introduced and defined the state of a quantum system, we move on to measurements of a quantum system prepared in state ρ.

If we conduct a quantum mechanical measurement of a quantum system prepared in state $\rho \in \mathcal{D}(\mathcal{H})$, we obtain measurement outcomes x_i with $i \in \{1, \ldots, n\}$, assuming finitely many. The measurement process can be described in two different but related ways depending on whether we are interested in the measurement outcomes.

Definition 2.1.3 (POVM)**.** *Let $\mathcal{X} = \{x_i\}$ be the finite set of all measurement outcomes with individual outcomes x_i for $i = \{1, \ldots, n\}$. A map $M : \mathcal{X} \to \mathcal{B}(\mathcal{H}), x_i \mapsto M_i$ is called* Positive Operator Valued Measure (POVM) *if*

(i) $M_i \geq 0$ where M_i are the positive operators associated to the i-th outcome with value x_i and

(ii) $\sum_{i=1}^{n} M_i = \mathbb{1}$.

The positive operators M_i *are called* effect operators *or for short* effects.

The probability of a measurement outcome x_i is obtained by

$$p_i = \operatorname{tr}(\rho M_i),$$

i.e., the effect operators associated to every outcome x_i via the POVM assign a probability to the quantum state ρ. A mapping $M : \mathcal{X} \to \mathcal{B}(\mathcal{H}), x_i \mapsto M_i$ is a POVM if and only if the mapping $\mathcal{X} \to [0,1], x_i \mapsto \operatorname{tr}(\rho M_i)$ with $x_i \in \mathcal{X}$ represents a probability measure for every $\rho \in \mathcal{D}(\mathcal{H})$ [18]. The probability of a certain outcome can be expressed with an effect operator. A measurement therefore serves as a connection between quantum mechanics and classical probabilities by associating those probabilities to quantum mechanical objects.
If, in addition, the positive operators $\{M_i\}_{i=1}^n$ satisfy $M_i^2 = M_i$, i.e., they are projectors, we obtain the special case of a *projective measurement*.
This yields the description of a measurement if we also care about the measurement outcomes. In standard quantum mechanics, a measurement is described by a self-adjoint operator O, which by spectral decomposition is given by

$$O = \sum_{i=1}^{n} \lambda_i P_i$$

with eigenvalues λ_i and orthogonal projections P_i onto the eigenspaces of O. The set $\{P_i\}_{i=1}^n$ forms a POVM and if we interpret the eigenvalues as measurement outcomes, we get

$$\operatorname{tr}(O\rho) = \operatorname{tr}\left(\sum_{i=1}^{n} \lambda_i P_i \rho\right) = \sum_{i=1}^{n} \lambda_i \operatorname{tr}(P_i \rho) = \sum_{i=1}^{n} \lambda_i p_i,$$

which is equal to the expectation value $\sum_{i=1}^n x_i p_i$ of the observable O for a quantum system prepared in the state $\rho \in \mathcal{D}(\mathcal{H})$.
Thus, every observable described mathematically by a self-adjoint operator induces a projective measurement due to the spectral theorem. In general, POVMs are not necessarily connected to an observable and are therefore more general than projective measurements. Hence, not every general POVM can be associated to a self-adjoint operator. However, the two are closely related because, according to *Naimark's Theorem* [19, Thm 2.42], every POVM can be seen as a projective measurement on a larger Hilbert space.

Two technical lemmas we use in the proofs are the Gentle Measurement and the Gentle Operator Lemma, which show that a measurement with a highly likely outcome can be performed with little disturbance to the measured quantum state.

Lemma 2.1.4 (Gentle Measurement [21, Lemma 9], [22, Lemma 5]). *Let $\rho \in \mathcal{D}(\mathcal{H})$ and $0 \leq T \leq \mathbb{1}$. We suppose that T has a high probability of detecting ρ, i.e., $\mathrm{tr}(T\rho) \geq 1 - \kappa$ with $\kappa \in [0, 1]$. Then, the post-measurement state*

$$\rho' := \frac{\sqrt{T}\rho\sqrt{T}}{\mathrm{tr}\,\rho T}$$

satisfies

$$\|\rho - \rho'\|_1 \leq 2\sqrt{\kappa}.$$

Lemma 2.1.5 (Gentle Operator [23, Lemma 9.4.2]). *Let $\rho \in \mathcal{D}(\mathcal{H})$ and $0 \leq T \leq \mathbb{1}$. We suppose that T has a high probability of detecting ρ, i.e., $\mathrm{tr}(T\rho) \geq 1 - \kappa$, with $\kappa \in [0, 1]$. Then,*

$$\left\|\rho - \sqrt{T}\rho\sqrt{T}\right\|_1 \leq 2\sqrt{\kappa}.$$

2.1.2 Composite systems

We defined states and measurements with respect to a single system. However, many interesting phenomena in quantum mechanics such as entanglement, which we define later, rely on composite quantum systems consisting of several subsystems. Hence, we consider $n \in \mathbb{N}$ distinct quantum systems described by the Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ with $\dim(\mathcal{H}_1) = d_1, \ldots, \dim(\mathcal{H}_n) = d_n$ and $d_1, \ldots, d_n \in \mathbb{N}$. The overall Hilbert space $\mathcal{H}$ is described by the tensor product of the Hilbert spaces modelling the subsystems, i.e., $\mathcal{H} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$ with $\dim(\mathcal{H}) = d = d_1 \cdot \ldots \cdot d_n$. For finite-dimensional Hilbert spaces, we can always choose $\mathcal{H} \cong \mathbb{C}^d$ and $\mathcal{H}_1 \cong \mathbb{C}^{d_1}, \ldots, \mathcal{H}_n \cong \mathbb{C}^{d_n}$ for $d_1, \ldots, d_n \in \mathbb{N}$ and $\mathcal{B}(\mathcal{H}) \cong \mathcal{M}_d$. Accordingly, states and measurements are defined on the larger Hilbert space by $\rho_{1,\ldots,n} \in \mathcal{D}(\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n) \subset \otimes_{i=1}^n \mathcal{M}_{d_i}$. A special case thereof are bipartite systems, where $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ with $\dim(\mathcal{H}) = d_1 \cdot d_2$. Let $\rho_{1,2} \in \mathcal{D}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ be the state describing a bipartite system. Note that each space has an ONB $\{\varphi_{1,i}\}_{i=1}^{d_1}$ and $\{\varphi_{2,j}\}_{j=1}^{d_2}$ and an ONB of the composite system $\mathcal{H}_1 \otimes \mathcal{H}_2$ is given by $\{\varphi_{1,i} \otimes \varphi_{2,j}\}_{i,j=1}^{d_1,d_2}$. Analogously, measurements can be defined w.r.t. this larger Hilbert space as well.
If we consider a composite system $\mathcal{H}_1 \otimes \mathcal{H}_2$, it is convenient to be able to study just one of the two subsystems. Therefore, we define the *partial trace* over a subsystem, where we apply the trace operation to only one of the two subsystems, i.e., $\mathrm{tr}_2 : \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$ with $\mathrm{tr}_2(\cdot) := \mathrm{id} \otimes \mathrm{tr}(\cdot)$ is applied to the state of the joint system $\rho_{1,2}$ where $\mathrm{tr} : \mathcal{B}(\mathcal{H}_2) \to \mathbb{C}$ represents the unnormalized trace operation.

Definition 2.1.6 (Partial Trace). *Let $\mathcal{H}_1 \otimes \mathcal{H}_2$ be a bipartite system with $d_1, d_2 \in \mathbb{N}$. The* partial trace *over system 2 is defined as the linear map $\mathrm{tr}_2(\cdot) : \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$ satisfying*

$$\mathrm{tr}[\mathrm{tr}_2(\rho_{1,2})A_1] = \mathrm{tr}[\rho_{1,2}(A_1 \otimes \mathrm{id}_2)] \qquad \forall\ \rho_{1,2} \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2), A_1 \in \mathcal{B}(\mathcal{H}_1).$$

Thus, we obtain the state of a subsystem from the state $\rho_{1,2}$ of the composite system by taking the partial trace $\mathrm{tr}_2(\rho_{1,2}) = (\mathrm{id} \otimes \mathrm{tr})(\rho_{1,2}) =: \rho_1$. We call ρ_1

the *reduced state* on the first system. The partial trace over the first subsystem is defined analogously.

A phenomenon that characterizes quantum mechanics and distinguishes the quantum world from the classical one is entanglement. A composed system is *entangled* if the subsystems are not separable. A quantum state is called *separable* if it can be written as a convex combination of product states, i.e.,

$$\rho = \sum_{i=1}^{l} \lambda_i \rho_1^i \otimes \rho_2^i$$

with $\lambda_i > 0$ and $\sum_i \lambda_i = 1$ and $\{\rho_k^i\}_{i=1}^l \subset \mathcal{D}(\mathcal{H}_k)$ for $k \in \{1, 2\}$.

Example 2.1.7 (Maximally entangled states)**.** *An extreme case of a state of a bipartite system is one that has maximal entanglement. Let $\{|i\rangle\}_{i=1}^d$ be an ONB of $\mathcal{H}$. Then, the corresponding maximally entangled state on $\mathcal{H} \otimes \mathcal{H}$ is defined as*

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle .$$

Note that $(A \otimes \mathbb{1}) |\Omega\rangle = (\mathbb{1} \otimes A^T) |\Omega\rangle$ for all $A \in \mathcal{B}(\mathcal{H})$, where the transposition is taken in the Schmidt basis of Ω.
Computing the reduced state, we get the maximally mixed state for both marginals, i.e., $\mathrm{tr}_1(|\Omega\rangle\langle\Omega|) = \mathrm{tr}_2(|\Omega\rangle\langle\Omega|) = \frac{\mathbb{1}_d}{d}$. Since the maximally entangled state is a pure state, this example reveals that the maximally mixed state can appear as the reduced state of a pure state.

In general, any mixed state can be written as a reduced state of a pure state of a composite system (see Ref. [18, Subsection 2.4.2]).

Definition 2.1.8 (Purification [18, Definition 2.74])**.** *Let $\rho \in \mathcal{D}(\mathcal{H}_1)$ be of dimension $d_1 \in \mathbb{N}$. We call a pure state on a composite system $\mathcal{H}_1 \otimes \mathcal{H}_2$, $|\psi\rangle\langle\psi| \in \mathcal{D}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, a* purification *of ρ if $\rho = \mathrm{tr}_2(|\psi\rangle\langle\psi|)$.*

Remark 2.1.9. *To purify a state $\rho \in \mathcal{D}(\mathcal{H}_1)$, we can add a system with dimension $d_2 = \mathrm{rank}\, \rho$. In particular, we can choose the additional system to be of dimension d_1. Note that purifications are non-unique and the dimension of the additional system is not limited [18, p. 103].*

2.1.3 Quantum channels

We now consider time evolution of quantum systems. Depending on whether we interpret the evolution as part of the preparation or the measurement process, two pictures emerge. If the state evolves in time, the evolution is part of the preparation process. This is known as *Schrödinger Picture*. If the observables evolve as part of the measurement process, we call this *Heisenberg Picture*. Such an evolution process corresponds to a linear map which transforms quantum states to quantum

states and effects to effects. Since states are positive operators, the map should preserve positivity. If we consider acting on one quantum subsystem of a bipartite system with a quantum operation T, the overall operation is $T \otimes \mathrm{id}$. Since we act with T on the first system and trivially on the second, we require the operation $T \otimes \mathrm{id}$ to preserve positivity as well.

Definition 2.1.10. *Let $d_1, d_2 \in \mathbb{N}$. A linear map $T : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ is called*

- positive *if and only if $T(A) \geq 0 \ \forall \ A \in \mathcal{B}(\mathcal{H}_1) \geq 0$,*
- completely positive (CP) *if $(T \otimes \mathrm{id}) : \mathcal{B}(\mathcal{H}_1 \otimes \mathbb{C}^n) \to \mathcal{B}(\mathcal{H}_2 \otimes \mathbb{C}^n)$ is positive for any $n \in \mathbb{N}$,*
- trace-preserving (TP) *if $\mathrm{tr}(T(A)) = \mathrm{tr}(A) \ \forall \ A \in \mathcal{B}(\mathcal{H}_1)$ and*
- unital *if $T(\mathbb{1}_{d_1}) = \mathbb{1}_{d_2}$.*

Furthermore, a map describing a quantum operation should be trace-preserving, since a state is characterized by $\mathrm{tr}(\rho) = 1$. These requirements yield the following definition for a map that describes the evolution of a quantum system.

Definition 2.1.11 (Quantum channel [18, Definition 4.5])**.** *Let $d_1, d_2 \in \mathbb{N}$. A map $T : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ is called* quantum channel *(Schrödinger picture) if*

(i) T is linear,

(ii) T is CP, i.e., $T \otimes \mathrm{id} : \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathcal{H}_2) \otimes \mathcal{B}(\mathbb{C}^n)$ is positive $\forall \ n \in \mathbb{N}$ and

(iii) T is TP, i.e., $\mathrm{tr}(T(A)) = \mathrm{tr}(A) \ \forall \ A \in \mathcal{B}(\mathcal{H}_1)$.

We denote the set of all channels mapping from $\mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ as $\mathrm{CPTP}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathrm{CPTP}(\mathcal{H})$ if $\dim(\mathcal{H}) = d = d_1 = d_2$.

Example 2.1.12. *We give some examples of quantum channels.*

a) The identity channel *leaving the state unchanged is denoted as* id*. This channel describes an ideal communication channel, since it leaves the input state invariant.*

b) The measurement channel $T_M(\cdot) = \sum_{i=1}^n \mathrm{tr}(M_i \ \cdot)|i\rangle\langle i|$ *with some ONB $\{|i\rangle\}_{i=1}^n$ shows that every POVM can be turned into a channel.*

c) The completely depolarizing channel $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ *is of the form $T(\cdot) = \mathrm{tr}(\cdot)\frac{\mathbb{1}_d}{d}$.*

d) From channels a) and c), we can construct further channels such as the depolarizing channel*, which is a mixture of the* id *channel and the completely depolarizing channel: $T(\cdot) = \lambda(\cdot) + (1-\lambda)\,\mathrm{tr}(\cdot)\frac{\mathbb{1}_d}{d}$ with $-\frac{1}{d^2-1} \leq \lambda \leq 1$.*

e) A unitary channel *with unitary $U \in \mathcal{U}_d := \mathcal{U}(\mathcal{H})$ has the form $T(\cdot) = U(\cdot)U^*$.*

We continue with useful representations of quantum channels.

Stinespring Dilation. A quantum channel describes an open system evolution which itself is not necessarily unitary. To construct a unitary evolution, we embed the open system (described by channel T) into a larger and closed, unitarily evolving system, i.e., we couple the system to an environment described by the state $|\varphi\rangle$ and let the coupled system evolve according to the unitary U. Afterwards, we trace out the environment.

Theorem 2.1.13 (Stinespring's dilation [18, Corollary 4.19])**.** *A linear map $T : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_1)$ is a quantum channel if and only if there exists a finite-dimensional Hilbert space $\mathcal{H}_E$, a unitary operator $U \in \mathcal{U}(\mathcal{H}_1 \otimes \mathcal{H}_E)$ and a pure quantum state $|\varphi\rangle \in \mathcal{D}(\mathcal{H}_E)$ such that $\forall\ \rho \in \mathcal{D}(\mathcal{H}_1)$*

$$T(\rho) = \mathrm{tr}_E[U(\rho \otimes |\varphi\rangle\langle\varphi|)U^*].$$

The environment space is called dilation space*. The* minimal dilation dimension*, i.e., the minimal* $\dim(\mathcal{H}_E)$*, is equal to* $\dim(\mathcal{H}_1) \cdot \dim(\mathcal{H}_1) = d_1^2$*.*

Choi-Jamiołkowski Isomorphism. We now establish a correspondence between channels and bipartite states which goes back to Choi and Jamiołkowski [24, 25].

Theorem 2.1.14 (Choi-Jamiołkowski Isomorphism [18, Theorem 4.48])**.** *Let $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) = \{T : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2) \mid T \text{ linear}\}$. The map*

$$\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

with

$$T \mapsto (\mathrm{id} \otimes T)(|\Omega\rangle\langle\Omega|) =: c_T \qquad \forall\ T \in CPTP(\mathcal{H}_1, \mathcal{H}_2)$$

defines an isomorphism between the spaces of linear maps from a d_1- to a d_2-dimensional system and linear operators on a $(d_1 d_2)$-dimensional Hilbert space $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, the so-called Choi-Jamiołkowski Isomorphism*.*
The inverse is given by

$$c_T \mapsto d_1\, \mathrm{tr}_1[(B^T \otimes \mathrm{id})c_T] = T(B)$$

for $B \in \mathcal{B}(\mathcal{H}_1)$. The corresponding matrix c_T on the bipartite system $\mathcal{H}_1 \otimes \mathcal{H}_2$ is called Choi matrix *corresponding to T.*

The following correspondences can be derived from this theorem.

Proposition 2.1.15 (Properties Choi matrix)**.** *Let $T : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ be a linear map and c_T its Choi matrix. Then, the following statements hold.*

(i) T is CP if and only if $c_T \geq 0$ [24], [18, Theorem 4.46].

(ii) T is TP if and only if $\mathrm{tr}_2(c_T) = \frac{\mathbb{1}_d}{d}$ [19, Theorem 2.26].

(iii) T is unital if and only if $\mathrm{tr}_1(c_T) = \frac{\mathbb{1}_d}{d}$ [26, Corollary 2].

Example 2.1.16 (Examples Channel-State Duality)**.** *We give some examples for Choi-Jamiołkowski states corresponding to some of the channels in Example 2.1.12.*

a) The Choi-Jamiołkowski state corresponding to the identity channel is the maximally entangled state $|\Omega\rangle\langle\Omega|$, which can directly be seen from Definition 2.1.14.

b) The completely depolarizing channel corresponds to the maximally mixed state $\frac{\mathbb{1}_{d^2}}{d^2}$.

c) Using linearity and Theorem 2.1.14, we immediately obtain the Choi-Jamiołkowski state of the depolarizing channel with a) and b) as $\lambda|\Omega\rangle\langle\Omega| + (1-\lambda)\frac{\mathbb{1}_{d^2}}{d^2}$.

The next and last way of representing quantum channels we present here is the *Kraus-representation.*

Kraus representation The Kraus representation corresponds to a spectral decomposition of the Choi-Jamiołkowski state into rank-one operators (and is therefore called *operator sum decomposition*). Kraus representations are only unique up to unitary equivalence.

Proposition 2.1.17 (Kraus representation [18, Proposition 4.21])**.** *Let $d_1, d_2 \in \mathbb{N}$. A linear map $T : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ is a quantum channel if and only if it can be written as*

$$T(\rho) = \sum_{i=1}^{n} K_i \rho K_i^* \qquad \text{with} \qquad \sum_{i=1}^{n} K_i^* K_i = \mathbb{1}$$

for all $\rho \in \mathcal{D}(\mathcal{H}_1)$ where $\{K_i\}_{i=1}^n$, $n \leq d_1 d_2$, with $K_i \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, is a set of Kraus operators.
The channel T additionally is unital *if and only if $\sum_{i=1}^n K_i K_i^* = \mathbb{1}$.*

Remark 2.1.18 (Relation Choi rank and dilation dimension [19, Corollary 2.21])**.** *Let* $\operatorname{rank}(c_T)$ *be the Choi rank. Then, there exists a Stinespring representation of T where the dimension of the dilation space is equal to* $\operatorname{rank}(c_T)$.

Finally, each of these three characterizations gives rise to a criterion for checking whether a map T is completely positive. This is summarized in the following theorem.

Theorem 2.1.19 (Characterization Complete Positivity [19, see Theorem 2.22])**.** *Let $d_1 \in \mathbb{N}$. A linear map $T : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_1)$ is CP if and only if one of the following three statements holds.*

(i) There exists a finite-dimensional Hilbert space $\mathcal{H}_E$ with $\dim(\mathcal{H}_E) = \operatorname{rank}(c_T)$*, an operator $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_E)$ and a pure quantum state $|\varphi\rangle \in \mathcal{H}_E$ such that $\forall\ \rho \in \mathcal{D}(\mathcal{H}_1)$*

$$T(\rho) = \operatorname{tr}_E[A(\rho \otimes |\varphi\rangle\langle\varphi|)A^*].$$

(ii) The corresponding Choi matrix is positive semidefinite, i.e., $c_T \geq 0$.

(iii) There exist operators $\{K_i\}_{i=1}^n$ *such that*

$$T(\rho) = \sum_{i=1}^{n} K_i \rho K_i^*$$

for all $\rho \in \mathcal{D}(\mathcal{H}_1)$ *and* $n = rank(c_T)$.

2.2 Quantifying information

To quantify information, we use the *Holevo information*, which relies on an *ensemble of states*, and the *von Neumann entropy*. The following definitions and results in this section are based on the book [27] if not stated differently.
A finitely supported probability distribution p on the set of quantum states $\mathcal{D}(\mathcal{H})$ that assigns probabilities p_i to ρ_i is called an *ensemble* $\{\rho_i, p_i\}$. We are interested in the information content both of a single state and of an ensemble of states. For a single state, this is quantified by the *entropy*. We define the *von Neumann entropy* as

$$S(\rho) = -\operatorname{tr} \rho \log \rho = \operatorname{tr}(f(\rho)),$$

where $f(\rho) = -\rho \log \rho$. The function $f(\cdot)$ is a uniformly continuous function on $[0, 1]$.
Note that this quantity can be upper and lower bounded by

$$0 \leq S(\rho) \leq \log d \tag{2.2.1}$$

for any state ρ on the d-dimensional Hilbert space $\mathcal{H}$. The minimum $S(\rho) = 0$ is achieved for pure states and the maximum for the maximally mixed state $\rho = \frac{\mathbb{1}_d}{d}$ [27, Section 5.2].
The continuity of the von Neumann entropy can be estimated by the following bound which can be found in Refs [28, Theorem 3.8] and [29, Theorem 1]. Suppose two density matrices ρ and σ on a d-dimensional Hilbert space and let $\epsilon := \frac{1}{2}\|\rho - \sigma\|_1$. Then,

$$|S(\rho) - S(\sigma)| \leq \epsilon \log(d-1) + h(\epsilon), \tag{2.2.2}$$

with $h(\epsilon) = H(\epsilon, 1-\epsilon) = -\epsilon \log \epsilon - (1-\epsilon)\log(1-\epsilon)$ being the binary entropy [28, Theorem 3.8].
Quantifying the information content of an ensemble is achieved by the *Holevo information* defined as

$$\chi\big(\{\rho_i, p_i\}\big) := S\big(\sum_i p_i \rho_i\big) - \sum_i p_i S(\rho_i). \tag{2.2.3}$$

For G being a compact group, there exists a unique left and right invariant probability measure, which we call *Haar measure* μ [30, Chapter VII.3]. We normalize the measure such that

$$\int_G d\mu = 1.$$

For more details on the Haar measure, we refer to Ref. [30, Chapter VII.3].

Furthermore, we define the notion of a t-design. Let $\{p_i\}_{j=1}^J$ be a probability distribution over a set of pure quantum states $\{|\psi_j\rangle, p_j\}_{j=1}^J$, $|\psi_j\rangle\langle\psi_j| \in \mathcal{D}_P(\mathcal{H})$ for all $j \in \{1, \ldots, J\}$. Given t copies of a state from this distribution, $\{|\psi_j\rangle, p_j\}_{j=1}^J$ is called a *quantum t-design* if

$$\sum_j p_j \left(|\psi_j\rangle\langle\psi_j|\right)^{\otimes t} = \int_{\mathcal{D}_P} \left(|\phi\rangle\langle\phi|\right)^{\otimes t} \mu(d\phi), \tag{2.2.4}$$

where the integral is taken over all pure states $|\phi\rangle\langle\phi| \in \mathcal{D}_P(\mathcal{H})$ with respect to the Haar measure. In words, the probability distribution $\{p_j\}_{j=1}^J$ cannot be distinguished from the Haar measure $\mu(d\phi)$ being the uniform probability distribution over all pure quantum states $|\phi\rangle\langle\phi| \in \mathcal{D}_P(\mathcal{H})$ [31, Definition 1].

To be able to study symmetries in quantum information theory, we – additionally to the basics in quantum information theory – need fundamental definitions and results from representation theory which we review in the following section.

2.3 Basic definitions and results from representation theory

Representation theory considers maps that associate invertible matrices to the group elements g of a group G. Instead of dealing with the group elements themselves, we work with the associated matrices. This is useful if G obeys a complicated structure because using a representation might be an easier or even the only way to obtain information about G [32, 33]. Moreover, a representation can be seen as an action of G on a vector space and symmetries, arising in a physical theory, for instance, are often associated with the action of groups. Studying representations of these groups therefore allows us to consider further symmetry properties and those can yield a simplification of the problem [34]. To study symmetries in quantum information theory, we introduce basic notations, concepts and tools from representation theory. There are many textbooks on representation theory such as [30, 32–35]. We closely follow Ref. [30] for the main definitions and theorems. From now on, G always denotes a compact Lie group (short: compact group). We start with a formal mathematical definition of a *representation*.

Definition 2.3.1 (Representation of a compact group)**.** *Let G be a compact group, $\mathcal{V} \cong \mathcal{H} \cong \mathbb{C}^d$ a complex vector space and $GL(\mathcal{V})$ the general linear group on $\mathcal{V}$. A continuous homomorphism from G to $GL(\mathcal{V})$ is called* representation *of the group G. The dimension $\dim(\mathcal{V}) = d$ is called the* degree *of the representation [30, p. 21].*

If $G \subset GL(\mathcal{H})$ is a closed subgroup, the representation $R : G \to GL(\mathcal{H})$ that maps every element to itself $g \mapsto g$ is called *defining representation* [32, p. 51]. In quantum mechanics, evolutions of systems can be described by unitaries. Thus, it is convenient to represent group elements by unitary operators. Therefore, we define *unitary representations*.

Definition 2.3.2 (Unitary representation [30, p. 156]). *Let G be a compact group. A* unitary representation *(short: representation) of G is a continuous homomorphism from G to the unitary operators $\mathcal{U}(\mathcal{H})$ on some complex, finite-dimensional Hilbert space $\mathcal{H}$.*

Theorem 2.3.3 (Weyl's unitary trick [30, Theorem VII.9.1]). *Let G be a compact Lie group and $V : G \to GL(\mathcal{H})$ be a representation. Suppose that $g \mapsto V_g$ is measurable and bounded in the sense that $\ell(V_g x)$ is measurable and bounded for each fixed $x \in \mathcal{H}$ and $\ell \in \mathcal{H}^*$. Furthermore, suppose that $V_g x$ is not almost everywhere in g zero for all $x \in \mathcal{H}$.*
Then, there exists an inner product on $\mathcal{H}$ such that all $V_g \in \mathcal{U}(\mathcal{H})$, i.e., V is a unitary representation.

This is why it suffices to consider unitary representations. Furthermore, we consider finite-dimensional unitary representations such that $\mathcal{H} \cong \mathbb{C}^d$ for some $d \in \mathbb{N}$. In the following, we always mean unitary representation if we briefly write representation.
We classify representations up to unitary equivalence and therefore define the notion of *unitarily equivalent*.

Definition 2.3.4 (Unitarily equivalent [30, p. 156]). *Let $U : G \to \mathcal{U}(\mathcal{H})$ and $V : G \to \mathcal{U}(\mathcal{H})$ be two representations of the same compact group G. We call U and V* unitarily equivalent (short: equivalent) *if and only if there exists a unitary $W \in \mathcal{U}(\mathcal{H})$ such that $V_g = W U_g W^{-1}$ $\forall$ $g \in G$.*

We illustrate the definitions and concepts from representation theory based on the example of the alternating group A_4. The reason why we use this particular group gets apparent later in this section.

Example 2.3.5 (Alternating group A_4: Introduction [35, Section 2.3]). *The alternating group A_4 is a subgroup of the symmetric group S_4 only consisting of the even permutations. The group consists of 12 elements which can be partitioned into 4 conjugacy classes, i.e., equivalence classes w.r.t. conjugation. Here, two group elements g_1 and g_2 are called* conjugate *if there exists an element $g \in G$ with $g_2 = g^{-1} g_1 g$.*
The simplest map that defines a representation is the map that associates the identity to every $g \in G$

$$\begin{aligned} \varphi : A_4 &\to GL(\mathbb{R}) \\ g &\mapsto 1. \end{aligned}$$

Let us define the *complex conjugate representation* $\bar{U}$ [30, p. 158] of the unitary representation U as

$$\begin{aligned} \bar{U} : G &\to GL(\mathcal{H}) \\ g &\mapsto \bar{U}_g, \end{aligned}$$

where $\bar{U}_g$ is the complex conjugate, i.e., the entrywise complex conjugate unitary matrix associated to $g \in G$ through the representation $\bar{U}$.

We can build more advanced representations using the direct sum and tensor product.

Lemma 2.3.6 (Direct sum and tensor product representation [30, p. 24, p. 29])**.** *Let $U : G \to GL(\mathcal{H}_U)$ and $V : G \to GL(\mathcal{H}_V)$ be two unitary representations of G. Then,*

i) $U \oplus V : G \to GL(\mathcal{H}_U \oplus \mathcal{H}_V)$ *with* $(U \oplus V)(g) := U_g \oplus V_g \ \forall \ g \in G$ *and*

ii) $U \otimes V : G \to GL(\mathcal{H}_U \otimes \mathcal{H}_V)$ *with* $(U \otimes V)(g) := U_g \otimes V_g \ \forall \ g \in G$

are also unitary representations.

Remark 2.3.7. *The corresponding unitary matrices of the direct sum and tensor product representation have the following special structure. Let U_g be the unitary matrix corresponding to $g \in G$, associated to the unitary representation U and let V_g be the unitary matrix corresponding to $g \in G$ associated to the unitary representation V. Then, the matrix representation of*

i) $U \oplus V$ *is* $\begin{pmatrix} U_g & 0 \\ 0 & V_g \end{pmatrix} \in \mathcal{M}_{d_U+d_V, d_U+d_V}$ *(block-structure) [30, p. 24] and of*

ii) $U \otimes V$ *is* $U_g \otimes V_g \in \mathcal{M}_{d_U \cdot d_V, d_U \cdot d_V}$ *[30, p. 29].*

We seek to consider representations that cannot be decomposed non-trivially as a direct sum. More complicated representations can be described by connecting those basic building blocks with a direct sum operation.
To define such elementary representations, we require the notion of an *U-invariant subspace.*

Definition 2.3.8 (Invariant subspace [30, p. 24])**.** *Let U be a unitary representation of G on $\mathcal{H}$. A subspace $\mathcal{H}' \subseteq \mathcal{H}$ is called* U-invariant *if for all $g \in G$ and $y \in \mathcal{H}'$, $U_g y \in \mathcal{H}'$.*

We are now ready to define the fundamental building blocks: irreducible representations.

Definition 2.3.9 (Irreducible representation [30, p. 157])**.** *A unitary representation U of G on a finite-dimensional vector space $\mathcal{H}$ is called* irreducible representation *(short:* irrep*) if and only if the only U-invariant subspaces of $\{U_g \mid g \in G\}$ are $\{0\}$ and $\mathcal{H}$.*

We can simplify any representation with the direct sum decomposition into irreps, whereas irreps themselves cannot be further decomposed. This is stated in the following corollary.

Corollary 2.3.10 (Characterization irrep [30, Corollary II.2.2])**.** *A representation U is an irrep if and only if it cannot be written as direct sum of non-trivial representations.*

In the case of A_4 this looks as follows.

Example 2.3.11 (Alternating group A_4: Irreducible Representations [35, Section 2.3])**.** *The group A_4 has four irreps $\varphi^{(1)}$, $\varphi^{(2)}$, $\varphi^{(3)}$, and $\varphi^{(4)}$. Three of them, $\varphi^{(1)}, \varphi^{(2)}, \varphi^{(3)}$, are one dimensional and $\varphi^{(4)}$ is three dimensional. One of the one-dimensional irreps, $\varphi^{(1)}$, is the trivial irrep which maps $g \mapsto 1$ (see Example 2.3.5). The irreps of the further two one-dimensional irreps are given by the second row of Table 2.1 and the third row, respectively. This holds because one-dimensional irreps coincide with their characters defined in Definition 2.3.21. Additionally, there is a three-dimensional irrep which can be described by permuting four four-dimensional basis vectors of a vector space according to the action of A_4. There is one one-dimensional subspace which results in a three-dimensional irrep.*

The following fundamental theorem expresses that any unitary representation of a compact group can be decomposed into a direct sum of irreps.

Theorem 2.3.12 (Direct sum decomposition [30, Theorem VII.9.3])**.** *Every representation of a compact group G is equivalent to a direct sum of irreps.*

This means that the representation itself, the image of the representation and the Hilbert space $\mathcal{H}$, the representation acts on, can be decomposed as direct sum of the corresponding maps, operators and spaces, respectively. When we decompose a representation into a direct sum of irreps, then some of the irreps might be equivalent. In that case, we group them together.
Let $\mathcal{H}$ be a d-dimensional Hilbert space and $U : G \to GL(\mathcal{H})$ a *reducible unitary representation*, i.e., there is a non-trivial decomposition as a direct sum of irreps. We equip every irrep with an index $k \in \{1, \dots, K\}$ for a total number K of irreps. Let $\dim \mathcal{H}_1 = b_1, \dots, \dim \mathcal{H}_K = b_K$.

$$\begin{aligned}
\mathcal{H} &\cong (\mathcal{H}_1^{(1)} \oplus \dots \oplus \mathcal{H}_1^{(n_1)}) \oplus \dots \oplus (\mathcal{H}_K^{(1)} \oplus \dots \oplus \mathcal{H}_K^{(n_K)}) \\
&= (\mathcal{H}_1)^{\oplus n_1} \oplus \dots \oplus (\mathcal{H}_K)^{\oplus n_K} \\
&= \bigoplus_{k=1}^{K} (\mathcal{H}_k)^{\oplus n_k} \\
&= \bigoplus_{k=1}^{K} n_k \mathcal{H}_k \\
&= \bigoplus_{k=1}^{K} (\mathcal{H}_k \otimes \mathcal{H}'_k),
\end{aligned}$$

where we call n_k, for $k \in \{1, \dots, K\}$, the *multiplicity* of the kth representation and $\mathcal{H}'_k$ is the corresponding Hilbert space of dimension n_k.
Analogously, for the unitary representation, we get the direct sum decomposition

$$U \cong \bigoplus_{k=1}^{K} U^{(k)} \otimes \mathbb{1}_{n_k}$$

with n_k being the multiplicity. The sum runs over all irreps of U.

For the image of the representation, i.e., the unitary matrices on $\mathcal{U}(\mathcal{H})$, we obtain the following structure

$$U_g \cong \bigoplus_{k=1}^{K} U_g^{(k)} \otimes \mathbb{1}_{n_k}$$

for all $g \in G$.

To study symmetries in quantum information theory in the next chapter, we consider representations of the form $\bar{U} \otimes V$. The following remark clarifies the relation between this tensor product representation and irreps.

Remark 2.3.13. *If U, V are irreps, then $\bar{U}$ an $\bar{V}$ are irreps as well [30, p. 30]. However, $\bar{U} \otimes V$ is usually no irrep [30, p. 29] as the following example illustrates.*

Example 2.3.14 (Alternating group A_4: Direct sum decomposition)**.** *If we consider more complicated representations, due to some particular symmetry, we can always decompose them as direct sums of simpler irreps. As an example, we consider $\bar{\varphi}^{(4)} \otimes \varphi^{(4)}$ which can be decomposed as*

$$\bar{\varphi}^{(4)} \otimes \varphi^{(4)} = \varphi^{(1)} \oplus \varphi^{(2)} \oplus \varphi^{(3)} \oplus 2\varphi^{(4)},$$

where $\varphi^{(4)}$ appears with multiplicity 2.

One reason why we chose the group A_4 as an example is that it provides a direct sum decomposition of a tensor representation with multiplicity larger than one. This becomes important when studying covariant programmable quantum processors. In the special case of an abelian group G, i.e., a group with commutative product, the following holds.

Corollary 2.3.15 (Degree irrep abelian group [30, Corollary II.4.3])**.** *If G is abelian, every irrep has degree 1.*

Example 2.3.16 (Alternating group A_4: non-abelian)**.** *The group A_4 is non-abelian. The irrep $\varphi^{(4)}$ is three-dimensional.*

The image of the representation, i.e., the set of unitaries $\{U_g\}_{g \in G}$, generates an algebra. Recall that $\mathcal{A}$ is an algebra if $A + B, AB, cA \in \mathcal{A}$ for all $A, B \in \mathcal{A}$ and $c \in \mathbb{C}$. If $AB = BA$ for all A, $B \in \mathcal{A}$, we call $\mathcal{A}$ an abelian algebra. Additionally, if there is an involution $* : \mathcal{A} \to \mathcal{A}$, which is the adjoint for matrix algebras, then $\mathcal{A}$ is called C*-algebra. To study symmetries in the next chapter, we define the commutant of the algebra $\mathcal{A}$ as

Definition 2.3.17 (Commutant)**.** *Let $\mathcal{A}$ be an algebra of operators on the Hilbert space $\mathcal{H}$. Its* commutant *is*

$$\mathcal{A}' := \{B \mid BA = AB \ \forall \ A \in \mathcal{A}\}.$$

If the algebra has a special structure, its commutant ultimately has a corresponding structure as well, which directly follows from the structure of $\mathcal{A}$ and the definition of a commutant.

Theorem 2.3.18 (Structure of the commutant [30, Thm IX.11.2]). *Let $U : G \to GL(\mathcal{H})$ be a unitary representation of G, which can be written as $\mathcal{H} = \bigoplus_{k=1}^{K} (\mathcal{H}_k \otimes \mathcal{H}'_k)$ such that $U_g = \bigoplus_{k=1}^{K} (U_g^{(k)} \otimes \mathbb{1}_{n_k})$ for all $g \in G$ where $U^{(k)}$, $k \in \{1, \dots, K\}$, are irreps of G. Furthermore, let $\mathcal{A}(U)$ be the operator algebra generated by the $\{U_g\}_{g \in G}$ and $\mathcal{A}'(U)$ the corresponding commutant. Then,*

$$\mathcal{A}(U) = \left\{ \bigoplus_{k=1}^{K} A_k \otimes \mathbb{1}_{n_k} \mid A_k \in \mathcal{B}(\mathcal{H}_k) \right\},$$
$$\mathcal{A}'(U) = \left\{ \bigoplus_{k=1}^{K} \mathbb{1}_{b_k} \otimes B_k \mid B_k \in \mathcal{B}(\mathcal{H}'_k) \right\}.$$

The particular structure of elements of the commutant is illustrated in Figure 2.1.

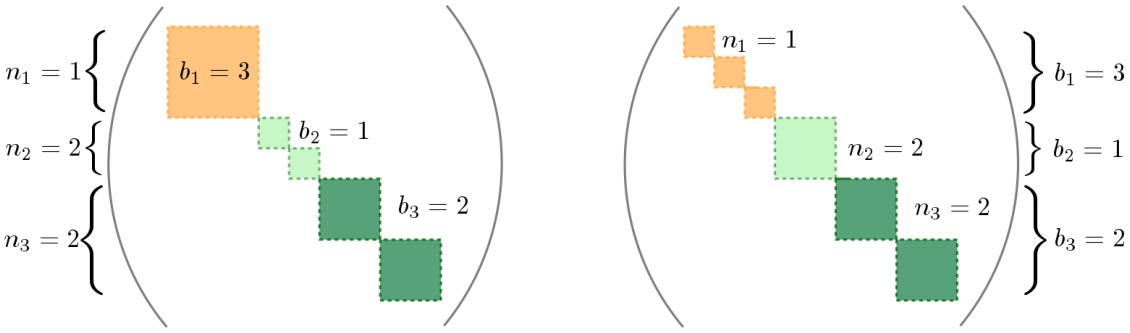

Figure 2.1: This figure illustrates the structure of elements of $\mathcal{A}(U)$ on the left-hand side and its commutant $\mathcal{A}'(U)$ on the right-hand side in the case of $K = 3$, i.e., 3 irreps in the direct sum decomposition [1, Figure 1].

Example 2.3.19 (Alternating group A_4: structure of the tensor representation). *Due to the direct sum decomposition of $\bar{\varphi}^{(4)} \otimes \varphi^{(4)}$ (see Example 2.3.14) we know that its image forms a block-diagonal matrix. There are three one-dimensional blocks on the diagonal and two three-dimensional ones.*
According to Theorem 2.3.18, the commutant consists of three one-dimensional blocks and three two-dimensional blocks.

We use *Schur's Lemma* as a tool to later prove statements concerning the covariant processor.

Lemma 2.3.20 (Schur's Lemma [30, Thm II.4.1, Thm II.4.2]). *Let U, V be two irreps of G on $\mathcal{H}_U$ and $\mathcal{H}_V$, respectively. The following statements hold:*

i) If $T_1 : \mathcal{H}_U \to \mathcal{H}_U$ is linear with $T_1 U_g = U_g T_1$ for all $g \in G$. Then,

$$T_1 = c\mathbb{1} \text{ for some } c \in \mathbb{C}.$$

ii) If $T_2 : \mathcal{H}_U \to \mathcal{H}_V$ such that $T_2 U_g = V_g T_2$ for all $g \in G$. Then, either $T = 0$ or U and V are unitarily equivalent and T_2 is unique up to a constant, i.e., if there is a T_2' satisfying $T_2' U_g = V_g T_2'$ for all $g \in G$, then $T_2' = cT_2$ for a constant c.

A convenient tool in the context of irreps is a map which associates a complex number to every element of the group. This number is related to a unitary representation through the trace operation and is defined as

Definition 2.3.21 (Character [30, p. 41])**.** *Let $U : G \to GL(\mathcal{H})$ be a representation such that $g \mapsto U_g$. The* character $\chi : G \to \mathbb{C}$ *maps every element of G to a complex number according to $g \mapsto \mathrm{tr}(U_g)$.*

In the case of a one-dimensional representation, the character is equal to the one-dimensional representation. The irreps and corresponding multiplicites that occur in the direct sum decomposition are determined by U and the character determines the direct sum decomposition into irreps [30, p. 41]. Note that the multiplicities can be obtained using the characters (see Corollary 2.3.25 below). Characters obey special properties illustrated in the following statements

Proposition 2.3.22 (Properties of characters [30, p. 41], [35, p. 13])**.** *Let $\chi : G \to \mathbb{C}$, $g \mapsto \mathrm{tr}(U_g)$ be a character.*

1. *If U, V are equivalent representations, then $\chi_U = \chi_V$.*
2. *For the direct sum representation, $\chi_{U\oplus V} = \chi_U + \chi_V$ holds.*
3. *For the tensor product representation, $\chi_{U\otimes V} = \chi_U \cdot \chi_V$ holds.*

Characters of groups are often gathered in *character tables*. We consider the alternating group again.

Example 2.3.23 (Alternating group A_4: character table [35, p. 20])**.** *In the following table we give the characters for the irreps of A_4. The rows correspond to the irreps, columns represent the four conjugacy classes and $\omega = e^{2\pi i/3}$.*

A_4	1	(123)	(132)	(12)(34)
$\varphi^{(1)}$	1	1	1	1
$\varphi^{(2)}$	1	ω	ω^2	1
$\varphi^{(3)}$	1	ω^2	ω	1
$\varphi^{(4)}$	3	0	0	-1

Table 2.1: Character table of A_4.

The following theorem and corollary state how characters can be used to study representations.

Theorem 2.3.24 (Orthogonality relations for characters, [30, Theorem VII.9.5])**.** *For all irreps α, β with characters χ_α, χ_β,*

$$\langle \chi_\alpha, \chi_\beta \rangle := \int_G \overline{\chi_\alpha(g)} \chi_\beta(g) d\mu(g) = \delta_{\alpha\beta},$$

where $\langle \chi_\alpha, \chi_\beta \rangle$ is the inner product on $L^2(G) = \{\chi : G \to \mathbb{C} \mid \int_G |\chi(g)|^2 d\mu(g) < \infty\}$ and μ denotes the Haar measure on G.

The multiplicities of irreps in the direct sum decomposition of a representation can be calculated as follows:

Corollary 2.3.25 (Multiplicity and characters [30, Corollary VII.9.6])**.** *Let U be a representation of a compact Lie group G and χ_U the corresponding character. Let χ_α be the character of an irrep α. Then,*

$$n_\alpha = \langle \chi_\alpha, \chi_U \rangle = \int_G \overline{\chi_\alpha(g)} \chi_U(g) d\mu(g)$$

is the multiplicity of α in the direct sum decomposition of U. Note that the n_α are uniquely determined by U.

These fundamental results of representation theory are crucial for studying symmetries in quantum information theory in the next chapter.

3 Symmetries in quantum information theory

In this chapter, we combine concepts from quantum information theory and representation theory, i.e., we consider states and channels that obey a certain symmetry. In Chapter 1, we have already motivated that the covariance symmetry property for channels plays an important role. These are channels whose action 'commutes' with the symmetry [36]. Hence, they can be defined as follows:

Definition 3.1 (UV-covariant quantum channel)**.** *Let G be a compact group and let U and V be representations of G on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $T : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ be a quantum channel. We call T UV-*covariant *if*

$$T(U_g A U_g^*) = V_g T(A) V_g^* \qquad \forall\ A \in \mathcal{B}(\mathcal{H}_1),\ \forall\ g \in G.$$

Such channels were studied in Ref. [16] for the case that U is an irrep, $V = U$, and $\bar{U} \otimes U$ is multiplicity-free.
The set of all UV-covariant channels is represented by

$$\mathcal{T}_{UV} \coloneqq \{T : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2) \mid T \text{ is a } UV\text{-covariant quantum channel}\}.$$

Examples of covariant quantum channels To illustrate the above definition, we provide examples of UV-covariant channels. For details, we refer to Ref. [37].

Example 3.2. *The following two examples can be found in Ref. [37, Subsection 3.2.2].*

1) A common example is UU-covariance w.r.t. the defining representation of the unitary group $\mathcal{U}(\mathcal{H})$, i.e.,

$$T(U_g \rho U_g^*) = U_g T(\rho) U_g^*\ \forall\ U_g \in \mathcal{U}(\mathcal{H}).$$

T is UU-covariant w.r.t. the defining representation of the unitary group if and only if there is an $\alpha \in [0, \frac{d^2}{d^2-1}]$ such that

$$T(\cdot) = \alpha \frac{\mathrm{tr}(\cdot)}{d} \mathbb{1} + (1 - \alpha)\mathrm{id}(\cdot).$$

2) Another example is OO-covariance w.r.t. the defining representation of the orthogonal group $\mathcal{O}(\mathcal{H})$, i.e.,

$$T(O_g \rho O_g^*) = O_g T(\rho) O_g^* \qquad \forall\ O_g \in \mathcal{O}(\mathcal{H}).$$

A channel T is OO-covariant w.r.t. the unitary representation O of the orthogonal group $\mathcal{O}(\mathcal{H})$ if and only if there are $\beta_1, \beta_2 \geq 0$ and $\beta_1 + \beta_2 \leq 1$ such that

$$T(\cdot) = (1-\beta_1-\beta_2)\mathrm{id}(\cdot) + \frac{\beta_1}{d-1}\big(\mathrm{tr}(\cdot)\mathbb{1} - (\cdot)^T\big) + \frac{2\beta_2}{d(d+1)-2}\left(\frac{d}{2}(\mathrm{tr}(\cdot)\mathbb{1} - (\cdot)^T) - \mathrm{id}(\cdot)\right)$$

where $(\cdot)^T$ is the transpose, i.e., the input state is transposed.

Due to the Choi-Jamiołkowski isomorphism (Theorem 2.1.14), there is a corresponding Choi-Jamiołkowski state to each channel $T \in \mathcal{T}_{UV}$. Thus, we define the set of all Choi-Jamiołkowski states corresponding to quantum channels $T \in \mathcal{T}_{UV}$ as

$$\mathcal{J}_{UV} := \{c_T \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \mid c_T := (\mathrm{id} \otimes T)(|\Omega\rangle\langle\Omega|) \forall\ T \in \mathcal{T}_{UV}\}.$$

The following lemma establishes a property of the Choi-Jamiołkowski states corresponding to UV-covariant quantum channels.

Lemma 3.3. *The covariance property of a channel $T \in \mathcal{T}_{UV}$ w.r.t. the unitary representations U and V of a group G is equivalent to the condition that the corresponding Choi-Jamiołkowski state $c_T \in \mathcal{J}_{UV}$ commutes with $\bar{U}_g \otimes V_g$ for every $g \in G$, i.e., $[c_T, \bar{U}_g \otimes V_g] = 0$.*

Proof. "⇒" Using $(\bar{U}_g \otimes U_g)|\Omega\rangle\langle\Omega|(\bar{U}_g \otimes U_g)^* = |\Omega\rangle\langle\Omega|$, we get the following equations

$$\begin{aligned}(\bar{U}_g \otimes V_g)^* c_T (\bar{U}_g \otimes V_g) &= (\bar{U}_g \otimes V_g)^*(\mathrm{id} \otimes T)(|\Omega\rangle\langle\Omega|)(\bar{U}_g \otimes V_g)\\ &= (\bar{U}_g \otimes V_g)^*(\mathrm{id} \otimes T)((\bar{U}_g \otimes U_g)|\Omega\rangle\langle\Omega|(\bar{U}_g \otimes U_g)^*)(\bar{U}_g \otimes V_g)\\ &= (\bar{U}_g \otimes V_g)^*(\bar{U}_g \otimes V_g)(\mathrm{id} \otimes T)(|\Omega\rangle\langle\Omega|)(\bar{U}_g \otimes V_g)^*(\bar{U}_g \otimes V_g)\\ &= \mathrm{id} \otimes T(|\Omega\rangle\langle\Omega|),\end{aligned}$$

where we use that T is covariant in the third equation.
"⇐" We assume that $[c_T, \bar{U}_g \otimes V_g] = 0$. Then,

$$\begin{aligned}c_T &= (\bar{U}_g \otimes V_g)^* c_T (\bar{U}_g \otimes V_g)\\ &= (\bar{U}_g \otimes V_g)^*(\mathrm{id} \otimes T)(|\Omega\rangle\langle\Omega|)(\bar{U}_g \otimes V_g)\\ &= (\bar{U}_g \otimes V_g)^*(\mathrm{id} \otimes T)((\bar{U}_g \otimes U_g)|\Omega\rangle\langle\Omega|(\bar{U}_g \otimes U_g)^*)(\bar{U}_g \otimes V_g),\end{aligned}$$

where we use $(\bar{U}_g \otimes U_g)|\Omega\rangle\langle\Omega|(\bar{U}_g \otimes U_g)^* = |\Omega\rangle\langle\Omega|$ again. Since

$$c_T = (\bar{U}_g \otimes V_g)^*(\mathrm{id} \otimes T)((\bar{U}_g \otimes U_g)|\Omega\rangle\langle\Omega|(\bar{U}_g \otimes U_g)^*)(\bar{U}_g \otimes V_g),$$

c_T is also the Choi-Jamiołkowski state of $V_g^* T(U_g(\cdot)U_g^*)V_g$. Since the Choi-Jamiołkowski map $T \mapsto c_T$ is an isomorphism (Theorem 2.1.14), the corresponding channels are equal and therefore, the channels are UV-covariant.
Thus, we showed that the UV-covariance property of T is equivalent to the corresponding Choi-Jamiołkowski state commuting with $\bar{U}_g \otimes V_g$. □

Let us consider the examples we presented in Example 3.2 together with the Examples 2.1.16 and expand them to their corresponding Choi-Jamiołkowski state and their symmetries.

Example 3.4. *We present the UU-covariant and the OO-covariant channel.*

1) Recall the UU-covariant channel from Example 3.2, which is a convex combination of the completely depolarizing channel $T_1 = \mathrm{tr}(\cdot)\frac{\mathbb{1}_d}{d}$ *and the* id*-channel* $T_2(\cdot) = (\cdot)$*. According to Lemma 3.3, the corresponding Choi-Jamiołkowski states satisfy* $[c_T, \bar{U}_g \otimes U_g] = 0$*. The states corresponding to* T_1 *and* T_2 *can be easily calculated*

$$c_{T_1} = (\mathrm{id} \otimes T_1)(|\Omega\rangle\langle\Omega|) = \frac{1}{d}\sum_{i,j=1}^{d} |i\rangle\langle j| \otimes \mathrm{tr}(|i\rangle\langle j|)\frac{\mathbb{1}}{d} = \frac{\mathbb{1}_{d^2}}{d^2},$$
$$c_{T_2} = (\mathrm{id} \otimes \mathrm{id})(|\Omega\rangle\langle\Omega|) = |\Omega\rangle\langle\Omega|.$$

Every state of the state space $\mathcal{J}_{UU}$ *can be written as* $\rho = a\mathbb{1} + b|\Omega\rangle\langle\Omega|$ *with appropriate* $a, b \in \mathbb{C}$ *such that* $\mathrm{tr}(\rho) = 1$ *and* $\rho \geq 1$*. This yields*

$$\mathcal{J}_{UU} = \left\{ \alpha\frac{\mathbb{1}_{d^2}}{d^2} + (1-\alpha)|\Omega\rangle\langle\Omega| \mid \alpha \in [0, \frac{d^2}{d^2-1}] \right\}$$

such that the elements satisfy the state properties. These states obeying the symmetry $[\bar{U}_g \otimes U_g, c_T] = 0$ *are called* isotropic states *[38, Example 2], [37, Subsection 3.1.3].*

2) From Example 3.2, we notice that an OO-covariant channel consists of

$$T_1(\cdot) = d\,\mathrm{tr}(\cdot)\mathbb{1},$$
$$T_2(\cdot) = \mathrm{id}(\cdot),$$
$$T_3(\cdot) = d(\cdot)^T,$$

where $(\cdot)^T$ *is the transpose. OO-covariance corresponds to* $[c_T, O_g \otimes O_g] = 0$ *for all* $g \in G$ *according to Lemma 3.3. We calculate the corresponding Choi-Jamiołkowski states again*

$$c_{T_1} = (\mathrm{id} \otimes T_1)(|\Omega\rangle\langle\Omega|) = \mathbb{1}_{d^2},$$
$$c_{T_2} = (\mathrm{id} \otimes T_2)(|\Omega\rangle\langle\Omega|) = |\Omega\rangle\langle\Omega| \text{ and}$$
$$c_{T_3} = (\mathrm{id} \otimes T_3)(|\Omega\rangle\langle\Omega|) = \mathbb{F},$$

where $\mathbb{F} = \sum_{i,j=1}^{d} |ij\rangle\langle ji|$ *is the* flip operator*. The state space is spanned by* $P_1 = |\Omega\rangle\langle\Omega|$*,* $P_2 = \frac{\mathbb{1}-\mathbb{F}}{2}$ *and* $P_3 = \frac{\mathbb{1}+\mathbb{F}}{2} - |\Omega\rangle\langle\Omega|$ *which yields*

$$\mathcal{J}_{OO} = \left\{ c_T : c_T = (1-\beta_1-\beta_2)P_1 + \beta_1\frac{P_2}{\mathrm{tr}(P_2)} + \beta_2\left(\frac{P_3}{\mathrm{tr}(P_3)}\right) \right\}$$

with $0 \leq \beta_1, \beta_2 \leq 1$*. These states are called* Werner states *[38, Example 1], [37, Subsection 3.1.2].*

These examples illustrate that studying symmetries in Quantum Information Theory through UV-covariant channels can be achieved by considering their corresponding Choi-Jamiołkowski states since UV-covariance corresponds to $[c_T, \bar{U}_g \otimes V_g] = 0$ (Lemma 3.3). Thus, we are interested in the struture and properties of elements that commute with $\bar{U}_g \otimes V_g$. Let us consider a representation of G of the form $\bar{U} \otimes V$ with U, V irreps of G. From Remark 2.3.13, we know that the representation $\bar{U} \otimes V$ is not necessarily irrep. By Theorem 2.3.12, this representation can be written as direct sum of irreps

$$\bar{U} \otimes V \cong \bigoplus_{k=1}^{K} n_k U^{(k)}$$

with irreps $U^{(k)}$ with $k \in \{1, \ldots, K\}$. The underlying Hilbert space decomposes as

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \cong \bigoplus_{k=1}^{K} \big(\mathcal{H}_k \otimes \mathcal{H}'_k\big),$$

where we sum over irreps of G. According to Remark 2.3.7, the involved direct sum leads to a block-diagonal structure of the corresponding matrices

$$\bar{U}_g \otimes V_g \cong \bigoplus_{k=1}^{K} U_g^{(k)} \otimes \mathbb{1}_{n_k},$$

where n_k is the multiplicity and $U_g^{(k)}$ corresponds to the dimension of the kth irrep of G, with $U_g^{(k)} \in \mathcal{B}(\mathcal{H}_k)$ being the corresponding matrices. Any $\bar{U}_g \otimes V_g$, $g \in G$, acts as the identity on the multiplicity spaces. According to Theorem 2.3.18, we obtain the following form for $(\bar{U}_g \otimes V_g)_{g \in G}$:

$$\mathcal{A}(\bar{U} \otimes V) = \Big\{ \bigoplus_{k=1}^{K} A_k \otimes \mathbb{1}_{n_k} \mid A_k \in \mathcal{B}(\mathcal{H}_k) \Big\},$$

where n_k is the multiplicity and b_k the dimension of the corresponding kth irrep $\forall\ k \in \{1, \ldots, K\}$ and A_k are the matrices corresponding to the kth irrep. Its commutant $\mathcal{G}'$ looks as follows

$$\mathcal{K} = \mathcal{A}'(\bar{U} \otimes V) = \Big\{ \bigoplus_{k=1}^{K} \mathbb{1}_{b_k} \otimes B_k \mid B_k \in \mathcal{B}(\mathcal{H}'_k) \Big\}. \tag{3.1}$$

If $\mathcal{K}$ is an abelian C^*-algebra, every $n_k = 1$ for all irreps $k \in \{1, \ldots, K\}$. In terms of matrices, we then obtain a diagonal matrix.

4 Programmable quantum processors

A programmable quantum processor serves as a quantum analogue of a computer with stored programs. Classically, the processor applies a function f (some operation), which is stored as a program state inside the machine, to an input (data) register x and outputs $f(x)$. Thus, a processor is a device which is able to apply different operations on the data register depending on the program state. Accordingly, a universal classical processor can perform any operation on an arbitrary input state.
The quantum analogon, a programmable quantum processor, implements a CPTP map that acts on some input state $\rho \in \mathcal{D}(\mathcal{H})$ which describes the state of a quantum system. The map is determined by the program state which represents the state of a second quantum system. Hence, the information about the implemented operation is encoded in a program quantum register and is not specified by classical parameters. Since such a processor is designed to implement a huge amount of different channels, we do not need to build a new processor each time we want to realize a new channel. We can simply change the program state which leads to more flexibility.
As an example, we consider a CNOT operation, that acts on a target and a control system which are the input and the program state in our case. If the program state is $|0\rangle$, the identity operation is applied to the input. If its state is $|1\rangle$, the NOT operation is performed, i.e., the input state is flipped. The device can at least implement two different operations: the identity and the NOT gate.
We call a programmable quantum processor which is able to perform any quantum operation on an arbitrary input state *universal*. Most of the literature focuses on the two fundamental quantum operations: unitary channels [4, 5, 7, 9, 39–44] and measurements [8, 10, 45–47]. The first model is supposed to implement any unitary quantum channel whereas the latter is able to implement the effect operators of any POVM depending on the program state.

To mathematically define the notion of a *programmable quantum processor*, we introduce the diamond norm, which we use to measure the distance between two quantum channels.

Definition 4.1 (Diamond norm)**.** *Let T be a linear map $\mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$. Its* diamond norm *is defined as*

$$\|T\|_\diamond = \sup_{\substack{k\geq 1 \\ \rho\in\mathcal{D}(\mathcal{H}_1\otimes\mathbb{C}^k)}} \|T\otimes \mathrm{id}_{\mathbb{C}^k}(\rho)\|_1,$$

where $\|\cdot\|_1 = \mathrm{tr}(\sqrt{T^*T})$ and $\mathbb{C}^k$ is the k-dimensional complex Hilbert space.

We define a programmable quantum processor with input $\rho \in \mathcal{D}(\mathcal{H}_1)$ that implements a class of channels with program states $\pi_\Phi \in \mathcal{D}(\mathcal{H}_P)$ of dimension d_P. This is schematically illustrated in Figure 4.1.

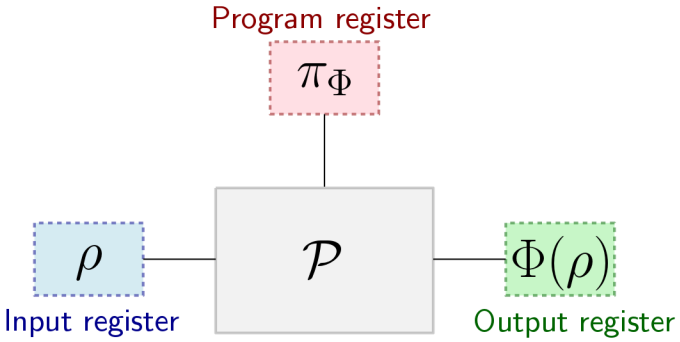

Figure 4.1: The figure shows a $\mathrm{PQP}_\mathcal{C}$ with its input, program and output register [1, Figure 2].

The following definition can be found in Ref. [1, Definition 3.1].

Definition 4.2 (ϵ-$\mathrm{PQP}_\mathcal{C}$)**.** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable Hilbert spaces. Then, we call $\mathcal{P} \in CPTP(\mathcal{H}_1 \otimes \mathcal{H}_P, \mathcal{H}_2)$, with finite-dimensional $\mathcal{H}_P$, an* ϵ-programmable quantum processor for a set $\mathcal{C} \subset \mathrm{CPTP}(\mathcal{H}_1, \mathcal{H}_2)$ *of channels (ϵ-$PQP_\mathcal{C}$), if for every quantum channel $\Phi \in \mathcal{C}$ there exists a state $\pi_\Phi \in \mathcal{D}(\mathcal{H}_P)$ such that*

$$\frac{1}{2}\|\mathcal{P}(\cdot\otimes\pi_\Phi) - \Phi(\cdot)\|_\diamond \leq \epsilon.$$

To address the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we refer to them as the input *and* output registers *and to $\mathcal{H}_P$ as the* program register*. We say that the processor $\mathcal{P}$* ϵ-implements *the class $\mathcal{C}$ of channels, leaving out the reference to ϵ in the case of exact programmability when $\epsilon = 0$.*

When $\mathcal{C} = \mathrm{CPTP}(\mathcal{H}_1, \mathcal{H}_2)$, we obtain a *universal* processor which we denote as ϵ-$\mathrm{PQP}_{\mathrm{CPTP}}$.

Another important special case which is addressed as *universal* in the literature (see for instance Refs. [14] and [15]), is that $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ and $\mathcal{C} = \mathcal{U}(\mathcal{H})$, the set of all unitary channels defined by conjugation with unitaries. However, since this is a restriction to unitaries, we call the corresponding processor *unitary-universal* and denote it as ϵ-$\mathrm{PQP}_\mathcal{U}$ instead of ϵ-UPQP, a *universal* programmable quantum processor. In the unitary case, the program states are assumed to be pure, in

accordance with the previous literature. If we allow a mixed program state π_P, we can always purify it on $\mathcal{H}_P \otimes \mathcal{H}_P$ and modify the processor to one acting on $\mathcal{H}_1 \otimes \mathcal{H}_P^{\otimes 2}$.

In its essence, the concept goes back to universal quantum gate arrays introduced by Nielsen and Chuang [4]. In their research, they also establish and prove the well-known *No-Programming Theorem* [4].

4.1 Exact programmability and the No-Programming Theorem

In the case of $\epsilon = 0$, we obtain the exact implementation of quantum channels by the programmable quantum processor. For $\mathcal{C} = \mathcal{U}_d$, the set of d-dimensional unitary channels, Nielsen and Chuang formulated their *No-Programming* Theorem in 1997, proving that it is not possible to construct a universal machine (universal gate array), which is able to program all unitary operations with finite-dimensional program register [4]. The main insight is that in order to program two different unitaries, two orthogonal program states are required. Since there are infinitely many unitaries the processor should be able to implement, we would need an infinite-dimensional program register.
Crucial for this is the independence of the processed program state from the input state, which we show first.

Let $|d\rangle \in \mathcal{H}_1$ be a pure input state and $|\psi_U\rangle \in \mathcal{H}_P$ the pure program state of the program space $\mathcal{H}_P$. The two registers are independent of each other and can thus be written as $|d\rangle \otimes |\psi_U\rangle$. They serve as input for the programmable quantum gate array which applies a unitary $U \in \mathcal{U}_d$ to the input state and transforms the program state into $|\psi'_U\rangle \in \mathcal{H}_P$. The array is modeled by a unitary operator $A \in \mathcal{U}(\mathcal{H}_1 \otimes \mathcal{H}_P)$ which acts as

$$A[|d\rangle \otimes |\psi_U\rangle] = (U\,|d\rangle) \otimes |\psi'_U\rangle$$

for all $|d\rangle \in \mathcal{H}_1$ and some state $|\psi_U\rangle$ of the program register. We say that the unitary operator U is *implemented* by A if a state $|\psi_U\rangle$ exists.
To see that $|\psi'_U\rangle$ is independent of $|d\rangle$, we suppose that we have two different input states $|d_1\rangle$ and $|d_2\rangle$ which lead to the following equalities

$$\begin{aligned} A[|d_1\rangle \otimes |\psi_U\rangle] &= (U\,|d_1\rangle) \otimes |\psi'_{U,1}\rangle\,, \\ A[|d_2\rangle \otimes |\psi_U\rangle] &= (U\,|d_2\rangle) \otimes |\psi'_{U,2}\rangle\,, \end{aligned} \tag{4.1.1}$$

where we suppose that $|\psi'_{U,1}\rangle$ and $|\psi'_{U,2}\rangle$ are two different program states after the application of A. If we take the inner product of the two equations, we get

$$\langle d_1|d_2\rangle = \langle d_1|d_2\rangle \cdot \langle \psi'_{U,1}|\psi'_{U,2}\rangle.$$

If $\langle d_1|d_2\rangle \neq 0$, we immediately conclude that $\langle \psi'_{U,1}|\psi'_{U,2}\rangle = 1$ and thus $|\psi'_{U,1}\rangle = |\psi'_{U_2}\rangle$. If $\langle d_1|d_2\rangle = 0$, we consider $|d_3\rangle = \frac{1}{\sqrt{2}}(|d_1\rangle + |d_2\rangle)$. Analogously to Eqs. (4.1.1),

we obtain

$$A[|d_3\rangle \otimes |\psi_U\rangle] = (U\,|d_3\rangle) \otimes |\psi'_{U,3}\rangle\,.$$

Since $\langle d_1|d_3\rangle \neq 0$ and $\langle d_2|d_3\rangle \neq 0$, we get $|\psi'_{U,1}\rangle = |\psi'_{U,3}\rangle$ and $|\psi'_{U,2}\rangle = |\psi'_{U,3}\rangle$ and hence,

$$|\psi'_{U,1}\rangle = |\psi'_{U,2}\rangle\,.$$

Therefore, the memory output $|\psi'_U\rangle$ is independent of the input $|d\rangle$.

Next, we recall the proof of the following no-go theorem from Ref. [4].

Theorem 4.1.1 (No-Programming Theorem [4])**.** *Let $U_1, \ldots, U_n$ be n unitary operators which are distinct (up to a global phase), i.e., $U_j \neq U_k e^{i\phi}$ for all $\phi \in [0, 2\pi)$, $k \neq j$, and implemented by some programmable quantum processor. The program register is at least n-dimensional, i.e., contains at least $\log_2 n$ qubits. The corresponding program states $|\psi_{U_1}\rangle, \ldots, |\psi_{U_n}\rangle$ are mutually orthogonal.*

Proof. We consider two program states $|\psi_{U_1}\rangle$ and $|\psi_{U_2}\rangle$ which lead to $U_1\,|d\rangle$ and $U_2\,|d\rangle$, respectively. The corresponding unitaries U_1 and U_2 are distinct up to a global phase. We get the following equations for the two inputs

$$A[|d\rangle \otimes |\psi_{U_1}\rangle] = U_1\,|d\rangle \otimes |\psi'_{U_1}\rangle\,,$$
$$A[|d\rangle \otimes |\psi_{U_2}\rangle] = U_2\,|d\rangle \otimes |\psi'_{U_2}\rangle\,.$$

Taking the inner product of the two equations yields

$$\langle \psi_{U_1}|\psi_{U_2}\rangle = \langle d|U_1^* U_2|d\rangle \langle \psi'_{U_1}|\psi'_{U_2}\rangle. \tag{4.1.2}$$

Assuming that $\langle \psi'_{U_1}|\psi'_{U_2}\rangle \neq 0$, we get

$$\frac{\langle \psi_{U_1}|\psi_{U_2}\rangle}{\langle \psi'_{U_1}|\psi'_{U_2}\rangle} = \langle d|\,U_1^* U_2\,|d\rangle\,. \tag{4.1.3}$$

Since merely the right-hand side of Eq. (4.1.3) depends on $|d\rangle$, this can only be true if $U_1^* U_2 = z\mathbb{1}$ for some $z \in \mathbb{C}$. Thus, U_1 and U_2 are the same up to a phase, i.e., $U_1 = e^{i\phi} U_2$ for some real ϕ. This contradicts the assumption that U_1 and U_2 are distinct up to a global phase. Therefore, $\langle \psi'_{U_1}|\psi'_{U_2}\rangle = 0$. Due to Eq. (4.1.2), we conclude that $\langle \psi_{U_1}|\psi_{U_2}\rangle = 0$ and hence, the program states are orthogonal.
Moreover, this shows that the number of program states equals the dimension of the program register and thus the program register requires at least n dimensions containing $\log_2 n$ qubits. □

The authors of Ref. [4] show that every unitary operation to be implemented requires an additional program register dimension. Thus, a deterministic programmable quantum processor with finite-dimensional program register is not possible. However, the question arises whether we can bypass this no-go theorem. The situation changes, if we impose certain symmetries, i.e., we are merely interested in implementing a particular class of channels with special features such as covariant quantum channels (see Definition 3.1) considered in Ref. [1]. Relaxing the model and allowing for an approximation with accuracy ϵ is another possibility yielding *approximate programmable quantum processors* (see Definition 4.2).

4.2 Approximate unitary-universal programmable quantum processors

Now, we consider Definition 4.2 with $\epsilon > 0$ and the special case $\mathcal{C} = \mathcal{U}(\mathcal{H})$. A tradeoff between the input dimension, the size of the program register and the precision of the implementation arises. Nielsen and Chuang's No-Programming Theorem set off the ground for extensive research seeking optimal upper and lower bounds on the size of the program register in terms of the input (and output) dimension d and the precision ϵ.

Two different approximations of this model can be found in the literature: probabilistic and deterministic approximate ϵ-PQP$_\mathcal{U}$s.

Probabilistic approximate ϵ-PQP$_\mathcal{U}$. In the probabilistic setting, unitary operators are implemented exactly but only with a certain success probability [4, 6, 7].

Deterministic approximate ϵ-PQP$_\mathcal{U}$. In the deterministic case, the processor exactly implements an approximate version of the output [14].

In the following, we focus on the deterministic approximation.

We are interested in upper and lower bounds on the program register size d_P of an ϵ-PQP$_\mathcal{U}$. The dimension of the program space depends on the accuracy of the approximation, i.e., the parameter ϵ and the size of the input register d. There are several bounds for the optimal program register size of ϵ-PQP$_\mathcal{U}$s in the literature. Concerning upper bounds, Refs. [48–50] conclude $d_P \leq 2^{\frac{4d^2 \log d}{\epsilon^2}}$. From Ref. [12, Lemma 1], where they use port-based teleportation working with copies of Choi-Jamiołkowski states, we can derive the upper bound $d^{\frac{2d^2}{\epsilon}}$. This was improved by Kubicki *et al.* in 2019 to $d_P \leq \left(\frac{\tilde{C}}{\epsilon}\right)^{d^2}$ for a constant $\tilde{C}$ [14]. Recently, Yang *et al.* [15] developed upper bounds that close the gap between upper and lower bounds in the vanishing error regime $\epsilon \to 0$. They deduce $d_P \leq \left(\frac{\Theta(d^2)}{\epsilon}\right)^{\frac{d^2-1}{2}}$ with Θ being the big-Θ notation for the asymptotic behavior of functions [15], i.e., $f(n) = \Theta(g(n))$ if $f(n) = \Omega(g(n))$ and $f(n) = O(g(n))$ for functions f and g where $f(n) = \Omega(g(n))$ if there exists a constant $c > 0$ such that $f(n) \geq cg(n)$. Furthermore, $f(n) = O(g(n))$ if there is a constant $c' > 0$ such that $|f(n)| \leq c'g(n)$ for large enough n.
An overview of the existing upper bounds on the program register dimension is shown in Table 4.1.

Optimality of the lower bounds (the polynomial growth cannot be improved) obtained in Ref. [8] is proven by Pérez-García: $d_P \geq \Theta\left(\left(\frac{1}{d}\right)^{\frac{d+1}{2}}\left(\frac{1}{\epsilon}\right)^{\frac{d-1}{2}}\right)$ for a constant K [10]. Lower bounds on the dimension of the program register are also obtained by Majenz in his PhD Thesis [51]. He shows $d_P \geq \Theta\left(\left(\frac{d}{\epsilon}\right)^2\right)$. Kubicki *et al.* improve this bound by an exponential factor with a newly established connection between ϵ-PQP$_\mathcal{U}$s and isometric embeddings between Banach spaces [14]. They characterize ϵ-PQP$_\mathcal{U}$s as an isometric embedding $\Phi : S_1(\mathcal{H}_1) \hookrightarrow \mathcal{B}(\mathcal{H}_P)$ with completely

$d_P \leq$	References
$d^{\frac{4d^2}{\epsilon^2}}$	Ishizaka & Hiroshima [48], Beigi & König [50], Christandl *et al.* [49]
$d^{\frac{2d^2}{\epsilon}}$	Pirandola *et al.* [12]
$\left(\frac{\tilde{C}}{\epsilon}\right)^{d^2}$	Kubicki *et al.* [14]
$\left(\frac{\Theta(d^2)}{\epsilon}\right)^{\frac{d^2-1}{2}}$	Yang *et al.* [15]

Table 4.1: Upper bounds on the program dimension of a finite-dimensional processor implementing unitary quantum channels.

bounded norm $\|\Phi\|_{cb} \leq 1$, i.e., complete contractions. Instead of ϵ-net arguments for the lower bounds (see Ref. [10]), Kubicki *et al.* [14] use type-constants, which give essential information about the geometry of the corresponding Banach spaces. Using properties of type constants, they manage to improve the existing lower bounds. The main idea for the lower bound is to study ϵ-embeddings between $S_1(\mathcal{H}_1)$ and the memory $\mathcal{B}(\mathcal{H}_P)$. They show $d_P \geq 2^{\frac{1-\epsilon}{3C}d-\frac{2}{3}log(d)}$ with constant C. A slight arithmetic improvement of the work by Yang *et al.* [15] yields lower bounds of the form $d_P \geq \left(1+\frac{\Theta(d^{-2})}{\sqrt{\epsilon}}\right)^{2\alpha}$ for any $\alpha < \frac{d^2-1}{2}$ using information-theoretic tools based on the Holevo information [2].
An overview of the existing lower bounds is shown in Table 4.2.

$d_P \geq$	References
$\Theta\left(\left(\frac{1}{d}\right)^{\frac{d+1}{2}}\left(\frac{1}{\epsilon}\right)^{\frac{d-1}{2}}\right)$	Pérez-García [10]
$\Theta\left(\left(\frac{d}{\epsilon}\right)^2\right)$	Majenz [51]
$2^{\frac{1-\epsilon}{3C}d-\frac{2}{3}\log d}$	Kubicki *et al.* [14]
$\left(1+\frac{\Theta(d^{-2})}{\sqrt{\epsilon}}\right)^{2\alpha}$	Yang *et al.* [15], with a slight arithmetic improvement [2]

Table 4.2: Lower bounds on the program dimension of a finite-dimensional processor implementing unitary quantum channels. The last row holds for $\alpha < \frac{d^2-1}{2}$.

4.3 Quantum teleportation

In this section, we consider ways of how an approximate unitary-universal programmable quantum processor could operate using quantum teleportation. Therefore, we introduce the standard teleportation protocol, which was originally introduced by Bennett *et al.* in 1993 [52] and can be found in many text books such as [17]. Thereafter, we present port-based teleportation followed by teleportation simulation.

Standard quantum teleportation. In general, quantum teleportation [52] is a protocol to deliver an unknown quantum state of a system C

$$|\varphi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \in \mathbb{C}^2$$

with $|\alpha|^2 + |\beta|^2 = 1$, from a sender (A) to a receiver (B). The protocol does not require a quantum channel but a classical communication channel and a shared entangled state

$$|\phi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

A does not know the state of the qubit, she only has one copy and therefore by the laws of quantum mechanics, she in general cannot determine the state without destroying it. Note that even if she did know the state, it would be out of scope to describe it classically to B because $|\varphi\rangle$ takes continuous values. Quantum teleportation is a possibility to use the shared entangled state $|\phi_{AB}\rangle$ to send $|\varphi\rangle$ from A to B with classical communication.
The three systems A, B and C are in the state

$$(\alpha\,|0\rangle + \beta\,|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

In the standard teleportation protocol, A possesses $|\varphi\rangle$, she wants to teleport, and one half of the shared entangled state $|\phi_{AB}\rangle$. She lets the two interact, i.e., applies a CNOT gate on CA and obtains

$$\frac{\alpha}{\sqrt{2}}(|000\rangle + |011\rangle) + \frac{\beta}{\sqrt{2}}(|110\rangle + |101\rangle).$$

Applying a Hadamard gate $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ on A, we obtain

$$\frac{\alpha}{2}(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + \frac{\beta}{2}(|0\rangle - |1\rangle)(|10\rangle + |01\rangle).$$

Rewriting this equation leads us to

$$\begin{aligned}\frac{1}{2}\big[\,&|00\rangle\,(\alpha\,|0\rangle + \beta\,|1\rangle) + |01\rangle\,(\alpha\,|1\rangle + \beta\,|0\rangle)\\ &+ |10\rangle\,(\alpha\,|0\rangle - \beta\,|1\rangle) + |11\rangle\,(\alpha\,|1\rangle - \beta\,|0\rangle)\big].\end{aligned} \tag{4.3.1}$$

We can directly see from Eq. (4.3.1) that if A performs a measurement of her qubits with outcome 00, then B's system is in the state $\alpha|0\rangle+\beta|1\rangle$. Since the output is classical, A sends her gained information to B via the classical channel between the two parties. Depending on the measurement outcome, B applies a unitary transformation on his part of $|\phi_{AB}\rangle$. Comparing B's state to the initial state $|\varphi\rangle$ A wanted to teleport, we realize that in the case of measurement outcome 00, B does not have to perform any operation because the state was already delivered correctly, i.e., B applies the identity operation. Together with the other cases, we obtain the following correspondence between the measurement outcomes and the states

$$\begin{aligned} 00 &\mapsto \alpha|0\rangle + \beta|1\rangle, \\ 01 &\mapsto \alpha|1\rangle + \beta|0\rangle, \\ 10 &\mapsto \alpha|0\rangle - \beta|1\rangle, \\ 11 &\mapsto \alpha|1\rangle - \beta|0\rangle. \end{aligned}$$

This leads to the following transformations, B has to apply

$$\begin{aligned} 00 &\triangleq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}_2, \\ 01 &\triangleq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =: X, \\ 10 &\triangleq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =: Z, \\ 11 &\triangleq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = XZ. \end{aligned}$$

These matrices correspond to applying $\mathbb{1}_2$, an X, Z or XZ gate.

This protocol can easily be turned into an ϵ-PQP$_{\mathcal{U}}$ where the processor performs the teleportation protocol. A programmable quantum processor is a machine which transforms a state controlled by program states, i.e., the information about the operation which should be carried out is contained in the program state. A programmable quantum processor stores and retrieves operations just like in the teleportation scheme, i.e., the standard teleportation scheme provides a probabilistic universal processor [4]. We want the processor to apply a unitary operator $U \in \mathcal{U}(\mathcal{H})$ to the input state $|\varphi\rangle$. To achieve this, we apply the unitary to B's part of the entangled state $(\mathrm{id} \otimes U)|\phi_{AB}\rangle$. Then, A performs the joint measurement. In the case of B applying the trivial operation because the state already is in the correct form, we get $U|\varphi\rangle$ as the output with a certain probability, i.e., the teleported state with U applied to it. Combining the actions of A and B to the operation a processor executes, this turns out to be a probabilistic ϵ-PQP$_{\mathcal{U}}$ [4].

Port-based teleportation (PBT). In the port-based teleportation protocol, the two agents A and B share a tensor product of n maximally entangled states of a

system C

$$|\phi_{AB}\rangle^{\otimes n} \text{ with } |\phi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \ , \ \mathcal{H}_{A_i} \cong \mathcal{H}_{B_i} \cong \mathbb{C}^2 \ , \ i = 1, \ldots, n,$$

i.e., B has multiple output ports such that B's part of the shared state consists of states $\{B_i\}_{i=1}^n$. We consider the qubit case for A_i, B_i, C_i, i.e., the corresponding Hilbert spaces $\mathcal{H}_{A_i}, \mathcal{H}_{B_i}, \mathcal{H}_{C_i} \cong \mathbb{C}^2$ for $i = 1, \ldots, n$. Applying the correction operation on B's side in the standard protocol corresponds to selecting one of the ports of the shared entangled state.
Here again, A performs a joint POVM with the set of effect operators $\{(E_i)_{\varphi A^n}\}_{i=1}^n$ (see Definition 2.1.3) the input state $|\varphi\rangle$ she wants to teleport and her part of the resource state A^n. She sends the outcome i of the measurement classically to B. Depending on the index which B receives, he selects the ith port. The rest is discarded.
This protocol cannot be achieved perfectly with finite resources. There are two approximate versions: deterministic PBT [49, 53] and probabilistic PBT [49, 54]. For a deterministic model, the shared state is fixed to be n maximally entangled states of local dimension 2. The protocol is given by combining the maximally entangled state with an optimal measurement.
In the probabilistic version, A's measurement provides an additional outcome indicating the success of the protocol with p being the success probability of the protocol, where the state is perfectly teleported.
Note that a probabilistic PBT protocol with success probability p can be converted into a deterministic one by sending a random port index to B in the case where A's measurement indicates an error [49].

To turn this into an ϵ-PQP$_{\mathcal{U}}$ [48, 55], we again encode the information about the unitary $U \in \mathcal{U}(\mathcal{H})$ into the entangled resource state, i.e., apply $(\mathrm{id} \otimes U^{\otimes n}) |\phi_{AB}\rangle^{\otimes n}$, i.e., U is applied to B's side on every port in advance. This corresponds to storing U in the program register. Then, the teleportation protocol is executed. Thus, A's measurement and B's selection can perform arbitrary unitaries since the shared entangled state $\phi_{AB}^{\otimes n}$ is changed into $|\psi_U\rangle = (\mathbb{1} \otimes U^{\otimes n}) |\phi_{AB}\rangle^{\otimes n}$. On B's side, the operation $U^{\otimes n}$ is already applied regardless of the port. This is possible because $U^{\otimes n}$ commutes with the selection of the port. Hence, after this process, we obtain an approximation of the input state with the unitary applied to it. If n is finite, the scheme must be approximate because otherwise, a deterministic universal programmable quantum processor with finite memory would exist contradicting the No-Programming Theorem 4.1.1. This is then an ϵ-PQP$_{\mathcal{U}}$ executing a port-based teleportation protocol.

Teleportation simulation. Another way of how programmable quantum processors can operate is via teleportation simulation, which uses the Choi-Jamiołkowski state of the channel, that is to be implemented, as a program.
In teleportation simulation, quantum channels are simulated by quantum teleportation [56]. One choice for the shared state is the Choi-Jamiołkowski state. From a processor's perspective, we use the Choi-Jamiołkowski state as program state and

the processor itself executes the standard teleportation protocol.
Details of how this works in the case of UV-covariant channels can be found in Subsection 5.1.3.

5 Covariant programmable quantum processors

This chapter is based on the following article

> Martina Gschwendtner, Andreas Bluhm, and Andreas Winter
> Programmability of covariant quantum channels
> *Quantum* 5:488, 2021 (24 pages)
> (see Article [1] in the bibliography)

The author of this thesis is the principal author of the above publication.

In this chapter, we investigate the programmability of a special class of channels respecting symmetry properties: UV-covariant quantum channels which we have already introduced in Chapter 3. They appear in different contexts such as channel discrimination, channel capacities and communication tasks (see Ref. [16] and the references therein). Since these channels form a special set, we ultimately are not in the setting of the *No-Programming Theorem* [4] anymore (see Section 4.1). Therefore, the question arises whether exact programmability is possible.

5.1 Exact programmability

Recall the definition of a programmable quantum processor (Definition 4.2). Here, we consider a programmable quantum processor with input $\rho \in \mathcal{D}(\mathcal{H}_1)$ that implements all UV-covariant channels $T \in \mathcal{T}_{UV}$ with program states $\pi_T \in \mathcal{B}(\mathcal{H}_P)$ of dimension d_P. This is schematically illustrated in Figure 4.1 in Chapter 4. In particular, when $\mathcal{C} = \mathcal{T}_{UV}$, we write CPQP_{UV} for the covariant programmable quantum processor and ϵ-CPQP_{UV} for the approximate version. Note that to allow for mixed states in the program register is natural, since the set $\mathcal{T}_{UV}$ is convex, whereas the set of pure states is not.

We call the CPQP_{UV}, which implements UV-covariant channels, covariant programmable quantum processor because it can be chosen $(U \otimes \mathbb{1}_{d_P})V$-covariant as well:

Proposition 5.1.1 (Covariant Processor [1, Proposition 3.2])**.** *Let $\mathcal{P}$ be a $CPQP_{UV}$. Then, there exists a $CPQP_{UV}$ $\mathcal{P}'$ with the same program dimension d_P that is $(U \otimes \mathbb{1}_{d_P})V$-covariant.*

Proof. We can construct the desired processor via twirling, i.e.,

$$\mathcal{P}'(A) = \int_G V_g^* \mathcal{P}[(U_g \otimes \mathbb{1}_{d_P}) A (U_g^* \otimes \mathbb{1}_{d_P})] V_g \mu(\mathrm{dg}) \qquad \forall\ \mathrm{A} \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_\mathrm{P}),$$

where μ is the Haar measure on G. We compute

$$\mathcal{P}'(\rho \otimes \pi_T) = \int_G V_g^* T^\epsilon (U_g \rho U_g^*) V_g d\mu(g)$$

for $T \in \mathcal{T}_{UV}$ and T^ϵ a quantum channel such that $\frac{1}{2}\|T - T^\epsilon\|_\diamond \leq \epsilon$, since $\mathcal{P}$ is an ϵ-CPQP$_{UV}$. Since

$$\int_G V_g^* T (U_g \rho U_g^*) V_g d\mu(g) = T(\rho)$$

for $T \in \mathcal{T}_{UV}$ by covariance, it holds that

$$\frac{1}{2}\|\mathcal{P}'(\cdot \otimes \pi_T) - T\|_\diamond \leq \epsilon$$

as well. This shows that $\mathcal{P}'$ is also a CPQP$_{UV}$ with the same program dimension. Using the invariance of the Haar measure (see Section 2.2), it can be verified that for any $g' \in G$,

$$V_{g'} \mathcal{P}'(A) V_{g'}^* = \mathcal{P}'[(U_{g'} \otimes \mathbb{1}_{d_P}) A (U_{g'}^* \otimes \mathbb{1}_{d_P})] \qquad \forall\ A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_P),$$

which shows that $\mathcal{P}'$ is $(U \otimes \mathbb{1}_{d_P})V$-covariant as desired. □

From Lemma 3.3, we know that $T \in \mathcal{T}_{UV}$ is equivalent to $[c_T, \bar{U}_g \otimes V_g] = 0$ for all $g \in G$. Due to this correspondence, we consider representations of the form $\bar{U} \otimes V$ (which are isomorphic to the adjoint representation of G if $V = U$) with $U_g \in \mathcal{U}(\mathcal{H}_1)$, $V_g \in \mathcal{U}(\mathcal{H}_2)$, $g \in G$ and the commutant (see Section 2.3 and Chapter 3)

$$\begin{aligned} \mathcal{K} := \mathcal{A}'(\bar{U} \otimes V) &= \{X \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \mid [X, \bar{U}_g \otimes V_g] = 0\ \forall\ g \in G\} \\ &= \left\{ \bigoplus_{k=1}^{K} \mathbb{1}_{b_k} \otimes B_k \mid B_k \in \mathcal{B}(\mathcal{H}_k') \right\}. \end{aligned}$$

Note that only $K \leq d_1 d_2$ irreps can appear in the commutant with multiplicity $n_k > 0$ where $n_k = \dim(\mathcal{H}_k')$ since $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite dimensional. We identify these elements with an index $k \in \{1, \ldots, K\}$ motivated by the fact that we want to relate the irreps occuring in the direct sum decomposition of $\bar{U} \otimes V$ to the number of extreme points of $\mathcal{J}_{UV}$, for instance.

If U is an irrep, all states in $\mathcal{K}$ are Choi-Jamiołkowski states of a quantum channel. This is proven in the following lemma which aligns with results in Refs. [57, p. 6] and [58, p. 7].

Lemma 5.1.2 ([1, Lemma 3.3])**.** *Let $\mathcal{K}$ be as defined above and let U be an irrep of a compact group G on $\mathcal{H}_1$. Let V be a representation of G on $\mathcal{H}_2$. Then $\mathcal{K} \cap \mathcal{D}(\mathcal{H}_1 \otimes \mathcal{H}_2) = \mathcal{J}_{UV}$. Moreover, if V is an irrep, any $T \in \mathcal{T}_{UV}$ is unital.*

Proof. "$\supseteq$" Let $c_T \in \mathcal{J}_{UV}$. Since all elements of $\mathcal{J}_{UV}$ are Choi-Jamiołkowski states corresponding to UV-covariant channels, they satisfy $c_T \geq 0$ and $\mathrm{tr}(c_T) = 1$ by definition. Hence, $c_T \in \mathcal{D}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. According to Lemma 3.3, $T \in \mathcal{T}_{UV}$ corresponds to $[c_T, \bar{U}_g \otimes V_g] = 0$ for all $g \in G$ and thus, $c_T \in \mathcal{K} \cap \mathcal{D}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

"$\subseteq$" Let us refer to $\mathcal{H}_1$ as system A and to $\mathcal{H}_2$ as system B. If we intersect $\mathcal{K}$ with the set of states $\mathcal{D}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, then every $\rho_{AB} \in \mathcal{K} \cap \mathcal{D}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ satisfies $\mathrm{tr}(\rho_{AB}) = 1$ and $\rho_{AB} \geq 0$ as well as $[\rho_{AB}, \bar{U}_g \otimes V_g] = 0$ for all $g \in G$. To obtain $\rho_{AB} \in \mathcal{J}_{UV}$, we additionally have to show the required property $\mathrm{tr}_B(\rho_{AB}) = \frac{\mathbb{1}_{d_1}}{d_1}$. Using $[\rho_{AB}, \bar{U}_g \otimes V_g] = 0$, we get

$$\mathrm{tr}_B(\rho_{AB}) = \mathrm{tr}_B\left((\bar{U}_g \otimes V_g)\rho_{AB}(\bar{U}_g \otimes V_g)^*\right) = \bar{U}_g \,\mathrm{tr}_B(\rho_{AB})\bar{U}_g^*$$

for any $g \in G$ which is equal to

$$\mathrm{tr}_B(\rho_{AB})\bar{U}_g = \bar{U}_g \,\mathrm{tr}_B(\rho_{AB}).$$

Due to Schur's Lemma 2.3.20, we infer:

$$\mathrm{tr}_B(\rho_{AB}) = \lambda \cdot \mathbb{1}_{d_1} \qquad \text{for some } \lambda \in \mathbb{C}.$$

Taking the trace on both sides results in

$$1 = \mathrm{tr}\left(\mathrm{tr}_B(\rho_{AB})\right) = \lambda \cdot \mathrm{tr}(\mathbb{1}_{d_1}) = \lambda \cdot d_1.$$

Hence, $\lambda = 1/d_1$. This yields $\mathrm{tr}_B(\rho_{AB}) = \frac{\mathbb{1}_{d_1}}{d_1}$ which we aimed to show.

With the same reasoning, we can also conclude that if V is an irrep,

$$\mathrm{tr}_A(c_T) = \frac{\mathbb{1}_{d_2}}{d_2}$$

for all $c_T \in \mathcal{J}_{UV}$, which implies that T is unital. □

In Ref. [5], the authors derived that channels implemented by a processor that is covariant with respect to the special unitary group $SU(\mathcal{H}_1)$ are unital. They use Schur's Lemma 2.3.20 as well. In the following, we consider the construction of covariant programmable quantum processors in the case where $\mathcal{K}$ is abelian.

5.1.1 Exact programmability using extreme points

In the case where the commutant $\mathcal{K}$ is abelian, it suffices to store the extreme points of $\mathcal{J}_{UV}$ which we show in this subsection. We present a correspondence between the commutant and the affiliated state space (Lemma 5.1.3). Moreover, we establish a connection between the state space and a CPQP_{UV} (Theorem 5.1.4). Using these statements, we prove the first main result for exact programmability (Corollary 5.1.5).

Lemma 5.1.3 ([1, Lemma 3.4])**.** *Let U be an irrep of a compact group G, and let V be another representation of G. Then, $\mathcal{K}$ is abelian if and only if $\mathcal{J}_{UV}$ is isomorphic to a simplex.*

Proof. "⇒" This statement is mentioned in Ref. [38] without proof. Let $B \in \mathcal{K}$ and let $\mathcal{K}$ be an abelian matrix algebra. This implies that $n_k = 1$ for all $k \in \{1, \ldots, K\}$ and let K be the number of irreps appearing in the direct sum decomposition of $\bar{U} \otimes V$. We obtain

$$B = \bigoplus_{k=1}^{K} \mathbb{1}_{b_k} \otimes x_k$$

where $x_k \in \mathbb{C}$. Moreover, $B \geq 0$ if and only if $x_k \geq 0$ for all $k \in \{1, \ldots, K\}$ and $\mathrm{tr}(B) = 1$ if and only if $\sum_k b_k x_k = 1$. According to Lemma 5.1.2, extreme points of $\mathcal{K} \cap \mathcal{D}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ are also extreme points of $\mathcal{J}_{UV}$. The extreme points of $\mathcal{K} \cap \mathcal{D}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ are of the form $x_i = 1/b_i$ for some $i \in \{1, \ldots, K\}$ and $x_l = 0$ for all $l \neq i$. We identify the K extreme points of $\mathcal{J}_{UV}$ with K points in $\mathbb{R}^K$. Therefore, $\mathcal{J}_{UV}$ is isomorphic to a $(K-1)$-simplex.

"⇐" We prove this statement by contraposition, i.e., we show if $\mathcal{K}$ is non-abelian, $\mathcal{J}_{UV}$ is not isomorphic to a simplex. If $\mathcal{K}$ is non-abelian, there is a $k \in \{1, \ldots, K\}$, such that the corresponding block is of dimension $n_k > 1$. Let us consider elements of the form

$$B_\varphi = \left(\frac{1}{b_k} |\varphi_k\rangle\langle\varphi_k| \otimes \mathbb{1}_{b_k} \right) \oplus \mathbf{0},$$

where $|\varphi_k\rangle\langle\varphi_k| \in \mathcal{D}(\mathcal{H}'_k)$. The normalization $\frac{1}{b_k}$ yields $\mathrm{tr}(B_\varphi) = 1$ and furthermore, $B_\varphi \geq 0$, i.e., $B \in \mathcal{D}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. These elements are extreme points of $\mathcal{J}_{UV}$ by Lemma 5.1.2. Thus, there are infinitely many extreme points of $\mathcal{J}_{UV}$. Hence, the set cannot be isomorphic to a simplex which has finitely many extreme points by definition. □

After showing this correspondence, we show a relation between the state space and the existence of a CPQP_{UV}.

Theorem 5.1.4 ([1, Theorem 5.3])**.** *Let U be an irrep of a compact group on $\mathcal{H}_1$ and V a representation of the same group on $\mathcal{H}_2$. If $\mathcal{J}_{UV}$ is isomorphic to a $(K-1)$-simplex, then there is a CPQP$_{UV}$ $\mathcal{P}$ that implements all $T \in \mathcal{T}_{UV}$ exactly with program dimension $d_P = K$.*

Proof. We fix a quantum channel $T \in \mathcal{T}_{UV}$ with its corresponding Choi-Jamiołkowski state $c_T \in \mathcal{J}_{UV}$. Since $\mathcal{J}_{UV}$ is isomorphic to a $(K-1)$-simplex, it is spanned by K extreme points c_{T_k} and can therefore be written as convex combination of these

$$c_T = \sum_{k=1}^{K} x_k c_{T_k},$$

where $x_k \in [0, 1]$ and $\sum_{k=1}^{K} x_k = 1$. By the Choi-Jamiołkowski isomorphism, there is a channel $T_k \in \mathcal{T}_{UV}$ corresponding to each of the extreme points c_{T_k} of $\mathcal{J}_{UV}$, i.e., T can be linearly decomposed:

$$T(\cdot) = \sum_{k=1}^{K} x_k T_k(\cdot)$$

with extreme points $T_k(\cdot)$. We encode the $\{x_k\}_{k\in\{1,\dots,K\}}$ in the program state as follows

$$|\psi_T\rangle = \sum_{k=1}^{K} \sqrt{x_k}\,|k\rangle \in \mathcal{D}_P(\mathcal{H}_P),$$

with an arbitrary orthonormal basis $\{|k\rangle\}_{k\in\{1,\dots,K\}}$ on $\mathcal{H}_P$. The following processor implements $T \in \mathcal{T}_{UV}$ exactly with a program register of dimension $d_P = K$:

$$\mathcal{P}(A\otimes B) = \mathrm{tr}_{\mathcal{H}_P}\left[\sum_{k=1}^{K} T_k(A)\otimes\langle k|B|k\rangle|k\rangle\langle k|\right] \qquad \forall\ A\in\mathcal{B}(\mathcal{H}_1),\ \forall\ B\in\mathcal{B}(\mathcal{H}_P)$$

and extended by linearity. We verify that this is indeed a CPQP_{UV}:

$$\mathcal{P}(\rho\otimes|\psi_T\rangle\langle\psi_T|) = \mathrm{tr}_{\mathcal{H}_P}\left[\sum_{k=1}^{K} T_k(\rho)\otimes\langle k|\psi_T\rangle\langle\psi_T|k\rangle|k\rangle\langle k|\right] = \sum_{k=1}^{K} x_k T_k(\rho) = T(\rho).$$

Hence, we showed that if $\mathcal{J}_{UV}$ is isomorphic to a simplex, there is a processor $\mathcal{P}$ that implements $T \in \mathcal{T}_{UV}$ exactly with program dimension $\dim(\mathcal{H}_P) = d_P = K$. □

The following corollary assures that if we want to know whether channels $T \in \mathcal{T}_{UV}$ are programmable exactly with finite-dimensional program register by their extreme points, we can consider the specific structure of the commutant $\mathcal{K}$.

Corollary 5.1.5. *Let U be an irrep of a compact group G on $\mathcal{H}_1$ and V a representation of G on $\mathcal{H}_2$. Furthermore, let $\mathcal{K}$ be the commutant of $\bar{U}\otimes V$. If $\mathcal{K}$ is abelian, then there is a CPQP$_{UV}$ $\mathcal{P}$ that implements all $T \in \mathcal{T}_{UV}$ exactly with $d_P \leq K$, where K is the number of irreps appearing in the direct sum decomposition of $\bar{U}\otimes V$.*

Proof. The proof directly follows from Lemma 5.1.3 and Theorem 5.1.4. □

We give examples of groups for which exact programmability holds.

Example 5.1.6 (Unitarily covariant and orthogonally covariant channels)**.** *We present the implementation of UU-covariant and OO-covariant channels.*

1) *We consider a CPQP$_{UU}$ that implements all UU-covariant channels where U is the defining representation of the group $G = \mathcal{U}(\mathcal{H})$. The tensor representation $\bar{U}\otimes U$ can be decomposed into a direct sum of a one-dimensional and a (d^2-1)-dimensional irrep both with multiplicity 1. Thus, elements of the commutant consist of a one-dimensional block with multiplicity equal to one and a one-dimensional block with multiplicity d^2-1, i.e., in this case $K=2$, $n_1 = b_1 = 1$, $n_2 = 1$ and $b_2 = d^2-1$. Due to Lemma 3.3, we know that the Choi-Jamiołkowski state $c_T \in \mathcal{J}_{\mathcal{T}_{UU}}$ satisfies $[c_T, \bar{U}_g\otimes U_g] = 0$. The state space therefore has the following form (see also Example 3.4):*

$$\mathcal{K}\cap\mathcal{D}(\mathcal{H}\otimes\mathcal{H}) = \left\{\hat{\alpha}\frac{\mathbb{1}}{d^2} + (1-\hat{\alpha})|\Omega\rangle\langle\Omega| \;\middle|\; \hat{\alpha}\in\left[0,\frac{d^2}{d^2-1}\right]\right\}$$

with extreme points $|\Omega\rangle\langle\Omega|$ *and* $\frac{1}{d^2-1}(\mathbb{1} - |\Omega\rangle\langle\Omega|)$*. Thus,* T *has the form* $T(\cdot) = \alpha\,\mathrm{tr}(\cdot)\frac{\mathbb{1}}{d} + (1-\alpha)\mathrm{id}(\cdot)$ *[37, 38]. Every* c_T *can be written as convex combination of the two extreme points and the set of all convex combinations*

$$\mathcal{K} \cap \mathcal{D}(\mathcal{H} \otimes \mathcal{H}) = \left\{ x|\Omega\rangle\langle\Omega| + (1-x)\frac{1}{d^2-1}(\mathbb{1} - |\Omega\rangle\langle\Omega|) \;\middle|\; x \in \big[0,1\big] \right\}$$

is isomorphic to a 1*-simplex. Thus, the* $CPQP_{UU}$ *can be implemented with program dimension* $d_P = 2$ *using the construction in Theorem 5.1.4.*

2) *Furthermore, as a second example, we consider a* $CPQP_{OO}$ *that implements all* OO*-covariant channels where* O *is the defining representation of the real orthogonal group* $G = \mathcal{O}(\mathcal{H})$*. Recall their structure from Example 3.2. The direct sum decomposition of* $O \otimes O$ *yields a one-dimensional irrep with multiplicity one, a* $(d-1)$*-dimensional irrep with multiplicity one and a* (d^2-d)*-dimensional irrep with multiplicity one. Thus, elements of the commutant consist of a one-dimensional block with multiplicity 1, a one-dimensional block with multiplicity* $d-1$ *and a one-dimensional block with multiplicity* d^2-d*. The corresponding Choi-matrices have already been provided in Example 3.4 for which we know that* $[c_T, O_g \otimes O_g] = 0$ *holds according to Lemma 3.3. Let* $\mathbb{F}$ *be the flip operator. The three minimal projections*

$$P_0 = |\Omega\rangle\langle\Omega|, \qquad P_1 = (\mathbb{1} - \mathbb{F})/2, \qquad P_2 = (\mathbb{1} + \mathbb{F})/2 - |\Omega\rangle\langle\Omega|$$

span the state space

$$\mathcal{J}_{OO} = \left\{ c_T : c_T = (1-\beta_1-\beta_2)P_1 + \beta_1\frac{P_2}{\mathrm{tr}(P_2)} + \beta_2\left(\frac{P_3}{\mathrm{tr}(P_3)}\right) \right\}$$

with $0 \leq \beta_1, \beta_2 \leq 1$*, which is isomorphic to a* 2*-simplex. Hence, the* $CPQP_{OO}$ *can be implemented with program dimension* $d_P = 3$*, again using the construction in Theorem 5.1.4.*

5.1.2 Structure of the commutant

In this subsection, we take a closer look at the special case where the direct sum decomposition of $\bar{U} \otimes V$ consists of at least one one-dimensional irrep with multiplicity $n_k > 1$. Based on the previous subsection, one could argue that it would not be possible to implement the corresponding UV-covariant channel exactly in this case. However, an argument why this situation never arises is provided in the following.

We assume $d_1 = d_2 = d$ and that there is a $k \in \{1, \ldots, K\}$ such that $n_k > 1$. Then, there would be elements in $\mathcal{J}_{UV}$ of the form

$$B_\varphi = |\varphi_k\rangle\langle\varphi_k| \oplus \mathbf{0} \in \mathcal{J}_{UV}$$

where $|\varphi_k\rangle\langle\varphi_k| \in \mathcal{D}(\mathcal{H}_k')$ is a rank-one projector and $\dim(\mathbf{0}) = d^2 - n_k$. By the Choi-Jamiołkowski isomorphism 2.1.14, there is a corresponding channel $T \in \mathcal{T}_{UV}$.

We consider a Kraus representation of the channel T (Proposition 2.1.17). Since T is a completely positive map and $\operatorname{rank}(B_\varphi) = 1$, the channel T can be written as $T(\cdot) = X(\cdot)X^*$ with one Kraus operator $X \in \mathcal{B}(\mathcal{H}_2)$. Since, additionally, T is trace-preserving, we know that $XX^* = \mathbb{1} = X^*X$, i.e., the Kraus operator is a unitary and T is a unitary channel

$$T(\cdot) = W(\cdot)W^*$$

with $W \in \mathcal{U}(\mathcal{H}_2)$.

Since there are infinitely many pure states, the processor would have to implement infinitely many corresponding unitary channels. This contradicts the *No-Programming Theorem* [4] according to which there is no processor that implements infinitely many unitary channels exactly with program dimension $d_P < \infty$. Hence, a processor $\mathcal{P}$ with $d_P < \infty$ cannot exist if $b_k = 1$ and $n_k > 1$ for some $k \in \{1, \ldots, K\}$. Note that the above argument does not work for $b_k > 1$, since then there might be no rank-1 elements in $\mathcal{K}$.

However, from a representation-theoretic perspective, this situation does not arise because a one-dimensional irrep in the direct sum decomposition of $\bar{U} \otimes V$ always has multiplicity ≤ 1 as the following proposition shows.

Proposition 5.1.7 ([1, Proposition 3.8])**.** *Let U be an irrep on $\mathcal{H}_1$ of a compact group G. Let V be another representation of G on $\mathcal{H}_2$ with dimension $d_2 \leq d_1$. In a direct sum decomposition of $\bar{U} \otimes V$, the one-dimensional irreps λ appear with multiplicity $n_\lambda \leq 1$.*

Proof. Let χ_U be the character of the irrep U, χ_V the character of V, and λ the character of the one-dimensional irrep λ. We use the following scalar product (see Theorem 2.3.24)

$$\langle \chi, \psi \rangle = \int_G \overline{\chi(g)} \psi(g) \mu(\mathrm{dg}),$$

where μ is the Haar measure on G. Here, ψ is the character of an arbitrary representation of G. Note that if U is an irrep, then this scalar product of the corresponding characters gives the multiplicity of U in the representation corresponding to ψ (see Corollary 2.3.25). We want to show that the multiplicity of λ in $\bar{U} \otimes V$ is ≤ 1. Note that the character of $\bar{U} \otimes V$ is $\bar{\chi}_U \cdot \chi_V$. Let $\hat{G}$ be the set of irreps of G and let

$$\chi_V = \sum_{\alpha \in \hat{G}} n_\alpha \chi_\alpha$$

be the decomposition of V into irreducible representations on the level of characters. Then,

$$\begin{aligned}\langle \lambda, \bar{\chi}_U \cdot \chi_V \rangle &= \int_G \overline{\lambda(g)} \big(\overline{\chi_U(g)} \chi_V(g)\big) \mu(\mathrm{dg}) \\ &= \int_G \overline{\lambda(g)\chi_U(g)} \chi_V(g) \mu(\mathrm{dg}) \\ &= \langle \lambda \cdot \chi_U, \chi_V \rangle. \\ &= \sum_{\alpha \in \hat{G}} n_\alpha \langle \lambda \cdot \chi_U, \chi_\alpha \rangle.\end{aligned}$$

Since λ is the character of a one-dimensional irrep, it is equal to the representation itself and $|\lambda(g)|^2 = 1$ for all $g \in G$. Thus, $\langle \lambda \cdot \chi_U, \lambda \cdot \chi_U \rangle = 1$ and the representation corresponding to $\lambda \cdot \chi_U$ is again irreducible. Note that the representation corresponding to $\lambda \cdot \chi_U$ has the same dimension as U.

The scalar product $\langle \lambda \cdot \chi_U, \chi_\alpha \rangle$ thus gives the multiplicity of $\lambda \cdot \chi_U$ in χ_α. Let α be such that $n_\alpha > 0$. If the dimension of the representation corresponding to χ_α is smaller than d_2, then $\langle \lambda \cdot \chi_U, \chi_\alpha \rangle = 0$ by Theorem 2.3.24. Thus, $\langle \lambda, \bar{\chi}_U \cdot \chi_V \rangle = 0$. If the dimension is d_2, then $\chi_V = \chi_\alpha$ for dimensional reasons and V is irreducible. The multiplicity of an irrep in another irrep can be at most equal to 1 which proves the assertion. □

Having provided a method for exact programmability in the case of the commutant being abelian, we give a different construction of covariant programmable quantum processors which is more widely applicable.

5.1.3 Exact programmability using teleportation

If $\mathcal{K}$ is not abelian, we need a different method to show exact programmability. This case appears for example for the finite group A_4 [35, p. 20]. We state the relevant example here again.

Example 5.1.8 ([1, Example 3.10])**.** *The alternating group A_4 is a subgroup of the symmetric group S_4 covering only the even permutations. The group has four irreps $\varphi^{(1)}$, $\varphi^{(2)}$, $\varphi^{(3)}$, and $\varphi^{(4)}$. Three of them, $\varphi^{(1)}$, $\varphi^{(2)}$, $\varphi^{(3)}$, are one dimensional and $\varphi^{(4)}$ is three dimensional. As an example, we consider $\bar{\varphi}^{(4)} \otimes \varphi^{(4)}$ which can be decomposed as*

$$\bar{\varphi}^{(4)} \otimes \varphi^{(4)} = \varphi^{(1)} \oplus \varphi^{(2)} \oplus \varphi^{(3)} \oplus 2\varphi^{(4)}$$

where $\varphi^{(4)}$ appears with multiplicity 2. Hence, there are three one-dimensional blocks on the diagonal and two three-dimensional ones, i.e., the commutant consists of one-dimensional blocks with multiplicity three and two-dimensional blocks with multiplicity three.

The task of formulating a processor is similar to *teleportation simulation*, i.e., the simulation of quantum channels by quantum teleportation (see the corresponding paragraph in Section 4.3). In the case of the Pauli group this goes back to Refs. [52,

59]. Important developments concerning the teleportation of covariant channels can be found in Refs. [26, 60, 61] and furthermore in the very recent Ref. [62]. We know from Refs. [56, p. 58] and [63, Proposition 2] that it is always possible to simulate UV-covariant channels exactly using the corresponding Choi-Jamiołkowski state.

This can easily be formulated as a processor which uses the Choi-Jamiołkowski state as program state and performs the teleportation protocol (see Section 4.3).

Therefore, the dimension of the program register is $d_1 d_2$. We make this precise in the next proposition already present in Refs. [56, 63], which we include here for convenience.

Proposition 5.1.9 ([1, Proposition 3.11])**.** *Let G be a compact group and let U be an irreducible representation on $\mathcal{H}_1$. Let V be another representation of G on $\mathcal{H}_2$. Then, there is a CPQP$_{UV}$ with program dimension $d_P \leq d_1 d_2$.*

Proof. Let $\rho \in \mathcal{D}(\mathcal{H}_1)$ be the state to be teleported. In this proof, we identify the input space as system A, $\mathcal{H}_1 \simeq \mathcal{H}_A$, and the program as a composite system with parts A' and B, $\mathcal{H}_P \simeq \mathcal{H}_{A'} \otimes \mathcal{H}_B =: \mathcal{H}_{A'B}$, where $\mathcal{H}_{A'} \simeq \mathcal{H}_A$ and $\mathcal{H}_B \simeq \mathcal{H}_2$ is isomorphic to the output space of T. We write d_1 for the dimension of A. Note that $\dim A = \dim A' = \dim A'' = d_1$. The Choi-Jamiołkowski states corresponding to the simulated channels serve as program states of the processor running the following protocol:

1. The processor measures according to the POVM

$$\{M_g\}_{g\in G} := \left\{ d_1^2 (\mathbb{1} \otimes \bar{U}_g)|\Omega\rangle\langle\Omega|_{AA'} (\mathbb{1} \otimes \bar{U}_g)^* \mu(\mathrm{dg}) \right\}_{\mathrm{g}\in\mathrm{G}}.$$

   Due to Schur's Lemma 2.3.20, this is a POVM. Note that this POVM can potentially be continuous.

2. On outcome g, apply $V_g^*(\cdot)V_g$ to the outcome of the protocol.

This construction of a processor implements UV-covariant channels $T \in \mathcal{T}_{UV}$ with the Choi-Jamiołkowski state as program state. The map is defined as

$$\mathcal{P}(X \otimes Y) := \int_G V_g^* \mathrm{tr}_{AA'}[(X \otimes Y)(M_g \otimes \mathbb{1}_B)] V_g \qquad \forall\ X \in \mathcal{B}(\mathcal{H}_A),\ \forall\ Y \in \mathcal{B}(\mathcal{H}_{A'B}) \tag{5.1.3.1}$$

and extended by linearity. Let us verify that indeed

$$\mathcal{P}(\rho^A \otimes c_T^{A'B}) = T(\rho). \tag{5.1.3.2}$$

In the following, we insert the definition of the Choi-Jamiołkowski state $c_T^{A'B} = (\mathrm{id}_{A'} \otimes T^{A''B})|\Omega\rangle\langle\Omega|_{A'A''}$ and we use $(X \otimes \mathbb{1})\,|\Omega\rangle = (\mathbb{1} \otimes X^T)\,|\Omega\rangle$, $X^T = \bar{X}$ for Hermitian X as well as $\mathrm{tr}_{A'}[(X \otimes \mathbb{1})|\Omega\rangle\langle\Omega|_{AA'}(Y \otimes \mathbb{1})] = 1/d_1 XY$. Here, A'' is again a system isomorphic to A. We calculate

$$
\begin{aligned}
&\mathrm{tr}_{AA'}[(\rho^A \otimes c_T^{A'B}) M_g \otimes \mathbb{1}_B] \\
&= d_1^2 \, d\mu(g) \, \mathrm{tr}_{AA'} \left[(\rho^A \otimes (\mathrm{id}_{A'} \otimes T^{A''B}) |\Omega\rangle\langle\Omega|_{A'A''}) \right. \\
&\qquad\qquad \left. (\mathbb{1}_A \otimes \bar{U}_g) |\Omega\rangle\langle\Omega|_{AA'} (\mathbb{1}_A \otimes \bar{U}_g)^* \otimes \mathbb{1}_B \right] \\
&= d_1^2 \, d\mu(g) \, \mathrm{tr}_{AA'} \left[\left((\mathrm{id}_{A'} \otimes T^{A''B}) |\Omega\rangle\langle\Omega|_{A'A''} \right) \right. \\
&\qquad\qquad \left. (\mathbb{1}_A \otimes \bar{U}_g)(\mathbb{1}_A \otimes \bar{\rho}^{A'}) |\Omega\rangle\langle\Omega|_{AA'} (\mathbb{1}_A \otimes \bar{U}_g)^* \otimes \mathbb{1}_B \right] \\
&= {d_1}^2 \, d\mu(g) \, \mathrm{tr}_{A'} \left((\mathrm{id}_{A'} \otimes T^{A''B}) |\Omega\rangle\langle\Omega|_{A'A''} (\bar{U}_g \bar{\rho} \bar{U}_g^*)_{A'} \otimes \mathbb{1}_B \right) \frac{1}{d_1} \\
&= \frac{d_1}{d_1} T(U_g \rho U_g^*) d\mu(g) \\
&= V_g T(\rho) V_g^* d\mu(g).
\end{aligned}
$$

Applying the unitary conjugation and integrating as in Eq. (5.1.3.1) yields Eq. (5.1.3.2). This shows that $\mathcal{P}$ is indeed a CPQP_{UV} with the desired program dimension.

□

We can benefit from the structure of the commutant $\mathcal{K}$ to reduce the program requirements further:

Theorem 5.1.10 ([1, Theorem 3.12])**.** *Let U be an irrep on $\mathcal{H}_1$ of a compact group G, V another representation of G on $\mathcal{H}_2$, and let $\mathcal{K}$ be of the form*

$$
\mathcal{K} = \left\{ \bigoplus_{k=1}^{K} \mathbb{1}_{b_k} \otimes B_k \mid B_k \in \mathcal{B}(\mathcal{H}'_k) \right\}.
$$

Then, there is a CPQP_{UV} with program dimension $d_P = \sum_{k=1}^{K} n_k$, where $n_k = \dim(\mathcal{H}'_k)$.

Proof. Since we have a programmable quantum processor $\mathcal{P}$, constructed in Proposition 5.1.9, available, we combine its protocol, i.e., the teleportation simulation part, with a compression map reducing the program dimension. Instead of $\mathcal{K}$, we consider a simpler matrix algebra with all multiplicities removed. These two matrix algebras are isomorphic. Let

$$
D^* : \mathcal{K} \to \bigoplus_{k=1}^{K} \mathcal{B}(\mathcal{H}'_k)
$$

be this isomorphism and let $\mathcal{H}_P = \bigoplus_{k=1}^{K} \mathcal{H}'_k$, which has dimension $d_P = \sum_{k=1}^{K} n_k$. Thus, D^* is a unital completely positive map with unital completely positive inverse C^*. Let $\tilde{C}$, $\tilde{D}$ be their dual maps. Both maps can be extended to quantum channels $C : \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_P)$, $D : \mathcal{B}(\mathcal{H}_P) \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by composing them with trace preserving conditional expectations onto the respective subalgebras. We define

$$
\mathcal{P}' = \mathcal{P} \circ (\mathrm{id} \otimes D)
$$

and it follows that $\mathcal{P}'(\rho \otimes C(c_T)) = T(\rho)$.

□

5.2 Approximate Programmability

Since exact universal programmable quantum processors with finite-dimensional program register are impossible due to the *No-Programming Theorem* [4], there is a great interest in approximate versions thereof. After considering exact programmability of UV-covariant channels in the last section, we now approximate the output such that the programmed output is ϵ-close to the ideal output. Therefore, we consider the concept of an ϵ-CPQP$_{UV}$ with $\epsilon > 0$, which is a processor that implements a channel T^ϵ instead of the exact result T. However, it should be ϵ-close to the ideal one in diamond norm. We are interested in upper and lower bounds on the program dimension of an ϵ-CPQP$_{UV}$. We show that the program register requirements for an ϵ-CPQP$_{UV}$ are not much lower than for an exact CPQP$_{UV}$. Furthermore, note that we can still benefit from the covariance property and the corresponding structure of the commutant $\mathcal{K}$.

5.2.1 Upper bounds on the program dimension

We construct generic upper bounds on the dimension of the program register d_P. We seek to establish an ϵ-net on the set of covariant channels $\mathcal{T}_{UV}$. Therefore, we make use of the following result about ϵ-nets from Ref. [64].

Lemma 5.2.1 (ϵ-nets in $\mathbb{R}^n$ [64, Lemma 9.5])**.** *Let $\epsilon \in (0,1)$ and let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. There is an ϵ-net $\mathcal{S}$ on the unit sphere $S^{n-1}_{\|\cdot\|}$ of $(\mathbb{R}^n, \|\cdot\|)$ of cardinality*

$$|\mathcal{S}| \leq \left(1 + \frac{2}{\epsilon}\right)^n.$$

That means, for all $x \in S^{n-1}_{\|\cdot\|}$, there is a $y \in \mathcal{S}$ such that $\|x - y\| \leq \epsilon$.

Due to the Choi-Jamiołkowski isomorphism 2.1.14, there is a $c_T \in \mathcal{J}_{UV}$ corresponding to each $T \in \mathcal{T}_{UV}$. According to Lemma 3.3, we know that $\mathcal{J}_{UV} \subseteq \mathcal{K} \cap \mathcal{D}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Thus, we benefit from the special block-diagonal structure of $\mathcal{K}$ to establish an ϵ-net on the set of UV-covariant channels.

Proposition 5.2.2 ([1, Proposition 4.2])**.** *Let*

$$\mathcal{K} = \left\{ \bigoplus_{k=1}^{K} \mathbb{1}_{b_k} \otimes B_k | B_k \in \mathcal{B}(\mathcal{H}'_k) \right\}.$$

For $\epsilon \in (0,1)$, there is a set $\mathcal{S}_{UV} \subseteq \mathcal{T}_{UV}$ such that for all $T \in \mathcal{T}_{UV}$, there is a $T^\epsilon \in \mathcal{S}_{UV}$ such that $\|T - T^\epsilon\|_\diamond \leq 2\epsilon$. Moreover,

$$|\mathcal{S}_{UV}| \leq \left(1 + \frac{2}{\epsilon}\right)^{d_n},$$

where $d_n = \sum_{k=1}^{K} n_k^2$.

Proof. Since $\mathcal{J}_{UV} \subseteq \mathcal{K}$ by Lemma 3.3, the real vector space $\mathrm{Lin}_{\mathbb{R}}\, \mathcal{J}_{UV}$ generated by $\mathcal{J}_{UV}$ has dimension at most d_n. Using the Choi-Jamiołkowski isomorphism 2.1.14, we infer that $\mathrm{Lin}_{\mathbb{R}}\, \mathcal{T}_{UV}$ is a real subspace of $\mathcal{B}(\mathcal{B}(\mathcal{H}_1), \mathcal{B}(\mathcal{H}_2))$ of dimension at most d_n. The restriction of the diamond norm turns $(\mathrm{Lin}_{\mathbb{R}}\, \mathcal{T}_{UV}, \|\cdot\|_\diamond)$ into a real normed space which is isometrically isomorphic to $(\mathbb{R}^{d_n}, \|\cdot\|)$ with induced norm $\|\cdot\| = \|I^{-1}(\cdot)\|_\diamond$, where $I : \mathrm{Lin}_{\mathbb{R}}\, \mathcal{J}_{UV} \to \mathbb{R}^{d_n}$ denotes the isomorphism. Lemma 5.2.1 ensures the existence of an ϵ-net $\mathcal{S}$ on the unit sphere of $(\mathrm{Lin}_{\mathbb{R}}\, \mathcal{T}_{UV}, \|\cdot\|_\diamond)$ and the unit sphere contains $\mathcal{T}_{UV}$ since $\|T\|_\diamond = 1$ for all $T \in \mathcal{T}_{UV}$.
To obtain $\mathcal{S}_{UV}$, we repeat the following steps. Take $\Phi \in \mathcal{S}$. If $\Phi \in \mathcal{T}_{UV}$, keep it and proceed to the next element. If there is no $T \in \mathcal{T}_{UV}$ such that $\|\Phi - T\|_\diamond \leq \epsilon$, remove Φ from the set and proceed to the next element. If there is a $T \in \mathcal{T}_{UV}$ such that $\|\Phi - T\|_\diamond \leq \epsilon$, exchange Φ by T and continue with the next element. This algorithm constructs $\mathcal{S}_{UV}$ with the desired properties. Indeed, for any $T \in \mathcal{T}_{UV}$, there is a $\Phi \in \mathcal{S}$ and a $\mathcal{T}^\epsilon \in \mathcal{S}_{UV}$ such that $\|T - \Phi\|_\diamond \leq \epsilon$ and $\|T^\epsilon - \Phi\|_\diamond \leq \epsilon$. Thus, $\|T^\epsilon - T\|_\diamond \leq 2\epsilon$. The upper bound on the cardinality follows since by construction $|\mathcal{S}_{UV}| \leq |\mathcal{S}|$. □

Using the previous proposition, we construct an ϵ-CPQP$_{UV}$.

Theorem 5.2.3 ([1, Theorem 4.3])**.** *For a compact group G and representations U on $\mathcal{H}_1$, V on $\mathcal{H}_2$ such that*

$$\mathcal{K} = \Big\{ \bigoplus_{k=1}^{K} \mathbb{1}_{b_k} \otimes B_k | B_k \in \mathcal{B}(\mathcal{H}'_k) \Big\},$$

there exists an ϵ-CPQP$_{UV}$ with program dimension

$$d_P \leq \left(1 + \frac{2}{\epsilon}\right)^{d_n}$$

where $d_n = \sum_{k=1}^{K} n_k^2$ and $n_k = \dim(\mathcal{H}'_k)$.

Proof. Let $\mathcal{S}_{UV} = \{T_1^\epsilon, \ldots, T_s^\epsilon\}$ be the set from Proposition 5.2.2. Then, we can define a processor by

$$\mathcal{P}(X \otimes Y) = \sum_{i=1}^{s} \langle i|Y|i\rangle T_i^\epsilon(X) \qquad \forall\ X \in \mathcal{B}(\mathcal{H}_1),\ \forall\ Y \in \mathcal{B}(\mathcal{H}_P)$$

and extending by linearity. Here, $\{|i\rangle\}_{i=1}^s$ is an orthonormal basis of $\mathcal{H}_P$. Choosing the program state for any $T \in \mathcal{T}_{UV}$ to be $|i\rangle\langle i|$ if $\|T - T_i^\epsilon\|_\diamond \leq 2\epsilon$, the map $\mathcal{P}$ can be checked to be an ϵ-CPQP$_{UV}$ using Proposition 5.2.2. □

5.2.2 Lower bounds on the program dimension

In addition to the upper bounds, we seek to provide lower bounds on the program dimension of an ϵ-CPQP$_{UV}$. The main idea is that all information about the UV-covariant channel $T \in \mathcal{T}_{UV}$ is contained in its corresponding Choi-Jamiołkowski state c_T. Thus, the program state $\pi_T \in \mathcal{D}(\mathcal{H}_P)$ has to store all information about c_T. Using the Holevo information to quantify the amount of information (see Eq. (2.2.3)), we obtain the following lower bounds:

Theorem 5.2.4 ([1, Theorem 4.4]). *Let $\epsilon \in [0,1)$, $\mathcal{P}^\epsilon \in CPTP(\mathcal{H}_1 \otimes \mathcal{H}_P, \mathcal{H}_2)$ be an ϵ-$CPQP_{UV}$, G be a compact group with an irrep U on $\mathcal{H}_1$ and V be a representation on $\mathcal{H}_2$. Then, the following lower bound for the program dimension d_P holds*

$$\frac{1}{2^{2h(\epsilon)}}\left(\sum_{k=1}^{K} n_k\right)^{(1-2\epsilon)}$$

with $h(\epsilon) = H(\epsilon, 1-\epsilon) = -\epsilon\log\epsilon - (1-\epsilon)\log(1-\epsilon)$ being the binary entropy, n_k the multiplicity of the irrep $k \in \{1,\ldots,K\}$ in the direct sum decomposition of $\bar{U} \otimes V$ and ϵ the approximation parameter of the ϵ-$CPQP_{UV}$.

Proof. Every channel T^ϵ, the processor is able to implement, corresponds to a Choi-Jamiołkowski state $c_{T^\epsilon} \in \mathcal{D}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ (see Theorem 2.1.14) which can be understood as the output state of the processor $\mathcal{P}^\epsilon \in \mathrm{CPTP}\ (\mathcal{H}_1 \otimes \mathcal{H}_P, \mathcal{H}_2)$ tensorized with the d_1-dimensional identity map $\mathrm{id} : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_1)$. The construction

$$[\mathrm{id} \otimes \mathcal{P}^\epsilon](|\Omega\rangle\langle\Omega|_{\mathcal{H}_1 \otimes \mathcal{H}_1} \otimes \cdot) : \mathcal{B}(\mathcal{H}_P) \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

provides a quantum channel which maps every program state π_T to a Choi-Jamiołkowski state c_{T^ϵ}. This is a completely positive map because the processor map itself and the identity map are completely positive.

Recall the definition of the Holevo information (Eq. (2.2.3))

$$\chi(\{\rho_i, p_i\}) := S\left(\sum_i p_i \rho_i\right) - \sum_i p_i S(\rho_i),$$

where S is the von Neumann entropy and $\{\rho_i, p_i\}$ is an ensemble of quantum states. Furthermore, recall the definition of a t-design (Eq. (2.2.4)). Let $p^{(k)} := (p_1^{(k)}, \ldots, p_{N_k}^{(k)})$ be a probability distribution and let $|\psi_j^{(k)}\rangle\langle\psi_j^{(k)}| \in \mathcal{D}_P(\mathcal{H}'_k)$, $j \in \{1,\ldots,N_k\}$, be a collection of states such that $\{|\psi_j^{(k)}\rangle, p_j^{(k)}\}$ is a 1-design [31, 65], i.e., its average is the same as the uniform average over pure states with respect to the Haar measure. Then, $p = (\lambda_1 p^{(1)}, \ldots, \lambda_K p^{(K)})$ is a probability distribution on Choi-Jamiołkowski states $c_T \in \mathcal{J}_{UV}$, where $p_j^{(k)}$ corresponds to $c_{T_{(k),j}} := 0 \oplus \mathbb{1}_{b_k}/b_k \otimes |\psi_j^{(k)}\rangle\langle\psi_j^{(k)}| \oplus 0$. Here, $\lambda_k = n_k/d_c$, where $d_c = \sum_{k=1}^K n_k$. It can be checked that the average state is

$$\frac{1}{d_c}\bigoplus_{k=1}^{K} \frac{\mathbb{1}_{b_k}}{b_k} \otimes \mathbb{1}_{n_k}.$$

The probability distribution also induces ensembles $\{c_{T^\epsilon_{(k),j}}, p_j^{(k)}\}$ and $\{\pi_{T_{(k),j}}, p_j^{(k)}\}$.

Applying the data-processing inequality [23, Subsection 10.7.2] (since the Holevo information is a mutual information), we get

$$\chi(\{\pi_{T_{(k),j}}, p_j^{(k)}\}) \geq \chi(\{c_{T^\epsilon_{(k),j}}, p_j^{(k)}\}).$$

Furthermore, note that the processor implements T up to accuracy ϵ, i.e.,

$$\frac{1}{2}\|T - T^\epsilon\|_\diamond \leq \epsilon$$

which can be related to the corresponding Choi-Jamiołkowski states [19, Eq. 3.414]

$$\frac{1}{2}\|c_T - c_T^\epsilon\|_1 \leq \frac{1}{2}\|T - T^\epsilon\|_\diamond \leq \epsilon.$$

Recall that c_T has a block-diagonal structure inherited from Eq. (3.1)

$$\mathcal{K} = \left\{ \bigoplus_{k=1}^{K} \mathbb{1}_{b_k} \otimes B_k \mid B_k \in \mathcal{B}(\mathcal{H}_k') \right\}.$$

Since $\bigoplus_{k=1}^{K} \mathcal{B}(\mathcal{H}_k')$ and $\mathcal{K}$ are isomorphic as matrix algebras, extending the dual map of this isomorphism to a map $C : \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_{d_c})$ (as in the proof of Theorem 5.1.10),

$$C : \bigoplus_{k=1}^{K} \mathbb{1}_{b_k} \otimes B_k \mapsto \bigoplus_{k=1}^{K} b_k B_k$$

discards the multiplicity spaces and thus reduces the dimensions from $d_1 d_2$ to $d_c = \sum_{k=1}^{K} n_k$. Since the trace distance is contractive under quantum channels,

$$\frac{1}{2}\|C(c_T) - C(c_{T^\epsilon})\|_1 \leq \frac{1}{2}\|c_T - c_{T^\epsilon}\|_1 \leq \epsilon.$$

With this trace-norm distance, we apply the Alicki-Fannes-Winter (AFW) inequality [28, 66], [67, Lemma 1] to bound the difference of the corresponding Holevo informations:

$$\begin{aligned}
&\left|\chi(\{C(c_{T_{(k),j}}), p_j^{(k)}\}) - \chi(\{C(c_{T^\epsilon_{(k),j}}), p_j^{(k)}\})\right| \\
&\qquad = \left| S\left(\sum_{kj} C(c_{T_{(k),j}}) p_j^{(k)}\right) - \sum_{kj} S(C(c_{T_{(k),j}})) p_j^{(k)} \right. \\
&\qquad\qquad \left. - S\left(\sum_{kj} C(c_{T^\epsilon_{(k),j}}) p_j^{(k)}\right) + \sum_{kj} S(C(c_{T^\epsilon_{(k),j}})) p_j^{(k)} \right| \\
&\qquad \leq \left| S\left(\sum_{kj} C(c_{T_{(k),j}}) p_j^{(k)}\right) - S\left(\sum_{kj} C(c_{T^\epsilon_{(k),j}}) p_j^{(k)}\right) \right| \\
&\qquad\qquad + \sum_{kj} \left| S(C(c_{T_{(k),j}})) - S(C(c_{T^\epsilon_{(k),j}})) \right| p_j^{(k)} \\
&\qquad \leq \ \epsilon \log d_c - h(\epsilon) + \epsilon \log d_c - h(\epsilon) = 2\epsilon \log d_c - 2h(\epsilon)
\end{aligned}$$

with $h(\epsilon) = H(\epsilon, 1-\epsilon) = -\epsilon \log \epsilon - (1-\epsilon)\log(1-\epsilon)$ the binary entropy, where we used the AFW inequality in the last step.

Thus, we obtain

$$\begin{aligned}\chi(\{\pi_{T_{(k),j}}, p_j^{(k)}\}) &\geq \chi(\{c_{T^{\epsilon}_{(k),j}}, p_j^{(k)}\}) \\ &\geq \chi(\{C(c_{T^{\epsilon}_{(k),j}}), p_j^{(k)}\}) \\ &\geq \chi(\{C(c_{T_{(k),j}}), p_j^{(k)}\}) - 2\epsilon \log d_c - 2h(\epsilon).\end{aligned}$$

Let us consider the term $\chi(\{C(c_{T_{(k),j}}), p_j^{(k)}\})$. Since the $C(c_{T_{(k),j}})$ of this ensemble are pure states, the second term of the Holevo information is zero. Moreover

$$\sum_{kj} C(c_{T_{(k),j}}) p_j^{(k)} = \frac{\mathbb{1}_{d_c}}{d_c}$$

and the von Neumann entropy of a maximally mixed state is $S\big(\frac{\mathbb{1}_{d_c}}{d_c}\big) = \log d_c$. Together with the inequality $\log d_P \geq \chi(\{\pi_{T_{(k),j}}, p_j^{(k)}\})$ (see Eq. (2.2.1)), we obtain

$$\begin{aligned}\log d_P &\geq \chi(\{\pi_{T_{(k),j}}, p_j^{(k)}\}) \\ &\geq (1-2\epsilon)\log d_c - 2h(\epsilon) \\ &= \log d_c^{(1-2\epsilon)} - 2h(\epsilon).\end{aligned}$$

This yields

$$d_P \geq \frac{1}{2^{2h(\epsilon)}} d_c^{(1-2\epsilon)} = \frac{1}{2^{2h(\epsilon)}} \Bigg(\sum_{k=1}^{K} n_k \Bigg)^{(1-2\epsilon)},$$

which proves the assertion. □

The derived bound on d_P increases with the block dimension n_k.

6 Quantum information theory in infinite dimension

In the previous chapters, we explored quantum information theory in finite dimension considering merely finite-dimensional Hilbert spaces. In quantum physics, many interesting properties, models, and phenomena require infinite-dimensional descriptions. This is why infinite-dimensional systems (continuous-variable systems) play a major role in quantum information processing and get more and more attention. So, let us dive into the infinite-dimensional world and generalize the previous setting to infinite dimension.

Working in infinite-dimensional systems is accompanied by constraints. Considering quantum channels, for instance, there is an energy constraint of the input signal for physical reasons. Also mathematically the necessity for constraints appears when dealing with infinite-dimensional Hilbert spaces. New features such as the discontinuity and unboundedness of the entropy of a quantum state as well as the emergence of continuous ensembles arise [27].

6.1 Infinite-dimensional setting

Let $\mathcal{H}$ be an infinite-dimensional and separable Hilbert space, i.e., with a countable orthonormal basis. The Banach space of all bounded operators with the operator norm $\|A\| = \sup_{x:\|x\|=1} \|Ax\|$ for $x \in \mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$ [27, p. 6, p. 233] is represented by

$$\mathcal{B}(\mathcal{H}) = \{A : \mathcal{H} \to \mathcal{H} \text{ linear} \mid \|A\| < \infty\}.$$

Let $\mathcal{T}(\mathcal{H})$ be the Banach space of trace-class operators

$$\mathcal{T}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) \mid \|A\|_1 = \operatorname{tr}\sqrt{A^*A} < \infty\}$$

with trace norm $\|\cdot\|_1$.
Furthermore, a quantum state (density operator) is a positive trace-class operator with unit trace in $\mathcal{T}(\mathcal{H})$. The set of all quantum states is represented by

$$\mathcal{D}(\mathcal{H}) = \{X \in \mathcal{T}(\mathcal{H}) \text{ such that } X \geq 0, \operatorname{tr}(X) = 1\}.$$

This state space is a closed convex subset of $\mathcal{T}(\mathcal{H})$. We also generalize the notion of a quantum channel to infinite dimension. Therefore, we first define *complete positivity*.

Definition 6.1.1 (Complete Positivity)**.** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two separable Hilbert spaces. A linear map $T : \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{H}_2)$ is called* completely positive *if the map*

$$T \otimes \mathrm{id} : \mathcal{T}(\mathcal{H}_1) \otimes \mathcal{B}(\mathbb{C}^n) \to \mathcal{T}(\mathcal{H}_2) \otimes \mathcal{B}(\mathbb{C}^n)$$

is positive.

A linear map which maps quantum states to quantum states is called *quantum channel*.

Definition 6.1.2 (Quantum Channel)**.** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable Hilbert spaces associated with system* 1 *and* 2*, respectively. A* quantum channel *from a system* 1 *to a system* 2 *is a linear map $T : \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{H}_2)$ which is completely positive (CP) and trace-preserving (TP).*

The set of all quantum channels is again represented by

$$CPTP(\mathcal{H}_1, \mathcal{H}_2) := \{T : \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{H}_2) \text{ linear, CP and TP}\}.$$

If $\mathcal{H} = \mathcal{H}_1 = \mathcal{H}_2$, we shortly write CPTP($\mathcal{H}$).

In the following, we consider a positive semidefinite, self-adjoint densely-defined energy operator (Hamiltonian) $H \geq 0$ with discrete spectrum describing a quantum system. Denoting its spectral decomposition as $H = \sum_{n=0}^{\infty} e_n P_n$ for all n, we additionally assume $\dim(P_n) < \infty$ (finite degeneracy of all eigenvalues).
To obtain a realistic setting while operating in infinite dimension, we assume an energy constraint. The bounded energy allows us to recover properties which are familiar from the finite-dimensional setting. We consider energy constraints on the input

$$\mathrm{tr}(\rho H_1) \leq E_1,$$

as well as on the output

$$\mathrm{tr}(\Phi(\rho) H_2) \leq E_2,$$

where H_1, $H_2 \geq 0$ are Hamiltonians with discrete spectrum and finite degeneracy of all eigenvalues on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and Φ is a quantum channel. We define energy-limited quantum channels as operations mapping energy-bounded states to energy-bounded states, i.e., those channels preserve energy-boundedness.

Definition 6.1.3 ((α, β)-energy-limited quantum channel [68])**.** *Given two positive semidefinite Hamiltonians $H_1 \geq 0$ and $H_2 \geq 0$ on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, a quantum channel $\Phi : \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{H}_2)$ is called* (α, β)-energy-limited *if for all $\rho \in \mathcal{D}(\mathcal{H}_1)$ with $\mathrm{tr}\, \rho H_1 < \infty$, it holds $\mathrm{tr}\, \Phi(\rho) H_2 < \infty$, and in fact*

$$\mathrm{tr}\, \Phi(\rho) H_2 \leq \alpha\, \mathrm{tr}\, \rho H_1 + \beta.$$

This can be expressed equivalently as $\Phi^*(H_2) \leq \alpha H_1 + \beta \mathbb{1}$, using the adjoint CPTP map $\Phi^* : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$.

As a metric to distinguish quantum channels, we introduced the diamond norm in Definition 4.1. However, in infinite dimension, there are some inconvenient features of the diamond norm. For instance, consider the attenuator channel, which is realized by a beamsplitter with transmissivity λ. Computing the diamond norm of two attenuator channels with parameters λ and λ' yields $\|\Phi_\lambda - \Phi_{\lambda'}\|_\diamond = 2$ for $\lambda \neq \lambda'$ [68]. Since the maximal distance the diamond norm can reach, is 2, all attenuator channels are in maximal distance from each other independent of their parameters. This is not realistic because to obtain this distance between two attenuator channels, we require highly energetic input states. However, for most realistic communication settings an energy-bound on the input is used. Thus, in the setting of an energy limit, Shirokov [69] and Winter [68] defined an energy-constrained diamond norm.

Definition 6.1.4 (Energy-constrained diamond norm [68, 69])**.** *Let $H_1 \geq 0$ be a positive semidefinite Hamiltonian on $\mathcal{H}_1$ with discrete spectrum and smallest eigenvalue 0, and $E > 0$. For a map $\Phi : \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{H}_2)$ that preserves self-adjointness, we define the* energy-constrained diamond norm *(more precisely: E-*constrained diamond norm*) as*

$$\|\Phi\|_\diamond^E = \sup_{\mathcal{H}_R} \sup_{\substack{\rho \in \mathcal{D}(\mathcal{H}_1 \otimes \mathcal{H}_R) \\ \operatorname{tr}(\rho(H_1 \otimes \mathbb{1}_R)) \leq E}} \|(\Phi \otimes \mathrm{id}_R)\rho\|_1 \tag{6.1.1}$$

with an additional quantum system R.

Note that an alternative, slightly different definition of the energy-constrained diamond norm was given in Refs. [70] and [56]. We can take the supremum in Eq. (6.1.1) over pure states by purification and using that the trace norm is contractive under the partial trace operation and without loss of generality, $\mathcal{H}_R \cong \mathcal{H}_1$ [68, 69]. Just like the diamond norm, the E-constrained version also has an operational meaning: how well we can distinguish between two quantum channels when we apply them to input states with bounded energy. In particular, we use a special property concerning the relation of diamond norms w.r.t. different energies: Let E and E' be two different energies with $0 < E < E'$. The corresponding diamond norms satisfy the inequalities

$$\|\Phi\|_\diamond^E \leq \|\Phi\|_\diamond^{E'} \leq \frac{E'}{E}\|\Phi\|_\diamond^E \tag{6.1.2}$$

for a map Φ that preserves self-adjointness. This and further useful properties of the E-constrained diamond norm can be found in Ref. [68].

Having specified the mathematically necessary constraints when the underlying Hilbert spaces are infinite dimensional, we briefly focus on two emerging features: first, the entropy of the quantum states is discontinuous and unbounded and secondly, continuous ensembles emerge.

To quantify information, we want to use the Holevo information based on the von Neumann entropy and continuous ensembles. Recall the definition of the von Neumann entropy

$$S(\rho) = -\operatorname{tr}(\rho \log \rho). \tag{6.1.3}$$

In infinite-dimensional systems, the von Neumann entropy is in general not continuous anymore (see continuity of the von Neumann entropy in finite dimension in Eq. (2.2.2)). However, if we assume an energy constraint $\operatorname{tr}\rho H < E$, the entropy in Eq. (6.1.3) is continuous on this set of energy-constrained states if and only if

$$\operatorname{tr}\exp(-\theta H) < \infty \tag{6.1.4}$$

for all $\theta > 0$ [27, Lemma 11.8.]. Since we always consider Hamiltonians that satisfy Eq. (6.1.4), we also have continuity in infinite dimension. In the case where we work with approximate versions of energy-bounded quantum states, we have to be careful because the entropy is then potentially discontinuous.

Furthermore, we consider a continuous generalization of ensembles which we have already defined in Section 2.2 for finite dimension. Let $\mathcal{X}$ be a domain in $\mathbb{R}^k$ that is locally compact with $x \in \mathcal{X}$. Furthermore, let $x \mapsto \rho_x$ be a continuous mapping with $\rho_x \in \mathcal{D}(\mathcal{H})$ and with finite von Neumann entropy $S(\rho_x) < \infty$. Considering a probability measure μ on $\mathcal{X}$, the Holevo information of the ensemble $\{\rho_x, \mu(dx)\}$ is well-defined as

$$\chi(\{\rho_x, \mu(dx)\}) = S\left(\int_{\mathcal{X}} \rho_x \mu(dx)\right) - \int_{\mathcal{X}} S(\rho_x)\mu(dx).$$

6.2 Bosonic Gaussian systems

In this chapter, we shed light on continuous-variable Bosonic systems, in particular on those which correspond to classical multidimensional Gaussian distributions. We mainly follow the review articles [71–73] if not stated differently.
Electromagnetic waves such as light waves are known to be the fundamental physical carrier of information. The electromagnetic field with its M quantized radiation modes can be described mathematically as an ensemble of oscillators. Thus, M quantum harmonic oscillators correspond to M Bosonic modes as a prototype of a continuous-variable system. We start by presenting the fundamentals in the one-mode case, generalizing this to the multimode case thereafter. The one-mode system can be either described by two *quadrature operators* x (position) and p (momentum) satisfying the canonical commutation relation

$$[x, p] = i\mathbb{1},$$

or equivalently by the *ladder operators* a (annihilation) and a^* (creation) which are Bosonic field operators satisfying the Bosonic commutation relation

$$[a, a^*] = \mathbb{1}.$$

These two descriptions are connected through the following equations

$$a = \frac{1}{\sqrt{2}}(x + ip), \qquad a^* = \frac{1}{\sqrt{2}}(x - ip).$$

The Hilbert space of the one-mode system is separable as well as infinite dimensional and is spanned by a countable basis $\{|n\rangle\}_{n=0}^{\infty}$ called the *Fock* or *number state basis*. Its elements are eigenstates of the number operator $N := a^*a$, i.e., $N|n\rangle = n|n\rangle$.
For M modes, $M \in \mathbb{N}$, there is a set of quadrature operators (x_j, p_j) corresponding to each mode $j \in \{1, \ldots, M\}$ satisfying the commutation relations

$$[x_j, p_k] = i\delta_{jk}\mathbb{1} \qquad \text{for } j, k = 1, \ldots, M.$$

Equivalently, there are Bosonic ladder operators a_j, a_j^* per mode $j \in \{1, \ldots, M\}$ satisfying

$$[a_j, a_k^*] = \delta_{jk}\mathbb{1} \qquad \text{for } j, k = 1, \ldots, M.$$

For $M \in \mathbb{N}$ modes the Hilbert space is the M-fold tensor power of the Hilbert space of the single-mode system with basis

$$\{|n_1, \ldots, n_M\rangle = |n_1\rangle \otimes \ldots \otimes |n_M\rangle\}_{(n_1,\ldots,n_M)\in\mathbb{N}_0^M}.$$

All the quantum operators can be written in one vector

$$X = (x_1, p_1, \ldots, x_M, p_M)^T.$$

The corresponding commutation relations are then of the following form

$$[X_j, X_k] = i\Omega_{j,k}\mathbb{1} \qquad \text{for } j, k = 1, \ldots, 2M$$

with $\Omega = \bigoplus_{j=1}^{M} \omega$ and $\omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
Analogously, the ladder operators can also be condensed in one vector

$$A = (a_1, a_1^*, \ldots, a_M, a_M^*)^T$$

which satisfies

$$[A_j, A_k] = \Omega_{j,k} \qquad \text{for } j, k = 1, \ldots, 2M.$$

Let us introduce the real symplectic space $(\mathbb{R}^{2M}, \Omega)$, which is the vector space $\mathbb{R}^{2M}$ equipped with the symplectic form and is called *phase space*. We define the *Weyl displacement operators*

$$\tilde{D}(\xi) := \exp(iX^T\Omega\xi), \tag{6.2.1}$$

where $\xi \in \mathbb{R}^{2M}$ [73, Eq. (11)]. Those operators establish a connection between operators and complex functions on phase space. An M-mode quantum state ρ can be represented by its *Wigner characteristic function*

$$\tilde{\chi}_\rho(\xi) = \mathrm{tr}(\rho\tilde{D}(\xi)),$$

where $\xi \in \mathbb{R}^{2M}$ and $\rho \in \mathcal{D}(\mathcal{H}^{\otimes M})$ [73, Eq. (12)].
The first and second moments are relevant quantities that characterize $\tilde{\chi}_\rho(\xi)$. The

first moment, denoted by $d_j(\rho)$ with $j = 1, \ldots, 2M$ is called *displacement vector* [73, Eq. (14)] and is defined as

$$d_j(\rho) = \mathrm{tr}(X_j \rho) \qquad j = 1, \ldots, 2M.$$

The second moment, the covariance matrix Γ, is defined as

$$\Gamma_{jk}(\rho) := \mathrm{tr}\left[\{X_j - d_j(\rho)\mathbb{1}, X_k - d_k(\rho)\mathbb{1}\}\rho\right]$$

for $j, k = 1, \ldots, 2M$, where $\{A, B\} := AB + BA$ is the anticommutator [73, Eq. (15)]. The covariance matrix is a real and symmetric $2M \times 2M$ matrix and satisfies the uncertainty principle [73, Eq. (17)], [72, Eq. (4.67)]

$$\Gamma(\rho) + i\Omega \geq 0.$$

There is a class of states that is completely characterized by the displacement vector together with the covariance matrix. These are called *Gaussian states*.

6.2.1 Gaussian quantum states

An M-mode Gaussian state is a quantum state with Gaussian characteristic function

$$\chi_{\rho_G(d,\Gamma)}(\xi) = \exp\left[\frac{1}{4}\xi^T(\Omega\Gamma\Omega^T)\xi + i(\Omega d)^T\xi\right] \qquad \text{for all } \xi \in \mathbb{R}^{2M}.$$

Since d and Γ determine the Gaussian state, we write $\rho_G(d, \Gamma)$ [73, Eq. (19)].

A special class of Gaussian states is formed by *coherent states* $\rho_G(\xi, \mathbb{1}_2)$ of a single mode with $\xi = (\xi_1, \xi_2)$. These are pure states which can be generated by displacing the vacuum state $|0\rangle$ [73, p. 627], [72, Eq. (4.97)], i.e.,

$$|\xi\rangle := D(\xi)\,|0\rangle\,.$$

All coherent states form an overcomplete basis of non-orthogonal vectors. Any pure state can be expanded in terms of all other coherent states using Ref. [27, p. 261]

$$\frac{1}{\pi}\int |\xi\rangle\langle\xi| d^2\xi = \mathbb{1}.$$

Furthermore, coherent states are the eigenstates of the annihilation operator [73, p. 627]

$$a\,|\xi\rangle = \frac{\xi_1 + i\xi_2}{\sqrt{2}}\,|\xi\rangle\,.$$

Another special class of Gaussian states are *thermal states* $\rho_G(0, (2N+1)\mathbb{1}_2)$ with $N = \mathrm{tr}(\rho a^* a)$ being the mean photon number. The von Neumann entropy of a one-mode thermal state is known to be [72, p. 4.79]

$$S(\rho_{G,th}^N) = g(N) := (N+1)\log(N+1) - N\log N. \tag{6.2.1.1}$$

Let us now dive into transformations of quantum states, i.e., quantum operations called *Gaussian channels*.

6.2.2 Gaussian quantum channels

Definition 6.2.1 (Gaussian channel [73, p. 639])**.** *A* Gaussian channel *is a quantum channel, i.e., a CPTP map* $\Phi : \mathcal{B}(\mathcal{H}^{\otimes n}) \to \mathcal{B}(\mathcal{H}^{\otimes m})$*, that maps every Gaussian state* ρ_G *to a Gaussian state, i.e.,* $\Phi(\rho_G)$ *is Gaussian as well.*

Gaussianity is a property of the channel, not the state, which is why we can input non-Gaussian states as well. The action of a Gaussian channel on a Gaussian state is uniquely determined by the action on its first and second moments. Gaussian channels that act on M modes are characterized by a vector $\eta \in \mathbb{R}^{2M}$ and two real $2M \times 2M$ matrices $\mathcal{K}$ and $\mathcal{N}$ which transform the displacement vector d and the covariance matrix Γ of the input state as follows [73, Eq. (109)], [72, Eq. (4.165)]

$$d \to \mathcal{K}d + \eta, \qquad \Gamma \to \mathcal{K}\Gamma\mathcal{K}^T + \mathcal{N}. \tag{6.2.2.1}$$

Since Γ is symmetric, $\mathcal{N}$ is required to be symmetric, i.e., $\mathcal{N} = \mathcal{N}^T$. Additionally, $\mathcal{K}$ and $\mathcal{N}$ have to satisfy the complete positivity-condition [73, Eq. (110)], [72, Eq. (4.166)]

$$\mathcal{N} + i\Omega - i\mathcal{K}\Omega\mathcal{K}^T \geq 0.$$

We can interpret $\mathcal{K}$ as causing the linear transformation of the canonical phase variables, while $\mathcal{N}$ introduces quantum or classical noise.

We introduce *gauge-covariant channels* as a special class of channels (see Ref. [27, Subsection 12.5.2]). Those channels commute with phase rotations, which is why they are also called *phase-insensitive*.

Definition 6.2.2 (Gauge-covariant channel)**.** *A channel* Φ *that maps a single Bosonic mode to a single Bosonic mode is called* gauge-covariant *if it satisfies*

$$\Phi(e^{i\phi N_1}\rho e^{-i\phi N_1}) = e^{i\phi N_2}\Phi(\rho)e^{-i\phi N_2},$$

where $\phi \in [0, 2\pi]$ *and* $N_j = a_j^* a_j$ *is the photon number operator in system* $j = 1, 2$*.*

Gauge-covariant channels can be described by a concatenation of an attenuation and an amplification channel [27, Prop. 12.42]. Being interested in additivity questions of the Holevo information, for instance, the parameter specifying the channel is considered to be real because the Holevo information is invariant under passive transformations. However, viewed from the perspective of a processor, phases yield different outputs we want to distinguish. Hence, we consider complex parameters. This results in an additional rotation in phase space. The rotation is described by the unitary operator

$$R(\varphi) = \exp(-i\varphi a^* a), \tag{6.2.2.2}$$

which transforms the covariance matrix according to

$$\hat{R}_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

We denote the corresponding rotation channel as

$$\mathcal{R}_\varphi(\cdot) = e^{-i\varphi N}(\cdot)e^{i\varphi N}.$$

The attenuator channel $\mathcal{T}_\lambda$ with parameter $0 \leq \lambda \leq 1$, $\lambda \in \mathbb{R}$, which is called *attenuation factor*, is described by the two matrices $\mathcal{K} = \sqrt{\lambda}\mathbb{1}_2$ and $\mathcal{N} = (1-\lambda)\mathbb{1}_2$. The amplifier channel $\mathcal{A}_\mu$ is described by the transforming matrices $\mathcal{K} = \sqrt{\mu}\mathbb{1}_2$ and $\mathcal{N} = (\mu - 1)\mathbb{1}_2$ for $\mu \in (1, \infty)$ [72, p. 4-37].

The following proposition specifies the structure of one-mode gauge-covariant channels using complex parameters. This results in a phase-rotation channel and a real-parameter attenuation and amplification channel, respectively, which can be found in Ref. [27]. Kok and Lovett provide a quantum optical explanation including the complex parameters in their book [74, p. 31].

Proposition 6.2.3 (Structure gauge-covariant Gaussian channel [2, Prop. 16])**.** *Any one-mode Bosonic gauge-covariant Gaussian channel Φ can be understood as a concatenation of a quantum-limited attenuator channel $\mathcal{T}_\lambda$, a rotation channel $\mathcal{R}_\varphi$ and a (diagonalizable) quantum-limited amplifier channel $\mathcal{A}_\mu$, i.e., $\Phi = \mathcal{A}_\mu \circ \mathcal{R}_\varphi \circ \mathcal{T}_\lambda$.*

Another class of Gaussian quantum channels are Gaussian unitary quantum channels which we introduce in the following subsection.

6.2.3 Gaussian unitary quantum channels

Special Gaussian unitary transformations are displacement operators

$$D(\tau) := \exp(\tau a^* - \tau^* a),$$

where $\tau = \frac{x+ip}{2}$ is the complex amplitude. This is the complex version of the Weyl displacement operators introduced in Eq. (6.2.1). They act on the annihilation operator as a shift

$$a \to D^*(\tau) a D(\tau) = a + \tau,$$

which yields translations in phase space. Another Gaussian unitary operator is the rotation in phase space described by the unitary operator (Eq. (6.2.2.2)) and also the squeezing operator

$$S(s) = \exp\left(\frac{s}{2}a^2 - \frac{s}{2}a^{*2}\right)$$

with squeezing factor $s \in [0, \infty)$. It has zero displacement and its covariance matrix is

$$\hat{S}_s = \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{s} \end{pmatrix}.$$

To study and define general Gaussian unitary channels, Gaussian unitaries, respectively, we introduce symplectic matrices. A $2M \times 2M$ matrix is called *symplectic* if it preserves the symplectic form

$$S \Omega S^T = \Omega.$$

Those matrices form the real symplectic group

$$Sp(2M, \mathbb{R}) := \{S \in \mathcal{M}_{2M \times 2M}(\mathbb{R}) \mid S \Omega S^T = \Omega\}.$$

A *Gaussian unitary*, which we denote as U_G, is equivalent to an affine symplectic map (S, η) that transforms the first and second moments according to

$$d \to Sd + \eta, \qquad \Gamma \to S\Gamma S^T.$$

This is obtained by setting $\mathcal{N} = 0$ and $\mathcal{K} = S$ in (6.2.2.1). We can always write

$$U_G = D(\tau)U_S, \tag{6.2.3.1}$$

where $D(\tau)$ is the displacement operator describing a translation in phase space and U_S is the Gaussian unitary associated to any $S \in Sp(2M, \mathbb{R})$ by the map $S \mapsto U_S$. The action of U_S on the quadrature operators is determined by the action of S on the quadrature operators by

$$U_S^* X_j U_S = \sum_{k=1}^{2M} S(s_{jk}) X_k.$$

A $2M \times 2M$ matrix can be decomposed as

$$S = S_1 \left[\bigoplus_{j=1}^{M} S(s_j) \right] S_2,$$

where S_1 and S_2 are orthogonal, symplectic matrices. This means that every symplectic matrix can be decomposed in a direct sum of squeezing operators modulo passive transformations that are characterized by $\eta = 0$ and $S^T S = \mathbb{1}_{2M \times 2M}$. Note that passive operations do not change the energy.
The simplest quantum channels are those that are reversible, i.e., unitary transformations $U^{-1} = U^*$, which transform

$$\rho_G \to U_G \rho_G U_G^*$$

or $|\varphi\rangle \to U |\varphi\rangle$ if pure states. Any channel can be conceived as a reduction of a unitary transformation acting on the state ρ plus some environment state ρ_E

$$\Phi(\rho) = \mathrm{tr}_E(U_G(\rho \otimes \rho_E)U_G^*).$$

For a Gaussian channel Φ, it is always possible to find a unitary dilation of this form with a Gaussian unitary U_G and a pure Gaussian environment state ρ_E. Note that this is the infinite-dimensional version of Stinespring's dilation we have already defined in finite dimension (see Theorem 2.1.13). In infinite dimension, we can choose an environment with $M_E \leq 2M$ modes.

The structure of a general Gaussian unitary channel which we use for lower bounds on the program register dimension in Ref. [2] can be described as follows: Gaussian unitary channels can be written as a phase-rotation channel, a squeezing unitary channel (introducing a squeezing factor) followed by another phase-rotation channel and a displacement operation [75, 76]. The first three components can be connected to symplectic matrices. Together with the displacement operation, we obtain a general Gaussian unitary channel (see Eq. (6.2.3.1)).

7 Infinite-dimensional programmable quantum processors

This chapter is based on the following article

Martina Gschwendtner and Andreas Winter
Infinite-Dimensional Programmable Quantum Processors
PRX Quantum 2:030308, 2021 (38 pages)
(see Article [2] in the bibliography)

The author of this thesis is the principal author of the above publication.

In this chapter, we generalize approximate programmable quantum processors to infinite dimension. We aim to approximately implement energy-limited quantum channels (see Definition 6.1.3) using a finite-dimensional program register. First of all, we define such a processor with an infinite-dimensional input state assuming an energy constraint. In finite dimension, we measure the accuracy of the implementation using the diamond norm whereas in infinite dimension we consider the E-constrained diamond norm due to the energy constraint. We then investigate the resource requirements, providing upper and lower bounds on the dimension of the program register for different classes of quantum channels. We start with energy-limited unitary quantum channels, followed by energy-limited one-mode gauge-covariant Gaussian quantum channels (Definition 6.2.2). These are physically important since they describe the effects of attenuation of signals and addition of noise in communication schemes and furthermore, they preserve thermal Gaussian states [73]. Finally, we consider the universal implementation of energy-limited Gaussian unitary channels of any finite number of Bosonic modes.

7.1 Setting, motivation and definition

Recall Definition 4.2 for a general definition of an ϵ-PQP$_{\mathcal{C}}$. Until now, we merely considered finite-dimensional Hilbert spaces. For the exact case $\epsilon = 0$, Nielsen and Chuang proved their *No-Programming Theorem* [4] stating that an exact and universal programmable quantum processor with finite-dimensional program register does not exist. Every implemented unitary operation requires an extra program space dimension which means that up to d unitary operations (distinct up to a global phase) can be implemented using a d-dimensional program register. With

a separable Hilbert space, however, one can choose the class $\mathcal{C}$ of implemented channels as a countable, dense set of unitaries and thus approximate every unitary channel arbitrarily well. This can concretely be done with the following construction which we call the *processor-encoding technique (PET)*. We use this several times in this chapter. If $\mathcal{C} = \{\Phi_i\}_{i=1}^K$ is a finite set of channels, we can construct the processor implementing those channels exactly with memory dimension equal to the cardinality K of the set by encoding the specifying index i from the set of channels into the program state π_Φ. Picking an arbitrary orthonormal system $\{|i\rangle\}_{i=1}^K$, the processor

$$\mathcal{P}(\rho \otimes \pi_\Phi) := \sum_{i=1}^{K} \Phi_i(\rho) \langle i| \pi_\Phi |i\rangle$$

implements the channels Φ_i from the set of channels $\mathcal{C}$ exactly

$$\mathcal{P}(\rho \otimes |i\rangle\langle i|) = \Phi_i(\rho).$$

The program dimension of the processor is equal to the cardinality K of $\mathcal{C}$ meeting the lower bound from the *No-Programming Theorem* (see Section 4.1) for unitary channels.

Therefore, we are interested in how the program register depends on the accuracy $\epsilon > 0$. In particular, we investigate programmable quantum processors that approximately implement (α, β)-energy-limited quantum channels between infinite-dimensional systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with Hamiltonians H_1 and H_2, respectively. We demand approximate programmability merely for (α, β)-energy-limited channels otherwise a finite-dimensional program register is not achievable: for every eigenstate $|\phi_n\rangle$ with energy e_n there is a well-defined CPTP map Φ_n mapping all of $\mathcal{D}(\mathcal{H}_1)$ to $|\phi_n\rangle\langle\phi_n|$. A processor which approximately implements any Φ_n, even with respect to $\|\cdot\|_\diamond^E$, can approximately prepare an arbitrary state $|\phi_n\rangle\langle\phi_n|$ from the ground state (or any other low-energy eigenstate) of $\mathcal{H}_1$, which intuitively requires an infinite-dimensional program register. In the case of energy-limited quantum channels, we use the E-constrained diamond norm and define:

Definition 7.1.1 (ϵ-EPQP$_\mathcal{C}$)**.** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable Hilbert spaces and consider a class of quantum channels $\mathcal{C} \subset CPTP(\mathcal{H}_1, \mathcal{H}_2)$. A quantum operation $\mathcal{P} \in CPTP(\mathcal{H}_1 \otimes \mathcal{H}_P, \mathcal{H}_2)$ is called an* ϵ-approximate E-constrained programmable quantum processor for $\mathcal{C}$ (ϵ-EPQP$_\mathcal{C}$) *if for all $\Phi \in \mathcal{C}$ there exists a state $\pi_\Phi \in \mathcal{D}(\mathcal{H}_P)$ such that*

$$\frac{1}{2}\|\mathcal{P}(\cdot \otimes \pi_\Phi) - \Phi\|_\diamond^E \leq \epsilon. \tag{7.1.1}$$

The dimension of the program register of an ϵ-EPQP$_\mathcal{C}$ is denoted as d_P^∞.

We consider the following classes of quantum channels in this chapter. Let $\mathcal{L}(\alpha, \beta)$ be the class of all (α, β)-energy-limited quantum channels. The corresponding processor is written as ϵ-EPQP$_{\mathcal{L}(\alpha,\beta)}$. The set of all (α, β)-energy-limited unitary channels $\mathcal{U}(\alpha, \beta)$ leads to the notation ϵ-EPQP$_{\mathcal{U}(\alpha,\beta)}$ for the corresponding processor. Furthermore, we consider the class of all (α, β)-energy-limited gauge-covariant

Gaussian channels abbreviated by $\mathcal{GCG}(\alpha,\beta)$ and the corresponding processor is an ϵ-EPQP$_{\mathcal{GCG}(\alpha,\beta)}$. Finally, all (α,β)-energy-limited Gaussian unitary channels are represented by $\mathcal{GU}(\alpha,\beta)$ and are implemented by an ϵ-EPQP$_{\mathcal{GU}(\alpha,\beta)}$.

Remark 7.1.2. *Note that the* No-Programming Theorem *[4] is not directly applicable here in the infinite-dimensional case of an ϵ-EPQP$_{\mathcal{U}(\alpha,\beta)}$. However, we can show that exact programmability with a finite-dimensional program dimension is not possible by reducing the infinite-dimensional case to the finite-dimensional one. If we merely consider unitaries of the form $U = U_0 + \sum_{n=2}^{\infty} P_n$, where U_0 is an arbitrary unitary on the support $\mathcal{H}_1 \subset \mathcal{H}$ of $P_0 + P_1$, i.e., $U_0 U_0^* = U_0^* U_0 = P_0 + P_1$, which are (α,β)-energy-limited for all $\alpha \geq 1$, $\beta \geq \epsilon_1 - \epsilon_0$, then a 0-EPQP$_{\mathcal{U}}$ would imply a 0-PQP$_{\mathcal{U}}$ for an input register $\mathcal{H}_1$, and since the latter is finite dimensional, the* No-Programming Theorem *applies.*

7.2 Recovering program states

To obtain lower bounds on the program dimension in the case of implementing energy-limited Gaussian channels, we make use of a lemma which is applicable to any unitary that a given processor implements approximately. It states that in this case the processor can be modified to another one that reuses the program state several times to approximately implement the same unitary several times sequentially or in parallel. For finite-dimensional unitaries, this was first shown by Yang *et al.* [15] using the diamond norm. In this section, we generalize this statement to the infinite-dimensional case and use this energy-constrained version in the next section to give lower bounds on the program dimension of approximate EPQP's implementing Gaussian unitaries.

We first recall some definitions from Shirokov [77].

Definition 7.2.1 (Completely bounded energy-constrained channel fidelity [77])**.** *Let $\mathcal{H}_1$ and $\mathcal{H}_R$ be two separable Hilbert spaces where $\mathcal{H}_R$ is a reference system. Furthermore, let $F(\rho,\sigma) := \operatorname{tr}\sqrt{\rho^{\frac{1}{2}}\sigma\rho^{\frac{1}{2}}}$ for $\rho,\sigma \in \mathcal{D}(\mathcal{H}_1)$ denote the fidelity. The* completely bounded energy-constrained channel fidelity *between two quantum channels $\mathcal{A}$ and $\mathcal{B}$ is defined as*

$$F_{cb}^{E}(\mathcal{A},\mathcal{B}) := \inf_{|\psi\rangle\langle\psi|} F\big((\mathcal{A}\otimes \mathrm{id}_R)(|\psi\rangle\langle\psi|), (\mathcal{B}\otimes \mathrm{id}_R)(|\psi\rangle\langle\psi|)\big),$$

such that $|\psi\rangle$ varies over pure states on $\mathcal{H}_1 \otimes \mathcal{H}_R$ with energy constraint $\langle\psi| H_1 |\psi\rangle \leq E$.

Without loss of generality, we take the infimum over all pure states $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_R$ with $\mathcal{H}_R \simeq \mathcal{H}_1$ [77].

We use the following two inequalities from Ref. [77]:

$$F_{cb}^{E}(\mathcal{A},\mathcal{B}) \geq 1 - \frac{1}{2}\|\mathcal{A}-\mathcal{B}\|_{\diamond}^{E} \tag{7.2.1}$$

and

$$\|\mathcal{A}-\mathcal{B}\|_{\diamond}^{E} \leq 2\sqrt{1-F_{cb}^{E}(\mathcal{A},\mathcal{B})^{2}}. \tag{7.2.2}$$

Since we operate on an ℓ-partite system, we define the *multiply energy-constrained diamond norm*.

Definition 7.2.2 (Multiply energy-constrained diamond norm [2])**.** *Let $\mathcal{H}^{(\ell)} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_\ell$, where each $\mathcal{H}_j$ carries its own Hamiltonian $H_j \geq 0$. For $E_j > 0$, we define the* multiply energy-constrained diamond norm *(more precisly, $(E_1,\ldots,E_\ell)$-*constrained diamond norm*) for a self-adjointness- and trace-preserving $\Phi : \mathcal{T}(\mathcal{H}^{(\ell)}) \to \mathcal{T}(\mathcal{H}')$ as*

$$\|\Phi\|_{\diamond}^{(E_1,\ldots,E_\ell)} := \sup_{\substack{\rho\in\mathcal{D}(\mathcal{H}^{(\ell)}\otimes\mathcal{H}_R)\\ \operatorname{tr}\rho_j H_j\leq E_j\\ \forall\ j=1,\ldots,\ell}} \|(\Phi\otimes\mathrm{id}_R)(\rho)\|_1$$

with $\rho_j \in \mathcal{D}(\mathcal{H}_j)$.

Remark 7.2.3. *Note that this norm is in fact equivalent to the E-constrained diamond norm: concretely, with $E_{\min} = \min_j E_j$ and $E_{sum} = \sum_j E_j$,*

$$\|\Phi\|_{\diamond}^{E_{\min}} \leq \|\Phi\|_{\diamond}^{(E_1,\ldots,E_\ell)} \leq \|\Phi\|_{\diamond}^{E_{sum}},$$

where we use $H = \sum_j H_j$ as the Hamiltonian on $\mathcal{H}^{(\ell)}$.

We now state and prove the lemma which is crucial for the information-theoretic lower bounds on the program dimension of a programmable quantum processor implementing energy-limited Gaussian quantum channels.

Lemma 7.2.4 ([2, Lemma 11])**.** *Consider a processor, i.e., a quantum channel $\mathcal{P} \in CPTP(\mathcal{H}\otimes\mathcal{H}_P,\mathcal{H})$, coming with a Hamiltonian $H \geq 0$, a number $E > 0$, and an integer $\ell \geq 1$. Then, there exists another processor $\widehat{\mathcal{P}} \in CPTP(\mathcal{H}^{\otimes\ell}\otimes\mathcal{H}_P,\mathcal{H}^{\otimes\ell})$ with the following property: for every unitary channel $\mathcal{U}(\cdot) = U(\cdot)U^*$ whose inverse $\mathcal{U}^*$ is (α',β')-energy-limited and such that there exists a pure state $|\psi_U\rangle$ on $\mathcal{H}_P$ with*

$$\frac{1}{2}\|\mathcal{P}(\cdot\otimes\psi_U)-\mathcal{U}\|_{\diamond}^{E} \leq \epsilon,$$

it holds that

$$\frac{1}{2}\left\|\widehat{\mathcal{P}}(\cdot\otimes\psi_U)-\mathcal{U}^{\otimes\ell}\right\|_{\diamond}^{(E,\ldots,E)} \leq 2\ell\epsilon',$$

where $\epsilon' = \left(1+\frac{\beta'}{E}\right)\sqrt{2\epsilon}$.

In words, whenever $\mathcal{P}$ ϵ-implements an (α,β)-energy-limited unitary channel $\mathcal{U}(\cdot) = U(\cdot)U^*$ with (α',β')-energy-limited inverse $\mathcal{U}^*$ (with respect to the E-constrained diamond norm), using a pure program state ψ_U, then $\widehat{\mathcal{P}}$ $2\ell\epsilon'$-implements $\mathcal{U}^{\otimes\ell}$ with respect to the $(E,\ldots,E)$-constrained diamond norm, using the same pure program state ψ_U.

Proof. Applying Eq. (7.2.1), we obtain for the energy-constrained completely bounded fidelity between the output of the processor and the target unitary,

$$F_{cb}^{E}\big(\mathcal{P}(\cdot\otimes\psi_P),\mathcal{U}\big)\geq 1-\epsilon.$$

Throughout the proof, we observe the convention to mark states for clarity with the index of the subsystem on which they act; similarly for channels, whenever it is not clear from the setting. Let $\mathcal{V}:\mathcal{B}(\mathcal{H})\otimes\mathcal{B}(\mathcal{H}_P)\to\mathcal{B}(\mathcal{H})\otimes\mathcal{B}(\mathcal{H}_Q)$ be a Stinespring dilation [78, Chapter 9] of $\mathcal{P}$, with $\mathcal{H}_Q$ being a suitable environment space.

Then, by Uhlmann's theorem for the completely bounded energy-constrained fidelity of quantum channels, stating that any two channels have isometric dilations with the same completely bounded energy-constrained fidelity [77, Prop. 1], there exists a state $|\phi_Q\rangle\in\mathcal{H}_Q$ such that

$$F_{cb}^{E}\big(\mathcal{V}\circ(\mathrm{id}_{\mathcal{H}}\otimes\psi_P),\mathcal{U}\otimes\phi_Q\big)\geq 1-\epsilon.$$

Note that here and in the following, we regard a state $\rho\in\mathcal{D}(\mathcal{H}_P)$ as a channel from a trivial system, represented by 1 (with one-dimensional Hilbert space $\mathbb{C}$), to $\mathcal{H}_P$. Using Eq. (7.2.2), we get

$$\frac{1}{2}\left\|\mathcal{V}\circ(\mathrm{id}_{\mathcal{H}}\otimes\psi_P)-\mathcal{U}\otimes\phi_Q\right\|_{\diamond}^{E}\leq\sqrt{1-(1-\epsilon)^2}\leq\sqrt{2\epsilon}. \tag{7.2.3}$$

We want to apply the processor several times, and for this we have to recover the program state. So, the first step of the proof is to show that we can modify the processor to a new map $\mathcal{P}'\in\mathrm{CPTP}(\mathcal{H}\otimes\mathcal{H}_P,\mathcal{H}\otimes\mathcal{H}_P)$ in such a way that apart from implementing $\mathcal{U}$, it also preserves the program state ψ_P (all with the appropriate approximations). For this purpose, choose a pseudoinverse of $\mathcal{V}$,

$$\mathcal{W}:=\mathcal{V}^*+\mathcal{R}=V^*(\cdot)V+\mathcal{R},$$

where $\mathcal{R}$ is a CP map designed to make $\mathcal{W}$ CPTP. Note that this is always possible because VV^* is a projection, and so $\mathcal{V}^*$ is CP and trace non-increasing. In fact, denoting $\Pi:=VV^*$ the projection operator onto the image of V in $\mathcal{H}\otimes\mathcal{H}_Q$, one choice is $\mathcal{R}(\xi)=\rho_0\,\mathrm{tr}\,\xi(\mathbb{1}-\Pi)$, with a fixed state $\rho_0\in\mathcal{D}(\mathcal{H}\otimes\mathcal{H}_P)$. We observe that indeed $\mathcal{W}\circ\mathcal{V}=\mathrm{id}_{\mathcal{H}}\otimes\mathrm{id}_{\mathcal{H}_P}$, and hence $(\mathcal{V}\circ\mathcal{W})\circ\mathcal{V}=\mathcal{V}$. Thus, we get from Eq. (7.2.3) that

$$\begin{aligned}\sqrt{2\epsilon}&\geq\frac{1}{2}\left\|\mathcal{V}\circ(\mathrm{id}_{\mathcal{H}}\otimes\psi_P)-\mathcal{V}\circ\mathcal{W}\circ(\mathcal{U}\otimes\phi_Q)\right\|_{\diamond}^{E}\\&=\frac{1}{2}\left\|\mathrm{id}_{\mathcal{H}}\otimes\psi_P-\mathcal{W}\circ(\mathcal{U}\otimes\phi_Q)\right\|_{\diamond}^{E}\\&\geq\frac{1}{2}\frac{E}{E+\beta'}\left\|\mathrm{id}_{\mathcal{H}}\otimes\psi_P-\mathcal{W}\circ(\mathcal{U}\otimes\phi_Q)\right\|_{\diamond}^{E+\beta'},\end{aligned}$$

where we obtain the second line by the invariance of the (E-constrained) diamond norm under multiplication from the left by an isometric channel ($\mathcal{V}$), and the last line by the equivalence of the energy-constrained diamond norms for different energy levels (see Eq. (6.1.2) and Ref. [68]).

Since $\mathcal{U}^*$ is (α', β')-energy-limited, we can lower bound the last expression in turn by

$$\frac{1}{2}\frac{E}{E+\beta'}\left\|\mathcal{U}^* \otimes \psi_P - \mathcal{W} \circ (\mathrm{id}_{\mathcal{H}} \otimes \phi_Q)\right\|_\diamond^{E/\alpha'}.$$

Defining the memory recovery map $\mathcal{M} : \mathcal{T}(\mathcal{H}_Q) \to \mathcal{T}(\mathcal{H}_P)$ by

$$\mathcal{M}(\rho) := \mathrm{tr}_{\mathcal{H}}\, \mathcal{W}(|0\rangle\langle 0|_{\mathcal{H}} \otimes \rho),$$

where $|0\rangle\langle 0|$ is a ground state (i.e., of zero energy) of the Hamiltonian H, we can now conclude that

$$\frac{1}{2}\|\mathcal{M}(\phi_Q) - \psi_P\|_1 \leq \left(1 + \frac{\beta'}{E}\right)\sqrt{2\epsilon} =: \epsilon'.$$

Note that this map depends only on the chosen Stinespring dilation $\mathcal{V}$ of $\mathcal{P}$, and thus we can define

$$\mathcal{P}' := (\mathrm{id}_{\mathcal{H}} \otimes \mathcal{M}) \circ \mathcal{V},$$

which, by the above reasoning, has the desired property that

$$\frac{1}{2}\|\mathcal{P}' \circ (\mathrm{id}_{\mathcal{H}} \otimes \psi_P) - \mathcal{U} \otimes \psi_P\|_\diamond^E \leq 2\epsilon', \tag{7.2.4}$$

via a simple application of the triangle inequality and the contractivity of the (E-constrained) diamond norm under multiplication from the left by CPTP maps.

The remaining argument requires careful notation. We have isomorphic copies $\mathcal{H}_j$ of $\mathcal{H}$, $j = 1, \ldots, \ell$, as well as the program register $\mathcal{H}_P$ in the big tensor-product space $\mathcal{H}^{(\ell)} \otimes \mathcal{H}_P$, and we use index j or P to indicate on which tensor factor a superoperator acts. We implicitly extend the action to the whole space by tensoring with the identity id on the other factors. This allows us to rewrite Eq. (7.2.4) as

$$\frac{1}{2}\|\mathcal{P}'_{jP} \circ (\mathrm{id}_j \otimes \psi_P) - \mathcal{U}_j \otimes \psi_P\|_\diamond^E \leq 2\epsilon',$$

for all $j = 1, \ldots, \ell$. By tensoring this with $\mathcal{U}_k$ (for $k < j$) and with id_k ($k > j$), and observing the definition of the multiply energy-constrained diamond norm (see Definition 7.2.2), this results in

$$\begin{aligned}\frac{1}{2}\Big\|\mathcal{P}'_{jP} \circ (\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_{j-1} \otimes \mathrm{id}_j \otimes \mathrm{id}_{j+1} \otimes \cdots \otimes \mathrm{id}_\ell \otimes \psi_P) \\ - \mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_{j-1} \otimes \mathcal{U}_j \otimes \mathrm{id}_{j+1} \otimes \cdots \otimes \mathrm{id}_\ell \otimes \psi_P\Big\|_\diamond^{(E,\ldots,E)} \leq 2\epsilon',\end{aligned} \tag{7.2.5}$$

for $j = 1, \ldots, \ell$. Note that we only impose energy constraints on the nontrivial systems $\mathcal{H}_j$, both in the simply and multiply constrained norm expressions.

Adding all the ℓ bounds to Eq. (7.2.5), and using the triangle inequality results in

$$\begin{aligned}\frac{1}{2}\Big\|\mathcal{P}'_{\ell P} \circ \cdots \circ \mathcal{P}'_{2P} \circ \mathcal{P}'_{1P} \circ (\mathrm{id}_1 \otimes \cdots \otimes \mathrm{id}_\ell \otimes \psi_P) \\ - \mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_\ell \otimes \psi_P\Big\|_\diamond^{(E,\ldots,E)} \leq 2\ell\epsilon'.\end{aligned}$$

This means that we can define our desired EPQP via $\widehat{\mathcal{P}} := \mathrm{tr}_P \circ \mathcal{P}'_{\ell P} \circ \cdots \circ \mathcal{P}'_{2P} \circ \mathcal{P}'_{1P}$, concluding the proof. □

7.3 Implementation of energy-limited unitary channels

In the following, we investigate bounds on the program dimension, the processor requires, to approximately implement all (α, β)-energy-limited unitary channels $\mathcal{U}(\alpha, \beta)$.

7.3.1 Upper bounds on the program dimension

To obtain upper bounds, we present a construction method which is based on an existing ϵ-PQP$_{\mathcal{U}}$ and can be seen as an expansion of a finite-dimensional programmable quantum processor to infinite dimension.

We construct a processor that maps any input state $\rho \in \mathcal{D}(\mathcal{H})$ with a certain energy

$$\operatorname{tr} \rho H \leq E$$

to $U\rho U^*$ with $U \in \mathcal{U}(\mathcal{H})$ and

$$\operatorname{tr} U\rho U^* H \leq \alpha E + \beta$$

approximately, using a program register $\mathcal{H}_P$ of dimension $d_P^\infty < \infty$. We express the approximation parameter γ of our γ-EPQP$_{\mathcal{U}(\alpha,\beta)}$ in terms of the approximation parameter ϵ of a finite-dimensional ϵ-PQP$_{\mathcal{U}}$. This is schematically illustrated in Figure 7.1.

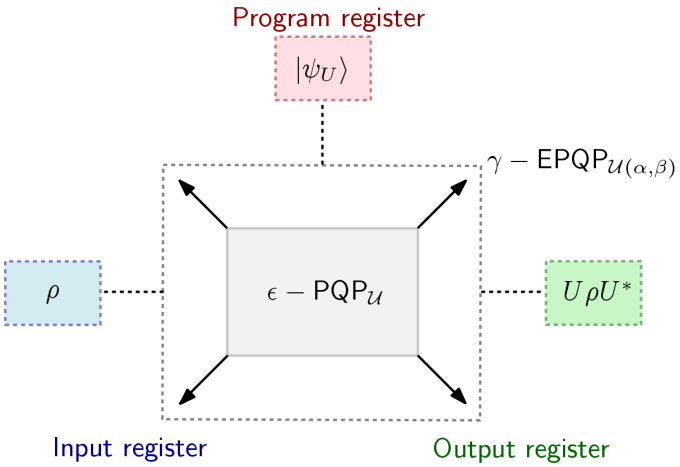

Figure 7.1: This is a schematic illustration of the construction method for an γ-EPQP$_{\mathcal{U}(\alpha,\beta)}$ based on an ϵ-PQP$_{\mathcal{U}}$. The parts of the figure that we construct in the proof are represented by dashed lines, the parts we assume to exist by regular lines [2, Figure 1].

We use the energy constraints to approximate the input and output system by a finite-dimensional subspace of $\mathcal{H}$.

Theorem 7.3.1 ([2, Theorem 9])**.** *Let H be a positive semidefinite Hamiltonian $H \geq 0$ on the separable Hilbert space $\mathcal{H}$ with discrete spectrum, finite degeneracy of all eigenvalues, smallest eigenvalue 0, and $E > 0$. Furthermore, let $\epsilon > 0$ and $d := \operatorname{rank}\{H \leq E/\epsilon^4\}$, the dimension of the subspace of $\mathcal{H}$ spanned by eigenvectors of H with eigenvalues $\leq \frac{E}{\epsilon^4}$. Assume that we have an ϵ-PQP$_{\mathcal{U}}$ $\mathcal{P}_d$ with d-dimensional input register and program register $\mathcal{H}_P$. Then, we can construct an infinite-dimensional γ-EPQP$_{\mathcal{U}(\alpha,\beta)}$ $\mathcal{P} \in CPTP(\mathcal{H} \otimes \mathcal{H}_P, \mathcal{H})$ such that for all (α, β)-energy-limited unitaries $U \in \mathcal{U}(\mathcal{H})$ there exists a unit vector $|\psi_U\rangle \in \mathcal{H}_P$ such that*

$$\frac{1}{2}\|\mathcal{P}(\cdot \otimes |\psi_U\rangle\langle\psi_U|) - U(\cdot)U^*\|_\diamond^E \leq \gamma := 4.5\epsilon\left(\alpha + \frac{\beta}{E}\right). \tag{7.3.1.1}$$

Proof. The construction of the infinite-dimensional processor consists of two components: a compression map that projects down to states on a finite-dimensional subspace $\mathcal{H}_d$ of $\mathcal{H}$ spanned by the lowest-lying energy eigenstates, and the application of the given finite-dimensional ϵ-PQP$_{\mathcal{U}}$ $\mathcal{P}_d$ to that subspace. Define P_δ, the projector onto the subspace $\mathcal{H}_\delta$ spanned by all eigenstates with eigenvalue $\leq E/\delta$, where $\delta \leq 1$. Consider $\mathcal{H}_{\delta^2}$, which has projector P_{δ^2} and define the compression map $\mathcal{K}$ onto $\mathcal{H}_\delta \subset \mathcal{H}_{\delta^2}$ as $\mathcal{K}(\rho) := P_\delta \rho P_\delta + \operatorname{tr}\rho(\mathbb{1} - P_\delta)|0\rangle\langle 0|$, where $|0\rangle$ is a ground state of the Hamiltonian H. Now we can define our infinite-dimensional processor as $\mathcal{P} = \mathcal{P}_d \circ (\mathcal{K} \otimes \mathrm{id}_{\mathcal{H}_P})$. Next, we need to describe how to use the processor $\mathcal{P}$ to implement an (α, β)-energy-limited unitary $U \in \mathcal{U}(\mathcal{H})$, namely what is the program state $|\psi_U\rangle$. To do this, consider the polar decomposition of $P_{\delta^2}UP_\delta$, which we can think of as an operator acting on $\mathcal{H}_\delta$, mapping to $\mathcal{H}_{\delta^2}$:

$$P_{\delta^2}UP_\delta = V_d\sqrt{P_\delta U^* P_{\delta^2} U P_\delta}, \tag{7.3.1.2}$$

where $V_d : \mathcal{H}_\delta \to \mathcal{H}_{\delta^2}$ is consequently an isometry. We obtain U_d as an extension of V_d to a unitary on $\mathcal{H}_d := \mathcal{H}_{\delta^2}$. By assumption, $\mathcal{P}_d$ can implement U_d approximately with error $\leq \epsilon$ in diamond norm, using a certain program state $|\phi_{U_d}\rangle$, and we let $|\psi_U\rangle := |\phi_{U_d}\rangle$. The rest of the proof is the demonstration that this construction satisfies the claimed approximation quality in E-constrained diamond norm. To start with, we show that

$$\|V_d - P_{\delta^2}UP_\delta\| \leq \delta\left(\alpha + \frac{\beta}{E}\right), \tag{7.3.1.3}$$

both operators in the difference being understood as operators on $\mathcal{H}_d$, and $\|\cdot\|$ denoting the operator norm. For this, thanks to Eq. (7.3.1.2) it is enough to show

$$\left(1 - \delta\left(\alpha + \frac{\beta}{E}\right)\right)P_\delta \leq P_\delta U^* P_{\delta^2} U P_\delta \leq P_\delta, \tag{7.3.1.4}$$

since the square root is operator monotonic, and thus implies

$$\left\|P_\delta - \sqrt{P_\delta U^* P_{\delta^2} U P_\delta}\right\| \leq \delta\left(\alpha + \frac{\beta}{E}\right).$$

The right-hand inequality in Eq. (7.3.1.4) follows trivially from $P_{\delta^2} \leq \mathbb{1}$ by conjugation with U^* and P_δ. The left-hand inequality amounts to showing $\langle\psi| U^* P_{\delta^2} U |\psi\rangle \geq 1 - \delta\left(\alpha + \frac{\beta}{E}\right)$ for all state vectors $|\psi\rangle \in \mathcal{H}_\delta$. Indeed, $|\psi\rangle\langle\psi|$ is supported on the subspace $\mathcal{H}_\delta$, all of whose state vectors have energy $\leq E/\delta$, in particular $\operatorname{tr}\left(|\psi\rangle\langle\psi| H\right) \leq E/\delta$. By our assumption that U is (α, β)-energy-limited, this implies $\operatorname{tr}\left(U|\psi\rangle\langle\psi|U^* H\right) \leq \alpha E/\delta + \beta$, and thus by Markov's inequality we get

$$\operatorname{tr}\left(U|\psi\rangle\langle\psi|U^*(\mathbb{1} - P_{\delta^2})\right) \leq \frac{\alpha E/\delta + \beta}{E/\delta^2} \leq \delta\left(\alpha + \frac{\beta}{E}\right),$$

proving the claim. To prove the bound (7.3.1.1), we consider an arbitrary state $\rho \in \mathcal{D}(\mathcal{H} \otimes \mathbb{C}^k)$ with energy bounded by E, i.e., $\operatorname{tr}\rho(H \otimes \mathbb{1}_{\mathbb{C}^k}) \leq E$. To start, by Markov's inequality this implies $\operatorname{tr}\rho(P_\delta \otimes \mathbb{1}) \geq 1 - \delta$, thus by the Gentle Operator Lemma 2.1.5, and the triangle inequality,

$$\left\|\rho - (\mathcal{K} \otimes \mathrm{id}_{\mathbb{C}^k})(\rho)\right\|_1 \leq 2\sqrt{\delta} + \delta, \tag{7.3.1.5}$$

and furthermore $(\mathcal{K} \otimes \mathrm{id}_{\mathbb{C}^k})(\rho)$ is a state on $\mathcal{H}_d \otimes \mathbb{C}^k$ that has energy bounded by $\operatorname{tr}(\mathcal{K} \otimes \mathrm{id}_{\mathbb{C}^k})(\rho)(H \otimes \mathbb{1}) \leq \operatorname{tr}\rho(H \otimes \mathbb{1}) \leq E$. Now, from the definition of $\mathcal{P}$ and the processor property of $\mathcal{P}_d$, we have

$$\|(\mathcal{P} \otimes \mathrm{id}_{\mathbb{C}^k})(\rho \otimes \psi_U) - (U_d \otimes \mathbb{1}_{\mathbb{C}^k})(\mathcal{K} \otimes \mathrm{id}_{\mathbb{C}^k})(\rho)(U_d \otimes \mathbb{1}_{\mathbb{C}^k})^*\|_1 \leq 2\epsilon. \tag{7.3.1.6}$$

Noting that $(U_d \otimes \mathbb{1}_{\mathbb{C}^k})(\mathcal{K} \otimes \mathrm{id}_{\mathbb{C}^k})(\rho)(U_d \otimes \mathbb{1}_{\mathbb{C}^k})^* = (V_d \otimes \mathbb{1}_{\mathbb{C}^k})(\mathcal{K} \otimes \mathrm{id}_{\mathbb{C}^k})(\rho)(V_d \otimes \mathbb{1}_{\mathbb{C}^k})^*$, because $(\mathcal{K} \otimes \mathrm{id}_{\mathbb{C}^k})(\rho)$ is supported on $\mathcal{H}_\delta \otimes \mathbb{C}^k$, we furthermore have

$$\begin{aligned}\|&(V_d \otimes \mathbb{1}_{\mathbb{C}^k})(\mathcal{K} \otimes \mathrm{id}_{\mathbb{C}^k})(\rho)(V_d \otimes \mathbb{1}_{\mathbb{C}^k})^* \\ &- (P_{\delta^2} U P_\delta \otimes \mathbb{1}_{\mathbb{C}^k})(\mathcal{K} \otimes \mathrm{id}_{\mathbb{C}^k})(\rho)(P_\delta U^* P_{\delta^2} \otimes \mathbb{1}_{\mathbb{C}^k})\|_1 \leq 2\delta\left(\alpha + \frac{\beta}{E}\right),\end{aligned} \tag{7.3.1.7}$$

where we invoke Eq. (7.3.1.3) twice. Continuing, we observe that we can drop the projection P_δ in the second term inside the norm, because $(\mathcal{K} \otimes \mathrm{id}_{\mathbb{C}^k})(\rho)$ is supported on $\mathcal{H}_\delta \otimes \mathbb{C}^k$. Next, by Eq. (7.3.1.5) we have

$$\begin{aligned}\|&(P_{\delta^2} U \otimes \mathbb{1}_{\mathbb{C}^k})(\mathcal{K} \otimes \mathrm{id}_{\mathbb{C}^k})(\rho)(U^* P_{\delta^2} \otimes \mathbb{1}_{\mathbb{C}^k}) \\ &- (P_{\delta^2} U \otimes \mathbb{1}_{\mathbb{C}^k})\rho(U^* P_{\delta^2} \otimes \mathbb{1}_{\mathbb{C}^k})\|_1 \leq 2\sqrt{\delta} + \delta.\end{aligned} \tag{7.3.1.8}$$

Finally, since $\operatorname{tr}\left((U \otimes \mathbb{1}_{\mathbb{C}^k})\rho(U \otimes \mathbb{1}_{\mathbb{C}^k})^* H\right) \leq \alpha E + \beta$, another application of Markov's inequality and the Gentle Operator Lemma 2.1.5 yields

$$\|(P_{\delta^2} U \otimes \mathbb{1}_{\mathbb{C}^k})\rho(U^* P_{\delta^2} \otimes \mathbb{1}_{\mathbb{C}^k}) - (U \otimes \mathbb{1}_{\mathbb{C}^k})\rho(U^* \otimes \mathbb{1}_{\mathbb{C}^k})\|_1 \leq 2\delta\sqrt{\alpha + \frac{\beta}{E}}. \tag{7.3.1.9}$$

It remains to put everything together: by the triangle inequality and the bounds from Eqs. (7.3.1.6), (7.3.1.7), (7.3.1.8) and (7.3.1.9), we obtain

$$\begin{aligned}\left\|(\mathcal{P} \otimes \mathrm{id}_{\mathbb{C}^k})(\rho \otimes \psi_U) - (U \otimes \mathbb{1}_{\mathbb{C}^k})\rho(U^* \otimes \mathbb{1}_{\mathbb{C}^k})\right\|_1 & \\ \leq 2\epsilon + 2\delta\left(\alpha + \frac{\beta}{E}\right) + 2\sqrt{\delta} + \delta + 2\delta\sqrt{\alpha + \frac{\beta}{E}} & \\ \leq 2\epsilon + 7\sqrt{\delta}\left(\alpha + \frac{\beta}{E}\right), &\end{aligned}$$

and choosing $\delta = \epsilon^2$ concludes the proof. □

With this construction, we can now import existing upper bounds for a finite-dimensional ϵ-PQP$_{\mathcal{U}}$ to provide an upper bound on the dimension of the program register of the infinite-dimensional γ-EPQP$_{\mathcal{U}(\alpha,\beta)}$. Upper bounds for the program dimension of a finite-dimensional ϵ-PQP$_{\mathcal{U}}$ were derived in previous works (see Section 4.2), most recently in Refs. [14] and [15].

Let $\mathcal{P}$ be an infinite-dimensional γ-EPQP$_{\mathcal{U}(\alpha,\beta)}$ and $\mathcal{P}_d$ a finite-dimensional ϵ-PQP$_{\mathcal{U}}$. Since our construction of an infinite-dimensional processor relies on a finite-dimensional one, we reformulate Eq. (7.3.1.1) as

$$\epsilon := \frac{\gamma}{4.5}\left(\alpha + \frac{\beta}{E}\right)^{-1}.$$

Table 7.1 gives an overview of existing bounds on the program dimension for finite-dimensional unitary processors in the literature. The bound in the second row of Table 7.1 was derived from Ref. [12, Lemma 1, Section II.C] which uses port-based teleportation working with copies of Choi-states. We get upper bounds for the infinite-dimensional γ-EPQP$_{\mathcal{U}(\alpha\beta)}$ $\mathcal{P}$ if we insert ϵ into the existing bounds which yields the third column in Table 7.1.

$d_P \leq$	References	$d_P^\infty \leq$
$d^{\frac{4d^2}{\epsilon^2}}$	Ishizaka & Hiroshima [48], Beigi & König [50], Christandl *et al.* [49]	$d^{\frac{81d^2(\alpha+\frac{\beta}{E})^2}{\gamma^2}}$
$d^{\frac{2d^2}{\epsilon}}$	Pirandola *et al.* [12]	$d^{\frac{9d^2(\alpha+\beta/E)}{\gamma}}$
$\left(\frac{\tilde{C}}{\epsilon}\right)^{d^2}$	Kubicki *et al.* [14]	$\left(\frac{4.5\tilde{C}(\alpha+\frac{\beta}{E})}{\gamma}\right)^{d^2}$
$\left(\frac{\Theta(d^2)}{\epsilon}\right)^{\frac{d^2-1}{2}}$	Yang *et al.* [15]	$\left(\frac{4.5\Theta(d^2)(\alpha+\frac{\beta}{E})}{\gamma}\right)^{\frac{d^2-1}{2}}$

Table 7.1: Upper bounds on the program dimension of an infinite-dimensional processor implementing unitary channels. The first column shows the upper bounds on the program dimension for the finite-dimensional case with the corresponding references in the second column. The third column presents the upper bounds on the program dimension for the infinite-dimensional processor. Note that d in the third column is the dimension of the subspace of $\mathcal{H}$ spanned by eigenvectors of H with eigenvalues $\leq \frac{E}{\epsilon^4}$ (see also [2, Table I]).

7.3.2 Lower bounds on the program dimension

Having provided upper bounds on the program dimension of an ϵ-EPQP$_{\mathcal{U}(\alpha,\beta)}$, we investigate lower bounds on the dimension of the program register by first presenting a method to obtain a finite-dimensional processor assuming an existing infinite-dimensional one in the following theorem which is illustrated in Figure 7.2.

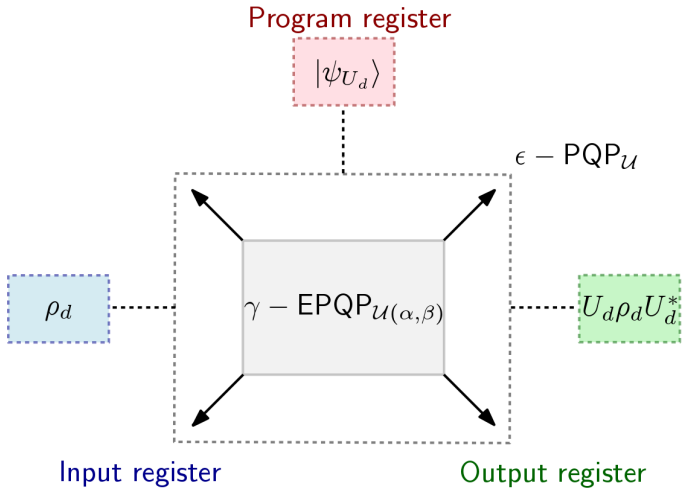

Figure 7.2: Assuming a γ-EPQP$_{\mathcal{U}(\alpha,\beta)}$, we can construct a finite-dimensional ϵ-PQP$_{\mathcal{U}}$ which is drawn in dashed lines including its input, output and program register [2, Figure 2].

Theorem 7.3.2 ([2, Theorem 10])**.** *Let $H \geq 0$ be the Hamiltonian with discrete spectrum describing the system on the separable Hilbert space $\mathcal{H}$, $E > 0$, $\gamma > 0$, and furthermore choose $d > 0$. Assume that we have an infinite-dimensional γ-EPQP$_{\mathcal{U}(\alpha,\beta)}$ $\mathcal{P} \in CPTP(\mathcal{H} \otimes \mathcal{H}_P, \mathcal{H})$ for all sufficiently large α and β. Then, there exists an ϵ-PQP$_{\mathcal{U}}$ $\mathcal{P}_d \in CPTP(\mathcal{H}_d \otimes \mathcal{H}_P, \mathcal{H}_d)$ such that for all $U_d \in \mathcal{U}(\mathcal{H}_d)$ there is a unit vector $|\psi_{U_d}\rangle \in \mathcal{H}_P$ with*

$$\frac{1}{2}\left\|\mathcal{P}_d(\cdot \otimes |\psi_{U_d}\rangle\langle\psi_{U_d}|) - U_d(\cdot)U_d^*\right\|_\diamond \leq \epsilon,$$

where $\epsilon = \gamma\frac{1}{E}\max\{E(d), E\}$ and $E(d)$ is the smallest energy such that the space spanned by the eigenstates of energies between 0 and $E(d)$ is of dimension d or larger.

Proof. We assume that an infinite-dimensional γ-EPQP$_{\mathcal{U}(\alpha,\beta)}$ processor $\mathcal{P} \in CPTP(\mathcal{H} \otimes \mathcal{H}_P, \mathcal{H})$ exists as described in the theorem, and construct a finite-dimensional one with the same program register, i.e., we aim to bound

$$\frac{1}{2}\left\|\mathcal{P}_d(\cdot \otimes |\psi_{U_d}\rangle\langle\psi_{U_d}|) - U_d(\cdot)U_d^*\right\|_\diamond \leq \epsilon := \epsilon(E, d, \gamma).$$

We start with fixing an isometric embedding V of the d-dimensional Hilbert space $\mathcal{H}_d$ into $\mathcal{H}$. Namely, with respect to the ordered spectral decomposition $H =$

$\sum_{n=0}^{\infty} e_n P_n$ of the Hamiltonian, let $n(d)$ be the smallest integer such that $d \leq \sum_{n=0}^{n(d)} \operatorname{rank} P_n$ and $E(d) := e_{n(d)}$ the largest occurring energy. Let

$$V : \mathcal{H}_d \hookrightarrow \left(\sum_{n_0}^{n(d)} P_n \right) \mathcal{H} =: \mathcal{H}' \subset \mathcal{H},$$

where the first embedding is an arbitrary isometry. This defines an isometric channel

$$\begin{aligned} \mathcal{V} : \mathcal{D}(\mathcal{H}_d) &\to \mathcal{D}(\mathcal{H}) \\ \rho &\mapsto V \rho V^*. \end{aligned}$$

Thanks to V we view $\mathcal{H}_d$ as a subspace of $\mathcal{H}$, and denote its orthogonal complement $\mathcal{H}_d^{\perp}$.

An arbitrary $U_d \in \mathcal{U}(\mathcal{H}_d)$ is extended to a unitary $U \equiv U_d \oplus \mathbb{1}_{\mathcal{H}_d^{\perp}} \in \mathcal{U}(\mathcal{H})$. We would like to implement this unitary using the processor $\mathcal{P}$, followed by a CPTP compression onto the subspace $\mathcal{H}_d$, using its projection operator Π_d:

$$\mathcal{K}(\rho) := \Pi_d \rho \Pi_d + \kappa_0 \operatorname{tr} \rho(\mathbb{1} - \Pi_d),$$

where κ_0 is an arbitrary state with energy zero in $\mathcal{H}_d$. Since we assume that there exists a γ-EPQP$_{\mathcal{U}(\alpha,\beta)}$ for all sufficiently large $\alpha > 0$ and $\beta > 0$, this unitary can be γ-implemented. (In fact, we could choose $\alpha = 1$ and $\beta = E(d)$, the largest occurring energy gap in $\mathcal{H}'$.) The processor is a concatenation of the isometric channel $\mathcal{V}$, the infinite-dimensional γ-EPQP$_{\mathcal{U}(\alpha,\beta)}$ $\mathcal{P}$ and the compression map $\mathcal{K}$, namely $\mathcal{P}' := \mathcal{K} \circ \mathcal{P} \circ (\mathcal{V} \otimes \mathrm{id}_{\mathcal{H}_P})$, which leads us to

$$\begin{aligned} \frac{1}{2} \| \mathcal{K} \circ \mathcal{P} \circ (\mathcal{V}(\cdot) \otimes \psi_{U_d}) - U_d(\cdot) U_d^* \|_{\diamond} & \\ &\leq \frac{1}{2} \| \mathcal{P} \circ (\mathcal{V}(\cdot) \otimes \psi_{U_d}) - U(\cdot) U^* \|_{\diamond}^{E(d)} \\ &\leq \frac{1}{E} \max\{E(d), E\} \frac{1}{2} \| \mathcal{P} \circ (\mathcal{V}(\cdot) \otimes \psi_{U_d}) - U(\cdot) U^* \|_{\diamond}^{E} \\ &\leq \gamma \frac{1}{E} \max\{E(d), E\}, \end{aligned}$$

where we use the contractivity of the diamond norm under postprocessing and that our subspace goes up to energy $E(d)$, so restricted to it the $E(d)$-constrained diamond norm equals the unconstrained diamond norm; then, that going to the E-constrained diamond norm blows up the error by a factor of at most $\frac{1}{E} \max\{E(d), E\}$. Finally, the infinite-dimensional processor makes an error of at most γ. Hence, this is the $\epsilon = \gamma \frac{1}{E} \max\{E(d), E\}$ we get for the resulting finite-dimensional processor. □

Thus, given an ϵ-EPQP$_{\mathcal{U}(\alpha,\beta)}$ for infinite-dimensional (α, β)-energy-limited unitaries, we can build a finite-dimensional ϵ-PQP$_{\mathcal{U}}$, whose program dimension is lower bounded through results from the literature, as shown in Table 7.2.

$d_P \geq$	References	$d_P^\infty \geq$
$\Theta\left(\left(\frac{1}{d}\right)^{\frac{d+1}{2}}\left(\frac{1}{\epsilon}\right)^{\frac{d-1}{2}}\right)$	Pérez-García [10]	$\Theta\left(\left(\frac{1}{d}\right)^{\frac{d+1}{2}}\left(\frac{E}{\gamma \max\{E(d), E\}}\right)^{\frac{d-1}{2}}\right)$
$\Theta\left(\left(\frac{d}{\epsilon}\right)^2\right)$	Majenz [51]	$\Theta\left(\left(\frac{dE}{\gamma \max\{E(d), E\}}\right)^2\right)$
$2^{\frac{1-\epsilon}{3C}d - \frac{2}{3}\log d}$	Kubicki *et al.* [14]	$2^{\frac{E-\gamma \max\{E(d),E\}}{3CE}d - \frac{2}{3}\log d}$
$\left(1 + \frac{\Theta(d^{-2})}{\sqrt{\epsilon}}\right)^{2\alpha}$	Yang *et al.* [15], with a slight arithmetic improvement	$\left(1 + \frac{\Theta(Ed^{-2})}{\sqrt{\gamma}\max\{E(d), E\}}\right)^{2\alpha}$

Table 7.2: Lower bounds on the program dimension of an infinite-dimensional processor implementing unitary channels. The first column shows the lower bounds on the program dimension for the finite-dimensional case with the corresponding references in the second column. The third column presents the lower bounds on the program dimension for the infinite-dimensional processor. The last row holds for any $\alpha < \frac{d^2-1}{2}$. The dimension d in the third column is the chosen dimension of the finite-dimensional processor (see also [2, Table II]).

7.4 Implementation of energy-limited Gaussian channels

In the previous section, we considered an infinite-dimensional input state with a certain maximal energy E and showed that there is a programmable quantum processor able to implement approximations of all (α, β)-energy-limited unitary channels with finite-dimensional program register. We provided upper and lower bounds on the program dimension of a processor that implements all energy-limited unitary channels for an infinite-dimensional input state up to a certain energy $E > 0$, i.e., $\mathrm{tr}(\rho H) \leq E$. In the following, the Hamiltonian is the photon-number operator $N := a^* a$. We now consider a special class of channels: energy-limited Gaussian quantum channels. As explained in the previous sections, we also assume an energy constraint $\mathrm{tr}(\rho H) \leq E$ on the input. Thus, we already know from Section 7.3 that there is a processor which implements an approximate version of all energy-limited Gaussian channels with finite-dimensional program register. These bounds also apply here.

In particular, we study the implementation of gauge-covariant Gaussian channels that are relevant in quantum optics, for instance, and furthermore, the implementation of energy-limited Gaussian unitary channels. Before considering these two classes of channels, we dedicate the next subsection to the relation of relevant ensembles we need for deriving the lower bounds.

7.4.1 Relating relevant ensembles of quantum states

The methodology for lower bounding the dimension of the program register, expounded in Ref. [15], relies on bounding the Holevo information of ensembles, it does not rely on the unitarity of the target channels, though. We choose a fiducial state $\rho_0 \in \mathcal{D}(\mathcal{H})$ of energy $\leq E$ as input for the processor and the channels $\Phi \in \mathcal{C}$, as well as a probability distribution $\mu(\mathrm{d}\Phi)$ on the class $\mathcal{C}$, so that by Definition 7.1.1 (Eq. (7.1.1)) we have for all $\Phi \in \mathcal{C}$,

$$\frac{1}{2}\|\mathcal{P}(\rho_0 \otimes \pi_\Phi) - \Phi(\rho_0)\|_1 \leq \epsilon.$$

We consider three ensembles of states: an ensemble on the program register $\mathcal{H}_P$, one on the output of the processor, $\omega_\Phi = \mathcal{P}(\rho_0 \otimes \pi_\Phi)$, and the ideal ensemble of a perfect processor:

$$\{\pi_\Phi, \mu(\mathrm{d}\Phi)\} \xrightarrow{\mathcal{P}(\rho_0 \otimes \cdot)} \{\omega_\Phi, \mu(\mathrm{d}\Phi)\} \overset{\epsilon}{\approx} \{\Phi(\rho_0), \mu(\mathrm{d}\Phi)\}.$$

The Holevo information of the ensemble on the left-hand side is upper bounded by $\log d_P^\infty$ (see Eq. (2.2.1)) and lower bounded by the middle one by data processing [23, Subsection 11.9.2]. We would like to apply a continuity bound for the von Neumann entropy to lower bound the Holevo information of the middle ensemble in terms of the Holevo information of the ensemble on the right-hand side. In finite dimension, this is straightforward using the Fannes inequality [79]. However, this is more subtle in infinite dimension, where analogous bounds exist when additionally the states obey an energy bound [67]. Assuming that $\mathcal{C} \subset \mathcal{L}(\alpha, \beta)$, this is indeed given for the states of the ideal ensemble: $\operatorname{tr} \Phi(\rho_0) H \leq \alpha E + \beta$. Since the approximate programmable quantum processor's output is not ideal, we have a priori no such bound for the actual output states ω_Φ. This issue can be solved by processing the middle ensemble further using the following lemma.

Lemma 7.4.1 ([2, Lemma 12])**.** *Consider two states $\rho, \sigma \in \mathcal{D}(\mathcal{H})$, where $\mathcal{H}$ carries a positive semidefinite, self-adjoint densely defined energy operator (Hamiltonian) $H \geq 0$ with discrete spectrum describing a quantum system, 0 being the smallest eigenvalue, and a number $E > 0$. If $\frac{1}{2}\|\rho - \sigma\|_1 \leq \eta$ and $\operatorname{tr} \rho H \leq E$, then there exists a state σ' with $\operatorname{tr} \sigma' H \leq \frac{E}{\eta}$ and*

$$\frac{1}{2}\|\sigma - \sigma'\|_1 \leq 3\sqrt{\eta}, \quad \frac{1}{2}\|\rho - \sigma'\|_1 \leq 4\sqrt{\eta}.$$

Proof. Take the subspace projector P_η onto the energy subspace of all eigenvalues $\leq \frac{E}{\eta}$, and construct the compression map

$$\mathcal{K}(\xi) = P_\eta \xi P_\eta + \kappa_0 \operatorname{tr} \xi (\mathbb{1} - P_\eta),$$

where κ_0 is an arbitrary state with support P_η, e.g., a ground state of H. Then, let $\sigma' := \mathcal{K}(\sigma)$. This does it, as can be seen as follows: $\operatorname{tr} \rho P_\eta \geq 1 - \eta$, hence by the trace norm assumption, $\operatorname{tr} \sigma P_\eta \geq 1 - 2\eta$, and now we can apply the Gentle Operator

Lemma 2.1.5 and get $\frac{1}{2}\|\sigma - P_\eta \sigma P_\eta\|_1 \leq \sqrt{2\eta}$, hence by the triangle inequality $\frac{1}{2}\|\sigma - \sigma'\|_1 \leq 2\eta + \sqrt{2\eta} \leq 3\sqrt{\eta}$.

Using the triangle inequality once more, we get the distance from ρ bounded by $4\sqrt{\eta}$. □

We can thus process the ensemble further, letting $\omega'_\Phi = \mathcal{K}(\omega_\Phi)$ with $\mathcal{K}$ being the compression map from Lemma 7.4.1:

$$\{\pi_\Phi, \mu(\mathrm{d}\Phi)\} \xrightarrow{\mathcal{P}(\rho_0 \otimes \cdot)} \{\omega_\Phi, \mu(\mathrm{d}\Phi)\} \xrightarrow{\mathcal{K}} \{\omega'_\Phi, \mu(\mathrm{d}\Phi)\} \overset{4\sqrt{\epsilon}}{\approx} \{\Phi(\rho_0), \mu(\mathrm{d}\Phi)\}.$$

Now both $\Phi(\rho_0)$ and ω'_Φ have their energy bounded by $\frac{\alpha E + \beta}{\epsilon} =: \widehat{E}$. Assuming not only a positive semidefinite Hamiltonian H_2, but also assuming finite degeneracy of all eigenvalues of its spectral decomposition, with finite Gibbs entropy at all temperatures on the output space $\mathcal{H}_2$, we then get the following chain of inequalities which lower bounds the program dimension:

$$\begin{aligned}
\log d_P^\infty &\geq \chi\big(\{\pi_\Phi, \mu(\mathrm{d}\Phi)\}\big) \\
&\geq \chi\big(\{\omega_\Phi, \mu(\mathrm{d}\Phi)\}\big) \\
&\geq \chi\big(\{\omega'_\Phi, \mu(\mathrm{d}\Phi)\}\big) \\
&\geq \chi\big(\{\Phi(\rho_0), \mu(\mathrm{d}\Phi)\}\big) - 16\sqrt{\epsilon} S\big(\gamma(\widehat{E}/4\sqrt{\epsilon})\big) - 2h(4\sqrt{\epsilon}) \\
&\geq \chi\big(\{\Phi(\rho_0), \mu(\mathrm{d}\Phi)\}\big) - 16\sqrt{\epsilon} S\left(\gamma\left(\frac{\alpha E + \beta}{4\sqrt{\epsilon}^3}\right)\right) - 2,
\end{aligned} \tag{7.4.1.1}$$

where $h(t) = -t \log t - (1-t)\log(1-t)$ is the binary entropy and we assume that $4\sqrt{\epsilon} \leq 1$. The first three inequalities rely on the definition of the program ensemble and the data processing inequality [23, Section 11.9.2] (twice) and the fourth inequality follows from applying Ref. [67, Lemma 15].

7.4.2 Energy-limited gauge-covariant Gaussian channels

We study the class of (α, β)-energy-limited gauge-covariant Gaussian channels $\mathcal{GCG}(\alpha, \beta)$. Recall that the corresponding programmable quantum processor is written as ϵ-EPQP$_{\mathcal{GCG}(\alpha,\beta)}$. We provide both upper and lower bounds on the dimension of the program register of an approximate programmable quantum processor that implements all (α, β)-energy-limited gauge-covariant Gaussian channels with an input and output state of a certain maximal energy.

7.4.2.1 Upper bounds gauge-covariant Gaussian channels

We establish an ϵ-net on $\mathcal{GCG}(\alpha, \beta)$ to get a discrete approximation of the output and thus obtain upper bounds on the program dimension d_P^∞. Afterwards, we construct a processor that implements the channels of the ϵ-net with program dimension equal to the cardinality of the ϵ-net using the *PET* (see Section 7.1). With this construction, we obtain upper bounds on the program dimension of a processor implementing all (α, β)-energy-limited gauge-covariant Gaussian channels.

Theorem 7.4.2 (Upper bounds [2, Theorem 17]). *Let $\epsilon > 0$ and $E > 0$. Then, there exists an infinite-dimensional ϵ-EPQP$_{\mathcal{GCG}(\alpha,\beta)}$ $\mathcal{P} \in CPTP(\mathcal{H}_1 \otimes \mathcal{H}_P, \mathcal{H}_2)$ whose program register is upper bounded as follows:*

$$d_P^{\infty} \leq \frac{CE^2(2E+2)(\beta+1)}{\epsilon^6}$$

for a constant C.

Proof. Since we consider gauge-covariant Gaussian channels, we use Proposition 6.2.3 which states that those channels can be described as a concatenation of an attenuator channel $\mathcal{T}_\lambda$, a rotation channel $\mathcal{R}_\varphi$ and a quantum-limited amplifier channel $\mathcal{A}_\mu$ and thus, we construct one ϵ-net on the set of attenuator channels, one on rotations and one on the amplifier channels. They are specified by one parameter each.

Let us consider the attenuator channels first. Recall that the parameter $0 \leq \lambda < 1$ is the attenuation parameter. Thus, we construct an ϵ_λ-net for λ with $\{\lambda_i\}_{i=1}^{|\mathcal{I}_\lambda|} \subset [0,1)$ such that for every λ there is an index $i \in \mathcal{I}_\lambda$ satisfying

$$|\lambda - \lambda_i| \leq \epsilon_\lambda.$$

The range of λ forms a compact interval. The cardinality of such a net is

$$|\mathcal{I}_\lambda| \leq \left(\frac{1}{\epsilon_\lambda} + 1\right).$$

Analogously, we construct an ϵ_φ-net for the parameter $\varphi \in [0, 2\pi]$ such that for every φ, there exists an index $j \in \mathcal{I}_\varphi$ with

$$|\varphi - \varphi_j| \leq \epsilon_\varphi$$

with cardinality

$$|\mathcal{I}_\varphi| \leq \left(\frac{2\pi}{\epsilon_\varphi} + 1\right).$$

We continue with the parameter describing the amplifier channel. Note that amplifier channels enlarge the energy. The larger the amplification factor, the higher the energy of the output. The (α, β)-energy-limitation of the considered channels yields a maximal amplification factor $\mu_{\max}$. Let us specify this parameter.
To obtain a necessary condition for the parameter μ, we consider the vacuum state $\rho_G(0, \mathbb{1}_2)$ as input state with zero energy. The attenuator channel with $\mathcal{K} = \sqrt{\lambda}\mathbb{1}_2$ and $\mathcal{N} = (1-\lambda)\mathbb{1}_2$ and $\eta = 0$ maps the vacuum state to the vacuum state. The amplifier channel with $\mathcal{K} = \sqrt{\mu}\mathbb{1}_2$ and $\mathcal{N} = (\mu-1)\mathbb{1}_2$, $\eta = 0$ maps it to $\rho_G(0, (2\mu-1)\mathbb{1}_2)$ with mean photon number

$$\operatorname{tr}(\rho_G(0, (2\mu-1)\mathbb{1}_2)) = \mu - 1$$

where we use $\operatorname{tr}(\rho_G a^* a) = \frac{1}{2}\operatorname{tr}(\Gamma) + \frac{1}{4}d^2 - \frac{1}{2}$ for a general $\rho_G(d, \Gamma)$ [72, Eq. (6.60)]. This yields the necessary condition

$$\mu \leq \beta + 1.$$

Hence, we choose

$$\mu_{max} = \beta + 1. \tag{7.4.2.1.1}$$

Due to the energy-constraint and $\mu_{\max}$, the values $\mu \in (1, \mu_{\max}]$ form a compact set and we construct an ϵ_μ-net $\{\mu_k\}_{k=1}^{|\mathcal{I}_\mu|} \subset (1, \mu_{\max}]$ such that for every μ there is an index $k \in \mathcal{I}_\mu$ such that

$$|\mu - \mu_k| \leq \epsilon_\mu.$$

The cardinality of this net reveals as

$$|\mathcal{I}_\mu| \leq \left(\frac{\mu_{\max} - 1}{\epsilon_\mu} + 1\right).$$

The overall cardinality for the parameter appears as follows

$$|\mathcal{I}_\Phi| = |\mathcal{I}_\lambda||\mathcal{I}_\varphi||\mathcal{I}_\mu| \leq \left(\frac{1}{\epsilon_\lambda}+1\right)\left(\frac{2\pi}{\epsilon_\varphi}+1\right)\left(\frac{\mu_{\max}-1}{\epsilon_\mu}+1\right) \leq \frac{16\mu_{\max}}{\epsilon_\lambda\epsilon_\varphi\epsilon_\mu} \leq \frac{16(\beta+1)}{\epsilon_\lambda\epsilon_\varphi\epsilon_\mu}$$

where we use Eq. (7.4.2.1.1) in the last inequality. Since we are interested in the cardinality of ϵ-nets in $\mathcal{GCG}(\alpha, \beta)$, we lift the parameter nets to nets on the set of channels. We use the E-diamond norm distance (see Definition 6.1.4).
Firstly, for the attenuator channel, we know from Ref. [80, Example 5] that

$$\|\mathcal{T}_\lambda - \mathcal{T}_{\lambda_i}\|_\diamond^E \leq 4\sqrt{2}\sqrt{E\epsilon_\lambda}.$$

Secondly, concerning the rotation channel, we use the result by Becker and Datta [80, Proposition 3.2] for the one-parameter unitary semigroup of rotations

$$\|\mathcal{R}_\varphi - \mathcal{R}_{\varphi_j}\|_\diamond^E \leq 4\sqrt{E}\sqrt{|\varphi - \varphi_j|} = 4\sqrt{E\epsilon_\varphi}.$$

Thirdly, the norm of the distance of the amplifier channels can be bounded as [80, Example 5]

$$\|\mathcal{A}_\mu - \mathcal{A}_{\mu_k}\|_\diamond^E \leq 4\sqrt{2}\sqrt{(2E+2)\epsilon_\mu}.$$

For the (α, β)-energy-limited gauge-covariant Gaussian channels we overall obtain

$$\begin{aligned}\|\Phi - \Phi_i\|_\diamond^E &\leq \|\mathcal{A}_\mu \circ \mathcal{R}_\varphi \circ \mathcal{T}_\lambda - \mathcal{A}_{\mu_k} \circ \mathcal{R}_{\varphi_j} \circ \mathcal{T}_{\lambda_i}\|_\diamond^E \\ &\leq \|\mathcal{T}_\lambda - \mathcal{T}_{\lambda_i}\|_\diamond^E + \|\mathcal{R}_\varphi - \mathcal{R}_{\varphi_j}\|_\diamond^E + \|\mathcal{A}_\mu - \mathcal{A}_{\mu_k}\|_\diamond^E \\ &\leq 4\sqrt{2}\sqrt{E}\sqrt{\epsilon_\lambda} + 4\sqrt{E}\sqrt{\epsilon_\varphi} + 4\sqrt{2}\sqrt{2E+2}\sqrt{\epsilon_\mu} =: \epsilon.\end{aligned}$$

We express ϵ_λ, ϵ_φ and ϵ_μ in terms of the ϵ-parameter that specifies the accuracy of the processor:

$$\epsilon_\lambda = \frac{\epsilon^2}{C_\lambda E}, \qquad \epsilon_\varphi = \frac{\epsilon^2}{C_\varphi E}, \qquad \epsilon_\mu = \frac{\epsilon^2}{C_\mu(2E+2)}.$$

Inserting these expressions into $|\mathcal{I}_\Phi|$ we get

$$|\mathcal{I}_\Phi| \leq \frac{16\ (\beta+1)\ C_\lambda\ C_\varphi\ C_\mu\ E(2E+2)}{\epsilon^6} = \frac{CE^2(2E+2)\ (\beta+1)}{\epsilon^6}.$$

We use the *PET* to construct an ϵ-EPQP$_{\mathcal{GCG}(\alpha,\beta)}$ with program dimension

$$d_P^\infty = |\mathcal{I}_\Phi| \leq \frac{CE^2(2E+2)(\beta+1)}{\epsilon^6},$$

concluding the proof. □

7.4.2.2 Lower bounds gauge-covariant Gaussian channels

To obtain lower bounds on the program dimension of a processor implementing all gauge-covariant Gaussian channels, we consider the decomposition of Proposition 6.2.3, from which we get three different building blocks for the ϵ-net for the upper bounds: attenuation, amplification and phase rotation. For lower bounds, the third part is particularly relevant because it yields ϵ-divergence.

Phase rotation. To lower-bound the program dimension d_P^∞ of an ϵ-EPQP$_{\mathcal{GCG}(\alpha,\beta)}$, we proceed in two steps. First, we apply Lemma 7.2.4 to the phase-rotation channels $\mathcal{R}_\varphi(\cdot) = e^{-i\varphi N}(\cdot)e^{i\varphi N}$, which are $(1,0)$-limited. This results in a modified processor implementing the rotation ℓ times in parallel. The second step is motivated by the fact that all information for the implementation of $\mathcal{R}_\varphi$ is contained in the program state, which has to contain almost the same information as the ℓ-tensor-power phase rotation. We design an ensemble on the output space to obtain lower bounds on the program dimension by bounding the Holevo information of the ensemble. The following theorem states the resulting lower bounds.

Theorem 7.4.3 (Lower bounds phase rotations [2, Theorem 18])**.** *Let $\epsilon > 0$ and $E > 0$. Then, for every infinite-dimensional ϵ-EPQP$_{\mathcal{GCG}(\alpha,\beta)}$ $\mathcal{P} \in CPTP(\mathcal{H} \otimes \mathcal{H}_P, \mathcal{H})$, with $\alpha \geq 1$ and $\beta \geq 0$, its program register can be lower bounded as follows:*

$$d_P^\infty \geq \frac{1}{8192e} \frac{\delta^2 E}{(\sqrt{2}E+1)^\delta} \left(\frac{1}{\sqrt{2\epsilon}}\right)^{1-\delta} \geq C\delta^2 \left(\frac{E}{\sqrt{\epsilon}}\right)^{1-\delta},$$

for any $0 < \delta < 1$, and the latter for $E \geq 1$ and an absolute constant C.

Proof. The gauge-covariant Gaussian channels contain the one-parameter group of phase-rotation unitaries $\mathcal{R}_\varphi$. Applying Lemma 7.2.4 to these unitaries, we obtain

$$\frac{1}{2}\left\|\widehat{\mathcal{P}}(\cdot \otimes \psi_R) - \mathcal{R}_\varphi^{\otimes\ell}\right\|_\diamond^{(E,\ldots,E)} \leq 2\ell\sqrt{2\epsilon}. \tag{7.4.2.2.1}$$

Next, we create a state of high total energy on which we act with $\mathcal{R}_\varphi^{\otimes\ell}$ to generate an ensemble, where φ is uniformly distributed on $[0, 2\pi]$. The ideal output is the phase rotation $\mathcal{R}_\varphi^{\otimes\ell}$ on ℓ modes. Since the photon number is the sum of those on the subsystems, the unitary is generated by the total photon number, i.e., $R_\varphi^{\otimes\ell} = e^{i\varphi(N_1+\ldots+N_\ell)}$. Since this yields a phase multiplication $e^{i\varphi n}$ on each degenerate subspace of total photon number n, the ℓ-fold tensor-product unitary is diagonal in the photon-number basis and we use only one state from each eigenspace.

For each total number n of photons, we define a unique way of distributing them across the ℓ modes in an as equilibrated way as possible. For instance, choose a partition of n into non-negative integers, $n = n_1 + \ldots + n_\ell$ and define $|\text{“}n\text{”}\rangle$ as the unit norm symmetrization of $|n_1\rangle \cdots |n_\ell\rangle$,

$$|\text{“}n\text{”}\rangle :\propto \sum_{\pi \in S_\ell} |n_{\pi(1)}\rangle \cdots |n_{\pi(\ell)}\rangle \,.$$

This state evidently has total photon number n, and the expected photon number in each mode is $\frac{n}{\ell}$.

As input state to the processor we choose

$$|\nu\rangle = \sum_{n=0}^{\infty} c_n \, |\text{“}n\text{”}\rangle \,,$$

with amplitudes such that $\langle\nu| N |\nu\rangle = \sum_n n|c_n|^2 \leq \ell E$. The following calculations of the Holevo information take place on the subspace spanned by $|\text{“}n\text{”}\rangle$. Note that on that subspace, the total number operator $N = N_1 + \ldots + N_\ell$ is isomorphic to a number operator, hence the time evolution of $\mathcal{R}_\varphi^{\otimes \ell}$ leaves this “virtual Fock space” $\mathcal{H}_V \subset \mathcal{H}^{\otimes \ell}$ invariant.

Since all information about the output of the processor is contained in the program state,

$$\log d_P^\infty \geq \chi\left(\left\{\widehat{\mathcal{P}}(|\nu\rangle\langle\nu| \otimes \psi_\varphi), \frac{\mathrm{d}\varphi}{2\pi}\right\}\right),$$

following the approach explained in Subsection 7.4.1. We compare these ensemble states $\omega_\varphi = \widehat{\mathcal{P}}(|\nu\rangle\langle\nu| \otimes \psi_\varphi)$ with the ideal ones $\mathcal{R}_\varphi^{\otimes \ell}(|\nu\rangle\langle\nu|)$, which are supported on $\mathcal{H}_V$ with projector P_V. Thus, defining the compression map onto that subspace,

$$\mathcal{K}_V(\xi) = P_V \xi P_V + \kappa_0 \operatorname{tr} \xi(\mathbb{1} - P_V),$$

and letting $\omega'_\varphi = \mathcal{K}_V(\omega_\varphi)$, we have, by Eq. (7.4.2.2.1) and the contractivity of the trace norm, that

$$\frac{1}{2}\left\|\mathcal{R}_\varphi^{\otimes \ell}(|\nu\rangle\langle\nu|) - \omega'_\varphi\right\|_1 \leq \frac{1}{2}\left\|\mathcal{R}_\varphi^{\otimes \ell}(|\nu\rangle\langle\nu|) - \omega_\varphi\right\|_1 \leq 2\ell\sqrt{2\epsilon},$$

and on the other hand by data processing for the Holevo information [23, Subsection 11.9.2],

$$\log d_P^\infty \geq \chi\left(\left\{\omega'_\varphi, \frac{\mathrm{d}\varphi}{2\pi}\right\}\right).$$

As mentioned before, the Hamiltonian restricted to the virtual Fock space $\mathcal{H}_V$ is isomorphic to a normal number operator $H_V = \sum_{n=0}^\infty n|\text{“}n\text{”}\rangle\langle\text{“}n\text{”}|$, and the ideal ensemble states have energy $\operatorname{tr} \mathcal{R}_\varphi^{\otimes \ell}(|\nu\rangle\langle\nu|)H_V = \operatorname{tr} |\nu\rangle\langle\nu| H_V \leq \ell E$. Now we invoke Lemma 7.4.1 with $\eta = 2\ell\sqrt{2\epsilon}$, yielding the compression map $\mathcal{K}$ onto the subspace with energy $\leq \frac{\ell E}{2\ell\sqrt{2\epsilon}} = \frac{E}{2\sqrt{2\epsilon}}$, which gives rise to states $\omega''_\varphi = \mathcal{K}(\omega_\varphi)$ with

$$\frac{1}{2}\left\|\mathcal{R}_\varphi^{\otimes \ell}(|\nu\rangle\langle\nu|) - \omega''_\varphi\right\|_1 \leq 4\sqrt{2\ell\sqrt{2\epsilon}},$$

and so finally

$$\begin{aligned}\log d_P^{\infty} &\geq \chi\left(\left\{\omega_{\varphi}'', \frac{\mathrm{d}\varphi}{2\pi}\right\}\right) \\ &\geq \chi\left(\left\{\mathcal{R}_{\varphi}^{\otimes\ell}(|\nu\rangle\langle\nu|), \frac{\mathrm{d}\varphi}{2\pi}\right\}\right) - 16\sqrt{2}\sqrt{\ell\sqrt{2\epsilon}}\, g\left(\frac{E}{16\sqrt{\epsilon}\sqrt{\ell\sqrt{2\epsilon}}}\right) - 2,\end{aligned}$$

using Eq. (7.4.1.1) in Subsection 7.4.1, where $g(N) = (N+1)\log(N+1) - N\log N$ is the formula for the von Neumann entropy of the thermal (Gibbs) state of mean photon number N (see Eq. (6.2.1.1)). On the other hand, it is easily seen that

$$\chi\left(\left\{\mathcal{R}_{\varphi}^{\otimes\ell}(|\nu\rangle\langle\nu|), \frac{\mathrm{d}\varphi}{2\pi}\right\}\right) = H(\{|c_n|^2\}),$$

which itself is maximized for the thermal distribution with mean photon number $\sum_n n|c_n|^2 = \ell E$, and the maximum is $g(\ell E)$. Using the elementary upper and lower bounds [67, p. 10]

$$\log N \leq g(N) \leq \log(N+1) + \log e,$$

and letting $\sqrt{\ell} = \frac{\delta}{16\sqrt{2\sqrt{2\epsilon}}}$, with $0 < \delta < 1$, we thus get

$$\begin{aligned}\log d_P^{\infty} &\geq \log(\ell E) - \delta\log\left(\frac{\sqrt{2}E}{\delta\sqrt{\epsilon}} + 1\right) - \delta\log e - 2 \\ &\geq (1-\delta)\log\frac{1}{\sqrt{2\epsilon}} + \log\frac{\delta^2 E}{512} - \delta\log(\sqrt{2}E+1) - \log(16e).\end{aligned}$$

The final form of the bound is hence

$$d_P^{\infty} \geq \frac{1}{8192e}\frac{\delta^2 E}{(\sqrt{2}E+1)^{\delta}}\left(\frac{1}{\sqrt{2\epsilon}}\right)^{1-\delta} \geq C\delta^2\left(\frac{E}{\sqrt{\epsilon}}\right)^{1-\delta},$$

as claimed. □

Thus, merely considering the phase-rotation part of the decomposition of gauge-covariant Gaussian channels results in lower bounds on the program dimension that diverge with ϵ. This confirms that divergence of the upper bounds is not an artifact of the ϵ-net construction.

Additionally, we give lower bounds for the special case of attenuation channels.

Attenuation. We want to explore what kinds of lower bounds we can obtain from looking at attenuation only, i.e., on ϵ-EPQP$_{\mathcal{T}}$, which denotes processors implementing the attenuator channels $\mathcal{T} = \{\mathcal{R}_{\varphi} \circ \mathcal{T}_{\lambda} : \varphi \in \{0, \pi\},\ \lambda \in [0;1]\}$. Note that we allow a single phase rotation of angle π, which in itself cannot give an unbounded lower bound. We could give the subsequent argument without it, but including it makes the following discussion a little nicer.

Theorem 7.4.4 (Lower bounds attenuation [2, Theorem 19]). *Let* $0 < \epsilon < \frac{1}{1024}$ *and* $E \geq 2^e - 1$. *Then, for every infinite-dimensional* ϵ*-EPQP*$_{\mathcal{T}}$ $\mathcal{P} \in CPTP(\mathcal{H} \otimes \mathcal{H}_P, \mathcal{H})$, *its program register is lower bounded as follows:*

$$d_P^\infty \geq 2^{-16} \frac{1}{\sqrt{\ln \log(E+1)}} (E+1)^{\frac{1}{2} - 16\sqrt{\epsilon}}.$$

Proof. As in the proof of the previous theorem, we test the processor on a concrete input state ρ_χ which we choose to be a coherent state $\rho_\chi = |\zeta\rangle\langle\zeta|$ with the highest allowed energy (photon number) E, i.e., $\zeta = \sqrt{2E}$. With this fixed input state, the processor generates the output states $\{\Phi(\rho_\chi)\}$, $\Phi \in \mathcal{T}$. These output states ideally are precisely the coherent states $|\xi\rangle\langle\xi|$ with $-\zeta \leq \xi \leq \zeta$. Rather than describing the ensemble of program states ρ_λ, which through the processor lead to unique output states $\rho_\xi = \mathcal{P}(|\zeta\rangle\langle\zeta| \otimes \rho_\lambda)$ (that approximate the coherent state $|\xi\rangle\langle\xi|$), we give instead directly a distribution over the ξ. We choose the truncated Gaussian distribution with variance σ^2

$$p_E(\xi) := \begin{cases} \frac{1}{1-\eta} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\xi^2}{2\sigma^2}} & \text{if } |\xi| \leq \sqrt{2E}, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\eta = 1 - \int_{-\sqrt{2E}}^{+\sqrt{2E}} d\xi \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\xi^2}{2\sigma^2}} = \operatorname{erfc}(\sqrt{E}/\sigma) \leq e^{-E/\sigma^2},$$

with the complementary error function $\operatorname{erfc}(x)$ and its well-known upper bound $\operatorname{erfc}(x) \leq e^{-x^2}$, see Ref. [81]. Note furthermore that, denoting the density of the centered normal distribution with variance σ^2 by $p(\xi) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\xi^2}{2\sigma^2}}$, we have

$$\frac{1}{2} \|p - p_E\|_{L^1} = \eta.$$

Following the method described at the end of Section 7.2, using data processing, we start off from the inequality

$$\log d_P^\infty \geq \chi(\{\rho_\xi, p_E(\xi) d\xi\}) = S\left(\int_{-\sqrt{2E}}^{\sqrt{2E}} d\xi \, p_E(\xi) \rho_\xi \right) - \int_{-\sqrt{2E}}^{\sqrt{2E}} d\xi \, p_E(\xi) S\left(\rho_\xi\right), \tag{7.4.2.2.2}$$

recalling the definition of the Holevo information. The remaining calculation is about controlling the Holevo information on the right-hand side, which we do by first modifying the states from ρ_ξ to the compressed state $\rho'_\xi = \mathcal{K}(\rho_\xi)$, and finally to $|\xi\rangle\langle\xi|$, incurring a certain error. According to Eq. (7.4.1.1) we get from Eq. (7.4.2.2.2)

$$\log d_P^\infty \geq S\left(\int_{-\sqrt{2E}}^{\sqrt{2E}} d\xi \, p_E(\xi) |\xi\rangle\langle\xi| \right) - 16\sqrt{\epsilon} S\left(\gamma\left(\frac{E}{4\sqrt{\epsilon}^3} \right) \right) - 2, \tag{7.4.2.2.3}$$

keeping in mind that our channels are $(1, 0)$-energy-limited and that the attenuator output states $|\xi\rangle\langle\xi|$ are pure. It remains to calculate the entropy, which however

is challenging. To lower bound it in turn, we modify the distribution from the truncated Gaussian p_E to the full Gaussian p, incurring another certain Fannes error, but having the benefit of leaving us with a Gaussian state. We abbreviate the mixtures

$$\omega := \int_{-\infty}^{\infty} d\xi \, \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\xi^2}{2\sigma^2}} |\xi\rangle\langle\xi|,$$
$$\omega_E := \int_{-\sqrt{2E}}^{\sqrt{2E}} d\xi \, p_E(\xi) |\xi\rangle\langle\xi|,$$

to which we can apply [67, Lemma 15], noting that both states have energy bounded by $\sigma^2/2$ and E, respectively. We choose $\sigma^2 = \frac{1}{t}E$ with $t \geq 1$, making the energy bound E, and $\eta \leq e^{-t}$, thus

$$|S(\omega) - S(\omega_E)| \leq 2\eta g\left(\frac{E}{\eta}\right) + h(\eta) \leq 2\eta\left(\log\left(\frac{E}{\eta} + 1\right) + \log e\right) + 1,$$

where $g(N) := (N+1)\log(N+1) - N\log N$ and $h(\eta) = H(\eta, 1-\eta) = -\epsilon\log\epsilon - (1-\epsilon)\log(1-\epsilon)$ the binary entropy. On the other hand, ω is a Gaussian state having a diagonal covariance matrix with eigenvalues 1 and $1 + 2\sigma^2$. From this we can obtain its symplectic eigenvalue, which is $\sqrt{1+2\sigma^2}$, as one can see by considering a Gaussian squeezing unitary that transforms the state to a thermal Gaussian state. Hence,

$$S(\omega) = g\left(\frac{\sqrt{1+2\sigma^2} - 1}{2}\right) \geq \log\left(\frac{\sqrt{1+2\sigma^2} + 1}{2}\right),$$

where here and in the previous display equation we use the bounds $\log(x+1) \leq g(x) \leq \log(x+1) + \log e$. Putting it all together, using Eq. (7.4.2.2.3) and the above bounds, we obtain

$$\begin{aligned}
\log d_P^{\infty} &\geq g\left(\frac{\sqrt{1+2\sigma^2} - 1}{2}\right) - 2\eta g\left(\frac{E}{\eta}\right) - 16\sqrt{\epsilon} g\left(\frac{E}{4\sqrt{\epsilon}^3}\right) - 3 \\
&\geq \log\left(\frac{\sqrt{1+2\sigma^2} + 1}{2}\right) - 2e^{-t}\log\left(Ee^t + 1\right) - 2e^{-t}\log e \\
&\quad - 16\sqrt{\epsilon}\log\left(\frac{E}{4\sqrt{\epsilon}^3} + 1\right) - 16\sqrt{\epsilon}\log e - 3 \\
&\geq \frac{1}{2}\log\left(\frac{E+1}{2t}\right) - 2e^{-t}\log\big((E+1)e^t\big) - 16\sqrt{\epsilon}\log\left(\frac{E+1}{4\sqrt{\epsilon}^3}\right) \\
&\quad - 3 - 2e^{-t}\log e - 16\sqrt{\epsilon}\log e \\
&= \left(\frac{1}{2} - 16\sqrt{\epsilon} - 2e^{-t}\right)\log(E+1) - \frac{1}{2}\log t \\
&\quad - \frac{1}{2} - 3 - 2e^{-t}\log e - 2e^{-t}\log e^t + 16\sqrt{\epsilon}\log\left(\frac{4}{e}\sqrt{\epsilon}^3\right).
\end{aligned}$$

Now we look at the terms in the last line, showing that their sum can be lower bounded by -14. Namely, note that the function $-x \log x$ is monotonically increasing on the interval $[0; 1/e]$, and so $e^{-t} \log e^t \leq \frac{\log e}{e}$ as well as $-\sqrt{\epsilon} \log \sqrt{\epsilon} \leq \frac{5}{32}$, where we use that $\epsilon \leq \frac{1}{1024}$. Thus,

$$\log d_P^\infty \geq \left(\frac{1}{2} - 16\sqrt{\epsilon}\right) \log(E+1) - \frac{1}{2} \log t - 14 - 2e^{-t} \log(E+1),$$

and letting $t = \ln \log(E+1) \geq 1$, recalling the assumption on E, concludes the proof. □

Amplification and attentuation. In the case of $\alpha > 1$, we can also consider amplifier channels. Thus, one could try to use attenuators as well as amplifiers to construct an ensemble. According to Proposition 6.2.3, the output of the attenuator channel serves as input for the amplifier channel. Heuristically, it makes sense to input a coherent state with the highest energy. The attenuator channel maps coherent states to coherent states which means that we input a coherent state into the amplifier channel. The latter maps these coherent states to displaced thermal states. However, we did not find an appropriate ensemble where the mixture of ensemble states is still Gaussian (this is a heuristic to be able to calculate entropies in closed form) and the Holevo information is improved compared to the coherent states.

Possibly, such an ensemble does not exist, because the amplifier channel introduces noise, which means that to obtain an advantage, the ensemble must use different amplification strengths, otherwise data processing shows directly that the Holveo information is only smaller. On the other hand, using a distribution over amplifications would likely result in an ensemble mixture that is a convex combination of different thermal states, and no non-trivial convex combination of them can lead to a thermal state again.

Hence, to obtain lower bounds for gauge-covariant Gaussian channels, we consider the subset of attenuator channels. In other words, we obtain the same lower bounds for a processor that is merely able to implement attenuator channels than one that implements all Gaussian channels.

7.4.3 Energy-limited Gaussian unitary channels

In analogy to existing programmable quantum processors that implement unitary channels, we focus on the implementation of (α, β)-energy-limited Gaussian unitary channels. This set is denoted as $\mathcal{GU}(\alpha, \beta)$ and the corresponding processor as ϵ-EPQP$_{\mathcal{GU}(\alpha,\beta)}$. We again provide upper and lower bounds on the dimension of the program register.

7.4.3.1 Upper bounds Gaussian unitary channels

We determine upper bounds for the program dimension d_P^∞ of an ϵ-EPQP$_{\mathcal{GU}(\alpha,\beta)}$ in three steps. Firstly, we construct an ϵ-net on a suitable parameter set, secondly we

relate it to a set of channels and thirdly, we construct a processor with program register equal to the cardinality of the ϵ-net. The set of all (α, β)-energy-limited Gaussian unitary channels is represented by $\mathcal{GU}(\alpha, \beta)$ with elements

$$\mathcal{U}_G(\cdot) = U_G(\cdot)U_G^*.$$

Theorem 7.4.5 (Upper bounds [2, Theorem 20]). *Let $\epsilon > 0$ and $E > 0$. Then, for a system of M Bosonic modes, there exists an infinite-dimensional ϵ-EPQP$_{\mathcal{GU}(\alpha,\beta)}$ $\mathcal{P} \in CPTP(\mathcal{H} \otimes \mathcal{H}_P, \mathcal{H})$ whose program register is upper bounded as follows:*

$$d_P^\infty \leq \left(\frac{2352(M\alpha)^{3/2}(\sqrt{\alpha}+1)(E+1)}{\epsilon^2}\right)^{4M^2} \left(\frac{2\sqrt{2}(\sqrt{2\beta}+1)\sqrt{\alpha E+\beta+1}}{\epsilon}\right)^{2M}$$

for an absolute constant c_S.

Proof. A general M-mode Gaussian unitary can be decomposed into a generalized phase rotation, which is a passive operation that does not change the energy, followed by M separate single-mode squeezing transformations, followed by another passive generalized rotation, and finally a displacement, see Subsection 6.2.3 and Refs. [75, 76]. Thus, we construct two nets: one for the first three operations based on the symplectic group and another one on the displacements. Note that these sets are not compact but due to the energy constraint, we can cut the sets such that they become compact. Where we place the cutoff depends on α and β.

For the first net, let us consider the following compact subset of $Sp_{2M}(\mathbb{R})$

$$Sp_{2M}^{\sqrt{\alpha}+1}(\mathbb{R}) := \{S \in Sp_{2M}(\mathbb{R}), \|S - \mathbb{1}_{\mathbb{R}^{2M}}\|_\infty \leq \sqrt{\alpha}+1\}.$$

The cutoff yields all elements with maximal singular value $r \leq \sqrt{\alpha}$. Since the singular values of a symplectic matrix come in pairs x and $1/x$, this means that all singular values of the matrices in the above subset lie between $\frac{1}{\sqrt{\alpha}}$ and $\sqrt{\alpha}$. From the Bloch-Messiah decomposition, which shows that modulo passive Gaussian transformations, every Gaussian unitary described by a symplectic matrix is equivalent to a tensor product of single-mode squeezing operators [82], this means that we only have to consider the (α, β)-energy-limitation of those M single-mode squeezers. It is elementary to see that a squeezing unitary with singular values of the symplectic matrix $r \geq 1$ and $1/r$ is not (α, β)-energy-limited for any $r^2 > \alpha$ and $\beta \geq 0$. An ϵ-net $\mathcal{I}_S$ on this set is constructed and its cardinality $|\mathcal{I}_S|$ determined in Ref. [83, Lemma S16] as

$$|\mathcal{I}_S| \leq \left(\frac{3(\sqrt{\alpha}+1)}{\epsilon_S}\right)^{4M^2}.$$

Concerning the displacement, we must have $|d|^2 = |d_1|^2 + \ldots |d_l|^2 \leq 2\beta$ for an admissible displacement vector $d = (d_1, \ldots, d_M)$, where each $d_j = (d_{j1}, d_{j2})$ is a pair of singe-mode phase-space coordinates, otherwise the unitary channel $\mathcal{U}_d$ is

not (α, β)-energy-limited, i.e., $|d|^2 \leq 2\beta$. Hence, we construct a net on the ball of radius $\sqrt{2\beta}$ in $\mathbb{R}^{2M}$ with the Euclidean metric. The cardinality is known to be

$$|\mathcal{I}_d| \leq \left(1 + \frac{\sqrt{2\beta}}{\epsilon_d}\right)^{2M}.$$

Bringing these two cardinalities together results in an overall net cardinality

$$\begin{aligned} |\mathcal{I}_{S,d}| \leq |\mathcal{I}_S||\mathcal{I}_d| &\leq \left(\frac{3(\sqrt{\alpha}+1)}{\epsilon_S}\right)^{4M^2}\left(1 + \frac{\sqrt{2\beta}}{\epsilon_d}\right)^{2M} \\ &\leq \left(\frac{3(\sqrt{\alpha}+1)}{\epsilon_S}\right)^{4M^2}\left(\frac{\sqrt{2\beta}+1}{\epsilon_d}\right)^{2M}. \end{aligned}$$

Having established ϵ-nets on the parameter level, we require an upper bound for $\frac{1}{2}\|\mathcal{U}_G - \mathcal{U}_{G_i}\|_\diamond^E$. So we transfer both nets on the sets of parameters to the corresponding channels. The symplectic matrices correspond to the first three parts of the decomposition, a rotation followed by a squeezing and again a rotation.

Since we assume an energy-limitation on the set of Gaussian unitary channels, we can construct a compact subset of channels that contains these channels, as follows. In fact, we obtain it from a compact subset of the symplectic group and a compact subset of the displacement group.

For the former, we use Ref. [83, Eq. (4)], which states

$$\frac{1}{2}\|\mathcal{U}_S - \mathcal{U}_{S'}\|_\diamond^E \leq \sqrt{(\sqrt{6}+\sqrt{10}+5\sqrt{2}M)(E+1)}\, g\big(\|S^{-1}S'\|_\infty\big)\sqrt{\|S^{-1}S' - \mathbb{1}\|_2},$$

where $g(x) := \sqrt{\frac{\pi}{x+1}} + \sqrt{2x}$. Note $\|S^{-1}\| \leq \sqrt{\alpha}$, $\|S'\| \leq 1 + \sqrt{\alpha}$, hence the argument x of $g(x)$ is between 1 and $\alpha + \sqrt{\alpha} \leq 2\alpha$, thus $g(x) \leq \sqrt{\pi/2} + \sqrt{2x} \leq \left(2 + \sqrt{\pi/2}\right)\sqrt{\alpha} < 3.26\sqrt{\alpha}$. Furthermore, $\|S^{-1}S' - \mathbb{1}\|_2 = \|S^{-1}(S' - S)\|_2 \leq \sqrt{2M}\|S^{-1}(S' - S)\|_\infty \leq \sqrt{2M}\|S^{-1}\|_\infty\|S' - S\|_\infty \leq \sqrt{2M}\sqrt{\alpha}\epsilon_S$. Finally, $\sqrt{6} + \sqrt{10} + 5\sqrt{2}M \leq (\sqrt{6} + \sqrt{10} + 5\sqrt{2})M < 13M$. Hence, in simplified form the result says that

$$\begin{aligned} \frac{1}{2}\|\mathcal{U}_S - \mathcal{U}_{S_i}\|_\diamond^E &\leq \sqrt{13M}\sqrt{E+1}\cdot 3.26\sqrt{\alpha}\cdot \sqrt[4]{2M}\sqrt[4]{\alpha}\sqrt{\epsilon_S} \\ &\leq 14(M\alpha)^{3/4}\sqrt{E+1}\sqrt{\epsilon_S}. \end{aligned}$$

For the latter, we consider Ref. [83, Eq. (3)], which states

$$\frac{1}{2}\|\mathcal{D}_z - \mathcal{D}_w\|_\diamond^E \leq \sin\left(\min\left\{\|z-w\|f(E), \frac{\pi}{2}\right\}\right) \leq \sqrt{2}\sqrt{E+1}\|z-w\|,$$

where $f(E) := \frac{1}{\sqrt{2}}(\sqrt{E} + \sqrt{E+1}) \leq \sqrt{2}\sqrt{E+1}$ and as before we only use the simplified upper bound.

Note that with the action of $\mathcal{U}_S$ on the input, the input energy of the displacement part changes, i.e., we work with the $(\alpha E + \beta)$-energy-constrained diamond norm

here. Thus, we obtain

$$\begin{aligned}\frac{1}{2}\|\mathcal{U}_d - \mathcal{U}_{d_i}\|_\diamond^{\alpha E+\beta} &\leq \|d - d_i\| f(\alpha E + \beta) \\ &\leq \sqrt{2}\sqrt{\alpha E + \beta + 1}\epsilon_d.\end{aligned}$$

Overall, we obtain for the net of channels

$$\begin{aligned}\frac{1}{2}\|\mathcal{U}_G - \mathcal{U}_{G_i}\|_\diamond^E &\leq \frac{1}{2}\|\mathcal{U}_d \circ \mathcal{U}_S - \mathcal{U}_{d_i} \circ \mathcal{U}_{S_i}\|_\diamond^E \\ &\leq \frac{1}{2}\|\mathcal{U}_S - \mathcal{U}_{S_i}\|_\diamond^E + \frac{1}{2}\|\mathcal{U}_d - \mathcal{U}_{d_i}\|_\diamond^{\alpha E+\beta} \\ &\leq 14(M\alpha)^{3/4}\sqrt{E+1}\sqrt{\epsilon_S} + \sqrt{2}\sqrt{\alpha E + \beta + 1}\epsilon_d.\end{aligned}$$

We now choose ϵ_S and ϵ_d in terms of ϵ, E, α, and β as follows:

$$\epsilon_S = \frac{\epsilon^2}{784(M\alpha)^{3/2}(E+1)}, \qquad \epsilon_d = \frac{\epsilon}{2\sqrt{2}\sqrt{\alpha E + \beta + 1}}$$

which yields an ϵ-net with

$$|\mathcal{I}_{S,d}| \leq \left(\frac{2352(M\alpha)^{3/2}(\sqrt{\alpha}+1)(E+1)}{\epsilon^2}\right)^{4M^2}\left(\frac{2\sqrt{2}(\sqrt{2\beta}+1)\sqrt{\alpha E+\beta+1}}{\epsilon}\right)^{2M}.$$

With the *PET* (see Section 7.1), we construct a programmable quantum processor with $d_P^\infty = |\mathcal{I}_{S,d}|$ which proves the assertion. □

7.4.3.2 Lower bounds Gaussian unitary channels

The following theorem states lower bounds on the dimension of the program register of an ϵ-EPQP$_{\mathcal{U}(\alpha,\beta)}$.

Theorem 7.4.6 (Lower bounds generalized phase rotations [2, Theorem 21])**.** *Let $\epsilon > 0$ and $E > 0$ and consider an M-mode Bosonic system with Hilbert space $\mathcal{H}$. Then, for every infinite-dimensional ϵ-EPQP$_{\mathcal{U}(\alpha,\beta)}$ $\mathcal{P} \in CPTP(\mathcal{H} \otimes \mathcal{H}_P, \mathcal{H})$, its program register can be lower bounded as follows:*

$$d_P^\infty \geq \frac{1}{4(512e)^M}\left(\frac{\delta^2 E/M}{(\sqrt{2}E/M+1)^\delta}\right)^M\left(\frac{1}{\sqrt{2\epsilon}}\right)^{(1-\delta)M} \geq (C\delta^2)^M\left(\frac{E/M}{\sqrt{\epsilon}}\right)^{(1-\delta)M},$$

for any $0 < \delta < 1$, and the latter for $\frac{E}{M} \geq 1$ and an absolute constant C. □

Proof. Analogously to the proof of Theorem 7.4.3 for lower bounds in the gauge-covariant case, we proceed in two steps: first, applying Lemma 7.2.4 and then finding a suitable ensemble.

To obtain lower bounds on the dimension of the program register of an ϵ-EPQP$_{\mathcal{GU}(\alpha,\beta)}$ for an M-mode Bosonic system with Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_M$, Lemma 7.2.4 is applicable to suitably energy-limited unitaries.

The difficulty is mainly that of finding a good distribution over those unitaries and a fiducial state with which to calculate a lower bound in the scope of Subsection 7.4.1.

We consider M-fold phase rotations $\mathcal{R}_{\underline{\varphi}} = \mathcal{R}_{\varphi_1} \otimes \cdots \otimes \mathcal{R}_{\varphi_M}$ with $\underline{\varphi} = (\varphi_1, \ldots, \varphi_M) \in [0; 2\pi]^M$ in Lemma 7.2.4. Note that these are a subset of the passive linear transformations, and as such conserve energy, hence are $(1, 0)$-energy-limited. We get that the modified processor $\widehat{\mathcal{P}}$ approximately implements the $\mathcal{R}_{\underline{\varphi}}^{\otimes \ell}$:

$$\frac{1}{2} \left\| \widehat{\mathcal{P}}(\cdot \otimes \pi_{\underline{\varphi}}) - \mathcal{R}_{\underline{\varphi}}^{\otimes \ell} \right\|_{\diamond}^{(E,\ldots,E)} \leq 2\ell\sqrt{2\epsilon}.$$

In the ℓM-mode system $\mathcal{H}^{\otimes \ell}$ we address the modes $\mathcal{H}_{jk}$ by double indices, where $j = 1, \ldots, M$ are the original physical modes and $k = 1, \ldots, \ell$ the repetitions. The repetitions of the j-th mode have the Hilbert space $\mathcal{H}_j^{\otimes \ell} = \mathcal{H}_{j1} \otimes \cdots \otimes \mathcal{H}_{j\ell} =: \mathcal{H}_{j\bullet}$. As in the proof of Theorem 7.4.3, we choose a virtual Fock space $\mathcal{H}_{V_j} \subset \mathcal{H}_{j\bullet}$ spanned by symmetric number states $|\text{“}n\text{”}\rangle_j$, and let

$$|\nu_j\rangle := \sum_{n=0}^{\infty} c_n \, |\text{“}n\text{”}\rangle_j \,,$$

where $\{|c_n|^2\}$ is the probability distribution of the thermal state (of the virtual mode $\mathcal{H}_{V_j}$) of mean energy (i.e., photon number) $\frac{\ell E}{M}$. The rotation $\mathcal{R}_{\varphi_j}^{\otimes \ell}$ acts on $\mathcal{H}_{j\bullet}$ and leaves $\mathcal{H}_{V_j}$ invariant, in fact it puts phases $e^{i\varphi_j n}$ in the above superposition. Now, $|\nu\rangle := |\nu_1\rangle \otimes \cdots \otimes |\nu_M\rangle$ is our fiducial state. It has the property that on each copy $\mathcal{H}_{\bullet k} = \mathcal{H}_{1k} \otimes \cdots \otimes \mathcal{H}_{Mk}$ of the original M-mode system, its energy is E. We can evaluate the Holevo information of the ideal ensemble of uniformly distributed states $\mathcal{R}_{\underline{\varphi}}^{\otimes \ell}(|\nu\rangle\langle\nu|)$ as we did in Theorem 7.4.3:

$$\begin{aligned} \chi\left(\left\{ \mathcal{R}_{\underline{\varphi}}^{\otimes \ell}(|\nu\rangle\langle\nu|), \frac{\mathrm{d}^M \underline{\varphi}}{(2\pi)^M} \right\} \right) &= M g\left(\frac{\ell E}{M} \right) \geq M \log \frac{\ell E}{M} \\ &= M \left(\log \ell + \log E - \log M \right). \end{aligned} \tag{7.4.3.2.1}$$

By Lemma 7.2.4, we now have for all $\underline{\varphi}$

$$\frac{1}{2} \left\| \widehat{\mathcal{P}}(|\nu\rangle\langle\nu| \otimes \pi_{\underline{\varphi}}) - \mathcal{R}_{\underline{\varphi}}^{\otimes \ell}(|\nu\rangle\langle\nu|) \right\|_1 \leq 2\ell\sqrt{2\epsilon}.$$

Now we can transform the processor outputs $\omega_{\underline{\varphi}} = \widehat{\mathcal{P}}(|\nu\rangle\langle\nu| \otimes \pi_{\underline{\varphi}})$ as in the proof of Theorem 7.4.3, first by the compression maps $\mathcal{K}_j$ from $\mathcal{H}_{j\bullet}$ to $\mathcal{H}_{V_j}$, resulting in $\omega'_{\underline{\varphi}} = (\mathcal{K}_1 \otimes \cdots \otimes \mathcal{K}_M)\omega_{\underline{\varphi}}$ obeying the same approximation as before

$$\frac{1}{2} \left\| \omega'_{\underline{\varphi}} - \mathcal{R}_{\underline{\varphi}}^{\otimes \ell}(|\nu\rangle\langle\nu|) \right\|_1 \leq 2\ell\sqrt{2\epsilon}.$$

Second, by applying a compression map $\mathcal{K}$ from $\mathcal{H}_{V_1} \otimes \cdots \otimes \mathcal{H}_{V_M}$ to an E-constrained subspace, according to Lemma 7.4.1, resulting in $\omega''_{\underline{\varphi}} = \mathcal{K}(\omega'_{\underline{\varphi}})$ such that

$$\frac{1}{2} \left\| \omega''_{\underline{\varphi}} - \mathcal{R}_{\underline{\varphi}}^{\otimes \ell}(|\nu\rangle\langle\nu|) \right\|_1 \leq 4\sqrt{2\ell\sqrt{2\epsilon}},$$

while obeying an energy bound $\operatorname{tr} \omega''_{\underline{\varphi}} H \leq \frac{E}{2\sqrt{2\epsilon}}$.

As explained in Subsection 7.4.1, we now have

$$\begin{aligned}
\log d_P^\infty &\geq \chi\left(\left\{\omega''_{\underline{\varphi}}, \frac{\mathrm{d}^M \underline{\varphi}}{(2\pi)^M}\right\}\right) \\
&\geq \chi\left(\left\{\mathcal{R}_{\underline{\varphi}}^{\otimes \ell}(|\nu\rangle\langle\nu|), \frac{\mathrm{d}^M \underline{\varphi}}{(2\pi)^M}\right\}\right) \\
&\quad - 16\sqrt{2}\sqrt{\ell}\sqrt{\sqrt{2\epsilon}} M g\left(\frac{E}{16\sqrt{\epsilon}\sqrt{\ell}\sqrt{\sqrt{2\epsilon}} M}\right) - 2,
\end{aligned}$$

see the proof of Theorem 7.4.3. Inserting Eq. (7.4.3.2.1) for the Holevo information in the last line expresses everything in terms of the g function, which we can upper and lower bound as before. We choose $\sqrt{\ell} = \frac{\delta}{16\sqrt{2}\sqrt{\sqrt{2\epsilon}}}$ with $0 < \delta < 1$, and obtain

$$\begin{aligned}
\log d_P^\infty &\geq M \log \frac{\ell E}{M} - \delta M \log\left(\frac{\sqrt{2}E}{M\delta\sqrt{\epsilon}} + 1\right) - \delta M \log e - 2 \\
&\geq M(1-\delta)\log\frac{1}{\sqrt{2\epsilon}} + M \log \frac{\delta^2 E}{512 M} - \delta M \log\left(\frac{\sqrt{2}E}{M} + 1\right) - M \log e - 2.
\end{aligned}$$

□

Remark 7.4.7. *Note that since phase rotations are a subset of Gaussian unitaries, the bounds from Theorem 7.4.3 directly apply here. This shows that the program dimension diverges at least with the inverse square root of ϵ.*

8 Additional observations and future perspectives

We briefly summarize important findings including some additional observations and future research possibilities. The content is based on Refs. [1, Section 5] and [2, Section VI]. Note that for $\epsilon = 0$, Theorem 5.2.4 yields exact lower bounds. These can be combined with the teleportation protocol and subsequent compression (see Theorem 5.1.10) which results in the optimal program register dimension

$$d_c = \sum_{k=1}^{K} n_k = d_P.$$

We define an ϵ-CPQP$_{UV}$ with mixed program states. However, if we insist on pure program states, we could use a suitable purification instead of π_Φ. In this case, we would obtain $d_P = d_c^2$ from Theorem 5.1.10. When the commutant $\mathcal{K}$ is abelian, Corollary 5.1.5 reveals that the upper bounds match the lower bounds for pure states as well. For a non-abelian commutant, this remains an open question.
This thesis presents two main settings for programmable quantum processors. Considering the implementation of a particular subset of all quantum channels – the UV-covariant quantum channels – results in a very different situation than the one for universal programming. While UV-covariant quantum channels are always exactly implementable with finite-dimensional program register for an irreducible U, implementation of all unitary channels is only possible in an approximate manner with a finite-dimensional program register. One could thus ask whether there are groups and representations where exact programmability is possible if U is not irreducible.
Another open aspect is the improvement of the upper bounds in the approximate case. We give exact upper bounds, approximate ones using ϵ-nets, however, rely on orthogonal program states which is not the case for the teleportation protocol using Choi-Jamiołkowski states as program states. If we have non-orthogonal program states, a different compression method for the program states is required.
Concerning tightness of bounds, further research could explore whether the upper and lower bounds we obtain for the infinite-dimensional version of the programmable quantum processor are tight. We choose Bosonic systems as a specific setting and establish upper and lower bounds on the program register of a programmable quantum processor implementing energy-limited Gaussian channels. For the unitary case, we provide upper bounds which scale polynomially in the reciprocal of the approximation parameter with quadratic order in the number of modes. Due to the energy constraint, we use ϵ-nets on compact sections of the

Gaussian unitary group. To obtain lower bounds, we define the multiply energy-constrained diamond norm which we use to generalize the method of Yang *et al.* [15] such that the same program state can be recovered to implement the same unitary several times. This finally leads to lower bounds that diverge with the reciprocal of the approximation parameter (polynomially of a degree at least half of the number of Bosonic modes). The recovering-trick of the program state cannot be applied if we aim to implement all energy-constrained single-mode gauge-covariant Gaussian quantum channels. We still obtain lower bounds which tend to a constant function of the energy constraint whereas the upper bounds we establish diverge with the accuracy of implementation. Further research is required to close this gap. An infinite-dimensional program register might even be necessary.
Another future perspective is to further generalize the processor towards implementing any kind of energy-limited Gaussian channel. This could potentially be approached by considering their Gaussian unitary dilation. Therefore, a version of Stinespring's Theorem, which provides an energy-limited dilation for every energy-limited Gaussian channel, would necessarily be required. Thus, the contribution of generalizing the concept of approximate programmable quantum processors in Chapter 7 opens up new horizons for programmable quantum processors in infinite dimension.

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