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Cohomological and Derived Persistence Theory of Functions

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Preface

Summary. *Relative interlevel set cohomology (RISC)* is an invariant of real-valued continuous functions stemming from *(zigzag) persistent homology* by Edelsbrunner, Letscher, Zomorodian, Carlsson, de Silva, and Morozov. We provide a proof to a structure theorem for RISC inspired by Crawley-Boevey, Höppner, and Lenzing, a theory of interleavings in the sense of Bubenik, de Silva, Scott, and Scoccola, as well as a functorial equivalence to *derived level set persistence* by Curry. Finally, we harness RISC to provide an abelian categorification of *extended persistence diagrams* and a *Mayer–Vietoris principle*. We note that Parts III and IV can be read independently of Part II.

Zusammenfassung (German Summary). *Relative Zwischenniveaumengen Kohomologie (RISC)* ist eine Invariante reellwertiger stetiger Funktionen, die aus *(Zickzack) Persistenter Homologie* von Edelsbrunner, Letscher, Zomorodian, Carlsson, de Silva und Morozov hervorgeht. Wir liefern einen Beweis für einen Struktursatz für RISC, der von Crawley-Boevey, Höppner und Lenzing inspiriert ist, eine Theorie von Verschachtelungen im Sinne von Bubenik, de Silva, Scott und Scoccola sowie eine Äquivalenz zur *abgeleiteten Niveaumengen Persistenz* von Curry. Schließlich nutzen wir RISC, um eine abelsche Kategorifizierung von *erweiterten Persistenzdiagrammen* und ein *Mayer–Vietoris Prinzip* zu liefern. Die Teile III und IV können unabhängig von Teil II gelesen werden.

Collaboratory Contributions. This thesis is a succession of the two preprints [BBF21, BF22]. Other papers the author has contributed to during graduate studies are [BBF24, BBF22].

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Contents

Part I

Structure of Relative Interlevel Set Cohomology (RISC)

In this Part I, which grew out of the preprint [BBF21], we revisit the construction of the *Mayer–Vietoris pyramid* introduced by [CdM09]. As our continuous counterpart of the Mayer–Vietoris pyramid differs in several aspects from [CdM09, BEMP13], we refer to it as *relative interlevel set cohomology (RISC)*. This name points to the fact that RISC records a broad set of relative cohomology groups that may be associated to a real-valued function and induced maps between these. As a mathematical object, the RISC associated to a continuous function is a presheaf on a particular poset $\mathbb{M}_{\mathrm{f}}$, whose shape is an infinite strip in the Euclidean plane, taking values in $\mathrm{Vect}_{\mathbb{F}}$ the category of vector spaces over some field $\mathbb{F}$ fixed throughout. Now the RISC is not an arbitrary presheaf $\mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ as it satisfies certain exactness properties. Under mild tameness assumptions, we show that RISC, or any presheaf $\mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ satisfying corresponding exactness properties, decomposes into a direct sum of indecomposables, whose supports are maximal axis-aligned rectangles in $\mathbb{M}_{\mathrm{f}}$, which are fully determined by a single vertex inside the interior $\operatorname{int} \mathbb{M}_{\mathrm{f}}$ of $\mathbb{M}_{\mathrm{f}}$. More specifically, our Structure Theorem 2.6 states that for a presheaf $F \colon \mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ satisfying the necessary tameness and exactness properties, the multiplicity of the summand associated to the vertex $v \in \operatorname{int} \mathbb{M}_{\mathrm{f}}$ is the dimension $\dim_{\mathbb{F}} \mathrm{Nat}(F, S_v)$, where $S_v \colon \mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ is the simple presheaf associated to v. This perspective is related to [CGR$^+$22] and in Section 2.5.2 we show that the data provided this way is equivalent to the *extended persistence diagram* by [CSEH09].

Tameness Assumptions. The strong tameness assumptions in [BEMP13, Theorem 1] were weakened by [BLO20, Theorem 10.1] to all layers of the pyramid being pointwise finite-dimensional (pfd). The assumption to our Structure Theorem 2.6 is that all open interlevel sets have finite-dimensional cohomology in each degree. We call a continuous function $\mathbb{F}$-*tame* if it satisfies this property with respect to the field $\mathbb{F}$. We show in Lemma 1.38 below that a function is $\mathbb{F}$-tame iff its RISC with coefficients in $\mathbb{F}$ is pfd. Moreover, in Corollary 2.31 we show that our tameness assumption is in some sense equivalent to another tameness assumption referred to as *q-tameness* by [CdSGO16, Section 1.1].

Proof of Structure Theorem. In order to show the Structure Theorem 2.6 we harness a construction by [HL81, Theorem 1, Proposition 5] used in their proof that any projective functor on a poset taking values in a *module category* is *free*. More specifically, in order to obtain a decomposition of a presheaf $F \colon \mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ we use their *reduction modulo radical* to provide an *approximating* natural transformation from a direct sum of indecomposable presheaves to F. Then we use a *functorial filtration* inspired by [Rin75, CB15] to show this approximation is a natural isomorphism. This approximation by reduction modulo radical is also used in our proof of equivalence to the *level set barcode* in Section 2.5.1. We will also revisit the Structure Theorem 2.6 as well as the reduction modulo radical in Section 10.2, where we reinterpret Theorem 2.6 as Corollary 10.18, which states that all projective presheaves of an a posteriori abelian subcategory of presheaves $\mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ containing any pfd relative interlevel set cohomology is *free* in some sense. So with this reinterpretation of the Structure Theorem 2.6 we come full circle using the reduction modulo radical in a similar way as the authors of this construction [HL81].

Homology vs. Cohomology. We note that our construction of the relative interlevel set cohomology, which applies to any cohomology theory, has an analogous homological construction, which is dual in the following sense. For a homology theory valued in graded vector spaces sending weak equivalences to isomorphisms, we consider the corresponding dual cohomology theory. The resulting invariant is pfd iff this is also the case for the corresponding invariant defined in terms of homology. Moreover, as the duality of vector spaces restricts to an equivalence on finite-dimensional vector spaces, any decomposition of this RISC yields a decomposition of the corresponding "homological" invariant. This way, homological decompositions can be obtained by duality, which is why they are not treated explicitly in this exposition.

Related Work. The restriction of RISC to the subposet corresponding to the south face of the pyramid yields essentially the same data as *Mayer–Vietoris systems* introduced by [BGO19]. In this restricted setting the authors also provide a structure theorem as well as a stability result. While this restriction to the south face retains all information on the level of objects, it results in some loss of information on the level of morphisms as we will see in Section 2.6. In particular, different RISC interleavings can restrict to identical interleavings of Mayer–Vietoris systems, as we show in Example 6.25.

As noted above the present exposition is heavily inspired by [CdM09, BEMP13]. In particular, the RISC is a patchwork of the layers of the Mayer–Vietoris pyramid introduced by [CdM09] and when identifying any indexing point of the Mayer–Vietoris pyramid with its corresponding points in other layers, then we obtain a Möbius strip from this patchwork. So conceptually the RISC is a continuously indexed diagram on a poset in the shape of a covering of the Möbius strip, more specifically the universal covering as we will see later. We mention this relation to the Möbius strip here, as it has been discussed in [CdM09] skipping the connection to covering spaces or the universal covering in particular. We also note that the construction by [CdM09] is in the discretely indexed setting, which is somewhat more elementary than the continuously indexed setting as the indexing category can be constructed from vertices, arrows, and relations without any need for higher syzygies in this context. While the results by [BEMP13] are shown in the continuously indexed setting, the individual layers of the Mayer–Vietoris pyramid are dealt with independently, so there is no need to ensure consistent choices of basis vectors between homological degrees. This additional degree of freedom manifests itself in the fourth case of a proof by exhaustion of their [BEMP13, Basis Flip Lemma], where the authors choose *some* complementary subspace to the image of an intrinsically defined map. (In their own words, the authors consider "the orthogonal complement", which we interpreted here as some complement as there is no natural scalar product on homology groups.) We note however, that the results by [BEMP13] already suggest that a globally consistent choice of basis vectors should exist as the supports of indecomposables of each layer align exactly as shown in their [BEMP13, Figures 4 and 6].

We also note that [CdM09, BEMP13] both rely on Gabriels theorem, while our proof of the Structure Theorem 2.6 solely relies on elementary category theory and linear algebra and in conjunction with the results of Chapter 9 and Remark 9.17 in particular it even provides a new perspective on the (derived) structure theory of A_n-quiver representations.

The tameness assumptions from [CdM09, BEMP13] were also weakened in [CdSKM19] by using measure theory. Roughly speaking, the authors bypass the step involving interleavings of generalized persistence modules [BdSS15] and map functions directly to measures, which they compare with the bottleneck distance of persistence diagrams [CSEH07]. As mentioned above, the authors of [BLO20, Section 8.1] describe a way of obtaining a decomposition for *relative interlevel set homology (RISH)* when pfd using their theory of *rectangle-decomposable* modules, which has been developed in parallel to the approach provided here. We compare their approach and our approach at the beginning of Section 2.2.

We also note that one may obtain *interlevel set cohomology* from RISC by restriction to a subposet of $\mathbb{M}$. Thus, under the assumption that $f\colon X \to \mathbb{R}$ is $\mathbb{F}$-tame, its interlevel set cohomology decomposes as well. Similar results have been shown by [CO20, Section 9.3], [BCB20, Section 5], and [BGO19, Theorem 2.19].

Overview. In Section 1.1 we introduce the relative interlevel set cohomology (RISC) as an invariant of $\mathbb{F}$-tame real-valued continuous functions. Given a continuous function $f\colon X \to \mathbb{R}$, the study of *interlevel set persistent cohomology* concerns the cohomology (with field coefficients) of preimages $f^{-1}(I)$ of open intervals I. This construction can be extended to the relative cohomology of pairs $f^{-1}(I, C)$, where $I \subseteq \mathbb{R}$ is an open interval and $C \subseteq I$ is the complement $C = \mathbb{R} \setminus J$ of some closed interval J. Now taking the difference

$$(I, C) \mapsto I \setminus C$$

yields a bijection between the set of all such pairs (I, C) with $I \neq C$ and the set of non-empty intervals in $\mathbb{R}$. We refer to such pairs (I, C) as *interval pairs*. Moreover, for any pair of open subspaces (U, V) of $\mathbb{R}$ with $U \setminus V = I \setminus C$ the cohomologies of $f^{-1}(U, V)$ and $f^{-1}(I, C)$ are naturally isomorphic by excision. From our perspective the pair (I, C) is a particularly convenient choice to represent the interval $I \setminus C$, see also Proposition 1.2 below. Furthermore, given any pair of open subspaces (U, V) of $\mathbb{R}$ such that any connected component of U contains finitely[1] many connected components of V, the cohomology of $f^{-1}(U, V)$ is naturally isomorphic to a product of cohomologies for pairs $f^{-1}(I, C)$ as above. More specifically, for each such factor the difference $I \setminus C$ is a connected component of $U \setminus V$. We parametrize the set of all such pairs (I, C) as well as the cohomological degrees by a lattice $\mathbb{M}_\mathrm{f}$. As it turns out, any continuous function $f\colon X \to \mathbb{R}$ induces a presheaf $\mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$, with some of the internal maps induced

[1] We make this finiteness assumption here to exclude pathological situations, where $U \setminus V$ has non-open connected components as for example in a Cantor set. With this finiteness assumption, the present situation is amenable to an inductive argument using relative Mayer–Vietoris sequences in cohomology dual to [tom08, Theorem 10.7.3].

by inclusions and the other maps being differentials of a corresponding Mayer–Vietoris sequence and where $\mathrm{Vect}_{\mathbb{F}}$ denotes the category of vector spaces over some fixed field $\mathbb{F}$. The existence of these differentials is one of our motives to consider preimages of open subsets as opposed to closed subsets of $\mathbb{R}$. We refer to this functor as the *relative interlevel set cohomology (RISC)* of f when $f\colon X \to \mathbb{R}$ is $\mathbb{F}$-tame and we show that it satisfies certain exactness properties. We call such functors *cohomological*; this is Definition 1.26. Furthermore, we show in Lemma 1.38 and Proposition 1.41 that the RISC is pfd and *sequentially continuous* (Definition 1.39). As a byproduct, any cohomology class from the RISC determines a natural transformation from an indecomposable and vice versa. Dually, natural transformations from a corresponding homological construction to a *sequentially cocontinuous* indecomposable of a certain kind are one-to-one with elements of the dual space of a corresponding homology group. However, this dual space is naturally isomorphic to cohomology. Thus, cohomology even appears in the analogous construction of a decomposition of the corresponding homological invariant. This is part of the reason why we work with cohomology in place of homology. As noted above, this is no limitation in the present setting.

In Chapter 2 we show in Theorem 2.6 that any pfd sequentially continuous cohomological functor decomposes into a direct sum of indecomposables of a certain type. Each indecomposable can be characterized by its support, which is a maximal axis-aligned rectangle as shown in Fig. 2.1a. A posteriori, the upper left vertex of this rectangle gives the corresponding vertex in the *extended persistence diagram* as we define it in Definition 2.2. This close relationship between the indecomposables and the extended persistence diagram as well as the fact that there is just one type of indecomposable is our main motivation for gluing the layers of the Mayer–Vietoris pyramid to a single diagram.

1 Relative Interlevel Set Cohomology (RISC)

1.1 Constructing RISC

The authors of [CdM09] introduced their Mayer–Vietoris pyramid as an invariant of real-valued functions, which is a commutative diagram of graded vector spaces in the shape of a pyramid. The authors of [BEMP13] provided a continuously-indexed variant independent of the set of critical values. Moreover, [CdM09] proposed gluing the different layers of the Mayer–Vietoris along their edges. The following construction is a formalization of this idea in the continuously indexed setting. As suggested by Fig. 1.1 we glue the layers of the *cohomological Mayer–Vietoris pyramid* to one presheaf on a poset $\mathbb{M}_\mathrm{f} \subset \mathbb{R}^\circ \times \mathbb{R}$ in the shape of an infinite strip, where $\mathbb{R}^\circ$ is the opposite poset of $\mathbb{R}$. We provide a formal definition of the cohomological Mayer–Vietoris pyramid in the text below and we will name this presheaf $h(f) \colon \mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$ associated to a continuous function $f \colon X \to \mathbb{R}$ the *relative interlevel set cohomology (RISC)* of f. Here $\mathbb{M}_\mathrm{f}^\circ$ denotes the opposite poset of $\mathbb{M}_\mathrm{f} \subset \mathbb{R}^\circ \times \mathbb{R}$. Just like the Mayer–Vietoris pyramid vanishes on the west and east edges, the relative interlevel set cohomology $h(f) \colon \mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$ vanishes on the boundary $\partial\mathbb{M}_\mathrm{f} = l_0 \cup l_1$, more on this later.

1.1.1 The Strip $\mathbb{M}$ Indexing RISC

We start with specifying the indexing poset $\mathbb{M}_\mathrm{f}$ for relative interlevel set cohomology. To this end, let $\mathbb{R}$ and $\mathbb{R}^\circ$ denote the posets given by the orders $\leq$ and $\geq$ on $\mathbb{R}$, respectively. Then we may form the product poset $\mathbb{R}^\circ \times \mathbb{R}$, which is a lattice and whose underlying set is the Euclidean plane. Let l_0 and l_1 be two lines of slope -1 in $\mathbb{R}^\circ \times \mathbb{R}$ with l_1 sitting above l_0 as shown in Fig. 1.2. Then the indexing poset $\mathbb{M}$ is the sublattice of $\mathbb{R}^\circ \times \mathbb{R}$ given by the convex hull of l_0 and l_1. For the specific choices of l_0 and l_1 with l_1 and l_0 intersecting the x-axis at $r > 0$ and $-r$ respectively, we write $\mathbb{M}_r := \mathbb{M}$ and $\mathbb{M}_\mathrm{f} := \mathbb{M}_\pi$. (The "f" in $\mathbb{M}_\mathrm{f}$ stands for "full" as we use $\mathbb{M}_\mathrm{f}$ to parametrize the *full relative interlevel set cohomology* of any given function; other indexing posets $\mathbb{M}_r$, $r > 0$ can be convenient indexing posets for other variants of persistent homology such as *relative sublevel set cohomology* indexed by $\mathbb{M}_{\frac{\pi}{2}}$, which is subject to future work.) The degree-shift in cohomology will correspond to the central[1] automorphism $\Sigma \colon \mathbb{M} \to \mathbb{M}$ with the following defining property (also see Fig. 1.2):

[1]By a central automorphism we mean an automorphism that commutes with any other lattice automorphism of $\mathbb{M}$.

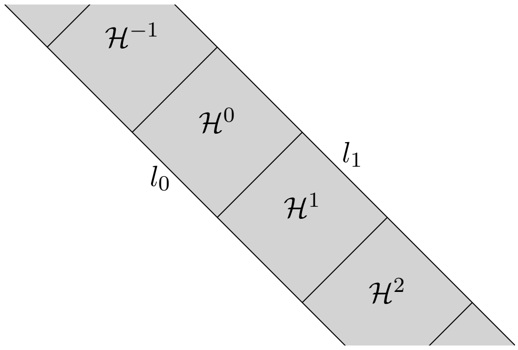


Figure 1.1: The different layers of the Mayer–Vietoris pyramid glued together to form a presheaf on the indexing poset $\mathbb{M} \subset \mathbb{R}^{\circ} \times \mathbb{R}$. Here we also rotated the Mayer–Vietoris pyramid by 45 degrees counter-clockwise before the gluing so that we could use the product partial order on $\mathbb{R}^{\circ} \times \mathbb{R}$.

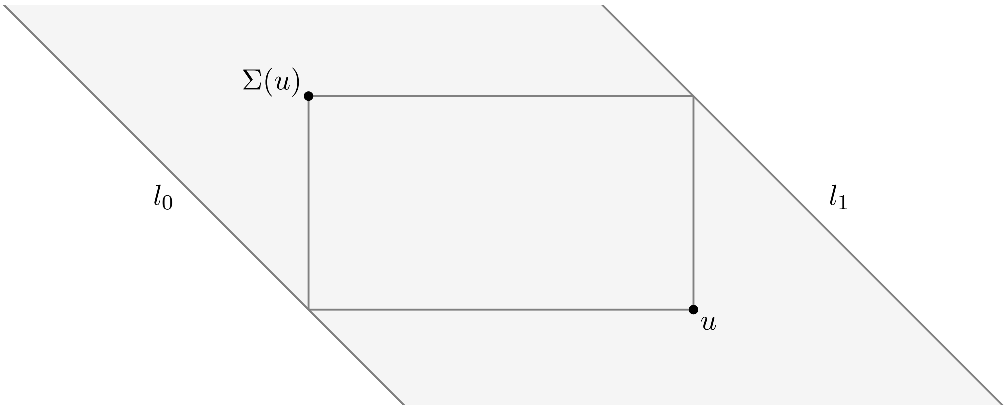


Figure 1.2: The maximal axis-aligned rectangle with lower right vertex u and upper left vertex $\Sigma(u)$.

> For any $u \in \mathbb{M}$ there is a (unique) maximal axis-aligned rectangle contained in $\mathbb{M}$ with lower right vertex u and upper left vertex $\Sigma(u)$.

We also note that $\Sigma \colon \mathbb{M} \to \mathbb{M}$ is a glide reflection along the bisecting line between l_0 and l_1, and the amount of translation is the distance of l_0 and l_1. Moreover, as a space, $\mathbb{M}/\langle\Sigma\rangle$ is a Möbius strip; see also [CdM09].

Remark 1.1 (Justification of Our Notation for $\Sigma \colon \mathbb{M} \to \mathbb{M}$). As noted above, the RISC $h(f) \colon \mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ associated to a function $f \colon X \to \mathbb{R}$ will be a presheaf vanishing on the boundary $\partial\mathbb{M}_{\mathrm{f}}$. In particular, the RISC $h(f)$ maps any arrow $u \preceq v$ between boundary points $u, v \in \partial\mathbb{M}$ to an isomorphism. Now let W be the wide subcategory of $\mathbb{M}$ that is the (disjoint) union of the full subcategory on the boundary points $\partial\mathbb{M}$ and the discrete subcategory on the interior points $\operatorname{int}\mathbb{M}$. Then $\mathbb{M}$ is a *homotopical category* with weak equivalences W in the sense of [Rie14, Definition 2.1.1]. We consider the simplicial localization $L\mathbb{M}$ of $\mathbb{M}$ at the weak equivalences W. For any two comparable points

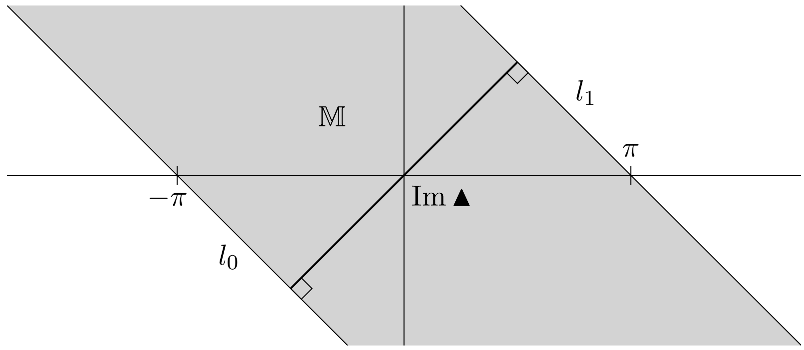


Figure 1.3: The strip $\mathbb{M}_\mathrm{f}$ and the image of the embedding $\blacktriangle\colon \overline{\mathbb{R}} \to \mathbb{M}_\mathrm{f}$.

$u \preceq v \in \mathbb{M}$ with $[u, v] \subset \operatorname{int}\mathbb{M}$ the simplicial mapping set $\operatorname{Hom}_{L\mathbb{M}}(u, v)$ has two connected components. One of them containing (the image of) the arrow $u \preceq v$ and one containing all admissible zigzags with at least one vertex a boundary point in $\partial\mathbb{M}$. As it turns out, both of these connected components are weakly contractible. In the more general case, where we have $\Sigma^n(u) \preceq v$ and $\big[\Sigma^n(u), v\big] \subset \operatorname{int}\mathbb{M}$ for some $n \in \mathbb{N}_0$, the simplicial mapping set $\operatorname{Hom}_{L\mathbb{M}}(u, v)$ is weakly equivalent to S^n. In any other case the simplicial mapping set is weakly contractible. Moreover, any boundary point in $\partial\mathbb{M}$ is an $(\infty, 1)$-zero object of $L\mathbb{M}$ and any axis-aligned rectangle in $\mathbb{M}$ yields an $(\infty, 1)$-bicartesian square in $L\mathbb{M}$. In particular, the maximal axis-aligned rectangle shown in Fig. 1.2 and defining $\Sigma(u)$ for $u \in \mathbb{M}$ yields an $(\infty, 1)$-bicartesian square in $L\mathbb{M}$ with the two boundary points $(\infty, 1)$-zero objects. Thus, $\Sigma(u)$ is the *suspension* of u in $L\mathbb{M}$ and u is the *loop space object* of $\Sigma(u)$. Adopting the terminology by [Hel68] we may also say that $\mathbb{M}$ is *strictly stable* with suspension $\Sigma\colon \mathbb{M} \to \mathbb{M}$, see also Definition 1.8 below. We note however, that $L\mathbb{M}$ is not a stable $(\infty, 1)$-category as only vertical and horizontal arrows have a kernel and a cokernel, see also Remark 1.27. For a note on the dg-category corresponding to $L\mathbb{M}$ and its relation to commonly used concepts of representation theory see Remark 7.18 below.

1.1.2 Parametrizing Interval Pairs Using the Strip $\mathbb{M}_\mathrm{f}$

A posteriori the region of $\mathbb{M}_\mathrm{f}$ indexing relative interlevel set cohomology in degree 0 will provide a fundamental domain D with respect to the action of $\langle\Sigma\rangle \cong \mathbb{Z}$ on $\mathbb{M}_\mathrm{f}$, which we specify now. To this end, we embed the extended reals $\overline{\mathbb{R}} := [-\infty, \infty]$ into the strip $\mathbb{M}_\mathrm{f}$ by precomposing the diagonal map $\Delta\colon \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (t, t)$ with the homeomorphism $\arctan\colon \overline{\mathbb{R}} \to [-\pi/2, \pi/2]$, yielding a map

$$\blacktriangle = \Delta \circ \arctan\colon \overline{\mathbb{R}} \to \mathbb{M}_\mathrm{f},\ t \mapsto (\arctan t, \arctan t)$$

such that $\operatorname{Im}\blacktriangle$ is a perpendicular line segment through the origin joining l_0 and l_1, see Fig. 1.3. We specify the fundamental domain as shown in Fig. 1.4 by

$$D := ({\downarrow}\operatorname{Im}\blacktriangle) \setminus \Sigma^{-1}({\downarrow}\operatorname{Im}\blacktriangle),$$

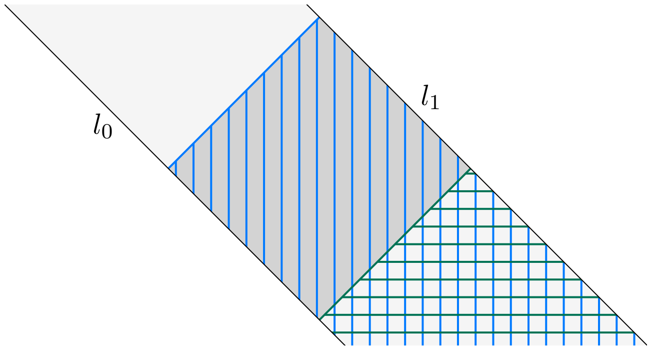


Figure 1.4: The fundamental domain $D := \downarrow \operatorname{Im} \blacktriangle \setminus \Sigma^{-1}(\downarrow \operatorname{Im} \blacktriangle)$.

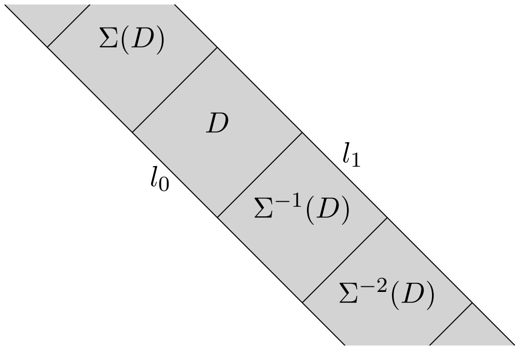


Figure 1.5: The tessellation of $\mathbb{M}_{\mathrm{f}}$ induced by Σ and D.

where $\downarrow \operatorname{Im} \blacktriangle$ is the downset of the image of $\blacktriangle$. Fig. 1.5 shows the tessellation of $\mathbb{M}_{\mathrm{f}}$ induced by Σ and D.

The purpose of this fundamental domain D is to parametrize all *interval pairs* (I, C), these are all pairs (I, C) where $I \subseteq \mathbb{R}$ is an open interval and $C \subseteq I$ is the complement of some closed interval. To specify such a pair for each point in D the following proposition characterizes a monotone map ρ from $\mathbb{M}_{\mathrm{f}}$ to the poset of pairs of open subspaces of $\mathbb{R}$, which is locally constant on $\mathbb{M}_{\mathrm{f}} \setminus D$; a schematic image of ρ is shown in Fig. 1.6.

Proposition 1.2. *Let $\mathcal{P}$ denote the set of pairs of open subspaces of $\mathbb{R}$. Then there is a unique monotone map*

$$\rho = (\rho_1, \rho_0) \colon \mathbb{M}_{\mathrm{f}} \to \mathcal{P}$$

with the following four properties:

(1) For any $t \in \mathbb{R}$ we have $(\rho \circ \blacktriangle)(t) = (\mathbb{R}, \mathbb{R} \setminus \{t\})$.

(2) For any $u \in \partial \mathbb{M}_{\mathrm{f}}$ we have $\rho_1(u) = \rho_0(u)$.

(3) For any axis-aligned rectangle contained in $\uparrow D$ the corresponding joins and meets are preserved by ρ_1.

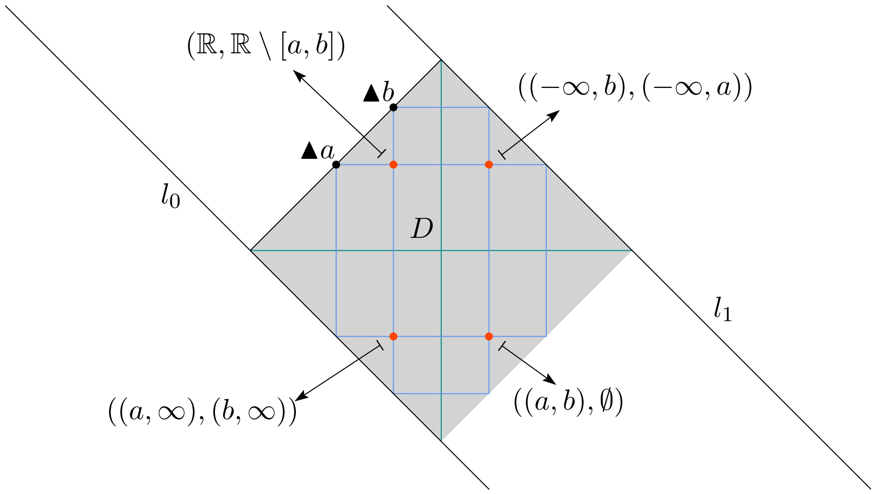


Figure 1.6: A schematic image of ρ.

(4) For any axis-aligned rectangle contained in $\downarrow D$ *the corresponding joins and meets are preserved by* ρ_0.

The reason we consider $\rho\colon \mathbb{M}_\mathrm{f} \to \mathcal{P}$ as a monotone map defined on $\mathbb{M}_\mathrm{f}$ rather than $D \subset \mathbb{M}_\mathrm{f}$ is that the additional points in $\mathbb{M}_\mathrm{f} \setminus D$ will be useful when we extend RISC to a locally persistent functor in Part II. We also note that we have the explicit formula

$$\rho(u) = \mathbb{R} \cap \blacktriangle^{-1} \left(\operatorname{int}(\downarrow \Sigma(u)), \mathbb{M}_\mathrm{f} \setminus \uparrow u\right),$$

where $\operatorname{int}(\downarrow \Sigma(u))$ is the interior of the downset of $\Sigma(u)$.

1.1.3 The Mayer–Vietoris Pyramid

The next step in the construction of relative interlevel set cohomology is the construction of the (continuous) Mayer–Vietoris pyramid due to [CdM09, BEMP13]. Now in the present monograph we work with open subsets and cohomology, which requires the use of contravariant functors. However, in some respect contravariant functors are notationally not as convenient as covariant functors. For this reason, we introduce the following abstraction.

Definition 1.3 (Mayer–Vietoris Functors, [Osb79, Section 1])**.** Let $\mathcal{T}$ be the category of pairs $X = (X_1, X_0)$ with X_1 a topological space, and $X_0 \subseteq X_1$ an open subspace and let $\mathcal{W}$ be some additive (or pointed) category. We name $\mathcal{T}$ the *category of pairs of spaces and open subsets*. A *Mayer–Vietoris functor* $\mathcal{T} \to \mathcal{W}$ is a functor

$$\mathcal{H}_\bullet\colon \mathcal{T} \to \mathcal{W}^{\mathbb{Z}}$$

to the category $\mathcal{W}^{\mathbb{Z}}$ of $\mathbb{Z}$-graded[2] objects in $\mathcal{W}$ vanishing on pairs with identical compo-

[2] see also Definition 1.11 below

nents, together with a natural *boundary operator*

$$\partial_{(X;A,B)} \colon \mathcal{H}_\bullet(X) \longrightarrow \mathcal{H}_{\bullet-1}(A \cap B)$$

for any *triad* $(X; A, B)$ of pairs of $\mathcal{T}$, i.e. we have componentwise $A, B \subseteq X = A \cup B$ and $A \cap B$ denotes the componentwise intersection of $A = (A_1, A_0)$ and $B = (B_1, B_0)$.

Example 1.4. If we have a cohomology theory $\mathcal{H}^\bullet$ taking values in the category $\mathrm{Vect}_{\mathbb{F}}$ of vector spaces over a field $\mathbb{F}$, then we may view $\mathcal{H}^\bullet$ as a Mayer–Vietoris functor

$$\mathcal{H}_\bullet \colon \mathcal{T} \to (\mathrm{Vect}_{\mathbb{F}}^{\circ})^{\mathbb{Z}}, \, X \mapsto \mathcal{H}^\bullet(X).$$

By excision any triad in $\mathcal{T}$ is excisive in each component in the sense of [tom08, Section 17.1.4], hence the boundary operators ∂ can be obtained from the corresponding Mayer–Vietoris sequences as described in [tom08, Section 17.1.4].

While we use this abstraction by [Osb79, Section 1] primarily for notational convenience, it also makes transparent, which of our constructions are "purely categorical" and which are (linear) algebraic. Another obvious application would be that of a homology theory. However, for field coefficients any homology theory can be turned into a cohomology theory without any loss of information. Moreover, from the present perspective cohomology theories are more practical when working with open subsets. Now let $f \colon X \to \mathbb{R}$ be a continuous function and let $\mathcal{H}_\bullet \colon \mathcal{T} \to \mathcal{W}^{\mathbb{Z}}$ be a Mayer–Vietoris functor in the sense of Definition 1.3. Then we obtain the *Mayer–Vietoris pyramid* as a functor

$$F' \colon D \to \mathcal{W}^{\mathbb{Z}}, u \mapsto \big(\mathcal{H}_\bullet \circ f^{-1} \circ \rho\big)(u),$$

taking values in the category of $\mathbb{Z}$-graded objects in $\mathcal{W}$. Now the degree-shift yields the autofunctor

$$\Sigma \colon \mathcal{W}^{\mathbb{Z}} \to \mathcal{W}^{\mathbb{Z}}, \, M_\bullet \mapsto M_{\bullet-1}$$

on graded objects in $\mathcal{W}$. As mentioned in Remark 1.12 below, this notation is inspired by the suspension functor on chain complexes.

1.1.4 Construction of the RISC Presheaf

The Mayer–Vietoris pyramid $F' \colon D \to \mathcal{W}^{\mathbb{Z}}$ is a functor taking values in the category of $\mathbb{Z}$-graded objects in $\mathcal{W}$ on the fundamental domain D. We now glue its layers to a single functor $\mathbb{M}_{\mathrm{f}} \to \mathcal{W}$ using the boundary operator ∂ of $\mathcal{H}_\bullet$ as a Mayer–Vietoris functor in the sense of Definition 1.3.

As an intermediate step, we extend F' to a functor

$$F \colon \mathbb{M}_{\mathrm{f}} \to \mathcal{W}^{\mathbb{Z}},$$

which is $\mathbb{Z}$-equivariant or *strictly stable* in the sense that

$$F \circ \Sigma = \Sigma \circ F. \tag{1.1}$$

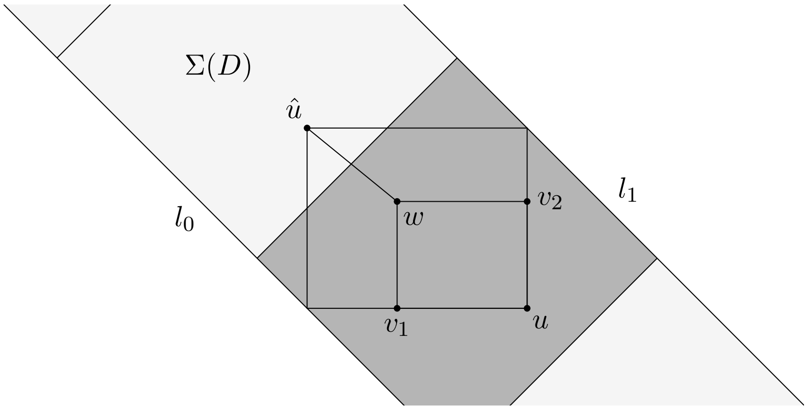


Figure 1.7: The axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in D$ determined by $(w, \hat{u}) \in R_D$.

This means that we are attaching a full copy of the Mayer–Vietoris pyramid to each tile of the strip $\mathbb{M}$ (Fig. 1.1). Note that on the level of objects of $\mathcal{W}^{\mathbb{Z}}$, the resulting functor F carries no new information in comparison to F', and we discard the redundant information by evaluating at degree 0:

$$\mathrm{ev}_0 \colon \mathcal{W}^{\mathbb{Z}} \to \mathcal{W},\, M_\bullet \mapsto M_0.$$

Now in order to obtain such a strictly stable functor F from F' we need to glue consecutive layers using boundary operators. To this end, let

$$R_D := \{(w, \hat{u}) \in D \times \Sigma(D) \mid w \preceq \hat{u} \preceq \Sigma(w)\}.$$

As shown in Fig. 1.7, any pair $(w, \hat{u}) \in R_D$ determines an axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in D$ with $\Sigma(u) = \hat{u}$. Moreover, as this rectangle is contained in D, the corresponding join $w = v_1 \vee v_2$ and meet $u = v_1 \wedge v_2$ are preserved by ρ by Proposition 1.2.(3-4). Furthermore, since taking preimages is a homomorphism of boolean algebras, f^{-1} also preserves joins and meets, which in this case are the componentwise unions and intersections. This means that $f^{-1}(\rho(u))$ is the componentwise intersection of $f^{-1}(\rho(v_1))$ and $f^{-1}(\rho(v_2))$, while $f^{-1}(\rho(w))$ is their union. With this we obtain the triad $f^{-1}(\rho(w); \rho(v_1), \rho(v_2))$ in the category $\mathcal{T}$ of pairs of spaces and open subsets in the sense of Definition 1.3. Thus, we have the boundary operators

$$\partial'_{(w,\hat{u})} \colon \left(\mathcal{H}_\bullet \circ f^{-1} \circ \rho\right)(w) \to \left(\mathcal{H}_{\bullet-1} \circ f^{-1} \circ \rho\right)(u)$$

of the Mayer–Vietoris functor $\mathcal{H}_\bullet$ for the triad $f^{-1}(\rho(w); \rho(v_1), \rho(v_2))$. We now view R_D as a subposet of $D \times \Sigma(D)$. Moreover, let $\mathrm{pr}_1 \colon R_D \to D$ and $\mathrm{pr}_2 \colon R_D \to \Sigma(D)$ be the projections to the first and the second component, respectively. Then ∂' is a natural transformation as in the diagram

$$
\begin{tikzcd}
R_D \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\swarrow \partial'"] & D \arrow[d, "F'"] \\
\Sigma(D) \arrow[r, "\Sigma \circ F' \circ \Sigma^{-1}"'] & \mathcal{W}^{\mathbb{Z}}.
\end{tikzcd}
$$

As a result of Proposition 1.23 below, the functor $F' \colon D \to \mathcal{W}^{\mathbb{Z}}$ and ∂ determine a unique functor $F \colon \mathbb{M}_{\mathrm{f}} \to \mathcal{W}^{\mathbb{Z}}$ that is strictly stable in the sense that $\Sigma \circ F = F \circ \Sigma$. To obtain a functor of type $\mathbb{M}_{\mathrm{f}} \to \mathcal{W}$ from F we post-compose F with the evaluation at 0:

$$h^{\circ}(f) \coloneqq \mathrm{ev}_0 \circ F \colon \mathbb{M}_{\mathrm{f}} \to \mathcal{W}. \tag{1.2}$$

Definition 1.5 (Relative Interlevel Set Cohomology (RISC))**.** If we are in the situation of Example 1.4, with $\mathcal{W}$ being the opposite category of the category of vector spaces over some field $\mathbb{F}$ and $\mathcal{H}_\bullet$ being the categorical dual of a cohomology theory $\mathcal{H}^\bullet$ taking values in $\mathrm{Vect}_{\mathbb{F}}$, then we name

$$h(f) \coloneqq h^{\circ\circ}(f) = (\mathrm{ev}_0 \circ F)^{\circ} \colon \mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$$

the *relative interlevel set cohomology (RISC) of f induced by $\mathcal{H}^\bullet$.* Moreover, if $\mathcal{H}^\bullet$ is singular cohomology with coefficients in $\mathbb{F}$, then we name $h(f)$ the *relative interlevel set cohomology (RISC) of f with coefficients in* $\mathbb{F}$.

Remark 1.6. In [BBF24] we provide a construction analogous to Definition 1.5 for piecewise linear (PL) functions on finite simplicial complexes using singular homology in place of a general cohomology theory; the reason being that PL functions on finite simplicial complexes are sufficiently tame to obtain a well-behaved theory using homology and preimages of closed subsets. We name this construction *relative interlevel set homology (RISH)*. Moreover, in [BBF24, Appendix B] we also provide more general construction of RISH that is relative to a subspace of the domain of a function analogous to the relative homology of spaces. We note there is a closely analogous construction in the present context as well, which does not require any new insight beyond above construction or [BBF24, Appendix B]. In order to constrain the cognitive load, we limited the present exposition to the special case of absolute relative interlevel set cohomology.

1.1.5 Extending RISC to a Contravariant Functor

In the following we extend h to a contravariant functor from the category of spaces over the reals $\mathbb{R}$, henceforth called $\mathbb{R}$-*spaces*, to the category of presheaves on $\mathbb{M}_{\mathrm{f}}$. For notational convenience, let $\mathcal{H}_\bullet \colon \mathcal{T} \to \mathcal{W}^{\mathbb{Z}}$ be a Mayer–Vietoris functor in the sense of

Definition 1.3. For a commutative triangle

$$\begin{tikzcd} X \arrow[rr, bend left, "\varphi"] \arrow[dr, "f\circ\varphi"'] & & Y \arrow[dl, "f"] \\ & \mathbb{R} & \end{tikzcd} \tag{1.3}$$

of topological spaces, the map φ yields a continuous map of pairs

$$\left(\varphi^{-1} \circ f^{-1} \circ \rho\right)(u) \to \left(f^{-1} \circ \rho\right)(u),$$

which is natural in $u \in D$. By the functoriality of the Mayer–Vietoris functor $\mathcal{H}_\bullet$, the naturality of its boundary operator ∂, and the naturality part of Proposition 1.23, the collection of these maps induces a natural transformation

$$h^\circ(\varphi)_f \colon h^\circ(f \circ \varphi) \to h^\circ(f). \tag{1.4}$$

As $\varphi\colon X \to Y$ determines a morphism in the category of $\mathbb{R}$-spaces as in (1.3) for any function $Y \to \mathbb{R}$, we use f as a subscript in (1.4) to disambiguate the corresponding natural transformations. Combining (1.2) and (1.4) we obtain the functor

$$h^\circ \colon \mathrm{Top}/\mathbb{R} \to \mathcal{W}^{\mathbb{M}_\mathrm{f}}.$$

Again, whenever $\mathcal{W} = \mathrm{Vect}_\mathbb{F}^\circ$ and $\mathcal{H}_\bullet$ is the categorical dual of a cohomology theory $\mathcal{H}^\bullet$ taking values in $\mathrm{Vect}_\mathbb{F}$, then we obtain a contravariant functor

$$h \coloneqq h^{\circ\circ} \colon (\mathrm{Top}/\mathbb{R})^\circ \to \mathrm{Vect}_\mathbb{F}^{\mathbb{M}_\mathrm{f}^\circ}$$

henceforth called *relative interlevel set cohomology (RISC)*.

Remark 1.7. The subscript notation in (1.4) is not random and we can explain it using the notions from Appendix A.2. More specifically, we describe the data that specifies the functor h° as a *lax cocone* in the strict 2-category of categories under the contravariant functor

$$\mathrm{Hom}(-, \mathbb{R}) \colon \mathrm{Top}^\circ \to \mathrm{Set} \hookrightarrow \mathrm{Cat}$$

with vertex $\mathcal{W}^{\mathbb{M}_\mathrm{f}}$ in the sense of Definition A.2. (Here we embed the category of sets into the category of categories via the functor $\mathrm{Set} \hookrightarrow \mathrm{Cat}$ that maps a set to the corresponding discrete category.) While such a description does not gain us anything at this point, it will be convenient in Part II, where we substitute *locally persistent categories* introduced by [Sco20] for ordinary categories. Now in order to specify such a lax cocone, we have to provide a functor $G(X) \colon \mathrm{Hom}(X, \mathbb{R}) \to \mathcal{W}^{\mathbb{M}_\mathrm{f}}$ for any topological space X in particular, where we view $\mathrm{Hom}(X, \mathbb{R})$ as a discrete category. To this end, we may just specify $G(X) = h^\circ|_{\mathrm{Hom}(X,\mathbb{R})}$. Now suppose we have a continuous map $\varphi\colon X \to Y$. To comply with Definition A.2, we have to specify a natural transformation

$$G(\varphi) \colon h^\circ \circ \mathrm{Hom}(\varphi, \mathbb{R}) \Longrightarrow h^\circ|_{\mathrm{Hom}(Y,\mathbb{R})}.$$

As $\mathrm{Hom}(Y, \mathbb{R})$ is discrete, this is just a family of morphisms $h^\circ(f \circ \varphi) \to h^\circ(f)$ in $\mathcal{W}^{\mathbb{M}_\mathrm{f}}$, one for each continuous function $f \colon Y \to \mathbb{R}$. Now we already provided such a morphism in (1.4). Denoting this family of morphisms by

$$h^\circ(\varphi) \colon h^\circ \circ \mathrm{Hom}(\varphi, \mathbb{R}) \Rightarrow h^\circ|_{\mathrm{Hom}(Y,\mathbb{R})}$$

we may specify $G(\varphi) = h^\circ(\varphi)$. This way our convention to use f as an index in (1.4) is consistent with the notational conventions for natural transformations. Moreover, by the functoriality of the Mayer–Vietoris functor $\mathcal{H}_\bullet$ and the naturality of its boundary operator ∂, the diagram of natural transformations (A.2) commutes. Thus, the data just provided forms a lax cocone under the contravariant functor

$$\mathrm{Hom}(-, \mathbb{R}) \colon \mathrm{Top}^\circ \to \mathrm{Set} \hookrightarrow \mathrm{Cat}$$

with vertex $\mathcal{W}^{\mathbb{M}_\mathrm{f}}$ in the sense of Definition A.2. Now in order to connect this back to the functor $h^\circ \colon \mathrm{Top}/\mathbb{R} \to \mathcal{W}$ specified above, we also define a lax cocone $\bar{G}$ under $\mathrm{Hom}(-, \mathbb{R})$ with vertex $\mathrm{Top}/\mathbb{R}$, which is initial, by

$$\begin{aligned} \bar{G}(X) \colon \mathrm{Hom}(X, \mathbb{R}) \to \mathrm{Top}/\mathbb{R},\ f \mapsto f \\ \bar{G}(\varphi \colon X \to Y) \colon \mathrm{Hom}(\varphi, \mathbb{R}) \Rightarrow \bar{G}(Y),\ f \mapsto \bar{G}(\varphi)_f = \varphi. \end{aligned} \tag{1.5}$$

As the lax cocone specified by (1.5) is initial, the lax cocone with vertex $\mathcal{W}^{\mathbb{M}_\mathrm{f}}$ that we denoted as G induces a functor $\mathrm{Top}/\mathbb{R} \to \mathcal{W}^{\mathbb{M}_\mathrm{f}}$. As it turns out, this functor $\mathrm{Top}/\mathbb{R} \to \mathcal{W}^{\mathbb{M}_\mathrm{f}}$ induced by the lax cocone G under $\mathrm{Hom}(-, \mathbb{R})$ with vertex $\mathcal{W}^{\mathbb{M}_\mathrm{f}}$ is precisely the functor $h^\circ \colon \mathrm{Top}/\mathbb{R} \to \mathcal{W}^{\mathbb{M}_\mathrm{f}}$ we specified above.

1.2 Stable Functors on the Strip $\mathbb{M}$

In the following we provide the missing ingredient Proposition 1.23 that we used in Section 1.1.4 for the construction of RISC. We will also deduce the related Lemma 1.24, which will be useful when we construct interleavings of RISC in Part II. As we will also need these results for other choices of a fundamental domain D in Part IV, we will deduce these results in a more general setting.

Definition 1.8 (Strictly Stable Category, [Hel68]). A *strictly stable category* is a pair of a category $\mathcal{C}$ and a *suspension* automorphism of categories $\Sigma \colon \mathcal{C} \to \mathcal{C}$.

We adopt the convention by [Hel68] to denote the suspension automorphisms of different strictly stable categories all by the same letter Σ.

Example 1.9. The poset $\mathbb{M} \subset \mathbb{R}^\circ \times \mathbb{R}$, seen as a thin category, is a strictly stable category when endowed with the suspension automorphism $\Sigma \colon \mathbb{M} \to \mathbb{M}$.

Now let $\mathcal{C}_1$ and $\mathcal{C}_2$ be strictly stable categories endowed with automorphisms $\Sigma \colon \mathcal{C}_i \to \mathcal{C}_i$, $i = 1, 2$.

Definition 1.10 (Strictly Stable Functor)**.** A functor $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ is *strictly stable* if it commutes with Σ, i.e. $\Sigma \circ F = F \circ \Sigma$. Similarly, a natural transformation $\eta\colon F \to G$ is *strictly stable* if it commutes with Σ, i.e. $\Sigma \circ \eta = \eta \circ \Sigma$, where the symbol $\circ$ is used to denote right and left whiskering, respectively.

With these definitions strictly stable categories form a strict 2-category. Moreover, by dropping the associated automorphism, we obtain a strict *forgetful* 2-functor from strictly stable categories to categories. We will now construct a right adjoint to this forgetful functor. To this end, let $\mathcal{W}$ be an ordinary category, then we may associate a strictly stable category $\mathcal{W}^{\mathbb{Z}}$ to $\mathcal{W}$.

Definition 1.11 (Category of Graded Objects)**.** For an ordinary category $\mathcal{W}$ we define the strictly stable *category of $\mathbb{Z}$-graded objects in $\mathcal{W}$*, denoted as $\mathcal{W}^{\mathbb{Z}}$, to be the category whose objects are maps $M_\bullet\colon n \mapsto M_n$ from $\mathbb{Z}$ to the class of objects in $\mathcal{W}$ and whose morphisms, compositions, and identities are defined pointwise in $\mathcal{W}^{\mathbb{Z}}$. As the associated automorphism we choose

$$\Sigma\colon \mathcal{W}^{\mathbb{Z}} \to \mathcal{W}^{\mathbb{Z}},\ M_\bullet \mapsto \Sigma(M_\bullet) \coloneqq M_{\bullet-1}.$$

Remark 1.12. If $\mathcal{W}$ is an abelian category and $M_\bullet$ is an object of $\mathcal{W}^{\mathbb{Z}}$, then $\Sigma(M_\bullet) = M_{\bullet-1}$ is naturally quasi-isomorphic to the suspension of $M_\bullet$ as a chain complex with vanishing boundary operators.

This construction yields a strict 2-functor from categories to strictly stable categories.

Lemma 1.13. *The forgetful functor and $(-)^{\mathbb{Z}}$ form a strict 2-adjunction with the evaluation at 0 as the counit*

$$\mathrm{ev}_0\colon \mathcal{W}^{\mathbb{Z}} \to \mathcal{W},\quad M_\bullet \mapsto M_0.$$

In a "contravariant context", we will also write $M^\bullet$, $\Omega(M^\bullet) \coloneqq M^{\bullet-1}$, and $\mathrm{ev}^0\colon \mathcal{W}^{\mathbb{Z}} \to \mathcal{W}^{\mathbb{Z}}$ in place of $M_\bullet$, Σ, and ev_0, respectively.

Corollary 1.14. *Let $\mathrm{Func}_\Sigma\big(\mathbb{M}, \mathcal{W}^{\mathbb{Z}}\big)$ be the category of strictly stable functors $\mathbb{M} \to \mathcal{W}^{\mathbb{Z}}$, and let $\mathrm{Func}\big(\mathbb{M}, \mathcal{W}^{\mathbb{Z}}\big)$ and $\mathrm{Func}(\mathbb{M}, \mathcal{W})$ be the corresponding categories of ordinary functors. Then the post-composition of the forgetful functor $\mathrm{Func}_\Sigma\big(\mathbb{M}, \mathcal{W}^{\mathbb{Z}}\big) \to \mathrm{Func}\big(\mathbb{M}, \mathcal{W}^{\mathbb{Z}}\big)$ with*

$$\mathrm{Func}(\mathbb{M}, \mathrm{ev}_0)\colon \mathrm{Func}\big(\mathbb{M}, \mathcal{W}^{\mathbb{Z}}\big) \to \mathrm{Func}(\mathbb{M}, \mathcal{W})$$

yields an isomorphism of categories.

It will be useful to have a counterpart to Corollary 1.14 in the context of contravariant functors $\mathbb{M}^\circ \to \mathcal{W}$ for easier reference.

Corollary 1.15. *Let $\mathrm{Func}_\Sigma\big(\mathbb{M}^\circ, \mathcal{W}^{\mathbb{Z}}\big)$ be the category of strictly stable contravariant functors $\mathbb{M}^\circ \to \mathcal{W}^{\mathbb{Z}}$, and let $\mathrm{Func}\big(\mathbb{M}^\circ, \mathcal{W}^{\mathbb{Z}}\big)$ and $\mathrm{Func}(\mathbb{M}^\circ, \mathcal{W})$ be the corresponding categories of ordinary contravariant functors. Then the post-composition of the forgetful functor $\mathrm{Func}_\Sigma\big(\mathbb{M}^\circ, \mathcal{W}^{\mathbb{Z}}\big) \to \mathrm{Func}\big(\mathbb{M}^\circ, \mathcal{W}^{\mathbb{Z}}\big)$ with*

$$\mathrm{Func}\big(\mathbb{M}^\circ, \mathrm{ev}^0\big)\colon \mathrm{Func}\big(\mathbb{M}^\circ, \mathcal{W}^{\mathbb{Z}}\big) \to \mathrm{Func}\big(\mathbb{M}^\circ, \mathcal{W}\big)$$

yields an isomorphism of categories.

Now let $\mathcal{A}$ be an additive (or pointed) category, let $C \subset \mathbb{M}$ be a convex subposet, and let $F\colon C \to \mathcal{A}$ be a functor vanishing on $C \cap \partial\mathbb{M}$.

Lemma 1.16. *Let $u, v \in C$ with $u \preceq v \npreceq \Sigma(u)$. Then $F(u \preceq v) = 0$.*

Now let $\Sigma\colon \mathcal{A} \to \mathcal{A}$ be an automorphism of $\mathcal{A}$ preserving zero (or terminal) objects and let $F\colon \mathbb{M} \to \mathcal{A}$ be a strictly stable functor with respect to Σ vanishing on $\partial\mathbb{M}$, let D be a convex subposet of $\mathbb{M}$ that is a fundamental domain with respect to the action of $\langle\Sigma\rangle$, and let $F' := F|_D$.

Definition 1.17. We set $R_D := \big\{(v, w) \in D \times \Sigma(D) \,\big|\, v \preceq w \preceq \Sigma(v)\big\}$.

If we view R_D as a subposet of $D \times \Sigma(D)$ with the product order, we obtain the two functors $F' \circ \mathrm{pr}_1 = F \circ \mathrm{pr}_1$ and $\Sigma \circ F' \circ \Sigma^{-1} \circ \mathrm{pr}_2 = F \circ \mathrm{pr}_2$, where $\mathrm{pr}_1\colon R_D \to D$ and $\mathrm{pr}_2\colon R_D \to \Sigma(D)$ are the projections to the first and the second component, respectively. The following definition provides a natural transformation $\partial(F, D)$ as in the diagram

$$\begin{tikzcd} R_D \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\swarrow\partial(F,D)"] & D \arrow[d, "F"] \\ \Sigma(D) \arrow[r, "F"'] & \mathcal{A}. \end{tikzcd}$$

Definition 1.18. We set $\partial(F, D)\colon F \circ \mathrm{pr}_1 \Rightarrow F \circ \mathrm{pr}_2, (v, w) \mapsto F(v \preceq w)$.

In the following statement we will use $w \preceq \Sigma(v) \in \Sigma^{n+1}(D)$ as a shorthand for $w, \Sigma(v) \in \Sigma^{n+1}(D)$, $n \in \mathbb{Z}$, and $w \preceq \Sigma(v)$.

Lemma 1.19. *Suppose that $\partial' := \partial(F, D)$. Then*

$$F(v \preceq w) = \begin{cases} (\Sigma^n \circ F' \circ \Sigma^{-n})(v \preceq w) & v, w \in \Sigma^n(D) \\ (\Sigma^n \circ \partial')_{(\Sigma^{-n}(v), \Sigma^{-n}(w))} & w \preceq \Sigma(v) \in \Sigma^{n+1}(D) \\ 0 & \textit{otherwise} \end{cases} \tag{1.6}$$

for all $v \preceq w \in \mathbb{M}$.

This lemma shows that F is determined by its restriction $F|_D$ and the natural transformation $\partial(F, D)$.

Now let $F'\colon D \to \mathcal{A}$ be an arbitrary functor vanishing on $D \cap \partial\mathbb{M}$, let ∂' be a natural transformation as in the diagram

$$\begin{tikzcd} R_D \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\swarrow\partial'"] & D \arrow[d, "F'"] \\ \Sigma(D) \arrow[r, "\Sigma\circ F'\circ\Sigma^{-1}"'] & \mathcal{A}, \end{tikzcd}$$

and let $F\colon \mathbb{M} \to \mathcal{A}$ be defined by equation (1.6). We aim to show that F is a functor.

Lemma 1.20. *Let $v \preceq w \npreceq \Sigma(v)$, then $F(v \preceq w) = 0$.*

Proof. If $v, w \in \Sigma^n(D)$ for some n, the statement follows from Lemma 1.16 and the defining equation (1.6). In any other case the result follows directly from the construction (1.6). □

Lemma 1.21. *Let $u \preceq v \preceq w \preceq \Sigma(u)$, then*

$$F(u \preceq w) = F(v \preceq w) \circ F(u \preceq v).$$

Proof. Without loss of generality we assume $u \in D$. Since $D \cup \Sigma(D)$ is convex we have $v, w \in D \cup \Sigma(D)$. If $w \in D$ we are done, since D is convex and F' is a functor. Suppose $w \in \Sigma(D)$ and $v \in D$. Then

$$\begin{aligned} F(u \preceq w) &= \partial'_{(u,w)} \\ &= \partial'_{(v,w)} \circ F'(u \preceq v) \\ &= F(v \preceq w) \circ F(u \preceq v) \end{aligned}$$

by the naturality of ∂' in the first argument. Similarly, if $v, w \in \Sigma(D)$, then

$$\begin{aligned} F(u \preceq w) &= \partial'_{(u,w)} \\ &= (\Sigma \circ F' \circ \Sigma^{-1})(v \preceq w) \circ \partial'_{(u,v)} \\ &= F(v \preceq w) \circ F(u \preceq v) \end{aligned}$$

follows from the naturality of ∂' in its second argument. □

Lemma 1.22. *The data for F yields a functor.*

Proof. For all $u \preceq v \preceq w \in \mathbb{M}$ we have to show the equation

$$F(u \preceq w) = F(v \preceq w) \circ F(u \preceq v).$$

If $w \preceq \Sigma(u)$, then we are done by Lemma 1.21. Otherwise, Lemma 1.20 implies that $F(u \preceq w) = 0$ and thus we have to show

$$0 = F(v \preceq w) \circ F(u \preceq v).$$

In case $v \npreceq \Sigma(u)$ or $w \npreceq \Sigma(v)$, Lemma 1.20 applies to the right-hand side of this equation as well.

Now suppose $v \preceq \Sigma(u)$ and $w \preceq \Sigma(v)$. Since $\partial\,({\downarrow}\,\Sigma(u))$ divides $\mathbb{M}$ into two connected components there is some point $v' \in [v \preceq w] \cap \partial\,({\downarrow}\,\Sigma(u))$. Two applications of Lemma 1.21 yield

$$\begin{aligned} F(v \preceq w) \circ F(u \preceq v) &= F(v' \preceq w) \circ F(v \preceq v') \circ F(u \preceq v) \\ &= F(v' \preceq w) \circ F(u \preceq v'). \end{aligned}$$

We are done if we can show that $F(u \preceq v') = 0$.

Now $F|_{[u,\Sigma(u)]}$ is a functor by Lemma 1.21. Moreover, $u \preceq v'$ factors through a point in $\partial\mathbb{M}$ by our choice of v'. And since $F|_{\partial\mathbb{M}} = 0$ we obtain $F(u \preceq v') = 0$ and thus the desired result. □

Lemma 1.22 and Lemma 1.19 in conjunction imply the following.

Proposition 1.23. *For any functor $F' \colon D \to \mathcal{A}$ vanishing on $D \cap \partial\mathbb{M}$ together with a natural transformation*

$$\begin{tikzcd} R_D \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & D \arrow[d, "F'"] \arrow[dl, phantom, "\Swarrow \partial'"] \\ T(D) \arrow[r, "\Sigma \circ F' \circ \Sigma^{-1}"'] & \mathcal{A}, \end{tikzcd}$$

there is a unique strictly stable functor $F \colon \mathbb{M} \to \mathcal{A}$ with

$$F|_D = F', \quad F|_{\partial\mathbb{M}} = 0, \quad \text{and} \quad \partial(F, D) = \partial'.$$

Moreover, this construction is natural in the data provided by $F' \colon D \to \mathcal{A}$ and ∂'.

The next Lemma 1.24 shows how we may extend partially defined natural transformations.

Lemma 1.24. *Let $F, G \colon \mathbb{M} \to \mathcal{A}$ be strictly stable functors vanishing on $\partial\mathbb{M}$ and let $D, E \subset \mathbb{M}$ be convex fundamental domains with $D \subset E \cup \Sigma(E)$ and both $\operatorname{int}\mathbb{M} \setminus (D \cap E)$ and $\operatorname{int}\mathbb{M} \setminus (D \cap \Sigma(E))$ disconnected. Moreover, let $\eta \colon F|_{D \cap E} \to G|_{D \cap E}$ and $\nu \colon F|_{D \cap \Sigma(E)} \to G|_{D \cap \Sigma(E)}$ be natural transformations with the following two properties:*

(1) For any $u \in D \cap E$ and $v \in D \cap \Sigma(E)$ with $u \preceq v \preceq \Sigma(u)$ the diagram

$$\begin{tikzcd} F(v) \arrow[r, "\nu_v"] & G(v) \\ F(u) \arrow[u, "F(u \preceq v)"] \arrow[r, "\eta_u"'] & G(u) \arrow[u, "G(u \preceq v)"'] \end{tikzcd}$$

commutes.

(2) For any $v \in D \cap \Sigma(E)$ and $w \in \Sigma(D \cap E)$ with $v \preceq w \preceq \Sigma(v)$ the diagram

$$\begin{tikzcd} F(w) \arrow[r, "(\Sigma \circ \eta \circ \Sigma^{-1})_w"] & G(w) \\ F(v) \arrow[u, "F(v \preceq w)"] \arrow[r, "\nu_v"'] & G(v) \arrow[u, "G(v \preceq w)"'] \end{tikzcd}$$

commutes.

Then the natural transformations η and ν extend uniquely to a single strictly stable natural transformation from F to G.

In Chapter 8 (more specifically Section 8.2), where we relate RISC and (sheaf-theoretical) derived level set persistence, it will be useful to have a contravariant counterpart to Proposition 1.23 for easier reference. Thinking of the equation $\Sigma \circ F = F \circ \Sigma$ in the covariant setting as the shadow of an induced $(\infty, 1)$-natural transformation on the level of $(\infty, 1)$-categories, the analogous equation in the contravariant setting takes the form $F \circ \Sigma = \Omega \circ F$, so that we may think of $\Omega \colon \mathcal{D} \to \mathcal{D}$ as the shadow of a choice of loop space objects. Suppose now that $\mathcal{A}$ is an additive (or pointed) category and that $\Omega \colon \mathcal{A} \to \mathcal{A}$ is an automorphism of $\mathcal{D}$ preserving zero (or terminal) objects. For a strictly stable contravariant functor $F \colon \mathbb{M}^\circ \to \mathcal{A}$ we set $\delta(F, D) \colon F \circ \mathrm{pr}_2 \Rightarrow F \circ \mathrm{pr}_1, (v, w) \mapsto F(v \preceq w)$ in analogy to Definition 1.18.

Proposition 1.25. *For any contravariant functor $F' \colon D^\circ \to \mathcal{A}$ vanishing on $D \cap \partial\mathbb{M}$ together with a natural transformation*

$$\begin{tikzcd}
R_D^\circ \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\overset{\delta'}{\Nearrow}"] & D^\circ \arrow[d, "F'"] \\
\Sigma(D)^\circ \arrow[r, "\Omega \circ F' \circ \Sigma^{-1}"'] & \mathcal{A},
\end{tikzcd}$$

there is a unique strictly stable contravariant functor $F \colon \mathbb{M}^\circ \to \mathcal{A}$ with

$$F|_D = F', \quad F|_{\partial\mathbb{M}} = 0, \quad \textit{and} \quad \delta(F, D) = \delta'.$$

Moreover, this construction is natural in the data provided by $F' \colon D^\circ \to \mathcal{A}$ and δ'.

1.3 Cohomological Presheaves on $\mathbb{M}$

One of the distinguishing properties of RISC is that of being *cohomlogical*, in other literature this property is also referred to as *(middle) exactness*. In this Section 1.3 we provide a useful characterization of this property in the context of presheaves on $\mathbb{M}$. As with the previous section, these considerations are independent of the particular choice of fundamental domain D.

Let $F \colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ be a presheaf vanishing on $\partial\mathbb{M}$. Moreover, suppose there is a convex fundamental domain $D \subset \mathbb{M}$ with respect to $\langle \Sigma \rangle$, such that for any axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in D$ as shown in Fig. 1.8, the long sequence

$$\begin{tikzcd}
 & & \cdots \arrow[r] & F(\Sigma(u)) \arrow[dlll, rounded corners] \\
 & F(w) \arrow[r] & F(v_1) \oplus F(v_2) \arrow[r, "(1\ \ -1)"] & F(u) \arrow[dlll, rounded corners] \\
 F(\Sigma^{-1}(w)) \arrow[r] & \cdots & &
\end{tikzcd} \tag{1.7}$$

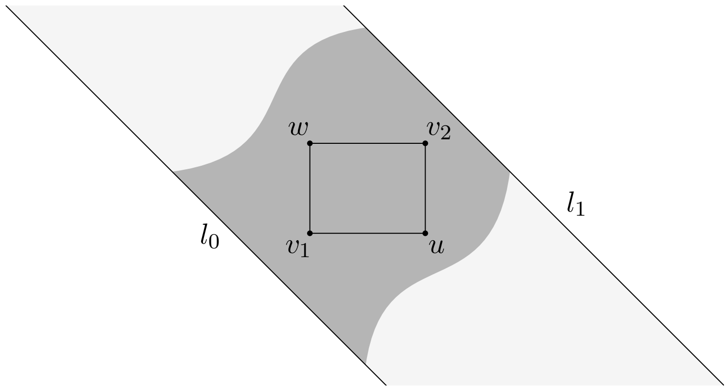


Figure 1.8: An axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in D$ contained in the fundamental domain D, which is shaded in grey.

is exact. We show that (1.7) is exact for any axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in \mathbb{M}$. We begin with the special case that $v_2 \in \partial\mathbb{M}$; then we have $F(v_2) \cong \{0\}$ and we set $v := v_1$. As shown in Fig. 1.9 the union of the orbits of u, v, and w form a subposet, which is isomorphic to $\mathbb{Z}$. As D is convex, the intersection of D and this subposet consists of three consecutive points; in Fig. 1.9 these are $\Sigma^2(v)$, $\Sigma^2(w)$, and $\Sigma^3(u)$. Moreover, these three consecutive points describe an axis-aligned rectangle contained in D. Thus, the restriction of F to this subposet yields the long exact sequence

$$\cdots \longrightarrow F(\Sigma(u)) \longrightarrow F(w) \longrightarrow F(v) \longrightarrow F(u) \longrightarrow F(\Sigma^{-1}(w)) \longrightarrow \cdots . \tag{1.8}$$

Definition 1.26 (Cohomological Presheaf)**.** We say that a presheaf $F \colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ vanishing on $\partial\mathbb{M}$ is *cohomological* if for any axis-aligned rectangle with one corner lying on l_1 and the other corners $u \preceq v \preceq w \in \mathbb{M}$, the long sequence (1.8) is exact.

Remark 1.27. This notion of a cohomological presheaf is inspired by the theory of *triangulated categories*. A *cohomological* functor on a triangulated category yields a long exact sequence for any *distinguished triangle*, see for example [KS90, Definitions 1.5.2 and 1.5.1]. In Remark 1.1 we considered the simplicial localization $L\mathbb{M}$ of $\mathbb{M}$ at the arrows between boundary points and we observed that any axis-aligned rectangle in $\mathbb{M}$ yields an $(\infty, 1)$-bicartesian square in $L\mathbb{M}$. In particular, each of the axis-aligned squares in Fig. 1.9, each with a vertex on the boundary $\partial\mathbb{M}$, are $(\infty, 1)$-bicartesian, hence the orbits of u, v, and w yield a long fiber-cofiber sequence in $L\mathbb{M}$. Moreover, any cofiber sequence in $L\mathbb{M}$ is part of one such long fiber-cofiber sequence

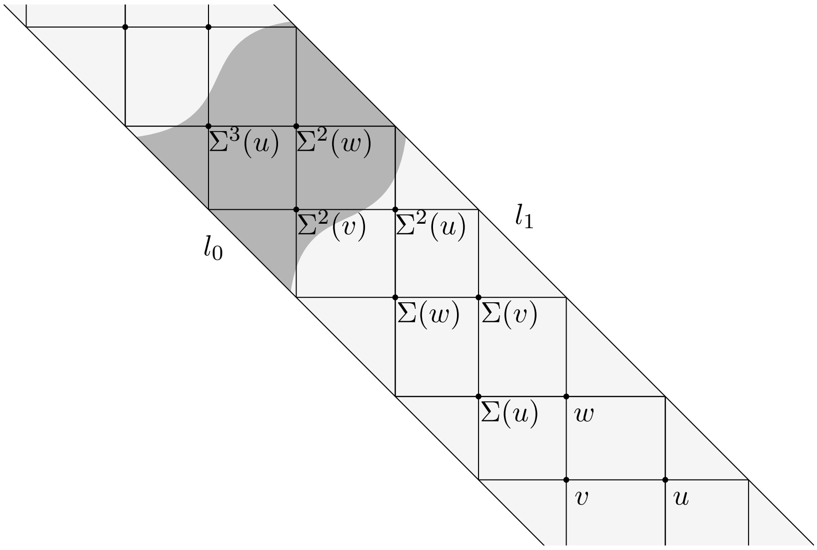


Figure 1.9: The linear subposet given by the orbits of u, v, and w. The region shaded in dark grey is our fundamental domain D.

in $L\mathbb{M}$. Thus, we may rephrase Definition 1.26 to say that a presheaf $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ vanishing on $\partial\mathbb{M}$ is cohomological if the induced presheaf $\mathrm{Ho}(L\mathbb{M})^\circ \to \mathrm{Vect}_\mathbb{F}$ on the homotopy category $\mathrm{Ho}(L\mathbb{M})$ sends (images of) cofiber sequences to exact sequences; almost as in the definition of cohomological functors on triangulated categories. As the homotopy category $\mathrm{Ho}(L\mathbb{M})$ is not triangulated, it may also be instructive to think of Definition 1.26 as an offspring of the notion of a cohomology theory on topological spaces; the origin of the term cohomological functor in spite of the homotopy category of the category of topological spaces not being triangulated either. As a side note, the infinite stairs in Fig. 1.9 are also the preimage of a particular type of geodesic triangle on the Möbius strip $\mathbb{M}/\langle\Sigma\rangle$.

Now suppose we have an arbitrary axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in \mathbb{M}$ as shown in Fig. 1.10. Together with the additional point $p \in \mathbb{M}$ we obtain two axis-aligned rectangles with one corner on l_1. As F is cohomological, the commutative diagram

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & F(\Sigma(v_1)) & \longrightarrow & F(\Sigma(p)) & \longrightarrow & F(w) & \longrightarrow & F(v_1) & \longrightarrow & F(p) & \longrightarrow & \dots \\
 & & \downarrow & & \| & & \downarrow & & \downarrow & & \| & & \\
\dots & \longrightarrow & F(\Sigma(u)) & \longrightarrow & F(\Sigma(p)) & \longrightarrow & F(v_2) & \longrightarrow & F(u) & \longrightarrow & F(p) & \longrightarrow & \dots
\end{array}$$

has exact rows. Thus, by the Barratt–Whitehead Lemma [BW56, Lemma 7.4] the long sequence (1.7) is exact. From this we obtain the following.

Proposition 1.28. *For a presheaf $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ vanishing on $\partial\mathbb{M}$ the following are equivalent.*

(1) *There is a convex fundamental domain $D \subset \mathbb{M}$ such that for any axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in D$ the long sequence* (1.7) *is exact.*

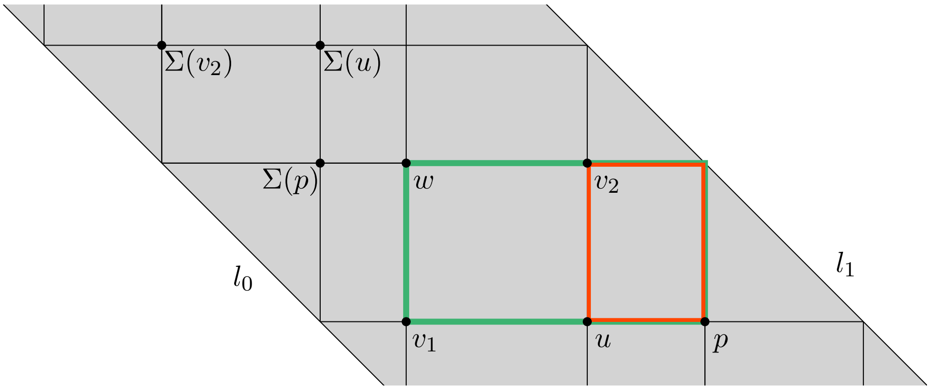


Figure 1.10: Axis-aligned rectangles determined by u, v_1, v_2, and w.

(2) The presheaf F is cohomological.

(3) For any axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in \mathbb{M}$ the long sequence (1.7) *is exact.*

(4) For any axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in \mathbb{M}$ the square

$$\begin{tikzcd} F(w) \arrow[r] \arrow[d] & F(v_2) \arrow[d] \\ F(v_1) \arrow[r] & F(u) \end{tikzcd}$$

is middle exact in the sense of Definition B.1.

Proof. Above we have shown that (1) implies (2) and that (2) implies (3). It is clear that (3) implies both (1) and (4). Moreover, if we consider Fig. 1.9, then we see that any three consecutive points of the sub-$\langle\Sigma\rangle$-set generated by u, v, and w describe an axis-aligned rectangle with one vertex on the boundary $\partial\mathbb{M}$ and thus (4) implies (2). □

Homological Functors. We may dualize Definition 1.26 and Proposition 1.28 in the sense of Remark 2.32 as follows.

Definition 1.29. We say that a functor $F \colon \mathbb{M} \to \mathrm{Vect}_{\mathbb{F}}$ vanishing on $\partial\mathbb{M}$ is *homological* if for any axis-aligned rectangle with one corner lying on l_1 and the other corners $u \preceq v \preceq w \in \mathbb{M}$, the long sequence

$$\cdots \longrightarrow F(\Sigma^{-1}(w)) \longrightarrow F(u) \longrightarrow F(v) \longrightarrow F(w) \longrightarrow F(\Sigma(u)) \longrightarrow \cdots$$

is exact.

Proposition 1.30. *For a functor $F\colon \mathbb{M} \to \mathrm{Vect}_{\mathbb{F}}$ vanishing on $\partial\mathbb{M}$ the following are equivalent.*

(1) There is a convex fundamental domain $D \subset \mathbb{M}$ such that for any axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in D$ the long sequence

$$\begin{tikzcd} \cdots \arrow[r] & F(\Sigma^{-1}(w)) \arrow[dll, out=0, in=180] & \\ F(u) \arrow[r] & F(v_1) \oplus F(v_2) \arrow[r, "(1\ \ -1)"] & F(w) \arrow[dll, out=0, in=180] \\ F(\Sigma(u)) \arrow[r] & \cdots & \end{tikzcd} \tag{1.9}$$

is exact.

(2) The functor F is homological.

(3) For any axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in \mathbb{M}$ the long sequence (1.9) *is exact.*

(4) For any axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in \mathbb{M}$ the square

$$\begin{tikzcd} F(u) \arrow[r] \arrow[d] & F(v_1) \arrow[d] \\ F(v_2) \arrow[r] & F(w) \end{tikzcd}$$

is middle exact.

Exactness and RISC. Now suppose $\mathcal{H}^\bullet$ is a cohomology theory taking values in the category $\mathrm{Vect}_{\mathbb{F}}$ of vector spaces over a field $\mathbb{F}$, let $f\colon X \to \mathbb{R}$ be a continuous function, and let $h(f)\colon \mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ be its RISC induced by $\mathcal{H}^\bullet$.

Proposition 1.31. *The RISC $h(f)\colon \mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ is cohomological in the sense of Definition 1.26.*

Proof. By construction any axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in D$ as shown in Fig. 1.7 yields a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & h(f)(\Sigma(u)) \arrow[dll, out=0, in=180] & \\ h(f)(w) \arrow[r] & h(f)(v_1) \oplus h(f)(v_2) \arrow[r, "(1\ \ -1)"] & h(f)(u) \arrow[dll, out=0, in=180] \\ h(f)(\Sigma^{-1}(w)) \arrow[r] & \cdots\,. & \end{tikzcd}$$

By Proposition 1.28.(1) this is one way of characterizing cohomological presheaves on $\mathbb{M}$. $\square$

We note that the characterization Proposition 1.30.(4) has been stated by [CdM09] and proven by [CdM09, BEMP13] in the special case of an axis-aligned rectangle contained within a tile of the tessellation shown in Fig. 1.5.

1.4 Mayer–Vietoris Sequences in RISC

Now suppose we have a continuous function $f\colon X \to \mathbb{R}$ as well as an open cover $A \cup B = X$. In the following we describe a counterpart to the Mayer–Vietoris sequence relating the relative interlevel set cohomologies $h(f)$, $h(f|_A)$, $h(f|_B)$, $h(f|_{A\cap B})\colon \mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$ induced by some cohomology theory $\mathcal{H}^\bullet$ taking values in $\mathrm{Vect}_\mathbb{F}$. Now by the functoriality of RISC deduced in Section 1.1.5 and the exactness of the Mayer–Vietoris sequence we have the pointwise exact sequence

$$h(f) \to h(f|_A) \oplus h(f|_B) \xrightarrow{\varphi} h(f|_{A\cap B})$$

of presheaves $\mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$, where $\varphi\colon h(f|_A) \oplus h(f|_B) \to h(f|_{A\cap B})$ is the natural transformation that restricts pointwise to the naturally induced map on the first summand $h(f|_A)$ and to the negated naturally induced map on the second summand $h(f|_B)$. As it turns out, this exact sequence can be extended as follows.

Theorem 1.32 (Mayer–Vietoris Sequence)**.** *For a continuous function $f\colon X \to \mathbb{R}$ and an open cover $A \cup B = X$ there is a pointwise exact sequence*

$$h(f|_A) \circ \Sigma \oplus h(f|_B) \circ \Sigma \xrightarrow{\varphi\circ\Sigma} h(f|_{A\cap B}) \circ \Sigma \xrightarrow{d} h(f) \to h(f|_A) \oplus h(f|_B) \xrightarrow{\varphi} h(f|_{A\cap B}) \tag{1.10}$$

of presheaves $\mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$, where $\varphi\colon h(f|_A) \oplus h(f|_B) \to h(f|_{A\cap B})$ is the natural transformation that restricts pointwise to the naturally induced map on the first summand $h(f|_A)$ and to the negated naturally induced map on the second summand $h(f|_B)$.

Remark 1.33. As mentioned in Remark 1.6 above, there is a construction of RISC relative to a subspace of the domain of a function closely analogous to [BBF24, Appendix B], which we omitted in the present exposition to constrain the cognitive load. Moreover, we provide a relative Mayer–Vietoris sequence for RISH with [BBF24, Lemma 3.2]. We note that the proof of Theorem 1.32 generalizes to a relative Mayer–Vietoris sequence for RISC analogous to [BBF24, Lemma 3.2] in a straightforward way.

Construction of the Mayer–Vietoris Differential. In order to verify Theorem 1.32, we need to specify a natural *differential*

$$d\colon h(f|_{A\cap B}) \circ \Sigma \to h(f).$$

Now there are unique strictly stable presheaves $h^\#(f)$, $h^\#(f|_{A\cap B})\colon \mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}^\mathbb{Z}$ such that

$$\mathrm{ev}^0 \circ h^\#(f) = h(f) \quad \text{and} \quad \mathrm{ev}^0 \circ h^\#(f|_{A\cap B}) = h(f|_{A\cap B})$$

by Corollary 1.15. So natural transformations $d\colon h(f|_{A\cap B}) \circ \Sigma \to h(f)$ are one-to-one with strictly stable natural transformations

$$d^{\#}\colon h^{\#}(f|_{A\cap B}) \circ \Sigma = \Omega \circ h^{\#}(f|_{A\cap B}) \longrightarrow h^{\#}(f),$$

which are uniquely determined by their restrictions $d^{\#}|_D\colon \Omega \circ h^{\#}(f|_{A\cap B})|_D \to h^{\#}(f)|_D$ to the fundamental domain D. Moreover, we may easily specify such a natural transformation on D pointwise using the differential of the Mayer–Vietoris sequence in cohomology by [tom08, Section 17.1.4]:

$$\Delta^{\bullet}\colon \Omega \circ h^{\#}(f|_{A\cap B})|_D \to h^{\#}(f)|_D.$$

However, in order to obtain a strictly stable natural transformation $d^{\#}$ by the naturality part of Proposition 1.25 from $\Delta^{\bullet}$, we need the commutativity of the diagram

$$\begin{tikzcd}
\Omega \circ h^{\#}(f|_{A\cap B}) \circ \mathrm{pr}_2 \arrow[r, "\Delta^{\bullet-1}"] \arrow[d, "\Omega \circ \delta(h^{\#}(f|_{A\cap B}), D)"'] & h^{\#}(f) \circ \mathrm{pr}_2 \arrow[d, "\delta(h^{\#}(f), D)"] \\
\Omega \circ h^{\#}(f|_{A\cap B}) \circ \mathrm{pr}_1 \arrow[r, "\Delta^{\bullet}"'] & h^{\#}(f) \circ \mathrm{pr}_1
\end{tikzcd} \tag{1.11}$$

of graded presheaves $R_D^{\circ} \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{Z}}$ and natural transformations (of degree 0). While the square (1.11) does not commute, we have the following result, which is just as useful.

Lemma 1.34. *The square* (1.11) *is anti-commutative in the sense that the two composite diagonal natural transformations sum up to* 0 *pointwise.*

Proof. Let $(w, \hat{u}) \in R_D$. As shown in Fig. 1.7, the pair $(w, \hat{u}) \in R_D$ determines an axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in D$ with $\Sigma(u) = \hat{u}$. Moreover, evaluating diagram (1.11) at the point $(w, \hat{u}) \in R_D$ we obtain the diagram

$$\begin{tikzcd}
\mathcal{H}^{\bullet-2}\big(A \cap B \cap f^{-1}\rho(u)\big) \arrow[r, "\Delta^{\bullet-1}"] \arrow[d, "\Omega(\delta'')"'] & \mathcal{H}^{\bullet-1}\big(f^{-1}\rho(u)\big) \arrow[d, "\delta'"] \\
\mathcal{H}^{\bullet-1}\big(A \cap B \cap f^{-1}\rho(w)\big) \arrow[r, "\Delta^{\bullet}"'] & \mathcal{H}^{\bullet}\big(f^{-1}\rho(w)\big),
\end{tikzcd} \tag{1.12}$$

where $\Delta^{\bullet}$ is the differential of the Mayer–Vietoris sequence for the triad $(X; A, B) \cap f^{-1}\rho(w)$, the map $\Delta^{\bullet-1}$ is the (shifted) differential of the Mayer–Vietoris sequence for the triad $(X; A, B) \cap f^{-1}\rho(u)$, the map δ' is the differential for the triad $f^{-1}(\rho(w); \rho(v_1), \rho(v_2))$, and δ'' is the differential for the triad $A \cap B \cap f^{-1}(\rho(w); \rho(v_1), \rho(v_2))$. We have to show that (1.12) is an anti-commutative square. To this end, we use the concrete description of the differentials of Mayer–Vietoris sequences by [tom08, Section 17.1.4] in terms of inclusion-induced maps of certain spaces similar to homotopy pushouts and the suspension *suspension isomorphism* [tom08, Section 17.1.2]. Following [tom08] we also write $I = [0,1]$, $0 = \{0\}$, $1 = \{1\}$, and we write products as juxtaposition, e.g. $IX = I \times X$. Now let

$$\begin{aligned} N_i &\coloneqq 0f^{-1}\rho_i(v_1) \cup If^{-1}\rho_i(u) \cup 1f^{-1}\rho_i(v_2) \quad \text{for } i = 1, 0, \\ N_{1,0} &\coloneqq 0f^{-1}\rho_1(v_1) \cup If^{-1}\rho_0(u) \cup 1f^{-1}\rho_1(v_2), \\ \text{and} \quad M &\coloneqq 0IA \cup II(A \cap B) \cup 1IB \end{aligned}$$

so we have subsets $N_1, N_0, N_{1,0} \subseteq If^{-1}\rho_1(w)$ and $M, IN_1, IN_0, IN_{1,0} \subseteq IIX$. In particular, we may consider the intersections $M \cap IN_i \subseteq If^{-1}\rho_i(w)$ for $i = 1, 0$. Moreover, we have the commutative diagram

$$\begin{tikzcd} M \cap IN_i \arrow[r, two heads, "\mathrm{pr}_{2,3}"] \arrow[d, two heads, "\mathrm{pr}_{1,3}"'] \arrow[dr, dashed, two heads] & N_i \arrow[d, two heads, "\mathrm{pr}_2"] \\ M \cap If^{-1}\rho_i(w) \arrow[r, two heads, "\mathrm{pr}_2"'] & f^{-1}\rho_i(w) \end{tikzcd}$$

of homotopy equivalences for $i = 1, 0$, where the subscripts to the projections indicate which factors we project onto. From this we obtain the zigzag of pairs of spaces

$$\begin{tikzcd} f^{-1}\rho(w) \\ (M \cap IN_1, M \cap IN_0) \arrow[u, two heads] \arrow[d, hook] \\ (M \cap IN_1, M \cap IN_{1,0}) \arrow[d, hook] \\ (M \cap IN_1, (0IA \cup 1IB) \cap M \cap IN_1 \cup M \cap IN_{1,0}) \\ (II(A \cap B) \cap IN_1, \partial II(A \cap B) \cap M \cap IN_1 \cup II(A \cap B) \cap IN_{1,0}) \arrow[u, hook] \\ \big(I^2, \partial I^2\big) \times A \cap B \cap f^{-1}\rho(u). \arrow[u, hook] \end{tikzcd} \tag{1.13}$$

with each arrow pointing upwards inducing an isomorphism in cohomology. By homotopy invariance and excision each arrow pointing upwards in (1.13) induces an isomorphism in cohomology, hence we obtain a linear map

$$\theta \colon \mathcal{H}^\bullet\big(\big(I^2, \partial I^2\big) \times A \cap B \cap f^{-1}\rho(u)\big) \longrightarrow \mathcal{H}^\bullet\big(f^{-1}\rho(w)\big).$$

Furthermore, we have suspension isomorphisms

$$\begin{aligned} \sigma_1 \colon \mathcal{H}^{\bullet-1}\big((I, \partial I) \times A \cap B \cap f^{-1}\rho(u)\big) &\to \mathcal{H}^\bullet\big(\big(I^2, \partial I^2\big) \times A \cap B \cap f^{-1}\rho(u)\big) \\ \text{and} \quad \sigma_2 \colon \mathcal{H}^{\bullet-2}\big(A \cap B \cap f^{-1}\rho(u)\big) &\to \mathcal{H}^{\bullet-1}\big((I, \partial I) \times A \cap B \cap f^{-1}\rho(u)\big) \end{aligned}$$

by [tom08, Section 17.1.2]. Now the first suspension isomorphism σ_1 corresponds to peeling off the first factor of I coming from the triad $(X; A, B)$ and the second suspension isomorphism σ_2 corresponds to peeling off the second factor of I coming from the triad of pairs $f^{-1}(\rho(w); \rho(v_1), \rho(v_2))$. So by composing these suspension isomorphisms with θ we obtain the equation

$$\theta \circ \sigma_1 \circ \sigma_2 = \Delta^\bullet \circ \Omega(\delta'') \tag{1.14}$$

of linear maps $\mathcal{H}^{\bullet-2}\big(A \cap B \cap f^{-1}\rho(u)\big) \to \mathcal{H}^\bullet\big(f^{-1}\rho(w)\big)$. Now let

$$\nu \colon I \times I \times A \cap B \cap f^{-1}\rho_1(u) \to I \times I \times A \cap B \cap f^{-1}\rho_1(u),\ (s, t, p) \mapsto (t, s, p)$$

be the homeomorphism swapping the components of the first two factors of I. Then ν induces the linear self-map

$$\mathcal{H}^\bullet(\nu) \colon \mathcal{H}^\bullet\big((I^2, \partial I^2) \times A \cap B \cap f^{-1}\rho(u)\big) \to \mathcal{H}^\bullet\big((I^2, \partial I^2) \times A \cap B \cap f^{-1}\rho(u)\big)$$

for the cohomology of the pair $\big(I^2, \partial I^2\big) \times A \cap B \cap f^{-1}\rho(u)$ so we may consider the composite linear map

$$\theta \circ \mathcal{H}^\bullet(\nu) \circ \sigma_1 \circ \sigma_2 \colon \mathcal{H}^{\bullet-2}\big(A \cap B \cap f^{-1}\rho(u)\big) \longrightarrow \mathcal{H}^\bullet\big(f^{-1}\rho(w)\big).$$

As the factors of I are being swapped in this linear map before we apply the map θ, the suspension isomorphism σ_2 now corresponds to peeling off the factor of I coming from the triad $(X; A, B)$ and σ_1 corresponds to peeling off the factor of I coming from the triad of pairs $f^{-1}(\rho(w); \rho(v_1), \rho(v_2))$. As a result, we have the equation

$$\theta \circ \mathcal{H}^\bullet(\nu) \circ \sigma_1 \circ \sigma_2 = \delta' \circ \Delta^{\bullet-1} \tag{1.15}$$

of linear maps $\mathcal{H}^{\bullet-2}\big(A \cap B \cap f^{-1}\rho(u)\big) \to \mathcal{H}^\bullet\big(f^{-1}\rho(w)\big)$. By combining the equations (1.14) and (1.15) with Corollary 1.36 below we obtain the equation

$$\Delta^\bullet \circ \Omega(\delta'') + \delta' \circ \Delta^{\bullet-1} = \theta \circ \sigma_1 \circ \sigma_2 + \theta \circ \mathcal{H}^\bullet(\nu) \circ \sigma_1 \circ \sigma_2 = 0. \qquad \square$$

Now let $\tilde{\Delta}^\bullet \colon \Omega \circ h^\#(f|_{A \cap B})|_D \to h^\#(f)|_D$ be the natural transformation of graded presheaves $D^\circ \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{Z}}$ defined degree-wise by

$$\tilde{\Delta}^n := (-1)^n \cdot \Delta^n \quad \text{for } n \in \mathbb{Z}.$$

Then we have the commutative square

$$\begin{tikzcd}
\Omega \circ h^\#(f|_{A \cap B}) \circ \mathrm{pr}_2 \arrow[r, "\tilde{\Delta}^{\bullet-1}"] \arrow[d, "\Omega \circ \delta(h^\#(f|_{A \cap B}), D)"'] & h^\#(f) \circ \mathrm{pr}_2 \arrow[d, "\delta(h^\#(f), D)"] \\
\Omega \circ h^\#(f|_{A \cap B}) \circ \mathrm{pr}_1 \arrow[r, "\tilde{\Delta}^\bullet"'] & h^\#(f) \circ \mathrm{pr}_1
\end{tikzcd}$$

of graded presheaves $D^\circ \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{Z}}$ by Lemma 1.34, hence there is a unique strictly stable natural transformation

$$d^\# \colon \Omega \circ h^\#(f|_{A\cap B}) = h^\#(f|_{A\cap B}) \circ \Sigma \to h^\#(f)$$

of strictly stable presheaves $\mathbb{M}_{\mathrm{f}}^\circ \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{Z}}$ with $d^\#|_D = \tilde{\Delta}^\bullet$ by the naturality part of Proposition 1.25. Thus, we may specify a natural *differential* for a Mayer–Vietoris sequence in RISC by

$$d := \mathrm{ev}^0 \circ d^\# \colon h(f|_{A\cap B}) \circ \Sigma \to h(f).$$

Proof of Theorem 1.32. Now that we specified the differential $d \colon h(f|_{A\cap B}) \circ \Sigma \to h(f)$, the exactness of the sequence (1.10) follows pointwise from the exactness of the Mayer–Vietoris sequence in cohomology by [tom08, Section 17.1.4]. □

An Auxiliary Fact for Cohomology Theories. We now provide the missing ingredient Corollary 1.36 to the proof of Lemma 1.34 above. As previously we write $I = [0,1]$ for the unit interval. Moreover, let $\sigma \colon I^2 \to I^2$, $(s,t) \mapsto (t,s)$ be the homeomorphism swapping the two components of the square I^2.

Lemma 1.35. *For any cohomology theory $\mathcal{K}^\bullet$ we have the equation*

$$\mathrm{id}_{\mathcal{K}^\bullet(I^2,\partial I^2)} + \mathcal{K}^\bullet(\sigma) = 0.$$

Proof. Let $\tilde{\mathcal{K}}^\bullet$ be the reduced cohomology theory on pointed spaces corresponding to $\mathcal{K}^\bullet$. By the universal property of the quotient space, there is a unique continuous map $\bar{\sigma} \colon I^2/\partial I^2 \to I^2/\partial I^2$ such that the diagram

$$\begin{tikzcd} I^2 \arrow[r, "\sigma"] \arrow[d, "q"'] & I^2 \arrow[d, "q"] \\ I^2/\partial I^2 \arrow[r, dashed, "\bar{\sigma}"'] & I^2/\partial I^2 \end{tikzcd}$$

commutes, hence we have the commutative diagram

$$\begin{tikzcd} \mathcal{K}^\bullet(I^2,\partial I^2) \arrow[r, "\mathcal{K}^\bullet(\sigma)"] & \mathcal{K}^\bullet(I^2,\partial I^2) \\ \tilde{\mathcal{K}}^\bullet(I^2/\partial I^2, *) \arrow[u, "\mathcal{K}^\bullet(q)"] \arrow[r, "\tilde{\mathcal{K}}^\bullet(\bar{\sigma})"'] & \tilde{\mathcal{K}}^\bullet(I^2/\partial I^2, *) \arrow[u, "\mathcal{K}^\bullet(q)"'] \end{tikzcd}$$

with both vertical arrows representing isomorphisms. Thus, it suffices to show the equation

$$\mathrm{id}_{\tilde{\mathcal{K}}^\bullet(I^2/\partial I^2)} + \tilde{\mathcal{K}}^\bullet(\bar{\sigma}) = 0.$$

Now let

$$\gamma \colon I^2/\partial I^2 \to I^2/\partial I^2 \vee I^2/\partial I^2,\ (s,t) \to \begin{cases} (2s,t)_1 & s \leq \frac{1}{2} \\ (2s-1,t)_2 & s \geq \frac{1}{2} \end{cases}$$

be the *comultiplication on* $I^2/\partial I^2$ and let $\langle \mathrm{id}, \bar{\sigma} \rangle \colon I^2/\partial I^2 \vee I^2/\partial I^2 \to I^2/\partial I^2$ be the unique continuous map characterized by the commutativity of the diagram

$$\begin{tikzcd}
I^2/\partial I^2 \arrow[r, "\mathrm{inc}_1"] \arrow[dr, equal] & I^2/\partial I^2 \vee I^2/\partial I^2 \arrow[d, dashed, "\langle \mathrm{id}, \bar{\sigma} \rangle"] & I^2/\partial I^2 \arrow[l, "\mathrm{inc}_2"'] \arrow[dl, "\bar{\sigma}"] \\
& I^2/\partial I^2, &
\end{tikzcd}$$

where $\mathrm{inc}_i \colon I^2/\partial I^2 \to I^2/\partial I^2 \vee I^2/\partial I^2$ denotes the corresponding inclusion into the first or second component for $i = 1, 2$. Then we have the homotopy commutative hexagon

$$\begin{tikzcd}
& I^2/\partial I^2 \arrow[dl, equal] \arrow[dr, equal] \arrow[dd, "\gamma"] & \\
I^2/\partial I^2 \arrow[dd, equal] & & I^2/\partial I^2 \arrow[dd, equal] \\
& I^2/\partial I^2 \vee I^2/\partial I^2 \arrow[ul, "\mathrm{pr}_1"] \arrow[ur, "\mathrm{pr}_2"'] \arrow[dd, "\langle \mathrm{id}, \bar{\sigma} \rangle"] & \\
I^2/\partial I^2 \arrow[ur, "\mathrm{inc}_1"] \arrow[dr, equal] & & I^2/\partial I^2 \arrow[ul, "\mathrm{inc}_2"'] \arrow[dl, "\bar{\sigma}"] \\
& I^2/\partial I^2, &
\end{tikzcd} \tag{1.16}$$

where $\mathrm{pr}_i \colon I^2/\partial I^2 \to I^2/\partial I^2 \vee I^2/\partial I^2$ denotes the corresponding projection onto the first or second component for $i = 1, 2$. Moreover, as $\gamma \colon I^2/\partial I^2 \to I^2/\partial I^2 \vee I^2/\partial I^2$, may serve as a model for the comultiplication on the based 2-sphere, we have the equation of *mapping degrees*:

$$\deg(\langle \mathrm{id}, \bar{\sigma} \rangle \circ \gamma) = \deg(\mathrm{id}) + \deg(\bar{\sigma}) = 1 + \deg(\bar{\sigma}). \tag{1.17}$$

Furthermore, we have $\deg(\bar{\sigma}) = -1$ by [tom08, Proposition 6.5.6] and hence $\deg(\langle \mathrm{id}, \bar{\sigma} \rangle \circ \gamma) = 0$ in conjunction with (1.17). As a result, the composition of the two center vertical maps in (1.16) is null-homotopic. Thus, if we apply $\tilde{\mathcal{K}}^\bullet$ to (1.16), then we obtain the commutative hexagon

$$
\begin{tikzcd}
& \tilde{\mathcal{K}}^\bullet\big(I^2/\partial I^2\big) \arrow[dl, equal] \arrow[dr, "\tilde{\mathcal{K}}^\bullet(\bar{\sigma})"] \arrow[dd, "\tilde{\mathcal{K}}^\bullet(\langle \mathrm{id}, \bar{\sigma} \rangle)"] & \\
\tilde{\mathcal{K}}^\bullet\big(I^2/\partial I^2\big) \arrow[dd, equal] & & \tilde{\mathcal{K}}^\bullet\big(I^2/\partial I^2\big) \arrow[dd, equal] \\
& \tilde{\mathcal{K}}^\bullet\big(I^2/\partial I^2 \vee I^2/\partial I^2\big) \arrow[ul, "\tilde{\mathcal{K}}^\bullet(\mathrm{inc}_1)"'] \arrow[ur, "\tilde{\mathcal{K}}^\bullet(\mathrm{inc}_2)"] \arrow[dd, "\tilde{\mathcal{K}}^\bullet(\gamma)"] & \\
\tilde{\mathcal{K}}^\bullet\big(I^2/\partial I^2\big) \arrow[ur, "\tilde{\mathcal{K}}^\bullet(\mathrm{pr}_1)"] \arrow[dr, equal] & & I^2/\partial I^2 \arrow[ul, "\tilde{\mathcal{K}}^\bullet(\mathrm{pr}_2)"'] \arrow[dl, equal] \\
& I^2/\partial I^2 &
\end{tikzcd}
$$

with exact diagonals by homotopy invariance and exactness of $\tilde{\mathcal{K}}^\bullet$ and the center column a complex. In conjunction with [tom08, Hexagon Lemma 11.1.3] we obtain that $\tilde{\mathcal{K}}^\bullet(\bar{\sigma}) = -\mathrm{id}_{\tilde{\mathcal{K}}^\bullet(I^2/\partial I^2)}$. □

Corollary 1.36. *For a topological space X with $A \subseteq X$ an open subset and*

$$\nu \coloneqq \sigma \times X \colon I \times I \times X \to I \times I \times X,\ (s,t,p) \mapsto (t,s,p)$$

the homeomorphism swapping the components of the first two factors of I, we have the equation

$$\mathrm{id}_{\mathcal{H}^\bullet((I^2/\partial I^2)\times(X,A))} + \mathcal{H}^\bullet(\nu) = 0. \tag{1.18}$$

Proof. In order to compute the linear map

$$\mathcal{H}^\bullet(\nu) \colon \mathcal{H}^\bullet\big(\big(I^2/\partial I^2\big) \times (X,A)\big) \to \mathcal{H}^\bullet\big(\big(I^2/\partial I^2\big) \times (X,A)\big)$$

using Lemma 1.35 we define the following cohomology theory $\mathcal{K}^\bullet$ in terms of the pair (X,A): We define the $\mathbb{Z}$-graded vector space

$$\mathcal{K}^\bullet(Y,B) \coloneqq \mathcal{H}^\bullet((Y,B) \times (X,A))$$

for any pair of spaces (Y,B) as well as the associated differential

$$\tilde{\delta}_{(Y,B)} \colon \mathcal{K}^{\bullet-1}(B) \to \mathcal{K}^\bullet(Y,B)$$

as the composition

$$\begin{CD}
\mathcal{K}^{\bullet-1}(B) \\
@| \\
\mathcal{H}^{\bullet-1}(B \times X, B \times A) \\
@A{\wr}AA \\
\mathcal{H}^{\bullet-1}(B \times X \cup Y \times A, Y \times A) \\
@VV{\delta_{(Y \times X,\, B \times X \cup Y \times A,\, Y \times A)}}V \\
\mathcal{H}^{\bullet}((Y, B) \times (X, A)) \\
@| \\
\mathcal{K}^{\bullet}(Y, B)
\end{CD}$$

of the inverse of the excision isomorphism

$$\mathcal{H}^{\bullet-1}(B \times X \cup Y \times A, Y \times A) \longrightarrow \mathcal{H}^{\bullet-1}(B \times X, B \times A)$$

and the differential

$$\delta_{(Y \times X,\, B \times X \cup Y \times A,\, Y \times A)} \colon \mathcal{H}^{\bullet-1}(B \times X \cup Y \times A, Y \times A) \longrightarrow \mathcal{H}^{\bullet}((Y, B) \times (X, A))$$

for the triple $(Y \times X, B \times X \cup Y \times A, Y \times A)$. Provided that $\mathcal{K}^{\bullet}$ is a cohomology theory, which is easily checked, the equation (1.18) follows directly from Lemma 1.35. $\square$

1.5 Tameness of RISC

Suppose $\mathcal{H}^{\bullet}$ is a cohomology theory taking values in $\mathrm{Vect}_{\mathbb{F}}$ and mapping weak equivalences to isomorphisms. We now view any relative interlevel set cohomology as being induced by $\mathcal{H}^{\bullet}$.

Definition 1.37 (Tame Functions)**.** We say that a continuous function $f \colon X \to \mathbb{R}$ is $\mathcal{H}^{\bullet}$-*tame* if all open interlevel sets of f have finite-dimensional cohomology in each degree, i.e.

$$\dim_{\mathbb{F}} \mathcal{H}^n\big(f^{-1}(I)\big) < \infty$$

for any integer $n \in \mathbb{Z}$ and any open interval $I \subseteq \mathbb{R}$. Moreover, if $\mathcal{H}^{\bullet}$ is singular cohomology with coefficients in $\mathbb{F}$, we say that f is $\mathbb{F}$-*tame*.

Lemma 1.38. *The function $f \colon X \to \mathbb{R}$ is $\mathcal{H}^{\bullet}$-tame iff the functor $h(f) \colon \mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ is pointwise finite-dimensional (pfd).*

Proof. As $\mathcal{H}^n\big(f^{-1}(I)\big)$ appears as a value of $h(f) \colon \mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ for any open interval $I \subseteq \mathbb{R}$ and any integer $n \in \mathbb{Z}$, the function f is $\mathcal{H}^{\bullet}$-tame if $h(f)$ is pfd. Now suppose $f \colon X \to \mathbb{R}$ is $\mathcal{H}^{\bullet}$-tame, let $u \in T^{-n}(D)$ for some $n \in \mathbb{Z}$, let X_u be the absolute component of $\left(f^{-1} \circ \rho \circ T^n\right)(u)$, and let A_u be the relative component. Then we obtain the exact sequence

$$\mathcal{H}^{n-1}(X_u) \to \mathcal{H}^{n-1}(A_u) \to h(f)(u) \to \mathcal{H}^n(X_u) \to \mathcal{H}^n(A_u).$$

Now X_u is an open interlevel set of f and A_u is the disjoint union of at most two open interlevel sets. As f is $\mathcal{H}^\bullet$-tame all four cohomology groups surrounding $h(f)(u)$ in above exact sequence are finite-dimensional. As a result, $h(f)(u)$ is finite-dimensional as well. $\square$

We end this Section 1.5 by showing that $h(f)\colon \mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$ satisfies the following form of continuity, when $f\colon X \to \mathbb{R}$ is $\mathcal{H}^\bullet$-tame.

Definition 1.39 (Sequentially Continuous Presheaves)**.** We say that a presheaf $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ is *sequentially continuous* if for any increasing sequence $(u_k)_{k=1}^\infty$ in $\mathbb{M}$ converging to u the natural map

$$F(u) \to \varprojlim_k F(u_k) \tag{1.19}$$

is an isomorphism. Dually, a covariant functor $F\colon \mathbb{M} \to \mathrm{Vect}_\mathbb{F}$ is *sequentially continuous* if for any decreasing sequence $(u_k)_{k=1}^\infty$ in $\mathbb{M}$ converging to u the natural map (1.19) is an isomorphism, see also Remark 2.32 below.

Remark 1.40. Any presheaf $\mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ is sequentially continuous iff it maps sequential colimits in $\mathbb{M}$ to their corresponding sequential limits in $\mathrm{Vect}_\mathbb{F}$.

Proposition 1.41. *If $f\colon X \to \mathbb{R}$ is $\mathcal{H}^\bullet$-tame, then the functor $h(f)\colon \mathbb{M}_\mathrm{f}^\circ \to \mathrm{vect}_\mathbb{F}$ is sequentially continuous.*

Proof. Let $(u_k)_{k=1}^\infty$ be an increasing sequence in $\mathbb{M}_\mathrm{f}$ converging to u. Without loss of generality we assume that $(u_k)_{k=1}^\infty$ is contained in a single tile $\Sigma^{-n}(D)$ of the tessellation induced by Σ and D as shown in Fig. 1.5. We write X_k for the absolute component of $(f^{-1} \circ \rho \circ \Sigma^n)(u_k)$ and A_k for the relative component. With this we have the commutative diagram

$$\begin{tikzcd}
\mathcal{H}^{n-1}(\bigcup_k X_k) \arrow[r] \arrow[d] & \mathcal{H}^{n-1}(\bigcup_k A_k) \arrow[r] \arrow[d] & h(f)(u) \arrow[r] \arrow[d] & \mathcal{H}^n(\bigcup_k X_k) \arrow[r] \arrow[d] & \mathcal{H}^n(\bigcup_k A_k) \arrow[d] \\
\varprojlim_k \mathcal{H}^{n-1}(X_k) \arrow[r] & \varprojlim_k \mathcal{H}^{n-1}(A_k) \arrow[r] & \varprojlim_k h(f)(u_k) \arrow[r] & \varprojlim_k \mathcal{H}^n(X_k) \arrow[r] & \varprojlim_k \mathcal{H}^n(A_k).
\end{tikzcd} \tag{1.20}$$

Now for each $k \in \mathbb{N}$ the subspace $X_k \subseteq X$ is an open interlevel set of $f\colon X \to \mathbb{R}$. Similarly, A_k is a disjoint union of at most two open interlevel sets. As $f\colon X \to \mathbb{R}$ is

$\mathcal{H}^\bullet$-tame the inverse sequences

$$\begin{aligned}
&\left(\mathcal{H}^{n-2}(X_{k+1}) \to \mathcal{H}^{n-2}(X_k)\right)_{k=1}^{\infty}, && (1.21)\\
&\left(\mathcal{H}^{n-2}(A_{k+1}) \to \mathcal{H}^{n-2}(A_k)\right)_{k=1}^{\infty}, && (1.22)\\
&\left(\mathcal{H}^{n-1}(X_{k+1}) \to \mathcal{H}^{n-1}(X_k)\right)_{k=1}^{\infty}, && (1.23)\\
&\left(\mathcal{H}^{n-1}(A_{k+1}) \to \mathcal{H}^{n-1}(A_k)\right)_{k=1}^{\infty}, && (1.24)\\
&(\mathcal{H}^{n}(X_{k+1}) \to \mathcal{H}^{n}(X_k))_{k=1}^{\infty}, && (1.25)\\
\text{and}\quad &(\mathcal{H}^{n}(A_{k+1}) \to \mathcal{H}^{n}(A_k))_{k=1}^{\infty} && (1.26)
\end{aligned}$$

are pfd. As the inverse sequences (1.23), (1.24), (1.25), and (1.26) are pfd and as inverse limits of finite-dimensional vector spaces are exact, both rows of (1.20) are exact. Moreover, as (1.21), (1.22), (1.23), and (1.24) are pfd, they satisfy the Mittag-Leffler condition. As a result, the four vertical maps surrounding $h(f)(u) \to \varprojlim_k h(f)(u_k)$ in (1.20) are isomorphisms by [May99, Section 19.4]. With this it follows from the five lemma that $h(f)(u) \to \varprojlim_k h(f)(u_k)$ is an isomorphism as well. □

2 Decomposition of RISC and Other Cohomological Presheaves

For this entire Chapter 2 we assume that $\mathcal{H}^\bullet$ is a cohomology theory taking values in $\mathrm{Vect}_\mathbb{F}$ and mapping weak equivalences to isomorphisms. Having defined the relative interlevel set cohomology induced by $\mathcal{H}^\bullet$ as a presheaf $h(f)\colon \mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$, we now define the *extended persistence diagram* of an $\mathbb{F}$-tame function $f\colon X \to \mathbb{R}$, originally due to [CSEH09], as the 0*-th Betti function* of $h(f)$. As we will see with Lemma 2.14 below, the 0-th Betti function $\beta^0(F)\colon \operatorname{int}\mathbb{M} \to \mathbb{N}_0$ of a presheaf $F\colon \mathbb{M}^\circ \to \mathrm{vect}_\mathbb{F}$ supported in $\operatorname{int}\mathbb{M}$ counts for each point $v \in \operatorname{int}\mathbb{M}$, the maximal number $\beta^0(F)(v)$ of linearly independent vectors in $F(v)$ born at v; for the functor $h(f)$, these are cohomology classes. Now for any point $v \in \mathbb{M}$, there is an *associated simple presheaf*

$$S_v\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F},\ u \mapsto \begin{cases} \mathbb{F} & u = v \\ \{0\} & \text{otherwise} \end{cases} \tag{2.1}$$

with all internal maps necessarily trivial.

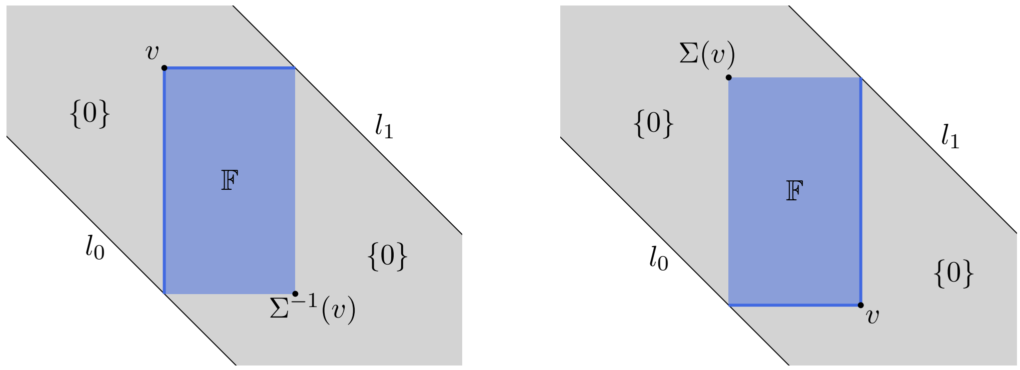
Definition 2.1 (0-th Betti Function)**.** For any presheaf $F\colon \mathbb{M}^\circ \to \mathrm{vect}_\mathbb{F}$ vanishing on $\partial\mathbb{M}$ we define its 0*-th Betti function* by

$$\beta^0(F)\colon\ \operatorname{int}\mathbb{M} \to \mathbb{N}_0,\ v \mapsto \dim_\mathbb{F} \operatorname{Nat}(F, S_v).$$

We name this the 0-th Betti function as it counts the maximal number of linearly independent natural transformations to each simple presheaf $\mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ supported on $\operatorname{int}\mathbb{M}$ in the same way that the 0-th Betti function on a topological space X counts the maximal number of linearly independent 1-forms on the vector space generated by its points compatible with paths in X. In a more restricted setting, we extend this notion to *higher Betti functions* with Definition 11.1 in Section 11.1.

Definition 2.2 (Extended Persistence Diagram)**.** Suppose $f\colon X \to \mathbb{R}$ is an $\mathcal{H}^\bullet$-tame continuous function. The *extended persistence diagram of f induced by $\mathcal{H}^\bullet$* is $\operatorname{Dgm}(f;\mathcal{H}^\bullet) := \beta^0(h(f))$. When $\mathcal{H}^\bullet$ is singular cohomology with coefficients in $\mathbb{F}$, we also define the *extended persistence diagram of f (over $\mathbb{F}$)* as $\operatorname{Dgm}(f) := \operatorname{Dgm}(f;\mathbb{F}) := \beta^0(h(f))$.

Originally the extended persistence diagram was defined differently by [CSEH09]; see Section 2.5.2 for details on the connection between these two definitions. The multiset defined by [CSEH09] and the multiset defined here are equivalent. We note that $\operatorname{Dgm}(f)$ is supported in the downset $\downarrow \operatorname{Im} \blacktriangle \subseteq \mathbb{M}_\mathrm{f}$, which is shaded in gray in Fig. 2.7 below.



(a) The indecomposable $B_v \colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$. (b) The indecomposable $B^v \colon \mathbb{M} \to \mathrm{Vect}_\mathbb{F}$.

Figure 2.1: The contravariant and the covariant indecomposables.

Definition 2.2 is consistent with [BBF24, Definition 2.2] in the sense that both definitions yield the same multiset for X a finite simplicial complex and f piecewise linear. The main goal of this Chapter 2 is to prove the Structure Theorem 2.6 below, which states that any sequentially continuous pfd cohomological functor $\mathbb{M}^\circ \to \mathrm{vect}_\mathbb{F}$ decomposes into the following type of indecomposables, see also Fig. 2.1a.

Definition 2.3 (Block Presheaf)**.** For $v \in \mathbb{M}$ we define the associated *block presheaf* as

$$B_v \colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F},\ u \mapsto \begin{cases} \mathbb{F} & u \in (\downarrow v) \cap \operatorname{int}\left(\uparrow \Sigma^{-1}(v)\right) \\ \{0\} & \text{otherwise,} \end{cases}$$

where $\operatorname{int}\left(\uparrow \Sigma^{-1}(v)\right)$ is the interior of the upset of $\Sigma^{-1}(v)$ in $\mathbb{M}$. The internal maps are identities whenever both domain and codomain are $\mathbb{F}$, otherwise they are zero.

Lemma 2.4 (Yoneda Lemma)**.** *Let $G \colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ be a presheaf vanishing on $\partial\mathbb{M}$ and let $v \in \operatorname{int}\mathbb{M}$. Then the evaluation at $1 \in \mathbb{F} = B_v(v)$ yields a linear isomorphism*

$$\mathrm{Nat}(B_v, G) \cong G(v),$$

where $\mathrm{Nat}(B_v, G)$ denotes the vector space of natural transformations from B_v to G.

Remark 2.5. For any presheaf $G \colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ vanishing on $\partial\mathbb{M}$ there is a unique $\mathbb{F}$-linear presheaf $G^\flat \colon (\mathbb{F}\mathbb{M}/\partial\mathbb{M})^\circ \to \mathrm{Vect}_\mathbb{F}$ such that the diagram

$$\begin{tikzcd} (\mathbb{F}\mathbb{M}/\partial\mathbb{M})^\circ \arrow[d, dashed, "G^\flat"'] & \mathbb{M}^\circ \arrow[l] \arrow[dl, "G"] \\ \mathrm{Vect}_\mathbb{F} & \end{tikzcd} \tag{2.2}$$

commutes. Moreover, for $v \in \mathbb{M}$ we have

$$B_v^\flat \cong \mathrm{Hom}_{\mathbb{F}\mathbb{M}/\partial\mathbb{M}}(-, v) \tag{2.3}$$

as presheaves $(\mathbb{F}\mathbb{M}/\partial\mathbb{M})^\circ \to \mathrm{Vect}_\mathbb{F}$. Furthermore, the correspondence (2.2) extends to an $\mathbb{F}$-linear isofunctor of presheaf categories. Considering this isomorphism of $\mathbb{F}$-linear categories as well as the canonical isomorphism (2.3), the previous Yoneda Lemma 2.4 "is equivalent" to the (strong) Yoneda Lemma for enriched categories [Kel05, Section 2.4]. (Here we put "is equivalent" in quotes, as any two true statements are in some sense equivalent independent of their contents.) In the beginning of Chapter 7 we relate the $\mathbb{F}$-linear category $\mathbb{F}\mathbb{M}/\partial\mathbb{M}$ to the mesh category associated to an A_n-quiver. Moreover, in Remark 7.18 we relate the quotient category $\mathbb{F}\mathbb{M}/\partial\mathbb{M}$ to the simplicial localization described in Remark 1.1.

Theorem 2.6 (Structure Theorem)**.** *Any sequentially continuous pfd cohomological presheaf* $F\colon \mathbb{M}^\circ \to \mathrm{vect}_\mathbb{F}$ *decomposes as*

$$F \cong \bigoplus_{v \in \operatorname{int} \mathbb{M}} B_v^{\oplus \mu(v)}, \tag{2.4}$$

where $\mu := \beta^0(F)\colon \operatorname{int}\mathbb{M} \to \mathbb{N}_0$*.*

This structure theorem, which we prove in Section 2.2 below, has the following two corollaries.

Corollary 2.7. *Any sequentially continuous pfd cohomological presheaf* $\mathbb{M}^\circ \to \mathrm{vect}_\mathbb{F}$ *is projective in the full subcategory of presheaves* $\mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ *vanishing on* $\partial\mathbb{M}$*.*

Proof. This follows from the Structure Theorem 2.6 and the Yoneda Lemma 2.4. □

Corollary 2.8. *If* $f\colon X \to \mathbb{R}$ *is a* $\mathbb{F}$*-tame function, then its RISC* $h(f)\colon \mathbb{M}_\mathrm{f}^\circ \to \mathrm{vect}_\mathbb{F}$ *decomposes as*

$$h(f) \cong \bigoplus_{v \in \operatorname{int} \mathbb{M}_\mathrm{f}} B_v^{\oplus \mu(v)},$$

where $\mu := \mathrm{Dgm}(f)\colon \operatorname{int}\mathbb{M}_\mathrm{f} \to \mathbb{N}_0$*.*

Proof. This follows from the Structure Theorem 2.6 in conjunction with Proposition 1.31, Lemma 1.38, and Proposition 1.41. □

In Fig. 2.2 we show a geometric simplicial complex in $\mathbb{R}^3$ and the two indecomposables of the RISC of its height function. Moreover, Fig. 2.3 shows a compact subset of $\mathbb{R}^2$, which we name the *infinite brush*, as well as its RISC having an infinite number of direct summands. Even though this direct sum is infinite with bounded support, it is still pfd and hence the existence of a decomposition as in Fig. 2.3 can be seen as a consequence of Theorem 2.6.

Sequentially Continuous Pfd Cohomological Presheaves on $\mathbb{M}$ Concretely. The Structure Theorem 2.6 above describes the structure of all sequentially continuous pfd cohomological presheaves. Conversely, we may ask ourselves, when is a direct sum as on the right-hand side of (2.4) sequentially continuous, pfd, and cohomological? Now

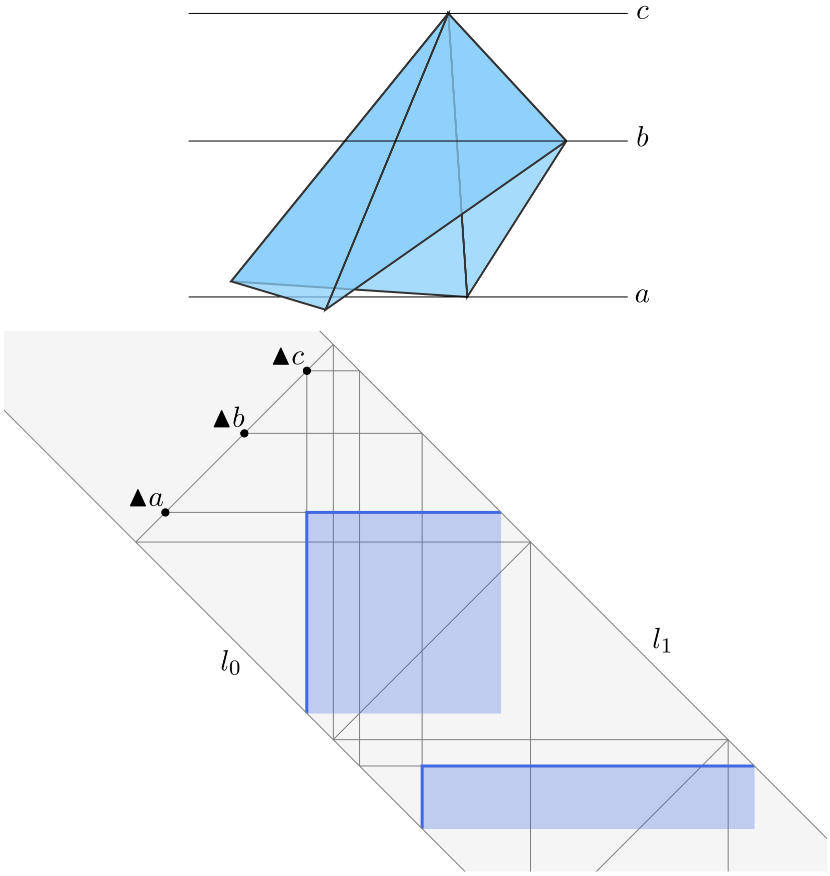


Figure 2.2: A geometric simplicial complex in $\mathbb{R}^3$ at the top and the two indecompsables of the RISC of its height function at the bottom.

it's easy to see that any presheaf like this is cohomological no matter what. Moreover, if $\bigoplus_{v \in \operatorname{int} \mathbb{M}} B_v^{\oplus \mu(v)}$ is pfd, then $\bigoplus_{v \in \operatorname{int} \mathbb{M}} B_v^{\oplus \mu(v)} = \prod_{v \in \operatorname{int} \mathbb{M}} B_v^{\mu(v)}$. Thus, $\bigoplus_{v \in \operatorname{int} \mathbb{M}} B_v^{\oplus \mu(v)}$ is both sequentially continuous and cohomological if it is pfd by the commutativity of limits. So the remaining question is, when a direct sum as on the right-hand side of (2.4) is pfd. Now for any $u, v \in \operatorname{int} \mathbb{M}$ we have

$$B_v(u) = \begin{cases} \mathbb{F} & v \in R_u \\ \{0\} & v \notin R_u, \end{cases} \tag{2.5}$$

where $R_u \coloneqq (\uparrow u) \cap \operatorname{int}(\downarrow T(u)) \subset \operatorname{int} \mathbb{M}$. Thus, for a function $\mu\colon \operatorname{int} \mathbb{M} \to \mathbb{N}_0$ and for any $u \in \operatorname{int} \mathbb{M}$ we have the equality

$$\left(\bigoplus\nolimits_{v \in \operatorname{int} \mathbb{M}} B_v^{\oplus \mu(v)}\right)(u) = \bigoplus_{v \in \operatorname{int} \mathbb{M}} B_v^{\oplus \mu(v)}(u) = \bigoplus_{v \in R_u} \mathbb{F}^{\oplus \mu(v)},$$

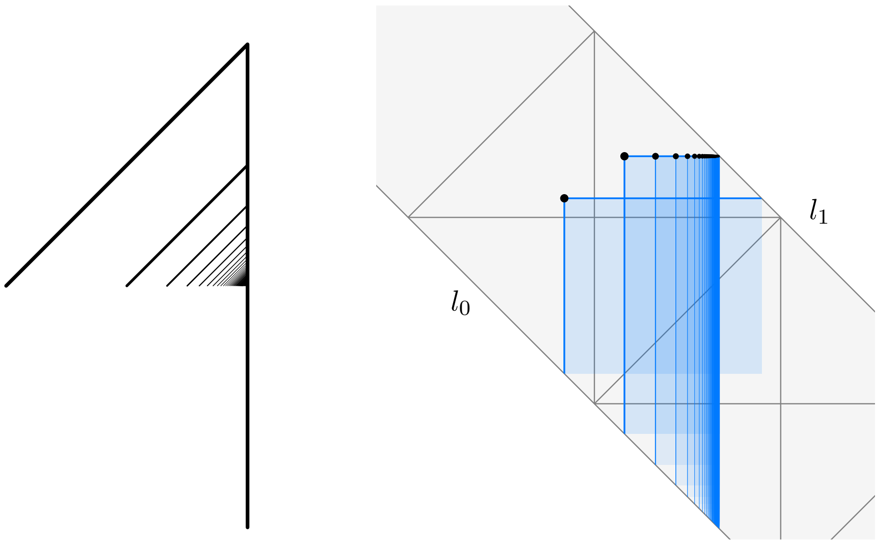


Figure 2.3: A compact subset of $\mathbb{R}^2$, which we name the *infinite brush*, on the left-hand side and the RISC of its height function on the right-hand side.

hence $\bigoplus_{v \in \operatorname{int} \mathbb{M}} B_v^{\oplus \mu(v)}$ is pfd iff

$$\sum_{v \in R_u} \mu(v) < \infty \tag{2.6}$$

for any $u \in \operatorname{int} \mathbb{M}$. We summarize this result as a proposition.

Proposition 2.9. *For any function $\mu \colon \operatorname{int} \mathbb{M} \to \mathbb{N}_0$ the direct sum $\bigoplus_{v \in \operatorname{int} \mathbb{M}} B_v^{\oplus \mu(v)}$ is pfd iff* (2.6) *is a valid estimate for any $u \in \operatorname{int} \mathbb{M}$.*

Corollary 2.10. *The 0-th Betti function $\beta^0(F) \colon \operatorname{int} \mathbb{M} \to \mathbb{N}_0$ of any sequentially continuous pfd cohomological presheaf $F \colon \mathbb{M}^\circ \to \mathrm{vect}_{\mathbb{F}}$ satisfies the estimate*

$$\sum_{v \in R_u} \beta^0(F)(v) < \infty \quad \text{for any } u \in \operatorname{int} \mathbb{M}.$$

Proof. This follows in conjunction with the Structure Theorem 2.6. □

As a further consequence of the Structure Theorem 2.6 natural transformations between sequentially continuous pfd cohomological presheaves have a very concrete description.

Remark 2.11 (Structure of Natural Transformations). Suppose we have maps of sets $p \colon I \to \operatorname{int} \mathbb{M}$ and $q \colon J \to \operatorname{int} \mathbb{M}$ with $I \cap J = \emptyset$ and such that the sets $p^{-1}(R_u)$ and $q^{-1}(R_u)$ are finite for all $u \in \operatorname{int} \mathbb{M}$. Writing $B_k = B_{p(k)}$ for $k \in I \cup J$ the direct sums $\bigoplus_{i \in I} B_i$ and $\bigoplus_{j \in J} B_j$ are pfd by Proposition 2.9. In particular, we have

$\bigoplus_{i \in I} B_i = \prod_{i \in I} B_I$ and thus

$$\begin{aligned}
\mathrm{Nat}\left(\bigoplus_{j \in J} B_j, \bigoplus_{i \in I} B_i\right) &\cong \prod_{j \in J} \prod_{i \in I} \mathrm{Nat}(B_j, B_i) \\
&\cong \prod_{j \in J} \prod_{i \in I} B_i(q(j)) \\
&\cong \prod_{j \in J} \prod_{\substack{i \in I \\ p(i) \in R_j}} \mathbb{F} \\
&\cong \prod_{j \in J} \prod_{i \in p^{-1}(R_j)} \mathbb{F}
\end{aligned} \tag{2.7}$$

by the Yoneda Lemma 2.4 and (2.5), where $R_j \coloneqq R_{q(j)}$ for $j \in J$. As a result of (2.7) and our assumption that $p^{-1}(R_u)$ is finite for all $u \in \operatorname{int} \mathbb{M}$, any natural transformation $\bigoplus_{j \in J} B_j \to \bigoplus_{i \in I} B_i$ is represented by an $I \times J$-matrix with finitely many non-zero entries in every column.

2.1 Approximation by Reduction Modulo Radical

In order to prove the Structure Theorem 2.6, we first construct an *approximating* natural transformation

$$\varphi\colon \bigoplus_{v \in \operatorname{int} \mathbb{M}} B_v^{\oplus \mu(v)} \longrightarrow F,$$

where $\mu \coloneqq \beta^0(F)\colon \operatorname{int} \mathbb{M} \to \mathbb{N}_0$, and then we show that φ is a natural isomorphism in Proposition 2.20 below. To this end, we harness a construction by [HL81, Theorem 1, Proposition 5] used in their proof that any projective functor on a poset taking values in a *module category* is *free*; see also Corollary 10.18 and Remark 10.19 for a similar use of their *reduction modulo radical* in the present monograph. Then we show in Section 2.2 that φ is a natural isomorphism using a *functorial filtration* inspired by [Rin75, CB15].

We start by introducing some notation. First, we note that a presheaf $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ vanishing on $\partial\mathbb{M}$ can equivalently be thought of as a bifunctor $F\colon \mathbb{R} \times \mathbb{R}^\circ \to \mathrm{Vect}_\mathbb{F}$ supported on the interior $\operatorname{int} \mathbb{M} \subset \mathbb{R} \times \mathbb{R}$, which is covariant in its first argument and contravariant in the second. For axis-parallel internal maps of a functor $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ we use similar notation as with bifunctors. More specifically, if we have $u, v \in \mathbb{M}$ with $u \preceq v$, $u = (x, y)$, and $v = (s, t)$, then we use notation as in the commutative diagram

$$\begin{tikzcd}
F(v) = F(s,t) \arrow[r, "F(s \leq x, t)"] \arrow[d, "F(s, y \leq t)"'] \arrow[dr, "F(u \preceq v)"] & F(x,t) \arrow[d, "F(x, y \leq t)"] \\
F(s,y) \arrow[r, "F(s \leq x, y)"'] & F(x,y) = F(u).
\end{tikzcd}$$

We adopt the terminology by [OS21] when we define the *Hilbert function* of a presheaf $F\colon \mathbb{M}^{\circ} \to \mathrm{vect}_{\mathbb{F}}$ supported in $\operatorname{int} \mathbb{M}$ by

$$\mathrm{Hilb}(F)\colon \operatorname{int} \mathbb{M} \to \mathbb{N}_0,\ u \mapsto \dim_{\mathbb{F}} F(u).$$

Following [CGR+22, Paragraph 3.1] we define the *radical* of $F\colon \mathbb{M}^{\circ} \to \mathrm{vect}_{\mathbb{F}}$ to be the subpresheaf

$$\operatorname{rad} F\colon \mathbb{M}^{\circ} \to \mathrm{vect}_{\mathbb{F}},\ u \mapsto \sum_{v \succ u} \operatorname{Im} F(u \preceq v),$$

where v ranges over all $v \in \mathbb{M}$ with $v \succ u$. Moreover, we note that

$$(\operatorname{rad} F)(x,y) = \left(\bigcup_{s<x} \operatorname{Im} F(s \leq x, y)\right) + \left(\bigcup_{t>y} \operatorname{Im} F(x, y \leq t)\right).$$

While we have no need for this statement, we show in Appendix B.2 that the radical $\operatorname{rad} F$ is indeed the intersection of all maximal subpresheaves of F for the sake of completeness. The following lemma relating the radical of a presheaf to simple presheaves associated to points in $\mathbb{M}$ as in (2.1) suffices for the present monograph.

Lemma 2.12. *Let $\alpha\colon F \to S_v$ be a homomorphism of presheaves on $\mathbb{M}$ for some $v \in \mathbb{M}$, where $S_v\colon \mathbb{M}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ is the simple presheaf associated to v as in* (2.1). *Then the triangle*

$$\begin{tikzcd} \operatorname{rad} F \arrow[r, hook] \arrow[dr, "0"'] & F \arrow[d, "\alpha"] \\ & S_v \end{tikzcd}$$

commutes.

Corollary 2.13. *Any maximal subpresheaf of a presheaf $F\colon \mathbb{M}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ contains the radical* $\operatorname{rad} F\colon \mathbb{M}^{\circ} \to \mathrm{vect}_{\mathbb{F}}$ *pointwise.*

Proof. This follows in conjunction with the universal property of kernels and the fact that any simple presheaf $\mathbb{M}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ is isomorphic to S_v for some $v \in \mathbb{M}$. $\square$

We note that the content of this Section 2.1 up to and including Corollary 2.13 is essentially the contents of the first paragraph of the proof of [HL81, Proposition 5] applied to our special case. In particular, we may adopt the terminology of [HL81] and refer to the pointwise quotient presheaf $F/\operatorname{rad} F$ as a *reduction modulo radical.* Following [CGR+22, Paragraph 3.1] we may write the 0-th Betti function as a difference of Hilbert functions or as the Hilbert function of the reduction modulo radical:

Lemma 2.14. *If $F\colon \mathbb{M}^{\circ} \to \mathrm{vect}_{\mathbb{F}}$ is a presheaf vanishing on $\partial\mathbb{M}$, then we have*

$$\beta^0(F) = \mathrm{Hilb}(F) - \mathrm{Hilb}(\operatorname{rad} F) = \mathrm{Hilb}(F/\operatorname{rad} F).$$

Proof. Let $v \in \operatorname{int} \mathbb{M}$. Then we have

$$\begin{aligned}
\beta^0(F)(v) &= \dim_{\mathbb{F}} \operatorname{Nat}(F, S_v) \\
&= \dim_{\mathbb{F}} \operatorname{Nat}(F/\operatorname{rad} F, S_v) \\
&= \dim_{\mathbb{F}} \operatorname{Hom}_K((F/\operatorname{rad} F)(v), \mathbb{F}) \\
&= \dim_{\mathbb{F}} (F/\operatorname{rad} F)(v) \\
&= \operatorname{Hilb}(F/\operatorname{rad} F)(v) \\
&= \operatorname{Hilb}(F)(v) - \operatorname{Hilb}(\operatorname{rad} F)(v).
\end{aligned}$$

Here the second equality follows from Lemma 2.12 and the third equality from the fact that the top $F/\operatorname{rad} F$ is *semisimple* in the sense that all internal maps between values at different points of $\mathbb{M}$ are zero. □

Decomposition of the Reduction Modulo Radical. In the next step we construct a decomposition of the reduction modulo radical $F/\operatorname{rad} F$ into simple presheaves of the form (2.1) for any pfd presheaf $F\colon \mathbb{M}^\circ \to \operatorname{vect}_{\mathbb{F}}$.

Definition 2.15. A *reduced dual frame* for a pfd presheaf $F\colon \mathbb{M}^\circ \to \operatorname{vect}_{\mathbb{F}}$ is a map of sets $p\colon I \to \mathbb{M}$ with

$$\# p^{-1}(v) = \dim_{\mathbb{F}} \operatorname{Nat}(F, S_v) \quad \text{for any } v \in \mathbb{M} \tag{2.8}$$

together with an I-tuple (or choice function)

$$(\alpha_i)_{i\in I} \in \prod_{i\in I} \operatorname{Nat}\big(F, S_{p(i)}\big)$$

such that $(\alpha_i)_{i\in p^{-1}(v)}$ is a basis of $\operatorname{Nat}(F, S_v)$ for any $v \in \mathbb{M}$.

Now suppose we have a pfd presheaf $F\colon \mathbb{M}^\circ \to \operatorname{vect}_{\mathbb{F}}$ as well as a reduced dual frame for F provided to us as a map of sets $p\colon I \to \mathbb{M}$ and an I-tuple

$$(\alpha_i)_{i\in I} \in \prod_{i\in I} \operatorname{Nat}\big(F, S_{p(i)}\big).$$

For $i \in I$ we now also write $S_i := S_{p(i)}\colon \mathbb{M}^\circ \to \operatorname{Vect}_{\mathbb{F}}$ and $B_i := B_{p(i)}\colon \mathbb{M}^\circ \to \operatorname{Vect}_{\mathbb{F}}$. Then there is a unique natural transformation

$$\theta\colon F \longrightarrow \prod_{i\in I} S_i$$

such that the diagram

$$\begin{tikzcd}
F \arrow[d, dashed, "\theta"'] \arrow[dr, "\alpha_i"] & \\
\prod_{i\in I} S_i \arrow[r, "\mathrm{pr}_i"'] & S_i
\end{tikzcd}$$

commutes for any $i \in I$.

Definition 2.16. We name $\theta\colon F \longrightarrow \prod_{i\in I} S_i$ the *reducing projection* of the reduced dual frame provided to us by $p\colon I \to \mathbb{M}$ and the I-tuple $(\alpha_i)_{i\in I} \in \prod_{i\in I} \operatorname{Nat}(F, S_i)$.

By Lemma 2.12 there is a unique natural transformation

$$\bar{\theta}\colon F/\operatorname{rad} F \to \prod_{i\in I} S_i$$

such that the triangle

$$\begin{tikzcd} F \arrow[r, "\theta"] \arrow[d, "\mathrm{pr}"'] & \prod_{i\in I} S_i \\ F/\operatorname{rad} F \arrow[ur, dashed, "\bar{\theta}"'] & \end{tikzcd}$$

commutes, where $\mathrm{pr}\colon F \to F/\operatorname{rad} F$ is the pointwise projection to the quotient space. By the following Proposition 2.17 the induced natural transformation $\bar{\theta}$ is an isomorphism. Moreover, as $F\colon \mathbb{M}^\circ \to \mathrm{vect}_{\mathbb{F}}$ is pfd, the direct product $\prod_{i\in I} S_i$ is pfd as well, hence $\bigoplus_{i\in I} S_i = \prod_{i\in I} S_i$ and thus $\bar{\theta}$ provides a choice of decomposition into simple presheaves.

Proposition 2.17. *For any pfd presheaf $F\colon \mathbb{M}^\circ \to \mathrm{vect}_{\mathbb{F}}$ and any reduced dual frame for F its reducing projection $\theta\colon F \to \prod_{i\in I} S_i$ is a natural epimorphism with* $\ker\theta = \operatorname{rad} F$.

Proof. It suffices to show $\bar{\theta}\colon F/\operatorname{rad} F \to \prod_{i\in I} S_i$ is an isomorphism. By Lemma 2.12 there is a unique natural transformation

$$\bar{\alpha}_i\colon F/\operatorname{rad} F \to S_i$$

such that the triangle

$$\begin{tikzcd} F \arrow[r, "\alpha_i"] \arrow[d, "\mathrm{pr}"'] & S_i \\ F/\operatorname{rad} F \arrow[ur, dashed, "\bar{\alpha}_i"'] & \end{tikzcd}$$

commutes for any $i \in I$. Now let $u \in \mathbb{M}$. By the left-exactness of $\operatorname{Nat}(-, S_u)$ and by Lemma 2.12 we have the exact sequence

$$\begin{tikzcd} 0 \arrow[d] \\ \operatorname{Nat}(F/\operatorname{rad} F, S_u) \arrow[d, "\operatorname{Nat}(\mathrm{pr}, S_u)"] \\ \operatorname{Nat}(F, S_u) \arrow[d, "0"] \\ \operatorname{Nat}(\operatorname{rad} F, S_u), \end{tikzcd}$$

hence $\operatorname{Nat}(\mathrm{pr}, S_u)$ is a linear isomorphism. Thus, the subset

$$\{\bar{\alpha}_i\}_{i \in p^{-1}(u)} \subset \operatorname{Nat}(F/\operatorname{rad} F, S_u)$$

is a basis as well and hence the I-tuple

$$(\bar{\alpha}_i)_{i \in I} \in \prod_{i \in I} \operatorname{Nat}(F/\operatorname{rad} F, S_i)$$

provides a reduced dual frame for the reduction modulo radical $F/\operatorname{rad} F$ in conjunction with the map $p \colon I \to \mathbb{M}$. Moreover, by the commutativity of

$$\begin{tikzcd}
\operatorname{Nat}\left(F/\operatorname{rad} F, \prod_{i \in I} S_i\right) \arrow[r, "\sim"] \arrow[d] & \prod_{i \in I} \operatorname{Nat}(F/\operatorname{rad} F, S_i) \arrow[d] \\
\operatorname{Nat}\left(F, \prod_{i \in I} S_i\right) \arrow[r, "\sim"] & \prod_{i \in I} \operatorname{Nat}(F, S_i)
\end{tikzcd}$$

the natural transformation $\bar{\theta} \colon F/\operatorname{rad} F \to \prod_{i \in I} S_i$ is the reducing projection of the reduced dual frame provided to us by $p \colon I \to \mathbb{M}$ and the I-tuple $(\bar{\alpha}_i)_{i \in I} \in \prod_{i \in I} \operatorname{Nat}(F/\operatorname{rad} F, S_i)$. As the reduction modulo radical $F/\operatorname{rad} F$ is semisimple in the sense that all internal maps between values at different points of $\mathbb{M}$ are zero, the natural transformation $\bar{\theta} \colon F/\operatorname{rad} F \to \prod_{i \in I} S_i$ being a natural isomorphism reduces to the fact, that a basis for the dual space of a finite-dimensional vector space induces an isomorphism to $\mathbb{F}^n$ for an appropriate $n \in \mathbb{N}_0$. $\square$

Lifting to an Approximation. Now suppose that $F \colon \mathbb{M}^\circ \to \mathrm{vect}_{\mathbb{F}}$ vanishes on $\partial\mathbb{M}$ and let $\upsilon_u \colon B_u \to S_u$ be the natural transformation mapping $B_u(u) = \mathbb{F}$ identically onto $S_u(u) = \mathbb{F}$ for any $u \in \operatorname{int} \mathbb{M}$ and $\upsilon_i \colon B_i \to S_i$ similarly for $i \in I$. (Using the terminology of [HL81, Kra15] we may also say that υ_u is "the" *projective cover* of S_u in the category of presheaves vanishing on $\partial\mathbb{M}$.) Then we have the canonical natural transformation

$$\bigoplus_{i \in I} \upsilon_i \colon \bigoplus_{i \in I} B_i \longrightarrow \bigoplus_{i \in I} S_i.$$

To simplify notation we now write $H := \bigoplus_{i \in I} B_i$ and $\upsilon := \bigoplus_{i \in I} \upsilon_i$. It is easy to see that $\upsilon \colon H \to \bigoplus_{i \in I} S_i$ is surjective and that

$$\operatorname{rad} H = \ker \upsilon. \tag{2.9}$$

As H is projective in the category of presheaves vanishing on $\partial\mathbb{M}$ by the Yoneda Lemma 2.4, there is a lift $\varphi \colon H \to F$ as in the commutative triangle

$$\begin{tikzcd}
 & F \arrow[d, two heads, "\theta"] \\
H \arrow[ur, dashed, "\varphi"] \arrow[r, "\upsilon"'] & \bigoplus_{i \in I} S_i.
\end{tikzcd} \tag{2.10}$$

Definition 2.18. We name a lift $\varphi\colon H \to F$ as in diagram (2.10) an *approximation of F by reduction modulo radical lifting* the reduced dual frame $(\alpha_i)_{i\in I} \in \prod_{i\in I} \operatorname{Nat}(F, S_i)$.

Proposition 2.19. *Any approximation $\varphi\colon H \to F$ of F by reduction modulo radical induces a pointwise isomorphism on reductions modulo radical $\bar{\varphi}\colon H/\operatorname{rad} H \xrightarrow{\sim} F/\operatorname{rad} F$.*

Proof. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\operatorname{rad} H \arrow[r, "\operatorname{rad}\varphi"] \arrow[d, hook] & \operatorname{rad} F \arrow[d, hook] \\
H \arrow[r, "\varphi"] \arrow[d, "\upsilon"'] & F \arrow[d, "\theta"] \\
\bigoplus_{i\in I} S_i \arrow[r, equal] \arrow[d] & \bigoplus_{i\in I} S_i \arrow[d] \\
0 & 0
\end{tikzcd}$$

with exact columns by (2.9) and Proposition 2.17. □

2.2 Pointwise Bijectivity by Functorial Filtration

In the previous Section 2.1 we constructed the approximation $\varphi\colon H \to F$ by reduction modulo radical following [HL81, Theorem 1, Proposition 5]. Now in [HL81] it is assumed that $F\colon \mathbb{M}^\circ \to \operatorname{vect}_{\mathbb{F}}$ be projective, whereas here in the Structure Theorem 2.6 we assume that F is pfd, sequentially continuous, and cohomological; so projectivity is implied a posteriori as Corollary 2.7. For this reason, we need an additional argument to show that the approximation $\varphi\colon H \to F$ is indeed a natural isomorphism. More specifically, we use a *functorial filtration* (2.12) inspired by [Rin75] and introduced by [CB15] in the context of $\mathbb{R}$-indexed persistence modules. We also note that the authors of [BLO20, Section 8.1] extend the functorial filtration by [CB15] to the two-parameter setting with four indices in total in order to obtain their structure theorem [BLO20, Theorem 8.2] for *relative interlevel set homology (RISH)*, which has been developed independently to the proof we provide here. However, as we will see this is not needed in the context of RISC and the original approach by [CB15] can be used without increasing the dimension of the indexing set of the functorial filtration. Morally, the reason why two parameters are sufficient here is that just like ordinary persistence modules RISC is an invariant of an object indexed by one parameter, namely real-valued functions. This perspective is developed further in Chapter 9, where we provide an equivalence between certain presheaves on $\mathbb{M}$ and *tame derived sheaves* on the reals $\mathbb{R}$.

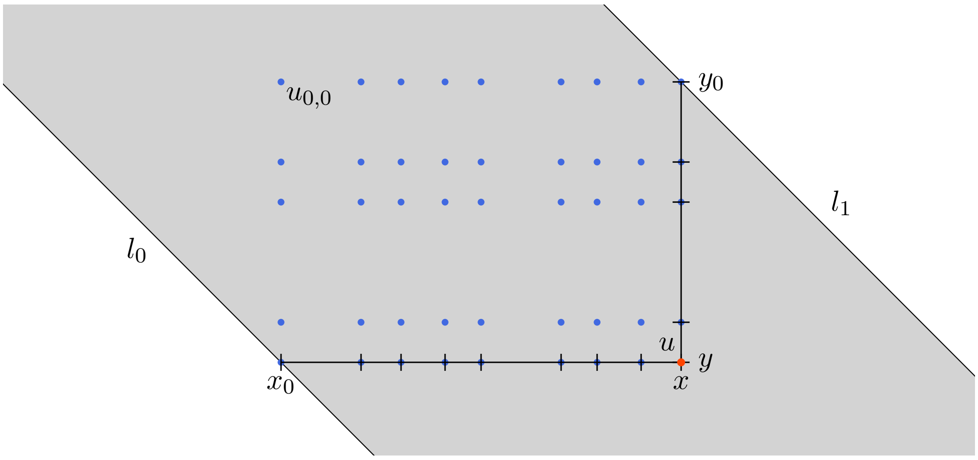


Figure 2.4: The horizontal and vertical filtrations of $F(u)$ by images.

Using the previous Proposition 2.19 we now show that the approximation $\varphi\colon H \to F$ by reduction modulo radical is a natural isomorphism whenever F is sequentially continuous and cohomological.

Proposition 2.20. *If $F\colon \mathbb{M}^\circ \to \mathrm{vect}_{\mathbb{F}}$ is a sequentially continuous cohomological functor, as defined in Definition 1.39 and Definition 1.26, then any approximation $\varphi\colon H \to F$ by reduction modulo radical is a natural isomorphism.*

In conjunction with (2.8) this proposition immediately implies the Structure Theorem 2.6. Now in order to prove Proposition 2.20, we have to show pointwise that

$$\varphi_u\colon H(u) \longrightarrow F(u)$$

is an isomorphism for all $u := (x, y) \in \operatorname{int} \mathbb{M}$. To this end, we fix some notation, which we use in the proof of Proposition 2.20 and auxiliary lemmas. As depicted in Fig. 2.4, let x_0 be the x-coordinate of the intersection of l_0 and the horizontal line through u, let $x_1 < x_2 < \cdots < x_{k-1}$ be the points of discontinuity of the function

$$(x_0, x) \to \mathbb{N}_0,\ s \mapsto \operatorname{rank} F(s \leq x, y), \tag{2.11}$$

and let $x_k := x$. Similarly, let y_0 be the intersection of l_1 and the vertical line through u, let $y_1 > y_2 > \cdots > y_{l-1}$ be the points of discontinuity of the function

$$(y, y_0) \to \mathbb{N}_0,\ t \mapsto \operatorname{rank} F(x, y \leq t),$$

and let $y_l := y$. Moreover, we set $u_{(i,j)} := (x_i, y_j)$ for $i = 0, \ldots, k$ and $j = 0, \ldots, l$, then we have $u = u_{(k,l)}$ and $\Sigma(u) = u_{(0,0)}$. With some abuse of notation, we may also drop the parentheses and write $u_{i,j}$ in place of $u_{(i,j)}$. Furthermore, let $\preceq$ be the colexicographic order on $I := \{0, \ldots, k\} \times \{0, \ldots, l\}$, which is defined by

$$(i, j) \preceq (i', j') \quad :\Leftrightarrow \quad j < j' \ \vee\ (j = j' \ \wedge\ i < i').$$

For any presheaf $G\colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ vanishing on $\partial\mathbb{M}$ and $\zeta \in I$ we set

$$G_\zeta \coloneqq \sum_{\xi \preceq \zeta} \operatorname{Im} G(u \preceq u_\xi)$$

to obtain the natural filtration

$$\bigcup_{\zeta \in I} G_\zeta = G(u).$$

For a pair $\zeta \coloneqq (i,j) \in I$ we will drop the parentheses in the index and write $G_{i,j} = G_\zeta$ in place of $G_{(i,j)}$. With this notation we may write the filtration $\bigcup_{\zeta\in I} G_\zeta$ as

$$\begin{aligned}\{0\} &= G_{0,0} = G_{1,0} = G_{2,0} = \cdots = G_{k,0} \\ &= G_{0,1} \subseteq G_{1,1} \subseteq G_{2,1} \subseteq \cdots \subseteq G_{k,1} \\ &= G_{0,2} \subseteq G_{1,2} \subseteq G_{2,2} \subseteq \cdots \subseteq G_{k,2} \\ &\;\;\vdots \\ &= G_{0,l} \subseteq G_{1,l} \subseteq G_{2,l} \subseteq \cdots \subseteq G_{k,l} = G(u),\end{aligned} \tag{2.12}$$

see also Fig. 2.5. We may describe this filtration more concretely using the equations

$$G_{i,0} = 0 \qquad \text{for } i = 0,\ldots,k, \tag{2.13}$$

$$G_{k,j-1} = G_{0,j} \qquad \text{for } j = 1,\ldots,l, \text{ and} \tag{2.14}$$

$$G_{i,j} = \operatorname{Im} G(u \preceq u_{i,j}) + \operatorname{Im} G(u \preceq u_{k,j-1}) \qquad \text{for } j = 1,\ldots,l \text{ and } i = 0,\ldots,k. \tag{2.15}$$

Proof of Proposition 2.20. We show that

$$\varphi_\xi\colon H_\xi \to F_\xi$$

is a linear isomorphism for all $\xi \in I$ by induction on ξ. By (2.13) the map $\varphi_{i,0}$ is an isomorphism for all $i = 0,\ldots,k$. Moreover, $\varphi_{0,j}$ is an isomorphism if $\varphi_{k,j-1}$ is an isomorphism for all $j = 1,\ldots,l$ by (2.14). Thus, in order to complete our proof by induction, it suffices to show that $\varphi_{i,j}\colon H_{i,j} \to F_{i,j}$ is an isomorphism whenever $\varphi_{i-1,j}\colon H_{i-1,j} \to F_{i-1,j}$ is an isomorphism for all $i = 1,\ldots,k$ and $j = 1,\ldots,l$. To this end, we consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H_{i-1,j} \arrow[r] \arrow[d, "\varphi_{i-1,j}"'] & H_{i,j} \arrow[r] \arrow[d, "\varphi_{i,j}"] & H_{i,j}/H_{i-1,j} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F_{i-1,j} \arrow[r] & F_{i,j} \arrow[r] & F_{i,j}/F_{i-1,j} \arrow[r] & 0 \end{tikzcd}$$

with exact rows. By the five lemma, it suffices to show that the vertical map on the right-hand side is an isomorphism. To this end, we note that H is cohomological, as

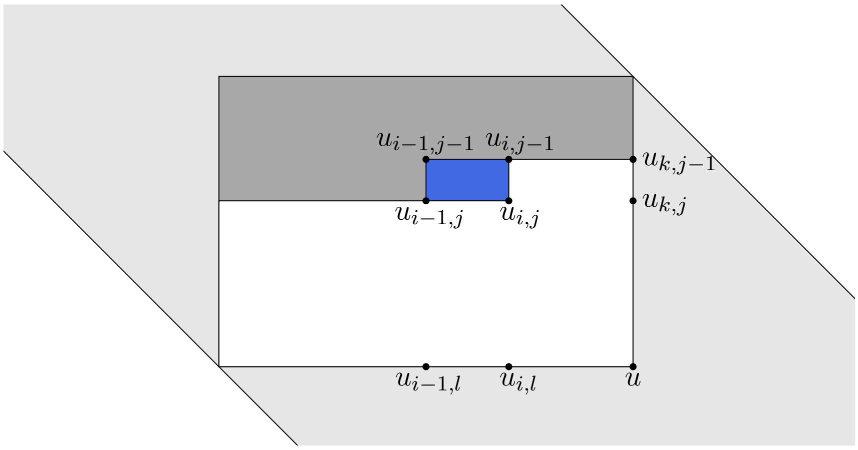


Figure 2.5: The filtration of $G(u)$ in terms of the colexicographic order on I. The large axis-aligned rectangle contains all points such that the corresponding image in $G(u)$ can be non-zero. The subspace $G_{i-1,j} \subseteq G(u)$ is the sum of the images in $G(u)$ corresponding to points in the region shaded in dark grey. If we add the images corresponding to points in the blue rectangle (or just the image corresponding to the lower right vertex $u_{i,j}$), then we obtain $G_{i,j}$ as the next step in the filtration.

it is a direct sum of cohomological functors. Thus, by Lemma 2.21 below, there is a commutative square

$$\begin{tikzcd}
\dfrac{H(u_{i,j})}{\operatorname{Im} H(u_{i,j} \preceq u_{i-1,j}) + \operatorname{Im} H(u_{i,j} \preceq u_{i,j-1})} \arrow[r, "\sim"] \arrow[d] & \dfrac{H_{i,j}}{H_{i-1,j}} \arrow[d] \\
\dfrac{F(u_{i,j})}{\operatorname{Im} F(u_{i,j} \preceq u_{i-1,j}) + \operatorname{Im} F(u_{i,j} \preceq u_{i,j-1})} \arrow[r, "\sim"] & \dfrac{F_{i,j}}{F_{i-1,j}},
\end{tikzcd}$$

where the two vertical maps are induced by φ. As the two horizontal maps are isomorphisms by Lemma 2.21, it remains to show that the vertical map on the left-hand side is an isomorphism. To this end, we consider the commutative square

$$\begin{tikzcd}
\dfrac{H(u_{i,j})}{\operatorname{Im} H(u_{i,j} \preceq u_{i-1,j}) + \operatorname{Im} H(u_{i,j} \preceq u_{i,j-1})} \arrow[r] \arrow[d] & \dfrac{H(u_{i,j})}{(\operatorname{rad} H)(u_{i,j})} \arrow[d, "\bar{\varphi}_{u_{i,j}}"] \\
\dfrac{F(u_{i,j})}{\operatorname{Im} F(u_{i,j} \preceq u_{i-1,j}) + \operatorname{Im} F(u_{i,j} \preceq u_{i,j-1})} \arrow[r] & \dfrac{F(u_{i,j})}{(\operatorname{rad} F)(u_{i,j})},
\end{tikzcd}$$

where the vertical maps are induced by φ and the horizontal maps are induced by the internal maps of H and F respectively. We have to show that the vertical map on the left-hand side is an isomorphism. The two vector spaces on the right-hand side are the values of the reductions modulo radical of H and F at $u_{i,j}$, respectively. By Proposition 2.19 above the linear map

$$\bar{\varphi}_{u_{i,j}} : (H/\operatorname{rad} H)(u_{i,j}) \to (F/\operatorname{rad} F)(u_{i,j})$$

is an isomorphism. Moreover, the horizontal map at the top and the horizontal map at the bottom are isomorphisms by Corollary 2.26 and Lemma 2.27, respectively. Thus, the vertical map on the left-hand side is an isomorphism as well. As $u \in \operatorname{int} \mathbb{M}$ was arbitrary, $\varphi\colon H \to F$ is a natural isomorphism. □

2.2.1 Auxiliary Lemmas

Lemma 2.21. *For any cohomological functor $G\colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ and any pair of indices $(i,j) \in \{1,\dots,k\} \times \{1,\dots,l\}$ there is an isomorphism*

$$\frac{G(u_{i,j})}{\operatorname{Im} G(u_{i,j} \preceq u_{i-1,j}) + \operatorname{Im} G(u_{i,j} \preceq u_{i,j-1})} \xrightarrow{\sim} \frac{G_{i,j}}{G_{i-1,j}}$$

natural in G.

Proof. We consider the commutative diagram

$$\begin{array}{ccccc}
G(u_{i-1,j-1}) & \longrightarrow & G(u_{i,j-1}) & \longrightarrow & G(u_{k,j-1}) = G(x, y_{j-1}) \\
\downarrow & & \downarrow & & \downarrow \\
G(u_{i-1,j}) & \longrightarrow & G(u_{i,j}) & \longrightarrow & G(u_{k,j}) = G(x, y_j) \\
\downarrow & & \downarrow & & \downarrow \\
G(u_{i-1,l}) & \longrightarrow & G(u_{i,l}) & \longrightarrow & G(u) = G(x, y).
\end{array}$$

By Proposition 1.28.4 all squares in this diagram are middle exact, see also Fig. 2.5. Thus, by Proposition B.3 the map $G(u \preceq u_{i,j})$ induces an isomorphism

$$\frac{G(u_{i,j})}{\operatorname{Im} G(u_{i,j} \preceq u_{i-1,j}) + \operatorname{Im} G(u_{i,j} \preceq u_{i,j-1})} \xrightarrow{\sim} \frac{\operatorname{Im} G(u \preceq u_{i,j}) + \operatorname{Im} G(u \preceq u_{k,j-1})}{\operatorname{Im} G(u \preceq u_{i-1,j}) + \operatorname{Im} G(u \preceq u_{k,j-1})}.$$

Moreover, by (2.15) the codomain of this isomorphism is $G_{i,j}/G_{i-1,j}$ and thus we may write this isomorphism also as

$$\frac{G(u_{i,j})}{\operatorname{Im} G(u_{i,j} \preceq u_{i-1,j}) + \operatorname{Im} G(u_{i,j} \preceq u_{i,j-1})} \xrightarrow{\sim} \frac{G_{i,j}}{G_{i-1,j}}.$$ □

Before we prove Corollary 2.26 and Lemma 2.27 we need to establish three auxiliary results. To this end, we note that the inclusion

$$\{x_i \mid i = 0,\dots,k\} \hookrightarrow [x_0, x]$$

has the upper adjoint

$$r_1\colon [x_0, x] \to \{x_i \mid i = 0, \ldots, k\},\ s \mapsto \max\{x_i \mid x_i \leq s\}.$$

Similarly

$$r_2\colon [y, y_0] \to \{y_j \mid j = 0, \ldots, l\},\ t \mapsto \min\{y_j \mid t \leq y_j\}$$

is the lower adjoint of the inclusion

$$\{y_j \mid j = 0, \ldots, l\} \hookrightarrow [y, y_0].$$

Lemma 2.22. *We have*

$$\begin{aligned} \operatorname{Im} F(r_1(s) \leq x, y) &= \operatorname{Im} F(s \leq x, y) \qquad && \textit{for all } s \in [x_0, x] \textit{ as well as} \\ \operatorname{Im} F(x, y \leq r_2(t)) &= \operatorname{Im} F(x, y \leq t) && \textit{for all } t \in [y, y_0]. \end{aligned}$$

Proof. We prove the first equation, the second can be shown in an analogous manner. To this end, we consider the filtration

$$\bigcup_{s \leq x} \operatorname{Im} F(s \leq x, y) = F(x, y) = F(u).$$

For $s_0 \in [x_0, x)$ the canonical map

$$F(s_0, y) \longrightarrow \varprojlim_{s > s_0} F(s, y) \tag{2.16}$$

is an isomorphism by the sequential continuity of F. As a result, the image of the canonical map

$$\varprojlim_{s > s_0} F(s, y) \longrightarrow F(x, y) = F(u)$$

and the image $\operatorname{Im} F(s_0 \leq x, y)$ are the same. Moreover, the image of (2.16) and the intersection

$$\bigcap_{s > s_0} \operatorname{Im} F(s \leq x, y)$$

are identical, hence

$$\operatorname{Im} F(s_0 \leq x, y) = \bigcap_{s > s_0} \operatorname{Im} F(s \leq x, y).$$

As a result of this equation, the function (2.11) is upper semi-continuous, i.e. the superlevel sets of (2.11) are closed. Moreover, as $x_1, \ldots, x_{k-1}$ are by definition the points of discontinuity of (2.11), we have

$$\operatorname{Im} F(x_i \leq x, y) = \operatorname{Im} F(s \leq x, y)$$

for all $i = 0, \ldots, k-1$ and $s \in [x_i, x_{i+1})$. Using the upper adjoint $r_1\colon [x_0, x] \to \{x_i \mid i = 0, \ldots, k\}$ we can state this last equation without explicit quantification over $\{0, \ldots, k-1\}$ as

$$\operatorname{Im} F(r_1(s) \leq x, y) = \operatorname{Im} F(s \leq x, y) \quad \text{for all } s \in [x_0, x].$$

□

Now suppose we have

$$\begin{aligned} v \coloneqq (v_1, v_2) &\in [u, \Sigma(u)] = [x_0, x] \times [y, y_0] \\ \text{and} \qquad (s,t) &\in [v, \Sigma(u)] = [x_0, v_1] \times [v_2, y_0]. \end{aligned}$$

We consider the commutative square

$$\begin{tikzcd} F(s,t) \arrow[r] \arrow[d] & F(v_1,t) \arrow[d, "F(v_1, v_2 \leq t)"] \\ F(s,v_2) \arrow[r, "F(s \leq v_1, v_2)"'] & F(v) \end{tikzcd}$$

and we define

$$F_v(s,t) \coloneqq \operatorname{Im} F(s \leq v_1, v_2) + \operatorname{Im} F(v_1, v_2 \leq t).$$

Moreover, let

$$r \coloneqq r_1 \times r_2 \colon [u, \Sigma(u)] \to \{u_\xi \mid \xi \in I\},$$

i.e. r is the lower adjoint to the inclusion $\{u_\xi \mid \xi \in I\} \hookrightarrow [u, \Sigma(u)]$. Then we have

$$(F_v \circ r)(s,t) \subseteq F_v(s,t).$$

Furthermore, in the special case that $v = u$ we have equality:

$$(F_u \circ r)(s,t) = F_u(s,t);$$

as a result of Lemma 2.22. By the following lemma this is true even when $v \neq u$.

Lemma 2.23. *We have* $(F_v \circ r)(s,t) = F_v(s,t)$.

Remark 2.24. In general it may very well happen that

$$\begin{aligned} \operatorname{Im} F(r_1(s) \leq v_1, v_2) &\neq \operatorname{Im} F(s \leq v_1, v_2) \\ \text{or} \quad \operatorname{Im} F(v_1, v_2 \leq r_2(t)) &\neq \operatorname{Im} F(v_1, v_2 \leq t). \end{aligned}$$

Thus, it is crucial to consider the two summands of $F_v(s,t)$ in conjunction.

Proof. It suffices to show that

$$F_v(r_1(s), t) = F_v(s,t) \tag{2.17}$$

$$\text{and} \quad F_v(s, r_2(t)) = F_v(s,t), \tag{2.18}$$

independent of s and t, since this implies that

$$(F_v \circ r)(s,t) = F_v(r_1(s), r_2(t)) = F_v(s, r_2(t)) = F_v(s,t).$$

We show (2.18), our proof of (2.17) is similar. Now

$$F_v(s, r_2(t)) = \operatorname{Im} F(s \leq v_1, v_2) + \operatorname{Im} F(v_1, v_2 \leq r_2(t)),$$

which is a subspace of

$$\operatorname{Im} F(s \leq v_1, v_2) + \operatorname{Im} F(v_1, v_2 \leq t) = F_v(s,t) \subseteq F(v).$$

Moreover, this inclusion

$$F_v(s, r_2(t)) \hookrightarrow F_v(s,t)$$

induces a canonical map

$$\pi_v \colon \frac{F(v)}{F_v(s, r_2(t))} \longrightarrow \frac{F(v)}{F_v(s,t)}.$$

Now in order to prove (2.18), it suffices to show that π_v is an isomorphism. In the special case that $v = u$, we already have $F_u(s, r_2(t)) = F_u(s,t)$ by the second equation from Lemma 2.22, hence

$$\pi_u = \mathrm{id} \colon \frac{F(u)}{F_u(s, r_2(t))} \xrightarrow{=} \frac{F(u)}{F_u(s,t)}.$$

Our approach is to reduce the general case for π_v to this special case of $\pi_u = \mathrm{id}$. To this end, we consider the commutative diagram

$$\begin{tikzcd}
F(s, r_2(t)) \arrow[r] \arrow[d] & F(v_1, r_2(t)) \arrow[r] \arrow[d] & F(x, r_2(t)) \arrow[d] \\
F(s,t) \arrow[r] \arrow[d] & F(v_1, t) \arrow[r] \arrow[d] & F(x,t) \arrow[d] \\
F(s, v_2) \arrow[r] \arrow[d] & F(v) \arrow[r] \arrow[d] \arrow[dr, "F(u \preceq v)"] & F(x, v_2) \arrow[d] \\
F(s,y) \arrow[r] & F(v_1, y) \arrow[r] & F(u).
\end{tikzcd} \tag{2.19}$$

By Proposition 1.28.4 all axis-aligned squares and rectangles in this diagram are middle exact. Now $F(u \preceq v)$ maps $\operatorname{Im} F(v_1, v_2 \leq r_2(t))$ to a subspace of $\operatorname{Im} F(x, y \leq r_2(t))$, hence $F_v(s, r_2(t))$ is mapped to a subspace of $F_u(s, r_2(t))$. Similarly, $F(u \preceq v)$ maps $F_v(s,t)$ to a subspace of $F_u(s,t)$. As a result we obtain the commutative diagram

$$\begin{tikzcd}
F_v(s, r_2(t)) \arrow[r, "F(u \preceq v)"] \arrow[d, hook] & F_u(s, r_2(t)) \arrow[d, equal] \\
F_v(s,t) \arrow[r, "F(u \preceq v)"'] \arrow[d, hook] & F_u(s,t) \arrow[dd, hook] \\
F(v) \arrow[dr, "F(u \preceq v)"'] & \\
& F(u)
\end{tikzcd}$$

from which we obtain the induced commutative square

$$\begin{tikzcd}
\dfrac{F(v)}{F_v(s, r_2(t))} \arrow[r] \arrow[d, "\pi_v"'] & \dfrac{F(u)}{F_u(s, r_2(t))} \arrow[d, equal, "\pi_u = \mathrm{id}"] \\
\dfrac{F(v)}{F_v(s, t)} \arrow[r] & \dfrac{F(u)}{F_u(s, t)}.
\end{tikzcd}$$

As all axis-aligned squares and rectangles of (2.19) are middle exact, the two horizontal maps of this square are isomorphisms by Proposition B.3, hence π_v is an isomorphism as well. $\square$

Lemma 2.25. *The restriction of $\beta^0(F)$ to $(\uparrow u) \cap \operatorname{int}(\downarrow \Sigma(u))$ is supported on the grid $\{u_\xi \mid \xi \in I\}$.*

Proof. Let $v := (v_1, v_2) \in (\uparrow u) \cap \operatorname{int}(\downarrow \Sigma(u)) \setminus \{u_\xi \mid \xi \in I\}$. We have to show that $\beta^0(F)(v) = 0$. As $v \notin \{u_\xi \mid \xi \in I\}$ we have $v \neq r(v)$, which implies $v \prec r(v)$. Thus, we have $v_1 > r_1(v_1)$ or $v_2 < r_2(v_2)$. Without loss of generality we assume that $v_2 < r_2(v_2)$. Now let $j = 0, \ldots, l-1$ be such that $y_j = r_2(v_2) > v_2$. Considering the commutative diagram

$$\begin{tikzcd}
F(x_0, y_j) \arrow[r] \arrow[d] & F(v_1, y_j) \arrow[d] \\
F(x_0, v_2) \arrow[r] & F(v)
\end{tikzcd}$$

we see that

$$\begin{aligned}
(F_v \circ r)(x_0, v_2) &= F_v(x_0, y_j) \\
&= \operatorname{Im} F(x_0 \leq v_1, v_2) + \operatorname{Im} F(v_1, v_2 \leq y_j) \\
&\subseteq \sum_{w \succ v} \operatorname{Im} F(v \preceq w) \\
&= (\operatorname{rad} F)(v).
\end{aligned}$$

Moreover, Lemma 2.23 implies that

$$\begin{aligned}
(F_v \circ r)(x_0, v_2) &= F_v(x_0, v_2) \\
&= \operatorname{Im} F(x_0 \leq v_1, v_2) + \operatorname{Im} F(v_1, v_2 \leq v_2) \\
&= \operatorname{Im} F(x_0 \leq v_1, v_2) + F(v) \\
&= F(v).
\end{aligned}$$

The previous two chains of equations (and an inclusion) taken together we obtain

$$F(v) \subseteq (\operatorname{rad} F)(v) \subseteq F(v),$$

hence

$$\begin{aligned}\beta^0(F)(v) &= \dim_{\mathbb{F}} F(v) - \dim_{\mathbb{F}}(\operatorname{rad} F)(v) \\ &= \dim_{\mathbb{F}} F(v) - \dim_{\mathbb{F}} F(v) = 0\end{aligned}$$

by Lemma 2.14. □

Corollary 2.26. *The canonical map*

$$\frac{H(u_{i,j})}{\operatorname{Im} H(u_{i,j} \preceq u_{i-1,j}) + \operatorname{Im} H(u_{i,j} \preceq u_{i,j-1})} \longrightarrow \frac{H(u_{i,j})}{(\operatorname{rad} H)(u_{i,j})} \tag{2.20}$$

is a linear isomorphism.

Proof. We consider the restriction of $\beta^0(F)$ to the blue rectangle in Fig. 2.5. By Lemma 2.25, this restriction can be non-zero only at the vertices $u_{i-1,j-1}$, $u_{i,j-1}$, $u_{i-1,j}$, or $u_{i,j}$. Thus, any indecomposable summand of H, which is not born at $u_{i,j}$ and yet alive at $u_{i,j}$, is alive at $u_{i-1,j}$ or $u_{i,j-1}$, hence

$$\operatorname{Im} H(u_{i,j} \preceq u_{i-1,j}) + \operatorname{Im} H(u_{i,j} \preceq u_{i,j-1}) = \sum_{w \succ u_{i,j}} \operatorname{Im} H(u_{i,j} \preceq w) = (\operatorname{rad} H)(u_{i,j}).$$

As a result, the canonical map (2.20) is an identity. □

Lemma 2.27. *The canonical map*

$$\frac{F(u_{i,j})}{\operatorname{Im} F(u_{i,j} \preceq u_{i-1,j}) + \operatorname{Im} F(u_{i,j} \preceq u_{i,j-1})} \longrightarrow \frac{F(u_{i,j})}{(\operatorname{rad} F)(u_{i,j})} \tag{2.21}$$

is a linear isomorphism.

Proof. Let $v := u_{i,j}$ and let $R := [v, u_{i-1,j-1}] \setminus \{v\}$, i.e. R is the blue rectangle in Fig. 2.5 except for the vertex $v = u_{i,j}$. Then we have the inclusion

$$\operatorname{Im} F(v \preceq w) \subseteq F_v(w)$$

for any $w \in R$, and thus

$$(\operatorname{rad} F)(u_{i,j}) = \sum_{w \succ u_{i,j}} \operatorname{Im} F(u_{i,j} \preceq w) = \sum_{w \in R} \operatorname{Im} F(v \preceq w) = \sum_{w \in R} F_v(w).$$

Moreover, by Lemma 2.23

$$\begin{aligned}F_v(w) &= (F_v \circ r)(w) = F_v(u_{i-1,j-1}) \\ &= \operatorname{Im} F(v \preceq u_{i-1,j}) + \operatorname{Im} F(v \preceq u_{i,j-1}) \\ &= \operatorname{Im} F(u_{i,j} \preceq u_{i-1,j}) + \operatorname{Im} F(u_{i,j} \preceq u_{i,j-1})\end{aligned}$$

for any $w \in R$ and as a result the denominators of the domain and the codomain of (2.21) are identical, hence (2.21) is the identity. □

2.3 Decomposition of q-Tame Cohomological Presheaves

In this Section 2.3 we generalize the Structure Theorem 2.6 from pfd cohomological presheaves to *q-tame* [CdSGO16, Section 1.1] cohomological presheaves.

Definition 2.28 (q-Tame Presheaf)**.** We say that a presheaf $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ is *q-tame* if $F(u \preceq v)$ has finite rank for all $u \prec v \in \mathbb{M}$.

Proposition 2.29. *Let $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ be a cohomological presheaf which is q-tame. Then F is pfd.*

Proof. Let $(x,y) \in \operatorname{int}\mathbb{M}$. We show that $F(x,y)$ is finite-dimensional. To this end, let $\delta > 0$ be such that $(x-\delta, y), (x+\delta, y+\delta) \in \mathbb{M}$. Now let

$$x_0 \coloneqq x - \delta, \qquad x_1 \coloneqq x, \qquad x_2 \coloneqq x + \delta,$$
$$y_1 \coloneqq y, \quad \text{and} \quad y_2 \coloneqq y + \delta.$$

We consider the commutative diagram

$$\begin{tikzcd}
F(x_0,y_2) \arrow[r] \arrow[d] & F(x_1,y_2) \arrow[r] \arrow[d] & F(x_2,y_2) \arrow[d] \\
F(x_0,y_1) \arrow[r] \arrow[d, equal] & F(x_1,y_1) \arrow[r] \arrow[d, equal] & F(x_2,y_1) \arrow[d, equal] \\
F(x_0,y_1) \arrow[r] & F(x_1,y_1) \arrow[r] & F(x_2,y_1).
\end{tikzcd}$$

By Proposition 1.28.4 all squares in this diagram are middle exact, hence

$$\frac{F(x_1,y_1)}{\operatorname{Im} F(x_0 \le x_1, y_1) + \operatorname{Im} F(x_1, y_1 \le y_2)} \cong \frac{\operatorname{Im} F(x_1 \le x_2, y_1) + \operatorname{Im} F(x_2, y_1 \le y_2)}{\operatorname{Im} F(x_0 \le x_2, y_1) + \operatorname{Im} F(x_2, y_1 \le y_2)}$$

by Proposition B.3. As F is q-tame, the numerator on the right-hand side and both denominators are finite-dimensional. Thus, the numerator $F(x_1,y_1) = F(x,y)$ on the left has to be finite-dimensional as well. □

This proposition has the following two corollaries.

Corollary 2.30. *Let $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ be a q-tame sequentially continuous cohomological presheaf. Then F is pfd and*

$$F \cong \bigoplus_{v \in \operatorname{int}\mathbb{M}} B_v^{\oplus \mu(v)},$$

where $\mu \coloneqq \beta^0(F)$.

Proof. This follows in conjunction with Theorem 2.6. □

Corollary 2.31. *Any continuous function $f\colon X \to \mathbb{R}$ is $\mathbb{F}$-tame iff $h(f)\colon \mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$ is q-tame, in which case it decomposes as*

$$h(f) \cong \bigoplus_{v \in \operatorname{int} \mathbb{M}_\mathrm{f}} B_v^{\oplus \mu(v)},$$

where $\mu \coloneqq \mathrm{Dgm}(f)$.

Proof. If $h(f)\colon \mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$ is q-tame, then it is pfd by Proposition 2.29 (and vice versa). This in turn is equivalent to $f\colon X \to \mathbb{R}$ being $\mathbb{F}$-tame by Lemma 1.38. As a result we may use Corollary 2.8 to obtain a decomposition of $h(f)\colon \mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$ into block presheaves with multiplicities provided by $\mathrm{Dgm}(f)\colon \operatorname{int} \mathbb{M}_\mathrm{f} \to \mathbb{N}_0$ as in the statement. □

2.4 Decomposition of Homological Functors

We note that $\mathbb{M}$ is self-dual as a lattice. Thus, there is an obvious dual version of Theorem 2.6. We state this result here for completeness and for reference.

Remark 2.32. The reflection at the diagonal

$$\mathbb{M} \to \mathbb{M},\ (x, y) \mapsto (y, x)$$

is a self-duality of the lattice $\mathbb{M}$ in the sense that it is order-reversing and interchanges joins and meets. In particular, any covariant functor on $\mathbb{M}$ can be made into a contravariant functor and vice versa by precomposition with this reflection.

The following is dual to Definition 2.3 in the sense of this remark, see also Fig. 2.1b.

Definition 2.33 (Block)**.** For $v \in \operatorname{int} \mathbb{M}$ we define

$$B^v\colon \mathbb{M} \to \mathrm{Vect}_\mathbb{F},\ w \mapsto \begin{cases} \mathbb{F} & w \in (\uparrow v) \cap \operatorname{int}(\downarrow \Sigma(v)) \\ \{0\} & \text{otherwise,} \end{cases}$$

where $\operatorname{int}(\downarrow \Sigma(v))$ is the interior of the downset of $\Sigma(v)$ in $\mathbb{M}$. The internal maps are identities whenever both domain and codomain are $\mathbb{F}$, otherwise they are zero.

With this we may state the dual of Theorem 2.6 in the sense of Remark 2.32.

Theorem 2.34. *Any sequentially continuous homological functor $F\colon \mathbb{M} \to \mathrm{vect}_\mathbb{F}$ decomposes as*

$$F \cong \bigoplus_{v \in \operatorname{int} \mathbb{M}} (B^v)^{\oplus \mu(v)},$$

where $\mu : \operatorname{int} \mathbb{M} \to \mathbb{N}_0,\ v \mapsto \dim_\mathbb{F} F(v) - \dim_\mathbb{F} \sum_{u \prec v} \operatorname{Im} F(u \preceq v)$.

2.5 Connections of RISC to Level Set and Extended Persistence

We now use the Structure Theorem 2.6 to connect RISC to two other variants of persistence, namely level set persistence [CdM09] and extended persistence [CSEH09]. A posteriori this also implies that our Definition 2.2 of the extended persistence diagram is consistent with the original definition by [CSEH09].

2.5.1 The Level Set Barcode

In this Section 2.5.1 we connect the extended persistence diagram, as defined in Definition 2.2, to the *level set barcode*, as introduced by [CdM09] as *levelset zigzag persistence intervals*; see also [BEMP13, CdSKM19]. The extended persistence diagram is a multiset of interior points of $\mathbb{M}_\mathrm{f}$ while the level set barcode is a multiset of pairs of an integer degree and an interval in $\mathbb{R}$. To connect the two notions we describe a bijection between $\operatorname{int}\mathbb{M}_\mathrm{f}$ and $\mathbb{Z}\times\mathcal{I}$, where $\mathcal{I}\subset 2^{\mathbb{R}}$ is the set of non-empty intervals in $\mathbb{R}$. Specifically, for a point $u\in\operatorname{int}\mathbb{M}_\mathrm{f}$ and the unique integer $n\in\mathbb{Z}$ with $\Sigma^n(u)\in D$ we set

$$\nu(u) \coloneqq n \quad \text{and} \quad I(u) \coloneqq (\rho_1\circ\Sigma^n)(u)\setminus(\rho_0\circ\Sigma^n)(u),$$

where we think of $\nu(u)\in\mathbb{Z}$ as the degree associated to $u\in\operatorname{int}\mathbb{M}_\mathrm{f}$ and of $I(u)\in\mathcal{I}$ as the associated interval. With this we obtain the bijection

$$\mathfrak{b}\colon \operatorname{int}\mathbb{M}_\mathrm{f}\to\mathbb{Z}\times\mathcal{I},\ u\mapsto(\nu(u),I(u))\,. \tag{2.22}$$

Proposition 2.35. *For an $\mathbb{F}$-tame function $f\colon X\to\mathbb{R}$ satisfying the additional tameness assumption that the naturally induced map*

$$\varinjlim_{\delta>0}\mathcal{H}^n\big(f^{-1}(t-\delta,t+\delta)\big)\to\mathcal{H}^n\big(f^{-1}(t)\big) \tag{2.23}$$

is an isomorphism of vector spaces for any $t\in\mathbb{R}$ with extended persistence diagram $\mu\coloneqq\operatorname{Dgm}(f)\colon\operatorname{int}\mathbb{M}_\mathrm{f}\to\mathbb{N}_0$, the multiset

$$\mu\circ\mathfrak{b}^{-1}\colon\mathbb{Z}\times\mathcal{I}\xrightarrow{\mathfrak{b}^{-1}}\operatorname{int}\mathbb{M}_\mathrm{f}\xrightarrow{\mu}\mathbb{N}_0$$

is the multiset of levelset zigzag persistence intervals of $f\colon X\to\mathbb{R}$ in the sense of [CdM09, Section 2].

In order to see that the additional tameness assumption of Proposition 2.35 is needed, consider the line with two origins as an $\mathbb{F}$-tame example, where the naturally induced map (2.23) is not surjective for $t=0=n$. Now let $f\colon X\to\mathbb{R}$ be a continuous function satisfying the assumptions of Proposition 2.35. In order to show Proposition 2.35 we choose a *representation* $p\colon J\to\operatorname{int}\mathbb{M}_\mathrm{f}$ of the multiset $\mu\coloneqq\operatorname{Dgm}(f)\colon\operatorname{int}\mathbb{M}_\mathrm{f}\to\mathbb{N}_0$. This means that $p\colon J\to\operatorname{int}\mathbb{M}_\mathrm{f}$ is some map of sets with

$$\mu(v)=\#p^{-1}(v)\quad\text{for all } v\in\operatorname{int}\mathbb{M}_\mathrm{f}.$$

We have to show that

$$\mathfrak{b}\circ p\colon J\to\mathbb{Z}\times\mathcal{I},\ i\mapsto((\nu\circ p)(i),(I\circ p)(i))$$

is a representation of the multiset of levelset zigzag persistence intervals $f\colon X\to\mathbb{R}$ henceforth referred to as the levelset barcode of f. To this end, we choose a basis $\{\omega_i\}_{i\in p^{-1}(v)}$ for a complement of

$$\sum_{w\succ v}\operatorname{Im}h(f)(v\preceq w)=(\operatorname{rad}h(f))(v)$$

in $h(f)(v)$ for each $v \in \operatorname{int} \mathbb{M}_{\mathrm{f}}$. As in Section 2.1, we now write $S_i \coloneqq S_{p(i)} \colon \mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ and $B_i \coloneqq B_{p(i)} \colon \mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ for $i \in J$. By the Yoneda Lemma 2.4 there is a unique natural transformation

$$\varphi_i \colon B_i \to h(f)$$

sending $1 \in \mathbb{F} = (B_i \circ p)(i)$ to $\omega_i \in (h(f) \circ p)(i)$ for any $i \in J$. Moreover, the J-tuple

$$(\varphi_i)_{i\in J} \in \prod_{i \in J} \mathrm{Nat}(B_i, h(f))$$

induces a natural transformation

$$\varphi \colon \bigoplus_{i \in J} B_i \to h(f)$$

by the universal property of the direct sum.

Lemma 2.36. *The natural transformation* $\varphi \colon \bigoplus_{i \in J} B_i \to h(f)$ *is a natural isomorphism.*

Proof. For each $i \in J$ let

$$\bar{\omega}_i \in (h(f)/\operatorname{rad} h(f))(p(i))$$

be the image of $\omega_i \in (h(f) \circ p)(i)$ under the canonical projection $\mathrm{pr} \colon h(f) \to h(f)/\operatorname{rad} h(f)$. Then the subset

$$\{\bar{\omega}_i\}_{i \in p^{-1}(v)} \subset (h(f)/\operatorname{rad} h(f))(v)$$

is a basis for each $v \in \operatorname{int} \mathbb{M}_{\mathrm{f}}$. Now let

$$(\bar{\alpha}_i)_{i\in J} \in \prod_{i \in J} \mathrm{Nat}(h(f)/\operatorname{rad} h(f), S_i)$$

be the unique J-tuple such that the subset

$$\{\bar{\alpha}_{i,v}\}_{i \in p^{-1}(v)} \subset \mathrm{Hom}_{\mathbb{F}}((h(f)/\operatorname{rad} h(f))(v), \mathbb{F})$$

is the dual basis of $\{\bar{\omega}_i\}_{i \in p^{-1}(v)}$ for each $v \in \operatorname{int} \mathbb{M}_{\mathrm{f}}$. Or to be more precise, the J-tuple $(\bar{\alpha}_i)_{i\in J} \in \prod_{i \in J} \mathrm{Nat}(h(f)/\operatorname{rad} h(f), S_i)$ is uniquely characterized by the equations

$$\bar{\alpha}_{i,p(j)}(\bar{\omega}_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

for all $i, j \in J$. Moreover, we set

$$\alpha_i \coloneqq \mathrm{pr} \circ \bar{\alpha}_i \colon h(f) \to S_i$$

for any $i \in J$. Then the J-tuple

$$(\alpha_i)_{i\in J} \in \prod_{i \in J} \mathrm{Nat}(h(f), S_i)$$

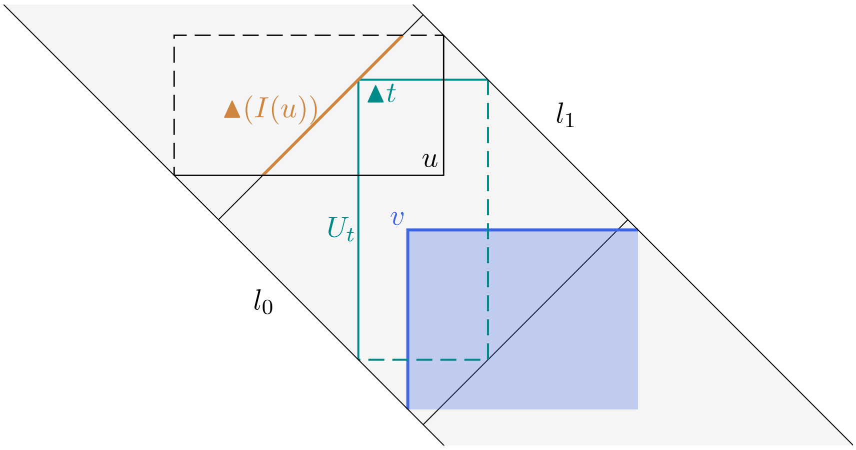


Figure 2.6: The axis-aligned rectangle U_t and the support of the indecomposable B_v.

is a reduced dual frame for the presheaf $h(f)\colon \mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ in the sense of Definition 2.15. Now let $\theta\colon h(f) \to \prod_{i\in J} S_i$ be the corresponding reducing projection as in Definition 2.16. Then we have the commutative triangle

$$\begin{tikzcd} & h(f) \arrow[d, two heads, "\theta"] \\ \bigoplus_{i\in J} B_i \arrow[ur, "\varphi"] \arrow[r] & \bigoplus_{i\in J} S_i \end{tikzcd}$$

by construction of $\varphi\colon \bigoplus_{i\in J} B_i \to h(f)$, hence φ is an approximation of $h(f)$ by reduction modulo radical of in the sense of Definition 2.18. Thus, the statement follows in conjunction with Proposition 2.20. □

For $n \in \mathbb{Z}$ and $t \in \mathbb{R}$ we set

$$J_{n,t} := \{i \in J \mid n = (\nu \circ p)(i) \ \text{ and } \ t \in (I \circ p)(i)\}\,.$$

As $t \in (\rho_1 \circ \Sigma^n \circ p)\,(i)$ for any $i \in J_{n,t}$, the corresponding cohomology class $\omega_i \in (\mathcal{H}^n \circ \rho \circ \Sigma^n \circ p)\,(i)$ has a pullback $\omega_i|_t \in \mathcal{H}^n\big(f^{-1}(t)\big)$.

Lemma 2.37. *The subset* $\{\omega_i|_t\}_{i\in J_{n,t}} \subset \mathcal{H}^n\big(f^{-1}(t)\big)$ *is a basis for any* $t \in \mathbb{R}$ *and* $n \in \mathbb{Z}$*.*

Proof. Let $t \in \mathbb{R}$, $n \in \mathbb{Z}$, and

$$U_t := \{u \in D \mid t \in I(u)\} = (\downarrow \blacktriangle t) \cap \operatorname{int}\left(\uparrow \left(\Sigma^{-1} \circ \blacktriangle\right)(t)\right),$$

see also Fig. 2.6. Then we have

$$J_{n,t} = (\Sigma^n \circ p)^{-1}\,(U_t). \tag{2.24}$$

We now deduce the chain of isomorphisms

$$\begin{aligned}\mathcal{H}^n\big(f^{-1}(t)\big) &\cong \varinjlim_{u\in U_t} \big(\mathcal{H}^n \circ f^{-1} \circ \rho_1\big)(u) \\ &\cong \varinjlim_{u\in U_t} \big(\mathcal{H}^n \circ f^{-1} \circ \rho\big)(u) \\ &= \varinjlim_{u\in U_t} \big(h(f) \circ \Sigma^{-n}\big)(u).\end{aligned} \tag{2.25}$$

The first isomorphism of (2.25) follows from $\{t\} = \bigcap_{u\in U_t} \rho_1(u)$, the fact that $\{(t-\delta, t+\delta) \mid \delta > 0\}$ is a final subset of $\{\rho_1(u) \mid u \in U_t\}$, and our additional tameness assumption that (2.23) is an isomorphism. The second isomorphism of (2.25) follows from the fact that U_t has a final subset on which ρ_0 is empty, hence $(\rho_1, \emptyset)$ and ρ agree on this subset. Now considering (2.25), it suffices to show that the subset of *germs*

$$\{[\omega_i]_t\}_{i\in J_{n,t}} \subset \varinjlim_{u\in U_t} \big(h(f) \circ \Sigma^{-n}\big)(u)$$

is a basis. Moreover, by Lemma 2.36 we have the natural isomorphism

$$\varphi \circ \Sigma^{-n} \colon \bigoplus_{i\in J} B_{(\Sigma^n \circ p)(i)} \to h(f) \circ \Sigma^{-n}$$

of presheaves $\mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$. Now for $i \in J$, the naturally induced map

$$\varinjlim_{u\in U_t} \big(\varphi_i \circ \Sigma^{-n}\big)_u \colon \varinjlim_{u\in U_t} B_{(\Sigma^n \circ p)(i)}(u) \longrightarrow \varinjlim_{u\in U_t} \big(h(f) \circ \Sigma^{-n}\big)(u)$$

sends the germ $[1]_t \in \varinjlim_{u\in U_t} B_{(\Sigma^n \circ p)(i)}(u)$ to the germ $[\omega_i]_t \in \varinjlim_{u\in U_t} \big(h(f) \circ \Sigma^{-n}\big)(u)$. Thus, it suffices to show that we have the equivalence

$$[1]_t = [0]_t \in \varinjlim_{u\in U_t} B_{(\Sigma^n \circ p)(i)}(u) \iff i \in J_{n,t} \tag{2.26}$$

for any $i \in J$. Now for $v \in \operatorname{int} \mathbb{M}_\mathrm{f}$ we have

$$\varinjlim_{u\in U_t} B_v(u) \cong \begin{cases} \mathbb{F} & v \in U_t \\ \{0\} & v \notin U_t, \end{cases}$$

see also Fig. 2.6. So in particular we have

$$\varinjlim_{u\in U_t} B_{(\Sigma^n \circ p)(i)}(u) \cong \begin{cases} \mathbb{F} & i \in J_{n,t} \\ \{0\} & i \notin J_{n,t} \end{cases}$$

for any $i \in J$ by (2.24), which implies the equivalence (2.26). $\square$

Proof of Proposition 2.35. We show that

$$\mathfrak{b} \circ p \colon J \to \mathbb{Z} \times \mathcal{I},\ i \mapsto ((\nu \circ p)(i), (I \circ p)(i))$$

is a representation of the levelset barcode of $f \colon X \to \mathbb{R}$. For each $i \in J$ we obtain a degree $n := (\nu \circ p)(i)$ as well as an interval $(I \circ p)(i)$. Moreover, for each $t \in (I \circ p)(i)$ we have a basis element $\omega_i|_t \in \mathcal{H}^n\big(f^{-1}(t)\big)$ in the n-th cohomology of the fiber of t by Lemma 2.37. Thus, the entire family $\{\omega_i\}_{i \in J}$ induces a simultaneous decomposition of the cohomology vector spaces of all fibers of f in such a way that any two basis elements associated to the same $i \in J$ arise as pullbacks of the same cohomology class ω_i. For a homology theory dual to $\mathcal{H}^\bullet$ it is now easy to see that the multiset "$\{(\mathfrak{b} \circ p)(i)\}_{i \in J}$" is indeed the corresponding levelset barcode of $f \colon X \to \mathbb{R}$, i.e. the multiset of *levelset zigzag persistence intervals* of f in the sense of [CdM09, Section 2]. □

2.5.2 Extended Persistence

We now describe how the extended persistence diagram as defined by [CSEH09] corresponds to our Definition 2.2. We note that the connection between the levelset barcode and extended persistence is well-understood [CdM09, BEMP13]. Here we provide a correspondence between RISC and extended persistence for $\mathbb{F}$-tame functions but without the additional tameness assumption of Proposition 2.35.

Extended Persistent Cohomology as a Restriction of RISC. Let $f \colon X \to \mathbb{R}$ be an $\mathbb{F}$-tame function. We consider the left-hand side of Fig. 2.7 and the restriction of $h(f) \colon \mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$ to the subposet of $\mathbb{M}_\mathrm{f}$, which is shaded in blue in this figure. Now the horizontal blue line segment to the upper left coincides with the horizontal green line of Fig. 1.6. Thus, each point on this horizontal blue line segment is assigned a cohomology vector space in degree 0 of one of the following three types of preimages of $f \colon X \to \mathbb{R}$:

- An open sublevel set of f,
- all of X,
- or a pair with X as the first component and an open superlevel set as the second component.

As a result, the restriction of $h(f) \colon \mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$ to the horizontal blue line segment to the upper left of Fig. 2.7 coincides with the extended persistent cohomology of $f \colon X \to \mathbb{R}$ in degree 0 up to isomorphism of posets. (In the original definition, which is for piecewise linear functions, closed sublevel sets and closed superlevel sets are used. When considering continuous functions with weake tameness assumptions, it is not uncommon to consider preimages of open subsets in place of closed sets.) Similarly, any point on the vertical blue line segment in the center is assigned the cohomology space of some pair of preimages in degree 1 and any point on the horizontal blue line at the lower right is assigned the cohomology of some pair in degree 2 and so forth.

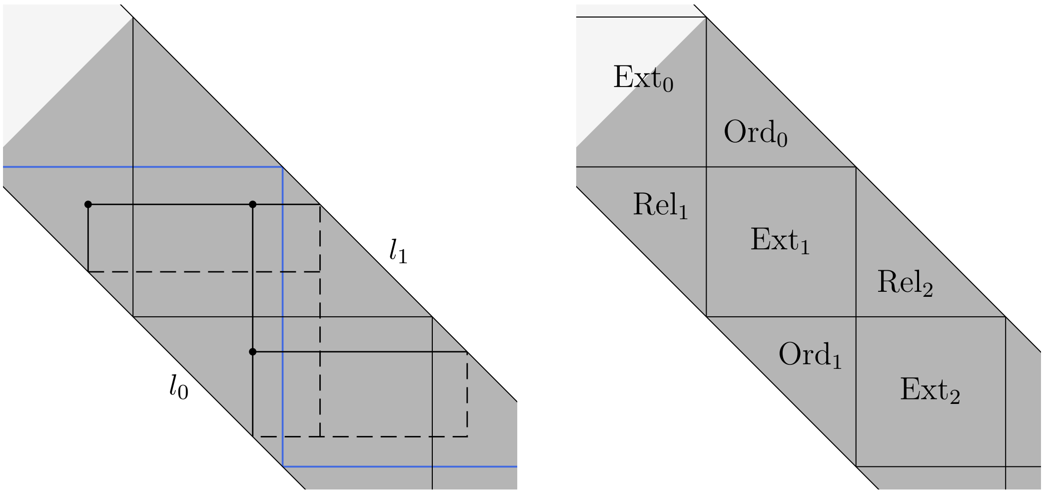


Figure 2.7: In the graphic on the left we see the subposet of $\mathbb{M}_{\mathrm{f}}$ corresponding to extended persistent cohomology shaded in blue as well as three vertices contained in the domains corresponding to 1-dimensional relative, extended, and ordinary persistent cohomology. On the right we see the regions in the strip $\mathbb{M}_{\mathrm{f}}$ corresponding to the ordinary, relative, and extended subdiagrams [CSEH09]. In both figures the support $\downarrow \operatorname{Im} \blacktriangle$ of RISC and of extended persistence diagrams is shaded in grey.

Ordinary, Relative, and Extended Subdiagrams. We now show that the restriction of the multiset $\operatorname{Dgm}(f)\colon \operatorname{int} \mathbb{M}_{\mathrm{f}} \to \mathbb{N}_0$ to each of the different regions on the right-hand side in Fig. 2.7 yield the corresponding subdiagram of the extended persistence diagram defined in [CSEH09] up to reparametrization. Now the RISC $h(f)\colon \mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ decomposes into block presheaves as in Definition 2.3 by Theorem 2.6. Moreover, the support of each such block presheaf intersects exactly one of the blue line segments on the left-hand side in Fig. 2.7. We consider the vertical blue line segment in the center of the graphic on the left in Fig. 2.7 first, which carries the extended persistent cohomology of $f\colon X \to \mathbb{R}$ in degree 1. Any choice of decomposition of $h(f)\colon \mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ yields a decomposition of its restriction to this line segment and thus of extended persistent cohomology in degree 1. Moreover, the support of the block presheaf assigned to any of the black dots in the graphic on the left of Fig. 2.7 intersects this vertical line segment.

First assume that the black dot on the lower right "is part" of the multiset $\operatorname{Dgm}(f)$. Then the restriction of the associated block presheaf to the vertical blue line segment is a direct summand of the extended persistent cohomology of $f\colon X \to \mathbb{R}$ in degree 1 and the intersection of its support with the blue line segment is the life span of the corresponding feature in the sense that the point of intersection of the upper edge marks the birth of a cohomology class of an open sublevel set that dies as soon as it is pulled back to the open sublevel set corresponding to the point of intersection of the lower edge. Moreover, this life span is encoded by the position of this black dot. Now this particular black dot on the lower right in the left graphic of Fig. 2.7 is contained in the triangular region labeled as Ord_1 in the graphic on the right-hand side of Fig. 2.7. Furthermore, any vertex of the extended persistence diagram $\mathrm{Dgm}(f)$ contained in the triangular region labeled Ord_1 describes a feature of $f\colon X \to \mathbb{R}$, which is born at some open sublevel set and also dies at some open sublevel set. Thus, up to reparametrization, the ordinary persistence diagram of $f\colon X \to \mathbb{R}$ in degree 1 is the restriction of $\mathrm{Dgm}(f)\colon \operatorname{int} \mathbb{M}_\mathrm{f} \to \mathbb{N}_0$ to the region labeled Ord_1.

Now suppose that the black dot on the upper left of the left-hand image in Fig. 2.7 "is part" of the multiset $\mathrm{Dgm}(f)\colon \operatorname{int} \mathbb{M}_\mathrm{f} \to \mathbb{N}_0$. Then the intersection of the support of the associated block presheaf with the vertical blue line segment describes the life span of a feature which is born at the cohomology of X relative to some open superlevel set in degree 1 and also dies at the cohomology of X relative to some smaller open superlevel set. Moreover, this is true for any vertex that "is part" of $\mathrm{Dgm}(f)$ and also contained in the triangular region labeled Rel_1 in the graphic on the right of Fig. 2.7. Thus, up to reparametrization, the relative subdiagram of $f\colon X \to \mathbb{R}$ in degree 1 is the restriction of $\mathrm{Dgm}(f)\colon \operatorname{int} \mathbb{M}_\mathrm{f} \to \mathbb{N}_0$ to the region labeled Rel_1.

Finally, the black dot to the upper right in the left-hand image of Fig. 2.7 (if it "is part" of $\mathrm{Dgm}(f)$), or any other vertex of $\mathrm{Dgm}(f)$ in the square region labeled Ext_1 in the graphic on the right of Fig. 2.7, describes a feature, which is born at the cohomology of X relative to some open superlevel set and dies at the cohomology of some open sublevel set of $f\colon X \to \mathbb{R}$. Thus, the extended subdiagram of $f\colon X \to \mathbb{R}$ in degree 1 is the restriction of $\mathrm{Dgm}(f)\colon \operatorname{int} \mathbb{M}_\mathrm{f} \to \mathbb{N}_0$ to the square region labeled Ext_1.

As we have analogous correspondences for each line segment of the subposet of $\mathbb{M}_\mathrm{f}$ shaded in blue in the left graphic of Fig. 2.7, we obtain a partition of the lower right part of the strip $\mathbb{M}_\mathrm{f}$ into regions, corresponding to ordinary, relative, and extended subdiagrams of $\mathrm{Dgm}(f)$ analogous to [CSEH09].

2.6 Interaction Across Cohomological Degrees

In the previous Section 2.5 we showed how closely RISC is related to level set persistence as well as extended persistence. More specifically, all three are functorial invariants retaining the same amount of information on the level of objects or isomorphism classes to be precise. Moreover, level set and extended persistence are functors taking values in the category of graded zigzag or extended persistence modules respectively with the homomorphisms induced by functoriality all homogeneous of degree 0. Moreover, these homogeneous homomorphisms can be obtained from the corresponding induced natural transformations of RISC by restriction. Similarly, the corresponding homomorphisms of *Mayer–Vietoris systems* introduced by [BGO19] can be obtained by restriction. In this Section 2.6 we provide an explicit example, demonstrating how this restriction of natural transformations to level set persistence, extended persistence, or Mayer–Vietoris systems entails a loss of information, which is in stark contrast to the observation that all information on the level of objects is retained. Moreover, for this particular example even sublevel set persistence retains almost all information on the level of objects.

Now let C_4 be the cyclic graph on four vertices $\{x_1, x_2, x_3, x_4\}$. Moreover, let $x_5 \neq x_i$

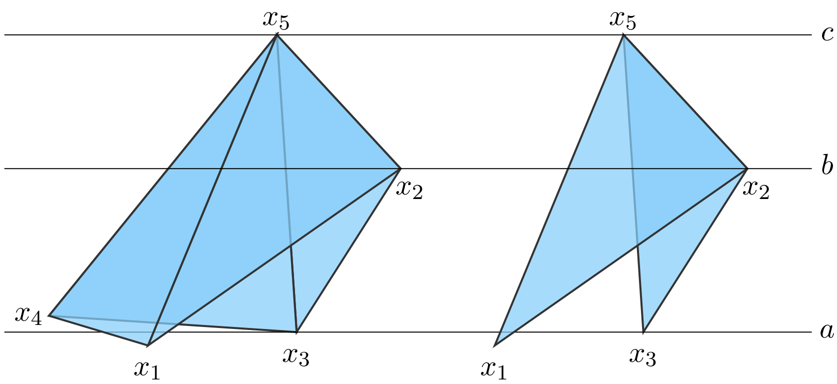


Figure 2.8: The simplicial complex $X := x_5 * C_4$ (left) and its subcomplex $A \subset X$ (right). The function $f\colon X \to \mathbb{R}$ is the height function on X for this particular embedding $X \hookrightarrow \mathbb{R}^3$.

for $i = 1, \ldots, 4$ be an additional vertex and let $X := x_5 * C_4$ be the simplicial cone over C_4 with x_5 as the tip of the cone. Or in other words, X is a simplicial complex with maximal simplices $\{x_1, x_2, x_5\}, \{x_2, x_3, x_5\}, \{x_3, x_4, x_5\}, \{x_4, x_1, x_5\}$ shown on the left-hand side in Fig. 2.8. Furthermore, let $a < b < c \in \mathbb{R}$ and let $f\colon X \to \mathbb{R}$ be the unique simplexwise linear function with

$$f(x_1) = f(x_3) = f(x_4) = a, \quad f(x_2) = b, \quad \text{and} \quad f(x_5) = c,$$

as indicated in Fig. 2.8. Moreover, let $A \subset X$ be the full subcomplex of X spanned by the four vertices $\{x_1, x_2, x_3, x_5\}$ as shown on the right-hand side in Fig. 2.8 and let $j\colon A \hookrightarrow X$ be the inclusion. Then we have the commutative triangle

$$\begin{tikzcd} A \arrow[rr, hook, bend left, "j"] \arrow[dr, "f|_A"'] & & X \arrow[dl, "f"] \\ & \mathbb{R}, & \end{tikzcd}$$

hence there is an induced natural transformation $h(j)_f\colon h(f) \to h(f|_A)$ of presheaves $\mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ as described in Section 1.1.5. We now compute the RISC of $h(f), h(f|_A)\colon \mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ as well as the natural transformation $h(j)_f\colon h(f) \to h(f|_A)$.

First we note that we have the generating homology classes

$$[\![x_1]\!] \in H_0\big(X, X \setminus f^{-1}[a,c]\big) \quad \text{and} \quad [\![x_1]\!] \in H_0\big(A, A \setminus f^{-1}[a,c]\big)$$

each. As the pair $(\mathbb{R}, \mathbb{R} \setminus [a,c])$ is maximal for $\big(H_0 \circ f^{-1}\big)(\mathbb{R}, \mathbb{R} \setminus [a,c])$ to be non-trivial, we have a vertex at the point $w := \mathfrak{b}^{-1}(0, [a,c]) \in D$ in the extended persistence diagram $\mathrm{Dgm}(f) = \mathrm{Dgm}(f; \mathbb{F})$ of $f\colon X \to \mathbb{R}$, where $\mathfrak{b}\colon \operatorname{int} \mathbb{M}_{\mathrm{f}} \to \mathbb{Z} \times \mathcal{I}$ is the bijection from the interior of $\mathbb{M}_{\mathrm{f}}$ to degrees and level set bars defined by (2.22). See Fig. 2.9 for the position

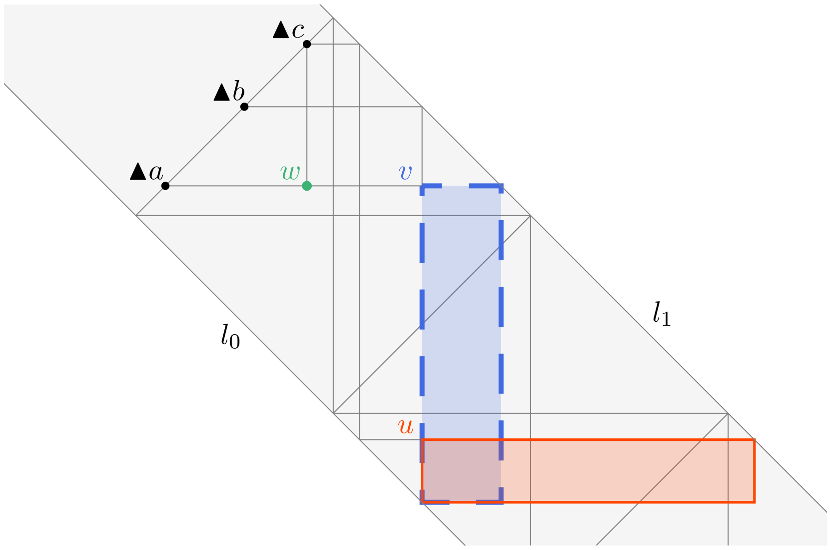


Figure 2.9: The vertices $u, v, w \in \operatorname{int} \mathbb{M}_{\mathrm{f}}$ as well as the supports of the indecomposables $B_u, B_v \colon \mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$.

of $w \in \operatorname{int} \mathbb{M}_{\mathrm{f}}$. Similarly, we have $\mathrm{Dgm}(f|_A)(w) = 1$. Moreover, we have the homology class

$$[\![x_3]\!] - [\![x_1]\!] \in H_0\big(A \cap f^{-1}(-\infty, b), \{x_1, x_3\}\big) = H_0\big(A \cap f^{-1}(-\infty, b), A \cap f^{-1}(-\infty, a)\big)$$

extending $[\![x_1]\!] \in H_0\big(A \cap f^{-1}(-\infty, b), A \cap f^{-1}(-\infty, a)\big)$ to a basis. Now let $\omega_0 \in H^0\big(A \cap f^{-1}(-\infty, b), A \cap f^{-1}(-\infty, a)\big)$ be the unique cohomology class satisfying the equations

$$\langle \omega_0, [\![x_1]\!] \rangle = 0 \quad \text{and} \quad \langle \omega_0, [\![x_3]\!] - [\![x_1]\!] \rangle = 1,$$

where $\langle -, - \rangle$ denotes the Kronecker product. The lifespan of ω_0 as an element of the presheaf $h(f|_A) \colon \mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ with its birth at $v := \mathfrak{b}^{-1}(0, [a, b)) \in \operatorname{int} \mathbb{M}_{\mathrm{f}}$ is shown in Fig. 2.9. As a result we have the natural isomorphism

$$B_w \oplus B_v \xrightarrow{\sim} h(f|_A) \tag{2.27}$$

of presheaves mapping $(0, 1) \in \big(B_w \oplus B_v\big)(v)$ to $\omega_0 \in h(f|_A)(v)$ as well as the equality

$$\mathrm{Dgm}(f|_A) = \mathbf{1}_{\{w,v\}} \colon \operatorname{int} \mathbb{M}_{\mathrm{f}} \to \mathbb{N}_0$$

of the extended persistence diagram $\mathrm{Dgm}(f|_A)$ and the indicator function $\mathbf{1}_{\{w,v\}}$; see also Fig. 2.9. Furthermore, we have the generating homology class

$$[\![x_1, x_2]\!] + [\![x_2, x_3]\!] \in \big(H_1 \circ f^{-1}\big)((-\infty, b), (-\infty, c)).$$

Let $\omega_1 \in \big(H^1 \circ f^{-1}\big)((-\infty, b), (-\infty, c))$ be the unique cohomology class satisfying

$$\langle \omega_1, [\![x_1, x_2]\!] + [\![x_2, x_3]\!] \rangle = 1.$$

The lifespan of ω_1 with its birth at $u := \mathfrak{b}^{-1}(1, [b, c)) \in \operatorname{int} \mathbb{M}_{\mathrm{f}}$ is shown in Fig. 2.9. Thus, we have the natural isomorphism

$$B_w \oplus B_u \xrightarrow{\sim} h(f) \tag{2.28}$$

of presheaves mapping $(0, 1) \in \big(B_w \oplus B_u\big)(u)$ to $\omega_1 \in h(f)(u)$ as well as the equality

$$\operatorname{Dgm}(f) = \mathbf{1}_{\{w,\, u\}} \colon \operatorname{int} \mathbb{M}_{\mathrm{f}} \to \mathbb{N}_0.$$

We now show that the diagram of presheaves

$$\begin{array}{ccc} B_w \oplus B_u & \xrightarrow{\quad\sim\quad} & h(f) \\ \Big\downarrow {\scriptstyle B_w \oplus B_{u \preceq v}} & & \Big\downarrow {\scriptstyle h(j)_f} \\ B_w \oplus B_v & \xrightarrow{\quad\sim\quad} & h(f|_A) \end{array} \tag{2.29}$$

commutes. Now the correctness of the leftmost summand B_w in (2.29) is easy to see, as this natural transformation just corresponds to the identification of 0-cycles within the single connected component X. Moreover, the map

$$H_1\big(A \cap f^{-1}(-\infty, b), A \cap f^{-1}(-\infty, c)\big) \to \big(H_1 \circ f^{-1}\big)((-\infty, b), (-\infty, c))$$

induced by the componentwise inclusion

$$\big(A \cap f^{-1}(-\infty, b), A \cap f^{-1}(-\infty, c)\big) \hookrightarrow f^{-1}((-\infty, b), f^{-1}(-\infty, c))$$

maps $[\![x_1, x_2]\!] + [\![x_2, x_3]\!] \mapsto [\![x_1, x_2]\!] + [\![x_2, x_3]\!]$, hence we have the equation

$$\langle h(j)_f(\omega_1), [\![x_1, x_2]\!] + [\![x_2, x_3]\!]\rangle = 1. \tag{2.30}$$

Moreover, the internal map

$$h(f|_A)(u \preceq v) \colon h(f|_A)(v) \to h(f|_A)(u) = H^1\big(A \cap f^{-1}(-\infty, b), A \cap f^{-1}(-\infty, c)\big)$$

is equal to the 0-th differential of the long exact sequence for the triple $(f \circ j)^{-1}((-\infty, c), (-\infty, b), (-\infty, a))$, hence we have

$$\langle h(f|_A)(u \preceq v)(\omega_0), [\![x_1, x_2]\!] + [\![x_2, x_3]\!]\rangle = \langle \omega_0, [\![x_3]\!] - [\![x_1]\!]\rangle = 1, \tag{2.31}$$

where we use the more naive sign convention for the differential of the singular cochain complex. By combining the equations (2.30) and (2.31) we see that the second direct summand $B_{u \preceq v} \colon B_u \to B_v$ of the vertical natural transformation on the left in (2.29) is correct as well.

Now the pointwise restrictions of $B_{u \preceq v} \colon B_u \to B_v$ to the indexing posets for extended persistence, zigzag persistence (for some sequence of regular values), or Mayer–Vietoris systems are necessarily zero as the intersection of supports of B_u and B_v is disjoint from these regions. For extended and level set persistence, this is also evident from the fact that the corresponding indecomposables live in different degrees and the induced homomorphisms are homogeneous of degree 0. Thus, we may also say that RISC captures interactions between indecomposables that are associated with different degrees in extended and level set persistence. We will put this into perspective in Chapter 9 and Example 8.28 in particular with the derived category $D^+(\mathbb{R})$ of sheaves on the reals and derived homomorphisms induced by non-split extensions.

Part II

Interleavings of RISC

While the Mayer–Vietoris pyramid introduced by [CdM09] has mostly been used to compare and relate different invariants in topological data analysis (TDA) to one another, we introduced RISC as a functorial invariant in its own right in the previous Part I. Now one of the first questions that comes up, whenever an invariant is introduced in the context of TDA, is in which sense the invariant is *stable*. To this end, the widely adopted concept of an *interleaving* has been introduced by [CdSGO16] and generalized to indexing categories other than $(\mathbb{R}, \leq)$ by [BdSS15]. In particular, the authors of [BdSS15, Section 2.5] introduce the notion of a *(super)linear family* on a poset as well as interleavings in terms of such (super)linear families. We provide an appropriate (super)linear family on the indexing poset $\mathbb{M}_\mathrm{f}$ in Example 5.33 below. However, in order to show that any pair of δ-close functions induces a δ-*interleaving* on the level of RISC (for $\delta \geq 0$), we cannot adopt the framework for *inverse-image stability* by [BdSS15, Section 3.2] in a straightforward way; the reason being, that the canonical (super)linear family on $\mathbb{M}_\mathrm{f}$ does not preserve the "cohomological degree". As a result, the interleaving homomorphisms will map some cohomology classes to a cohomology class of one degree higher. In order to resolve these subtleties in the present context, we adopt the framework of *locally persistent categories* introduced by [Sco20]. To this end, we will introduce a new calculus of *weighted diagrams* in locally persistent categories in Chapter 5 and we will also relate locally persistent categories to *weighted categories* in Section 6.2. Furthermore, we provide Example 6.25, which shows that the induced interleavings of RISC capture more information than the corresponding interleavings of extended persistence or Mayer–Vietoris systems in the sense of [BS14, Seciton 6] and [BGO19, Definition 2.24] respectively.

The content of this Part II, which grew out of the preprint [BBF21], is purely categorical, as it only depends on the existence and naturality of the boundary operator from the Mayer–Vietoris sequence. For this reason and for notational convenience, we assume that

$$\mathcal{H}_\bullet \colon \mathcal{T} \to \mathcal{W}^\mathbb{Z}$$

is a Mayer–Vietoris functor with boundary operator ∂ in the sense of Definition 1.3 and that

$$h^\circ \colon \mathrm{Top}/\mathbb{R} \to \mathcal{W}^\mathbb{M}$$

is the induced functor from spaces over the reals to $\mathcal{W}$-valued functors on $\mathbb{M}$ as described in Section 1.1.5.

3 The Weighted Category of Functions

Up until this point, we were only concerned with the functoriality of RISC. However, functions on topological spaces can be related in more ways than by homomorphisms of $\mathbb{R}$-spaces as in the diagram (1.3). For example, if $f, g\colon X \to \mathbb{R}$ are two continuous functions on the same non-empty compact space X, then we may measure their distance $\|g - f\|_\infty$. Now the norm $\|g - f\|_\infty$ is just a shorthand for $\max\{-\inf(g - f), \sup(g - f)\}$ and as a pair these two values offer more information on the similarities of f and g than the norm $\|g - f\|_\infty$ on its own. For instance, if $\inf(g - f) = \sup(g - f)$, then g is just a shift of f by that same constant value. For this reason, we relate two such functions by the pair of values

$$\mathbf{d}(f, g) := (\inf(g - f), \sup(g - f)).$$

We may think of $\mathbf{d}(f, g)$ as an element of the monoidal poset $\mathbb{R}^\circ \times \mathbb{R}$ satisfying the triangle inequality

$$\mathbf{d}(f_1, f_3) \preceq \mathbf{d}(f_2, f_3) + \mathbf{d}(f_1, f_2) \tag{3.1}$$

for any continuous $f_i\colon X \to \mathbb{R}$, $i = 1, 2, 3$. In order to extend $\mathbf{d}$ to functions on non-compact non-empty spaces we freely adjoin a supremum as follows.

Definition 3.1 (Closure by ∞)**.** For a monoidal poset V we define its *closure by* ∞ to be the monoidal poset

$$\tilde{V} := V \sqcup \{\infty\}$$

with ∞ freely adjoined as a supremum. The monoidal operation is extended to $\tilde{V}$ in the obvious way. We say that an element $a \in \tilde{V}$ is *finite* if $a \in V$.

For $V := \mathbb{R}^\circ \times \mathbb{R}$ we may then extend $\mathbf{d}$ to functions on non-compact non-empty spaces by an assignment taking values in $\tilde{V}$. Now for any monoidal poset V we may think of the opposite monoidal poset $\tilde{V}^\circ$ as a thin monoidal category. From this perspective the assignment $\mathbf{d}$ endows the set of continuous functions $\operatorname{Hom}(X, \mathbb{R})$ on a fixed non-empty topological space X with the structure of a small $\tilde{V}^\circ$-enriched category (for $V := \mathbb{R}^\circ \times \mathbb{R}$). This idea, that a set together with an assignment like $\mathbf{d}$ satisfying the triangle inequality can be understood as an enriched category, is due to [Law73], who introduced *Lawvere metric spaces*.

Now in addition to the triangle inequality, metric spaces satisfy yet another property, namely symmetry, and we have a similar property here. Let

$$\dagger\colon \mathbb{R}^\circ \times \mathbb{R} \to \mathbb{R}^\circ \times \mathbb{R},\ (x, y) \mapsto (-y, -x) \tag{3.2}$$

be the reflection at the antidiagonal in $\mathbb{R}^2$, which is an involution of the monoidal poset $\mathbb{R}^\circ \times \mathbb{R}$, and let $\tilde{\dagger}\colon \tilde{V} \to \tilde{V}$ be its extension to $\tilde{V}$ mapping $\infty \mapsto \infty$ for $V = \mathbb{R}^\circ \times \mathbb{R}$. Then we have

$$\mathbf{d}(g, f) = \tilde{\dagger}\mathbf{d}(f, g)$$

for any two functions $f, g\colon X \to \mathbb{R}$. In the language of $\tilde{V}^\circ$-enriched categories, we may say that the base change $\tilde{\dagger}_\bullet \mathrm{Hom}(X, \mathbb{R})$ of $\mathrm{Hom}(X, \mathbb{R})$ along $\tilde{\dagger}\colon \tilde{V}^\circ \to \tilde{V}^\circ$ is equal to its $\tilde{V}^\circ$-enriched opposite category $\mathrm{Hom}(X, \mathbb{R})^\circ$.

Now for a continuous map $\varphi\colon X \to Y$ of non-empty spaces we have the inequality

$$\mathbf{d}(f \circ \varphi, g \circ \varphi) \preceq \mathbf{d}(f, g)$$

for any two continuous functions $f, g\colon Y \to \mathbb{R}$. Thus, we may view

$$\mathrm{Hom}(\varphi, \mathbb{R})\colon \mathrm{Hom}(Y, \mathbb{R}) \to \mathrm{Hom}(X, \mathbb{R})$$

as a $\tilde{V}^\circ$-enriched functor. With some abuse of notation we now write Top for the category of non-empty topological spaces. Also, for this entire Part II, whenever we speak of a continuous function on a topological space, non-emptiness is implicitly assumed. In Remark 1.7 we considered $\mathrm{Hom}(-, \mathbb{R})$ as a contravariant functor from topological spaces to Set $\hookrightarrow$ Cat. Using the $\tilde{V}^\circ$-enrichments of function sets we may now consider it a contravariant functor

$$\mathrm{Hom}(-, \mathbb{R})\colon \mathrm{Top}^\circ \to \tilde{V}^\circ\text{-Cat},$$

where $\tilde{V}^\circ$-Cat denotes the strict 2-category of $\tilde{V}^\circ$-enriched categories. We also saw in Remark 1.7 how the category of $\mathbb{R}$-spaces can be obtained as the initial lax cocone under $\mathrm{Hom}(-, \mathbb{R})$, so we may now repeat this construction here with $\tilde{V}^\circ$-Cat in place of Cat. However, the underlying ordinary category of any $\tilde{V}^\circ$-enriched category is a thin category. So this construction could not possibly yield an enrichment of Top/$\mathbb{R}$. We need a context for enrichment that can enrich Top/$\mathbb{R}$ and also contains $\tilde{V}^\circ$ as a subcategory so that we can retain the information provided by $\mathbf{d}$. (Note that in Remark 1.7 we did not actually construct the initial lax cocone under $\mathrm{Hom}(-, \mathbb{R})$ as a functor to the category of sets either, but as a functor to the strict 2-category of categories.) Later we will use *persistent sets* as a context for enrichment leading to locally persistent categories introduced by [Sco20]. However, it will be beneficial to use *V-weighted sets* as a context for enrichment as an intermediate step, which produces *V-weighted categories*. Note that we may also view $\tilde{V}^\circ$ as V° with an initial object ∞ freely adjoined. When passing from $\tilde{V}^\circ$ to V-weighted sets, we take this construction one step further by freely adjoining arbitrary coproducts to V°.

Definition 3.2 (Weighted Sets and Weighted Categories)**.** Let V be an additively written monoidal poset. A *V-weighted set* is a map of sets $w\colon M \to V$. A *homomorphism of V-weighted sets* $w_i\colon M_i \to V$, $i = 1, 2$ is a map $\varphi\colon M_1 \to M_2$ such that $w_2(\varphi(m)) \preceq w_1(m)$ for all $m \in M_1$. The *convolution* of two such V-weighted sets is

$$w_1 \star w_2\colon M_1 \times M_2 \to V,\ (m_1, m_2) \mapsto w_1(m_1) + w_2(m_2)$$

and can be extended to homomorphisms of V-weighted sets in the obvious way. We denote the monoidal category of V-weighted sets determined by this data by $V\mathrm{Set}$. Moreover, we also define the forgetful functor

$$\mathrm{dom}\colon V\mathrm{Set} \to \mathrm{Set},\ (w\colon M \to V) \mapsto M.$$

A *V-weighted category* is a category enriched in $V\mathrm{Set}$. Furthermore, we have the base change along the forgetful functor

$$\mathrm{dom}_\bullet\colon V\mathrm{Set}\text{-}\mathrm{Cat} \to \mathrm{Cat},\ \mathcal{C} \mapsto \mathrm{dom}_\bullet\,\mathcal{C},$$

which applies $\mathrm{dom}\colon V\mathrm{Set} \to \mathrm{Set}$ to the corresponding V-weighted morphism sets.

We note that weighted categories have appeared in the literature on persistence theory in [BdS17, Section 2.2]. If we now unravel the definition of a $V\mathrm{Set}$-enriched category, then we realize that a V-weighted category $\mathcal{C}$ is just an ordinary category, namely $\mathrm{dom}_\bullet\,\mathcal{C}$, where each morphism φ in $\mathcal{C}$ is assigned a weight $w(\varphi) \in V$ in such a way that the triangle inequality

$$w(\psi \circ \varphi) \preceq w(\psi) + w(\varphi)$$

is satisfied for any two composable morphisms φ and ψ in $\mathcal{C}$ and any identity $\mathrm{id}_X\colon X \to X$ in $\mathcal{C}$ has weight $w(\mathrm{id}_X) \preceq 0$. However, in the sense of enriched categories $\mathrm{dom}_\bullet\,\mathcal{C}$ is not the underlying ordinary category $\mathcal{C}_0$, whose sets of morphisms consist of the morphisms φ in $\mathcal{C}$ with weight $w(\varphi) \preceq 0$. We also have the fully faithful strong monoidal functor

$$\tilde{V}^\circ \to V\mathrm{Set},\ \begin{cases} \infty \mapsto (\emptyset \to V) \\ v \mapsto (\{\bullet\} \to V,\ \bullet \mapsto v) \end{cases}$$

and by taking the base change along this functor we may view any small $\tilde{V}^\circ$-enriched category as a small V-weighted category. We now construct a lax cocone under

$$\mathrm{Hom}(-,\mathbb{R})\colon \mathrm{Top}^\circ \to \tilde{V}^\circ\text{-}\mathrm{Cat} \hookrightarrow V\mathrm{Set}\text{-}\mathrm{Cat}$$

(for $V = \mathbb{R}^\circ \times \mathbb{R}$) in the sense of Definition A.2. We start with its vertex, denoted $(\mathrm{Top}/\mathbb{R})_w$, whose class of objects is the class of all real-valued continuous functions. For two functions $f\colon X \to \mathbb{R}$ and $g\colon Y \to \mathbb{R}$ we define the set of V-weighted homomorphisms by

$$\mathrm{dom}(f,g) \coloneqq \{\varphi\colon X \to Y \mid \mathbf{d}(f, g \circ \varphi) \in V\}$$

and their weights by

$$w_{f,g}\colon\ \mathrm{dom}(f,g) \to V,\ \varphi \mapsto w_{f,g}(\varphi) \coloneqq \mathbf{d}(f, g \circ \varphi).$$

The composition in $(\mathrm{Top}/\mathbb{R})_w$ is defined by the composition of continuous maps and indeed, for homomorphisms $\varphi \in \mathrm{dom}(f_1,f_2)$ and $\psi \in \mathrm{dom}(f_2,f_3)$ we have the triangle inequality

$$\begin{aligned} w_{1,3}(\psi \circ \varphi) &= \mathbf{d}(f_1, f_3 \circ \psi \circ \varphi) \\ &\preceq \mathbf{d}(f_2 \circ \varphi, f_3 \circ \psi \circ \varphi) + \mathbf{d}(f_1, f_2 \circ \varphi) \\ &\preceq \mathbf{d}(f_2, f_3 \circ \psi) + \mathbf{d}(f_1, f_2 \circ \varphi) \\ &= w_{2,3}(\psi) + w_{1,2}(\varphi) \end{aligned}$$

by (3.1) writing $w_{1,2} = w_{f_1,f_2}$ and similarly for $w_{1,3}$ and $w_{2,3}$. This concludes our construction of $(\mathrm{Top}/\mathbb{R})_w$ as a V-weighted category. As mentioned already, simply omitting the weights we have an ordinary category with morphisms $\mathrm{dom}(f, g)$ between two continuous functions f and g, which we denote as $\mathrm{dom}_\bullet(\mathrm{Top}/\mathbb{R})_w$ (as it is the base change along the forgetful functor dom from $\mathbb{R}^\circ \times \mathbb{R}$-weighted sets to the category of sets). This is very convenient, as it allows us to use reasoning from ordinary category theory to relate two functions with different domains. Moreover, this construction of $(\mathrm{Top}/\mathbb{R})_w$ is an instance of the following notion of a *V-weighted category of elements*.

Definition 3.3 (Weighted Category of Elements)**.** For an additively written monoidal poset V and a contravariant functor $F\colon \mathcal{B}^\circ \to \tilde{V}^\circ\text{-Cat}$ taking values in small $\tilde{V}^\circ$-enriched categories, we say that F has *pairwise disjoint values* if $F(A) \cap F(B) = \emptyset$ for any objects $A \neq B$ of $\mathcal{B}$. Now suppose that $F\colon \mathcal{B}^\circ \to \tilde{V}^\circ\text{-Cat}$ is such a contravariant functor with pairwise disjoint values. Moreover, for any object A of $\mathcal{B}$ and any two elements $a, a' \in F(A)$ let $\mathbf{d}(a, a') \in \tilde{V}^\circ$ be the corresponding *element of morphisms* in $\tilde{V}^\circ$ with respect to the $\tilde{V}^\circ$-enrichment of the values of F. (Note that this notation is not ambiguous as F has pairwise disjoint values.) Then we define the *V-weighted category of elements of F* to be the V-weighted category $\mathrm{el}(F)$ whose objects are the elements $a \in F(A)$ for some object A of $\mathcal{B}$ and whose V-weighted set of morphisms from $a \in F(A)$ to $b \in F(B)$ is defined by

$$\mathrm{dom}(a, b) \coloneqq \{\varphi\colon A \to B \mid \mathbf{d}(a, F(\varphi)(b)) \in V\}$$

and their weights by

$$w_{a,b}\colon \mathrm{dom}(a, b) \to V,\ \varphi \mapsto w_{a,b}(\varphi) \coloneqq \mathbf{d}(a, F(\varphi)(b)).$$

The composition in $\mathrm{el}(F)$ is defined by the composition in $\mathcal{B}$.

For two elements $a \in F(A)$, $b \in F(B)$, and a morphism $\varphi \in \mathrm{dom}(a, b)$, we say that $\varphi\colon a \to b$ is *vertical* if $A = B$ and $\varphi = \mathrm{id}_A$ and we say that φ is *horizontal* if $a = F(\varphi)(b)$.

We define a lax cocone under $F\colon \mathcal{B}^\circ \to \tilde{V}^\circ\text{-Cat} \hookrightarrow V\mathrm{Set}\text{-Cat}$ with vertex $\mathrm{el}(F)$ by

$$\begin{aligned} G'(A)\colon F(A) \to \mathrm{el}(F),\ &\begin{cases} a \mapsto a \\ \bullet \mapsto \mathrm{id}_A \end{cases} \\ G'(\varphi\colon A \to B)\colon F(\varphi) \Rightarrow G'(B),\ &b \mapsto G'(\varphi)_b \coloneqq \varphi. \end{aligned} \tag{3.3}$$

Remark 3.4. Any morphism $\varphi\colon a \to b$ of a V-weighted category of elements $\mathrm{el}(F)$ as in Definition 3.3 factors uniquely into a vertical morphism of the same weight and a horizontal morphism:

$$\begin{tikzcd} a \arrow[d, "\mathrm{id}_A"'] \arrow[dr, "\varphi"] & \\ F(\varphi)(b) \arrow[r, "\varphi"'] & b, \end{tikzcd}$$

where $a \in F(A)$. Moreover, any horizontal morphism φ is uniquely determined by its underlying morphism $\varphi\colon A \to B$ in $\mathcal{B}$ and its codomain $b \in F(B)$; the domain is always $F(\varphi)(b)$.

Lemma 3.5. *The horizontal morphisms of a V-weighted category of elements* $\mathrm{el}(F)$ *as in Definition 3.3 form a wide subcategory of the underlying ordinary category* $\mathrm{el}(F)_0$ *of* $\mathrm{el}(F)$ *as a* $V\mathrm{Set}$*-enriched category.*

Proof. As all horizontal morphisms have weight 0, they are contained in the underlying ordinary category of $\mathrm{el}(F)$. Now suppose $\varphi\colon a \to b$ and $\psi\colon b \to c$ are horizontal morphisms of $\mathrm{el}(F)$. We have to show that $\psi \circ \varphi\colon a \to c$ is a horizontal morphism as well. As φ and ψ are horizontal, we have $a = F(\varphi)(b)$ and $b = F(\psi)(c)$. Moreover, by the functoriality of $F\colon \mathcal{B}^\circ \to \tilde{V}^\circ\text{-Cat}$ we have

$$a = F(\varphi)(b) = F(\varphi)(F(\psi)(c)) = F(\psi \circ \varphi)(c),$$

hence $\psi \circ \varphi\colon F(\psi \circ \varphi)(c) \to c$ is a horizontal morphism as well. □

Definition 3.6 (Horizontal Subcategory)**.** We define the *horizontal subcategory* $\mathrm{el}(F)_h$ of a V-weighted category of elements $\mathrm{el}(F)$ as in Definition 3.3 to be the subcategory of horizontal morphisms.

Thus, we have inclusions of subcategories $\mathrm{el}(F)_h \hookrightarrow \mathrm{el}(F)_0 \hookrightarrow \mathrm{dom}_\bullet \mathrm{el}(F)$ by Lemma 3.5.

Remark 3.7. The subcategory of horizontal morphisms in $(\mathrm{Top}/\mathbb{R})_w$ is $\mathrm{Top}/\mathbb{R}$, which also happens to be the underlying ordinary category of $(\mathrm{Top}/\mathbb{R})_w$. To see this, we note that for two functions $f, g\colon X \to \mathbb{R}$ on a non-empty space X we always have $\inf(g - f) \leq \sup(g - f)$, hence $\mathbf{d}(f, g) \not\leq 0$ in $\mathbb{R}^\circ \times \mathbb{R}$.

As it turns out, the lax cocone defined by (3.3) is initial as well. However, as we will not need this result we omit it. Instead, we will use Theorem 6.12 to characterize *V-persistent functors* whose domain is a base change of the V-weighted category of elements $\mathrm{el}(F)$. However, the following Lemma 3.8 is easy to see.

Lemma 3.8. *Forgetting the structure provided by the elements of morphisms* $\mathbf{d}$ *on the values of* $F\colon \mathcal{B}^\circ \to \tilde{V}^\circ\text{-Cat}$*, we consider the contravariant functor*

$$\bar{F}\colon \mathcal{B}^\circ \to \mathrm{Class} \hookrightarrow \mathrm{Cat},\ A \mapsto \bar{F}(A),$$

where $\bar{F}(A)$ *is the underlying class of* $F(A)$*. Then we may restrict the lax cocone* G' *as defined in* (3.3) *to the initial lax cocone* $\bar{G}$ *defined by*

$$\begin{aligned} \bar{G}(A)&\colon \bar{F}(A) \to \mathrm{el}(F)_h,\ a \mapsto a \\ \bar{G}(\varphi\colon A \to B)&\colon \bar{F}(\varphi) \Rightarrow \bar{G}(B),\ \ b \mapsto \bar{G}(\varphi)_b \coloneqq \varphi = G'(\varphi)_b. \end{aligned}$$

Remark 3.9. If we instantiate Lemma 3.8 for the contravariant functor

$$F = \mathrm{Hom}(-, \mathbb{R})\colon \mathrm{Top}^\circ \to \tilde{V}^\circ,$$

then we recover the initial lax cocone (1.5) by Remark 3.7.

The following general considerations on V-weighted categories will be useful later.

Lemma 3.10. *The weight of an identity in a V-weighted category is an idempotent of V.*

Proof. We write V additively. Let $\mathcal{C}$ be a V-weighted category, let X be an object of $\mathcal{C}$, and let $v \coloneqq w(\mathrm{id}_X) \in V$. Then we have $v = w(\mathrm{id}_X) \preceq 0$ and hence

$$v + v \preceq v + 0 = v. \tag{3.4}$$

Moreover, by the triangle inequality we have

$$v = w(\mathrm{id}_X) = w(\mathrm{id}_X \circ \mathrm{id}_X) \preceq w(\mathrm{id}_X) + w(\mathrm{id}_X) = v + v. \tag{3.5}$$

By combining the estimates (3.4) and (3.5) and the anti-symmetry of the partial order $\preceq$ on V we obtain $v + v = v$. $\square$

Corollary 3.11. *For an additively written monoidal poset V without any non-trivial idempotents and a V-weighted category $\mathcal{C}$, the identities of $\mathcal{C}$ have weight* 0.

4 The Locally Persistent Category of Functors on $\mathbb{M}_{\mathrm{f}}$

In the previous Chapter 3 we enriched the category of $\mathbb{R}$-spaces to a V-weighted category with $V = \mathbb{R}^\circ \times \mathbb{R}$ as a monoidal poset. We next provide an enrichment of functors $\mathbb{M}_{\mathrm{f}} \to \mathcal{W}$ as well, and then we also enrich the covariant functor $h^\circ \colon \mathrm{Top}/\mathbb{R} \to \mathcal{W}^{\mathbb{M}_{\mathrm{f}}}$ used in the construction of RISC in Section 1.1.4. However, as a context for enrichment the formalism of V-weighted sets is insufficient for this purpose. Instead, we can use the related concept of a *locally persistent category* introduced by [Sco20], which we will now explain.

Definition 4.1 (Locally Persistent Category, [Sco20, Definition 5.1.1])**.** Let V be an additively written monoidal poset. The data for a *locally V-persistent category* $\mathcal{C}$ consists of

- a class of *objects* $\mathrm{Ob}(\mathcal{C})$,
- for any two objects X and Y of $\mathcal{C}$ a functor $\mathrm{Hom}(X, Y) \colon V \to \mathrm{Set}$ named their *V-persistent set of morphisms*,
- for any object X of $\mathcal{C}$ a distinguished element $\mathrm{id}_X \in \mathrm{Hom}(X, X)$ named the *identity at X*,
- and for any three objects X, Y, and Z of $\mathcal{C}$ a natural transformation
$$- \,{}_{\_}\!\diamond_{\_}\, - \colon \mathrm{Hom}(Y, Z) \boxtimes \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z) \circ \nabla \tag{4.1}$$
named *composition*, where $\nabla \colon V \times V \to V$, $(u, v) \mapsto u + v$ is the addition in V and $\mathrm{Hom}(Y, Z) \boxtimes \mathrm{Hom}(X, Y)$ is the functor
$$V \times V \to \mathrm{Set},\ (v, u) \mapsto \mathrm{Hom}(Y, Z)(v) \times \mathrm{Hom}(X, Y)(u).$$

If we instantiate the natural transformation (4.1) at $(v, u) \in V \times V$, then we obtain a map of sets

$$- \,{}_v\!\diamond_u\, - \colon \mathrm{Hom}(Y, Z)(v) \times \mathrm{Hom}(X, Y)(u) \to \mathrm{Hom}(X, Z)(v + u),\ (\psi, \varphi) \mapsto \psi \,{}_v\!\diamond_u\, \varphi.$$

The coherence conditions for such data to determine a locally V-persistent category can be provided as equations:

$$(\zeta \,{}_w\!\diamond_v\, \psi) \,{}_{w+v}\!\diamond_u\, \varphi = \zeta \,{}_w\!\diamond_{v+u}\, (\psi \,{}_v\!\diamond_u\, \varphi) \quad \text{and} \quad \mathrm{id}_Y \,{}_0\!\diamond_u\, \varphi = \varphi = \varphi \,{}_u\!\diamond_0\, \mathrm{id}_X \tag{4.2}$$

or

$$(\zeta \diamond \psi) \diamond \varphi = \zeta \diamond (\psi \diamond \varphi) \quad \text{and} \quad \mathrm{id} \diamond \varphi = \varphi = \varphi \diamond \mathrm{id}$$

when suppressing any subscript notation for better readability.

Definition 4.1 can be easily extended to define a strict 2-category of locally V-persistent categories. We omit the details, as we will provide a more conceptual characterization of locally persistent categories in Section 5.3 below, where the corresponding notions of 1- and 2-morphisms are immediate. For a locally V-persistent category $\mathcal{C}$, we may also define ordinary sets of morphisms $\mathrm{Hom}_0(X, Y) \coloneqq \mathrm{Hom}(X, Y)(0)$ and an ordinary composition

$$- \circ -\colon \mathrm{Hom}_0(Y, Z) \times \mathrm{Hom}_0(X, Y) \to \mathrm{Hom}_0(X, Z),\ (\psi, \varphi) \mapsto \psi \circ \varphi \coloneqq \psi \ {}_0{\diamond}_0\ \varphi. \tag{4.3}$$

Together with the identity elements $\mathrm{id}_X \in \mathrm{Hom}_0(X, X)$ this data determines an ordinary category.

Definition 4.2 (Underlying Ordinary Category)**.** For a locally V-persistent category $\mathcal{C}$ we name the ordinary category determined by the evaluation of each V-persistent set of morphisms of $\mathcal{C}$ at $0 \in V$, the composition (4.3), and its identities, the *underlying ordinary category* of $\mathcal{C}$. We denote the underlying ordinary category of $\mathcal{C}$ by $\mathcal{C}_0$. For a V-peristent functor $F\colon \mathcal{C} \to \mathcal{D}$ we name the induced functor $F_0\colon \mathcal{C}_0 \to \mathcal{D}_0$ between the underlying ordinary categories the *underlying ordinary functor*.

For an ordinary category $\mathcal{C}$, a *V-persistent enrichment* of $\mathcal{C}$ is a locally V-persistent category $\tilde{\mathcal{C}}$ such that $\tilde{\mathcal{C}}_0 = \mathcal{C}$.

Similarly, a *V-persistent enrichment* of a functor between ordinary categories requires V-persistent enrichments for its domain and codomain and an extension of this functor to a V-persistent functor.

4.1 The Enrichment of Functors on $\mathbb{M}_\mathrm{f}$

As already alluded to in the beginning of this Chapter 4, we intend to provide an $\mathbb{R}^\circ \times \mathbb{R}$-persistent enrichment of the covariant functor $h^\circ\colon \mathrm{Top}/\mathbb{R} \to \mathcal{W}^{\mathbb{M}_\mathrm{f}}$ that we used as an abstraction for convenience in the construction of RISC in Section 1.1.4 in the sense of Definition 4.2. In this Section 4.1 we take a first step in this direction by providing an $\mathbb{R}^\circ \times \mathbb{R}$-persistent enrichment of the category of functors $\mathbb{M}_\mathrm{f} \to \mathcal{W}$. We should note that the content of this Section 4.1 already appeared in [BdSS15] in slightly different terminology.

Enrichment of Functors on Posets in General. Let V be an additively written monoidal poset, let M be another poset, and let

$$V \times M \to M,\ (a, m) \mapsto a.m \tag{4.4}$$

be an action of V on M as a monoidal poset. Or, in more categorical terms, we view V as a monoid object in the category of posets Pos and we assume that M is a module object over V. Moreover, let $\mathcal{W}$ be some category. We now show that the action (4.4) induces a V-persistent enrichment on the category of functors $M \to \mathcal{W}$ in the sense of Definition 4.2. To this end, we assume we have functors $F, G\colon M \to \mathcal{W}$. We define

$$\mathrm{Hom}(F,G)\colon V \to \mathrm{Set},\ a \mapsto \mathrm{Nat}(F, G(a.\_)).$$

As G and $\mathrm{Nat}(F, \_)$ are both functors, $\mathrm{Hom}(F,G)$ is indeed a persistent set in the sense of Definition 5.1. Next we have to define a composition. To this end, let $H\colon M \to \mathcal{W}$ be another functor, let $a, b \in V$, and let $\varphi \in \mathrm{Hom}(F,G)(a) = \mathrm{Nat}(F, G(a.\_))$ and $\psi \in \mathrm{Hom}(G,H)(b) = \mathrm{Nat}(G, H(b.\_))$ be natural transformations. We define $\psi\ {}_b\!\diamond_a\, \varphi \in \mathrm{Hom}(F,H)(b+a) = \mathrm{Nat}(F, H(b.a.\_))$ to be the composition

$$\begin{tikzcd} F \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi\,{}_b\diamond_a\varphi"'] & G(a.\_) \arrow[r, "\psi_{a.\_}"] & H(b.a.\_) \end{tikzcd}$$

of natural transformations. It is easy to check that the composition $\diamond$ just defined and the identity natural transformations satisfy the coherence conditions (4.2) and hence we obtain a V-persistent enrichment of the category of functors $M \to \mathcal{W}$. The interleavings in the sense of [Sco20, Definition 3.1.8] and Definition 5.31 obtained from this enrichment are closely related to the interleavings introduced by [BdSS15, Definition 3.4]; see also Example 5.33 below.

The Action of $\mathbb{R}^\circ \times \mathbb{R}$ on $\mathbb{M}_{\mathrm{f}}$. As we saw in the previous paragraph, it suffices to provide an action $(\mathbb{R}^\circ \times \mathbb{R}) \times \mathbb{M}_{\mathrm{f}} \to \mathbb{M}_{\mathrm{f}}$ of posets to obtain an $\mathbb{R}^\circ \times \mathbb{R}$-persistent enrichment of the category of functors $\mathbb{M}_{\mathrm{f}} \to \mathcal{W}$. To this end, let $\mathrm{Aut}(\mathbb{M}_{\mathrm{f}})$ be the automorphism group of $\mathbb{M}_{\mathrm{f}}$ in the category of lattices. Then there is a unique group homomorphism

$$\alpha\colon \mathbb{R}^2 \to \mathrm{Aut}(\mathbb{M}_{\mathrm{f}}) \quad \text{with} \quad \mathrm{ev}_0 \circ \alpha = \arctan \times \arctan,$$

where $\mathrm{ev}_0\colon \mathrm{Aut}(\mathbb{M}_{\mathrm{f}}) \to \mathbb{M}_{\mathrm{f}}$, $\varphi \mapsto \varphi(0)$ is the evaluation at the origin, see also Fig. 4.1. We provide an explicit description of α in Remark 4.3 below. As α is monotone, the uncurrying $\alpha^\flat$ of α determines an action of $\mathbb{R}^\circ \times \mathbb{R}$ on $\mathbb{M}_{\mathrm{f}}$:

$$\alpha^\flat\colon (\mathbb{R}^\circ \times \mathbb{R}) \times \mathbb{M}_{\mathrm{f}} \to \mathbb{M}_{\mathrm{f}},\ (a, u) \mapsto a.u \coloneqq \alpha^\flat(a, u).$$

In conjunction with the construction from the previous paragraph we obtain an $\mathbb{R}^\circ \times \mathbb{R}$-persistent enrichment of the category of functors $\mathbb{M}_{\mathrm{f}} \to \mathcal{W}$. With some abuse of notation we will write $\mathcal{W}^{\mathbb{M}_{\mathrm{f}}}$ for both, the locally $\mathbb{R}^\circ \times \mathbb{R}$-persistent category and for the underlying ordinary category.

Remark 4.3 (An Explicit Description of α). Let $a \coloneqq (a_1, a_2) \in \mathbb{R}^\circ \times \mathbb{R}$. We construct $\alpha(a)\colon \mathbb{M}_{\mathrm{f}} \to \mathbb{M}_{\mathrm{f}}$ as follows. Consider the stratification of $\mathbb{R}$ by the 0-strata $\frac{\pi}{2} + \pi\mathbb{Z}$. We refer to a 1-stratum as *even* if it contains some point of $2\pi\mathbb{Z}$, and *odd* otherwise. We define $g_a\colon \mathbb{R} \to \mathbb{R}$ as the unique map that preserves the strata and satisfies

$$\tan(g_a(x)) = \begin{cases} \tan(x) + a_2 & \text{if } x \text{ lies in an even 1-stratum,} \\ \tan(x) - a_1 & \text{if } x \text{ lies in an odd 1-stratum.} \end{cases}$$

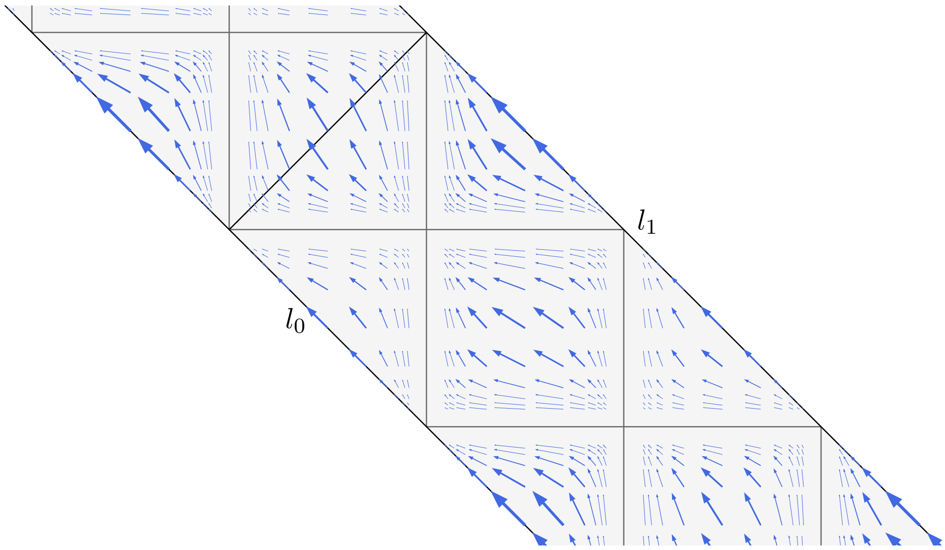


Figure 4.1: The lattice automorphism $\alpha(a)\colon \mathbb{M}_\mathrm{f} \to \mathbb{M}_\mathrm{f}$ for $a = (-0.315, 0.525)$.

With $\sigma\colon \mathbb{R} \to \mathbb{R},\ t \mapsto \pi - t$ being the reflection at $\pi/2$ we have

$$\alpha(a) = (\sigma \circ g_a \circ \sigma) \times g_a,$$

completing our explicit description of α.

The following property of the action $\alpha^\flat\colon (\mathbb{R}^\circ \times \mathbb{R}) \times \mathbb{M}_\mathrm{f} \to \mathbb{M}_\mathrm{f}$ will be useful later.

Lemma 4.4. *For any $a \in \mathbb{R}^\circ \times \mathbb{R}$ and any $u \in \mathbb{M}_\mathrm{f}$ we have $a.\Sigma(u) = \Sigma(a.u)$.*

Proof. This follows directly from the fact that $\Sigma\colon \mathbb{M}_\mathrm{f} \to \mathbb{M}_\mathrm{f}$ is central in the sense that Σ commutes with any other lattice automorphism and that $\alpha(a)\colon \mathbb{M}_\mathrm{f} \to \mathbb{M}_\mathrm{f}$ is a lattice automorphism by definition of $\alpha\colon \mathbb{R}^2 \to \mathrm{Aut}(\mathbb{M}_\mathrm{f})$. $\square$

5 Weighted Diagrams in Locally Persistent Categories

In the previous Chapter 4 we defined locally persistent categories introduced by [Sco20]. Now in order to reason about locally persistent categories it can be convenient to have a concrete description of *small* locally persistent categories as well as *persistent functors* with small domains. Moreover, the theory of enriched categories will be useful to obtain such concrete descriptions. A characterization of locally persistent categories in terms of enriched categories and Day convolution has been given by [Sco20, Definition 5.1.1]. However, working with Day convolution directly can be tedious in comparison to cartesian (i.e. categorical) products defined as limits. For this reason, we take a slightly different approach.

5.1 The Category of Persistent Sets

Definition 5.1 (Persistent Set)**.** A *persistent set* is a functor $S\colon V \to \mathrm{Set}$ from some partially ordered set V (seen as a thin category) to the category of sets. We also say that S is a *persistent set over* V in this case.

A *homomorphism* of persistent sets from $S\colon V \to \mathrm{Set}$ to $T\colon W \to \mathrm{Set}$ is a pair of a monotone map $\varphi\colon V \to W$ and a natural transformation $S \to T \circ \varphi$.

Denoting the category of persistent sets by PSet and the category of posets by Pos we have a functor

$$\Phi\colon \mathrm{PSet} \to \mathrm{Pos},\ (S\colon V \to \mathrm{Set}) \mapsto V.$$

While our main interest is in persistent sets over $\mathbb{R}^\circ \times \mathbb{R}$, it will be convenient to work with the category of persistent sets over any poset as a whole, as this allows us to avoid the use of Day convolution. This is similar to the way in which lax monoidal functors can be used in place of monoid objects with respect to Day convolution in stable homotopy theory. Clearly, the functor $\Phi\colon \mathrm{PSet} \to \mathrm{Pos}$ preserves binary products, as well as terminal objects. In the following we obtain several other properties of Φ, which we will be use in Section 5.3 to provide a convenient characterization of locally persistent categories. Choosing a terminal object $*$ in PSet, i.e. a functor assigning a singleton set to the unique element of some singleton poset, we have a forgetful functor $\mathrm{Hom}(*, -)\colon \mathrm{PSet} \to \mathrm{Set}$. Unfortunately, this forgetful functor is not monadic, for similar reasons that the forgetful functor $\mathrm{Hom}(\Phi(*), -)\colon \mathrm{Pos} \to \mathrm{Set}$ is not monadic either; see for example [Gri07, Section XVI.9]. However, using the functor Φ we can satisfy a relative notion of monadicity, as we will see in the remainder of this subsection.

Definition 5.2. For a cartesian monoidal category $\mathcal{B}$ with terminal object $*$ we define the *prototypical category over* $\mathcal{B}$ denoted as $\mathrm{Set}/\mathrm{Hom}(*,-)$ to be the comma category of the cospan of functors

$$\mathrm{Set} \xrightarrow{\mathrm{id}_{\mathrm{Set}}} \mathrm{Set} \xleftarrow{\mathrm{Hom}(*,-)} \mathcal{B}$$

as a cartesian monoidal category. More explicitly, the objects of $\mathrm{Set}/\mathrm{Hom}(*,-)$ are triples of a set M, an object B of $\mathcal{B}$, and a map of sets $M \to \mathrm{Hom}(*,B)$. For two such triples (S,B,f) and (T,C,g), a morphism $(M,B,f) \to (N,C,g)$ is a pair of a morphism $\varphi\colon B \to C$ in $\mathcal{B}$ and a map of sets $\tilde{\varphi}\colon M \to N$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "\tilde{\varphi}"] \arrow[d, "f"'] & N \arrow[d, "g"] \\ \mathrm{Hom}(*,B) \arrow[r, "\mathrm{Hom}(*,\varphi)"'] & \mathrm{Hom}(*,C) \end{tikzcd}$$

commutes.

For the category of posets Pos the forgetful functor $\mathrm{Hom}(\Phi(*),-)\colon \mathrm{Pos} \to \mathrm{Set}$ is naturally isomorphic to the usual forgetful functor $\mathrm{Pos} \to \mathrm{Set}$ assigning to a poset its underlying set. So the prototypical category over Pos has as objects triples of a set M, a poset V, and a map of sets $M \to V$. To simplify notation we will write $\mathrm{Set}/\mathrm{Pos} = \mathrm{Set}/\mathrm{Hom}(\Phi(*),-)$ in this case.

Definition 5.3 (Relative Monadicity)**.** Consider a strong monoidal functor $\Phi\colon \mathcal{M} \to \mathcal{B}$ between cartesian monoidal categories with $*$ a terminal object of $\mathcal{M}$. Then we have the functor

$$\mathrm{Hom}(*,-) \mathbin{/\!\!/} \mathcal{B}\colon \mathcal{M} \to \mathrm{Set}/\mathrm{Hom}(\Phi(*),-),\ X \mapsto (\mathrm{Hom}(*,X), \Phi(X), \Phi_{*,X}).$$

We say that $\mathcal{M}$ is *monadic relative to* $\mathcal{B}$ *(with respect to the functor* Φ*)* if $\mathrm{Hom}(*,-) \mathbin{/\!\!/} \mathcal{B}$ is a monadic functor whose left adjoint is strong monoidal.

In our setting the functor $\mathrm{Hom}(*,-) \mathbin{/\!\!/} \mathrm{Pos}\colon \mathrm{PSet} \to \mathrm{Set}/\mathrm{Pos}$ has a very concrete description. Up to natural isomorphism, the functor $\mathrm{Hom}(*,-) \mathbin{/\!\!/} \mathrm{Pos}$ sends a persistent set $S\colon V \to \mathrm{Set}$ to the naturally induced map of sets

$$\mathrm{Hom}(*,S) \mathbin{/\!\!/} \mathrm{Pos}\colon \coprod_{v \in V} S(v) \to V \tag{5.1}$$

from the disjoint union $\coprod_{v \in V} S(v)$ to V. Its left adjoint is

$$(w\colon M \to V) \mapsto \begin{cases} V \to \mathrm{Set}, \\ v \mapsto w^{-1}(\downarrow v), \end{cases} \tag{5.2}$$

where $\downarrow v$ denotes the downset of any $v \in V$.

Definition 5.4 (Free Persistent Set)**.** We say that a persistent set is *free* if it is in the essential image of the left adjoint (5.2) to $\mathrm{Hom}(*,-) /\!\!/ \mathrm{Pos}\colon \mathrm{PSet} \to \mathrm{Set}/\mathrm{Pos}$.

Precomposing the functor $\mathrm{Hom}(*,-) /\!\!/ \mathrm{Pos}\colon \mathrm{PSet} \to \mathrm{Set}/\mathrm{Pos}$ with its left adjoint (5.2) we obtain the monad

$$\Xi\colon \mathrm{Set}/\mathrm{Pos} \to \mathrm{Set}/\mathrm{Pos},\ (w\colon M \to V) \mapsto \Xi(w), \tag{5.3}$$

where

$$\Xi(w)\colon \bigcup_{v \in V} w^{-1}(\downarrow v) \times \{v\} \to V,\ (m,v) \mapsto v$$

for any map of sets $w\colon M \to V$. When we instantiate the monad multiplication $\mu\colon \Xi \circ \Xi \to \Xi$ at $w\colon M \to V$, then we obtain the fiber-preserving map

$$\mu_w\colon \Xi(\Xi(w)) \to \Xi(w),\ ((m,u),v) \mapsto (m,v).$$

The unit $\eta\colon \mathrm{id} \to \Xi$ instantiated at w is the fiber-preserving map

$$\eta_w\colon w \to \Xi(w),\ m \mapsto (m, w(m)).$$

Next we describe the induced *semantical comparison functor* from PSet to the Eilenberg–Moore category $(\mathrm{Set}/\mathrm{Pos})^{\Xi}$ of Ξ. To this end, let $S\colon V \to \mathrm{Set}$ be a persistent set. Then the Ξ-algebra structure homomorphism on $w_S \coloneqq \mathrm{Hom}(*,S) /\!\!/ \mathrm{Pos}$ is the fiber-preserving map

$$\Xi(w_S) \to w_S,\ ((m,u),v) \mapsto S(u \preceq v)(m).$$

Proposition 5.5. *The category of persistent sets* PSet *is monadic relative to the category of posets* Pos *(with respect to* $\Phi\colon \mathrm{PSet} \to \mathrm{Pos}$*).*

If we unravel Definition 5.3 and Proposition 5.5 for a fixed poset V, then we obtain the well known result that the category of Set-valued functors on V is monadic over the category of sets over V, see for example [Tri22]. We could then use the relation between indexed categories and fibrations to deduce Proposition 5.5 from this result. For simplicity, we provide a more explicit proof instead.

Proof. We describe a quasi-inverse to the aforementioned semantical comparison functor from PSet to the Eilenberg–Moore category $(\mathrm{Set}/\mathrm{Pos})^{\Xi}$ of Ξ. To this end, let $w\colon M \to V$ be a Ξ-algebra with structure homomorphism $\alpha\colon \Xi(w) \to w$. We define the data for a persistent set by

$$S_w\colon V \to \mathrm{Set},\ \begin{cases} v \mapsto w^{-1}(v), \\ (u \preceq v) \mapsto (m \mapsto \alpha(m,v)). \end{cases}$$

Now the two properties required for S_w to be a persistent set correspond to the unit and the action property for the monad algebra $\alpha\colon \Xi(w) \to w$: We have

$$S_w(v \preceq v)(m) = \alpha(m,v) = \alpha(\eta_w(m)) = m$$

and

$$\begin{aligned}S_w(v_2 \preceq v_3)(S_w(v_1 \preceq v_2)(m)) &= \alpha(\alpha(m, v_2), v_3)\\ &= \alpha(\mu_w((m, v_2), v_3))\\ &= \alpha(m, v_3)\\ &= S_w(v_1 \preceq v_3)(m)\end{aligned}$$

for any $v, v_1 \preceq v_2 \preceq v_3 \in V$ and $m \in M$. Similarly, if we have Ξ-algebras (w_1, α_1) and (w_2, α_2) with corresponding persistent sets S_1 and S_2 as well as a morphism $(\varphi, \tilde{\varphi})\colon w_1 \to w_2$ in Set/Pos, then the coherence condition for $(\varphi, \tilde{\varphi})$ to be a homomorphism of Ξ-algebras is equivalent to the naturality of the pointwise induced map $S_1 \to S_2 \circ \varphi$ corresponding to $(\varphi, \tilde{\varphi})$. □

Corollary 5.6. *The full subcategory of free persistent sets in the sense of Definition 5.4 is isomorphic to the Kleisli category of the monad* $\Xi\colon \mathrm{Set/Pos} \to \mathrm{Set/Pos}$.

Now for each poset V, the over category Set/V is a subcategory of Set/Pos, which is invariant under $\Xi\colon \mathrm{Set/Pos} \to \mathrm{Set/Pos}$.

Lemma 5.7. *For a poset V the functor* $\Xi\colon \mathrm{Set/Pos} \to \mathrm{Set/Pos}$ *preserves colimits in* Set/V.

Proof. Colimits in Set/V can be obtained by taking the corresponding colimit of domains and then the induced map from this colimit to V. Thus, it suffices to show that Ξ preserves the domains of colimits. Moreover, for a map $w\colon M \to V$ the domain of $\Xi(w)$ can be obtained by first taking the preimage $w^{-1}(\downarrow v)$ for each $v \in V$, which is cocontinuous in Set/V, and then forming the coproduct of all of these preiamges, which is cocontinuous as well. □

5.2 Categories Enriched Relative to a Monoid Object

In order to provide a concrete description of small *locally persistent categories* [Sco20] using Proposition 5.5, it will be convenient to have a relative notion of an enriched category. For this entire subsection, let $\mathcal{M}$ and $\mathcal{B}$ be cartesian monoidal categories, let $*$ be a terminal object of $\mathcal{M}$, and let $\Phi\colon \mathcal{M} \to \mathcal{B}$ be a strong monoidal functor. Now suppose we have an $\mathcal{M}$-enriched category $\mathcal{C}$. Then we may consider the base change $\Phi_\bullet\mathcal{C}$, which is a $\mathcal{B}$-enriched category. So if we have another $\mathcal{B}$-enriched category $\mathcal{D}$, then we may consider $\mathcal{B}$-enriched functors $\Phi_\bullet\mathcal{C} \to \mathcal{D}$. Conceptually, we may think of such functors as "enriched functors $\mathcal{C} \to \mathcal{D}$ above Φ". Now a relatively simple kind of $\mathcal{B}$-enriched category is one with a single object, which is the same thing as a monoid object V in $\mathcal{B}$. So in the following Definition 5.8 we may think of the $\mathcal{B}$-enriched functor $I_{\mathcal{C}}\colon \Phi_\bullet\mathcal{C} \to V$ as an "enriched functor $\mathcal{C} \to V$ above Φ".

Definition 5.8 (Relative Enrichment)**.** Let V be a monoid object in $\mathcal{B}$ seen as a $\mathcal{B}$-enriched category with a single object. An *$\mathcal{M}$-enriched category relative to V (with* *respect to the functor* Φ*)* is an $\mathcal{M}$-enriched category $\mathcal{C}$ such that each of the morphism objects of $\Phi_\bullet\mathcal{C}$ in $\mathcal{B}$ is equal to V and moreover, this pointwise equality satisfies the coherence conditions for a $\mathcal{B}$-enriched functor $I_\mathcal{C}\colon \Phi_\bullet\mathcal{C} \to V$.

For $\mathcal{M}$-enriched categories $\mathcal{C}$ and $\mathcal{D}$ relative to V, an $\mathcal{M}$-enriched functor $F\colon \mathcal{C} \to \mathcal{D}$ is *relative to* V if the diagram

$$\begin{tikzcd} \Phi_\bullet\mathcal{C} \arrow[rr, bend left, "\Phi_\bullet F"] \arrow[dr, "I_\mathcal{C}"'] & & \Phi_\bullet\mathcal{D} \arrow[dl, "I_\mathcal{D}"] \\ & V & \end{tikzcd}$$

commutes, which is equivalent to $\Phi_\bullet F$ being the identity id_V on morphism objects, which are all equal to V.

For two $\mathcal{M}$-enriched functors $F, G\colon \mathcal{C} \to \mathcal{D}$ relative to V, an $\mathcal{M}$-natural transformation $\eta\colon F \Rightarrow G$ is *relative to* V if the induced $\mathcal{B}$-natural transformation $I_\mathcal{D} \circ \Phi_\bullet\eta$ from $I_\mathcal{C}\colon \Phi_\bullet\mathcal{C} \to V$ to itself is the identity $\mathcal{B}$-natural transformation.

We denote the strict 2-category of $\mathcal{M}$-enriched categories relative to V, $\mathcal{M}$-enriched functors relative to V, and $\mathcal{M}$-natural transformations relative to V by $(\mathcal{M}\text{-Cat})_V$.

Remark 5.9. If V is a commutative monoid object in $\mathcal{B}$ and $\mathcal{C}$ is an $\mathcal{M}$-enriched category relative to V, then the dual $\mathcal{M}$-enriched category $\mathcal{C}^\circ$ (in the sense of ordinary enriched categories) is again an $\mathcal{M}$-enriched category relative to V. This constitutes a strict *oppositization* 2-endofunctor on the strict 2-category of $\mathcal{M}$-enriched categories relative to V.

Now for ordinary enriched categories, the prototypical example is that of a Set-enriched category, which is just an ordinary category. As each enriched category has an underlying ordinary category coming from the base change to the category of sets, we may ask ourselves what the corresponding V-relative notion may be. Here we make a case for the idea that the prototypical choice of Φ should be the *codomain functor*

$$\mathrm{codom}\colon \mathrm{Set}^\to \to \mathrm{Set}$$

from the arrow category of Set to Set sending any map of sets to its codomain. In this case, a monoid object in the base category Set is just a monoid V and a $\mathrm{Set}^\to$-enriched category relative to V is just an ordinary category, where each morphism is assigned a "grade" in V and the composition is compatible with the monoid operation of V.

5.2.1 Base Change to Prototypical Relative Enrichment

Now suppose we have an $\mathcal{M}$-enriched category $\mathcal{C}$ relative to a monoid object V of $\mathcal{B}$ (with respect to the functor Φ). Then the base change of $\mathcal{C}$ along the functor $\mathrm{Hom}(*, -) \mathbin{/\!\!/} \mathcal{B}$ from Definition 5.3 yields a category enriched in $\mathrm{Set}/\mathrm{Hom}(\Phi(*), -)$ relative to V. Instead, we intend to obtain a $\mathrm{Set}^\to$-enriched category relative to $\mathrm{Hom}(\Phi(*), V)$. To this end, we use the following lemma.

Lemma 5.10. *The commutative diagram*

$$
\begin{tikzcd}
(S,B,f) \arrow[r, mapsto] & f \\
\mathrm{Set}/\mathrm{Hom}(\Phi(*),-) \arrow[r] \arrow[d] & \mathrm{Set}^{\rightarrow} \arrow[d, "\mathrm{codom}"] \\
\mathcal{B} \arrow[r] & \mathrm{Set} \\
B \arrow[r, mapsto] & \mathrm{Hom}(\Phi(*),B)
\end{tikzcd}
$$

induces an isomorphism of strict 2*-categories between* $\mathrm{Set}/\mathrm{Hom}(\Phi(*),-)$*-categories relative to* V *and* $\mathrm{Set}^{\rightarrow}$*-categories relative to* $\mathrm{Hom}(\Phi(*),V)$*.*

Thus, we can always turn an $\mathcal{M}$-category relative to V into a $\mathrm{Set}^{\rightarrow}$-category relative to $\mathrm{Hom}(\Phi(*),V)$ by first taking the base change along $\mathrm{Hom}(*,-) /\!\!/ \mathcal{B}$ and then the isomorphism of 2-categories from Lemma 5.10.

Definition 5.11. For an $\mathcal{M}$-enriched category $\mathcal{C}$ relative to V, we name the $\mathrm{Set}^{\rightarrow}$-enriched category obtained from the base change along $\mathrm{Hom}(*,-) /\!\!/ \mathcal{B}$ and Lemma 5.10 the *underlying* $\mathrm{Set}^{\rightarrow}$*-enriched category relative to* $\mathrm{Hom}(\Phi(*),V)$ of $\mathcal{C}$.

Passing from an $\mathcal{M}$-enriched category relative to V to its underlying $\mathrm{Set}^{\rightarrow}$-enriched category relative to $\mathrm{Hom}(\Phi(*),V)$ constitutes a strict 2-functor. Moreover, once we have a $\mathrm{Set}^{\rightarrow}$-enriched category relative to some additively written monoid V, we may take another step to obtain an ordinary category by taking the fibers of $0 \in V$ as the new set of morphisms. In more explicit terms, the ordinary category obtained from an $\mathcal{M}$-enriched category $\mathcal{C}$ relative to some monoid object V in $\mathcal{B}$ from both of these steps, has as morphisms those morphisms of the underlying ordinary category of $\mathcal{C}$ that are mapped by $\Phi_{*,-}$ to the unit of V.

5.2.2 Monadicity Over Weighted Quivers

Now generally speaking, $\mathcal{M}$-enriched categories are not very concrete. So in order to simplify working with them, it can be useful to have a concrete description of small $\mathcal{M}$-enriched categories relative to a monoid object V of $\mathcal{B}$. In this subsection we provide a concrete description of small $\mathcal{M}$-enriched categories relative to V in the special case that $\mathcal{M}$ is monadic relative to $\mathcal{B}$ in the sense of Definition 5.3. To this end, we use the result that ordinary small categories are monadic over quivers as a guiding example, see for example [Per19, Section 5.5.1].

Definition 5.12 (Weighted Quiver)**.** For an additively written monoid V, a V*-weighted quiver* is a quiver $Q = (Q_0, Q_1)$ with *vertices* Q_0 and *arrows* Q_1 together with a *weight function* $w\colon Q_1 \to V$ on its arrows.

A *homomorphism of V-weighted quivers* is a homomorphism of quivers preserving their weights.

In other words, the category of V-weighted quivers is the comma category of the cospan of functors

$$\begin{array}{ccc} (Q_0, Q_1) \longmapsto Q_1 & & \\ \mathrm{Quiv} \xrightarrow[\mathrm{pr}_1]{} \mathrm{Set} \longleftarrow \mathbf{1} & & \\ V \longleftarrow\!\shortmid \ast. & & \end{array} \tag{5.4}$$

A homomorphism of V-weighted quivers is *rooted* if it is an identity on vertices. A diagram of V-weighted quivers is *rooted* if all of its homomorphisms are rooted. A connected colimit is *rooted* if it is a connected colimit of a connected rooted diagram.

Lemma 5.13. *For an additively written monoid V the category of V-weighted quivers is cocomplete. Moreover, in order to compute a colimit we may compute the colimits of vertices and arrows separately in* Set *and then take the induced maps from arrows to vertices and from arrows to V.*

Proof. As a functor category the category of quivers is cocomplete. Moreover, the functor $\mathrm{pr}_1 \colon \mathrm{Quiv} \to \mathrm{Set}$, $(Q_0, Q_1) \mapsto Q_1$ is cocontinuous. Thus, the comma category $V\mathrm{Quiv}$ is cocomplete by [RB88, Theorem 3, Section 5.2]. Moreover, by the proof of [RB88, Theorem 3], the two objects of a colimit in a comma category are the corresponding colimits of components and the morphism of a colimit is the one induced by the universal property of colimits. □

We now show that small $\mathcal{M}$-enriched categories relative to a monoid object V of $\mathcal{B}$ are monadic over $\mathrm{Hom}(\Phi(\ast), V)$-weighted quivers whenever $\mathcal{M}$ is monadic relative to $\mathcal{B}$ in the sense of Definition 5.3. To this end, we start with the "prototypical case" of $\mathrm{Set}^{\to}$-enriched categories relative to an additively written monoid V and then we reduce the general case to this one. In order for small $\mathrm{Set}^{\to}$-categories relative to V to be monadic over V-weighted quivers, we need, in particular, a pair of adjoint functors between small $\mathrm{Set}^{\to}$-categories relative to V and V-weighted quivers. Suppose we have a small $\mathrm{Set}^{\to}$-enriched category relative to V. Then we may simply forget the composition and which morphisms are identities to obtain a V-weighted quiver. This way we obtain a forgetful functor

$$U_0 \colon (\mathrm{Set}^{\to}\text{-cat})_V \longrightarrow V\mathrm{Quiv}$$

from the category of small $\mathrm{Set}^{\to}$-categories relative to V, denoted as $(\mathrm{Set}^{\to}\text{-cat})_V$, to the category of V-weighted quivers denoted as $V\mathrm{Quiv}$. Conversely, suppose we have a V-weighted quiver with underlying quiver Q. Then we may form the path category associated to Q, see for example [Per19, Definition 4.2.21]. Now in order to obtain a $\mathrm{Set}^{\to}$-enriched category relative to V from this, we have to assign a weight in V to each path in Q in such a way that the composition of paths is compatible with the addition in V. To this end, we may simply assign to each path in Q the sum[1] of the weights of

[1] As we do not assume V to be commutative at this point, we have to take care that we sum the weights in the same order as the corresponding arrows appear in the path.

its arrows. This way we obtain a functor

$$F_0\colon V\mathrm{Quiv} \longrightarrow (\mathrm{Set}^{\to}\text{-cat})_V$$

in the opposite direction. Moreover, there is a natural homomorphism $Q \to U_0(F_0(Q))$ of V-weighted quivers sending each vertex of Q to itself and each arrow to the corresponding path of length 1.

Lemma 5.14. *The functors F_0 and U_0 form a monadic pair of adjoint functors with the aforementioned natural homomorphism as a unit.*

We omit the proof of Lemma 5.14 as it is very similar to the proof of monadicity of small categories over quivers, see for example [Per19, Section 5.5.1].

Definition 5.15 (Weighted Paths Monad)**.** We name $\mathrm{P} \coloneqq U_0 \circ F_0$ the *V-weighted paths monad* on V-weighted quivers.

Lemma 5.16. *The V-weighted paths monad $\mathrm{P} = U_0 \circ F_0$ preserves rooted epimorphisms.*

Proof. Let $f\colon Q \to R$ be a rooted epimorphism. We have to show that for any path $p = (a_1, \ldots, a_n)$ in R there is a path p' in Q with $\mathrm{P}(f)(p') = p$. To this end, we may choose an arrow a'_i in Q with $f(a'_i) = a_i$ for each $i = 1, \ldots, n$ as f is an epimorphism. Moreover, as f is rooted, $p' \coloneqq (a'_1, \ldots, a'_n)$ is a path in Q. □

Now the V-weighted paths monad P does not preserve arbitrary colimits. However, coequalizers of rooted pairs conforming to the following notion are preserved.

Definition 5.17. For a category $\mathcal{D}$ a *reflexive symmetric pair* is a pair of parallel morphisms $f, g\colon X \to Y$ in $\mathcal{D}$ such that there is a *section* $s\colon Y \to X$ and a *symmetry* $\sigma\colon X \to X$ in $\mathcal{D}$ satisfying the equations

$$f \circ s = \mathrm{id}_Y = g \circ s, \quad f \circ \sigma = g, \quad \text{and} \quad g \circ \sigma = f. \tag{5.5}$$

Lemma 5.18. *Reflexive symmetric pairs are preserved by any functor.*

Proof. The equations (5.5) are preserved by any functor. □

Lemma 5.19. *Suppose $f, g\colon X \to Y$ is a reflexive symmetric pair in the category of sets. Then the relation $\sim$ on Y defined by $f(x) \sim g(x)$ for $x \in X$ is reflexive and symmetric.*

Proof. Let $s\colon Y \to X$ be a section and let $\sigma\colon X \to X$ be a symmetry for $f, g\colon X \to Y$. Now suppose we have $y \in Y$ and let $x \coloneqq s(y)$. Then we also have $y = f(x)$ and $y = g(x)$ and hence $y \sim y$. Moreover, suppose we have $y = f(x)$ and $y' = g(x)$ for some $x \in X$ and let $x' = \sigma(x)$. Then we also have $y' = g(x) = f(x')$ and $y = f(x) = g(x')$ and hence $y' \sim y$. □

Lemma 5.20. *The V-weighted paths monad $\mathrm{P} = U_0 \circ F_0$ preserves coequalizers of rooted reflexive symmetric pairs.*

Proof. By Lemma 5.13 a colimit of V-weighted quivers can be obtained from the corresponding colimits of vertices and arrows. In particular, the vertices of rooted connected colimits are preserved by any endofunctor. It remains to show that P preserves the arrows of rooted reflexive symmetric coequalizers as well. To this end, let $f, g\colon Q \to R$ be a rooted reflexive symmetric pair in $V\mathrm{Quiv}$ with section $s\colon R \to Q$, symmetry $\sigma\colon Q \to Q$, and $q\colon R \to S$ their coequalizer. By Lemma 5.18 the pair $\mathrm{P}(f), \mathrm{P}(g)\colon \mathrm{P}(Q) \to \mathrm{P}(R)$ is reflexive symmetric as well. We now define the relation $\sim$ on the arrows R_1 of R by

$$f(a) \sim g(a) \quad \text{for arrows} \quad a \in Q_1,$$

as well as the relation $\approx$ on the arrows of $\mathrm{P}(R)$, which are paths in R, by

$$\mathrm{P}(f)(p) \approx \mathrm{P}(g)(p) \quad \text{for paths } p \text{ of } Q.$$

By Lemma 5.19 the relations $\sim$ and $\approx$ are reflexive and symmetric. Thus, we have to show that $\mathrm{P}(q)\colon \mathrm{P}(R) \to \mathrm{P}(S)$ is a quotient map for the transitive closure of $\approx$. By Lemma 5.16 the homomorphism $\mathrm{P}(q)\colon \mathrm{P}(R) \to \mathrm{P}(S)$ is an epimorphism. So it remains to be shown that for any two paths p and p' in R that are identified by $\mathrm{P}(q)$, there is a chain of paths in R from p to p' with any two consecutive paths related by $\approx$. To this end, let a and a' each be the first arrow of p and p' respectively. Then we have $q(a) = q(a')$. Moreover, as the arrows S_1 of S form a quotient for the transitive closure of $\sim$, there is a chain of arrows in R from a to a' with any two consecutive arrows related by $\sim$. Furthermore, as $\sim$ is symmetric this chain of arrows from a to a' yields a chain of paths with respect to the relation $\approx$ from p to another path p'' with a' as its first arrow and the other arrows coming from p. Continuing inductively we may replace each of the arrows of p with the corresponding arrow of p'. □

We now reduce the general case to Lemma 5.14. To this end, we denote the category of small $\mathcal{M}$-enriched categories relative to V by $(\mathcal{M}\text{-cat})_V$. Now let

$$\tilde{U}_1\colon (\mathcal{M}\text{-cat})_V \longrightarrow (\mathrm{Set}^{\to}\text{-cat})_{V'}$$

be the functor that assigns to a small $\mathcal{M}$-category relative to V its underlying $\mathrm{Set}^{\to}$-category relative to $V' \coloneqq \mathrm{Hom}(\Phi(*), V)$ in the sense of Definition 5.11. We intend to show that the composition $U_0 \circ \tilde{U}_1\colon (\mathcal{M}\text{-cat})_V \longrightarrow V'\mathrm{Quiv}$ is monadic, if $\mathcal{M}$ is monadic relative to $\mathcal{B}$ in the sense of Definition 5.3. Assuming $\mathcal{M}$ is monadic relative to $\mathcal{B}$ from now on, we first show that $\tilde{U}_1$ is monadic and then we use a criterion due to [Bec69] for the composition of monadic functors, which is Proposition C.4. Now for $\tilde{U}_1$ to be monadic, it needs to have a left adjoint

$$\tilde{F}_1\colon (\mathrm{Set}^{\to}\text{-cat})_{V'} \longrightarrow (\mathcal{M}\text{-cat})_V$$

in particular. To this end, we define $\tilde{F}_1$ to be the composition of the inverse of the isomorphism from Lemma 5.10 followed by the base change $\mathcal{F}_\bullet$ along $\mathcal{F}$. As the adjunction $\mathcal{F} \dashv \mathrm{Hom}(*, -) \mathbin{/\!\!/} \mathcal{B}$ is monadic, the adjunction $\tilde{F}_1 \dashv \tilde{U}_1$ is monadic as well by Lemma C.5. Now in order to meet the assumptions of Proposition C.4, we need to extend the data we have so far to a *distributive adjoint situation over* V'Quiv in the sense of Definition C.3. (This will be useful later as well.) To this end, let $\mathcal{M}_V$ be the subcategory of $\mathcal{M}$ that consists of all objects of $\mathcal{M}$ that map to V under $\Phi\colon \mathcal{M} \to \mathcal{B}$ and all morphisms that map to id_V under Φ. Moreover, let $\mathcal{M}_V$Quiv be the category of $\mathcal{M}_V$-quivers, whose objects consist of a set of vertices Q_0 and a Q_0^2-indexed family of objects in $\mathcal{M}_V$ and homomorphisms defined in the obvious way. Then we have a forgetful functor

$$\tilde{U}_0\colon (\mathcal{M}\text{-cat})_V \to \mathcal{M}_V\mathrm{Quiv},$$

which is also monadic by [Wol73, Theorem 2.13] whenever $\mathcal{M}_V$ is cocomplete. We make no use of [Wol73, Theorem 2.13] but we take note of this result for attribution. For a distributive adjoint situation, we still need to provide an additional monadic adjunction

$$\begin{tikzcd} V'\mathrm{Quiv} \arrow[r, bend right, "F_1"'] \arrow[r, phantom, "\top"] & \mathcal{M}_V\mathrm{Quiv}. \arrow[l, bend right, "U_1"'] \end{tikzcd}$$

To obtain $U_1\colon \mathcal{M}_V\mathrm{Quiv} \to V'\mathrm{Quiv}$ we may simply extend the forgetful functor

$$\Phi_{*,-}\colon \mathcal{M}_V \to \mathrm{Set}/V',\ X \mapsto \begin{cases} \mathrm{Hom}(*, X) \to \mathrm{Hom}(\Phi(*), V) = V' \\ \qquad\qquad f \mapsto \Phi(f) \end{cases}$$

to Q_0^2-indexed families. Similarly, the left adjoint can be obtained by extending the functor

$$\mathrm{Set}/V' \to \mathcal{M}_V,\ (w\colon S \to V') \mapsto \mathcal{F}(S, V, w)$$

to Q_0^2-indexed families. Altogether we obtain the diagram

$$\begin{tikzcd} & (\mathcal{M}\text{-cat})_V \arrow[dl, bend left, "\tilde{U}_1"] \arrow[dr, "\tilde{U}_0"] & \\ (\mathrm{Set}^{\to}\text{-cat})_{V'} \arrow[ur, bend left, "\tilde{F}_1"] \arrow[ur, phantom, "\perp"] \arrow[dr, bend left, "U_0"] \arrow[dr, phantom, "\perp"] & & \mathcal{M}_V\mathrm{Quiv} \arrow[dl, bend left, "U_1"] \arrow[dl, phantom, "\perp"] \\ & V'\mathrm{Quiv} \arrow[ul, bend left, "F_0"] \arrow[ur, bend left, "F_1"] & \end{tikzcd} \tag{5.6}$$

of functors and adjunctions with $U_0 \circ \tilde{U}_1 = U_1 \circ \tilde{U}_0$.

Lemma 5.21. *The commutative square* (5.6) *is a distributive adjoint situation over* V'Quiv *in the sense of Definition C.3.*

Proof. We have yet to show that the square

$$\begin{tikzcd}
(\mathcal{M}\text{-cat})_V \arrow[r, "\tilde{U}_1"] \arrow[d, "\tilde{U}_0"'] & (\mathrm{Set}^{\to}\text{-cat})_{V'} \arrow[d, "U_0"] \\
\mathcal{M}_V\mathrm{Quiv} \arrow[r, "U_1"'] & V'\mathrm{Quiv}
\end{tikzcd}$$

is a commutative Beck–Chevalley square in the sense of Definition C.1. This in turn follows from the first triangle identity for the adjunction $\mathcal{F} \dashv \mathrm{Hom}(*, -) \mathbin{/\!\!/} \mathcal{B}$ and the commutativity of (5.6). $\square$

Theorem 5.22. *The composition*

$$U_0 \circ \tilde{U}_1 \colon (\mathcal{M}\text{-cat})_V \longrightarrow V'\mathrm{Quiv}$$

is monadic whenever $\mathcal{M}$ is monadic relative to $\mathcal{B}$ in the sense of Definition 5.3.

Proof. This follows directly from Lemma 5.21 and Proposition C.4. $\square$

Corollary 5.23. *The category of small $\mathcal{M}$-categories relative to V is isomorphic to the category of monad algebras over $U_1 \circ F_1 \circ \mathrm{P}$ whenever $\mathcal{M}$ is monadic relative to $\mathcal{B}$ (with the monad structure on $U_1 \circ F_1 \circ \mathrm{P}$ induced by the distributive adjoint situation* (5.6)*).*

Proof. This follows directly from Theorem 5.22 and the commutativity of (5.6). $\square$

5.3 Locally Persistent Categories in Terms of Relative Enrichments

In this subsection we use the notion of relative enrichments from the previous Section 5.2 to provide a convenient characterization of locally persistent categories that does not require the use of Day convolution. Now the context that is needed to speak about relative enrichments involves cartesian monoidal categories $\mathcal{M}$ and $\mathcal{B}$ as well as a strong monoidal functor $\Phi \colon \mathcal{M} \to \mathcal{B}$. We introduced such context in Section 5.1 with $\mathcal{B} = \mathrm{Pos}$ being the category of posets, $\mathcal{M} = \mathrm{PSet}$ being the category of persistent sets over varying posets, and

$$\Phi \colon \mathrm{PSet} \to \mathrm{Pos},\ (S \colon V \to \mathrm{Set}) \mapsto V$$

being the functor that assigns to a persistent set $S \colon V \to \mathrm{Set}$ its indexing poset V.

Lemma 5.24. *For an additively written monoidal poset V, a locally V-persistent category is precisely a* PSet*-enriched category relative to V (with respect to the functor Φ) in the sense of Definition 5.8.*

Definition 5.25. A V*-persistent functor* between locally V-persistent categories is a PSet-enriched functor relative to V.

A V*-persistent natural transformation* between V-persistent functors is a PSet-natural transformation relative to V.

We denote the strict 2-category of locally V-persistent categories, V-persistent functors, and V-persistent natural transformations by $(\mathrm{PSet}\text{-}\mathrm{Cat})_V$.

Remark 5.26. Suppose V is a commutative monoidal poset and that $\mathcal{C}$ is a locally V-persistent category. As noted in Remark 5.9, the opposite PSet-enriched category $\mathcal{C}^\circ$ is again a locally V-persistent category. As it turns out, the underlying ordinary category of $\mathcal{C}^\circ$ is identical to the opposite category of $\mathcal{C}_0$. We denote this category by $\mathcal{C}_0^\circ$.

As a note of caution, we note that the underlying ordinary category of a locally V-persistent category $\mathcal{C}$ in the sense of Definition 4.2 is different from the underlying ordinary category of $\mathcal{C}$ as a PSet-enriched category. We now explain how the underlying ordinary category in the sense of Definition 4.2 and the underlying $\mathrm{Set}^{\to}$-enriched category in the sense of Definition 5.11 are related. To this end, let $\mathcal{C}$ be a locally V-persistent category. Then we form the base change along the forgetful functor $\mathrm{Hom}(*, -) /\!\!/ \mathrm{Pos}$ to obtain the underlying Set/Pos-enriched category $(\mathrm{Hom}(*, -) /\!\!/ \mathrm{Pos})_\bullet\mathcal{C}$ relative to V. Moreover, by Proposition 5.5 the category of persistent sets PSet is monadic over Set/Pos with $\Xi\colon \mathrm{Set}/\mathrm{Pos} \to \mathrm{Set}/\mathrm{Pos}$ from (5.3) being the induced monad. Thus, the morphism objects of $(\mathrm{Hom}(*, -) /\!\!/ \mathrm{Pos})_\bullet\mathcal{C}$ have naturally induced Ξ-algebra structure maps, which are compatible with the composition. The corresponding $(\mathrm{Set}/\mathrm{Pos})^\Xi$-enriched category is the base change of $\mathcal{C}$ along the semantical comparison functor, hence we may also think of locally V-persistent categories as $(\mathrm{Set}/\mathrm{Pos})^\Xi$-enriched categories relative to V in the sense of Definition 5.8. This perspective will be useful when we connect locally V-persistent categories to V-weighted categories in the sense of Definition 3.2 in Section 6.2. On the other hand, we may also apply the isomorphism from Lemma 5.10 to $(\mathrm{Hom}(*, -) /\!\!/ \mathrm{Pos})_\bullet\mathcal{C}$ to obtain the underlying $\mathrm{Set}^{\to}$-enriched category relative to V in the sense of Definition 5.11. (Here we abuse notation by writing V for the monoidal poset as well as the underlying monoid $\mathrm{Hom}(\Phi(*), V)$.) Now the morphism objects of a $\mathrm{Set}^{\to}$-enriched category relative to V are just V-valued maps and we may obtain an ordinary category by taking the fibers of $0 \in V$ for each of these V-valued maps, which is naturally isomorphic to the underlying ordinary category $\mathcal{C}_0$ in the sense of Definition 4.2.

5.4 Weighted Diagrams in Locally Persistent Categories

By Proposition 5.5, Theorem 5.22, and Lemma 5.24 the category of small locally V-persistent categories is monadic over V-weighted quivers. In this Section 5.4 we use this result to describe a calculus of diagrams for locally V-persistent categories. We take note of a similar result mentioned by [Per21] in his talk, which inspired us to study the notion of monadicity for locally V-persistent categories in the first place. The notion of a V-weighted quiver that we introduced in Definition 5.12 ignores the partial order defined on V; more precisely, V is just a monoid in Definition 5.12. On the other hand, [Per21] considers quivers whose arrows carry weights in $[0, \infty)$. While homomorphisms of V-weighted quivers are required to be weight-preserving in Definition 5.12, the homomorphisms considered by [Per21] may as well decrease the edge weights. Paolo Perrone then states that small $[0, \infty)$-weighted categories are monadic over the category *weighted graphs* as defined in his talk [Per21].

Now suppose we have a V-weighted quiver $Q = (Q_0, Q_1)$ with weight function $w\colon Q_1 \to V$. Then we may apply the functor $\Xi\colon \mathrm{Set}/\mathrm{Pos} \to \mathrm{Set}/\mathrm{Pos}$ defined at (5.3) to the weight function $w\colon Q_1 \to V$. As the elements of the domain of $\Xi(w)$ are pairs of an arrow in Q and an element of V, there is a natural map from the domain of $\Xi(w)$ to Q_0. This way we obtain a functor

$$\Xi'\colon V\mathrm{Quiv} \to V\mathrm{Quiv},$$

which is naturally isomorphic to the composition of functors $U_1 \circ F_1$, where $F_1 \dashv U_1$ is the corresponding adjoint pair from the distributive adjoint situation (5.6). Thus, the composition $\Xi' \circ \mathrm{P}$ is naturally isomorphic to the monad $U_1 \circ F_1 \circ \mathrm{P}$ from Corollary 5.23. This immediately implies the following.

Proposition 5.27. *The category of small locally V-persistent categories is isomorphic to the category of monad algebras over $\Xi' \circ \mathrm{P}$ with respect to the monad structure inherited from $U_1 \circ F_1 \circ \mathrm{P}$.*

Lemma 5.28. *The functor $\Xi' \circ \mathrm{P}$ preserves coequalizers of rooted reflexive symmetric pairs.*

Proof. By Lemma 5.20 the V-weighted paths monad P preserves coequalizers of rooted reflexive symmetric pairs. Moreover, the functor $\Xi\colon \mathrm{Set}/\mathrm{Pos} \to \mathrm{Set}/\mathrm{Pos}$ preserves colimits in Set/V by Lemma 5.7 and hence $\Xi'\colon V\mathrm{Quiv} \to V\mathrm{Quiv}$ preserves arbitrary colimits. □

Definition 5.29 (Rooted Homomorphism)**.** We say that a homomorphism in the Eilenberg–Moore category of $\Xi' \circ \mathrm{P}$ is *rooted* if the underlying homomorphism of V-weighted quivers is rooted in the sense of Definition 5.12. In other words, a homomorphism of $\Xi' \circ \mathrm{P}$-algebras is rooted iff the corresponding functor is an identity on objects.

The following Proposition 5.30 provides a convenient recipe to construct small locally V-persistent categories.

Proposition 5.30. *The Eilenberg–Moore category of $\Xi' \circ \mathrm{P}$, and hence the category of small locally V-persistent categories, has coequalizers of rooted reflexive symmetric pairs.*

Proof. This follows directly from Lemma 5.28 and [Lin69, Propostion 3]. □

We now explain how Proposition 5.30 yields a calculus of diagrams for locally V-persistent categories. Say for example we have $a, a', b, b', c \in V$ with $b' + a \preceq c \succeq a' + b$ and consider the diagram

$$\begin{tikzcd} \bullet \arrow[r, "a"'] \arrow[d, "b"] \arrow[dr, phantom, "c"] & \bullet \arrow[d, "b'"'] \\ \bullet \arrow[r, "a'"] & \bullet \end{tikzcd} \tag{5.7}$$

describing a V-weighted quiver R and a single *V-weighted* 2-*cell for* R of weight c. Then we define a locally V-persistent category from (5.7) as follows. We define a V-weighted quiver Q that has two arrows of weight c corresponding to the 2-cell in (5.7):

$$\begin{tikzcd} \bullet \arrow[r, "a"'] \arrow[d, "b"] \arrow[dr, bend left=15, "c"] \arrow[dr, bend right=15, "c"'] & \bullet \arrow[d, "b'"'] \\ \bullet \arrow[r, "a'"] & \bullet. \end{tikzcd} \tag{5.8}$$

We now define a rooted reflexive symmetric pair $f, g\colon Q \to \Xi'(\mathrm{P}(R))$ in the sense of Definitions 5.12 and 5.17 in the Kleisli category of $\Xi' \circ \mathrm{P}$. Specifically, $f\colon Q \to \Xi'(\mathrm{P}(R))$ maps each vertex to itself, each axis-aligned arrow of Q in (5.8) to the corresponding path of length 1 in $\Xi'(\mathrm{P}(R))$, the upper diagonal arrow to the unique path of weight c in $\Xi'(\mathrm{P}(R))$ from the vertex on the upper left, to the vertex on the upper right, and then to the vertex on the lower right. Similarly, the lower diagonal arrow is mapped by f to the corresponding path of weight c through the vertex on the lower left. The Kleisli homomorphism $g\colon Q \to \Xi'(\mathrm{P}(R))$ is almost the same as f except that it maps the upper diagonal arrow in (5.8) to the path through the vertex on the lower left and the lower diagonal arrow to the path through the vertex on the upper right. Moreover, the arrow-wise inclusion of R into Q yields a section for the pair $f, g\colon Q \to \Xi'(\mathrm{P}(R))$ and the endomorphism of Q swapping the two diagonal arrows a symmetry in the sense of Definition 5.17. Then we take the coequalizer of the corresponding reflexive symmetric pair $f^\flat, g^\flat\colon \Xi'(\mathrm{P}(Q)) \to \Xi'(\mathrm{P}(R))$ of free $\Xi' \circ \mathrm{P}$-algebras in the Eilenberg–Moore category, which exists by Proposition 5.30, to obtain a locally V-persistent category. In order to specify a V-persistent functor to another locally V-persistent category it suffices to “fill” the diagram (5.7) as follows

$$\begin{tikzcd} X_1 \arrow[r, "\varphi", "a"'] \arrow[d, "\psi"', "b"] \arrow[dr, phantom, "c"] & X_2 \arrow[d, "b'"', "\psi'"] \\ X_3 \arrow[r, "a'", "\varphi'"'] & X_4 \end{tikzcd} \tag{5.9}$$

with the “data”

$$\begin{aligned} \varphi &\in \mathrm{Hom}(X_1, X_2)(a), & \varphi' &\in \mathrm{Hom}(X_3, X_4)(a'), \\ \psi &\in \mathrm{Hom}(X_1, X_3)(b), & \psi' &\in \mathrm{Hom}(X_2, X_4)(b'), \end{aligned}$$

subject to the equation

$$\mathrm{Hom}(X_1, X_4)(b' + a \preceq c)(\psi' \;{}_{b'}\!\diamond_a \varphi) = \mathrm{Hom}(X_1, X_4)(a' + b \preceq c)(\varphi' \;{}_{a'}\!\diamond_b \psi).$$

This method for constructing small locally V-persistent categories generalizes. For any V-weighted quiver and a compatible set of V-weighted 2-cells as in (5.7), we may define a rooted reflexive symmetric pair in the Kleisli category of $\Xi' \circ \mathrm{P}$ and then take the corresponding coequalizer in the Eilenberg–Moore category of $\Xi' \circ \mathrm{P}$, which exists by Proposition 5.30. Moreover, if V has joins, then we can also paste such diagrams. Say for example we have two such squares side by side:

$$\begin{tikzcd}
X_1 \arrow[r, "\varphi", "a"'] \arrow[d] \arrow[dr, phantom, "c"] & X_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "d"] & X_3 \arrow[d] \\
X_4 \arrow[r] & X_5 \arrow[r, "b", "\psi"'] & X_6.
\end{tikzcd}$$

Then we may first thicken the rows as follows

$$\begin{tikzcd}
X_1 \arrow[rr] \arrow[d, equal] & & X_3 \arrow[dd] \\
X_1 \arrow[r] \arrow[dd] \arrow[dr, phantom, "b+c"] & X_2 \arrow[d] \arrow[r, phantom, "d+a"] & \\
& X_5 \arrow[r] & X_6 \arrow[d, equal] \\
X_4 \arrow[rr] & & X_6
\end{tikzcd}$$

and then eliminate the zigzag path in the middle:

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "(b+c) \vee (d+a)"] & X_3 \arrow[d] \\
X_4 \arrow[r] & X_6.
\end{tikzcd}$$

A similar calculus for diagrams in persistent categories has been introduced by [Sco20, Section 3.2.1]. The main difference is that in [Sco20, Section 3.2.1] the 2-cells carry no weights and are assumed to commute "as soon as it makes sense". By putting weights on the 2-cells we have some more flexibility.

5.5 Interleavings in Locally Persistent Categories

Interleavings in locally persistent categories were defined by [Sco20, Definition 3.1.8] and then also rephrased as diagrams in [Sco20, Notation 3.2.3]. Here we use the calculus of diagrams introduced in the previous Section 5.4 to define the notion of an interleaving. For simplicity, we assume that V is commutative, which happens to be the case in all of our applications. Considering a V-weighted diagram of the form

$$\begin{tikzcd} X \arrow[rr, bend left, "\varphi", "a"'] & w & Y \arrow[ll, bend left, "\psi", "b"'] \end{tikzcd} \tag{5.10}$$

with $0, b + a \preceq w$ in a V-persistent category $\mathcal{C}$, we may interpret this as follows. Removing objects and weighted morphisms from (5.10) we have the diagram

$$\begin{tikzcd} \bullet \arrow[rr, bend left, "a"'] & w & \bullet. \arrow[ll, bend left, "b"'] \end{tikzcd} \tag{5.11}$$

Now there are two possibilities how we might add a 2-cell of weight w to the underlying V-weighted quiver of diagram (5.11). Reusing the labels of (5.10), one possibility is to place a 2-cell of weight w between the path "$\psi \;{}_b\!\diamond_a \varphi$" and the path of length 0 at the left vertex. And the other possibility is a 2-cell between the path "$\varphi \;{}_a\!\diamond_b \psi$" and the 0-path at the right vertex in (5.11). We interpret a diagram of the same form as (5.11) or (5.10) to have both of these 2-cells of weight w. Interpreting (5.10) in the same way as (5.9) yields

$$\varphi \in \operatorname{Hom}(X, Y)(a) \quad \text{and} \quad \psi \in \operatorname{Hom}(Y, X)(b)$$

as well as the equations

$$\begin{aligned} &\operatorname{Hom}(X, X)(0 \preceq w)(\mathrm{id}_X) = \operatorname{Hom}(X, X)(b + a \preceq w)(\psi \;{}_b\!\diamond_a \varphi) \\ \text{and} \qquad &\operatorname{Hom}(Y, Y)(0 \preceq w)(\mathrm{id}_Y) = \operatorname{Hom}(Y, Y)(b + a \preceq w)(\varphi \;{}_a\!\diamond_b \psi). \end{aligned} \tag{5.12}$$

Definition 5.31 (Interleaving)**.** We refer to a weighted diagram of the form (5.10) as an *interleaving* of X and Y.

Remark 5.32. A common requirement on interleavings is the condition that $a = b$ and $w = b + a = 2a$. We omit these constraints for convenience, so there is no need to "symmetrize" an interleaving when an "asymmetric" interleaving is more natural, see for example the interleaving (6.5) below.

As with other diagrams we may also paste two such interleaving diagrams

$$\begin{tikzcd} X \arrow[rr, bend left, "\varphi", "a"'] & w & Y \arrow[ll, bend left, "\psi", "b"'] \arrow[rr, bend left, "\varphi'", "a'"'] & w' & Z \arrow[ll, bend left, "\psi'", "b'"'] \end{tikzcd}$$

to obtain the interleaving

$$\begin{tikzcd} X \arrow[rr, bend left, "\varphi'\ {}_{a'}\!\diamond_a \varphi", "a'+a"'] \arrow[rr, phantom, "w+w'"] & & Z. \arrow[ll, bend left, "b+b'", "\psi\ {}_b\!\diamond_{b'} \psi'"'] \end{tikzcd}$$

The pasting of such interleaving diagrams is the sticking point when proving the triangle inequality for interleaving distances; see for example [Sco20, Lemma 3.1.10] or [dSMS18, Theorem 2.7]. In the following Example 5.33 we unravel this notion of an interleaving in the context of the V-persistent enrichment of the functor category $\mathbb{M}_\mathrm{f} \to \mathcal{W}$ provided in Section 4.1.

Example 5.33 (δ-Interleavings in the sense of [BdSS15]). With some abuse of notation (we use the letter Ω differently in Section 1.1) we define

$$\Omega_\delta := \alpha(-\delta, \delta) \colon \mathbb{M}_\mathrm{f} \to \mathbb{M}_\mathrm{f},\ u \mapsto \Omega_\delta(u) := (-\delta, \delta).u$$

for any $\delta \in \mathbb{R}$. Then the mapping $\delta \mapsto \Omega_\delta$ describes a *(super)linear family* on $\mathbb{M}_\mathrm{f}$ in the sense of [BdSS15, Section 2.5]. Now suppose we have $\delta \geq 0$ and functors $F, G \colon \mathbb{M}_\mathrm{f} \to \mathcal{W}$. Moreover, let $a := (-\delta, \delta)$ and cosider an interleaving of F and G in the sense of Definition 5.31:

$$\begin{tikzcd} F \arrow[rr, bend left, "\varphi", "a"'] \arrow[rr, phantom, "2a"] & & G. \arrow[ll, bend left, "a", "\psi"'] \end{tikzcd} \tag{5.13}$$

As we discussed in Section 5.5, we consider this diagram as a diagram with two 2-cells, each with a weight of $2a$. Now the first 2-cell relates the identity id_F at F to the composition $\psi\ {}_a\!\diamond_a \varphi$, which is the composition of natural transformations

$$F \xrightarrow{\varphi} G \circ \Omega_\delta \xrightarrow{\psi_{a.\_}} F \circ \Omega_{2\delta}$$

with weight $2a$. By (5.12) the constraint imposed on id_F and $\psi\ {}_a\!\diamond_a \varphi$ by the first 2-cell is the equation $\mathrm{Hom}(F, F)(0 \preceq 2a)(\mathrm{id}_F) = \psi\ {}_a\!\diamond_a \varphi$. Moreover, we have $\mathrm{Hom}(F, F)(0 \preceq 2a)(\mathrm{id}_F) = F \circ \alpha(0 \preceq 2a)$ by definition of the $\mathbb{R}^\circ \times \mathbb{R}$-persistent enrichment of $\mathcal{W}^{\mathbb{M}_\mathrm{f}}$. Thus, the first 2-cell in (5.13) tells us that the diagram of presheaves and natural transformations

$$\begin{tikzcd} F \arrow[dr, "\varphi"] \arrow[dd, bend right, "F\circ\alpha(0\preceq 2a)"'] & \\ & G \circ \Omega_\delta \arrow[dl, "\psi_{a.\_}"] \\ F \circ \Omega_{2\delta} & \end{tikzcd} \tag{5.14}$$

commutes. Analogous considerations for the second 2-cell in (5.13) yield a similar diagram, which we may combine with (5.14) to obtain an interleaving diagram of the usual form:

$$\begin{tikzcd}
F \arrow[dr, "\varphi"] \arrow[dd, bend right, "F\circ\alpha(0\preceq 2a)"'] & & G \arrow[dl, "\psi"'] \arrow[dd, bend left, "G\circ\alpha(0\preceq 2a)"] \\
F \circ \Omega_\delta \arrow[dr, "\varphi_{a\cdot_}"] & & G \circ \Omega_\delta \arrow[dl, "\psi_{a\cdot_}"'] \\
F \circ \Omega_{2\delta} & & G \circ \Omega_{2\delta}.
\end{tikzcd}$$

6 Enrichment of RISC to a Persistent Functor

As already noted at the beginning of Section 4.1, we intend to provide an $\mathbb{R}^\circ \times \mathbb{R}$-persistent enrichment of $h^\circ\colon \mathrm{Top}/\mathbb{R} \to \mathcal{W}^{\mathbb{M}_\mathrm{f}}$. So in particular, we need $\mathbb{R}^\circ \times \mathbb{R}$-persistent enrichments of the domain and codomain of h°. We already provided an enrichment of the codomain $\mathcal{W}^{\mathbb{M}_\mathrm{f}}$ in Section 4.1. For the domain $\mathrm{Top}/\mathbb{R}$ we provided an enrichment to an $\mathbb{R}^\circ \times \mathbb{R}$-weighted category in Chapter 3. We now describe how any $\mathbb{R}^\circ \times \mathbb{R}$-weighted category can be turned into a locally $\mathbb{R}^\circ \times \mathbb{R}$-persistent category. To this end, we first provide an embedding of V-weighted sets into V-persistent sets for a monoidal poset V in the next section.

6.1 Embedding Weighted Sets into Persistent Sets

Now let V be an additively written monoidal poset.

Definition 6.1 (Associated Persistent Set)**.** For a V-weighted set $w\colon M \to V$ in the sense of Definition 3.2 we define the *associated persistent set* over V to be the persistent set

$$P(w)\colon V \to \mathrm{Set},\ v \mapsto w^{-1}(\downarrow v).$$

For two V-weighted sets $w_i\colon M_i \to V$, $i = 1, 2$ we define the associated *natural coherence transformation*

$$P(w_1) \boxtimes P(w_2) \to P(w_1 \star w_2) \circ \nabla,\ (m_1, m_2) \mapsto (m_1, m_2), \tag{6.1}$$

where $\boxtimes$ denotes the product in PSet, which is different from the product in the subcategory of persistent sets over V, and $\nabla\colon V \times V \to V$ denotes the addition in V.

We also have a unique natural transformation from the terminal persistent set

$$* \longrightarrow P(\bullet \mapsto 0) \circ \eta, \tag{6.2}$$

where $\eta\colon \Phi(*) \to V$ denotes the zero map.

The data provided in Definition 6.1 makes $P\colon V\mathrm{Set} \to \mathrm{PSet}$ a lax monoidal functor. Ignoring the monoidal structures on the domain and codomain of P, we may also view P as a fully faithful functor $P\colon V\mathrm{Set} \to \mathrm{PSet}/V$, where $\mathrm{PSet}/V \subset \mathrm{PSet}$ is the subcategory of persistent sets over V and ordinary natural transformations. We also note that the restriction of $P\colon V\mathrm{Set} \to \mathrm{PSet}/V$ to the wide subcategory Set/V and the restriction of the left adjoint (5.2) to the functor $\mathrm{Hom}(*, -) /\!\!/ \mathrm{Pos}\colon \mathrm{PSet} \to \mathrm{Set}/\mathrm{Pos}$ to Set/V are identical, hence the image of $P\colon V\mathrm{Set} \to \mathrm{PSet}/V$ is the full subcategory of PSet/V on free persistent sets over V in the sense of Definition 5.4, which is isomorphic to the Kleisli category of the restricted monad $\Xi|_{\mathrm{Set}/V}\colon \mathrm{Set}/V \to \mathrm{Set}/V$ by Proposition 5.5.

6.2 Base Change to Persistent Sets

Now let V be an additively written monoidal poset without non-trivial idempotents and let $\mathcal{C}$ be a V-weighted category. Then the base change $P_\bullet\mathcal{C}$ is a locally V-persistent category in the sense of Definition 4.1. We take note of the following two useful properties of this base change.

Lemma 6.2. *The base change*

$$P_\bullet\colon V\mathrm{Set}\text{-}\mathrm{Cat} \longrightarrow (\mathrm{PSet}\text{-}\mathrm{Cat})_V,\, \mathcal{C} \mapsto P_\bullet\mathcal{C}$$

provides an embedding (in a strict 2-categorical sense) of the strict 2-category of V-weighted categories into the strict 2-category of locally V-persistent categories. Moreover, for a V-weighted category $\mathcal{C}$, the underlying ordinary category $(P_\bullet\mathcal{C})_0$ in the sense of Definition 4.2 and the underlying ordinary category $\mathcal{C}_0$ of $\mathcal{C}$ as a $V\mathrm{Set}$-enriched category are identical.

Proof. As $P\colon V\mathrm{Set} \to \mathrm{PSet}/V$ is fully faithful it provides a one-to-one correspondence relating the data describing $V\mathrm{Set}$-enriched functors $\mathcal{C} \to \mathcal{D}$ and the data describing PSet-enriched functors $P_\bullet\mathcal{C} \to P_\bullet\mathcal{D}$ relative to V. Now it is easy to see that the coherence homomorphisms (6.1) and (6.2) in PSet are *opcartesian* over ∇ respectively η in the sense of [Shu08, Section 3]. Moreover, the coherence homomorphism (6.1) being opcartesian entails that the data for a $V\mathrm{Set}$-enriched functor $\mathcal{C} \to \mathcal{D}$ is compatible with compositions iff the corresponding data for a PSet-enriched functor $P_\bullet\mathcal{C} \to P_\bullet\mathcal{D}$ is compatible with compositions. Similarly, the coherence homomorphism (6.2) being opcartesian entails that compatibility of such data with identities is satisfied in $V\mathrm{Set}$-Cat iff it is satisfied in PSet-Cat.

As $P\colon V\mathrm{Set} \to \mathrm{PSet}/V$ is fully faithful and as (6.2) is *opcartesian* we obtain a one-to-one correspondence relating the data for a $V\mathrm{Set}$-natural transformation $F \Rightarrow G$ and the data for a PSet-natural transformation $P_\bullet F \Rightarrow P_\bullet G$ relative to V. As $P\colon V\mathrm{Set} \to \mathrm{PSet}/V$ is fully faithful and lax monoidal, such data is $V\mathrm{Set}$-natural as a transformation $F \Rightarrow G$ iff it is PSet-natural as a transformation $P_\bullet F \Rightarrow P_\bullet G$.

If we unravel Definition 4.2, then we see that for a V-weighted category $\mathcal{C}$ the morphisms of the underlying ordinary category $(P_\bullet\mathcal{C})_0$ are the morphisms of $\mathcal{C}$ of weight $\preceq 0$, which are the morphisms of $\mathcal{C}_0$. Moreover, the identities and restricted compositions are identical as well. □

As already noted in the previous Section 6.1, the functor $P\colon V\mathrm{Set} \to \mathrm{PSet}/V$ also yields an isomorphism between $V\mathrm{Set}$ and the Kleisli category of the monad

$\Xi|_{\mathrm{Set}/V}\colon \mathrm{Set}/V \to \mathrm{Set}/V$. Thus, we may also view V-weighted categories as locally V-persistent categories, whose V-persistent sets of morphisms are free in the sense of Definition 5.4. We now use this insight to characterize the base change $P_\bullet\mathcal{C}$ by a universal property characterizing V-persistent functors with domain $P_\bullet\mathcal{C}$. Clearly, the initial functor with codomain $P_\bullet\mathcal{C}$ is the identity functor $P_\bullet\mathcal{C} \to P_\bullet\mathcal{C}$. Now suppose we have objects X and Y of $\mathcal{C}$ with the V-weighted set of morphisms $w \coloneqq w_{X,Y}\colon\ \mathrm{dom}(X,Y) \to V$. Then we may instantiate the unit for the monad $\Xi|_{\mathrm{Set}/V}\colon \mathrm{Set}/V \to \mathrm{Set}/V$ at $w\colon\ \mathrm{dom}(X,Y) \to V$ to obtain the homomorphism $\eta_w\colon w \to \Xi(w)$ of sets over V, which is the transform of the identity $P(w) = P(w)$ under the adjunction between sets over V and persistent sets over V. If we consider the base change $(\mathrm{Hom}(*,-) \mathbin{/\!\!/} \mathrm{Pos})_\bullet P_\bullet\mathcal{C}$, then $\Xi(w)$ is the set of morphisms over V between X and Y in this Set/V-enriched category. So we might think that the collection of all of these units yields an enriched functor

$$\mathcal{C} \longrightarrow (\mathrm{Hom}(*,-) \mathbin{/\!\!/} \mathrm{Pos})_\bullet P_\bullet\mathcal{C}.$$

However, this does not make any sense, as $\mathcal{C}$ is a V-weighted category and not a Set/V-enriched category. Nevertheless, the collection of these units satisfies the coherence conditions of the following Definition 6.3.

Definition 6.3. For a V-weighted category $\mathcal{C}$ and a locally V-persistent category $\mathcal{D}$ we define a locally large category $\mathrm{Func}'(\mathcal{C},\mathcal{D})$ as follows. An *object* of $\mathrm{Func}'(\mathcal{C},\mathcal{D})$ consists of a map

$$G_0\colon\ \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D}),\ X \mapsto G_0(X)$$

from the objects of $\mathcal{C}$ to the objects of $\mathcal{D}$ as well as an $\mathrm{Ob}(\mathcal{C})^2$-indexed family of homomorphisms of sets over V, specifically for each pair of objects X and Y of $\mathcal{C}$ with the V-weighted set of morphisms

$$w \coloneqq w_{X,Y}\colon\ \mathrm{dom}(X,Y) \to V$$

and the V-persistent set of morphisms

$$S \coloneqq \mathrm{Hom}(G_0(X), G_0(Y))\colon V \longrightarrow \mathrm{Set}$$

between $G_0(X)$ and $G_0(Y)$, a homomorphism of sets over V

$$G_{1,X,Y}\colon w \longrightarrow \mathrm{Hom}(*,S) \mathbin{/\!\!/} \mathrm{Pos}$$

such that the following two *coherence conditions* hold:

(1) For any object X of $\mathcal{C}$ we have $G_{1,X,X}(\mathrm{id}_X) = \mathrm{id}_{G_0(X)}$. (Note that the identity at X is of weight 0 by Lemma 3.10.)

(2) For any commutative triangle

$$\begin{tikzcd} X \arrow[d, "\varphi"'] \arrow[dr, "\psi\circ\varphi"] & \\ Y \arrow[r, "\psi"'] & Z \end{tikzcd}$$

in $\mathcal{C}$ we have the valid diagram

$$\begin{tikzcd} G_0(X) \arrow[d, "G_{1,X,Y}(\varphi)"'] \arrow[dr, "G_{1,X,Z}(\psi\circ\varphi)"] \arrow[d, phantom, shift left=6ex, "w"] & \\ G_0(Y) \arrow[r, "G_{1,Y,Z}(\psi)"'] & G_0(Z) \end{tikzcd}$$

in the sense of Section 5.4, where $w \coloneqq w_{Y,Z}(\psi) + w_{X,Y}(\varphi)$.

For two such objects G and H, a *morphism* $G \to H$ of $\mathrm{Func}'(\mathcal{C}, \mathcal{D})$ is an $\mathrm{Ob}(\mathcal{C})$-indexed family of morphisms of $\mathcal{D}_0$, specifically for each object X of $\mathcal{C}$ a morphism $G_0(X) \to H_0(X)$ in $\mathcal{D}_0$ such that for any morphism $\varphi\colon X \to Y$ of $\mathcal{C}$ we have the valid diagram

$$\begin{tikzcd} G_0(X) \arrow[r] \arrow[d, "G_{1,X,Y}(\varphi)"'] \arrow[dr, phantom, "w_{X,Y}(\varphi)"] & H_0(X) \arrow[d, "H_{1,X,Y}(\varphi)"] \\ G_0(Y) \arrow[r] & H_0(Y). \end{tikzcd} \tag{6.3}$$

Identities and compositions in $\mathrm{Func}'(\mathcal{C}, \mathcal{D})$ are defined in the obvious way.

By extension this defines a strict 2-bifunctor

$$\mathrm{Func}'\colon (V\text{Set-Cat})^\circ \times (\text{PSet-Cat})_V \longrightarrow \mathrm{CAT},\ (\mathcal{C}, \mathcal{D}) \mapsto \mathrm{Func}'(\mathcal{C}, \mathcal{D})$$

taking values in the strict 2-category CAT of locally large categories.

Remark 6.4. We take note of the following abuse of notation in Definition 6.3. Considering the concrete formula (5.1) for the functor $\mathrm{Hom}(*, -) /\!\!/ \mathrm{Pos}\colon \mathrm{PSet} \to \mathrm{Set}/\mathrm{Pos}$, the domain of $\mathrm{Hom}(*, S) /\!\!/ \mathrm{Pos}$ is a disjoint union rather than a union of the values of a V-persistent set of morphisms $S \coloneqq \mathrm{Hom}(G_0(X), G_0(Y))\colon V \longrightarrow \mathrm{Set}$. Thus, its elements are not really elements of the V-persistent set of morphisms $S = \mathrm{Hom}(G_0(X), G_0(Y))$. Nevertheless, we treat them as such when stating the coherence conditions of Definition 6.3.

We now use the unit of the monad $\Xi|_{\mathrm{Set}/V}\colon \mathrm{Set}/V \to \mathrm{Set}/V$ to define an object $H_{\mathcal{C}} \coloneqq H$ of $\mathrm{Func}'(\mathcal{C}, P_\bullet\mathcal{C})$. To this end, we define $H_0\colon\ \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(P_\bullet\mathcal{C})$, $X \mapsto X$ to be the identity on the objects of $\mathcal{C}$ and for two objects X and Y of $\mathcal{C}$ with the V-weighted set of morphisms $w \coloneqq w_{X,Y}\colon\ \mathrm{dom}(X, Y) \to V$, we define $H_{1,X,Y} \coloneqq \eta_w\colon w \to \Xi(w)$, where η_w is the unit of the monad $\Xi|_{\mathrm{Set}/V}\colon \mathrm{Set}/V \to \mathrm{Set}/V$ at $w\colon\ \mathrm{dom}(X, Y) \to V$.

Lemma 6.5. *The data just defined as $H_{\mathcal{C}} = H$ satisfies the coherence conditions of Definition 6.3, hence it is indeed an object of the category* $\mathrm{Func}'(\mathcal{C}, P_\bullet\mathcal{C})$.

Definition 6.6. We name $H_{\mathcal{C}}$ the *universal object* of $\mathrm{Func}'(\mathcal{C}, P_\bullet\mathcal{C})$.

Now for two locally V-persistent categories $\mathcal{D}$ and $\mathcal{E}$, let $V\mathrm{PFunc}(\mathcal{D}, \mathcal{E})$ be the locally large category of V-persistent functors $\mathcal{D} \to \mathcal{E}$ and V-persistent natural transformations. Then the following Lemma 6.7 justifies the naming convention of Definition 6.6.

Lemma 6.7. *The strict 2-functor* $\mathrm{Func}'(\mathcal{C}, -)\colon (\mathrm{PSet\text{-}Cat})_V \to \mathrm{CAT}$ *and the evaluation at the universal object* $H_{\mathcal{C}}$ *yield an isomorphism of categories* $V\mathrm{PFunc}(P_\bullet\mathcal{C}, \mathcal{D}) \to \mathrm{Func}'(\mathcal{C}, \mathcal{D})$ *for any locally V-persistent category $\mathcal{D}$.*

Proof. As already noted in Section 5.3, the base change along the semantical comparison functor for the monadic functor $\mathrm{Hom}(*, -) \mathbin{/\!\!/} \mathrm{Pos}\colon \mathrm{PSet} \to \mathrm{Set}/\mathrm{Pos}$ sending each PSet-enriched category $\mathcal{D}$ to the category $(\mathrm{Hom}(*, -) \mathbin{/\!\!/} \mathrm{Pos})_\bullet\mathcal{D}$ enriched in Ξ-algebras, yields an equivalence from the strict 2-category of locally V-persistent categories to categories enriched in Ξ-algebras relative to V. Moreover, we may view V-weighted categories as categories enriched in the Kleisli category for the monad $\Xi|_{\mathrm{Set}/V}\colon \mathrm{Set}/V \to \mathrm{Set}/V$ as noted above. To make this more precise, we define a lax monoidal functor

$$\Omega\colon V\mathrm{Set} \to (\mathrm{Set}/\mathrm{Pos})_\Xi,$$

where $(\mathrm{Set}/\mathrm{Pos})_\Xi$ is the Kleisli category of the monad $\Xi\colon \mathrm{Set}/\mathrm{Pos} \to \mathrm{Set}/\mathrm{Pos}$ as defined in Definition C.6, to obtain the isomorphism of strict 2-categories

$$\Omega_\bullet\colon V\mathrm{Set\text{-}Cat} \to \big((\mathrm{Set}/\mathrm{Pos})_\Xi\text{-}\mathrm{Cat}\big)_V,\ \mathcal{C} \mapsto \Omega_\bullet\mathcal{C},$$

where $\big((\mathrm{Set}/\mathrm{Pos})_\Xi\text{-}\mathrm{Cat}\big)_V$ is the strict 2-category of $(\mathrm{Set}/\mathrm{Pos})_\Xi$-enriched categories relative to V in the sense of Definition 5.8. To this end, let $w_i\colon M_i \to V$, $i = 1, 2$ be V-weighted sets and let $\varphi\colon w_1 \to w_2$ be a homomorphism of V-weighted sets. Then we define

$$\Omega(w_i) \coloneqq w_i\colon M_i \to V, \quad i = 1, 2$$

and

$$\Omega(\varphi)\colon w_1 \to \Xi(w_2),\ M_1 \ni m \mapsto (\varphi(m), w_1(m)) \in \coprod_{v \in V} w_2^{-1}(\downarrow v).$$

It is easy to see that $\Omega_\bullet\colon V\mathrm{Set\text{-}Cat} \to \big((\mathrm{Set}/\mathrm{Pos})_\Xi\text{-}\mathrm{Cat}\big)_V$ is indeed an isomorphism of strict 2-categories. Moreover, in conjunction with Corollary 3.11 it follows that any $(\mathrm{Set}/\mathrm{Pos})_\Xi$-enriched category relative to V has *pure identities* in the sense of Definition C.7. Furthermore, we have the commutative diagram

$$\begin{tikzcd}
V\mathrm{Set} \arrow[r, "P"] \arrow[d, "\Omega"'] & \mathrm{PSet} \arrow[d, "\mathrm{Hom}(*,-)/\!\!/\mathrm{Pos}"] \\
(\mathrm{Set}/\mathrm{Pos})_\Xi \arrow[r, "\Xi"'] & (\mathrm{Set}/\mathrm{Pos})^\Xi
\end{tikzcd}$$

of lax monoidal functors and thus the commutative diagram

$$\begin{tikzcd}
V\text{Set-Cat} \arrow[r, "P_\bullet"] \arrow[d, "\Omega_\bullet"'] & \text{PSet-Cat} \arrow[d, "(\mathrm{Hom}(*,-)/\!\!/\mathrm{Pos})_\bullet"] \\
(\text{Set}/\text{Pos})_\Xi\text{-Cat} \arrow[r, "\Xi_\bullet"'] & (\text{Set}/\text{Pos})^\Xi\text{-Cat}
\end{tikzcd}$$

of base change 2-functors.

Now in Definition C.8 we define the strict 2-bifunctor

$$\mathrm{EFunc}' \colon \big((\text{Set}/\text{Pos})_\Xi\text{-Cat}_p\big)^\circ \times (\text{Set}/\text{Pos})^\Xi\text{-Cat} \longrightarrow \mathrm{CAT}$$

and we have Lemma C.10. Thus, it suffices to provide a 2-natural isomorphism

$$\mathrm{EFunc}'\big(\Omega_\bullet\mathcal{C}, (\mathrm{Hom}(*,-) /\!\!/ \mathrm{Pos})_\bullet\mathcal{D}\big) \to \mathrm{Func}'(\mathcal{C},\mathcal{D}) \tag{6.4}$$

of locally large categories for any V-weighted category $\mathcal{C}$ and locally V-persistent category $\mathcal{D}$ mapping the *universal object* $H'_{\Omega_\bullet\mathcal{C}}$ as in Definition C.9 to the universal object $H_\mathcal{C}$. To this end, let $\mathcal{C}$ be a V-weighted category and let $\mathcal{D}$ be a locally V-peristent category. Then in order to turn an object G' of the category $\mathrm{EFunc}'\big(\Omega_\bullet\mathcal{C}, (\mathrm{Hom}(*,-) /\!\!/ \mathrm{Pos})_\bullet\mathcal{D}\big)$ into an object G of the category $\mathrm{Func}'(\mathcal{C},\mathcal{D})$, we just need to identify commutative diagrams of the form

$$\begin{tikzcd}
\{\bullet\} \arrow[r] \arrow[d] & M \arrow[d, "w"] \\
\{0\} \arrow[r, hook] & V
\end{tikzcd}$$

with the corresponding elements of the fiber $w^{-1}(0)$ of $0 \in V$ under any map $w\colon M \to V$. The coherence conditions (1) and (2) of Definition 6.3 then correspond to the diagrams (C.4) and (C.5) respectively. Morphisms of the category $\mathrm{EFunc}'\big(\Omega_\bullet\mathcal{C}, (\mathrm{Hom}(*,-) /\!\!/ \mathrm{Pos})_\bullet\mathcal{D}\big)$ are turned into morphisms of the category $\mathrm{Func}'(\mathcal{C},\mathcal{D})$ in much the same way. The diagram (6.3) then corresponds to the diagram (C.6). Moreover, it is easy to see that for $\mathcal{D} = P_\bullet\mathcal{C}$, the universal object $H'_{\Omega_\bullet\mathcal{C}}$ is mapped to the universal object $H_\mathcal{C}$ under the isomorphism (6.4) we just described. $\square$

6.2.1 Interleavings in $P_\bullet(\mathrm{Top}/\mathbb{R})_w$

Before we consider interleavings in $P_\bullet(\mathrm{Top}/\mathbb{R})_w$, we consider interleavings in V-weighted categories in general as well as in a weighted category of elements in the sense of Definition 3.3.

Interleavings in V-Weighted Categories. Suppose that V is commutative and that $\varphi\colon X \to Y$ is a weighted isomorphism in $\mathcal{C}$, i.e. an isomorphism in $\mathrm{dom}_\bullet\,\mathcal{C}$. Then we have the interleaving

$$
\begin{tikzcd}
X \arrow[r, bend left, "\varphi", "w(\varphi)"'] \arrow[r, phantom, "w(\varphi^{-1})+w(\varphi)"] & Y \arrow[l, bend left, "\varphi^{-1}", "w(\varphi^{-1})"']
\end{tikzcd}
\tag{6.5}
$$

in $P_\bullet\mathcal{C}$. Moreover, it is easy to see that any other interleaving in $P_\bullet\mathcal{C}$ can be obtained from an interleaving of the form (6.5) by precomposition with a V-persistent functor between locally V-persistent *index* categories, hence an interleaving of the form (6.5) is minimal.

Interleavings in a Weighted Category of Elements. Now let $F\colon \mathcal{B}^\circ \to \tilde{V}^\circ\text{-Cat}$ be a contravariant functor taking values in small $\tilde{V}^\circ$-enriched categories with pairwise disjoint values. Then we have the V-weighted category of elements $\mathrm{el}(F)$ as defined in Definition 3.3 so we may consider interleavings in $P_\bullet\mathrm{el}(F)$. Moreover, for a weighted isomorphism $\varphi\colon a \to b$ of $\mathrm{el}(F)$ with $a \in F(A)$ and $b \in F(B)$, the inverse $\varphi^{-1}\colon b \to a$ also is an inverse to $\varphi\colon A \to B$ as a morphism $\varphi^{-1}\colon B \to A$ in the base category $\mathcal{B}$. Thus, the induced $\tilde{V}^\circ$-enriched functor $F(\varphi)\colon F(B) \to F(A)$ is a $\tilde{V}^\circ$-enriched isofunctor with inverse $F(\varphi^{-1})\colon F(A) \to F(B)$, hence we have

$$
w_{b,a}(\varphi^{-1}) = \mathbf{d}(b, F(\varphi^{-1})(a)) = \mathbf{d}(F(\varphi)(b), a)
\tag{6.6}
$$

and as a further result we may factor the corresponding instance of the interleaving (6.5) as

$$
\begin{tikzcd}
a \arrow[r, bend left, "\mathrm{id}_A", "\mathbf{d}(a,F(\varphi)(b))"'] \arrow[r, phantom, "\mathbf{d}(F(\varphi)(b),a)+\mathbf{d}(a,F(\varphi)(b))"] & F(\varphi)(b) \arrow[l, bend left, "\mathrm{id}_A", "\mathbf{d}(F(\varphi)(b),a)"'] \arrow[r, bend left, "\varphi", "0"'] \arrow[r, phantom, "0"] & b \arrow[l, bend left, "\varphi^{-1}", "0"']
\end{tikzcd}
\tag{6.7}
$$

into an interleaving of vertical morphisms on the left and an interleaving induced by the horizontal isomorphism $\varphi\colon F(\varphi)(b) \to b$ on the right.

Interleavings of Functions on a Fixed Domain. Now we consider $\mathrm{Hom}(X, \mathbb{R})$ for a non-empty topological space as a V-weighted category. Then any morphism $\bullet\colon f \to g$ of $\mathrm{Hom}(X, \mathbb{R})$ is an isomorphism of weight $\mathbf{d}(f, g) \in V = \mathbb{R}^\circ \times \mathbb{R}$ and hence induces an interleaving

$$\begin{tikzcd} f \arrow[rr, bend left, "\bullet", "\mathbf{d}(f,g)"'] \arrow[rr, phantom, "(\mathrm{id}_V+\dagger)\mathbf{d}(f,g)"] & & g \arrow[ll, bend left, "\dagger\mathbf{d}(f,g)"', "\bullet"] \end{tikzcd} \tag{6.8}$$

in $P_\bullet\mathrm{Hom}(X,\mathbb{R})$, where $\dagger\colon V \to V$ is the reflection at the antidiagonal as in (3.2). As $\dagger\colon V \to V$ is an involution of the monoidal poset V, the weight $(\mathrm{id}_V + \dagger)\mathbf{d}(f,g)$ of "the" 2-cell in (6.8) is invariant under $\dagger$, hence $(\mathrm{id}_V + \dagger)\mathbf{d}(f,g)$ lies on the antidiagonal of $\mathbb{R}^2$ to the upper left of the origin and is thus determined by a single non-negative parameter.

Now we may as well consider two functions $f, g\colon X \to \mathbb{R}$ as objects of the V-weighted category $(\mathrm{Top}/\mathbb{R})_w$. Then a morphism $\bullet\colon f \to g$ of $\mathrm{Hom}(X,\mathbb{R})$ corresponds to the homomorphism $\mathrm{id}_X\colon f \to g$ in $(\mathrm{Top}/\mathbb{R})_w$, see also the lax cocone we defined in Eq. (3.3). Thus, by postcomposing diagram (6.8) with the V-weighted "inclusion" functor

$$G'(X)\colon \mathrm{Hom}(X,\mathbb{R}) \to (\mathrm{Top}/\mathbb{R})_w$$

we obtain the interleaving

$$\begin{tikzcd} f \arrow[rr, bend left, "\mathrm{id}_X", "\mathbf{d}(f,g)"'] \arrow[rr, phantom, "(\mathrm{id}_V+\dagger)\mathbf{d}(f,g)"] & & g. \arrow[ll, bend left, "\dagger\mathbf{d}(f,g)"', "\mathrm{id}_X"] \end{tikzcd} \tag{6.9}$$

Interleavings of Functions on Distinct Domains. Now suppose we have a homeomorphism $\varphi\colon X \to Y$ as well as continuous functions $f\colon X \to \mathbb{R}$ and $g\colon Y \to \mathbb{R}$ with $\mathbf{d}(f, g\circ\varphi) \in V = \mathbb{R}^\circ \times \mathbb{R}$. Then we may specialize the diagrams of interleavings (6.7) and (6.9) to simplify the corresponding interleaving (6.5) to the diagram of interleavings

$$\begin{tikzcd} f \arrow[rr, bend left, "\mathrm{id}_X", "\mathbf{d}(f,g\circ\varphi)"'] \arrow[rr, phantom, "(\mathrm{id}_V+\dagger)\mathbf{d}(f,g\circ\varphi)"] & & g\circ\varphi \arrow[ll, bend left, "\dagger\mathbf{d}(f,g\circ\varphi)"', "\mathrm{id}_X"] \arrow[r, bend left, "\varphi", "0"'] \arrow[r, phantom, "0"] & g \arrow[l, bend left, "0"', "\varphi^{-1}"] \end{tikzcd} \tag{6.10}$$

in $P_\bullet(\mathrm{Top}/\mathbb{R})_w$. By our considerations for interleavings in V-weighted categories above, any interleaving in $P_\bullet(\mathrm{Top}/\mathbb{R})_w$ can be obtained from a pasting of interleavings as in (6.10) by precomposition with a V-persistent functor between locally V-persistent index categories.

6.3 Persistent Functors from a V-Weighted Category of Elements

As we now have the functor $P\colon V\mathrm{Set} \to \mathrm{PSet}$ from Definition 6.1, we may use the base change along P to turn the $\mathbb{R}^\circ \times \mathbb{R}$-weighted category $(\mathrm{Top}/\mathbb{R})_w$ into a locally $\mathbb{R}^\circ \times \mathbb{R}$-persistent category $P_\bullet(\mathrm{Top}/\mathbb{R})_w$. Moreover, we also know from Lemma 6.7 that in order to provide an $\mathbb{R}^\circ \times \mathbb{R}$-persistent functor

$$P_\bullet(\mathrm{Top}/\mathbb{R})_w \to \mathcal{D}$$

for some locally $\mathbb{R}^\circ \times \mathbb{R}$-persistent category $\mathcal{D}$, such as the opposite category of the category of presheaves $\mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$, it suffices to provide an object of the category $\mathrm{Func}'((\mathrm{Top}/\mathbb{R})_w, \mathcal{D})$. Now $(\mathrm{Top}/\mathbb{R})_w$ is not an arbitrary $\mathbb{R}^\circ \times \mathbb{R}$-weighted category as it is also the V-weighted category of elements of the contravariant functor

$$\mathrm{Hom}(-, \mathbb{R})\colon \mathrm{Top}^\circ \to \tilde{V}^\circ\text{-Cat}$$

for $V = \mathbb{R}^\circ \times \mathbb{R}$ in the sense of Definition 3.3. In this Section 6.3 we show that for a monoidal poset V without non-trivial idempotents, a contravariant functor

$$F\colon \mathcal{B}^\circ \to \tilde{V}^\circ\text{-Cat}$$

with pairwise disjoint values, and a locally V-persistent category $\mathcal{D}$, the objects of $\mathrm{Func}'(\mathrm{el}(F), \mathcal{D})$, and thus V-persistent functors $P_\bullet\mathrm{el}(F) \to \mathcal{D}$, can be simplified even further. To this end, suppose we have an object $G = (G_0, G_1)$ of $\mathrm{Func}'(\mathrm{el}(F), \mathcal{D})$. As any morphism of $\mathrm{el}(F)$ factors into a vertical and a horizontal morphism by Remark 3.4, the second coherence condition of Definition 6.3 implies that G_1 is uniquely determined by its restriction to the horizontal and vertical morphisms of $\mathrm{el}(F)$. We now describe the restrictions of G_1 to the vertical and horizontal morphisms of $\mathrm{el}(F)$ in terms of the strict 2-functor

$$\mathrm{Func}'(-, \mathcal{D})\colon V\mathrm{Set}\text{-Cat} \to \mathrm{CAT}$$

and the lax cocone $G' \in \mathrm{LCocone}(F, \mathrm{el}(F))$ under F specified in Definition 3.3.

Restrictions to Vertical Morphisms. Now each vertical morphism of $\mathrm{el}(F)$ is a morphism of $F(A)$ for some object A of $\mathcal{B}$. In order to describe the restriction of G_1 to the vertical morphisms of $F(A)$, we may apply the strict 2-functor

$$\mathrm{Func}'(-, \mathcal{D})\colon V\mathrm{Set}\text{-Cat} \to \mathrm{CAT}$$

to the V-weighted functor

$$G'(A)\colon F(A) \to \mathrm{el}(F),$$

which is a leg of the lax cocone $G' \in \mathrm{LCocone}(F, \mathrm{el}(F))$, to obtain the functor

$$\mathrm{Func}'(G'(A), \mathcal{D})\colon \mathrm{Func}'(\mathrm{el}(F), \mathcal{D}) \to \mathrm{Func}'(F(A), \mathcal{D}).$$

Then we may evaluate $\mathrm{Func}'(G'(A), \mathcal{D})$ on the object G of the category $\mathrm{Func}'(\mathrm{el}(F), \mathcal{D})$ to obtain the object

$$G_A := \mathrm{Func}'(G'(A), \mathcal{D})(G) \tag{6.11}$$

of the category $\mathrm{Func}'(F(A), \mathcal{D})$. By Definition 6.3 of the strict 2-bifunctor Func' the family $G_{A,1}$ is the restriction of G_1 to the vertical morphisms of $\mathrm{el}(F)$ contained in $F(A)$.

As there is at most one vertical morphism between any two objects of $\mathrm{el}(F)$ we may somewhat simplify the notation for the restrictions $G_{A,1}$ for $A \in \mathrm{Ob}(\mathcal{B})$ of G_1 to vertical morphisms. To this end, suppose we have two elements $a, a' \in F(A)$ whose element of morphisms $\mathbf{d}(a, a')$ of $F(A)$ is finite in the sense that $\mathbf{d}(a, a') \in V$, which is equivalent to the existence of a vertical morphism $a \to a'$ of $\mathrm{el}(F)$. In this case we set

$$\check{G}(a, a') := G_{A,1,a,a'}(\bullet) = G_{1,a,a'}(\mathrm{id}_A). \tag{6.12}$$

Note that as $F \colon \mathcal{B}^\circ \to \tilde{V}^\circ\text{-Cat}$ has pairwise disjoint values it is also fine to omit the object A in our notation (6.12).

Restriction to Horizontal Morphisms. Now suppose we have a morphism $\varphi \colon A \to B$ of $\mathcal{B}$. Then we have the V-weighted functor $F(\varphi) \colon F(B) \to F(A)$ and by applying the strict 2-functor $\mathrm{Func}'(-, \mathcal{D})$ to $F(\varphi)$ we obtain the functor

$$\mathrm{Func}'(F(\varphi), \mathcal{D}) \colon \mathrm{Func}'(F(A), \mathcal{D}) \to \mathrm{Func}'(F(B), \mathcal{D}).$$

We may evaluate this functor on the object G_A of the category $\mathrm{Func}'(F(A), \mathcal{D})$ to obtain the object

$$\varphi_! G_A := \mathrm{Func}'(F(\varphi), \mathcal{D})(G_A)$$

of the category $\mathrm{Func}'(F(B), \mathcal{D})$. (This notation is consistent with the theory of *opfibrations*, see for example [Shu08, Section 3].) So we now have two objects of the category $\mathrm{Func}'(F(B), \mathcal{D})$, the object G_B and $\varphi_! G_A$. Moreover, $\varphi \colon A \to B$ is a horizontal morphism $\varphi \colon F(\varphi)(b) \to b$ for any $b \in F(B)$. We will now see that the images of all of these horizontal morphisms under G_1 assemble to a morphism

$$G_\varphi \colon \varphi_! G_A \to G_B$$

of the category $\mathrm{Func}'(F(B), \mathcal{D})$ as we defined it in Definition 6.3. To this end, we define

$$G_{\varphi,b} := G_{1,F(\varphi)(b),b}(\varphi)$$

to be the image of the horizontal morphism $\varphi \colon F(\varphi)(b) \to b$ under G_1. So we now have a family

$$G_\varphi = \big(G_{\varphi,b} \colon G_0(F(\varphi)(b)) \to G_0(b)\big)_{b \in F(B)}$$

of morphisms of $\mathcal{D}_0$ as defined in Definition 4.2 indexed by the elements $b \in F(B)$. We note that we are abusing notation again as described in Remark 6.4 by considering G_φ a family of morphisms of $\mathcal{D}_0$. Moreover, we have

$$G_0(b) = G_{B,0}(b)$$

by definition (6.11) of G_B for $b \in F(B)$. Furthermore, we have

$$G_0(F(\varphi)(b)) = G_{A,0}(F(\varphi)(b)) = (\varphi_! G_A)_0(b).$$

So the family $G_\varphi = \big(G_{\varphi,b}\big)_{b\in F(B)}$ does indeed provide the data for a morphism $G_\varphi \colon \varphi_! G_A \to G_B$. We now show that the family G_φ also satisfies the necessary coherence condition (6.3).

Lemma 6.8. *The family $G_\varphi = \big(G_{\varphi,b}\colon (\varphi_! G_A)_0(b) \to G_{B,0}(b)\big)_{b\in F(B)}$ of morphisms of $\mathcal{D}_0$ is a morphism $G_\varphi \colon \varphi_! G_A \to G_B$ of the category* $\mathrm{Func}'(F(B), \mathcal{D})$.

Proof. Suppose we have elements $b, b' \in F(B)$ with the finite element of morphisms $\mathbf{d}(b,b') \in V$ of $F(B)$ and let $a := F(\varphi)(b)$ and $a' := F(\varphi)(b')$ be the corresponding elements of $F(A)$. If we view $F(B)$ as a V-weighted category, then we have a single morphism $\bullet\colon b \to b'$ of weight $w_{b,b'}(\bullet) = \mathbf{d}(b,b')$. So we have to show that

$$\begin{tikzcd}
(\varphi_! G_A)_0(b) \arrow[r, "G_{\varphi,b}"] \arrow[d, "(\varphi_! G_A)_{1,b,b'}(\bullet)"'] \arrow[dr, phantom, "\mathbf{d}(b,b')"] & G_{B,0}(b) \arrow[d, "G_{B,1,b,b'}(\bullet)"] \\
(\varphi_! G_A)_0(b') \arrow[r, "G_{\varphi,b'}"'] & G_{B,0}(b')
\end{tikzcd} \tag{6.13}$$

is a valid diagram in the sense of Section 5.4. Now we have

$$G_{B,1,b,b'}(\bullet) = G_{1,b,b'}(G'(B)(\bullet)) = G_{1,b,b'}(\mathrm{id}_B)$$

as well as

$$(\varphi_! G_A)_{1,b,b'}(\bullet) = G_{A,1,a,a'}(F(\varphi)(\bullet)) = G_{A,1,a,a'}(\bullet) = G_{1,a,a'}(G'(A)(\bullet)) = G_{1,a,a'}(\mathrm{id}_A)$$

by (3.3). Moreover, by the second coherence condition of Definition 6.3 for G we have the valid diagram

$$\begin{tikzcd}
G_0(a) \arrow[r, "G_{1,a,b}(\varphi)"] \arrow[d, "G_{1,a,a'}(\mathrm{id}_A)"'] \arrow[dr, "G_{1,a,b'}(\varphi)" description] & G_0(b) \arrow[d, "G_{1,b,b'}(\mathrm{id}_B)"] \\
G_0(a') \arrow[r, "G_{1,a',b'}(\varphi)"'] & G_0(b').
\end{tikzcd} \tag{6.14}$$

with $\mathbf{d}(b,b')$ in the upper triangle and $\mathbf{d}(a,a')$ in the lower triangle.

Furthermore, as $F(\varphi)\colon F(B) \to F(A)$ is a $\tilde{V}^{\circ}$-enriched functor we have $\mathbf{d}(a,a') \preceq \mathbf{d}(b,b')$. Thus, by pasting the two triangles of (6.14) we obtain the diagram (6.13). ☐

Restriction to Horizontal Morphisms as a Lax Cocone. By Lemma 6.7 there is a V-persistent functor $\tilde{G}\colon P_\bullet \mathrm{el}(F) \to \mathcal{D}$ such that

$$\mathrm{Func}'\big(\mathrm{el}(F), \tilde{G}\big)\big(H_{\mathrm{el}(F)}\big) = G,$$

where $H_{\mathrm{el}(F)}$ is the universal object of $\mathrm{Func}'(\mathrm{el}(F), P_\bullet \mathrm{el}(F))$ in the sense of Definition 6.6. Then we may consider the underlying ordinary functor $\tilde{G}_0\colon \mathrm{el}(F)_0 \to \mathcal{D}_0$. Moreover, we may restrict $\tilde{G}_0$ to the horizontal subcategory $\mathrm{el}(F)_h$ by Lemma 3.5:

$$\tilde{G}_h \coloneqq \tilde{G}_0|_{\mathrm{el}(F)_h}\colon \mathrm{el}(F)_h \to \mathcal{D}_0.$$

Forgetting the structure provided by the elements of morphisms $\mathbf{d}$ on the values of $F\colon \mathcal{B}^\circ \to \tilde{V}^\circ\text{-Cat}$ we now consider the functor

$$\bar{F}\colon \mathcal{B}^\circ \to \mathrm{Class} \hookrightarrow \mathrm{Cat},\ A \mapsto \bar{F}(A),$$

where $\bar{F}(A)$ is the underlying class of $F(A)$ for any object A of $\mathcal{B}$. By Lemma 3.8 the lax cocone $G' \in \mathrm{LCocone}(F, \mathrm{el}(F))$ defined by (3.3) restricts to an initial lax cocone $\bar{G} \in \mathrm{LCocone}\big(\bar{F}, \mathrm{el}(F)_h\big)$ under $\bar{F}\colon \mathcal{B}^\circ \to \mathrm{Class} \hookrightarrow \mathrm{Cat}$ with vertex $\mathrm{el}(F)_h$. Thus, if we apply the map

$$\mathrm{LCocone}\big(\bar{F}, \tilde{G}_h\big)\colon \mathrm{LCocone}\big(\bar{F}, \mathrm{el}(F)_h\big) \to \mathrm{LCocone}\big(\bar{F}, \mathcal{D}_0\big)$$

to the initial lax cocone $\bar{G}$, then we obtain the lax cocone

$$\vec{G} \coloneqq \mathrm{LCocone}\big(\bar{F}, \tilde{G}_h\big)\big(\bar{G}\big) \in \mathrm{LCocone}\big(\bar{F}, \mathcal{D}_0\big)$$

under $\bar{F}$ with vertex $\mathcal{D}_0$. As it turns out, we have $\vec{G}(A) = G_{A,0}$ for any object A of $\mathcal{B}$ and $\vec{G}(\varphi) = G_\varphi$ for any morphism φ of $\mathcal{B}$. We summarize this as a lemma.

Lemma 6.9. *The lax cocone $\vec{G} \in \mathrm{LCocone}\big(\bar{F}, \mathcal{D}_0\big)$ under $\bar{F}\colon \mathcal{B}^\circ \to \mathrm{Class} \hookrightarrow \mathrm{Cat}$ that is induced by the functor $\mathrm{el}(F)_h \to \mathcal{D}_0$ corresponding to the object G of $\mathrm{Func}'(\mathrm{el}(F), \mathcal{D})$ is the restriction of G to horizontal morphisms.*

Both Restrictions as an Element of a Lax Limit. If we now consider the notion of a 0-truncated lax limit as in Definition A.1, then we see that the restrictions to vertical morphisms G_A indexed by the objects A of $\mathcal{B}$ and the restrictions to horizontal morphisms G_φ indexed by the morphisms φ of $\mathcal{B}$ provide precisely the data needed for an element of the 0-truncated lax limit $\lim_0 \mathrm{Func}'(F(-), \mathcal{D}))$ of the functor

$$\mathrm{Func}'(F(-), \mathcal{D}))\colon \mathcal{B} \to \mathrm{CAT}_0.$$

We just have to show that the restrictions to horizontal morphisms also satisfy the coherence conditions of Definition A.1.

Lemma 6.10. *The restrictions to vertical morphisms G_A and to horizontal morphisms G_φ for objects A and morphisms φ of $\mathcal{B}$ respectively provide an element of the 0-truncated lax limit $\lim_0 \mathrm{Func}'(F(-), \mathcal{D}))$.*

Proof. We consider a pair of composable morphisms $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ in $\mathcal{B}$ as well as the induced homorphism

$$\psi_! G_\varphi \coloneqq \mathrm{Func}'(F(\psi), \mathcal{D}))(G_\varphi) \colon (\psi \circ \varphi)_! G_A \to \psi_! G_B$$

in the category $\mathrm{Func}'(F(C), \mathcal{D})$. We have to show that the diagram

$$\begin{tikzcd}
(\psi \circ \varphi)_! G_A \arrow[d, "\psi_! G_\varphi"'] \arrow[dr, "G_{\psi\circ\varphi}"] & \\
\psi_! G_B \arrow[r, "G_\psi"'] & G_C
\end{tikzcd} \tag{6.15}$$

commutes. Now the restriction to horizontal morphisms is a lax cocone under $\bar{F} \colon \mathcal{B}^\circ \to \mathrm{Class} \hookrightarrow \mathrm{Cat}$ with vertex $\mathcal{D}_0$ by Lemma 6.9. Thus, we have the commutative diagram

$$\begin{tikzcd}
G_{A,0} \circ \bar{F}(\psi \circ \varphi) \arrow[d, Rightarrow, "\psi_! G_\varphi"'] \arrow[dr, Rightarrow, "G_{\psi\circ\varphi}"] & \\
G_{B,0} \circ \bar{F}(\psi) \arrow[r, Rightarrow, "G_\psi"'] & G_C
\end{tikzcd} \tag{6.16}$$

of functors $\bar{F}(C) \to \mathcal{D}_0$ and natural transformations by the coherence conditions of Definition A.2. Now the only difference between (6.15) and (6.16) is in the objects, the morphisms are identical. As the composition in $\mathrm{Func}'(F(C), \mathcal{D})$ agrees with the composition of natural transformations between functors $\bar{F}(C) \to \mathcal{D}_0$ as well, the commutativity of (6.16) implies the commutativity of (6.15). $\square$

We may also describe the element of the 0-truncated lax limit $\lim_0 \mathrm{Func}'(F(-), \mathcal{D})$ corresponding to vertical and horizontal restrictions of the object G of the category $\mathrm{Func}'(\mathrm{el}(F), \mathcal{D})$ at a more conceptual level. More specifically, by Remark A.4 we may view the lax cocone $G' \in \mathrm{LCocone}(F, \mathrm{el}(F))$ defined by (3.3) as an element of the 0-truncated lax limit $\lim_0 V\mathrm{Func}(F(-), \mathrm{el}(F))$ of the functor $V\mathrm{Func}(F(-), \mathrm{el}(F)) \colon \mathcal{B} \to \mathrm{CAT}_0$, where $V\mathrm{Func}$ denotes the bifunctor assigning to any two V-weighted categories the locally large category of V-weighted functors and V-natural transformations. Thus, in order to obtain an element of the 0-truncated lax limit $\lim_0 \mathrm{Func}'(F(-), \mathcal{D})$ from the object G of the category $\mathrm{Func}'(\mathrm{el}(F), \mathcal{D})$, it suffices to provide a natural transformation

$$V\mathrm{Func}(F(-), \mathrm{el}(F)) \Longrightarrow \mathrm{Func}'(F(-), \mathcal{D})$$

of functors $\mathcal{B} \to \mathrm{CAT}_0$ in terms of G. Now the pointwise application of the functor $\mathrm{Func}'(?, \mathcal{D})$ to V-weighted functors yields a natural transformation

$$\Delta(\mathrm{Func}'(?, \mathcal{D})) \colon V\mathrm{Func}(F(-), \mathrm{el}(F)) \Rightarrow \mathrm{Func}(\mathrm{Func}'(\mathrm{el}(F), \mathcal{D}), \mathrm{Func}'(F(-), \mathcal{D})) \tag{6.17}$$

and the pointwise evaluation at G a natural transformation

$$\Delta(\mathrm{ev}_G)\colon \mathrm{Func}(\mathrm{Func}'(\mathrm{el}(F),\mathcal{D}),\mathrm{Func}'(F(-),\mathcal{D}) \Longrightarrow \mathrm{Func}'(F(-),\mathcal{D}). \tag{6.18}$$

Thus, by composing the natural transformations (6.17), and (6.18) we obtain a natural transformation $V\mathrm{Func}(F(-),\mathrm{el}(F)) \Longrightarrow \mathrm{Func}'(F(-),\mathcal{D})$.

Proposition 6.11. *The map from the objects of* $\mathrm{Func}'(\mathrm{el}(F),\mathcal{D})$ *to the* 0*-truncated lax limit* $\lim_0 \mathrm{Func}'(F(-),\mathcal{D})$ *obtained by restriction to vertical and horizontal morphisms as just described is a bijection.*

Proof. Injectivity follows from the coherence conditions of Definition 6.3 and the fact that any morphism of $\mathrm{el}(F)$ factors into a vertical morphism of the same weight and a horizontal morphism by Remark 3.4. Thus, it suffices to show that any element of the 0-truncated lax limit $\lim_0 \mathrm{Func}'(F(-),\mathcal{D})$, provided as an assignment of an object G_A of $\mathrm{Func}'(F(A),\mathcal{D})$ for each object A of $\mathcal{B}$ and a homorphism

$$G_\varphi\colon \varphi_! G_A \to G_B$$

in $\mathrm{Func}'(F(B),\mathcal{D})$ for each morphism $\varphi\colon A \to B$ of $\mathcal{B}$, determines an object $G = (G_0, G_1)$ of $\mathrm{Func}'(\mathrm{el}(F),\mathcal{D})$. To simplify our notation, we set

$$\check{G}(a,a') \coloneqq G_{A,1,a,a'}(\bullet)$$

for elements $a, a' \in F(A)$ and an object A of $\mathcal{B}$ as in (6.12). We start with defining the map G_0 on the level of objects

$$G_0(a) \coloneqq G_{A,0}(a)$$

for $a \in F(A)$ and A an object of $\mathcal{B}$. Now suppose we have a morphism $\varphi\colon a \to b$ of $\mathrm{el}(F)$ for elements $a \in F(A)$, $b \in F(B)$ and objects A, B of $\mathcal{B}$. Then let $a' \coloneqq F(\varphi)(b)$ and let $w \coloneqq w_{a,b}(\varphi) = \mathbf{d}(a,a')$ be the weight of $\varphi\colon a \to b$. As we have the factorization

$$\begin{tikzcd}
a \arrow[d, "\mathrm{id}_A"'] \arrow[dr, "\varphi"] & \\
F(\varphi)(b) \arrow[r, "\varphi"'] & a
\end{tikzcd}$$

into a vertical and horizontal morphism, we may set

$$G_{1,a,b}(\varphi) \coloneqq G_\varphi(b) \;{}_0\!\diamond_w\; \check{G}(a,a') \in \mathrm{Hom}(G_0(a), G_0(b))(w),$$

which we may also express as the V-weighted diagram

$$\begin{tikzcd}
G_0(a) \arrow[d, "\check{G}(a,a')"'] \arrow[dr, "G_{1,a,b}(\varphi)"] & \\
G_0(a') \arrow[r, "G_\varphi(b)"'] \arrow[ur, phantom, "w_{a,b}(\varphi)"] & G_0(b)
\end{tikzcd} \tag{6.19}$$

in the locally V-persistent category $\mathcal{D}$. With $G = (G_0, G_1)$ we now have the data for an object of the category $\mathrm{Func}'(\mathrm{el}(F), \mathcal{D})$. It remains to be shown that the coherence conditions of Definition 6.3 are satisfied. To this end, let $a \xrightarrow{\varphi} b \xrightarrow{\psi} c$ be a pair of composable morphisms of $\mathrm{el}(F)$ for elements $a \in F(A)$, $b \in F(B)$, $c \in F(C)$ and objects A, B, C of $\mathcal{B}$. Moreover, let

$$a' \coloneqq F(\varphi)(b), \quad b' \coloneqq F(\psi)(c), \quad \text{and} \quad a'' \coloneqq F(\psi \circ \varphi)(c).$$

We have to show that

$$\begin{tikzcd}
G_0(a) \arrow[d, "G_{1,a,b}(\varphi)"'] \arrow[dr, "G_{1,a,c}(\psi\circ\varphi)"] \arrow[dr, phantom, "w_1", shift right=4ex] & \\
G_0(b) \arrow[r, "G_{1,b,c}(\psi)"'] & G_0(c)
\end{tikzcd} \tag{6.20}$$

is a valid diagram in the sense of Section 5.4, where

$$w_1 \coloneqq w_{b,c}(\psi) + w_{a,b}(\varphi) = \mathbf{d}(b, b') + \mathbf{d}(a, a').$$

To this end, we consider the diagram

$$\begin{tikzcd}
G_0(a) \arrow[d, "\check{G}(a{,}a')"] \arrow[dr, "G_{1,a,b}(\varphi)"] \arrow[dd, bend right=60, "\check{G}(a{,}a'')"'] \arrow[dr, phantom, "w_{a,b}(\varphi)", shift right=3ex] & & \\
G_0(a') \arrow[r, "G_\varphi(b)"] \arrow[d, "\check{G}(a'{,}a'')"] \arrow[l, phantom, "w_2"] \arrow[dr, phantom, "\mathbf{d}(b{,}b')"] & G_0(b) \arrow[d, "\check{G}(b{,}b')"] \arrow[dr, "G_{1,b,c}(\psi)"] \arrow[dr, phantom, "w_{b,c}(\psi)", shift right=3ex] & \\
G_0(a'') \arrow[r, "G_\varphi(b')"] \arrow[rr, bend right=30, "G_{\psi\circ\varphi}(c)"', "0"] & G_0(b') \arrow[r, "G_\psi(c)"] & G_0(c),
\end{tikzcd} \tag{6.21}$$

where $w_2 \coloneqq \mathbf{d}(a', a'') + \mathbf{d}(a, a')$. By definition of G_1 or by reinstantiating diagram (6.19) we see that the two right triangles of (6.21) are valid diagrams. The square in (6.21) is a valid diagram by (6.3) and our assumption that $G_\varphi \colon \varphi_! G_A \to G_B$ is a morphism in the category $\mathrm{Func}'(F(B), \mathcal{D})$. The degenerate triangle on the left is a valid diagram by our assumption that G_A is an object of the category $\mathrm{Func}'(F(A), \mathcal{D})$ and the degenerate triangle at the bottom is valid by the coherence condition (A.1) for elements of the 0-truncated lax limit $\lim_0 \mathrm{Func}'(F(-), \mathcal{D})$. Thus, we may paste the subdiagrams of (6.21) to obtain the valid diagram

$$\begin{tikzcd}
G_0(a) \arrow[dr, "G_{1,a,b}(\varphi)"] \arrow[dd, "\check{G}(a{,}a'')"'] & & \\
 & G_0(b) \arrow[dr, "G_{1,b,c}(\psi)"] \arrow[d, phantom, "w_1"] & \\
G_0(a'') \arrow[rr, "G_{\psi\circ\varphi}(c)"'] & {} & G_0(c)
\end{tikzcd}$$

since

$$w_2 = \mathbf{d}(a', a'') + \mathbf{d}(a, a') = \mathbf{d}(F(\varphi(b)), F(\varphi)(b')) + \mathbf{d}(a, a') \preceq \mathbf{d}(b, b') + \mathbf{d}(a, a') = w_1.$$

Flipping this diagram over we obtain the valid diagram

$$\begin{tikzcd}
 & G_0(a) \arrow[d, "\check{G}(a{,}a'')"] \arrow[ddl, bend right, "G_{1,a,b}(\varphi)"'] & \\
 & G_0(a'') \arrow[r, "G_{\psi\circ\varphi}(c)"] & G_0(c). \\
G_0(b) \arrow[ur, phantom, "w_1"] \arrow[urr, bend right, "G_{1,b,c}(\psi)"'] & &
\end{tikzcd} \tag{6.22}$$

Moreover, if we instantiate the diagram (6.19) for the morphism $\psi \circ \varphi \colon a \to c$ of $\mathrm{el}(F)$, then we obtain the diagram

$$\begin{tikzcd}
G_0(a) \arrow[d, "\check{G}(a{,}a'')"'] \arrow[dr, "G_{1,a,c}(\psi\circ\varphi)"] \arrow[dr, phantom, shift right=3ex, "w_{a,c}(\psi\circ\varphi)"] & \\
G_0(a'') \arrow[r, "G_{\psi\circ\varphi}(c)"'] & G_0(c),
\end{tikzcd} \tag{6.23}$$

which is valid by definition of G_1. By pasting the valid diagrams (6.23) and (6.22) we obtain the diagram (6.20). $\square$

By combining Lemma 6.7 and Proposition 6.11 we obtain the following result.

Theorem 6.12. *The restrictions to vertical and horizontal morphisms of a V-weighted category of elements* $\mathrm{el}(F)$ *as in Definition 3.3 yield a bijection from the class of V-persistent functors* $P_\bullet \mathrm{el}(F) \to \mathcal{D}$ *to the elements of the* 0*-truncated lax limit* $\lim_0 \mathrm{Func}'(F(-), \mathcal{D})$.

6.4 Construction of Enrichment

Now let $V = \mathbb{R}^\circ \times \mathbb{R}$. In Section 6.2 we saw how we can use the base change along the lax monoidal functor $P \colon V\mathrm{Set} \to \mathrm{PSet}$ to turn any V-weighted category into a locally V-persistent category in a faithful way. In particular, we may turn the V-weighted category $(\mathrm{Top}/\mathbb{R})_w$ of $\mathbb{R}$-spaces into the V-persistent category $P_\bullet(\mathrm{Top}/\mathbb{R})_w$, whose underlying ordinary category (Definition 4.2) is equal to the category of $\mathbb{R}$-spaces $\mathrm{Top}/\mathbb{R}$ by Remark 3.7 and Lemma 6.2. Thus, the base change $P_\bullet(\mathrm{Top}/\mathbb{R})_w$ is a V-persistent enrichment of $\mathrm{Top}/\mathbb{R}$ in the sense of Definition 4.2. Moreover, we provided a V-persistent enrichment of the category of functors $\mathbb{M}_\mathrm{f} \to \mathcal{W}$ in Section 4.1. So we now have the necessary enrichments of domain and codomain to provide a V-persistent enrichment of the covariant functor $h^\circ \colon \mathrm{Top}/\mathbb{R} \to \mathcal{W}^{\mathbb{M}_\mathrm{f}}$ that we used in the construction of RISC in Section 1.1.4.

6.4.1 Persistent Enrichments of Functors on $\mathbb{R}$-Spaces

In the previous Section 6.3 we characterized locally V-persistent functors from a V-weighted category of elements in the sense of Definition 3.3. We will now apply these results to $(\mathrm{Top}/\mathbb{R})_w$ as a V-weighted category of elements of the contravariant functor

$$F \coloneqq \mathrm{Hom}(-, \mathbb{R}) \colon \mathrm{Top}^\circ \to \tilde{V}^\circ\text{-Cat}$$

to obtain a characterization of V-persistent enrichments of ordinary functors with domain $\mathrm{Top}/\mathbb{R}$. The reason why we specialize to $(\mathrm{Top}/\mathbb{R})_w$ for this purpose is that in general the horizontal subcategory of a V-weighted category of elements $\mathrm{el}(F)$ in the sense of Definition 3.3 and Lemma 3.5 is a proper subcategory of the underlying ordinary category $\mathrm{el}(F)_0$, whereas for $(\mathrm{Top}/\mathbb{R})_w$ both categories are identical by Remark 3.7. In conjunction with Lemma 6.2 we see that $\mathrm{Top}/\mathbb{R}$ is not only the underlying ordinary category of $(\mathrm{Top}/\mathbb{R})_w$, but also the underlying ordinary category of the locally V-persistent category $P_\bullet(\mathrm{Top}/\mathbb{R})_w$ in the sense of Definition 4.2. Thus, it makes sense to speak of V-persistent enrichments of functors with domain $\mathrm{Top}/\mathbb{R}$.

Now suppose that $\mathcal{D}$ is a locally V-persistent category and let $K \colon \mathrm{Top}/\mathbb{R} \to \mathcal{D}_0$ be an ordinary functor. In order to provide any V-persistent functor $P_\bullet(\mathrm{Top}/\mathbb{R})_w \to \mathcal{D}$ it suffices to provide an element of the 0-truncated lax limit $\lim_0 \mathrm{Func}'(F(-), \mathcal{D})$ by Theorem 6.12. However, we intend to provide a V-persistent enrichment of $K \colon \mathrm{Top}/\mathbb{R} \to \mathcal{D}_0$ and not just any V-persistent functor $P_\bullet(\mathrm{Top}/\mathbb{R})_w \to \mathcal{D}$. So some of the data for an element of the 0-truncated lax limit $\lim_0 \mathrm{Func}'(F(-), \mathcal{D})$ providing a V-persistent enrichment of $K \colon \mathrm{Top}/\mathbb{R} \to \mathcal{D}_0$ is predetermined by K, while the other part of the data has to be provided in addition. In the following we show that the horizontal part of such an element of the lax limit is predetermined, while the vertical part has to be provided in addition. To this end, let

$$\bar{F}\colon \mathrm{Top}^\circ \to \mathrm{Class} \hookrightarrow \mathrm{Cat},\ X \mapsto \mathrm{Hom}(X, \mathbb{R}),$$

be the contravariant functor (viewing each class as a discrete category) corresponding to $F\colon \mathrm{Top}^\circ \to \tilde{V}^\circ\text{-Cat}$ as in Lemma 3.8. Then we have the initial lax cocone $\bar{G} \in \mathrm{LCocone}\big(\bar{F}, \mathrm{Top}/\mathbb{R}\big)$ defined by (1.5) as well as the map of classes

$$\mathrm{LCocone}\big(\bar{F}, K\big)\colon \mathrm{LCocone}\big(\bar{F}, \mathrm{Top}/\mathbb{R}\big) \to \mathrm{LCocone}\big(\bar{F}, \mathcal{D}_0\big).$$

By applying this map $\mathrm{LCocone}\big(\bar{F}, K\big)$ to the initial lax cocone $\bar{G} \in \mathrm{LCocone}\big(\bar{F}, \mathrm{Top}/\mathbb{R}\big)$ we obtain the lax cocone

$$\vec{G} \coloneqq \mathrm{LCocone}\big(\bar{F}, K\big)\big(\bar{G}\big) \in \mathrm{LCocone}\big(\bar{F}, \mathcal{D}_0\big) \tag{6.24}$$

corresponding to the functor $K\colon \mathrm{Top}/\mathbb{R} \to \mathcal{D}_0$.

Now suppose we already had a V-persistent enrichment $\tilde{K}\colon P_\bullet(\mathrm{Top}/\mathbb{R})_w \to \mathcal{D}$ of $K\colon \mathrm{Top}/\mathbb{R} \to \mathcal{D}_0$, i.e. $\tilde{K}_0 = K$, and let

$$G \coloneqq \mathrm{Func}'\big(\mathrm{el}(F), \tilde{K}\big)\big(H_{\mathrm{el}(F)}\big)$$

be the corresponding object of the category $\mathrm{Func}'(\mathrm{el}(F), \mathcal{D}) = \mathrm{Func}'((\mathrm{Top}/\mathbb{R})_w, \mathcal{D})$, where $H_{\mathrm{el}(F)}$ is the universal object of $\mathrm{Func}'(\mathcal{C}, P_\bullet\mathcal{C})$ in the sense of Definition 6.6. Then $\vec{G}$ is the restriction of G to horizontal homomorphisms in $(\mathrm{Top}/\mathbb{R})_w$ by Lemma 6.9 and Remark 3.7. Moreover, the restrictions of G to vertical and horizontal homomorphisms provide the element of the 0-truncated lax limit $\lim_0 \mathrm{Func}'(F(-), \mathcal{D})$ that determines $\tilde{K}$ by Theorem 6.12. Thus, the missing ingredient that is needed in order to enrich $K\colon \mathrm{Top}/\mathbb{R} \to \mathcal{D}_0$ to a V-persistent functor $P_\bullet(\mathrm{Top}/\mathbb{R})_w \to \mathcal{D}$ is an assignment $\check{G}$ as in (6.12) completing $\vec{G}$ to an element of the 0-truncated lax limit $\lim_0 \mathrm{Func}'(F(-), \mathcal{D})$. More specifically, for any two functions $f, g\colon X \to \mathbb{R}$ with $\mathbf{d}(f, g) \in V$ we seek a weighted morphism $\check{G}(f, g) \in \mathrm{Hom}(K(f), K(g))(\mathbf{d}(f, g))$ of weight $\mathbf{d}(f, g)$ of the locally V-persistent category $\mathcal{D}$ such that the assignment $\check{G}$ satisfies the two notions of *compatibility* of the following definition.

Definition 6.13. Let $\mathcal{D}$ be a locally V-persistent category, let $K\colon \mathrm{Top}/\mathbb{R} \to \mathcal{D}_0$ be an ordinary functor, and let $\vec{G}$ be the corresponding lax cocone under the contravariant functor

$$\mathrm{Hom}(-, \mathbb{R})\colon \mathrm{Top}^\circ \to \mathrm{Set} \hookrightarrow \mathrm{Cat}$$

with vertex $\mathcal{D}_0$. Moreover, suppose we have an assignment

$$\check{G}_X(f, g) \coloneqq \check{G}(f, g) \in \mathrm{Hom}(K(f), K(g))(\mathbf{d}(f, g))$$

for any two functions $f, g\colon X \to \mathbb{R}$ with $\mathbf{d}(f, g) \in V$. We say that the assignment $\check{G}$ is *compatible with vertical composition* if each assignment $\check{G}_X$ is an object of the category

$\operatorname{Func}'(\operatorname{Hom}(X,\mathbb{R}),\mathcal{D})$ for any non-empty space X. Moreover, we say that $\check{G}$ is *compatible with precomposition* if for any continuous map $\varphi\colon X \to Y$ the natural transformation

$$\vec{G}(\varphi)\colon \vec{G}(X)(- \circ \varphi) \Longrightarrow \vec{G}(Y)$$

between functors on the discrete category $\operatorname{Hom}(Y,\mathbb{R})$ is a momorphism

$$\vec{G}(\varphi)\colon \varphi_! \check{G}_X \longrightarrow \check{G}_Y$$

of the category $\operatorname{Func}'(\operatorname{Hom}(Y,\mathbb{R}),\mathcal{D})$.

Note that (as in the proof of Lemma 6.10) the coherence condition (A.1), which is necessary for $\check{G}$ and $\vec{G}$ to provide an element of the 0-truncated lax limit $\lim_0 \operatorname{Func}'(F(-),\mathcal{D}))$, is unconditionally satisfied as $\vec{G} \in \operatorname{LCocone}(\bar{F},\mathcal{D}_0)$ is a lax cocone. In summary, we obtain the following result.

Proposition 6.14. *Let $\mathcal{D}$ be a locally V-persistent category and let $K\colon \mathrm{Top}/\mathbb{R} \to \mathcal{D}_0$ be an ordinary functor. Moreover, suppose we have an assignment*

$$\check{G}(f,g) \in \operatorname{Hom}(K(f),K(g))(\mathbf{d}(f,g))$$

for any two functions $f,g\colon X \to \mathbb{R}$ with $\mathbf{d}(f,g) \in V$ that is compatible with both, vertical composition and precomposition in the sense of Definition 6.13. Then there is a unique V-persistent enrichment $\tilde{K}\colon P_\bullet(\mathrm{Top}/\mathbb{R})_w \to \mathcal{D}$ of $K\colon \mathrm{Top}/\mathbb{R} \to \mathcal{D}_0$ such that the restriction of $\tilde{K}$ to vertical homomorphisms agrees with $\check{G}$.

Proof. Let $\vec{G}$ be the lax cocone corresponding to $K\colon \mathrm{Top}/\mathbb{R} \to \mathcal{D}_0$ under the contravariant functor

$$\operatorname{Hom}(-,\mathbb{R})\colon \mathrm{Top}^\circ \to \mathrm{Set} \hookrightarrow \mathrm{Cat}$$

with vertex $\mathcal{D}_0$ as defined by (6.24). By compatibility of $\check{G}$ with vertical composition and precomposition the assignment $\check{G}$ and the lax cocone $\vec{G}$ provide an element of the 0-truncated lax limit $\lim_0 \operatorname{Func}'(F(-),\mathcal{D})$, hence there is a V-persistent functor $\tilde{K}\colon P_\bullet(\mathrm{Top}/\mathbb{R})_w \to \mathcal{D}$ that agrees with $\check{G}$ on vertical homomorphisms and with $\vec{G}$ on horizontal homomorphisms by Theorem 6.12. Moreover, $\vec{G}$ is the lax cocone corresponding to $\tilde{K}_0$ by Lemma 6.9 and Remark 3.7. As $\vec{G}$ also is the lax cocone corresponding to the functor $K\colon \mathrm{Top}/\mathbb{R} \to \mathcal{D}_0$ by definition (6.24) and as the lax cocone $\bar{G}$ with vertex $\mathrm{Top}/\mathbb{R}$ is initial, we have $\tilde{K}_0 = K$. $\square$

We now unravel Definition 6.3 to obtain characterizations of the two notions of compatibility introduced in Definition 6.13 in terms of V-weighted diagrams in the sense of Section 5.4.

Lemma 6.15. *The assignment $\check{G}$ is compatible with vertical composition iff for any three functions $f,f',f''\colon X \to \mathbb{R}$ with $\mathbf{d}(f,f'),\mathbf{d}(f',f'') \in V$ we to have the equation*

$$\check{G}(f,f) = \mathrm{id}_{K(f)} \tag{6.25}$$

as well as the valid diagram

$$\begin{tikzcd} K(f) \arrow[d, "\check{G}(f,f')"] \arrow[dd, bend right=60, "\check{G}(f,f'')"'] \\ K(f') \arrow[d, "\check{G}(f',f'')"] \arrow[l, phantom, "w"] \\ K(f''), \end{tikzcd} \tag{6.26}$$

where $w \coloneqq \mathbf{d}(f',f'') + \mathbf{d}(f,f')$.

Proof. Let $\check{G}_X$ be the restriction of $\check{G}$ to functions on the space X. Then the equations (6.25) and (6.26) correspond to the coherence conditions (1) and (2) of Definition 6.3. $\square$

We note that we draw the arrows of diagram (6.26) vertically as they represent images of vertical homomorphisms in $(\mathrm{Top}/\mathbb{R})_w$ retaining visual proximity to diagram (6.21) in the proof of Proposition 6.11.

Lemma 6.16. *The assignment $\check{G}$ is compatible with precomposition iff for any continuous map $\varphi\colon X \to Y$ and any two continuous functions $f,g\colon Y \to \mathbb{R}$ with $\mathbf{d}(f,g) \in V$ we have the valid diagram*

$$\begin{tikzcd} K(f\circ\varphi) \arrow[r, "\vec{G}(\varphi)_f"] \arrow[d, "\check{G}(f\circ\varphi,\, g\circ\varphi)"'] \arrow[dr, phantom, "\mathbf{d}(f,g)"] & K(f) \arrow[d, "\check{G}(f,g)"] \\ K(g\circ\varphi) \arrow[r, "\vec{G}(\varphi)_g"'] & K(g). \end{tikzcd} \tag{6.27}$$

Proof. Let $\check{G}_X$ and $\check{G}_Y$ be the restrictions of $\check{G}$ to functions on X and Y respectively. Then in order for

$$\vec{G}(\varphi)\colon \varphi_!\check{G}_X \longrightarrow \check{G}_Y$$

to be a momorphism of the category $\mathrm{Func}'(\mathrm{Hom}(Y,\mathbb{R}),\mathcal{D})$ we need to have that the diagram (6.27) is valid as it is diagram (6.3) of Definition 6.3 instantiated at the objects $\varphi_!\check{G}_X, \check{G}_Y$ of the category $\mathrm{Func}'(\mathrm{Hom}(Y,\mathbb{R}),\mathcal{D})$ and the objects $f,g\colon Y \to \mathbb{R}$ of the V-weighted category $\mathrm{Hom}(Y,\mathbb{R})$. $\square$

6.4.2 Persistent Enrichment of RISC

We now instantiate the results of the previous Section 6.4.1 to provide a V-persistent enrichment of the covariant functor $h^\circ\colon \mathrm{Top}/\mathbb{R} \to \mathcal{W}^{\mathbb{M}_\mathrm{f}}$ in the sense of Definition 4.2. More specifically, by instantiating Proposition 6.14 with h° for K we see that in order to provide a V-persistent enrichment of $h^\circ\colon \mathrm{Top}/\mathbb{R} \to \mathcal{W}^{\mathbb{M}_\mathrm{f}}$ it suffices to provide an *induced* natural transformation

$$\check{h}^\circ(f,g)\colon h^\circ(f) \to h^\circ(g)(a.\_)$$

for any two functions $f, g\colon X \to \mathbb{R}$ with $a \coloneqq \mathbf{d}(f,g) \in V$ such that $\check{h}^\circ$ is compatible with vertical composition and precomposition in the sense of Definition 6.13.

Construction of Induced Natural Transformations. Now let $f, g\colon X \to \mathbb{R}$ be continuous functions with $\mathbf{d}(f,g) \in \mathbb{R}^\circ \times \mathbb{R}$. In the following we construct the required natural transformation

$$\check{h}^\circ(f,g)\colon h(f)^\circ \to h(g)^\circ(\mathbf{d}(f,g).\_).$$

To this end, let $F\colon \mathbb{M}_\mathrm{f} \to \mathcal{W}^\mathbb{Z}$ be defined as in the construction of $h^\circ(f)\colon \mathbb{M}_\mathrm{f} \to \mathcal{W}$ in Section 1.1.4. Or, in other words, we define F to be the strictly stable functor corresponding to $h^\circ(f)$ under the isomorphism of categories from Corollary 1.14. Completely analogously we have a functor $G\colon \mathbb{M}_\mathrm{f} \to \mathcal{W}^\mathbb{Z}$ as we would use it in the construction of $h^\circ(g)\colon \mathbb{M}_\mathrm{f} \to \mathcal{W}$. We use Lemma 1.24 to construct a natural transformation $\varphi\colon F \to G(w.\_)$ and then we obtain $\check{h}^\circ(f,g)\colon h^\circ(f) \to h^\circ(g)(a.\_)$ as $\check{h}^\circ(f,g) \coloneqq \mathrm{ev}_0 \circ \varphi$ by whiskering with the evaluation at 0, analogously to the construction of $h^\circ(f)$ from F, see also Corollary 1.14. Now suppose $a = (x,y) \in \mathbb{R}^\circ \times \mathbb{R}$ with $\mathbf{d}(f,g) \preceq a$ and $(r,s) \subseteq \mathbb{R}$ is an open interval. (In the proof of Lemma 6.21 below, it will be necessary to consider a situation where $\mathbf{d}(f,g) \neq a$.) Then we have

$$f^{-1}(r,s) \subseteq g^{-1}(x+r, y+s)$$

and thus

$$\big(f^{-1} \circ \rho\big)(u) \subseteq \big(g^{-1} \circ \rho\big)(a.u)$$

for all $u \in \mathbb{M}_\mathrm{f}$, where $\rho = (\rho_1, \rho_0)$ is the assignment of points in $\mathbb{M}_\mathrm{f}$ to pairs of open subspaces of $\mathbb{R}$ characterized by Proposition 1.2. As in [BdSS15, Section 3], this induces a morphism

$$\big(\mathcal{H}_\bullet \circ f^{-1} \circ \rho\big)(u) \to \big(\mathcal{H}_\bullet \circ g^{-1} \circ \rho\big)(a.u)$$

of $\mathcal{W}$, which is natural in $u \in \mathbb{M}_\mathrm{f}$. Now we have

$$\begin{aligned} F(u) &= \big(\mathcal{H}_\bullet \circ f^{-1} \circ \rho\big)(u) && \text{for all } u \in D \text{ and} \\ G(a.u) &= \big(\mathcal{H}_\bullet \circ g^{-1} \circ \rho\big)(a.u) && \text{for all } u \in E \coloneqq \alpha(-a)(D), \end{aligned}$$

see also Fig. 6.1. Thus, we may restrict F and $G(a.\_)$ to the intersection $D \cap E$ to obtain the natural transformation

$$\eta\colon F|_{D\cap E} \to G(a.\_)|_{D\cap E}.$$

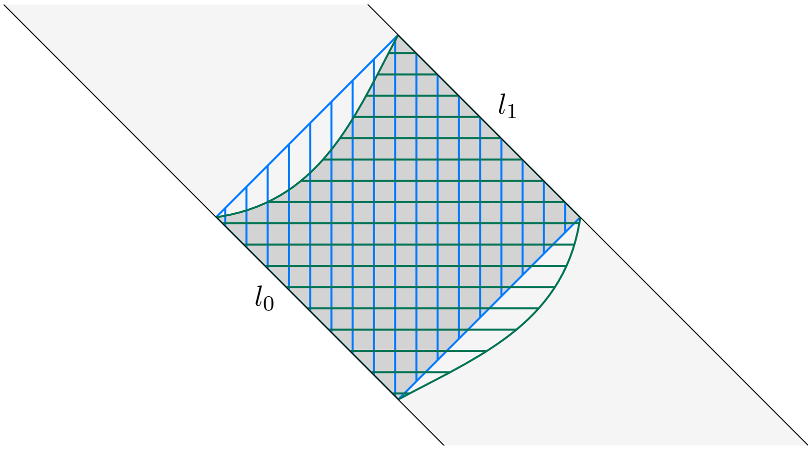


Figure 6.1: The fundamental domain D, its shift E by $\alpha(-a)$, and their intersection shaded in grey.

As shown in Fig. 6.1, the intersection $D \cap E$ is no fundamental domain in general, so η does not describe a natural transformation $F \to G(a.\_)$. In order to extend η to a natural transformation $\varphi\colon F \to G(a.\_)$ we use Lemma 1.24. So we need to supply the missing ingredient, which is a natural transformation

$$\nu\colon F|_{D\cap\Sigma(E)} \to G(a.\_)|_{D\cap\Sigma(E)}.$$

To this end, we consider the monotone map

$$\xi\colon u \mapsto \big((g^{-1} \circ \rho_1)(a.u), (f^{-1} \circ \rho_0)(u)\big)$$

from $\mathbb{M}_\mathrm{f}$ to the set of pairs of open subspaces of X. In some sense ξ interpolates between $f^{-1} \circ \rho$ and $\big(g^{-1} \circ \rho\big)(a.\_)$ since we have the chain of inclusions

$$f^{-1} \circ \rho \subseteq \xi \subseteq \big(g^{-1} \circ \rho\big)(a.\_)$$

pointwise in $\mathbb{M}_\mathrm{f}$ and moreover, ξ agrees with $f^{-1} \circ \rho$ when restricted to the region shaded in red in Fig. 6.2. In particular we have

$$\left(\mathcal{H}_\bullet \circ \xi\right)(u) = \left(\mathcal{H}_\bullet \circ f^{-1} \circ \rho\right)(u) = F(u) \tag{6.28}$$

for any point $u \in D$ contained in the red region. Furthermore, ξ and $\big(g^{-1} \circ \rho\big)(a.\_)$ agree, when restricted to the region shaded in blue. Thus, if $\Sigma^{-1}(v)$ is contained in the blue

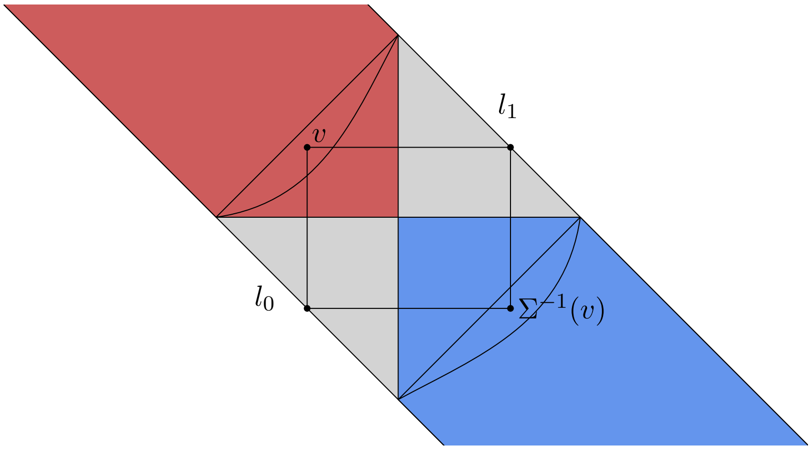


Figure 6.2: The maps ξ and $f^{-1} \circ \rho$ coincide on the red region, whereas ξ and $(g^{-1} \circ \rho)(a.\_)$ agree on the blue region.

region for some $v \in \Sigma(E)$, then

$$\begin{aligned}
\left(\mathcal{H}_{\bullet-1} \circ \xi \circ \Sigma^{-1}\right)(v) &= \left(\mathcal{H}_{\bullet-1} \circ \xi\right)\left(\Sigma^{-1}(v)\right) \\
&= \left(\mathcal{H}_{\bullet-1} \circ g^{-1} \circ \rho\right)\left(a.\Sigma^{-1}(v)\right) \\
&= \left(\Sigma \circ \mathcal{H}_{\bullet} \circ g^{-1} \circ \rho\right)\left(a.\Sigma^{-1}(v)\right) \\
&= (\Sigma \circ G)\left(a.\Sigma^{-1}(v)\right) \\
&= (\Sigma \circ G)\left(\Sigma^{-1}(a.v)\right) \\
&= \left(\Sigma \circ G \circ \Sigma^{-1}\right)(a.v) \\
&= G(a.v).
\end{aligned} \tag{6.29}$$

Here the fourth equality of (6.29) follows from $a.\Sigma^{-1}(v) \in D$ and the construction of G or rather $h^{\circ}(g) \colon \mathbb{M}_{\mathrm{f}} \to \mathcal{W}$, the fifth equality follows from Lemma 4.4, and the last equality from $G \colon \mathbb{M}_{\mathrm{f}} \to \mathcal{W}^{\mathbb{Z}}$ being strictly stable. By Proposition 1.2.(3-4) the map ξ preserves the joins and meets of any axis-aligned rectangle contained in $D \cup E$. Thus, any axis-aligned rectangle in $D \cup E$ gives rise to some triad of pairs of open subspaces of X. In particular, if $v \in D \cap \Sigma(E)$, then the axis-aligned rectangle shown in Fig. 6.2 gives rise to a triad T of pairs of open subspaces of X with $\xi(v)$ as its first component and $\left(\xi \circ \Sigma^{-1}\right)(v)$ the intersection of the other two components, hence we have the boundary operator

$$\partial_T \colon \left(\mathcal{H}_{\bullet} \circ \xi\right)(v) \to \left(\mathcal{H}_{\bullet-1} \circ \xi \circ \Sigma^{-1}\right)(v)$$

of the Mayer–Vietoris functor $\mathcal{H}_{\bullet}$ at the triad T. Combining this with the equations (6.28) and (6.29) we define

$$\nu_v := \partial_T \colon F(v) \to G(a.v).$$

The naturality of $\nu \colon F|_{D \cap \Sigma(E)} \to G(a.\_)|_{D \cap \Sigma(E)}$ follows from the naturality of the boundary operator ∂ of the Mayer–Vietoris functor $\mathcal{H}_{\bullet}$.

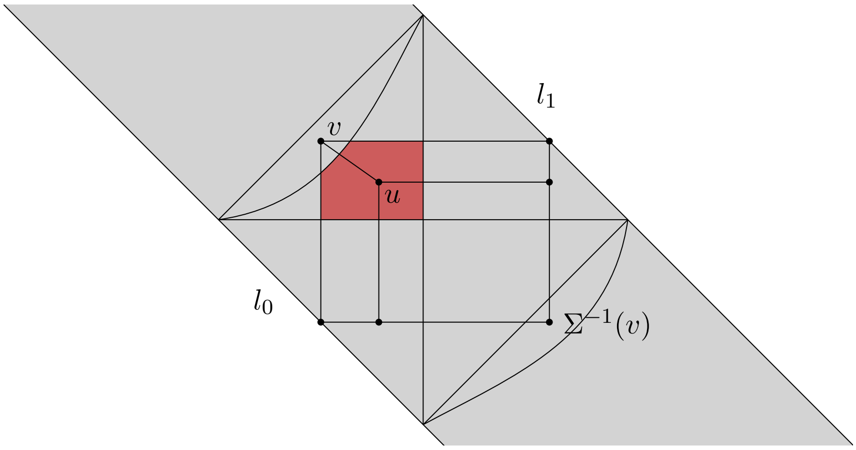


Figure 6.3: The points u and v with corresponding axis-aligned rectangles in $D \cup E$.

As $D \subset E \cup \Sigma(E)$, the natural transformations $\eta\colon F|_{D\cap E} \to G(a._)|_{D\cap E}$ and $\nu\colon F|_{D\cap\Sigma(E)} \to G(a._)|_{D\cap\Sigma(E)}$ provide the required data to fully determine a strictly stable natural transformation $\varphi\colon F \to G(a._)$. It remains to show that the necessary coherence conditions for comparable points of $\mathbb{M}_\mathrm{f}$ in different tiles of the tessellation induced by $D = D \cap E$ and $D \cap \Sigma(E)$ are satisfied as well.

Proposition 6.17. *There is a unique strictly stable natural transformation $\varphi\colon F \to G(a._)$ that agrees with η on $D \cap E$ and with ν on $D \cap \Sigma(E)$.*

Proof. We have to show that the natural transformations μ and ν satisfy the assumptions of Lemma 1.24. This in turn follows directly from Lemmas 6.18 and 6.19 below. ☐

Lemma 6.18. *The natural transformations η and ν satisfy property (1) of Lemma 1.24.*

Proof. It suffices to show that the solid square in

$$\begin{tikzcd} F(v) \arrow[r, "\nu_v = \partial_T"] & G(a.v) \\ F(u) \arrow[u, "F(u \preceq v)"] \arrow[r, "\eta_u"'] \arrow[ur, dashed, "\partial_{T'}"] & G(a.u) \arrow[u, "G(a.u \preceq a.v)"'] \end{tikzcd} \tag{6.30}$$

commutes for all u contained in the region shaded in red in Fig. 6.3. To this end, we consider the axis-aligned rectangle in Fig. 6.3 with u the upper left vertex. As ξ preserves the corresponding join u and the meet $\Sigma^{-1}(v)$, this axis-aligned rectangle gives rise to a triad T' with $\xi(u)$ as its first component and $\left(\xi \circ \Sigma^{-1}\right)(v)$ the intersection of the other two components, hence we have the boundary operator

$$\partial_{T'}\colon\ (\mathcal{H}_\bullet \circ \xi)\,(u) \to \left(\mathcal{H}_{\bullet-1} \circ \xi \circ \Sigma^{-1}\right)(v)$$

of the Mayer–Vietoris functor $\mathcal{H}_\bullet$ at the triad T'. By equations (6.28) and (6.29) the domain and codomain of $\partial_{T'}$ match up with the dashed arrow in (6.30). Now the upper

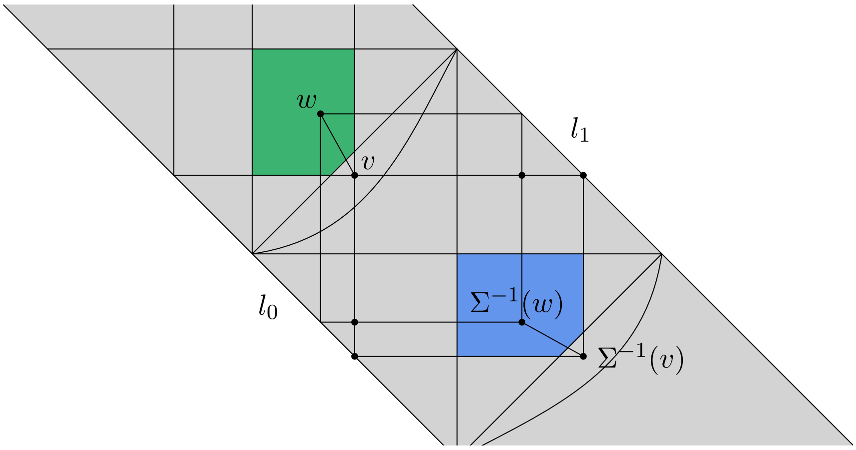


Figure 6.4: The points v and w with corresponding axis-aligned rectangles in $D \cup E$.

triangle in (6.30) commutes by the naturality of the boundary operator ∂ for the Mayer–Vietoris functor $\mathcal{H}_\bullet$. Moreover, we have pointwise $\xi \subseteq \big(g^{-1} \circ \rho\big)(a.\_)$ and thus, using the naturality of the boundary operator ∂ once again, we obtain the commutativity of the lower triangle in (6.30). $\square$

Lemma 6.19. *The natural transformations η and ν satisfy property (2) of Lemma 1.24.*

Proof. It suffices to show that the solid square in

$$\begin{tikzcd} F(w) \arrow[r, "(\Sigma\circ\eta\circ\Sigma^{-1})_w"] & G(a.w) \\ F(v) \arrow[u, "F(v\preceq w)"] \arrow[r, "\nu_v=\partial_T"'] \arrow[ur, dashed, "\partial_{T''}"] & G(a.v) \arrow[u, "G(a.v\preceq a.w)"'] \end{tikzcd} \tag{6.31}$$

commutes for all w contained in the region shaded in green in Fig. 6.4. To this end, we consider the axis-aligned rectangle shown in Fig. 6.4 with v and $\Sigma^{-1}(w)$ vertices. As ξ preserves the join v and the meet $\Sigma^{-1}(w)$, this axis-aligned rectangle gives rise to a triad T'' with $\xi(v)$ as its first component and $\left(\xi \circ \Sigma^{-1}\right)(w)$ the intersection of the other two components, hence we have the boundary operator

$$\partial_{T''} \colon \left(\mathcal{H}_\bullet \circ \xi\right)(v) \to \left(\mathcal{H}_{\bullet-1} \circ \xi \circ \Sigma^{-1}\right)(w)$$

of the Mayer–Vietoris functor $\mathcal{H}_\bullet$ at the triad T''. By equations (6.28) and (6.29) the domain and the codomain of $\partial_{T''}$ match up with the dashed arrow in (6.31). The lower triangle in (6.31) commutes by the naturality of the boundary operator ∂ for the Mayer–Vietoris functor $\mathcal{H}_\bullet$. Moreover, we have pointwise $f^{-1} \circ \rho \subseteq \xi$ and thus, once again using the naturality of the boundary operator ∂, we obtain the commutativity of the upper triangle in (6.31). $\square$

Now by Proposition 6.17 there is a unique strictly stable natural transformation $\varphi\colon F \to G(a.\_)$ that agrees with η on $D \cap E$ and with ν on $D \cap \Sigma(E)$.

Definition 6.20 (Induced Natural Transformation)**.** For any two continuous functions $f, g\colon X \to \mathbb{R}$ with $a \coloneqq \mathbf{d}(f, g) \in \mathbb{R}^\circ \times \mathbb{R}$ their *induced* natural transformation $\check{h}^\circ(f, g)\colon h(f)^\circ \to h(g)^\circ(a.\_)$ is obtained by whiskering with the evaluation at zero: $\check{h}^\circ(f, g) \coloneqq \mathrm{ev}_0 \circ \varphi$.

Compatibility with Vertical Composition. We now show that the induced natural transformations just defined respect vertical composition in the sense of Definition 6.13. Spelling this out for h°, this means that for any non-empty space X the restriction $\check{h}^\circ_X$ of $\check{h}^\circ$ to functions on X is an object of the category $\mathrm{Func}'\big(\mathrm{Hom}(X, \mathbb{R}), \mathcal{W}^{\mathbb{M}_\mathrm{f}}\big)$. To avoid any confusion, we should note that in the reasoning that follows, we will use two different types of diagrams, ordinary commutative diagrams of functors and natural transformations as well as V-weighted diagrams in the locally V-persistent category of functors $\mathbb{M}_\mathbf{f} \to \mathcal{W}$ in the sense of Section 5.4, which have V-weighted 2-cells.

Lemma 6.21 (Compatibility with Vertical Composition)**.** *The assignment $\check{h}^\circ$ of induced natural transformations is compatible with vertical composition in the sense of Definition 6.13.*

Proof. We show the characterization provided by Lemma 6.15. To this end, suppose we have continuous functions $f, f', f''\colon X \to \mathbb{R}$ with $\mathbf{d}(f, f'), \mathbf{d}(f', f'') \in \mathbb{R}^\circ \times \mathbb{R}$. Then we have the equation $\check{h}^\circ(f, f) = \mathrm{id}_{h^\circ(f)}$. Thus, it remains to be shown that the diagram

$$\begin{tikzcd}
 & h^\circ(f) \arrow[d, "\check{h}^\circ(f,f')"] \arrow[dd, bend right=60, "\check{h}^\circ(f,f'')"'] \\
 & h^\circ(f') \arrow[d, "\check{h}^\circ(f',f'')"] \arrow[l, phantom, "w"] \\
 & h^\circ(f'')
\end{tikzcd} \tag{6.32}$$

is valid in the locally V-persistent category of functors $\mathbb{M}_\mathrm{f} \to \mathcal{W}$. To this end, let $a \coloneqq \mathbf{d}(f, f')$, $b \coloneqq \mathbf{d}(f', f'')$, and $c \coloneqq \mathbf{d}(f, f'')$. Then we have $w = b + a$ in particular.

Now the composition of natural transformation

$$\begin{tikzcd} h^\circ(f) \arrow[r, "\check{h}^\circ(f,f')"] \arrow[rr, bend right, "\check{h}^\circ(f',f'')_b \diamond_a \check{h}^\circ(f,f')"'] & h^\circ(f')(a._) \arrow[r, "\check{h}^\circ(f',f'')_{a._}"] & h^\circ(f'')(w._) \end{tikzcd} \tag{6.33}$$

has weight w as a weighted morphism of $\mathcal{W}^{\mathbb{M}_\mathrm{f}}$. Whereas $\check{h}^\circ(f, f'')$ has weight $c \preceq w$ by the triangle inequality (3.1). So we need to shift $\check{h}^\circ(f, f'')$ to the same level w as the 2-cell in (6.32):

$$[c \preceq w].\check{h}^\circ(f, f'') \coloneqq \mathrm{Hom}(h^\circ(f), h^\circ(f''))(c \preceq w)\big(\check{h}^\circ(f, f'')\big).$$

If we unravel the enrichment of functors we described in Section 4.1, then we see that the shift $[c \preceq w].\check{h}^\circ(f, f'')$ is the composition of natural transformations

$$\begin{tikzcd} h^\circ(f) \arrow[d, "\check{h}^\circ(f,f'')"'] \arrow[dr, "{[c\preceq w].\check{h}^\circ(f,f'')}"] & \\ h^\circ(f'')(c._) \arrow[r, "h^\circ(f'')\circ\alpha(c\preceq w)"'] & h^\circ(f'')(w._). \end{tikzcd} \tag{6.34}$$

By considering the ordinary commutative diagrams (6.33) and (6.34) we see that the V-weighted diagram (6.32) is valid iff the ordinary diagram

$$\begin{tikzcd} h^\circ(f) \arrow[r, "\check{h}^\circ(f,f')"] \arrow[d, "\check{h}^\circ(f,f'')"'] & h^\circ(f')(a._) \arrow[d, "\check{h}^\circ(f',f'')_{a._}"] \\ h^\circ(f'')(c._) \arrow[r, "h^\circ(f'')\circ\alpha(c\preceq w)"'] & h^\circ(f'')(w._) \end{tikzcd} \tag{6.35}$$

commutes, which we will now confirm. To this end, let $h^\#(f)\colon \mathbb{M}_\mathrm{f} \to \mathcal{W}^\mathbb{Z}$ be the strictly stable functor corresponding to $h^\circ(f)\colon \mathbb{M}_\mathrm{f} \to \mathcal{W}$ under the isomorphism of categories from Corollary 1.14. Or in other words, $h^\#(f)\colon \mathbb{M}_\mathrm{f} \to \mathcal{W}^\mathbb{Z}$ is the functor that we would use in the construction of $h^\circ(f)\colon \mathbb{M}_\mathrm{f} \to \mathcal{W}$ as described in Section 1.1.4. We define $h^\#(f')$, $h^\#(f'')$, $\check{h}^\#(f, f')$, $\check{h}^\#(f', f'')$, and $\check{h}^\#(f, f'')$ analogously using the isomorphism of categories from Corollary 1.14. Then the commutativity of (6.35) is equivalent to the commutativity of

$$\begin{tikzcd} h^\#(f) \arrow[r, "\check{h}^\#(f,f')"] \arrow[d, "\check{h}^\#(f,f'')"'] & h^\#(f')(a._) \arrow[d, "\check{h}^\#(f',f'')_{a._}"] \\ h^\#(f'')(c._) \arrow[r, "h^\#(f'')\circ\alpha(c\preceq w)"'] & h^\#(f'')(w._) \end{tikzcd} \tag{6.36}$$

by Corollary 1.14. As all functors and natural transformations in (6.36) are strictly stable, it suffices to check the commutativity for any point $v \in D$. To this end, we partition D into the three regions

$$\begin{aligned} D_1 &\coloneqq D \cap \alpha(-w)(D), \\ D_2 &\coloneqq \alpha(-a)(D) \setminus \alpha(-w)(D), \quad \text{and} \\ D_3 &\coloneqq D \setminus \alpha(-a)(D). \end{aligned}$$

For $v \in D_1$ the commutativity of (6.36) follows from the functoriality of the Mayer–Vietoris functor $\mathcal{H}_\bullet$. For $v \in D_2 \cup D_3$ we consider the monotone map

$$\xi_0 \colon u \mapsto \Big(\big(f''^{-1} \circ \rho_1 \big)(w.u), \big(f^{-1} \circ \rho_0 \big)(u) \Big).$$

(Note that we use w for the shift in place of $c \preceq w$ in the definition of ξ_0.) Then the axis-aligned rectangle shown in Fig. 6.2 yields a triad T of pairs of open subspaces of X with $\xi_0(v)$ as its first component and $\left(\xi_0 \circ \Sigma^{-1}\right)(v)$ the intersection of the other two components, hence we have the boundary operator

$$\partial_T \colon \left(\mathcal{H}_\bullet \circ \xi_0\right)(v) \to \left(\mathcal{H}_{\bullet-1} \circ \xi_0 \circ \Sigma^{-1}\right)(v)$$

of the Mayer–Vietoris functor $\mathcal{H}_\bullet$ at the triad T. By equations (6.28) and (6.29) we have

$$\left(\mathcal{H}_\bullet \circ \xi_0\right)(v) = h^{\#}(f)(v) \quad \text{and} \quad \left(\mathcal{H}_{\bullet-1} \circ \xi_0 \circ \Sigma^{-1}\right)(v) = h^{\#}(f'')(w.v). \tag{6.37}$$

Moreover, by the naturality of the boundary operator ∂ for the Mayer–Vietoris functor $\mathcal{H}_\bullet$, the triangle

$$\begin{tikzcd} h^{\#}(f)(v) \arrow[d, "\check{h}^{\#}(f,f'')_v"'] \arrow[dr, "\partial_T"] & \\ h^{\#}(f'')(c.v) \arrow[r, "h^{\#}(f'')(c.v \preceq w.v)"'] & h^{\#}(f'')(w.v) \end{tikzcd}$$

commutes. It remains to show that the triangle

$$\begin{tikzcd} h^{\#}(f)(v) \arrow[r, "\check{h}^{\#}(f,f')_v"] \arrow[dr, "\partial_T"'] & h^{\#}(f')(a.v) \arrow[d, "\check{h}^{\#}(f',f'')_{a.v}"] \\ & h^{\#}(f'')(w.v) \end{tikzcd} \tag{6.38}$$

commutes. For $v \in D_3$ we consider the monotone map

$$\xi_1 \colon u \mapsto \Big(\big(f'^{-1} \circ \rho_1 \big)(a.u), \big(f^{-1} \circ \rho_0 \big)(u) \Big),$$

that we would use for the construction of $\check{h}^{\#}(f, f')$. By equations (6.28) and (6.29) we have

$$(\mathcal{H}_\bullet \circ \xi_1)(v) = h^{\#}(f)(v) \quad \text{and} \quad \left(\mathcal{H}_{\bullet-1} \circ \xi_1 \circ \Sigma^{-1}\right)(v) = h^{\#}(f')(a.v).$$

In conjunction with (6.37) this allows us to rewrite (6.38) as

$$\begin{tikzcd}
(\mathcal{H}_\bullet \circ \xi_1)(v) \arrow[r, "\check{h}^{\#}(f, f')_v"] \arrow[d, equal] & \left(\mathcal{H}_{\bullet-1} \circ \xi_1 \circ \Sigma^{-1}\right)(v) \arrow[d, "\check{h}^{\#}(f', f'')_{a.v}"] \\
(\mathcal{H}_\bullet \circ \xi_0)(v) \arrow[r, "\partial_T"'] & \left(\mathcal{H}_{\bullet-1} \circ \xi_0 \circ \Sigma^{-1}\right)(v)
\end{tikzcd}$$

Moreover, we have pointwise $\xi_1 \subseteq \xi_0$, and thus the commutativity of this square follows from the naturality of the boundary operator ∂ for the Mayer–Vietoris functor $\mathcal{H}_\bullet$. For $v \in D_2$ we consider the monotone map

$$\xi_2 \colon u \mapsto \Big(\Big(f''^{-1} \circ \rho_1\Big)(b.u), \Big(f'^{-1} \circ \rho_0\Big)(u)\Big),$$

that we would use for the construction of $\check{h}^{\#}(f', f'')$. By the equations (6.28), (6.29), and Lemma 4.4 we have

$$\begin{aligned}
&(\mathcal{H}_\bullet \circ \xi_2)(a.v) = h^{\#}(f')(a.v) \\
\text{and} \quad (\mathcal{H}_{\bullet-1} \circ \xi_2)\left(a.\Sigma^{-1}(v)\right) = &\left(\mathcal{H}_{\bullet-1} \circ \xi_2 \circ \Sigma^{-1}\right)(a.v) = h^{\#}(f'')(b.a.v).
\end{aligned}$$

In conjunction with (6.37) this allows us to rewrite (6.38) as

$$\begin{tikzcd}
(\mathcal{H}_\bullet \circ \xi_0)(v) \arrow[r, "\check{h}^{\#}(f, f')_v"] \arrow[d, "\partial_T"'] & (\mathcal{H}_\bullet \circ \xi_2)(a.v) \arrow[d, "\check{h}^{\#}(f', f'')_{a.v}"] \\
\left(\mathcal{H}_{\bullet-1} \circ \xi_0 \circ \Sigma^{-1}\right)(v) \arrow[r, equal] & (\mathcal{H}_{\bullet-1} \circ \xi_2)\left(a.\Sigma^{-1}(v)\right)
\end{tikzcd}$$

Moreover, we have pointwise $\xi_0 \subseteq \xi_2(a.\_)$, and thus the commutativity of this square follows once again from the naturality of the boundary operator ∂ for the Mayer–Vietoris functor $\mathcal{H}_\bullet$. $\square$

Compatibility with Precomposition. We now show that the induced natural transformations in the sense of Definition 6.20 respect precomposition in the sense of Definition 6.13. Spelling this out for h°, this means that for any continuous map $\varphi \colon X \to Y$ the $\operatorname{Hom}(Y, \mathbb{R})$-indexed family of natural transformations

$$\left(h^\circ(\varphi)_f \colon h^\circ(f \circ \varphi) \to h^\circ(f)\right)_{f \colon Y \to \mathbb{R}}$$

from Section 1.1.5, describes a morphism

$$h^\circ(\varphi)\colon \varphi_! \check{h}^\circ_X \longrightarrow \check{h}^\circ_Y$$

of the category $\operatorname{Func}'\big(\operatorname{Hom}(Y,\mathbb{R}), \mathcal{W}^{\mathbb{M}_\mathrm{f}}\big)$, where $\check{h}^\circ_X$ and $\check{h}^\circ_Y$ are the restrictions of $\check{h}^\circ$ of to $\operatorname{Hom}(X,\mathbb{R})$ and $\operatorname{Hom}(Y,\mathbb{R})$ respectively. As in the previous paragraph, we will use two different types of diagrams, ordinary commutative diagrams of functors and natural transformations as well as V-weighted diagrams in the locally V-persistent category of functors $\mathbb{M}_\mathbf{f} \to \mathcal{W}$ in the sense of Section 5.4, which have V-weighted 2-cells.

Lemma 6.22 (Compatibility with Precomposition)**.** *The assignment $\check{h}^\circ$ of induced natural transformations is compatible with precomposition in the sense of Definition 6.13.*

Proof. We show the characterization provided by Lemma 6.16. To this end, let $\varphi\colon X \to Y$ be a continuous map and let $f, g\colon Y \to \mathbb{R}$ be continuous functions with $\mathbf{d}(f,g) \in V$. We have to show that the diagram

$$\begin{tikzcd}
h^\circ(f\circ\varphi) \arrow[r, "h^\circ(\varphi)_f"] \arrow[d, "\check{h}^\circ(f\circ\varphi,\, g\circ\varphi)"'] \arrow[dr, phantom, "\mathbf{d}(f,g)"] & h^\circ(f) \arrow[d, "\check{h}^\circ(f,g)"] \\
h^\circ(g\circ\varphi) \arrow[r, "h^\circ(\varphi)_g"'] & h^\circ(g)
\end{tikzcd} \tag{6.39}$$

is valid in the locally V-persistent category of functors $\mathbb{M}_\mathrm{f} \to \mathcal{W}$. Now let $b \coloneqq \mathbf{d}(f,g)$ and let $a \coloneqq \mathbf{d}(f\circ\varphi, g\circ\varphi)$. Then we have $a \preceq b$. Moreover, the horizontal arrows in (6.39) both have weight 0. Furthermore, the vertical arrow on the right in (6.39) has weight $b = \mathbf{d}(f,g)$, which is the same weight as the 2-cell in (6.39). So the composition

$$\begin{tikzcd}
h^\circ(f\circ\varphi) \arrow[r, "h^\circ(\varphi)_f"] \arrow[dr, "\check{h}^\circ(f,g)\,{}_b\diamond_0\, h^\circ(\varphi)_f"'] & h^\circ(f) \arrow[d, "\check{h}^\circ(f,g)"] \\
 & h^\circ(g)(b._)
\end{tikzcd} \tag{6.40}$$

of natural transformations is already at the same level b as the 2-cell of (6.39). However, the weight of the vertical arrow on the left in (6.39) may have smaller weight $a \preceq b$. In order to adjust for this, we may consider the shift

$$[a \preceq b].\check{h}^\circ(f\circ\varphi, g\circ\varphi) \coloneqq \operatorname{Hom}(h^\circ(f\circ\varphi), h^\circ(g\circ\varphi))(a \preceq b)\big(\check{h}^\circ(f\circ\varphi, g\circ\varphi)\big),$$

which is the composition of natural transformations

$$\begin{tikzcd}
h^\circ(f\circ\varphi) \arrow[r, "\check{h}^\circ(f\circ\varphi,\, g\circ\varphi)"] \arrow[rr, bend right, "{[a\preceq b].\check{h}^\circ(f\circ\varphi,\, g\circ\varphi)}"'] & h^\circ(g\circ\varphi)(a._) \arrow[r, "h^\circ(g\circ\varphi)\circ\alpha(b\preceq a)"] & h^\circ(g\circ\varphi)(b._)
\end{tikzcd}$$

so we have the commutative triangle

$$\begin{tikzcd}
h^\circ(f \circ \varphi) \arrow[d, "\check{h}^\circ(f \circ \varphi, g \circ \varphi)"'] \arrow[ddr, "h^\circ(\varphi)_g \, {}_0\diamond_b \, [a \preceq b].\check{h}^\circ(f \circ \varphi, g \circ \varphi)"] & \\
h^\circ(g \circ \varphi)(a._) \arrow[d, "h^\circ(g \circ \varphi) \circ \alpha(a \preceq b)"'] & \\
h^\circ(g \circ \varphi)(b._) \arrow[r, "h^\circ(\varphi)_{g,b._}"'] & h^\circ(g)(b._)
\end{tikzcd} \tag{6.41}$$

of functors and natural transformations. By considering the ordinary commutative diagrams (6.40) and (6.41) we see that the V-weighted diagram (6.39) is valid iff the ordinary diagram

$$\begin{tikzcd}
h^\circ(f \circ \varphi) \arrow[r, "h^\circ(\varphi)_g"] \arrow[d, "\check{h}^\circ(f \circ \varphi, g \circ \varphi)"'] & h^\circ(f) \arrow[dd, "\check{h}^\circ(f, g)"] \\
h^\circ(g \circ \varphi)(a._) \arrow[d, "h^\circ(g \circ \varphi) \circ \alpha(a \preceq b)"'] & \\
h^\circ(g \circ \varphi)(b._) \arrow[r, "h^\circ(\varphi)_{g,b._}"'] & h^\circ(g)(b._)
\end{tikzcd} \tag{6.42}$$

of functors and natural transformations commutes, which we will now confirm. To this end, let $h^\#(f) \colon \mathbb{M}_\mathrm{f} \to \mathcal{W}^\mathbb{Z}$ be the strictly stable functor corresponding to $h^\circ(f) \colon \mathbb{M}_\mathrm{f} \to \mathcal{W}$ under the isomorphism of categories from Corollary 1.14. Or in other words, $h^\#(f) \colon \mathbb{M}_\mathrm{f} \to \mathcal{W}^\mathbb{Z}$ is the functor that we would use in the construction of $h^\circ(f) \colon \mathbb{M}_\mathrm{f} \to \mathcal{W}$ as described in Section 1.1.4. We define $h^\#(f \circ \varphi)$, $h^\#(g)$, $h^\#(g \circ \varphi)$, $\check{h}^\#(f, g)$, and $\check{h}^\#(f \circ \varphi, g \circ \varphi)$ analogously using the isomorphism of categories from

Corollary 1.14. Then the commutativity of (6.42) is equivalent to the commutativity of

$$\begin{tikzcd}
h^{\#}(f \circ \varphi) \arrow[rr, "h^{\#}(\varphi)_g"] \arrow[d, "\check{h}^{\#}(f\circ\varphi,\, g\circ\varphi)"'] & & h^{\#}(f) \arrow[dd, "\check{h}^{\#}(f,g)"] \\
h^{\#}(g \circ \varphi)(a._) \arrow[d, "h^{\#}(g\circ\varphi)\circ\alpha(a\preceq b)"'] & & \\
h^{\#}(g \circ \varphi)(b._) \arrow[rr, "h^{\#}(\varphi)_{g,b._}"'] & & h^{\#}(g)(b._)
\end{tikzcd} \tag{6.43}$$

by Corollary 1.14. As all functors and natural transformations in (6.43) are strictly stable, it suffices to check the commutativity for any point $v \in D$. To this end, we partition D into the two regions

$$D_1 \coloneqq D \cap \alpha(-b)(D) \quad \text{and} \quad D_2 \coloneqq D \setminus \alpha(-b)(D).$$

For $v \in D_1$ the commutativity of (6.36) follows from the functoriality of the Mayer–Vietoris functor $\mathcal{H}_\bullet$. For $v \in D_2$ we consider the monotone map

$$\xi' \colon u \mapsto \big(\big(\varphi^{-1} \circ g^{-1} \circ \rho_1\big)(b.u), \big(\varphi^{-1} \circ f^{-1} \circ \rho_0\big)(u)\big).$$

(Note that we use b for the shift in place of $a \preceq b$ in the definition of ξ'.) Then the axis-aligned rectangle shown in Fig. 6.2 yields a triad T of pairs of open subspaces of X with $\xi'(v)$ as its first component and $\left(\xi' \circ \Sigma^{-1}\right)(v)$ the intersection of the other two components, hence we have the boundary operator

$$\partial_T \colon \left(\mathcal{H}_\bullet \circ \xi'\right)(v) \to \left(\mathcal{H}_{\bullet-1} \circ \xi' \circ \Sigma^{-1}\right)(v)$$

of the Mayer–Vietoris functor $\mathcal{H}_\bullet$ at the triad T. By equations (6.28) and (6.29) we have

$$\left(\mathcal{H}_\bullet \circ \xi'\right)(v) = h^{\#}(f \circ \varphi)(v) \quad \text{and} \quad \left(\mathcal{H}_{\bullet-1} \circ \xi' \circ \Sigma^{-1}\right)(v) = h^{\#}(g \circ \varphi)(b.v).$$

Moreover, by the naturality of the boundary operator ∂ for the Mayer–Vietoris functor

$\mathcal{H}_\bullet$, the (degenerate) triangle

$$\begin{tikzcd}
h^{\#}(f \circ \varphi)(v) \arrow[d, "\check{h}^{\#}(f\circ\varphi,\, g\circ\varphi)_v"] \arrow[dd, bend right=60, "\partial_T"'] \\
h^{\#}(g \circ \varphi)(a.v) \arrow[d, "h^{\#}(g\circ\varphi)(a.v \preceq b.v)"] \\
h^{\#}(g \circ \varphi)(b.v)
\end{tikzcd} \tag{6.44}$$

commutes. By instantiating the diagram (6.43) at $v \in D_2$ and pasting it with (6.44) we may reduce our proof obligation to showing that the diagram

$$\begin{tikzcd}
h^{\#}(f \circ \varphi)(v) \arrow[r, "h^{\#}(\varphi)_{g,v}"] \arrow[d, "\partial_T"'] & h^{\#}(f)(v) \arrow[d, "\check{h}^{\#}(f,g)_v"] \\
h^{\#}(g \circ \varphi)(b.v) \arrow[r, "h^{\#}(\varphi)_{g,b.v}"'] & h^{\#}(g)(b.v)
\end{tikzcd} \tag{6.45}$$

commutes. To this end, we consider the monotone map

$$\xi \colon u \mapsto \big(\big(g^{-1} \circ \rho_1\big)(b.u), \big(f^{-1} \circ \rho_0\big)(u)\big)$$

as in the construction of the natural transformation $\check{h}^{\#}(f,g) \colon h^{\#}(f) \to h^{\#}(g)(b._)$. Then we have pointwise $\varphi \circ \xi' \subseteq \xi$, hence the commutativity of (6.45) follows from the naturality of the boundary operator ∂ of the Mayer–Vietoris functor $\mathcal{H}_\bullet$. $\square$

Persistent Enrichment of RISC. We may now combine the results of this Section 6.4.2 to obtain a V-persistent enrichment of the functor $h^\circ \colon \mathrm{Top}/\mathbb{R} \to \mathcal{W}^{\mathbb{M}_\mathrm{f}}$.

Theorem 6.23. *There is a unique V-persistent enrichment*

$$\tilde{h}^\circ \colon P_\bullet(\mathrm{Top}/\mathbb{R})_w \to \mathcal{W}^{\mathbb{M}_\mathrm{f}}$$

of the covariant functor $h^\circ \colon \mathrm{Top}/\mathbb{R} \to \mathcal{W}^{\mathbb{M}_\mathrm{f}}$ such that the restriction of $\tilde{h}^\circ$ agrees with $\check{h}^\circ$ on the vertical homomorphisms of $(\mathrm{Top}/\mathbb{R})_w$.

Proof. This follows immediately from Proposition 6.14 and Lemmas 6.21 and 6.22. $\square$

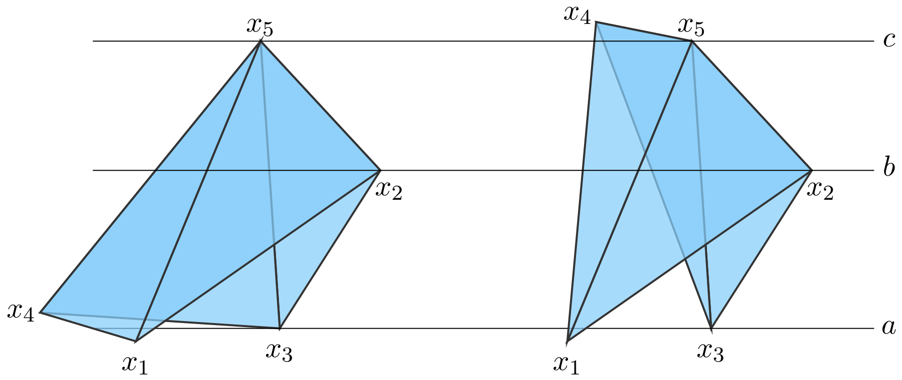


Figure 6.5: The function $f\colon X \to \mathbb{R}$ depicted as a height function (left) and the height function $g\colon X \to \mathbb{R}$ (right).

Remark 6.24. As $\mathbb{R}^\circ \times \mathbb{R}$ is a commutative monoidal poset, we may apply the oppositization 2-functor to a V-persistent enrichment $\tilde{h}^\circ\colon P_\bullet(\mathrm{Top}/\mathbb{R})_w \to \mathcal{W}^{\mathbb{M}_\mathrm{f}}$ as in Theorem 6.23 to obtain the V-persistent functor

$$\tilde{h} := \tilde{h}^{\circ\circ}\colon P_\bullet(\mathrm{Top}/\mathbb{R})_w^\circ \to (\mathcal{W}^\circ)^{\mathbb{M}_\mathrm{f}^\circ}$$

in the sense of Remark 5.9. Now suppose we have $\mathcal{W} = \mathrm{Vect}_\mathbb{F}^\circ$, i.e. $\mathcal{W}$ is the opposite category of the category of vector spaces over some field $\mathbb{F}$, and that $\mathcal{H}_\bullet$ is the categorical dual of a cohomology theory $\mathcal{H}^\bullet$ taking values in $\mathrm{Vect}_\mathbb{F}$. Then

$$\tilde{h}\colon P_\bullet(\mathrm{Top}/\mathbb{R})_w^\circ \to \mathrm{Vect}_\mathbb{F}^{\mathbb{M}_\mathrm{f}^\circ}$$

is a V-persistent enrichment of relative interlevel set cohomology $h\colon (\mathrm{Top}/\mathbb{R})^\circ \to \mathrm{Vect}_\mathbb{F}^{\mathbb{M}_\mathrm{f}^\circ}$ with respect $\mathcal{H}^\bullet$ by Remark 5.26. In this situation, we also write $\check{h}(f,g)\colon h(g)(a._) \to h(f)$ for the natural transformation of presheaves $\mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$ induced by a pair of functions $f, g\colon X \to \mathbb{R}$ with $a := \mathbf{d}(f,g) \in V$.

Example 6.25. In Section 2.6 we saw that the restriction of natural transformations of RISC, that are induced by ordinary horizontal homomorphisms of $\mathbb{R}$-spaces, to extended persistence or Mayer–Vietoris systems entails a loss of information. We now reuse part of our computations from Section 2.6 to see this loss of information occurs for homomorphisms that are part of an interleaving induced by a pair of functions $f, g\colon X \to \mathbb{R}$ as well. To this end, let $a < b < c \in \mathbb{R}$, $f\colon X \to \mathbb{R}$, and $A \subset X$ be as in Section 2.6. Moreover, let $g\colon X \to \mathbb{R}$ be the unique simplexwise linear function with $g|_A = f|_A\colon A \to \mathbb{R}$ and $g(x_5) = c$. Consider Fig. 6.5 for a visualization of $f, g\colon X \to \mathbb{R}$ as height functions of two different embeddings $X \hookrightarrow \mathbb{R}^3$. As we have $\mathbf{d}(f,g) = (0, c-a) \in V = \mathbb{R}^\circ \times \mathbb{R}$ we may instantiate the interleaving (6.9) for $f, g \colon X \to \mathbb{R}$:

$$\begin{tikzcd} f \arrow[rr, bend left, "\mathrm{id}_X", "{(0, c{-}a)}"'] \arrow[rr, phantom, "{(a{-}c, c{-}a)}"] & & g \arrow[ll, bend left, "\mathrm{id}_X", "{(a{-}c, 0)}"'] \end{tikzcd}. \tag{6.46}$$

As we discussed in Section 6.2.1, the weight of "the" 2-cell in (6.46) is determined by the single non-negative parameter $c - a \geq 0$. If we now post-compose the interleaving (6.46) with the V-persistent enrichment

$$\tilde{h} \colon P_{\bullet}(\mathrm{Top}/\mathbb{R})_w^{\circ} \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}_{\mathrm{f}}^{\circ}}$$

of RISC in the sense of Remark 6.24, then we obtain the interleaving

$$\begin{tikzcd} h(f) \arrow[rr, bend right, "\check{h}(g, f)"', "{(a{-}c, 0)}"] \arrow[rr, phantom, "{(a{-}c, c{-}a)}"] & & h(g) \arrow[ll, bend right, "\check{h}(f, g)"', "{(0, c{-}a)}"] \end{tikzcd} \tag{6.47}$$

in relative interlevel set cohomology with coefficients in $\mathbb{F}$. We now describe the induced natural transformation

$$\check{h}(g, f) \colon h(g) \circ \alpha(a - c,\, 0) \longrightarrow h(f),$$

which is one of the two weighted interleaving homomorphisms in (6.47).

Now recall from Section 2.6 that we have the natural isomorphism

$$B_w \oplus B_v \xrightarrow{\sim} h(f|_A) \tag{2.27 revisited}$$

with $u, v, w \in \operatorname{int} \mathbb{M}_{\mathrm{f}}$ as in Section 2.6. Using homotopy invariance of singular homology it is easy to see that the induced natural transformation $h(g) \to h(g|_A) = h(f|_A)$ of presheaves $\mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ is a natural isomorphism, hence we have the natural isomorphism

$$B_w \oplus B_v \xrightarrow{\sim} h(g)$$

by reusing (2.27). Moreover, the point $u \in \mathbb{M}_{\mathrm{f}}$ is a fixed point of $\alpha(a - c, 0) \colon \mathbb{M}_{\mathrm{f}} \to \mathbb{M}_{\mathrm{f}}$ and we have

$$w' \coloneqq \alpha(a - c, 0)(w) = (\alpha(a - c, 0) \circ \mathfrak{b}^{-1})(0, [a, c]) = \mathfrak{b}^{-1}(0, [2a - c, c])$$

so by whiskering the natural isomorphism

$$B_w \oplus B_u \xrightarrow{\sim} h(f), \tag{2.28 revisited}$$

from the right with $\alpha(a-c,0)$ we obtain the natural isomorphism

$$B_{w'} \oplus B_u \xrightarrow{\sim} h(f) \circ \alpha(a-c,0).$$

Completely analogously to the commutativity of (2.29) we obtain the commutativity of the diagram

$$\begin{tikzcd} B_{w'} \oplus B_u \arrow[r, "\sim"] \arrow[d, "B_{w' \preceq w} \oplus B_{u \preceq v}"'] & h(f) \circ \alpha(a-c,0) \arrow[d, "\check{h}(g,f)"] \\ B_w \oplus B_v \arrow[r, "\sim"] & h(g) \end{tikzcd} \tag{6.48}$$

of presheaves $\mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$. Moreover, as we already noted in Section 2.6 the pointwise restrictions of $B_{u \preceq v} \colon B_u \to B_v$ to the indexing posets for extended persistence and Mayer–Vietoris systems are necessarily zero as the intersection of supports of B_u and B_v is disjoint from these regions. Thus, the corresponding interleavings of extended persistence as in [BS14, Section 6] or of Mayer–Vietoris systems as in [BGO19, Section 2.3] do not see this non-trivial summand of the interleaving homomorphism $\check{h}(g,f) \colon h(f) \circ \alpha(a-c,\, 0) \longrightarrow h(g)$.

Part III

Equivalence of RISC and Derived Level Set Persistence

While much of persistence theory, such as [CdM09], is based on the representation theory of A_n-quivers, *derived level set persistence theory* has been developed by [Cur14, KS18] using the theory of sheaves. In this Part III we connect relative interlevel set cohomology to derived level set persistence theory through an equivalence of categories. Now one of the landmark results within the representation theory of quivers is the Happel functor [Hap88, Section I.5.6] providing a concrete description of derived categories of Dynkin quivers. In order to have a similar utility for sheaf-theoretical persistence theory, we provide a sheaf-theoretical counterpart of the Happel functor in Chapter 7. Then we factor RISC through the contravariant derived level set persistence functor in Section 8.3 by restriction of the Yoneda embedding along this sheaf-theoretical Happel functor. Finally, we show in Chapter 9 that this restricted Yoneda embedding yields an equivalence of categories between the category of *tame derived sheaves* on $\mathbb{R}$ and presheaves $\mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$ satisfying the necessary tameness and exactness properties; this is Corollary 9.4. Moreover, this result has the Corollary 9.5 providing a structure theorem for sheaves satisfying tameness assumptions that are weak enough to include sheaves that are not even weakly constructible in the sense of [KS90, Chapter VIII] as provided by Example 9.6.

Throughout this Part III, which grew out of the preprint [BF22], we make extensive use of homological algebra and sheaf theory. For most results needed from these two areas we will rely on chapters 1 and 2 of [KS90], respectively.

Auxiliary Refinements to the Theory of A_n-Quivers. One of the original outcomes of Happels theory is the equivalence of the derived categories of all A_n-quivers for a fixed $n \in \mathbb{N}$. In the present setting we view the real numbers $\mathbb{R}$ with the usual topology as a continuous counterpart to an A_n-quiver Q with alternating orientations of arrows, and we view sheaves on $\mathbb{R}$ as continuously indexed counterparts to representations of Q. Now when providing our continuously indexed counterpart to Happels functor we will not limit ourselves to the real numbers as a topological space and consider other topological spaces as well in the same way that Happel considered A_n-quivers with arbitrary orientations of arrows. As a result, we obtain an equivalence of categories between derived sheaves satisfying the necessary tameness assumptions on any of the topological spaces considered and the category of presheaves $\mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$ satisfying the necessary tameness and exactness properties; this is Theorem 9.1 which entails the aforementioned equivalence as Corollary 9.4. Thus, we obtain equivalences between categories of derived presheaves on topological spaces that are very different from $\mathbb{R}$ including spaces with totally ordered topologies, which are continuous counterparts to A_n-quivers with all arrows oriented in the same direction. This result is in stark contrast to [IRT20, Theorem 1.2.1], which entails that in general the derived categories of two *quivers of continuous type A* in the sense of [IRT19, Definition 1.1.1] are not equivalent. For more on this alternative approach to refining the representation theory of A_n-quivers to a continuous setting we refer to the entire series of articles [IRT19, Roc19, IRT20, Roc20]. In the discrete setting the functorial extension of the correspondence of barcodes (i.e., isomorphism classes) by [CdM09] through the equivalence by [Hap88] has also been investigated by [HIY22].

7 A Sheaf-Theoretical Happel Functor

Happel's Embedding. Dieter Happel provided an equivalence

$$\mathbb{F}\langle\mathbb{Z}Q\rangle \xrightarrow{\sim} \operatorname{ind}\big(D^b(\mathbb{F}Q)\big) \hookrightarrow D^b(\mathbb{F}Q) \tag{7.1}$$

between the *mesh category* $\mathbb{F}\langle\mathbb{Z}Q\rangle$ of the *repetition quiver* $\mathbb{Z}Q$ of a Dynkin quiver Q and the full subcategory of indecomposables $\operatorname{ind}\big(D^b(\mathbb{F}Q)\big)$ of the derived category $D^b(\mathbb{F}Q)$ in [Hap88, Section I.5.6]; see also [KS16, Theorem 2.2]. In this Chapter 7, we describe a continuous or sheaf-theoretical counterpart of the *Happel functor* (7.1) for Q an A_n-quiver. We illustrate the transition from representations of A_n-quivers to the continuous setting of sheaves on certain topological spaces at a specific example. To this end, let Q be the A_6-quiver tinted in blue in Fig. 7.1. We denote the path algebra of Q over the field $\mathbb{F}$ by $\mathbb{F}Q$ and the bounded derived category of finite-dimensional right $\mathbb{F}Q$-modules by $D^b(\mathbb{F}Q)$. Formally, the vertices of the *repetition quiver* $\mathbb{Z}Q$ of Q are provided by the product set $\mathbb{Z} \times Q_0$ of the integers and the vertices of Q and for each arrow $i \to j$ in Q we have arrows

$$(k, i) \to (k, j) \quad \text{and} \quad (k-1, j) \to (k, i) \quad \text{for } k \in \mathbb{Z}$$

in $\mathbb{Z}Q$. (Note that for an A_n-quiver Q, there is at most one arrow between any two vertices of both Q and $\mathbb{Z}Q$, hence there is no need to provide any names.) Then we have the embedding of quivers

$$Q \to \mathbb{Z}Q, \begin{cases} i \mapsto (0, i) \\ (i \to j) \mapsto ((0, i) \to (0, j)) \end{cases}$$

as shown in Fig. 7.1. Now let $\mathbb{F}(\mathbb{Z}Q)$ be the $\mathbb{F}$-linear path category of $\mathbb{Z}Q$ and let $\operatorname{mesh}(\mathbb{Z}Q)$ be the ideal in $\mathbb{F}(\mathbb{Z}Q)$ that is generated by the sums of paths between any two vertices of distance 2 henceforth called *mesh relations*. With this we define the *mesh category*

$$\mathbb{F}\langle\mathbb{Z}Q\rangle := \mathbb{F}(\mathbb{Z}Q)/\operatorname{mesh}(\mathbb{Z}Q).$$

The Strip $\mathbb{M}$ as a Continuous Counterpart to the Mesh Category. Before we provide a sheaf-theoretical counterpart of the Happel functor (7.1), we describe how the $\mathbb{F}$-linear category $\mathbb{F}\mathbb{M}/\partial\mathbb{M}$, obtained as the linearization of the poset $\mathbb{M}$ modulo boundary points, can be seen as a continuous counterpart to the mesh category $\mathbb{F}\langle\mathbb{Z}Q\rangle$. This requires modification of the mesh ideal for the following reason. If Q' is a refinement of Q splitting one of its arrows into two arrows and a vertex, then the canonical functor of path categories

$$\mathbb{F}(\mathbb{Z}Q') \to \mathbb{F}(\mathbb{Z}Q)$$

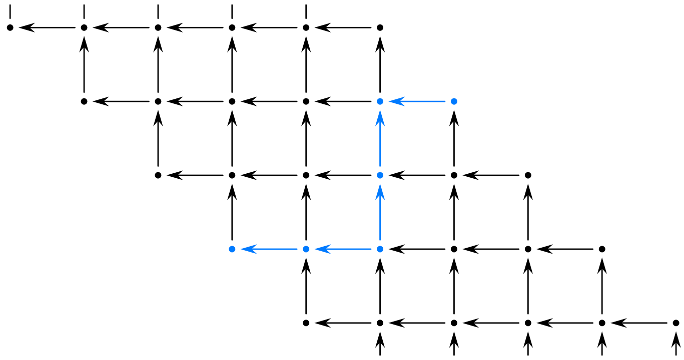


Figure 7.1: The A_6-quiver Q tinted in blue and embedded into its repetition quiver $\mathbb{Z}Q$.

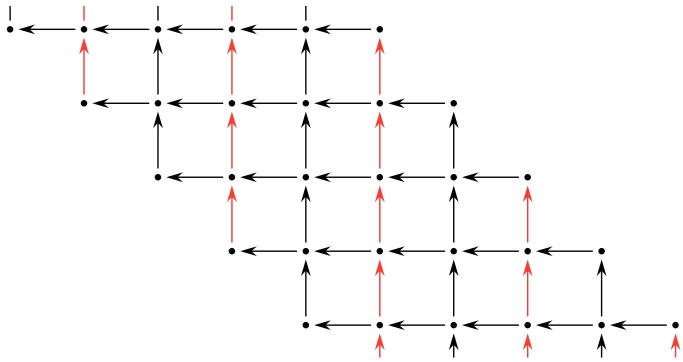


Figure 7.2: The autofunctor $\alpha\colon \mathbb{F}(\mathbb{Z}Q) \to \mathbb{F}(\mathbb{Z}Q)$ flips the sign of every arrow that is tinted in red.

does not map the mesh ideal $\operatorname{mesh}(\mathbb{Z}Q')$ into the other mesh ideal $\operatorname{mesh}(\mathbb{Z}Q)$. So considering the layout of the repetition quiver $\mathbb{Z}Q$ shown in Fig. 7.2, let

$$\alpha\colon \mathbb{F}(\mathbb{Z}Q) \to \mathbb{F}(\mathbb{Z}Q)$$

be the $\mathbb{F}$-linear autofunctor flipping the signs of arrows in every other column of $\mathbb{Z}Q$. Then α turns most of the mesh relations into commutativity relations. Now let $\operatorname{mesh}'(\mathbb{Z}Q) := \alpha(\operatorname{mesh}(\mathbb{Z}Q))$. Then α induces an $\mathbb{F}$-linear isofunctor

$$\bar{\alpha}\colon \mathbb{F}\langle\mathbb{Z}Q\rangle = \mathbb{F}(\mathbb{Z}Q)/\operatorname{mesh}(\mathbb{Z}Q) \to \mathbb{F}(\mathbb{Z}Q)/\operatorname{mesh}'(\mathbb{Z}Q).$$

However, as α does not turn any mesh relation into a commutativity relation we cannot write $\mathbb{F}(\mathbb{Z}Q)/\operatorname{mesh}'(\mathbb{Z}Q)$ as the linearization of a poset. So we have yet to account for those remaining relations consisting of paths of length 2 along the boundary of $\mathbb{Z}Q$. To this end, let $\mathbb{L}$ be the lattice whose Hasse diagram is the quiver shown in Fig. 7.3, which is the repetition quiver $\mathbb{Z}Q$ with additional vertices and arrows along the boundary tinted in green. Then we have the canonical $\mathbb{F}$-linear functor

$$\beta\colon \mathbb{F}(\mathbb{Z}Q) \to \mathbb{F}\mathbb{L}$$

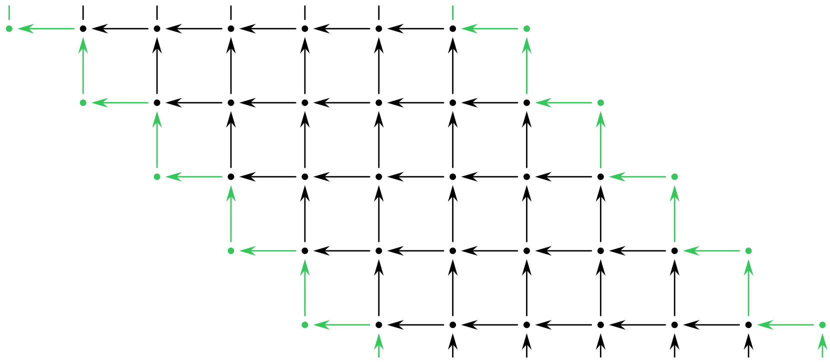


Figure 7.3: The vertices and arrows that need to be added to the repetition quiver $\mathbb{Z}Q$ to obtain the Hasse diagram of the lattice $\mathbb{L}$ are tinted in green.

mapping the modified mesh ideal $\mathrm{mesh}'(\mathbb{Z}Q)$ to the ideal generated by the *boundary vertices* $\partial\mathbb{L}$ tinted in green in Fig. 7.3; these are all vertices that have no more than two neighbors in the Hasse diagram of $\mathbb{L}$. Thus, β induces an $\mathbb{F}$-linear functor on quotient categories

$$\bar{\beta}\colon \mathbb{F}(\mathbb{Z}Q)/\mathrm{mesh}'(\mathbb{Z}Q) \to \mathbb{F}\mathbb{L}/\partial\mathbb{L}.$$

As it turns out, the functor $\bar{\beta}$ is fully faithful and merely adjoins zero objects to $\mathbb{F}(\mathbb{Z}Q)/\mathrm{mesh}'(\mathbb{Z}Q)$. As a result, we may uniquely extend the Happel functor (7.1) to an $\mathbb{F}$-linear functor preserving zero objects on $\mathbb{F}\mathbb{L}/\partial\mathbb{L}$ and then restrict it to an ordinary functor on $\mathbb{L}$ vanishing on $\partial\mathbb{L}$:

$$\begin{tikzcd}
\mathbb{F}\langle\mathbb{Z}Q\rangle \arrow[r, "\bar{\beta}\circ\bar{\alpha}"] \arrow[dr] & \mathbb{F}\mathbb{L}/\partial\mathbb{L} \arrow[d, dashed] & \mathbb{L} \arrow[l] \arrow[dl, dashed] \\
& D^b(\mathbb{F}Q). &
\end{tikzcd} \tag{7.2}$$

As we may view $\mathbb{L}$ as a discrete sublattice of $\mathbb{M}$ with $\partial\mathbb{L}$ contained in $\partial\mathbb{M}$, the poset $\mathbb{M}$ (together with the subposet $\partial\mathbb{M}$) can indeed be seen as a continuous counterpart to the mesh category.

A Continuously Indexed Counterpart to A_n-Quiver Representations. We now describe a continuously indexed counterpart to an A_n-quiver representation as a sheaf on a certain type of topological space defined in terms of $\mathbb{M}$. This analogy is based on the observation that in the above example of the A_6-quiver Q, all other arrow orientations for an A_6-quiver can be obtained from a shortest sub-quiver of the repetition quiver $\mathbb{Z}Q$ connecting the left boundary of $\mathbb{Z}Q$ to the right boundary with respect to the planar embedding of $\mathbb{Z}Q$ shown in Fig. 7.1. Moreover, any such sub-quiver Q' will also have a particular Happel functor

$$\mathbb{F}\langle\mathbb{Z}Q\rangle \xrightarrow{\sim} \mathrm{ind}\big(D^b(\mathbb{F}Q')\big) \hookrightarrow D^b(\mathbb{F}Q'),$$

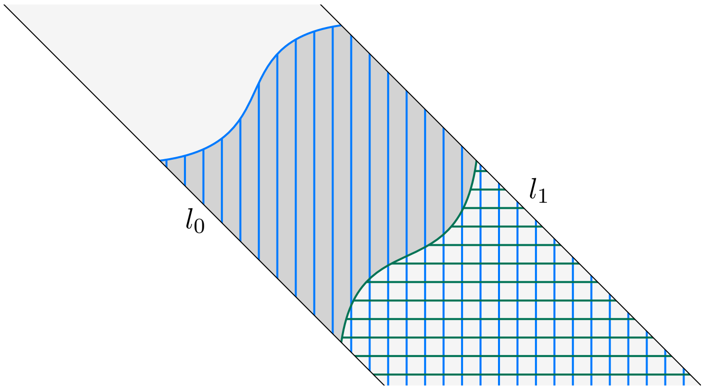


Figure 7.4: The boundary $q \coloneqq \partial C$ as well as the fundamental domain $D \coloneqq C \setminus \Sigma^{-1}(C)$.

which will be different for any two distinct sub-quivers even when their arrow orientations are the same. Now a convenient counterpart to this choice of a shortest sub-quiver connecting boundary components in the continuous setting is that of a closed proper downset $C \subsetneq \mathbb{M}$, i.e. a proper subposet C of $\mathbb{M}$ that is a downset and closed with respect to the Euclidean subspace topology of $\mathbb{M}$. The boundary $q \coloneqq \partial C$ of C in $\mathbb{M}$ is the underlying set of the topological space that we consider in place of the A_6-quiver Q as shown in Fig. 7.4. However, for no possible choice of $C \subsetneq \mathbb{M}$ the subspace $q \subset \mathbb{R}^2$ is a reasonable continuous counterpart to an A_n-quiver with all arrows pointing in the same direction. In place of the Euclidean topology, we consider the so called γ-topology on the plane $\mathbb{R}^2$ as defined in [KS90, Definition 3.5.1] for the cone $\gamma \coloneqq [0, \infty) \times (-\infty, 0] \subset \mathbb{R}^\circ \times \mathbb{R}$, which is the principal downset $\downarrow 0$ of the origin in $\mathbb{R}^\circ \times \mathbb{R}$. As in [KS90, Section 3.5] we denote the plane endowed with the γ-topology by $\mathbb{R}^2_\gamma$. Now let q_γ be the subspace of $\mathbb{R}^2_\gamma$, whose underlying set is q, and let

$$\phi_\gamma \colon q \to q_\gamma,\, t \mapsto t$$

be the natural continuous map. Moreover, let $\partial q \coloneqq q \cap \partial\mathbb{M}$ and let $j \colon \partial q \hookrightarrow q$ be the inclusion. Now a reasonable continuously indexed counterpart to a right $\mathbb{F}Q$-module is the notion of a sheaf on q_γ vanishing on ∂q. Conceptually, the reason we consider sheaves vanishing on ∂q is that in order to draw the above analogy from the mesh category $\mathbb{F}\langle\mathbb{Z}Q\rangle$ to $\mathbb{M}$, we had to add the boundary vertices of $\mathbb{L}$ in an intermediate step, and we have to account for that. Whenever C is the principal downset of a boundary point in $\partial\mathbb{M}$, then the topology of q_γ is a totally ordered poset, which resembles uniformly oriented A_n-quivers. For this reason, we believe that q_γ is a reasonable continuous counterpart to an A_n-quiver.

Coreflection to Sheaves Vanishing on the Boundary. Now let $\mathrm{Sh}(q_\gamma)$ be the category of $\mathbb{F}$-linear sheaves on q_γ and $\mathrm{Sh}(q_\gamma, \partial q)$ be the full subcategory of $\mathbb{F}$-linear sheaves in $\mathrm{Sh}(q_\gamma)$ vanishing on ∂q. Considering the unit $\eta^j \colon \mathrm{id} \to j_* \circ j^{-1}$ of the adjunction $j^{-1} \dashv j_*$ (viewed as a functor from $\mathrm{Sh}(q_\gamma)$ to homomorphisms in $\mathrm{Sh}(q_\gamma)$) and taking the kernel yields a coreflection $\flat = \ker \circ \eta^j \colon \mathrm{Sh}(q_\gamma) \to \mathrm{Sh}(q_\gamma, \partial q)$:

$$\begin{tikzcd} \mathrm{Sh}(q_\gamma, \partial q) \arrow[r, phantom, "\top"] \arrow[r, bend right] & \mathrm{Sh}(q_\gamma) \arrow[l, bend right, "\flat"'] \end{tikzcd}. \tag{7.3}$$

Lemma 7.1. *If F is a flabby sheaf on q_γ, then the unit $\eta^j_F \colon F \to j_* j^{-1} F$ is an epimorphism.*

Proof. Let $U \subseteq q_\gamma$ be an open subset. If $U \cap \partial q$ contains just a single point t, then we have the commutative diagram

$$\begin{tikzcd} F(U) \arrow[r, "\eta^j_{F,U}"] \arrow[dr, two heads] & (j_* j^{-1} F)(U) = (j^{-1}F)(\{t\}) \arrow[d, equal] \\ & F_t, \end{tikzcd}$$

hence η^j_F is an epimorphism at U. Now suppose $\partial q \subset U$. If ∂q is discrete, then we can find disjoint open subsets $V_0, V_1 \subset U$ with $\partial q \subset V_0 \cup V_1$ and each containing a single point of ∂q. Moreover, we have the commutative diagram

$$\begin{tikzcd} F(U) \arrow[r, "\eta^j_{F,U}"] \arrow[d, two heads] & (j_* j^{-1} F)(U) = (j^{-1}F)(\partial q) \arrow[d, equal] \\ F(V_0 \cup V_1) \arrow[r, "\eta^j_{F, V_0 \cup V_1}"] \arrow[d, "\cong"'] & (j_* j^{-1} F)(V_0 \cup V_1) \arrow[d, "\cong"] \\ F(V_0) \oplus F(V_1) \arrow[r, two heads, "\eta^j_{F,V_0} \oplus \eta^j_{F,V_1}"'] & (j_* j^{-1} F)(V_0) \oplus (j_* j^{-1} F)(V_1) \end{tikzcd}$$

with the lower horizontal map an epimorphism by our reasoning above. If ∂q is not discrete, then $U = q_\gamma$ and moreover, $\eta^j_{F, q_\gamma} \colon F(q_\gamma) \to (j_* j^{-1} F)(q_\gamma) = (j^{-1}F)(\partial q)$ is even an isomorphism. $\square$

Lemma 7.2. *The induced adjunction on the level of derived categories*

$$\begin{tikzcd} D^+(\mathrm{Sh}(q_\gamma, \partial q)) \arrow[r, phantom, "\top"] \arrow[r, bend right] & D^+(\mathrm{Sh}(q_\gamma)) \arrow[l, bend right, "R\flat"'] \end{tikzcd}$$

is coreflective as well.

Proof. By Lemma C.11 it suffices to show that all sheaves vanishing on ∂q are $\flat$-acyclic. To this end, we again view η^j as an exact functor from $\mathrm{Sh}(q_\gamma)$ to homomorphisms in $\mathrm{Sh}(q_\gamma)$. In particular the subcategory of flabby sheaves is η^j-injective

[KS90, Definition 1.8.2]. Suppose that F is a flabby sheaf on q_γ. Then $\eta^j(F) = \eta^j_F$ is an epimorphism by Lemma 7.1. Moreover, as $\mathrm{Sh}(q_\gamma)$ is additive, the full subcategory of all epimorphisms in the arrow category is injective with respect to taking kernels [KS90, Definition 1.8.2]. From this we obtain

$$R\flat \cong R\ker \circ R\eta^j \cong R\ker \circ \mathrm{D}^+(\eta^j)$$

by [KS90, Proposition 1.8.7] and by exactness of η^j.

Now suppose that G is an arbitrary sheaf vanishing on ∂q. Then $j^{-1}G = 0$, and so $\eta^j_G \colon G \to j_*j^{-1}G \cong 0$ is an epimorphism. We thus get

$$R\flat(G) \cong R\ker(\mathrm{D}^+(\eta^j)(G)) = R\ker(\eta^j_G) \cong \ker(\eta^j_G) = G.$$

In particular G is $\flat$-acyclic. (Here the second isomorphism of the previous equation follows from the fact that $\eta^j_G \colon G \to j_*j^{-1}G$ is a ker-injective object of the arrow category.) □

From this point onwards we write $D^+(q_\gamma)$ for the derived category $D^+(\mathrm{Sh}(q_\gamma))$ and we write $D^+(q_\gamma, \partial q)$ for the full subcategory of complexes of sheaves in $D^+(q_\gamma)$, whose cohomology sheaves vanish on ∂q. Then we obtain the following in conjunction with Lemma C.13.

Corollary 7.3. *The category $D^+(q_\gamma, \partial q)$ is a triangulated coreflective subcategory of $D^+(q_\gamma)$ with coreflector $R\flat \colon D^+(q_\gamma) \to D^+(q_\gamma, \partial q)$.*

Adjunctions to Sheaves on the Interior. Now let $\dot{q} \coloneqq q_\gamma \setminus \partial q = q_\gamma \setminus \partial\mathbb{M}$ and let $i \colon \dot{q} \hookrightarrow q_\gamma$ be the corresponding subspace inclusion. Then we have

$$i_! = \flat \circ i_* \colon \mathrm{Sh}(\dot{q}) \to \mathrm{Sh}(q_\gamma, \partial q). \tag{7.4}$$

Moreover, let $\ddot{q} \subseteq q_\gamma$ be the smallest open subset of q_γ containing $\dot{q}$ and let $i_1 \colon \dot{q} \hookrightarrow \ddot{q}$ and $i_2 \colon \ddot{q} \hookrightarrow q_\gamma$ be the corresponding inclusions.

Lemma 7.4. *The adjunction*

$$\begin{tikzcd} \mathrm{Sh}(\ddot{q}) \arrow[r, bend right, "i_1^{-1}"'] \arrow[r, phantom, "\top"] & \mathrm{Sh}(\dot{q}) \arrow[l, bend right, "i_{1*}"'] \end{tikzcd}$$

is an exact adjoint equivalence.

Proof. The inclusion $i_1 \colon \dot{q} \hookrightarrow \ddot{q}$ induces a lattice isomorphism between the topologies of $\dot{q}$ and $\ddot{q}$. Moreover, this isomorphism preserves all covers. □

Lemma 7.5. *For any sheaf F on q_γ the naturally induced map*

$$(\flat \circ \eta^i)_F \colon \flat F \to \flat i_* i^{-1} F = i_! i^{-1} F$$

is an isomorphism.

Proof. By Lemma 7.4 it suffices to check that

$$(\flat \circ \eta^{i_2})_F \colon \flat F \to \flat i_{2*} i_2^{-1} F$$

is an isomorphism. This in turn follows from Lemma D.1. $\square$

Composing the adjunctions

$$\begin{tikzcd} \mathrm{Sh}(q_\gamma, \partial q) \arrow[r, bend right] \arrow[r, phantom, "\top"] & \mathrm{Sh}(q_\gamma) \arrow[l, bend right, "\flat"'] \arrow[r, bend right, "i^{-1}"'] \arrow[r, phantom, "\top"] & \mathrm{Sh}(\dot{q}) \arrow[l, bend right, "i_*"'] \end{tikzcd}$$

we obtain the adjunction

$$\begin{tikzcd} \mathrm{Sh}(q_\gamma, \partial q) \arrow[r, bend right, "i^{-1}"'] \arrow[r, phantom, "\top"] & \mathrm{Sh}(\dot{q}). \arrow[l, bend right, "i_!"'] \end{tikzcd} \tag{7.5}$$

Lemma 7.6. *If ∂q is closed in q_γ, then* (7.5) *is an exact adjoint equivalence.*

Proof. By definition [KS90, Page 93] of the functor $(-)_{\dot{q}} \colon \mathrm{Sh}(q_\gamma) \to \mathrm{Sh}(q_\gamma, \partial q)$, $F \mapsto F_{\dot{q}}$ we have $\flat = (-)_{\dot{q}}$. With this the result follows from Lemma D.2. $\square$

Now whenever ∂q is closed in q_γ, we define the composition

$$\varepsilon^!_F \colon i_! i^{-1} F \xrightarrow{(\flat \circ \eta^i)^{-1}_F} \flat F \xrightarrow{\varepsilon_F} F$$

for any sheaf F on q_γ harnessing Lemma 7.5, where $\varepsilon_F \colon \flat F \to F$ is the counit of the adjunction (7.3).

Lemma 7.7. *If ∂q is closed in q_γ, then we have the $\mathbb{F}$-linear adjunction*

$$\begin{tikzcd} \mathrm{Sh}(q_\gamma) \arrow[r, bend right, "i^{-1}"'] \arrow[r, phantom, "\perp"] & \mathrm{Sh}(\dot{q}) \arrow[l, bend right, "i_!"'] \end{tikzcd}$$

with counit $\varepsilon^! \colon i_! \circ i^{-1} \to \mathrm{id}$.

Proof. Let F be a sheaf on $\dot{q}$, let G be a sheaf on q_γ, and let $\varphi \colon i_! F \to G$ be a sheaf homomorphism. We have to show there is a unique sheaf homomorphism $\varphi^{\#!} \colon F \to i^{-1} G$ such that the diagram

$$\begin{tikzcd} i_! F \arrow[r, "\varphi"] \arrow[d, "i_! \varphi^{\#!}"'] \arrow[dr, dashed, "\varphi^{\#}"] & G \\ i_! i^{-1} G & \flat G \arrow[l, "(\flat \circ \eta^i)_G"] \arrow[u, "\varepsilon_G"'] \end{tikzcd} \tag{7.6}$$

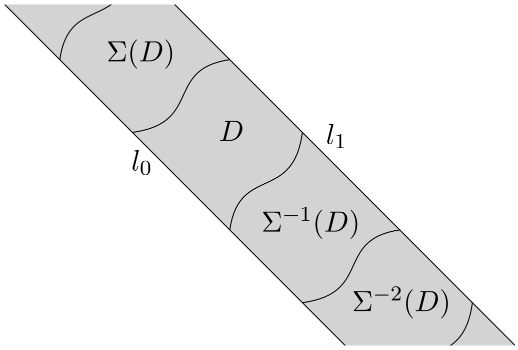


Figure 7.5: The tessellation of $\mathbb{M}$ induced by Σ and D.

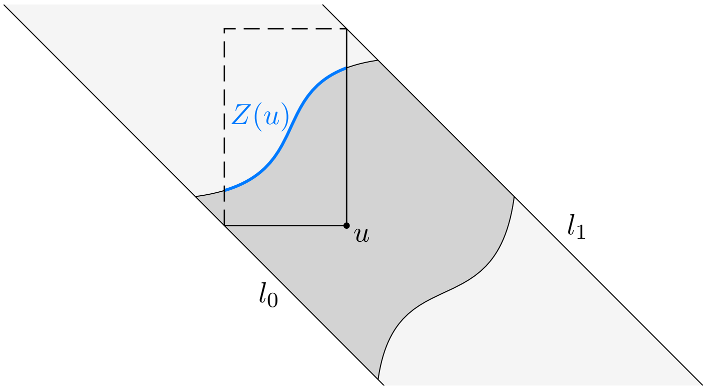


Figure 7.6: The locally closed subset $Z(u) \coloneqq q \cap (\uparrow u) \cap \operatorname{int}(\downarrow \Sigma(u))$.

commutes. Now the adjunction (7.3) yields a unique sheaf homomorphism $\varphi^{\#} : i_! F \to \flat G$ as indicated in (7.6). Moreover, the homomorphism $(\flat \circ \eta^i)_G : \flat G \to i_! i^{-1} G$ is an isomorphism by Lemma 7.5, and hence the vertical arrow on the left-hand side of (7.6) exists uniquely as indicated. Furthermore, $i_! : \operatorname{Sh}(\dot{q}) \to \operatorname{Sh}(q_\gamma)$ is fully faithful by Lemma 7.6 and this implies the claim. □

Construction of a Functor on a Fundamental Domain. The difference of downsets $D \coloneqq C \setminus \Sigma^{-1}(C)$ is a fundamental domain of $\mathbb{M}$ with respect to the action of $\langle \Sigma \rangle$ as shown in Fig. 7.4. Moreover, Σ and D induce the tessellation of $\mathbb{M}$ shown in Fig. 7.5. We now construct a functor

$$\iota_0 : D \to \operatorname{Sh}(q_\gamma, \partial q)$$

from D to the category of $\mathbb{F}$-linear sheaves on q_γ vanishing on ∂q. This will be our first step towards the construction of a sheaf-theoretical Happel functor

$$\iota : \mathbb{M} \to D^+(q_\gamma, \partial q),$$

which assigns an indecomposable object $\iota(u)$ of $D^+(q_\gamma, \partial q)$ to each $u \in \operatorname{int} \mathbb{M}$. For $u \in D$

we define the locally closed subset

$$Z(u) \coloneqq q \cap (\uparrow u) \cap \operatorname{int}(\downarrow \Sigma(u)).$$

Here $\uparrow u$ denotes the upset of u and $\operatorname{int}(\downarrow \Sigma(u))$ denotes the interior of the downset of $\Sigma(u)$. We now define a sheaf $\iota_0(u)$, which restricts to the sheaf of locally constant $\mathbb{F}$-valued functions on $Z(u)$ and that vanishes on $q \setminus Z(u)$. For an open subset $U \subseteq q_\gamma$ we set

$$\iota_0(u)(U) \coloneqq \{f \colon U \cap (\uparrow u) \to \mathbb{F} \mid f|_{U \cap (\uparrow u) \setminus \operatorname{int}(\downarrow \Sigma(u))} = 0\},$$

where functions of type $U \cap (\uparrow u) \to \mathbb{F}$ are required to be locally constant (or equivalently continuous with respect to the discrete topology on $\mathbb{F}$). The restriction homomorphisms of $\iota_0(u)$ are defined by restriction of functions.

Lemma 7.8. *For $u \in D$ the sheaf $\iota_0(u)$ restricts to the sheaf of locally constant $\mathbb{F}$-valued functions on $Z(u)$.*

Proof. For this proof let $\mathbb{F}_{Z(u)}$ denote the sheaf of locally constant functions on $Z(u)$. Using the subspace inclusion $j \colon Z(u) \hookrightarrow q_\gamma$ and the adjunction $j^{-1} \dashv j_*$ it suffices to provide an isomorphism $\iota_0(u) \to j_*\mathbb{F}_{Z(u)}$. Now for any open subset $U \subseteq q_\gamma$ and any locally constant function $f \colon U \cap Z(u) \to \mathbb{F}$ there is a unique locally constant extension $U \cap (\uparrow u) \to \mathbb{F}$ vanishing on $U \cap (\uparrow u) \setminus \operatorname{int}(\downarrow \Sigma(u))$. Thus, the restriction of functions yields the desired isomorphism $\iota_0(u) \to j_*\mathbb{F}_{Z(u)}$. $\square$

Lemma 7.9. *For $u \in D$ the sheaf $\iota_0(u)$ vanishes on $q \setminus Z(u)$.*

Proof. For an open subset $U \subseteq q_\gamma$ and a locally constant function $f \colon U \cap (\uparrow u) \to \mathbb{F}$ vanishing at a point $t \in U \cap (\uparrow u) \setminus \operatorname{int}(\downarrow \Sigma(u))$ the subset $f^{-1}(0)$ is a relatively open neighbourhood of t. Thus, we have $\iota_0(u)_t \cong \{0\}$. $\square$

Corollary 7.10. *For $u \in D$ we have $\iota_0(u) \cong \mathbb{F}_{Z(u)}$, where $\mathbb{F}_{Z(u)}$ is the sheaf defined at [KS90, Page 93].*

Proof. By the previous two lemmas the sheaf $\iota_0(u)$ satisfies the characterization [KS90, Proposition 2.3.6.(i)] of the sheaf $\mathbb{F}_{Z(u)}$ defined at [KS90, Page 93]. $\square$

In view of this Corollary 7.10 we will often abuse notation and write $\iota_0(u) = \mathbb{F}_{Z(u)}$ for $u \in D$. For $u \preceq v \in D$ the restriction of functions yields a sheaf homomorphism

$$\iota_0(u \preceq v) \colon \iota_0(u) \to \iota_0(v)$$

completing the construction of the desired functor

$$\iota_0 \colon D \to \operatorname{Sh}(q_\gamma, \partial q).$$

We note the following two properties of ι_0, which we will need later.

Lemma 7.11. *For $u, v \in D$ we have*

$$\operatorname{Hom}_{\operatorname{Sh}(q_\gamma)}(\iota_0(u), \iota_0(v)) = \langle \iota_0(u \preceq v) \rangle \cong \begin{cases} \mathbb{F} & [u,v] \subset \operatorname{int} \mathbb{M} \\ \{0\} & \textit{otherwise.} \end{cases}$$

Lemma 7.12. *Let F be an $\mathbb{F}$-linear sheaf on q_γ and let $u \in D$. Then we have a natural $\mathbb{F}$-linear isomorphism*

$$\operatorname{Hom}_{\operatorname{Sh}(q_\gamma)}(\iota_0(u), F) \cong \Gamma_{q \cap (\uparrow u)}(q \cap \operatorname{int}(\downarrow \Sigma(u)); F).$$

Proof. We have

$$\begin{aligned} \operatorname{Hom}_{\operatorname{Sh}(q_\gamma)}(\iota_0(u), F) &= \operatorname{Hom}_{\operatorname{Sh}(q_\gamma)}(\mathbb{F}_{Z(u)}, F) \\ &= \Gamma(q_\gamma; \mathscr{H}\!om(\mathbb{F}_{Z(u)}, F)) \\ &\cong \Gamma(q_\gamma; \Gamma_{Z(u)}(F)) \\ &= \Gamma(q_\gamma; \Gamma_{q \cap \operatorname{int}(\downarrow \Sigma(u))}(\Gamma_{q \cap (\uparrow u)}(F))) \\ &= \Gamma_{q \cap (\uparrow u)}(q \cap \operatorname{int}(\downarrow \Sigma(u)); F). \end{aligned}$$

Here $\mathscr{H}\!om$ denotes the internal homomorphism functor of sheaves as defined in [KS90, Definition 2.2.7]. The second equality follows from [KS90, (2.2.6)]. The isomorphism follows from [KS90, (2.3.16)]. The fourth equality follows from [KS90, Proposition 2.3.9.(ii)] and the last equality from [KS90, Proposition 2.3.9.(iii) and (2.3.12)]. □

Remark 7.13. In the previous example of an A_6-quiver Q, the construction analogous to the formation of the fundamental domain $D = C \setminus \Sigma^{-1}(C)$ yields the Auslander–Reiten quiver of Q. Within the series "Continuous Quivers of Type A" by K. Igusa, J. D. Rock, and G. Todorov there is an article dedicated to Auslander–Reiten theory in a continuous setting [Roc19].

Resolution by Sheaves that are Acyclic on Connected Opens. Now let $u \in D$. As $\phi_\gamma \colon q \to q_\gamma$ is continuous, the subset $Z(u) = \phi_\gamma^{-1}(Z(u)) \subseteq q$ is locally closed in q as well. Moreover, we have the pair of adjoint functors

$$\begin{tikzcd} \operatorname{Sh}(q_\gamma) \arrow[r, bend right, "\phi_\gamma^{-1}"'] \arrow[r, phantom, "\top"] & \operatorname{Sh}(q) \arrow[l, bend right, "\phi_{\gamma *}"'] \end{tikzcd}$$

between categories of $\mathbb{F}$-linear sheaves on q and q_γ. Thus, we have

$$(\phi_\gamma^{-1} \circ \iota_0)(u) = \phi_\gamma^{-1} \mathbb{F}_{Z(u)} \cong \mathbb{F}_{Z(u)}.$$

Furthermore, the adjunction unit

$$(\eta \circ \iota_0)_u : \mathbb{F}_{Z(u)} \to \phi_{\gamma *} \phi_\gamma^{-1} \mathbb{F}_{Z(u)} \cong \phi_{\gamma *} \mathbb{F}_{Z(u)} \tag{7.7}$$

is an isomorphism, hence ϕ_γ^{-1} is fully faithful when restricted to the essential image of ι_0. It will be convenient to have a description of $\phi_\gamma^{-1} \circ \iota_0$ in terms of complexes of sheaves that are $\Gamma(I;-)$-acyclic [KS90, Exercise I.19] for any connected open subset $I \subseteq q$. More specifically, we resolve each sheaf $\mathbb{F}_{Z(u)}$ in $\mathrm{Sh}(q)$ for $u \in D$ by sheaves of the form $\mathbb{F}_A$, where $A \subseteq q$ is closed with at most two connected components as opposed to being merely locally closed. Such a sheaf $\mathbb{F}_A$ is $\Gamma(I;-)$-acyclic as $\mathbb{F}_A|_I$ can be written as the pushforward of the locally constant sheaf on $A \cap I$ along the inclusion $A \cap I \hookrightarrow I$ and by the homotopy invariance of sheaf cohomology [KS90, Proposition 2.7.5]. To this end, we have the following generalization of [BBF24, Proposition 2.1.1].

Proposition 7.14. *Let $\mathcal{P}$ denote the set of pairs of closed subspaces of q. Then there is a unique monotonic map*

$$\rho\colon D^\circ \to \mathcal{P}$$

with the following three properties:

(1) For any $t \in q$ we have $\rho(t) = (q \cap (\uparrow t), \partial q \cap (\uparrow t))$, where $\partial q := q \cap \partial\mathbb{M}$ and $\partial\mathbb{M} := l_0 \cup l_1$.

(2) For any $u \in D \cap \partial\mathbb{M}$ the two components of $\rho(u)$ are identical.

(3) For any axis-aligned rectangle contained in D° the corresponding joins and meets are preserved by ρ. (A join in D° is a meet in D and vice versa.)

More concretely, the map ρ can be described by the formula

$$\rho(u) = (\rho_0(u), \rho_1(u)), \quad \text{where} \quad \begin{cases} \rho_0(u) = q \cap (\uparrow u) & \text{and} \\ \rho_1(u) = q \setminus \mathrm{int}(\downarrow \Sigma(u)), \end{cases} \tag{7.8}$$

for any $u \in D^\circ$. Moreover, we have

$$Z(u) = \rho_0(u) \setminus \rho_1(u). \tag{7.9}$$

Now for $u \in D$ let $\kappa_0(u)$ be the complex of sheaves with

$$(\kappa_0(u))^n = \begin{cases} \mathbb{F}_{\rho_0(u)} & n = 0 \\ \mathbb{F}_{\rho_1(u)} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

and the differential

$$\delta_u^0\colon \mathbb{F}_{\rho_0(u)} \to \mathbb{F}_{\rho_1(u)}$$

being the homomorphism which induces the identity on all stalks isomorphic to $\mathbb{F}$. Similarly, we may extend κ_0 to a functor

$$\kappa_0\colon D \to \mathrm{C}^+(q) := \mathrm{C}^+(\mathrm{Sh}(q))$$

from D to the category of complexes of $\mathbb{F}$-linear sheaves on q. In conjunction with (7.9) we obtain

$$H^0 \circ \kappa_0 \cong \phi_\gamma^{-1} \circ \iota_0 \quad \text{and} \quad H^1 \circ \kappa_0 \cong 0. \tag{7.10}$$

This equation yields the following two lemmas.

Lemma 7.15. *For any axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in D$ as shown in Fig. 7.7 the sequence*

$$0 \to \iota_0(u) \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} \iota_0(v_1) \oplus \iota_0(v_2) \xrightarrow{\begin{pmatrix}1 & -1\end{pmatrix}} \iota_0(w) \to 0 \tag{7.11}$$

is exact. Here we use 1 as a shorthand for the corresponding induced map, e.g. $\iota_0(u \preceq v_1)\colon \iota_0(u) \xrightarrow{1} \iota_0(v_1)$.

Proof. By Proposition 7.14.(3) the sequence of complexes

$$0 \to \kappa_0(u) \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} \kappa_0(v_1) \oplus \kappa_0(v_2) \xrightarrow{\begin{pmatrix}1 & -1\end{pmatrix}} \kappa_0(w) \to 0$$

is exact. In conjunction with (7.10) and the Snake Lemma we obtain the exactness of

$$0 \to \phi_\gamma^{-1}\iota_0(u) \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} \phi_\gamma^{-1}\iota_0(v_1) \oplus \phi_\gamma^{-1}\iota_0(v_2) \xrightarrow{\begin{pmatrix}1 & -1\end{pmatrix}} \phi_\gamma^{-1}\iota_0(w) \to 0.$$

As we can check the exactness of (7.11) on the level of stalks, this is sufficient. □

Lemma 7.16. *For any $u \in D$ the derived unit*

$$\left(\eta^{\mathrm{D}^+(q_\gamma)} \circ \iota_0\right)_u : \mathbb{F}_{Z(u)} \to R\phi_{\gamma *}\phi_\gamma^{-1}\mathbb{F}_{Z(u)} \cong R\phi_{\gamma *}\mathbb{F}_{Z(u)}$$

is an isomorphism of the derived category $\mathrm{D}^+(q_\gamma)$, where we view $\mathbb{F}_{Z(u)}$ as a complex concentrated in degree 0.

Proof. As $(\eta \circ \iota_0)_u$ is an isomorphism, it suffices to show that $\phi_\gamma^{-1}\mathbb{F}_{Z(u)} \cong \mathbb{F}_{Z(u)}$ is $\phi_{\gamma *}$-acyclic by Lemma C.11 from Appendix C.4. To this end, it suffices to check that $\mathbb{F}_{Z(u)} \in \mathrm{Sh}(q)$ is $\Gamma(U; -)$-acyclic for each open subset U of some basis $\mathcal{B}$ of q_γ by Lemma C.12. Now let $\mathcal{B} := \{q \cap \mathrm{int}(\downarrow v) \mid \Sigma(u) \not\preceq v\}$ be our choice of a basis and let $U = q \cap \mathrm{int}(\downarrow v) \in \mathcal{B}$. As U is connected, $\kappa_0(u)$ is a $\Gamma(U; -)$-acyclic resolution of $\mathbb{F}_{Z(u)}$. Moreover, as $\Sigma(u) \not\preceq v$, the differential $\Gamma(U; \delta_u^0)$ is surjective, hence $\mathbb{F}_{Z(u)}$ is $\Gamma(U; -)$-acyclic. □

Extension to a Sheaf-Theoretical Happel Functor. The short exact sequence (7.11) yields a particular distinguished triangle

$$\iota_0(u) \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} \iota_0(v_1) \oplus \iota_0(v_2) \xrightarrow{\begin{pmatrix}1 & -1\end{pmatrix}} \iota_0(w) \xrightarrow{\partial'} \iota_0(u)[1] \tag{7.12}$$

in the derived category $\mathrm{D}^+(q_\gamma)$, where we view each sheaf as a complex concentrated in degree 0. Now let $\Sigma := (-)[1]$ and let

$$R_D := \{(w, \hat{u}) \in D \times \Sigma(D) \mid w \preceq \hat{u} \preceq \Sigma(w)\}$$

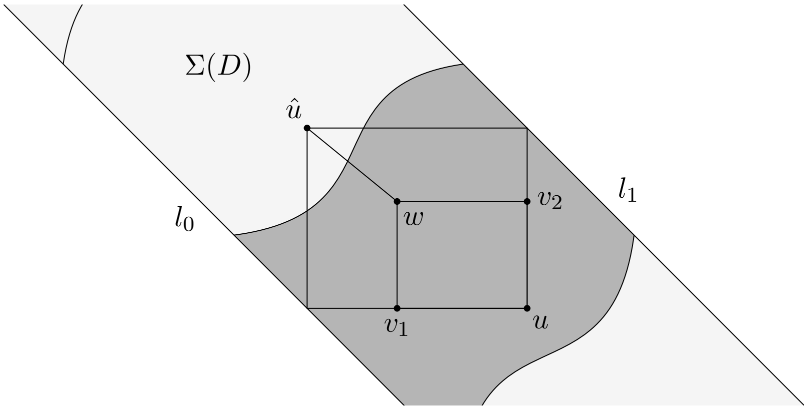


Figure 7.7: An axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in D$ and the corresponding pair $(w, \hat{u}) \in R_D$.

as in Definition 1.17. As shown in Fig. 7.7, any pair $(w, \hat{u}) \in R_D$ determines an axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in D$ with $T(u) = \hat{u}$. Moreover, the collection of homomorphisms ∂' for the corresponding triangles (7.12) forms a natural transformation as in the diagram

$$\begin{tikzcd}
R_D \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\swarrow \partial'"] & D \arrow[d, "\iota_0"] \\
\Sigma(D) \arrow[r, "\Sigma \circ \iota_0 \circ \Sigma^{-1}"'] & \mathrm{D}^+(q_\gamma),
\end{tikzcd}$$

where we consider R_D as a subposet of $D \times \Sigma(D)$ with the product order and with $\mathrm{pr}_1 \colon R_D \to D$ and $\mathrm{pr}_2 \colon R_D \to \Sigma(D)$ being the canonical projections from R_D to the first and second component, respectively. Thus, Proposition 1.23 yields a functor

$$\iota \colon \mathbb{M} \to \mathrm{D}^+(q_\gamma, \partial q)$$

that is a strictly stable extension of $\iota_0 \colon D \to \mathrm{Sh}(q_\gamma, \partial q) \hookrightarrow \mathrm{D}^+(q_\gamma, \partial q)$ in the sense that

$$\Sigma \circ \iota = \iota(-)[1] = \iota \circ \Sigma \quad \text{and} \quad \iota|_D = \iota_0, \tag{7.13}$$

see also Definitions 1.8 and 1.10.

Definition 7.17. We name the strictly stable functor $\iota \colon \mathbb{M} \to \mathrm{D}^+(q_\gamma, \partial q)$ satisfying the equations (7.13), which uniquely exists by Proposition 1.23, the *sheaf-theoretical Happel functor.*

As $\iota \colon \mathbb{M} \to D^+(q_\gamma, \partial q)$ vanishes on $\partial\mathbb{M}$ by construction, there is a unique transform to an $\mathbb{F}$-linear functor:

$$\begin{tikzcd}
\mathbb{F}\mathbb{M}/\partial\mathbb{M} \arrow[d, dashed, "\iota^\flat"'] & \mathbb{M} \arrow[l] \arrow[dl, "\iota"] \\
D^+(q_\gamma, \partial q). &
\end{tikzcd} \tag{7.14}$$

Comparing (7.2) and (7.14) we see that $\iota\colon \mathbb{M} \to D^+(q_\gamma, \partial q)$ is indeed a sheaf-theoretical counterpart of the Happel functor (7.1). Moreover, the following Proposition 7.19 implies that $\iota^\flat\colon \mathbb{F}\mathbb{M}/\partial\mathbb{M} \to D^+(q_\gamma, \partial q)$ is fully faithful as well and Corollary 9.2 below implies that the essential image of $\iota^\flat$ is the full subcategory of indecomposable *tame derived sheaves* (Definition 8.30) in $D^+(q_\gamma, \partial q)$.

Remark 7.18. In Remark 1.1 we considered the simplicial localization $L\mathbb{M}$ of $\mathbb{M}$ at the arrows between boundary points. Now for any two points $u, v \in \mathbb{M}$ the simplicial mapping set $\operatorname{Hom}_{L\mathbb{M}}(u, v)$ contains the weakly contractible subcomplex $I(u, v)$ containing all simplices of $\operatorname{Hom}_{L\mathbb{M}}(u, v)$ with at least one vertex a boundary point in $\partial\mathbb{M}$. For these (and other) simplicial sets S, let $\mathbb{F}S$ denote the Moore complex of the corresponding simplicial vector space over $\mathbb{F}$ in the sense of [Sta24, Tag 0194]. By passing to this Moore complex for each of these simplicial mapping sets we obtain the dg-category $\mathbb{F}_\bullet(L\mathbb{M})$ over $\mathbb{F}$ with the 2-sided dg-ideal

$$\mathbb{F}I\colon (u, v) \mapsto \mathbb{F}I(u, v).$$

As it turns out, the homotopy category $\operatorname{Ho}\big(\mathbb{F}_\bullet(L\mathbb{M})/\mathbb{F}I\big)$ of the quotient dg-category $\mathbb{F}_\bullet(L\mathbb{M})/\mathbb{F}I$ is isomorphic to $\mathbb{F}\mathbb{M}/\partial\mathbb{M}$. We also note that the dg-category $\mathbb{F}_\bullet(L\mathbb{M})/\mathbb{F}I$ has the added advantage over the simplicial localization $L\mathbb{M}$ of being fibrant with respect to the Dwyer–Kan model structure on dg-categories.

Generation of Homomorphisms by ι. We close this Chapter 7 showing the following counterpart to Lemma 7.11.

Proposition 7.19. *For $u, v \in \mathbb{M}$ we have*

$$\operatorname{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(u), \iota(v)) = \langle \iota(u \preceq v)\rangle \cong \begin{cases} \mathbb{F} & [u, v] \subset \operatorname{int}\mathbb{M} \\ \{0\} & \textit{otherwise.} \end{cases}$$

The long exact sequence from the following lemma is one ingredient to our proof of this proposition.

Lemma 7.20. *For all $u, w \in D$ there is a long exact sequence*

$$\begin{array}{l} \longrightarrow \operatorname{Hom}_{\mathrm{D}^+(q)}(\mathbb{F}_{Z(u)}, \mathbb{F}_{Z(w)}[3]) \longrightarrow 0 \longrightarrow \cdots, \\[2ex] \longrightarrow \operatorname{Hom}_{\mathrm{D}^+(q)}(\mathbb{F}_{Z(u)}, \mathbb{F}_{Z(w)}[2]) \longrightarrow 0 \longrightarrow 0 \longrightarrow \\[2ex] \longrightarrow \operatorname{Hom}_{\mathrm{D}^+(q)}(\mathbb{F}_{Z(u)}, \mathbb{F}_{Z(w)}[1]) \to \operatorname{coker}\Gamma\left(U; \delta^0_w\right) \to \operatorname{coker}\Gamma\left(V; \delta^0_w\right) \longrightarrow \\[2ex] 0 \longrightarrow \operatorname{Hom}_{\mathrm{D}^+(q)}(\mathbb{F}_{Z(u)}, \mathbb{F}_{Z(w)}) \longrightarrow \Gamma\left(U; \mathbb{F}_{Z(w)}\right) \longrightarrow \Gamma\left(V; \mathbb{F}_{Z(w)}\right) \longrightarrow \end{array} \tag{7.15}$$

where $U \coloneqq q \cap \operatorname{int}(\downarrow \Sigma(u))$ *and* $V \coloneqq q \setminus (\uparrow u)$.

Proof. We have $V = U \setminus Z(u)$ and thus there is a distinguished triangle

$$R\Gamma_{Z(u)}\left(\mathbb{F}_{Z(w)}\right) \to R\Gamma_U\left(\mathbb{F}_{Z(w)}\right) \to R\Gamma_V\left(\mathbb{F}_{Z(w)}\right) \to R\Gamma_{Z(u)}\left(\mathbb{F}_{Z(w)}\right)[1]$$

by [KS90, Equation (2.6.32)]. Applying the *cohomological* functor [KS90, Definition 1.5.2] $H^0(q; -)$ to this triangle we obtain the long exact sequence

$$\begin{tikzcd}
{} \arrow[r] & H^2_{Z(u)}\left(q;\mathbb{F}_{Z(w)}\right) \arrow[r] & H^2_U\left(q;\mathbb{F}_{Z(w)}\right) \arrow[r] & \cdots . \\
{} \arrow[r] & H^1_{Z(u)}\left(q;\mathbb{F}_{Z(w)}\right) \arrow[r] & H^1_U\left(q;\mathbb{F}_{Z(w)}\right) \arrow[r] & H^1_V\left(q;\mathbb{F}_{Z(w)}\right) \arrow[ulll, rounded corners] \\
0 \arrow[r] & H^0_{Z(u)}\left(q;\mathbb{F}_{Z(w)}\right) \arrow[r] & H^0_U\left(q;\mathbb{F}_{Z(w)}\right) \arrow[r] & H^0_V\left(q;\mathbb{F}_{Z(w)}\right) \arrow[ulll, rounded corners]
\end{tikzcd}$$

Now by [KS90, Equation (2.6.9)] we have

$$H^n_{Z(u)}\left(q;\mathbb{F}_{Z(w)}\right) \cong \operatorname{Hom}_{\mathrm{D}^+(q)}\left(\mathbb{F}_{Z(u)}, \mathbb{F}_{Z(w)}[n]\right).$$

By substituting the corresponding terms in above exact sequence we obtain an exact sequence whose left column coincides with the sequence (7.15). Moreover, by [KS90, Proposition 2.3.9.(iii) and Remark 2.6.9] we have

$$H^n_U\left(q;\mathbb{F}_{Z(w)}\right) \cong H^n\left(U;\mathbb{F}_{Z(w)}|_U\right) \cong H^n\left(U;\mathbb{F}_{Z(w)}\right)$$

and similarly

$$H^n_V\left(q;\mathbb{F}_{Z(w)}\right) \cong H^n\left(V;\mathbb{F}_{Z(w)}\right)$$

for all $n \in \mathbb{N}_0$. Furthermore, as U is open and connected and V is a disjoint union of at most two connected open subsets of q, the complex $\kappa_0(w)$ is a resolution of $\mathbb{F}_{Z(w)}$ by $\Gamma(U;-)$-acyclic and $\Gamma(V;-)$-acyclic sheaves by (7.10). Thus, we may replace each term in the center and right column of above exact sequence by $H^n(\Gamma(U;\kappa_0(w))$ and $H^n(\Gamma(V;\kappa_0(w))$, respectively. Simplifying these terms using (7.10) for each $n \in \mathbb{N}_0$ we obtain the long exact sequence (7.15). □

To make use of the long exact sequence from the previous lemma, we need to understand the map $\operatorname{coker}\Gamma\left(U;\delta^0_w\right) \to \operatorname{coker}\Gamma\left(V;\delta^0_w\right)$ from this sequence.

Lemma 7.21. *Suppose we have* $u, w \in D$ *with* $w \preceq u \npreceq \Sigma(w)$ *and* U *and* V *as in the previous lemma. Then the linear map* $\operatorname{coker}\Gamma\left(U;\delta^0_w\right) \to \operatorname{coker}\Gamma\left(V;\delta^0_w\right)$ *is an isomorphism.*

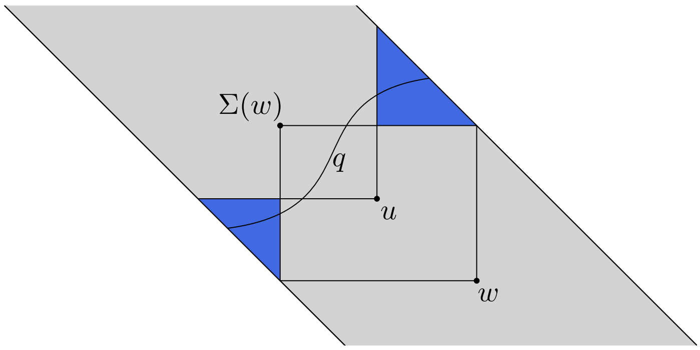


Figure 7.8: Here $u \in [w, \Sigma(w)]$ and $V \cap \rho_1(w)$ has two connected components.

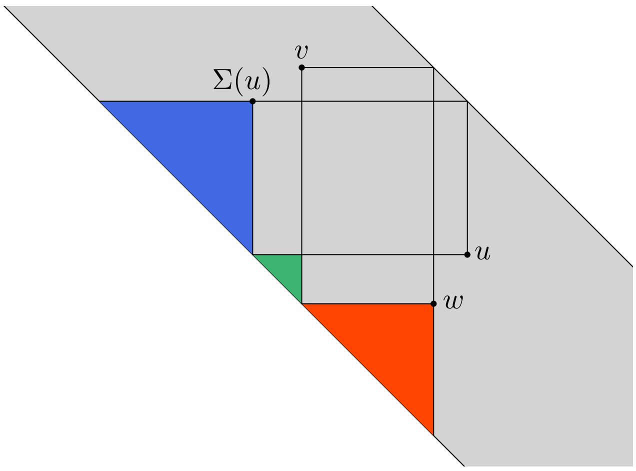


Figure 7.9: Here $u \preceq v$ factors through a point in $\partial\mathbb{M}$.

Proof. We distinguish between two cases. If $q \subset (\uparrow w)$, then two connected components of $\rho_1(w)$ are contained in the same connected component of $\rho_0(w)$. As $u \notin [w, \Sigma(w)]$ this property still holds after we intersect both sets with U or V. Thus, when we apply the cokernel to the map of maps $\Gamma\left(U; \delta_w^0\right) \to \Gamma\left(V; \delta_w^0\right)$, we obtain the identity $K \xrightarrow{\mathrm{id}} K$ (up to isomorphism of maps). If $q \not\subset (\uparrow w)$ both cokernels vanish. □

Using the previous two lemmas we show the following "vanishing theorem".

Lemma 7.22. *If we have $u, v \in \mathbb{M}$ with $[u, v] \cap \partial\mathbb{M} \neq \emptyset$, then*

$$\mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(u), \iota(v)) \cong \{0\}.$$

Proof. By Lemma 7.16 it suffices to show that

$$\mathrm{Hom}_{\mathrm{D}^+(q)}(\phi_\gamma^{-1}\iota(u), \phi_\gamma^{-1}\iota(v)) \cong \{0\}.$$

Without loss of generality we assume $u \in D$. For

$$v \notin D \cup \Sigma(D) \cup \Sigma^2(D)$$

we have

$$\mathrm{Hom}_{\mathrm{D}^+(q)}(\phi_\gamma^{-1}\iota(u), \phi_\gamma^{-1}\iota(v)) \cong \mathrm{Hom}_{\mathrm{D}^+(q)}(\mathbb{F}_{Z(u)}, \mathbb{F}_{Z(w)}[n])$$

for some $w \in D$ and $n \in \mathbb{Z} \setminus \{0, 1, 2\}$, and thus the statement follows directly from Lemma 7.20 in this case. If $v \in D$, then the statement follows from Lemma 7.11.

Now suppose we have $v \in \Sigma^2(D)$, then let $w := \Sigma^{-2}(v)$ and let U and V be as in the previous two lemmas. We have to show that

$$\mathrm{Hom}_{\mathrm{D}^+(q)}(\phi_\gamma^{-1}\iota(u), \phi_\gamma^{-1}\iota(v)) \cong \mathrm{Hom}_{\mathrm{D}^+(q)}(\mathbb{F}_{Z(u)}, \mathbb{F}_{Z(w)}[2]) \cong \{0\}.$$

By Lemma 7.20 it suffices to show that the map

$$\psi\colon\ \operatorname{coker}\Gamma\left(U; \delta_w^0\right) \to \operatorname{coker}\Gamma\left(V; \delta_w^0\right)$$

is an epimorphism. Now in order for $\operatorname{coker}\Gamma\left(V; \delta_w^0\right)$ to be non-zero, there need to be at least two connected components in $V \cap \rho_1(w)$ which lie in the same connected component of the superset $V \cap \rho_0(w)$. Moreover, for $V \cap \rho_1(w)$ to have at least two connected components we need to have $w \preceq u$. If $u \not\preceq \Sigma(w)$, then ψ is an isomorphism by Lemma 7.21. If $u \preceq \Sigma(w)$, then the two components of $V \cap \rho_1(w)$ lie in different components of $V \cap \rho_0(w)$ as illustrated by Fig. 7.8 and thus $\operatorname{coker}\Gamma\left(V; \delta_w^0\right) \cong \{0\}$, hence ψ is surjective.

Now suppose we have $v \in \Sigma(D)$, then let $w := \Sigma^{-1}(v)$ and let U and V be as in the previous two lemmas. We have to show that

$$\mathrm{Hom}_{\mathrm{D}^+(q)}(\phi_\gamma^{-1}\iota(u), \phi_\gamma^{-1}\iota(v)) \cong \mathrm{Hom}_{\mathrm{D}^+(q)}(\mathbb{F}_{Z(u)}, \mathbb{F}_{Z(w)}[1]) \cong \{0\}.$$

By Lemma 7.20 it suffices to show that the map

$$\varphi\colon \Gamma\left(U; \mathbb{F}_{Z(w)}\right) \to \Gamma\left(V; \mathbb{F}_{Z(w)}\right)$$

is an epimorphism and that

$$\psi\colon\ \operatorname{coker}\Gamma\left(U; \delta_w^0\right) \to \operatorname{coker}\Gamma\left(V; \delta_w^0\right)$$

is injective. If $u \not\preceq v$, then $Z(w)$ is not a closed subset of V and thus $\Gamma\left(V; \mathbb{F}_{Z(w)}\right) \cong \{0\}$, hence φ is surjective. Moreover, ψ is an isomorphism by Lemma 7.21. Finally, we consider the case $u \preceq v$. By symmetry, we may assume without loss of generality that the y-coordinate of v is at least as large as the y-coordinate of $\Sigma(u)$, see also Fig. 7.9. In this case $U \cap \rho_1(w)$ has at most one connected component and thus $\operatorname{coker}\Gamma\left(U; \delta_w^0\right) \cong \{0\}$, hence ψ is injective. Moreover, q has to intersect at least one of the three colored triangles in Fig. 7.9. (It may happen that the blue and the red triangle overlap, in which case the green triangle is empty.) Here each triangle is closed at the right edge and open at the top edge in the sense that for each triangle, the vertical edge on the right is part

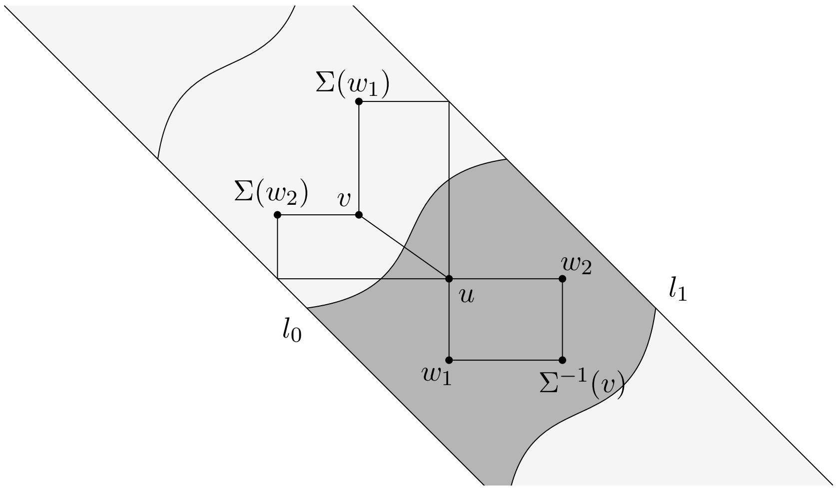


Figure 7.10: The axis-aligned rectangle in D determined by u and v.

of the triangle, but not the horizontal edge at the top. If q intersects the blue triangle, then $Z(w)$ and V are disjoint, hence $\Gamma\left(V;\mathbb{F}_{Z(w)}\right) \cong \{0\}$ and thus φ is surjective. If q intersects the green triangle, then $V \cap Z(w)$ is not a non-empty closed subset of V, hence we have $\Gamma\left(V;\mathbb{F}_{Z(w)}\right) \cong \{0\}$ in this case as well and thus φ is surjective. If q intersects the red triangle, then $U \cap Z(w)$ is a closed subset of U and $V \cap Z(w)$ has at most one connected component, hence φ is surjective. $\square$

Now we can deduce Proposition 7.19 from this "vanishing theorem".

Proof of Proposition 7.19. Without loss of generality we assume $u \in D$. If

$$[u, v] \cap \partial\mathbb{M} \neq \emptyset,$$

then the statement follows from the previous Lemma 7.22. Now suppose we have $[u, v] \subset \operatorname{int}\mathbb{M}$. If $v \in D$, the statement follows from Lemma 7.11. For $v \in \Sigma(D)$ let $w_1, w_2 \in D$ be as in Fig. 7.10. Now by construction of ι we have $\iota(u \preceq v) = \partial'$, where ∂' is the boundary map of the triangle

$$\iota\left(\Sigma^{-1}(v)\right) \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} \iota(w_1) \oplus \iota(w_2) \xrightarrow{\begin{pmatrix}1 & -1\end{pmatrix}} \iota(u) \xrightarrow{\partial'} \iota(v)[1]$$

associated to the short exact sequence

$$0 \to \iota_0\left(\Sigma^{-1}(v)\right) \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} \iota_0(w_1) \oplus \iota_0(w_2) \xrightarrow{\begin{pmatrix}1 & -1\end{pmatrix}} \iota_0(u) \to 0.$$

Applying the cohomological functor $\mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(u), -)$ to this triangle we obtain the exact sequence

$$\begin{array}{c}
\mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(u), \iota(w_1)) \oplus \mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(u), \iota(w_2)) \\
\Big\downarrow {\scriptstyle (1\ \ -1)} \\
\mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(u), \iota(u)) \\
\Big\downarrow {\scriptstyle \mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(u), \iota(u \preceq v))} \\
\mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(u), \iota(v)) \\
\Big\downarrow {\scriptstyle \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \\
\mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(u), (\iota \circ \Sigma)(w_1)) \oplus \mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(u), (\iota \circ \Sigma)(w_2)).
\end{array}$$

By the previous Lemma 7.22 this exact sequence starts with $\{0\}$ and ends with $\{0\}$, hence $\mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(u), \iota(u \preceq v))$ is an isomorphism. Moreover,

$$\mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(u), \iota(u)) = \langle \iota(u \preceq u) \rangle = \langle \mathrm{id}_{\iota(u)} \rangle \cong \mathbb{F}$$

by Lemma 7.11 and thus

$$\mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(u), \iota(v)) = \langle \iota(u \preceq v) \rangle \cong \mathbb{F}. \qquad \square$$

8 Induced Cohomological Presheaves

In this Chapter 8 we use the sheaf-theoretical Happel functor

$$\iota\colon \mathbb{M} \to \mathrm{D}^+(q_\gamma, \partial q) \hookrightarrow \mathrm{D}^+(q_\gamma),$$

that we constructed in the previous Chapter 7 to define a cohomological functor on the triangulated category $\mathrm{D}^+(q_\gamma)$ preserving a fair amount of information. To this end, we consider a bounded below cochain complex F of sheaves on q_γ as an object of the derived category $\mathrm{D}^+(q_\gamma)$. Then we have the presheaf

$$h_\gamma(F) \coloneqq \mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(-), F)\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$$

on $\mathbb{M}$ valued in the category of vector spaces over $\mathbb{F}$. As $\iota(u) \cong 0$ for all $u \in \partial\mathbb{M}$ the presheaf $h_\gamma(F)$ vanishes on $\partial\mathbb{M}$.

Lemma 8.1. *The presheaf $h_\gamma(F)\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ has bounded above support.*

Proof. Since F is a bounded below chain complex of sheaves, there is an integer $n \in \mathbb{Z}$, such that $F^k \cong 0$ for all $k < -n$. Thus, we have $h_\gamma(F)(u) \cong \{0\}$ for all $u \in \mathbb{M} \setminus \Sigma^n(Q)$. $\square$

The following Lemma 8.2 shows that the presheaf $h_\gamma(F)\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ satisfies the exactness properties of Definition 1.26.

Lemma 8.2. *The presheaf $h_\gamma(F)\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ is cohomological.*

Proof. If we apply the cohomological functor $\mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(-, F)\colon D^+(q_\gamma) \to \mathrm{Vect}_\mathbb{F}$ to the distinguished triangle (7.12) for $u \preceq v_1, v_2 \preceq w \in D$ as in Fig. 7.7, then we obtain the long exact sequence

$$\cdots \longrightarrow h_\gamma(F)(\Sigma(u)) \longrightarrow h_\gamma(F)(w) \longrightarrow h_\gamma(F)(v_1) \oplus h_\gamma(F)(v_2) \xrightarrow{(1\ \ -1)} h_\gamma(F)(u) \longrightarrow h_\gamma(F)(\Sigma^{-1}(w)) \longrightarrow \cdots .$$

Thus, the presheaf $h_\gamma(F)\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ is cohomological by Proposition 1.28.(3). $\square$

Using the Yoneda Lemma 2.4, we may rephrase Proposition 7.19 as follows.

Corollary 8.3. *The unique natural transformation*

$$B_v \to (h_\gamma \circ \iota)(v) = \mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(-), \iota(v))$$

sending $1 \in \mathbb{F} = B_v(v)$ *to* $\mathrm{id}_{\iota(v)} \in \mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(v), \iota(v))$ *is a natural isomorphism.*

The following proposition provides a description of $h_\gamma(F)$ in terms of local sheaf cohomology.

Proposition 8.4. *Let* $u \in D$ *and let* $n \in \mathbb{Z}$*, then we have a natural isomorphism*

$$h_\gamma(F)(\Sigma^{-n}(u)) = \mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}((\iota \circ \Sigma^{-n})(u), F) \cong H^n_{q \cap (\uparrow u)}(q \cap \mathrm{int}(\downarrow \Sigma(u)); F).$$

Proof. By Lemma 7.12 we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}((\iota \circ \Sigma^{-n})(u), F) &= \mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota_0(u)[-n], F) \\ &= \mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota_0(u), F[n]) \\ &\cong H^n_{q \cap (\uparrow u)}(q \cap \mathrm{int}(\downarrow \Sigma(u)); F). \end{aligned}$$

□

We also note that the assignment $F \mapsto h_\gamma(F) = \mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(-), F)$ is functorial in F. Therefore, we obtain the cohomological functor $h_\gamma \colon D^+(q_\gamma) \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ}$. Moreover, we take note of the following.

Lemma 8.5. *For each complex of sheaves* F *of the derived category* $D^+(q_\gamma)$ *the counit* $\varepsilon_F \colon R\flat(F) \to F$ *of the adjunction*

$$\begin{tikzcd} D^+(q_\gamma, \partial q) \arrow[r, hook, bend right] \arrow[r, phantom, "\top"] & D^+(q_\gamma) \arrow[l, bend right, "R\flat"'] \end{tikzcd}$$

is mapped to a natural isomorphism $h_\gamma(\varepsilon_F) \colon h_\gamma(R\flat(F)) \to h_\gamma(F)$.

Proof. Let $u \in \mathbb{M}$, then $\iota(u)$ is an object of $D^+(q_\gamma, \partial q)$. Thus, the linear map

$$\mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(u), \varepsilon_F) \colon \mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(u), R\flat(F)) \to \mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(u), F)$$

is an isomorphism. □

Let $h_{\gamma,0} \colon D^+(q_\gamma, \partial q) \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ}$ be the restriction of h_γ to $D^+(q_\gamma, \partial q)$. With this we may rephrase Lemma 8.5 more concisely:

Corollary 8.6. *The commutative square*

$$\begin{tikzcd} D^+(q_\gamma, \partial q) \arrow[r, hook] \arrow[d, "h_{\gamma,0}"'] & D^+(q_\gamma) \arrow[d, "h_\gamma"] \\ \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ} \arrow[r, equal] & \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ} \end{tikzcd} \tag{8.1}$$

of categories and functors satisfies the dual Beck–Chevalley condition as in Definition C.2.

Corollary 8.7. *We have the diagram*

$$\begin{tikzcd}
 & D^+(\dot{q}) \arrow[dl, "Ri_!"'] \arrow[d, "Ri_*"] \arrow[dl, phantom, shift left=3ex, "\Searrow"] \\
D^+(q_\gamma, \partial q) \arrow[d, "h_{\gamma,0}"'] \arrow[dr, Rightarrow, "h_\gamma \circ \varepsilon"] & D^+(q_\gamma) \arrow[l, "R\flat"'] \arrow[d, "h_\gamma"] \\
\mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ} \arrow[r, equal] & \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ}
\end{tikzcd}$$

with all 2*-cells natural isomorphisms, where* $D^+(\dot{q})$ *is the bounded-below derived category of the category of sheaves on* $\dot{q}$ *with the subspace topology inherited from* q_γ*.*

Proof. The triangular 2-cell at the top is an isomorphism by (7.4) and Lemma C.12. The square 2-cell at the bottom is an isomorphism by Corollary 8.6 or Lemma 8.5. □

In addition to the commutative square (8.1) we may also consider the commutative square

$$\begin{tikzcd}
D^+(\dot{q}) \arrow[r, "Ri_*"] \arrow[d, "h_\gamma \circ Ri_*"'] & D^+(q_\gamma) \arrow[d, "h_\gamma"] \\
\mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ} \arrow[r, equal] & \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ}.
\end{tikzcd} \tag{8.2}$$

Lemma 8.8. *The commutative square* (8.2) *satisfies the Beck–Chevalley condition as in Definition C.1.*

Proof. By Lemma 7.4 it suffices to show that the square

$$\begin{tikzcd}
D^+(\ddot{q}) \arrow[r, "Ri_{2*}"] \arrow[d, "h_\gamma \circ Ri_{2*}"'] & D^+(q_\gamma) \arrow[d, "h_\gamma"] \\
\mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ} \arrow[r, equal] & \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ}
\end{tikzcd} \tag{8.3}$$

satisfies the Beck–Chevalley condition. Now (8.3) satisfies the Beck–Chevalley condition iff the square

$$\begin{tikzcd}
D^+(\ddot{q}) \arrow[r, "Ri_{2*}"] \arrow[d, "H^n_{Z(u)}(\ddot{q};-)"'] \arrow[dr, phantom, "\cong"] & D^+(q_\gamma) \arrow[d, "H^n_{Z(u)}(q_\gamma;-)"] \\
\mathrm{Vect}_{\mathbb{F}} \arrow[r, equal] & \mathrm{Vect}_{\mathbb{F}}
\end{tikzcd}$$

satisfies the Beck–Chevalley condition for any $u \in D$ and any $n \in \mathbb{Z}$. This in turn follows from Corollary D.4. □

Now if ∂q is closed in q_γ, then $D^+(q_\gamma, \partial q)$ and $D^+(\dot{q})$ are equivalent by Lemma 7.6. So dealing with both of these categories separately is somewhat redundant in this case.

Lemma 8.9. *If ∂q is closed in q_γ, then the square diagram*

$$\begin{tikzcd}
D^+(q_\gamma) \arrow[r, "i^{-1}"] \arrow[d, "h_\gamma"'] & D^+(\dot{q}) \arrow[d, "h_\gamma \circ Ri_*"] \\
\mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ} \arrow[r, equal] \arrow[ur, Rightarrow, "h_\gamma \circ \eta^i"] & \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ}
\end{tikzcd} \tag{8.4}$$

satisfies both (the ordinary and the dual) Beck–Chevalley conditions, where $\eta^i\colon \mathrm{id} \to Ri_* \circ i^{-1}$ is the unit of the adjunction $i^{-1} \dashv Ri_*$.

Proof. By Lemma 8.8 the natural transformation $h_\gamma \circ \eta^i\colon h_\gamma \to h_\gamma \circ Ri_* \circ i^{-1}$, which is the mate of the identity natural transformation in the square (8.2), is a natural isomorphism, hence (8.4) satisfies the dual Beck–Chevalley condition. Moreover, by Lemma 7.7 the functor $i^{-1}\colon D^+(q_\gamma) \to D^+(\dot{q})$ also has the left adjoint $Ri_!\colon D^+(\dot{q}) \to D^+(q_\gamma)$. Now $h_\gamma \circ Ri_! = h_\gamma \circ \flat \circ Ri_*$ and moreover, the natural transformation

$$h_\gamma \circ \varepsilon \circ Ri_*\colon h_\gamma \circ \flat \circ Ri_* \Rightarrow h_\gamma \circ Ri_*$$

is a natural isomorphism by Lemma 8.5. Thus, the square diagram (8.4) satisfies the Beck–Chevalley condition iff the composition of square diagrams

$$\begin{tikzcd}
D^+(q_\gamma) \arrow[r, "i^{-1}"] \arrow[d, "h_\gamma"'] & D^+(\dot{q}) \arrow[r, equal] \arrow[d, "h_\gamma \circ Ri_*"'] & D^+(\dot{q}) \arrow[d, "h_\gamma \circ Ri_!"] \arrow[dl, phantom, "\overset{h_\gamma \circ \varepsilon \circ Ri_*}{\Longleftarrow}"] \\
\mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ} \arrow[r, equal] \arrow[ur, Rightarrow, "h_\gamma \circ \eta^i"] & \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ} \arrow[r, equal] & \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ}
\end{tikzcd} \tag{8.5}$$

satisfies the Beck–Chevalley condition. Now for a sheaf F on q_γ we have the commutative diagram

$$\begin{tikzcd}
\flat F \arrow[r, "(\flat \circ \eta^i)_F"] \arrow[d, "\varepsilon_F"'] & i_! i^{-1} F \arrow[d, "(\varepsilon \circ i_* \circ i^{-1})_F"] \arrow[dl, "\varepsilon^!_F"'] \\
F \arrow[r, "\eta^i_F"'] & i_* i^{-1} F,
\end{tikzcd}$$

hence the composition of square diagrams (8.5) is the square diagram

$$\begin{tikzcd}
D^+(q_\gamma) \arrow[r, "i^{-1}"] \arrow[d, "h_\gamma"'] & D^+(\dot{q}) \arrow[d, "h_\gamma \circ Ri_!"] \arrow[dl, Rightarrow, "h_\gamma \circ \varepsilon^!"'] \\
\mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ} \arrow[r, equal] & \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ} .
\end{tikzcd} \tag{8.6}$$

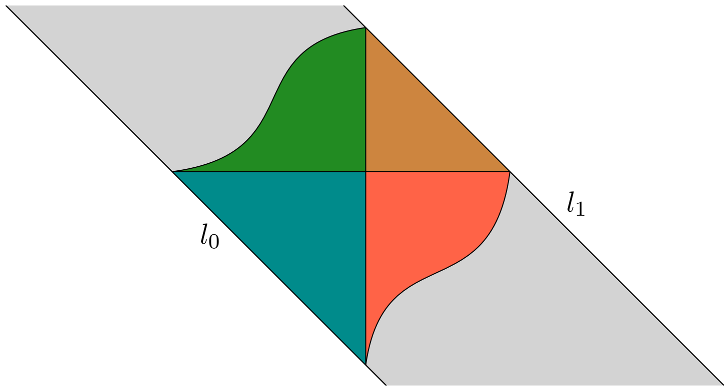


Figure 8.1: The fundamental domain D partitioned into four regions.

Moreover, the natural transformation $h_\gamma \circ \varepsilon^! \colon h_\gamma \circ Ri_! \circ i^{-1} \to h_\gamma$ is just the mate of the identity natural transformation of the commutative square

$$\begin{tikzcd} D^+(q_\gamma) \arrow[d, "h_\gamma"'] & D^+(\dot{q}) \arrow[l, "Ri_!"'] \arrow[d, "h_\gamma \circ Ri_!"] \\ \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ} \arrow[r, equal] & \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ} \end{tikzcd}$$

and thus (8.6) satisfies the Beck–Chevalley condition. □

Proposition 8.10. *The functor $h_{\gamma,0} \colon D^+(q_\gamma, \partial q) \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ}$ is conservative. In other words, if $\psi \colon G \to F$ is a homomorphism in $D^+(q_\gamma, \partial q)$ with $h_\gamma(\psi) \colon h_\gamma(G) \to h_\gamma(F)$ a natural isomorphism, then $\psi \colon G \to F$ is an isomorphism as well.*

Proof. Suppose $\psi \colon G \to F$ is a homomorphism in $D^+(q_\gamma, \partial q)$ with $h_\gamma(\psi) \colon h_\gamma(G) \to h_\gamma(F)$ a natural isomorphism and let $n \in \mathbb{Z}$ be arbitrary. It suffices to show that the induced linear map

$$H^n(U; \psi) \colon H^n(U; G) \to H^n(U; F)$$

is an isomorphism for each open subset U of some basis $\mathcal{B}$ of q_γ. To this end, let

$$\mathcal{B} \coloneqq \{q \cap \mathrm{int}(\downarrow (\Sigma(u)) \mid u \in D\}.$$

If ∂q is discrete as a subposet of $\mathbb{M}$, then $\mathcal{B}$ is a basis of q_γ. Otherwise, $\mathcal{B} \cup \{q\}$ is a basis of q_γ and both $H^n(F; q)$ and $H^n\,(G; q)$ vanish. So in either case, it suffices to check that $H^n(U; \psi)$ is an isomorphism for each $U \in \mathcal{B}$.

Now let $u \in D$ be contained in the region of D shaded in red in Fig. 8.1 and let $U \coloneqq q \cap \mathrm{int}(\downarrow (\Sigma(u))$. Then we have

$$H^n(U; \psi) \cong H^n_q(U; \psi) \cong \mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}((\iota \circ \Sigma^{-n})(u), \psi) \cong (h_\gamma(\psi) \circ \Sigma^{-n})_u$$

as linear maps by Proposition 8.4 and thus $H^n(U; \psi)$ is an isomorphism.

Now suppose that $u = (x, y) \in D$ is a point on the vertical line segment separating the red from the cyan region in Fig. 8.1, presuming the cyan region is non-empty. (Otherwise there is no need to treat this case.) Then for all $s < x$ the open subsets $q \cap \operatorname{int}(\downarrow (\Sigma(s, y))$ are identical. So let $U := q \cap \operatorname{int}(\downarrow (\Sigma(s, y))$ for some (and thus any) $s < x$. Now for $s < x$ we have the exact sequence

$$H^{n-1}(q \cap ((s, \infty) \times \mathbb{R}); \psi) \to H^n_{q \cap (\uparrow(s,y))}(U; \psi) \to H^n(U; \psi) \to H^n(q \cap ((s, \infty) \times \mathbb{R}); \psi)$$

in the category of linear maps by [KS90, Equation (2.6.32)]. As both F and G vanish on ∂q, taking the directed colimit of this sequence over all $s < x$ we obtain the exact sequence

$$0 \to \varinjlim_{s<x} H^n_{q \cap (\uparrow(s,y))}(U; \psi) \to H^n(U; \psi) \to 0.$$

Moreover, Proposition 8.4 implies

$$H^n_{q \cap (\uparrow(s,y))}(U; \psi) \cong \operatorname{Hom}_{\mathrm{D}^+(q_\gamma)}((\iota \circ \Sigma^{-n})(s, y), \psi) \cong (h_\gamma(\psi) \circ \Sigma^{-n})_{(s,y)},$$

hence $H^n_{q \cap (\uparrow(s,y))}(U; \psi)$ is an isomorphism for any $s < x$. Thus,

$$\varinjlim_{s<x} H^n_{q \cap (\uparrow(s,y))}(U; \psi) \cong H^n(U; \psi)$$

is an isomorphism as well.

Completely analogously one can show that $H^n(q \cap \operatorname{int}(\downarrow \Sigma(u)); \psi)$ is an isomorphism for any $u \in D$ contained in the region of D shaded in brown in $Fig.$ 8.1.

Now suppose that $u \in D$ is the point in the center of the crosshair shown in Fig. 8.1, presuming the green region is non-empty. Then for all $w \in \operatorname{int}(\uparrow u)$ we have $q \subset \operatorname{int}(\downarrow \Sigma(w))$ as well as the exact sequence

$$H^{n-1}(q \setminus (\uparrow w); \psi) \to H^n_{q \cap (\uparrow w)}(q; \psi) \to H^n(q; \psi) \to H^n(q \setminus (\uparrow w); \psi)$$

in the category of linear maps by [KS90, Equation (2.6.32)]. As both F and G vanish on ∂q, taking the directed colimit of this sequence over all $w \in \operatorname{int}(\uparrow u)$ we obtain the exact sequence

$$0 \to \varinjlim_{w \in \operatorname{int}(\uparrow u)} H^n_{q \cap (\uparrow w)}(q; \psi) \to H^n(q; \psi) \to 0.$$

Moreover, Proposition 8.4 implies

$$H^n_{q \cap (\uparrow w)}(q; \psi) \cong \operatorname{Hom}_{\mathrm{D}^+(q_\gamma)}((\iota \circ \Sigma^{-n})(w), \psi) \cong (h_\gamma(\psi) \circ \Sigma^{-n})_w,$$

hence $H^n_{q \cap (\uparrow w)}(q; \psi)$ is an isomorphism for all $w \in \operatorname{int}(\uparrow u)$. Thus,

$$\varinjlim_{w \in \operatorname{int}(\uparrow u)} H^n_{q \cap (\uparrow w)}(q; \psi) \cong H^n(q; \psi)$$

is an isomorphism as well. $\square$

8.1 Tameness and Induced Cohomological Presheaves

As before let $\dot{q} = q_\gamma \setminus \partial q = q_\gamma \setminus \partial\mathbb{M}$ and let $i\colon \dot{q} \hookrightarrow q_\gamma$ be the corresponding subspace inclusion.

Definition 8.11. We say that a bounded below complex F of sheaves (henceforth also *derived sheaf*) on q_γ is *tame* if $h_\gamma(F)\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ is pointwise finite-dimensional (pfd). Similarly, we say that an object F of $D^+(\dot{q})$ is *tame* if Ri_*F is tame. We denote the full subcategories of tame complexes in the derived categories $D^+(q_\gamma)$ and $D^+(\dot{q})$ by $D^+_t(q_\gamma)$ and $D^+_t(\dot{q})$ respectively. Furthermore, we denote the intersection of the two full subcategories $D^+_t(q_\gamma)$ and $D^+(q_\gamma, \partial q)$ by $D^+_t(q_\gamma, \partial q)$.

Lemma 8.12. *Let F be an object of $D^+(q_\gamma)$. If $\dim_\mathbb{F} H^n(I; F) < \infty$ for any connected open subset $I \subseteq q_\gamma$ and any integer $n \in \mathbb{Z}$, then F is tame.*

Proof. Let $u \in D$ and let $n \in \mathbb{Z}$. We show $\dim_\mathbb{F} \left(h_\gamma(F) \circ \Sigma^{-n}\right)(u) < \infty$. To this end, let $I \coloneqq q \cap \operatorname{int}(\downarrow \Sigma(u))$ and $J \coloneqq q \cap (\uparrow u)$. By Proposition 8.4 it suffices to show that $\dim_\mathbb{F} H^n_J(I; F) < \infty$. We consider the fragment

$$H^{n-1}(I; F) \to H^{n-1}(I \setminus J; F) \to H^n_J(I; F) \to H^n(I; F) \to H^n(I \setminus J; F)$$

of the corresponding long exact sequence for local sheaf cohomology, see for example [KS90, Equation (2.6.32) and Remark 2.6.10]. Now I is connected and $I \setminus J$ is the disjoint union of at most two connected open sets. By our assumptions on F all four cohomology spaces surrounding $H^n_J(I; F)$ in above exact sequence are finite-dimensional. As a result, $H^n_J(I; F)$ is finite-dimensional as well. □

Now let F be an object of $D^+(\dot{q})$.

Lemma 8.13. *Let $I \subseteq q_\gamma$ be a connected open subset. Then we have $H^n(I; Ri_*F) \cong H^n(I \setminus \partial q; F)$ for all integers $n \in \mathbb{Z}$.*

Proof. This follows from Lemma C.12 with $\Gamma(I; -)\colon \mathrm{Sh}(q_\gamma) \to \mathrm{Vect}_\mathbb{F}$ in place of $H\colon \mathcal{B} \to \mathcal{A}$ and $i_*\colon \mathrm{Sh}(\dot{q}) \to \mathrm{Sh}(q_\gamma)$ in place of $G\colon \mathcal{C} \to \mathcal{B}$. □

Corollary 8.14. *For any $n \in \mathbb{Z}$ we have $\dim_\mathbb{F} H^n(I; F) < \infty$ for all connected open subsets $I \subseteq \dot{q}$ iff we have $\dim_\mathbb{F} H^n(I; Ri_*F) < \infty$ for all connected open subsets $I \subseteq q_\gamma$.*

Lemma 8.15. *If $\dim_\mathbb{F} H^n(I; F) < \infty$ for all connected open subsets $I \subseteq \dot{q}$ and all integers $n \in \mathbb{Z}$, then F is tame. If ∂q is closed in q_γ, then the converse is true as well.*

Proof. The first implication follows from Corollary 8.14 and Lemma 8.12. Now suppose ∂q is closed in q_γ, let $I \subseteq \dot{q}$ be a connected open subset, let $n \in \mathbb{Z}$, and suppose that $h_\gamma(Ri_*F)\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ is pfd. As ∂q is closed in q_γ there is some point $u \in D$ with $\rho_1(u) = q \setminus I$ and $\rho_0(u) = q$. By Proposition 8.4, (7.8), and Lemma 8.13 this implies

$$\left(h_\gamma(Ri_*F) \circ \Sigma^{-n}\right)(u) \cong H^n(I; Ri_*F) \cong H^n(I; F),$$

hence $H^n(I; F)$ is finite-dimensional. □

Proposition 8.16. *Let F be an object of $D^+(q_\gamma)$ and suppose that $h_\gamma(F)\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ is pfd. Then $h_\gamma(F)$ is sequentially continuous.*

Proof. Let $(u_k)_{k=1}^\infty$ be an increasing sequence in $\mathbb{M}$ converging to $u \in \mathbb{M}$. Without loss of generality we assume that $(u_k)_{k=1}^\infty$ is contained in a single tile $\Sigma^{-n}(D)$ for some $n \in \mathbb{Z}$. As $h_\gamma(F)$ is pfd the projective system

$$\left\{ H^{n-1}_{q\cap(\uparrow u_k)}(q \cap \mathrm{int}(\downarrow \Sigma(u_k)); F) \right\}_{k=1}^\infty$$

satisfies the Mittag-Leffler condition by Proposition 8.4. Thus, the natural map

$$H^n_{q\cap(\uparrow u)}(q \cap \mathrm{int}(\downarrow \Sigma(u)); F) \to \varprojlim_k H^n_{q\cap(\uparrow u_k)}(q \cap \mathrm{int}(\downarrow \Sigma(u_k)); F)$$

is an isomorphism by [KS90, Proposition 2.7.1.(ii)]. In conjunction with Proposition 8.4 this implies the sequential continuity of $h_\gamma(F)$. $\square$

As already noted in the beginning of this Part IV we denote the category of finite-dimensional vector spaces over $\mathbb{F}$ by $\mathrm{vect}_\mathbb{F}$ and by $\mathcal{J}$ the full subcategory of pfd presheaves $\mathbb{M}^\circ \to \mathrm{vect}_\mathbb{F}$ that are cohomological, sequentially continuous, and have bounded above support. We summarize the results of this subsection so far:

Proposition 8.17. *Let F be an object of $D^+(q_\gamma)$ or of $D^+(\dot{q})$. Then F is tame iff $h_\gamma(F)\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ respectively $h_\gamma(Ri_*F)\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ is in $\mathcal{J}$.*

Proof. This follows directly from Lemmas 8.1, 8.2, and Proposition 8.16. $\square$

Compatibility with Coreflection and Sheaf Operations. Now by Corollary 8.6 the coreflector $R\flat\colon D^+(q_\gamma) \to D^+(q_\gamma, \partial q)$ restricts to a coreflector for the full subcategory inclusion $D^+_t(q_\gamma, \partial q) \hookrightarrow D^+_t(q_\gamma)$. With this we obtain the following.

Lemma 8.18. *The commutative square*

$$\begin{tikzcd} D^+_t(q_\gamma, \partial q) \arrow[r, hook] \arrow[d, hook] & D^+_t(q_\gamma) \arrow[d, hook] \\ D^+(q_\gamma, \partial q) \arrow[r, hook] & D^+(q_\gamma) \end{tikzcd}$$

of full replete subcategory inclusions satisfies the dual Beck–Chevalley condition.

Now let

$$h_{\gamma,t}\colon D^+_t(q_\gamma) \to \mathcal{J} \quad \text{and} \quad h_{\gamma,0,t}\colon D^+_t(q_\gamma, \partial q) \to \mathcal{J}$$

be the corresponding restrictions of $h_\gamma\colon D^+(q_\gamma) \to \mathrm{Vect}_\mathbb{F}^{\mathbb{M}^\circ}$.

Lemma 8.19. *The commutative square*

$$\begin{tikzcd} D_t^+(q_\gamma, \partial q) \arrow[r, hook] \arrow[d, "h_{\gamma,0,t}"'] & D_t^+(q_\gamma) \arrow[d, "h_{\gamma,t}"] \\ \mathcal{J} \arrow[r, equal] & \mathcal{J} \end{tikzcd}$$

satisfies the dual Beck–Chevalley condition.

Proof. This follows directly from Corollary 8.6. ◻

Lemma 8.20. *The derived adjunction*

$$\begin{tikzcd} D^+(q_\gamma) \arrow[r, bend right, "D^+(i^{-1})"'] \arrow[r, phantom, "\top"] & D^+(\dot{q}) \arrow[l, bend right, "Ri_*"'] \end{tikzcd}$$

restricts to an adjunction

$$\begin{tikzcd} D_t^+(q_\gamma) \arrow[r, bend right, "D^+(i^{-1})"'] \arrow[r, phantom, "\top"] & D_t^+(\dot{q}). \arrow[l, bend right, "Ri_*"'] \end{tikzcd}$$

Proof. By Definition 8.11 the derived functor $Ri_*\colon D^+(\dot{q}) \to D^+(q_\gamma)$ restricts to a functor $D_t^+(\dot{q}) \to D_t^+(q_\gamma)$. Now suppose F is an object of $D_t^+(q_\gamma)$. We have to show that $D^+(i^{-1})(F)$ is an object of $D_t^+(\dot{q})$. With some abuse of notation we write $i^{-1}F$ for $D^+(i^{-1})(F)$. By Definition 8.11 and Corollary 8.6 it suffices to show that $R\flat Ri_* i^{-1}F$ is an object of $D_t^+(q_\gamma)$. Now by Lemma C.12, (7.4), and Lemma 7.5 we have

$$R\flat Ri_* i^{-1}F \cong R(\flat \circ i_* \circ i^{-1})(F) = R(i_! \circ i^{-1})(F) \cong R\flat F,$$

which is in $D_t^+(q_\gamma)$ by Corollary 8.6 or Lemma 8.18. ◻

Lemma 8.21. *The commutative square*

$$\begin{tikzcd} D_t^+(\dot{q}) \arrow[r, "Ri_*"] \arrow[d, "h_{\gamma,t}\circ Ri_*"'] & D_t^+(q_\gamma) \arrow[d, "h_{\gamma,t}"] \\ \mathcal{J} \arrow[r, equal] & \mathcal{J} \end{tikzcd}$$

satisfies the Beck–Chevalley condition.

Proof. This follows directly from Lemmas 8.20 and 8.8. ◻

By Lemmas 8.18, 8.20, and C.12 we may compose the adjunctions

$$\begin{tikzcd} D_t^+(q_\gamma, \partial q) \arrow[r, bend right, hook] \arrow[r, phantom, "\top"] & D_t^+(q_\gamma) \arrow[l, bend right, "R\flat"'] \arrow[r, bend right, "D^+(i^{-1})"'] \arrow[r, phantom, "\top"] & D_t^+(\dot{q}) \arrow[l, bend right, "Ri_*"'] \end{tikzcd}$$

to obtain the adjunction

$$\begin{tikzcd} D_t^+(q_\gamma, \partial q) \arrow[r, bend right, "D^+(i^{-1})"'] \arrow[r, phantom, "\top"] & D_t^+(\dot{q}). \arrow[l, bend right, "Ri_!"'] \end{tikzcd} \tag{8.7}$$

Lemma 8.22. *If ∂q is closed in q_γ, then the adjunction* (8.7) *is an adjoint equivalence.*

Proof. This follows directly from Lemma 7.6. □

Lemma 8.23. *If ∂q is closed in q_γ, then the square diagram*

$$\begin{tikzcd} D_t^+(q_\gamma) \arrow[r, "i^{-1}"] \arrow[d, "h_{\gamma,t}"'] & D_t^+(\dot{q}) \arrow[d, "h_{\gamma,t}\circ Ri_*"] \\ \mathcal{J} \arrow[r, equal] \arrow[ur, phantom, "\Nearrow h_\gamma\circ\eta^i"] & \mathcal{J} \end{tikzcd}$$

satisfies both (the ordinary and the dual) Beck–Chevalley conditions.

Proof. This follows directly from Lemma 8.9. □

Tame Derived Sheaves as a Triangulated Subcategory. We end this Section 8.1 with a proof that $D_t^+(q_\gamma, \partial q)$ is a triangulated subcategory of $D^+(q_\gamma)$. To this end, we show the following auxiliary lemma.

Lemma 8.24. *The category of pfd sequentially continuous presheaves $\mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ is a weak Serre subcategory of $\mathrm{Vect}_\mathbb{F}^{\mathbb{M}^\circ}$, i.e., for any exact sequence*

$$F_1 \to F_2 \to F_3 \to F_4 \to F_5$$

with $F_i\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ pfd sequentially continuous for $i = 1, 2, 4, 5$ the presheaf $F_3\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ is pfd sequentially continuous as well.

Proof. The statement that $F_3\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ is pfd follows pointwise from the analogous property of the full subcategory inclusion $\mathrm{vect}_\mathbb{F} \hookrightarrow \mathrm{Vect}_\mathbb{F}$ of finite-dimensional vector spaces in $\mathrm{Vect}_\mathbb{F}$. Now let $(u_k)_{k=1}^\infty$ be an increasing sequence in $\mathbb{M}$ converging to u. As inverse limits of finite-dimensional vector spaces are exact, both rows of the commutative diagram

$$\begin{array}{ccccccccc} F_1(u) & \longrightarrow & F_2(u) & \longrightarrow & F_3(u) & \longrightarrow & F_4(u) & \longrightarrow & F_5(u) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \varprojlim_k F_1(u_k) & \longrightarrow & \varprojlim_k F_2(u_k) & \longrightarrow & \varprojlim_k F_3(u_k) & \longrightarrow & \varprojlim_k F_4(u_k) & \longrightarrow & \varprojlim_k F_5(u_k) \end{array} \tag{8.8}$$

are exact. Moreover, as the presheaves $F_i \colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ for $i = 1, 2, 4, 5$ are sequentially continuous, all four non-center vertical maps of (8.8) are isomorphisms. With this it follows from the five lemma that $F_3(u) \to \varprojlim_k F_3(u_k)$ is an isomorphism as well. □

Corollary 8.25. *The category $D_t^+(q_\gamma)$ is a triangulated subcategory of $D^+(q_\gamma)$.*

Proof. Suppose

$$F \to G \to H \to F[1]$$

is a distinguished triangle with F and G in $D_t^+(q_\gamma)$. We have to show that $h_\gamma(H) \colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ is pfd. As $h_\gamma \colon D^+(q_\gamma) \to \mathrm{Vect}_\mathbb{F}^{\mathbb{M}^\circ}$ is cohomological, we obtain the exact sequence

$$h_\gamma(F) \to h_\gamma(G) \to h_\gamma(H) \to h_\gamma(F[1]) \to h_\gamma(G[1])$$

with $h_\gamma(F)$, $h_\gamma(G)$, $h_\gamma(F[1])$, and $h_\gamma(G[1])$ pfd presheaves by assumption. In conjunction with Lemma 8.24 we obtain that $h_\gamma(H) \colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ is pfd as well. □

Corollary 8.26. *The category $D_t^+(q_\gamma, \partial q)$ is a triangulated subcategory of $D^+(q_\gamma)$.*

Proof. This follows from Corollaries 7.3 and 8.25. □

Corollary 8.27. *The category $D_t^+(\dot{q})$ is a triangulated subcategory of $D^+(\dot{q})$.*

Proof. The derived functor $Ri_* \colon D^+(\dot{q}) \to D^+(q_\gamma)$ is triangulated, hence the result follows from Definition 8.11 and Corollary 8.25. □

8.2 Alternative Construction of Induced Cohomological Presheaves

Let G be a bounded below complex of flabby sheaves on q_γ. In the following we provide an alternative construction of $h_\gamma(G) \colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$, which we use in Section 8.3 to draw the connection to *derived level set persistence.* To this end, we define the functor

$$\tilde{F}' \colon D^\circ \to C^+(\mathrm{Vect}_\mathbb{F}),\ u \mapsto \Gamma_{q \cap (\uparrow u)}(q \cap \mathrm{int}(\downarrow u); G),$$

where the internal maps are induced by inclusions. Post-composing $\tilde{F}'$ with the graded cohomology functor

$$H^\bullet \colon C^+(\mathrm{Vect}_\mathbb{F}) \to \mathrm{Vect}_\mathbb{F}^\mathbb{Z},\ C \mapsto H^\bullet(C)$$

we obtain the graded presheaf

$$F'\colon D^\circ \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{Z}},\ u \mapsto H^\bullet_{q\cap(\uparrow u)}(q \cap \operatorname{int}(\downarrow u); G)$$

as G is a complex of flabby sheaves. Now for $n \in \mathbb{Z}$ and $u \in D$ we have an isomorphism

$$\varphi_u^n\colon h_\gamma(G)(\Sigma^{-n}(u)) \xrightarrow{\cong} H^n_{q\cap(\uparrow u)}(q \cap \operatorname{int}(\downarrow u); G) = (F'(u))^n \tag{8.9}$$

by Proposition 8.4, which is natural in u. Moreover, by Corollary 1.15 there is a unique strictly stable presheaf $h_\gamma^\#(G)\colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{Z}}$ such that

$$\mathrm{ev}^0 \circ h_\gamma^\#(G) = h_\gamma(G),$$

where $\mathrm{ev}^0\colon \mathrm{Vect}_{\mathbb{F}}^{\mathbb{Z}} \to \mathrm{Vect}_{\mathbb{F}}$, $M^\bullet \mapsto M^0$ is the functor that sends a $\mathbb{Z}$-graded vector space $M^\bullet$ to its 0-th component M^0. Then the family of isomorphisms $\{\varphi_u^n\}_{u\in D, n\in\mathbb{Z}}$ from (8.9) assembles to a natural isomorphism

$$\varphi'\colon h_\gamma^\#(G)|_D \xrightarrow{\cong} F'.$$

Now the missing ingredient, in order to obtain a strictly stable presheaf $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{Z}}$ from $F'\colon D^\circ \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{Z}}$ is a natural transformation as in the diagram

$$\begin{tikzcd} R_D^\circ \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\overset{\delta'}{\Nearrow}"] & D^\circ \arrow[d, "F'"] \\ \Sigma(D)^\circ \arrow[r, "\Omega\circ F'\circ\Sigma^{-1}"'] & \mathrm{Vect}_{\mathbb{F}}^{\mathbb{Z}} \end{tikzcd} \tag{8.10}$$

according to Proposition 1.25, where $\Omega\colon \mathrm{Vect}_{\mathbb{F}}^{\mathbb{Z}} \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{Z}}$, $M^\bullet \mapsto M^{\bullet-1}$ is the up-shift of $\mathbb{Z}$-graded $\mathbb{F}$-vector spaces and where

$$R_D \coloneqq \{(w, \hat{u}) \in D \times \Sigma(D) \mid w \preceq \hat{u} \preceq \Sigma(w)\}$$

as in Definition 1.17. To this end, let $(w, \hat{u}) \in R_D$, let $u \coloneqq \Sigma^{-1}(\hat{u})$, and let $v_1, v_2 \in [u, w]$ be the lower left respectively the upper right vertex of $[u, w]$ as shown in Fig. 7.7. If we now instantiate Corollary D.7 with

$$\begin{aligned} X_1 &\coloneqq q \cap \operatorname{int}(\downarrow v_1), & A_1 &\coloneqq q \setminus (\uparrow v_1), \\ X_2 &\coloneqq q \cap \operatorname{int}(\downarrow v_2), \quad \text{and} & A_2 &\coloneqq q \setminus (\uparrow v_2), \end{aligned}$$

then we obtain the short exact sequence

$$0 \longrightarrow \tilde{F}'(w) \longrightarrow \tilde{F}'(v_1) \oplus \tilde{F}'(v_2) \xrightarrow{(1\ -1)} \tilde{F}'(u) = (\tilde{F}' \circ \Sigma^{-1})(\hat{u}) \longrightarrow 0 \tag{8.11}$$

of cochain complexes in $\mathrm{Vect}_{\mathbb{F}}$. By the zig-zag lemma, this short exact sequence yields a differential

$$\delta'_{(w,\hat{u})}\colon (\Omega \circ F' \circ \Sigma^{-1})(\hat{u}) \to F'(w).$$

Then δ' is a natural transformation as in (8.10). As it turns out, the diagram

$$\begin{tikzcd} h_\gamma^\#(G) \circ \mathrm{pr}_2 \arrow[r, Rightarrow, "\Omega\circ\varphi'\circ\Sigma^{-1}\circ\mathrm{pr}_2"] \arrow[d, Rightarrow, "\delta(h_\gamma^\#(G),D)"'] & \Omega \circ F' \circ \Sigma^{-1} \circ \mathrm{pr}_2 \arrow[d, Rightarrow, "\delta'"] \\ h_\gamma^\#(G) \circ \mathrm{pr}_1 \arrow[r, Rightarrow, "\varphi'\circ\mathrm{pr}_1"'] & F' \circ \mathrm{pr}_1 \end{tikzcd}$$

of presheaves on R_D commutes, where $\delta(h_\gamma^\#(G), D)$ is the natural transformation obtained from the internal maps of $h_\gamma^\#(G)$. By Proposition 1.25 this determines a unique strictly stable presheaf $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}^\mathbb{Z}$ as well as a unique strictly stable natural transformation $\varphi\colon h_\gamma^\#(G) \to F$. Moreover, as φ is strictly stable and as its restriction $\varphi|_D = \varphi'$ to the fundamental domain D is a natural isomorphism, φ is a natural isomorphism as well. With this we obtain the natural isomorphism

$$\mathrm{ev}^0 \circ \varphi\colon h_\gamma(G) \to \mathrm{ev}^0 \circ F.$$

8.3 Connection to Derived Level Set Persistence and RISC

The following form of derived level set persistence has been introduced by [Cur14, KS18]. Let $f\colon X \to \mathbb{R}$ be a continuous function. Then we may consider the derived pushforward $Rf_*\mathbb{F}_X$ as an object of the derived category $D^+(\mathbb{R})$ of $\mathbb{F}$-linear sheaves on $\mathbb{R}$. This construction can be made into a contravariant functor in the following way. For a commutative triangle

$$\begin{tikzcd} X \arrow[rr, bend left, "\varphi"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & \mathbb{R} & \end{tikzcd} \tag{8.12}$$

of topological spaces, we consider the unit

$$\eta^\varphi_{\mathbb{F}_Y}\colon \mathbb{F}_Y \to R\varphi_*\varphi^{-1}\mathbb{F}_Y \simeq R\varphi_*\mathbb{F}_X$$

of the derived adjunction $\varphi^{-1} \dashv R\varphi_*$ at $\mathbb{F}_Y$. Applying the functor Rg_* to $\eta^\varphi_{\mathbb{F}_X}$ we obtain the homomorphism

$$R\varphi_*\mathbb{F}_\varphi\colon Rg_*\mathbb{F}_Y \xrightarrow{(Rg_*\circ\eta^\varphi)_{\mathbb{F}_Y}} Rg_*R\varphi_*\mathbb{F}_X \simeq R(g\circ\varphi)_*\mathbb{F}_X = Rf_*\mathbb{F}_X,$$

see also [KS90, (2.7.4)]. This way we obtain the functor

$$R(-)_*\mathbb{F}_{(-)}\colon (\mathrm{Top}/\mathbb{R})^\circ \to D^+(\mathbb{R}),\ (f\colon X \to \mathbb{R}) \mapsto Rf_*\mathbb{F}_X$$

from the opposite category of the category of topological spaces over the reals $\mathrm{Top}/\mathbb{R}$ to the derived category $D^+(\mathbb{R})$. As a note of caution, we point out that this notation is not being used consistently across the literature as [BGO19] use the same notation for the functor

$$\mathrm{Top}/\mathbb{R} \to D^+(\mathbb{R}),\ (f\colon X \to \mathbb{R}) \mapsto \bigoplus_{n=0}^{\infty} R^n f_* \mathbb{F}_X[-i]. \tag{8.13}$$

Below we provide the Example 8.28, which also shows that the functor (8.13) and the functor we denote as $R(-)_*\mathbb{F}_{(-)}$ are not naturally isomorphic.

Now let

$$\blacktriangle := \Delta \circ \arctan\colon \overline{\mathbb{R}} = [-\infty, \infty] \to \mathbb{M}_\mathrm{f},\ t \mapsto (\arctan t, \arctan t).$$

In Chapter 7 we provided a sheaf-theoretical counterpart to Happels functor for any proper closed downset $C \subset \mathbb{M}_\mathrm{f}$. Now let $C :=\downarrow \mathrm{Im}\,\blacktriangle \subset \mathbb{M}_\mathrm{f}$ and $q := \partial C$ be the boundary of C in $\mathbb{M}_\mathrm{f}$ as in Chapter 7. Then the restriction $\blacktriangle|_\mathbb{R}\colon \mathbb{R} \to q_\gamma$ yields an embedding of $\mathbb{R}$ onto "the interior" $\dot{q} \subset q_\gamma$ as also illustrated by Fig. 1.3. The induced tessellation for this particular choice of $C \subset \mathbb{M}_\mathrm{f}$ is shown in Fig. 1.5. Moreover, we have $q_\gamma = \mathrm{Im}\,\blacktriangle \cong \overline{\mathbb{R}}$. Thus, by post-composing the functor $R(\blacktriangle|_\mathbb{R})_*\colon D^+(\mathbb{R}) \to D^+(q_\gamma)$ with h_γ we obtain the functor

$$h_\mathbb{R}\colon D^+(\mathbb{R}) \xrightarrow{R(\blacktriangle|_\mathbb{R})_*} D^+(q_\gamma) \xrightarrow{h_\gamma} \mathrm{Vect}_\mathbb{F}^{\mathbb{M}_\mathrm{f}^\circ}. \tag{8.14}$$

We also note that we have

$$\begin{aligned} h_\mathbb{R}(F) &= \mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(-), R(\blacktriangle|_\mathbb{R})_* F) \\ &\cong \mathrm{Hom}_{\mathrm{D}^+(\mathbb{R})}((\blacktriangle|_\mathbb{R})^{-1}\iota(-), F) \\ &= \mathrm{Hom}_{\mathrm{D}^+(\mathbb{R})}(\iota_\mathbb{R}(-), F) \end{aligned}$$

for any derived sheaf F on $\mathbb{R}$ by the adjointness $(\blacktriangle|_\mathbb{R})^{-1} \dashv R(\blacktriangle|_\mathbb{R})_*$, where $\iota_\mathbb{R}\colon \mathbb{M}_\mathrm{f} \to D^+(\mathbb{R})$ is the composition of functors

$$\begin{tikzcd} \mathbb{M}_\mathrm{f} \arrow[r, "\iota"] \arrow[rr, bend right, "\iota_\mathbb{R}"'] & D^+(q_\gamma, \partial q) \arrow[r, "(\blacktriangle|_\mathbb{R})^{-1}"] & D^+(\mathbb{R}). \end{tikzcd}$$

Moreover, reusing the notation of Section 2.5.1 we have

$$\iota_\mathbb{R}(u) \cong \mathbb{F}_{I(u)}[-\nu(u)] \tag{8.15}$$

by our construction of $\iota\colon \mathbb{M} \to \mathrm{D}^+(q_\gamma, \partial q)$ in Chapter 7. Furthermore, by composing $R(-)_*\mathbb{F}_{(-)}\colon (\mathrm{Top}/\mathbb{R})^\circ \to D^+(\mathbb{R})$ and $h_\mathbb{R}\colon D^+(\mathbb{R}) \to \mathrm{Vect}_\mathbb{F}^{\mathbb{M}_\mathrm{f}^\circ}$ we obtain a functor

$$(\mathrm{Top}/\mathbb{R})^\circ \xrightarrow{R(-)_*\mathbb{F}_{(-)}} D^+(\mathbb{R}) \xrightarrow{h_\mathbb{R}} \mathrm{Vect}_\mathbb{F}^{\mathbb{M}_\mathrm{f}^\circ}.$$

Now in Section 1.1 we have already provided the functor $h\colon (\mathrm{Top}/\mathbb{R})^\circ \to \mathrm{Vect}_\mathbb{F}^{\mathbb{M}_\mathrm{f}^\circ}$. In the following we show that these two functors are naturally isomorphic when restricted to functions on locally contractible spaces. More specifically, let lcContr denote the full subcategory of locally contractible topological spaces. Before we construct a natural isomorphism ζ as in the diagram

$$\begin{tikzcd}
(\mathrm{lcContr}/\mathbb{R})^{\circ} \arrow[d, "R(-)_*\mathbb{F}_{(-)}"'] \arrow[dr, "h"] & \\
D^+(\mathbb{R}) \arrow[r, "h_{\mathbb{R}}"'] \arrow[ur, phantom, "\overset{\zeta}{\Leftarrow}"] & \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}_{\mathrm{f}}^{\circ}},
\end{tikzcd}$$

we illustrate the behavior of all three of these functors with an example.

Example 8.28. Let $a < b < c \in \mathbb{R}$. Then we have the short exact sequence

$$0 \to \mathbb{F}_{[a,b)} \to \mathbb{F}_{[a,c)} \to \mathbb{F}_{[b,c)} \to 0$$

of sheaves on $\mathbb{R}$, which induces the distinguished triangle

$$\mathbb{F}_{[a,b)} \to \mathbb{F}_{[a,c)} \to \mathbb{F}_{[b,c)} \xrightarrow{\partial} \mathbb{F}_{[a,b)}[1]$$

in the derived category $D^+(\mathbb{R})$. Now let $u := \mathfrak{b}^{-1}(1, [b, c)) \in \operatorname{int} \mathbb{M}_{\mathrm{f}}$ and $v := \mathfrak{b}^{-1}(0, [a, b))$ as in Section 2.6, see also Fig. 2.9. By (8.15) and our construction of $\iota \colon \mathbb{M} \to \mathrm{D}^+(q_\gamma, \partial q)$ in Chapter 7 we have the commutative square

$$\begin{tikzcd}
\mathbb{F}_{[b,c)}[-1] \arrow[r, "\sim"] \arrow[d, "{\partial[-1]}"'] & \iota_{\mathbb{R}}(u) \arrow[d, "\iota_{\mathbb{R}}(u \preceq v)"] \\
\mathbb{F}_{[a,b)} \arrow[r, "\sim"] & \iota_{\mathbb{R}}(v)
\end{tikzcd}$$

with horizontal (derived) sheaf isomorphisms as indicated. Thus, we also have the commutative square

$$\begin{tikzcd}
B_u \arrow[r, "\sim"] \arrow[d, "B_{u \preceq v}"'] & h_{\mathbb{R}}\big(\mathbb{F}_{[b,c)}[-1]\big) \arrow[d, "{h_{\mathbb{R}}(\partial[-1])}"] \\
B_v \arrow[r, "\sim"] & h_{\mathbb{R}}\big(\mathbb{F}_{[a,b)}\big)
\end{tikzcd}$$

of presheaves $\mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ with horizontal natural isomorphisms as indicated by Corollary 8.3. As already noted in Section 2.6, the restriction of the natural transformation $B_{u \preceq v} \colon B_u \to B_v$ to a subposet of $\mathbb{M}_{\mathrm{f}}$ corresponding to zigzag persistence (for some sequence of regular values) vanishes as the intersection of supports of B_u and B_v is disjoint from any such subposet and similarly for *Mayer–Vietoris systems* [BGO19]. The corresponding operation on the level of sheaves is to take the graded cohomology sheaf turning any homomorphism in the derived category into a homogeneous homomorphism of degree 0. And indeed, the homomorphism of graded cohomology sheaves induced by $\partial[-1]\colon \mathbb{F}_{[b,c)}[-1] \to \mathbb{F}_{[a,b)}$ is zero as $\mathbb{F}_{[b,c)}[-1]$ and $\mathbb{F}_{[a,b)}$ are supported in different degrees. Now the functor from $D^+(\mathbb{R})$ to Mayer–Vietoris systems, that is the composition of $h_{\mathbb{R}}\colon D^+(\mathbb{R}) \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}_\mathrm{f}^\circ}$ and the restriction to the corresponding subposet of $\mathbb{M}_\mathrm{f}$, has been studied by the authors of [BGO19] and is denoted by Ψ in their article. Moreover, the homomorphism $\partial\colon \mathbb{F}_{[b,c)} \to \mathbb{F}_{[a,b)}[1]$ is also presented in [BGO19, Remark 4.9] as a witness that Ψ is not faithful, which we provided to the authors. This is in contrast to Corollary 9.4 below, which states that $h_{\mathbb{R}}$ restricts to an equivalence of categories from *tame derived sheaves* on $\mathbb{R}$ to $\mathcal{J}_\mathrm{f} = \mathcal{J}$. Now reconsidering the function $f\colon X \to \mathbb{R}$ as well as the simplicial inclusion $j\colon A \hookrightarrow X$ from Section 2.6, we have the commutative square

$$\begin{tikzcd}
\mathbb{F}_{[a,c]} \oplus \mathbb{F}_{[b,c)}[-1] \arrow[r, "\sim"] \arrow[d, "\mathbb{F}_{[a,c]}\oplus\partial[-1]"'] & Rf_*\mathbb{F}_X \arrow[d, "Rj_*\mathbb{F}_j"] \\
\mathbb{F}_{[a,c]} \oplus \mathbb{F}_{[a,b)} \arrow[r, "\sim"] & R(f\circ j)_*\mathbb{F}_A
\end{tikzcd}$$

with horizontal isomorphisms of the derived category $D^+(\mathbb{R})$ as indicated. Thus, the example of Section 2.6 shows that the loss of information on the level of homomorphisms incurred when passing from the derived category of sheaves to the graded category via cohomology or to Mayer–Vietoris systems via Ψ occurs in nature as well.

Construction of the Natural Isomorphism ζ. Now let $f\colon X \to \mathbb{R}$ be a continuous function with X a locally contractible topological space. Moreover, let $C^\bullet$ be the presheaf of singular cochains with coefficients in $\mathbb{F}$ on X and let $\epsilon\colon \mathbb{F}_X \to C^\bullet$ be the embedding of $\mathbb{F}_X$ as the subpresheaf of 0-cocycles of $C^\bullet$. By [Sel16] there is a complex $\mathcal{F}$ of flabby sheaves on X together with a quasi-isomorphism of complexes of presheaves $\tilde{\psi}\colon C^\bullet \to \mathcal{F}$ such that $\tilde{\psi}\circ\epsilon\colon \mathbb{F}_X \to \mathcal{F}$ is a quasi-isomorphism of complexes of sheaves. For $u \in D$ we consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\big(C^\bullet \circ f^{-1} \circ \blacktriangle^{-1}\big)(\mathrm{int}(\downarrow \Sigma(u)), \mathbb{M}_\mathrm{f} \setminus (\uparrow u)) \arrow[r, dashed] \arrow[d] & \Gamma_{(\blacktriangle\circ f)^{-1}(\uparrow u)}\big((\blacktriangle \circ f)^{-1}(\mathrm{int}(\downarrow \Sigma(u))); \mathcal{F}\big) \arrow[d] \\
\big(C^\bullet \circ f^{-1} \circ \blacktriangle^{-1}\big)(\mathrm{int}(\downarrow \Sigma(u))) \arrow[r, "\tilde{\psi}_{(\blacktriangle\circ f)^{-1}(\mathrm{int}(\downarrow\Sigma(u)))}"] \arrow[d] & \big(\mathcal{F} \circ f^{-1} \circ \blacktriangle^{-1}\big)(\mathrm{int}(\downarrow \Sigma(u))) \arrow[d] \\
\big(C^\bullet \circ f^{-1} \circ \blacktriangle^{-1}\big)(\mathbb{M}_\mathrm{f} \setminus (\uparrow u)) \arrow[r, "\tilde{\psi}_{(\blacktriangle\circ f)^{-1}(\mathbb{M}_\mathrm{f}\setminus(\uparrow u))}"] \arrow[d] & \big(\mathcal{F} \circ f^{-1} \circ \blacktriangle^{-1}\big)(\mathbb{M}_\mathrm{f} \setminus (\uparrow u)) \arrow[d] \\
0 & 0.
\end{tikzcd} \tag{8.16}$$

By definition of the relative singular cochain complex $\big(C^\bullet \circ f^{-1} \circ \blacktriangle^{-1}\big)(\mathrm{int}(\downarrow \Sigma(u)), \mathbb{M}_\mathrm{f} \setminus (\uparrow u))$ and the set of local sections

$\Gamma_{(\blacktriangle\circ f)^{-1}(\uparrow u)}\big((\blacktriangle\circ f)^{-1}(\operatorname{int}(\downarrow\Sigma(u)));\mathcal{F}\big)$ both columns are exact. In particular, the horizontal dashed arrow exists as indicated. Moreover, the lower two horizontal arrows are quasi-isomorphisms of cochain complexes as $\tilde{\psi}\colon C^\bullet\to\mathcal{F}$ is a quasi-isomorphism of complexes of presheaves. Thus, the dashed arrow is a quasi-isomorphism of cochain complexes as well. Taking the $\mathbb{Z}$-graded cohomology of the cochain complex $\big(C^\bullet\circ f^{-1}\circ\blacktriangle^{-1}\big)(\operatorname{int}(\downarrow\Sigma(u)),\mathbb{M}_{\mathrm{f}}\setminus(\uparrow u))$ we obtain

$$\big(H^\bullet\circ f^{-1}\circ\blacktriangle^{-1}\big)(\operatorname{int}(\downarrow\Sigma(u)),\mathbb{M}_{\mathrm{f}}\setminus(\uparrow u))=h^{\#}(f)(u),\tag{8.17}$$

where $h(f)\colon\mathbb{M}_{\mathrm{f}}^\circ\to\mathrm{Vect}_{\mathbb{F}}$ is the relative interlevel set cohomology (RISC) of $f\colon X\to\mathbb{R}$ as defined in Section 1.1 and $h^{\#}(f)\colon\mathbb{M}_{\mathrm{f}}^\circ\to\mathrm{Vect}_{\mathbb{F}}^{\mathbb{Z}}$ is the strictly stable presheaf with $\mathrm{ev}^0\circ h^{\#}(f)=h(f)$; such a strictly stable presheaf exists uniquely by Corollary 1.15. Moreover, we have

$$\begin{aligned}\Gamma_{(\blacktriangle\circ f)^{-1}(\uparrow u)}\big((\blacktriangle\circ f)^{-1}(\operatorname{int}(\downarrow\Sigma(u)));\mathcal{F}\big)&=\Gamma_{q\cap(\uparrow u)}(q\cap\operatorname{int}(\downarrow\Sigma(u));(\blacktriangle\circ f)_*\mathcal{F})\\&=\Gamma_{q\cap(\uparrow u)}(q\cap\operatorname{int}(\downarrow\Sigma(u));G),\end{aligned}$$

where $G:=(\blacktriangle\circ f)_*\mathcal{F}$. As both $\mathcal{F}$ and G are complexes of flabby sheaves, this implies that

$$H^\bullet_{(\blacktriangle\circ f)^{-1}(\uparrow u)}\big((\blacktriangle\circ f)^{-1}(\operatorname{int}(\downarrow\Sigma(u)));\mathcal{F}\big)\cong H^\bullet_{q\cap(\uparrow u)}(q\cap\operatorname{int}(\downarrow\Sigma(u));G)=F'(u),\tag{8.18}$$

where

$$F'\colon D^\circ\to\mathrm{Vect}_{\mathbb{F}}^{\mathbb{Z}},\ u\mapsto H^\bullet_{q\cap(\uparrow u)}(q\cap\operatorname{int}(\downarrow u);G)$$

as in the previous Section 8.2. Altogether, (8.17), the dashed arrow in (8.16), and (8.18) yield a natural isomorphism

$$\psi'\colon h^{\#}(f)|_D\xrightarrow{\sim}F'.$$

Now in order to apply Proposition 1.25 to extend ψ' to a strictly stable natural isomorphism

$$\psi\colon h^{\#}(f)\to F,$$

where $F\colon\mathbb{M}_{\mathrm{f}}^\circ\to\mathrm{Vect}_{\mathbb{F}}^{\mathbb{Z}}$ is defined as in the previous Section 8.2, we have to show that the diagram

$$\begin{tikzcd} h^{\#}(f)\circ\mathrm{pr}_2 \arrow[r, Rightarrow, "\Omega\circ\psi'\circ\Sigma^{-1}\circ\mathrm{pr}_2"] \arrow[d, Rightarrow, "\delta(h^{\#}(f),D)"'] & F\circ\mathrm{pr}_2 \arrow[d, Rightarrow, "\delta'"] \\ h^{\#}(f)\circ\mathrm{pr}_1 \arrow[r, Rightarrow, "\psi'\circ\mathrm{pr}_1"'] & F\circ\mathrm{pr}_1 \end{tikzcd}\tag{8.19}$$

of presheaves on R_D commutes, where $\delta(h^{\#}(f),D)$ is the natural transformation obtained from the internal maps of $h^{\#}(f)$. To this end, it suffices to check the commutativity of (8.19) for all pairs $(w,\hat{u})\in R_D$ with the same x- or the same y-coordinate and both within the square $\big(-\frac{\pi}{2},\frac{\pi}{2}\big)^2$. We treat the case, where w and $\hat{u}$ have the same

y-coordinate, the other case is similar. In this case there are real numbers $a < b < c$ with

$$\hat{u} = (\arctan a, \arctan b) \quad \text{and} \quad w = (\arctan c, \arctan b).$$

Then we have

$$\begin{aligned} \left(h_0^{\#}(f) \circ \mathrm{pr}_1\right)(w) &= \left(H^\bullet \circ f^{-1}\right)(\mathbb{R}, \mathbb{R} \setminus [a,b]) \\ \text{and} \quad \left(h_0^{\#}(f) \circ \mathrm{pr}_2\right)(\hat{u}) &= \left(H^{\bullet-1} \circ f^{-1}\right)((a,b)). \end{aligned}$$

Moreover,

$$\delta(h_0^{\#}(f), D)_{(w,\hat{u})} \colon \left(H^{\bullet-1} \circ f^{-1}\right)((a,b)) \to \left(H^\bullet \circ f^{-1}\right)(\mathbb{R}, \mathbb{R} \setminus [a,b]) \tag{8.20}$$

is the differential of the Mayer–Vietoris sequence associated to the square-shaped sublattice

$$\begin{tikzcd} f^{-1}(\mathbb{R}, \mathbb{R} \setminus [b,c]) & f^{-1}((-\infty,b),(-\infty,b)) \arrow[l, hook'] \\ f^{-1}((a,\infty),(c,\infty)) \arrow[u, hook] & f^{-1}((a,b),\emptyset) \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

of pairs of open subspaces of X. Now considering the right-hand side of (8.19) we have

$$\begin{aligned} (F \circ \mathrm{pr}_1)(w) &\cong H^\bullet_{f^{-1}([b,c])}(X; \mathcal{F}) \\ \text{and} \quad (F \circ \mathrm{pr}_2)(\hat{u}) &\cong H^{\bullet-1}\big(f^{-1}((a,b)); \mathcal{F}\big). \end{aligned}$$

Under these isomorphisms the map $\delta'_{(w,\hat{u})} \colon (F' \circ \mathrm{pr}_2)(\hat{u}) \to (F' \circ \mathrm{pr}_1)(w)$ corresponds to the differential associated to the short exact sequence

$$0 \to \Gamma_{f^{-1}([b,c])}(X; \mathcal{F}) \to \Gamma_{f^{-1}((-\infty,c])}\big(f^{-1}((a,\infty)); \mathcal{F}\big) \to \Gamma\big(f^{-1}((a,b)); \mathcal{F}\big) \to 0 \tag{8.21}$$

of cochain complexes in $\mathrm{Vect}_{\mathbb{F}}$. Now in order to show the commutativity of (8.19) at $(w, \hat{u}) \in R_D$ we show that the Mayer–Vietoris differential (8.20) can be realized as the differential associated to a short exact sequence of cochain complexes as well. To this end, we consider the sublattice

$$\begin{tikzcd} (\mathbb{R}, \mathbb{R} \setminus [b,c]) & (\mathbb{R} \setminus [b,c], \mathbb{R} \setminus [b,c]) \arrow[l, hook'] & ((-\infty,b),(-\infty,b)) \arrow[l, hook'] \\ ((a,\infty),(a,b) \cup (c,\infty)) \arrow[u, hook] & ((a,b) \cup (c,\infty),(a,b) \cup (c,\infty)) \arrow[l, hook'] \arrow[u, hook] & ((a,b),(a,b)) \arrow[l, hook'] \arrow[u, hook] \\ ((a,\infty),(c,\infty)) \arrow[u, hook] & ((a,b) \cup (c,\infty),(c,\infty)) \arrow[l, hook'] \arrow[u, hook] & ((a,b),\emptyset). \arrow[l, hook'] \arrow[u, hook] \end{tikzcd} \tag{8.22}$$

Now all inclusions of (8.22) other than those of the lower left square induce isomorphisms in singular cohomology by excision. Thus, the Mayer–Vietoris sequence associated to the outer square and the lower left square of (8.22), respectively, are isomorphic.

Moreover, the latter is the same as the long exact sequence associated to the triple $f^{-1}((a,\infty),(a,b)\cup(c,\infty),(c,\infty))$, which is the long exact sequence associated to the short exact sequence

$$\begin{array}{c} 0 \\ \big\downarrow \\ \big(C^\bullet \circ f^{-1}\big)((a,\infty),(a,b)\cup(c,\infty)) \\ \big\downarrow \\ \big(C^\bullet \circ f^{-1}\big)((a,\infty),(c,\infty)) \\ \big\downarrow \\ \big(C^\bullet \circ f^{-1}\big)((a,b)\cup(c,\infty),(c,\infty)) \\ \big\downarrow \\ 0 \end{array} \tag{8.23}$$

of cochain complexes in $\mathrm{Vect}_{\mathbb{F}}$. Furthermore, the short exact sequence (8.21) is isomorphic to the short exact sequence

$$\begin{array}{c} 0 \\ \big\downarrow \\ \Gamma_{f^{-1}([b,c])}\big(f^{-1}((a,\infty));\mathcal{F}\big) \\ \big\downarrow \\ \Gamma_{f^{-1}((a,c])}\big(f^{-1}((a,\infty));\mathcal{F}\big) \\ \big\downarrow \\ \Gamma_{f^{-1}((a,c])}\big(f^{-1}((a,b)\cup(c,\infty));\mathcal{F}\big) \\ \big\downarrow \\ 0 \end{array} \tag{8.24}$$

by the sheaf condition. Now the presheaf homomorphism $\tilde{\psi}\colon C^\bullet \to \mathcal{F}$ induces a homomorphism

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\big(C^\bullet \circ f^{-1}\big)((a,\infty),(a,b)\cup(c,\infty)) \arrow[r] \arrow[d] & \Gamma_{f^{-1}([b,c])}\big(f^{-1}((a,\infty));\mathcal{F}\big) \arrow[d] \\
\big(C^\bullet \circ f^{-1}\big)((a,\infty),(c,\infty)) \arrow[r] \arrow[d] & \Gamma_{f^{-1}((a,c])}\big(f^{-1}((a,\infty));\mathcal{F}\big) \arrow[d] \\
\big(C^\bullet \circ f^{-1}\big)((a,b)\cup(c,\infty),(c,\infty)) \arrow[r] \arrow[d] & \Gamma_{f^{-1}((a,c])}\big(f^{-1}((a,b)\cup(c,\infty));\mathcal{F}\big) \arrow[d] \\
0 & 0
\end{tikzcd} \tag{8.25}$$

of short exact sequences. Moreover, the horizontal arrow at the top of (8.25) induces the homomorphism $\psi'_w\colon h^{\#}(f)(w) \to F(w)$ under the aforementioned isomorphisms. Similarly, the horizontal arrow at the bottom of (8.25) induces the homomorphism $\big(\Omega \circ \psi' \circ \Sigma^{-1}\big)_{\hat{u}}\colon h^{\#}(f)(\hat{u}) \to F(\hat{u})$. Thus, the diagram

$$\begin{tikzcd}
h^{\#}(f)(\hat{u}) \arrow[r, "(\Omega\circ\psi'\circ\Sigma^{-1})_{\hat{u}}"] \arrow[d, "\delta(h^{\#}(f),D)_{(w,\hat{u})}"'] & F(\hat{u}) \arrow[d, "\delta'_{(w,\hat{u})}"] \\
h^{\#}(f)(w) \arrow[r, "\psi'_w"'] & F(w)
\end{tikzcd} \tag{8.26}$$

commutes. By a similar argument (8.26) commutes for any pair $(w,\hat{u}) \in R_D$ with identical x-coordinates and both points within the square $\big(-\frac{\pi}{2},\frac{\pi}{2}\big)^2$. We just need to keep track of an additional sign, as in this case the long exact sequence of the associated triple "passes through the second direct summand of the Mayer–Vietoris sequence". Now for an arbitrary pair $(w,\hat{u}) \in R_D$ we can always find a pair $(w',\hat{u}') \in R_D$ with the same x- or the same y-coordinate and both within the square $\big(-\frac{\pi}{2},\frac{\pi}{2}\big)^2$ and $w \preceq w' \preceq \hat{u}' \preceq \hat{u}$, hence the diagram (8.19) commutes. As a result, $\psi'\colon h^{\#}(f)|_D \xrightarrow{\sim} F'$ extends to a unique strictly stable natural transformation $\psi\colon h^{\#}(f) \to F$ by Proposition 1.25. Moreover, as ψ' is a natural isomorphism, ψ is a natural isomorphism as well. In conjunction with the previous Section 8.2 we obtain the cospan

$$h^{\#}(f) \overset{\psi}{\Longrightarrow} F \overset{\varphi}{\Longleftarrow} h^{\#}_{\gamma}(G)$$

of strictly stable presheaves and strictly stable natural isomorphisms. Post-composition respectively whiskering from the left with $\mathrm{ev}^0\colon \mathrm{Vect}_{\mathbb{F}}^{\mathbb{Z}} \to \mathrm{Vect}_{\mathbb{F}}$, $M^\bullet \mapsto M^0$ yields a natural isomorphism

$$h(f) \cong h_\gamma(G). \tag{8.27}$$

Moreover, as $\tilde{\psi} \circ \epsilon \colon \mathbb{F}_X \to \mathcal{F}$ is a quasi-isomorphism of complexes of sheaves, we have a quasi-isomorphism

$$G = (\blacktriangle \circ f)_* \mathcal{F} \simeq R(\blacktriangle \circ f)_* \mathbb{F}_X \simeq R\blacktriangle_* Rf_* \mathbb{F}_X$$

and hence

$$h_\gamma(G) \cong h_\gamma(R\blacktriangle_* Rf_* \mathbb{F}_X) \cong h_{\mathbb{R}}(Rf_* \mathbb{F}_X). \tag{8.28}$$

Combining (8.27) and (8.28) we obtain the natural isomorphism

$$\zeta_f \colon h(f) \xrightarrow{\cong} h_{\mathbb{R}}(Rf_* \mathbb{F}_X),$$

which is natural in $f \colon X \to \mathbb{R}$. In summary, we obtain the following.

Proposition 8.29. *There is a natural isomorphism* ζ *as in the diagram*

$$\begin{tikzcd}
(\mathrm{lcContr}/\mathbb{R})^\circ \arrow[d, "R(-)_*\mathbb{F}_{(-)}"'] \arrow[dr, "h"] & \\
D^+(\mathbb{R}) \arrow[r, "h_{\mathbb{R}}"'] \arrow[ur, phantom, "\overset{\zeta}{\Leftarrow}"] & \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}_{\mathrm{f}}^\circ}.
\end{tikzcd}$$

Restriction to Tame Objects. Now in Definition 1.37 we have also defined the notion of an $\mathbb{F}$-tame function. Analogously we may define the following notion for objects of $D^+(\mathbb{R})$.

Definition 8.30 (Tame Derived Sheaf)**.** We say that an object F of $D^+(\mathbb{R})$ is *tame* if $H^n(I; F)$ is finite-dimensional for any open interval $I \subseteq \mathbb{R}$ and any integer $n \in \mathbb{Z}$. Moreover, we denote the full subcategory of tame objects in $D^+(\mathbb{R})$ by $D^+_t(\mathbb{R})$.

By Lemma 8.15 and Proposition 8.17 the functor $h_{\mathbb{R}} \colon D^+(\mathbb{R}) \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}_{\mathrm{f}}^\circ}$ restricts to a functor $h_{\mathbb{R},t} \colon D^+_t(\mathbb{R}) \to \mathcal{J}_{\mathrm{f}}$, where $\mathcal{J}_{\mathrm{f}} = \mathcal{J}$ is the full subcategory of pfd presheaves $\mathbb{M}_{\mathrm{f}}^\circ \to \mathrm{vect}_{\mathbb{F}}$ that are cohomological, sequentially continuous, and have bounded above support. In conjunction with Proposition 8.29 we obtain the diagram

$$\begin{tikzcd}
(\mathrm{lcContr}/\mathbb{R})_t^\circ \arrow[d, "R(-)_*\mathbb{F}_{(-)}"'] \arrow[dr, "h"] & \\
D^+_t(\mathbb{R}) \arrow[r, "h_{\mathbb{R},t}"'] \arrow[ur, phantom, "\overset{\zeta}{\Leftarrow}"] & \mathcal{J}_{\mathrm{f}}.
\end{tikzcd} \tag{8.29}$$

In the next Chapter 9 we show with Corollary 9.4 that the lower horizontal arrow $h_{\mathbb{R},t} \colon D^+_t(\mathbb{R}) \to \mathcal{J}_{\mathrm{f}}$ in (8.29) is an equivalence of categories.

9 Equivalence of $D_t^+(q_\gamma, \partial q)$ and $\mathcal{J}$

In the previous Chapter 8 we considered the restricted Yoneda embedding

$$h_\gamma \colon \mathrm{D}^+(q_\gamma) \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ},\, F \mapsto h_\gamma(F) = \mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(-), F)$$

and we showed with Proposition 8.17 that h_γ maps any derived sheaf F that is tame in the sense of Definition 8.11 to a presheaf in $\mathcal{J}$, i.e. a pfd presheaf $\mathbb{M}^\circ \to \mathrm{vect}_{\mathbb{F}}$ that is cohomological, sequentially continuous, and has bounded above support. In this Chapter 9 we show that h_γ further restricts to an additive equivalence

$$h_{\gamma,0,t} \colon D_t^+(q_\gamma, \partial q) \to \mathcal{J},\, F \mapsto h_\gamma(F)$$

on all tame derived sheaves of $\mathrm{D}^+(q_\gamma)$ vanishing on ∂q, that is the triangulated subcategory $D_t^+(q_\gamma, \partial q) \subset \mathrm{D}^+(q_\gamma)$, see also Corollary 8.26.

Theorem 9.1. *The restricted Yoneda embedding*

$$h_{\gamma,0,t} \colon D_t^+(q_\gamma, \partial q) \to \mathcal{J},\, F \mapsto h_\gamma(F) = \mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(-), F)$$

is an $\mathbb{F}$-linear equivalence of categories.

Corollary 9.2. *Any tame derived sheaf F of $D_t^+(q_\gamma, \partial q)$ decomposes as*

$$F \cong \bigoplus_{v \in \operatorname{int} \mathbb{M}} \iota(v)^{\oplus \mu(v)},$$

where $\mu := (\beta^0 \circ h_\gamma)(F) \colon \operatorname{int} \mathbb{M} \to \mathbb{N}_0$.

Proof. This follows from Theorem 9.1 in conjunction with Theorem 2.6. □

We also point out, that it may be somewhat unusual to obtain a result like Corollary 9.2 in this way; see Remark 9.17 for a more detailed discussion of this aspect.

Corollary 9.3. *If ∂q is closed in q_γ, then the composition of $\mathbb{F}$-linear functors $D_t^+(\dot{q}) \xrightarrow{Ri_*} D_t^+(q_\gamma) \xrightarrow{h_\gamma} \mathcal{J}$ is an equivalence of categories.*

Proof. This statement follows from Theorem 9.1 in conjunction with Corollary 8.7 and Lemma 8.22. □

Corollary 9.4. *The restricted Yoneda embedding $h_{\mathbb{R},t} \colon D_t^+(\mathbb{R}) \to \mathcal{J}_{\mathrm{f}}$ is an $\mathbb{F}$-linear equivalence of categories.*

Proof. The map $\blacktriangle|_{\mathbb{R}} \colon \mathbb{R} \to \dot{q}$ is a homeomorphism. Moreover, by Lemma C.12 we have

$$Ri_* \circ D^+\left((\blacktriangle|_{\mathbb{R}})_*\right) \cong Ri_* \circ R(\blacktriangle|_{\mathbb{R}})_* \cong R(\blacktriangle|_{\mathbb{R}})_*,$$

where the symbol $\blacktriangle|_{\mathbb{R}}$ on the right-hand side denotes the embedding $\blacktriangle|_{\mathbb{R}} \colon \mathbb{R} \to q_\gamma$. With this the statement follows from Corollary 9.3 and Lemma 8.15. $\square$

Corollary 9.5. *Any tame derived sheaf F of $D^+_t(\mathbb{R})$ decomposes as*

$$F \cong \bigoplus_{v \in \operatorname{int} \mathbb{M}_{\mathrm{f}}} \iota_{\mathbb{R}}(v)^{\oplus \mu(v)},$$

where $\mu := (\beta^0 \circ h_{\mathbb{R}})(F) \colon \operatorname{int} \mathbb{M}_{\mathrm{f}} \to \mathbb{N}_0$.

Example 9.6. Let $X \subset \mathbb{R}^2$ be the *infinite brush* depicted by Fig. 2.3 on the left-hand side and let $f \colon X \to \mathbb{R}$ be its height function, which is $\mathbb{F}$-tame. On the right-hand side of Fig. 2.3 we see the RISC $h(f) \colon \mathbb{M}^\circ \to \mathrm{vect}_{\mathbb{F}}$ of f and its infinite direct sum decomposition. By Corollary 9.4 the 0-th Leray sheaf $R^0 f_* \mathbb{F}_X \cong f_* \mathbb{F}_X$ of $f \colon X \to \mathbb{R}$, aka sheaf-theoretical level set persistence in degree 0 since [Cur14], is the corresponding direct sum of indecomposable sheaves on $\mathbb{R}$. Considering the second diagonal line segment connecting l_0 and l_1 in Fig. 2.3 and the way in which it intersects the supports of the indecomposables, it is clear that $f_* \mathbb{F}_X$ is not *weakly constructible* in the sense of [KS90, Definition 8.1.3.(i)]. Harnessing Gabriel's Theorem the authors of [KS18, Theorem 1.17] provide a structure theorem for $\mathbb{R}$-*constructible* sheaves on $\mathbb{R}$, a specialization of weakly constructible sheaves. Now Corollary 9.5 also provides a structure theorem, but for tame sheaves on $\mathbb{R}$. As $f_* \mathbb{F}_X$ is tame but not constructible and as $\mathbb{F}_{\mathbb{Z}}$ is $\mathbb{R}$-constructible but not tame, the classes of $\mathbb{R}$-constructible sheaves and tame sheaves are incomparable.

Tame Derived Sheaves Concretely. Before we discuss the proof Theorem 9.1, we use Theorem 9.1 as well as the results of Chapter 2 to provide a concrete description of the categories $D^+_t(q_\gamma, \partial q)$ and $D^+_t(\mathbb{R})$. Now in Chapter 2 we already provided a concrete description of the category of sequentially continuous pfd cohomological presheaves $\mathbb{M}^\circ \to \mathrm{vect}_{\mathbb{F}}$, specifically, its isomorphisms classes are described by their 0-th Betti functions, which are functions $\mu \colon \operatorname{int} \mathbb{M} \to \mathbb{N}_0$ satisfying the estimate

$$\sum_{v \in R_u} \mu(v) < \infty \tag{2.6 revisited}$$

for any $u \in \operatorname{int} \mathbb{M}$, where $R_u := ({\uparrow}\, u) \cap \operatorname{int}({\downarrow}\, T(u)) \subset \operatorname{int} \mathbb{M}$. Now for such a presheaf to be in $\mathcal{J}$, it also needs to have bounded above support. Thus, the 0-th Betti functions of presheaves in $\mathcal{J}$ adhere to the following notion.

Definition 9.7 (Admissible Betti Function)**.** We say that a function $\mu \colon \operatorname{int} \mathbb{M} \to \mathbb{N}_0$ is an *admissible Betti function* if it has bounded above support and satisfies the estimate (2.6) for any $u \in \operatorname{int} \mathbb{M}$.

As the following Lemma 9.8 shows, not only do presheaves in $\mathcal{J}$ have admissible 0-th Betti functions, but also presheaves that are $\mathcal{J}$-presentable in the sense of Definition E.6.

Lemma 9.8. *The* 0*-th Betti function* $\beta^0(F)\colon \operatorname{int}\mathbb{M} \to \mathbb{N}_0$ *of any* $\mathcal{J}$*-presentable presheaf* $F\colon \mathbb{M}^\circ \to \mathrm{vect}_\mathbb{F}$ *is admissible in the sense of Definition 9.7.*

Proof. Since $F\colon \mathbb{M}^\circ \to \mathrm{vect}_\mathbb{F}$ is $\mathcal{J}$-presentable, there is some epimorphism $\psi\colon P \to F$ with P a presheaf in $\mathcal{J}$. Moreover, as $\mathrm{Nat}(-, S_u)$ is a left-exact functor for any $u \in \operatorname{int}\mathbb{M}$ we have the pointwise inequality $\beta^0(F) \leq \beta^0(P)$ of 0-th Betti functions. Furthermore, the 0-th Betti function $\beta^0(P)\colon \operatorname{int}\mathbb{M} \to \mathbb{N}_0$ is admissible by Corollary 2.10, hence the 0-th Betti function $\beta^0(F)$ is admissible as well. □

Lemma 9.9. *Any function* $\mu\colon \operatorname{int}\mathbb{M} \to \mathbb{N}_0$ *is an admissible Betti function iff the direct sum* $\bigoplus_{v \in \operatorname{int}\mathbb{M}} B_v^{\oplus \mu(v)}$ *is a presheaf in* $\mathcal{J}$*.*

Proof. Suppose $\mu\colon \operatorname{int}\mathbb{M} \to \mathbb{N}_0$ is an admissible Betti function. Then $\bigoplus_{v \in \operatorname{int}\mathbb{M}} B_v^{\oplus \mu(v)}$ is pfd by Proposition 2.9 and it has bounded above support. Moreover, any direct sums of block presheaves B_v, $v \in \operatorname{int}\mathbb{M}$ is cohomological no matter what. Thus, it remains to be shown that $\bigoplus_{v \in \operatorname{int}\mathbb{M}} B_v^{\oplus \mu(v)}$ is sequentially continuous. As it is pfd, we have $\bigoplus_{v \in \operatorname{int}\mathbb{M}} B_v^{\oplus \mu(v)} = \prod_{v \in \operatorname{int}\mathbb{M}} B_v^{\mu(v)}$. So sequential continuity follows from the commutativity of limits. Conversely, suppose $\bigoplus_{v \in \operatorname{int}\mathbb{M}} B_v^{\oplus \mu(v)}$ is a presheaf in $\mathcal{J}$. Then $\mu\colon \operatorname{int}\mathbb{M} \to \mathbb{N}_0$ has bounded above support and it satisfies the estimate (2.6) for any $u \in \operatorname{int}\mathbb{M}$ by Proposition 2.9, hence it is an admissible Betti function. □

Proposition 9.10. *For a function* $\mu\colon \operatorname{int}\mathbb{M} \to \mathbb{N}_0$ *the following are equivalent:*

(i) $\mu\colon \operatorname{int}\mathbb{M} \to \mathbb{N}_0$ *is an admissible Betti function.*

(ii) The direct sum $\bigoplus_{v \in \operatorname{int}\mathbb{M}} \iota(v)^{\oplus \mu(v)}$ *is a derived sheaf in* $D_t^+(q_\gamma, \partial q)$*.*

Furthermore, in the context of Section 8.3 where $\mathbb{M} = \mathbb{M}_\mathrm{f}$ *statements (i) and (ii) are equivalent to the following:*

(iii) The direct sum $\bigoplus_{v \in \operatorname{int}\mathbb{M}_\mathrm{f}} \iota_\mathbb{R}(v)^{\oplus \mu(v)}$ *is a tame derived sheaf on* $\mathbb{R}$*, i.e. a derived sheaf in* $D_t^+(\mathbb{R})$*.*

Proof. The equivalence of (i) and (ii) follows from Corollary 8.3, Theorem 10.1.(iii), and Lemma 9.9. The equivalence of (ii) and (iii) follows from Lemma 8.22. □

Corollary 9.11. *For any derived sheaf* F *of* $D_t^+(q_\gamma, \partial q)$ *or of* $D_t^+(\mathbb{R})$ *the* 0*-th Betti function* $(\beta^0 \circ h_\gamma)(F)$ *respectively* $(\beta^0 \circ h_\mathbb{R})(F)$ *is admissible.*

Proof. This follows in conjunction with Corollary 9.2 respectively Corollary 9.5. □

Remark 9.12 (Structure of Derived Sheaf Homomorphisms). As another application of Theorem 9.1 and Corollary 9.4, the Remark 2.11 can be transfered to the sheaf-theoretical setting. So the potentially infinite matrices, that we may use to describe the homomorphisms in $D_t^+(q_\gamma, \partial q)$ respectively $D_t^+(\mathbb{R})$, have finitely many non-zero entries in every column.

Auxiliary Results on Potentially Infinite Direct Sums of Derived Sheaves. Let $p\colon I \to \operatorname{int}\mathbb{M}$ be some map of sets such that the assignment

$$\operatorname{int}\mathbb{M} \to \mathbb{N}_0,\ u \mapsto \#p^{-1}(u) \tag{9.1}$$

is an admissible Betti function in the sense of Definition 9.7. Again we write $B_i \coloneqq B_{p(i)}\colon \mathbb{M}^\circ \to \mathrm{vect}_\mathbb{F}$ for any $i \in I$. Now by Corollary 8.3 we have

$$(h_\gamma \circ \iota \circ p)(i) \cong B_i \quad \text{for } i \in I. \tag{9.2}$$

Moreover, as (9.1) is an admissible Betti function, the direct sum $\bigoplus_{i\in I} B_i$ is in $\mathcal{J}$ by Lemma 9.9. Furthermore, as any presheaf in $\mathcal{J}$ is pfd by definition, the direct sum $\bigoplus_{i\in I} B_i$ is a biproduct. So considering (9.2), in order for there to be any chance that $h_{\gamma,0,t}\colon D_t^+(q_\gamma, \partial q) \to \mathcal{J}$ is an equivalence, the direct sum $\bigoplus_{i\in I}(\iota \circ p)(i)$ needs to exist as a biproduct in $D_t^+(q_\gamma, \partial q)$.

Proposition 9.13. *The direct sum $\bigoplus_{i\in I}(\iota \circ p)(i)$ together with the projections*

$$\mathrm{pr}_j\colon \bigoplus_{i\in I}(\iota \circ p)(i) \to (\iota \circ p)(j)$$

for $j \in I$ satisfies the universal property of the product in $D^+(q_\gamma, \partial q)$.

Our proof of this proposition is involved enough so that we defer it to its own Section 9.2.

Remark 9.14. If ∂q is not closed in q_γ, then it may well happen that the derived product $R\prod_{i\in I}(\iota \circ p)(i)$, which is the product in $D^+(q_\gamma)$, is not in $D^+(q_\gamma, \partial q)$. Only after applying the coreflection $R\flat$ to the derived product $R\prod_{i\in I}(\iota \circ p)(i)$ we obtain the product in $D^+(q_\gamma, \partial q)$, which conveniently is isomorphic to the direct sum $\bigoplus_{i\in I}(\iota \circ p)(i)$ by Proposition 9.13.

Corollary 9.15. *The direct sum $\bigoplus_{i\in I}(\iota \circ p)(i)$ is preserved by $h_\gamma\colon D^+(q_\gamma) \to \mathrm{Vect}_\mathbb{F}^{\mathbb{M}^\circ}$.*

Proof. As $\iota(u)$ is in $D^+(q_\gamma, \partial q)$ for any $u \in \operatorname{int}\mathbb{M}$, the functor

$$h_\gamma\colon D^+(q_\gamma) \to \mathrm{Vect}_\mathbb{F}^{\mathbb{M}^\circ},\ F \mapsto \mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(-), F)$$

maps the direct sum $\bigoplus_{i\in I}(\iota \circ p)(i)$ to the direct product $\prod_{i\in I}(h_\gamma \circ \iota \circ p)(i)\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ by Proposition 9.13. Moreover, we have $(h_\gamma \circ \iota \circ p)(i) \cong B_i$ by Corollary 8.3, hence

$$\prod_{i\in I}(h_\gamma \circ \iota \circ p)(i) \cong \prod_{i\in I} B_i.$$

Furthermore, the direct sum $\bigoplus_{i\in I} B_i$ is pfd by Lemma 9.9 and thus $\prod_{i\in I} B_i = \bigoplus_{i\in I} B_i$. Invoking Corollary 8.3 once more we obtain

$$\bigoplus_{i\in I} B_i \cong \bigoplus_{i\in I}(h_\gamma \circ \iota \circ p)(i). \qquad \square$$

Corollary 9.16. *The direct sum* $\bigoplus_{i\in I}(\iota\circ p)(i)$ *is tame in the sense of Definition 8.11.*

Proof. By Corollary 9.15 the direct sum $\bigoplus_{i\in I}(\iota\circ p)(i)$ is mapped to $\bigoplus_{i\in I}(h_\gamma\circ\iota\circ p)(i)$. Moreover, we have $(h_\gamma\circ\iota\circ p)(i)\cong B_i$ by Corollary 8.3, hence

$$\bigoplus_{i\in I}(h_\gamma\circ\iota\circ p)(i)\cong\bigoplus_{i\in I}B_i,$$

which is pfd by Lemma 9.9. □

Decompositions of Tame Derived Sheaves. Now any presheaf in $\mathcal{J}$ decomposes into block presheaves by Theorem 2.6. Moreover, block presheaves correspond to images of $\iota\colon\mathbb{M}\to D^+(q_\gamma,\partial q)$ under $h_{\gamma,0,t}\colon D^+_t(q_\gamma,\partial q)\to\mathcal{J}$. Thus, if $h_{\gamma,0,t}$ is an equivalence, then any derived sheaf of $D^+_t(q_\gamma,\partial q)$ necessarily decomposes into sheaves in the image of ι. We phrased this as Corollary 9.2 to Theorem 9.1. However, for technical reasons we prove the existence of decompositions first, and then we use this to show Theorem 9.1.

Remark 9.17. We also point out, that our proof for decompositions of tame derived sheaves in $D^+_t(q_\gamma,\partial q)$ is rather unconventional. A common technique for decomposing objects of a derived category is to decompose the objects of the corresponding abelian category first. Such technique is employed in [KS18, Thoerem 1.17], where the authors provide a decomposition of $\mathbb{R}$-constructible sheaves based on results by [Gab72], [Cur14, Section 15.3], and [Gui16, Corollary 7.3]. Then the authors of [KS18] show that the category of $\mathbb{R}$-constructible sheaves is hereditary so the existence of decompositions in the abelian category of $\mathbb{R}$-constructible sheaves extends to the derived category. Here we provide a self-contained proof working directly with the derived category. Rather than reducing the existence of decompositions in the derived category to the existence of decompositions in the underlying abelian category, we harness the functor $D^+_t(q_\gamma,\partial q)\xrightarrow{h_\gamma}\mathcal{J}\hookrightarrow\operatorname{pres}(\mathcal{J})$ to reduce the existence of decompositions in $D^+_t(q_\gamma,\partial q)$ to the existence of decompositions of projectives in the category of $\mathcal{J}$-presentable presheaves $\operatorname{pres}(\mathcal{J})$, which is the *a priori abelianization* of $D^+_t(q_\gamma,\partial q)$; also see Remark 10.19.

Now suppose F is an object of $D^+_t(q_\gamma,\partial q)$ and let $p\colon I\to\operatorname{int}\mathbb{M}$ be some map of sets such that

$$\#p^{-1}(u)=\left(\beta^0\circ h_\gamma\right)(F)(u)=\dim_{\mathbb{F}}\operatorname{Nat}(h_\gamma(F),S_u)$$

for any $u\in\operatorname{int}\mathbb{M}$. By Theorem 2.6 there is a natural isomorphism $\varphi\colon\bigoplus_{i\in I}B_i\to h_\gamma(F)$. Now for each $j\in I$ let $\psi_j\colon(\iota\circ p)(j)\to F$ be the image of $1\in\mathbb{F}=B_j(p(j))$ under the composition of linear maps

$$\mathbb{F}=B_j(p(j))\longrightarrow\bigoplus_{i\in I}B_i(p(j))\xrightarrow{\varphi_{p(j)}}h_\gamma(F)(p(j))=\operatorname{Hom}_{D^+(q_\gamma)}((\iota\circ p)(j),F).$$

Then the I-tuple

$$(\psi_i)_{i\in I}\in\prod_{i\in I}\operatorname{Hom}_{D^+(q_\gamma)}((\iota\circ p)(i),F)$$

induces a homomorphism

$$\psi\colon \bigoplus_{i\in I} (\iota \circ p)(i) \to F$$

in the derived category $D^+(q_\gamma)$. Now let $G \coloneqq \bigoplus_{i\in I}(\iota \circ p)(i)$. We aim to show that $\psi\colon G \to F$ is an isomorphism. To this end, the following lemma will serve us as a stepping stone.

Lemma 9.18. *The natural transformation $h_\gamma(\psi)\colon h_\gamma(G) \to h_\gamma(F)$ is a natural isomorphism.*

Proof. We consider the commutative diagram

$$\begin{tikzcd}
\bigoplus_{i\in I}(h_\gamma \circ \iota \circ p)(i) \arrow[r] & h_\gamma(G) \arrow[d, "h_\gamma(\psi)"] \\
\bigoplus_{i\in I} B_i \arrow[u] \arrow[r, "\cong", "\varphi"'] & h_\gamma(F)
\end{tikzcd}$$

of presheaves $\mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ and natural transformations. By Corollary 9.15 the horizontal arrow at the top is a natural isomorphism. Moreover, the vertical arrow on the left is a natural isomorphism by Corollary 8.3. As a result $h_\gamma(\psi)\colon h_\gamma(G) \to h_\gamma(F)$ is an isomorphism as well. □

In conjunction with Proposition 8.10 we obtain the following.

Corollary 9.19. *The homomorphism $\psi\colon G \to F$ is an isomorphism.*

Proof of Equivalence. We now show Theorem 9.1.

Proof of Theorem 9.1. First we show that $h_{\gamma,0,t}$ is essentially surjective. To this end, let $F\colon \mathbb{M}^\circ \to \mathrm{vect}_\mathbb{F}$ be a presheaf in $\mathcal{J}$. By the Structure Theorem 2.6 the presheaf F is naturally isomorphic to a direct sum $\bigoplus_{i\in I} B_{p(i)}\colon \mathbb{M}^\circ \to \mathrm{vect}_\mathbb{F}$ for some map of sets $p\colon I \to \operatorname{int}\mathbb{M}$ with $\# p^{-1}(u) = \beta^0(F)(u)$ for any $u \in \operatorname{int}\mathbb{M}$. Now $\bigoplus_{i\in I} B_{p(i)} \cong \bigoplus_{i\in I}(h_\gamma \circ \iota \circ p)(i)$ by Corollary 8.3. Moreover, $h_\gamma\left(\bigoplus_{i\in I}(\iota \circ p)(i)\right) \cong \bigoplus_{i\in I}(h_\gamma \circ \iota \circ p)(i)$ by Corollary 9.15, hence $h_{\gamma,0,t}\colon D_t^+(q_\gamma, \partial q) \to \mathcal{J}$ is essentially surjective.

Next we show $h_{\gamma,0,t}$ is fully faithful. To this end, let F and G be objects of $D_t^+(q_\gamma, \partial q)$, then we have

$$F \cong \bigoplus_{i\in I}(\iota \circ p_1)(i) \quad \text{and} \quad G \cong \bigoplus_{j\in J}(\iota \circ p_2)(j)$$

for some maps of sets $p_1\colon I \to \operatorname{int}\mathbb{M}$ and $p_2\colon J \to \operatorname{int}\mathbb{M}$ with

$$\# p_1^{-1}(u) = (\beta^0 \circ h_\gamma)(F)(u) \quad \text{and} \quad \# p_2^{-1}(u) = (\beta^0 \circ h_\gamma)(G)(u) \tag{9.3}$$

for any $u \in \operatorname{int}\mathbb{M}$ by Corollary 9.19. Moreover, by Proposition 9.13 the direct sum $\bigoplus_{j\in J}(\iota \circ p_2)(j)$ satisfies the universal property of the corresponding product in

$D^+_t(q_\gamma, \partial q)$. Furthermore, the functor $h_\gamma \colon D^+(q_\gamma) \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ}$ preserves the direct sums in (9.3) by Corollary 9.15. As any pfd direct sum of presheaves is a direct product as well we obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{D^+(q_\gamma)}(F, G) \arrow[r, "\cong"] \arrow[d, "h_\gamma"'] & \prod_{i \in I, j \in J} \mathrm{Hom}_{D^+(q_\gamma)}((\iota \circ p)(i), (\iota \circ p)(j)) \arrow[d, "\prod_{i \in I, j \in J} h_\gamma"] \\
\mathrm{Nat}(h_\gamma(F), h_\gamma(G)) \arrow[r, "\cong"] & \prod_{i \in I, j \in J} \mathrm{Nat}((h_\gamma \circ \iota \circ p)(i), (h_\gamma \circ \iota \circ p)(j))
\end{tikzcd}$$

with both horizontal maps isomorphisms. Thus, it suffices to show that the restriction of $h_\gamma \colon D^+(q_\gamma) \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ}$ to the full subcategory on the image of ι is fully faithful. To this end, we note that for a homomorphism $\psi \colon \iota(u) \to \iota(v)$, where $u, v \in \operatorname{int} \mathbb{M}$, the natural transformation

$$h_\gamma(\psi) \colon (h_\gamma \circ \iota)(u) \to (h_\gamma \circ \iota)(v)$$

sends the identity

$$\mathrm{id}_{\iota(u)} \in (h_\gamma \circ \iota)(u)(u) = \mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(u), \iota(u))$$

to

$$\psi \in (h_\gamma \circ \iota)(v)(u) = \mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(u), \iota(v)).$$

By Corollary 8.3 and the Yoneda Lemma 2.4 this describes a one-to-one correspondence between $\mathrm{Hom}_{\mathrm{D}^+(q_\gamma)}(\iota(u), \iota(v))$ and $\mathrm{Nat}((h_\gamma \circ \iota)(u), (h_\gamma \circ \iota)(v))$. □

9.1 Partial Faithfulness

To complete our proof of Theorem 9.1 we still need to provide a proof of Proposition 9.13. Now in order to prove Proposition 9.13 we will first show with Proposition 9.25 below that the functor $h_{\gamma,0} \colon D^+(q_\gamma, \partial q) \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ}$ is at least partially full and partially faithful in the sense that the family of homomorphisms with codomain $(\iota \circ \Sigma^{-n})(t)$ is in one-to-one correspondence with corresponding natural transformations with codomain $(h_\gamma \circ \iota \circ \Sigma^{-n})(t)$ for any $n \in \mathbb{Z}$ and $t \in q \setminus \partial q$. To this end, let $t \in q \setminus \partial q$ and let $n \in \mathbb{Z}$. For any $u \in \Sigma^{-n}(D)$ we define the following subsets of q:

$$\begin{aligned}
I(u) &:= q \cap \operatorname{int}\left(\downarrow \Sigma^{n+1}(u)\right) = q \setminus (\rho_1 \circ \Sigma^n)(u), \\
C(u) &:= q \setminus (\uparrow \Sigma^n(u)) = q \setminus (\rho_0 \circ \Sigma^n)(u), \\
\text{and} \quad Z(u) &:= I(u) \setminus C(u) = (\rho_0 \circ \Sigma^n)(u) \setminus (\rho_1 \circ \Sigma^n)(u).
\end{aligned}$$

Moreover, we define the functors

$$\begin{aligned}
\mathbb{F}_I[-n] &\colon \Sigma^{-n}(D) \to C^+(q_\gamma),\ u \mapsto \mathbb{F}_{I(u)}[-n], \\
\mathbb{F}_C[-n] &\colon \Sigma^{-n}(D) \to C^+(q_\gamma),\ u \mapsto \mathbb{F}_{C(u)}[-n], \\
\text{and} \quad \mathbb{F}_Z[-n] &\colon \Sigma^{-n}(D) \to C^+(q_\gamma),\ u \mapsto \mathbb{F}_{Z(u)}[-n],
\end{aligned}$$

where $C^+(q_\gamma) \coloneqq C^+(\mathrm{Sh}(q_\gamma))$ is the category of bounded below cochain complexes of sheaves on q_γ. We note that

$$\mathbb{F}_Z[-n] = \iota|_{\Sigma^{-n}(D)}. \tag{9.4}$$

By [KS90, Proposition 2.3.6.(v)] we have the short exact sequence

$$0 \to \mathbb{F}_C[-n] \to \mathbb{F}_I[-n] \to \mathbb{F}_Z[-n] \to 0$$

of functors $\Sigma^{-n}(D) \to C^+(q_\gamma)$. Thus, we obtain the sequence

$$\mathbb{F}_C[-n] \to \mathbb{F}_I[-n] \to \mathbb{F}_Z[-n] \to \mathbb{F}_C[1-n] \tag{9.5}$$

of functors $\Sigma^{-n}(D) \to D^+(q_\gamma)$, which is pointwise a distinguished triangle in $D^+(q_\gamma)$, see also [KS90, Equation (2.6.33)]. Now let q_0 be the unique point of intersection of q and l_0. Similarly, let q_1 be the unique point of intersection of q and l_1. We note that $t \in q$ lies on the vertical line through q_0 iff $q_0 \preceq t$. Similarly, t lies on the horizontal line through q_1 iff $q_1 \preceq t$. For $u \in \Sigma^{-n}(D)$ we write $C_0(u) \subseteq C(u)$ for the connected component of $C(u)$ containing q_0 if it exists, and otherwise we set $C_0(u) = \emptyset$. We define $C_1(u) \subseteq C(u)$ analogously as well as the functors

$$\begin{aligned} &\mathbb{F}_{C_0}[-n] \colon \Sigma^{-n}(D) \to C^+(q_\gamma),\, u \mapsto \mathbb{F}_{C_0(u)}[-n] \\ \text{and}\quad &\mathbb{F}_{C_1}[-n] \colon \Sigma^{-n}(D) \to C^+(q_\gamma),\, u \mapsto \mathbb{F}_{C_1(u)}[-n]. \end{aligned}$$

Now let $m = n, n-1$ and let

$$P \colon (\Sigma^{-n}(D))^\circ \to \mathrm{vect}_{\mathbb{F}},\, u \mapsto \mathrm{Hom}_{D^+(q_\gamma)}\left(\mathbb{F}_{I(u)}[-n], (\iota \circ \Sigma^{-n})(t)\right).$$

Then P is isomorphic to the presheaf

$$P' \colon (\Sigma^{-n}(D))^\circ \to \mathrm{vect}_{\mathbb{F}}, u \mapsto \begin{cases} \mathbb{F} & u \in \mathrm{int}\left(\uparrow \Sigma^{-n-1}(t)\right) \\ \{0\} & \text{otherwise,} \end{cases}$$

whose internal maps are identities whenever both domain and codomain are $\mathbb{F}$ and zero otherwise. For any presheaf $F \colon (\Sigma^{-n}(D))^\circ \to \mathrm{Vect}_{\mathbb{F}}$ we have

$$\mathrm{Nat}(F, P') \cong \left(\varinjlim_{u \in \mathrm{int}(\uparrow \Sigma^{-n-1}(t))} F(u) \right)^*.$$

As directed colimits and the dual space functor are exact, the functor $\mathrm{Nat}(-, P') \cong \mathrm{Nat}(-, P)$ is exact as well. Now suppose F is an object of $D^+(q_\gamma)$. In the following we use $\mathrm{Hom}(\mathbb{F}_{C_0}[-m], F)$ as a shorthand for the functor

$$(\Sigma^{-n}(D))^\circ \to \mathrm{Vect}_{\mathbb{F}},\, u \mapsto \mathrm{Hom}_{D^+(q_\gamma)}\left(\mathbb{F}_{C_0(u)}[-m], F\right)$$

and similarly for $\mathbb{F}_{C_1}[-m]$, $\mathbb{F}_I[-n]$, $\mathbb{F}_C[-n]$, or $\mathbb{F}_Z[-n]$ in place of $\mathbb{F}_{C_0}[-m]$.

Lemma 9.20. *We have*

$$\operatorname{Nat}\left(\operatorname{Hom}(\mathbb{F}_{C_0}[-m], F), P\right) \cong \begin{cases} \varinjlim_{U \ni q_0} H^m(U; F) & q_0 \preceq t \\ \{0\} & \text{otherwise.} \end{cases}$$

Similarly we have

$$\operatorname{Nat}\left(\operatorname{Hom}(\mathbb{F}_{C_1}[-m], F), P\right) \cong \begin{cases} \varinjlim_{U \ni q_1} H^m(U; F) & q_1 \preceq t \\ \{0\} & \text{otherwise.} \end{cases}$$

Corollary 9.21. *For any object F of $D^+(q_\gamma, \partial q)$ we have*

$$\operatorname{Nat}\left(\operatorname{Hom}(\mathbb{F}_C[-m], F), P\right) \cong \{0\}.$$

Proof. We have

$$\begin{aligned} \operatorname{Nat}\left(\operatorname{Hom}(\mathbb{F}_C[-m], F), P\right) &= \operatorname{Nat}\left(\operatorname{Hom}(\mathbb{F}_{C_0}[-m] \oplus \mathbb{F}_{C_1}[-m], F), P\right) \\ &= \operatorname{Nat}\left(\operatorname{Hom}(\mathbb{F}_{C_0}[-m], F) \oplus \operatorname{Hom}(\mathbb{F}_{C_1}[-m], F), P\right) \\ &= \operatorname{Nat}\left(\operatorname{Hom}(\mathbb{F}_{C_0}[-m], F), P\right) \oplus \operatorname{Nat}\left(\operatorname{Hom}(\mathbb{F}_{C_1}[-m], F), P\right) \\ &= \{0\} \oplus \{0\} \end{aligned}$$

by Lemma 9.20. □

Lemma 9.22. *The functor* $\operatorname{Hom}(\mathbb{F}_I[-n], -)$ *induces an isomorphism*

$$\operatorname{Hom}_{D^+(q_\gamma)}(F, (\iota \circ \Sigma^{-n})(t)) \xrightarrow{\cong} \operatorname{Nat}(\operatorname{Hom}(\mathbb{F}_I[-n], F), P).$$

Proof. The set of open neighbourhoods of $t \in q_\gamma$ of the form $I(u)$ for some $u \in \Sigma^{-n}(D)$ forms a neighbourhood basis for t. Thus, the induced map

$$\varinjlim_{u \in \Sigma^{-n}(D)\,:\, t \in I(u)} H^n(I(u); F) \xrightarrow{\cong} \varinjlim_{U \ni t} H^n(U; F)$$

is an isomorphism. With this we consider the commutative diagram

$$\begin{tikzcd} \operatorname{Hom}_{D^+(q_\gamma)}(F, (\iota \circ \Sigma^{-n})(t)) \arrow[r, "\operatorname{Hom}(\mathbb{F}_I[-n],-)"] \arrow[d] & \operatorname{Nat}(\operatorname{Hom}(\mathbb{F}_I[-n], F), P) \arrow[d] \\ \left(\varinjlim_{U \ni t} H^n(U; F)\right)^* \arrow[r, "\cong"] & \left(\varinjlim_{u \in \Sigma^{-n}(D)\,:\, t \in I(u)} H^n(I(u); F)\right)^* . \end{tikzcd}$$

Here the vertical map on the left-hand side takes any homomorphism $\psi \colon F \to (\iota \circ \Sigma^{-n})(t)$ to the family of maps $\{H^n(U; \psi) \colon H^n(U; F) \to \mathbb{F}\}_{U \ni t}$ and then to the naturally induced map of type $\varinjlim_{U \ni t} H^n(U;F) \to \mathbb{F}$. By Lemma D.5 this map is an isomorphism. The vertical map on the right-hand side takes any natural transformation $\eta\colon \mathrm{Hom}(\mathbb{F}_I[-n], F) \to P$ to the family of maps $\{\eta_u\colon H^n(I(u);F) \to \mathbb{F}\}_{u \in \Sigma^{-n}(D)\,:\, t \in I(u)}$ and then to the naturally induced map of type

$$\varinjlim_{u \in \Sigma^{-n}(D)\,:\, t \in I(u)} H^n(I(u);F) \to \mathbb{F}.$$

As $P(u) \cong \{0\}$ for all $u \in \Sigma^{-n}(D)$ with $t \notin I(u)$, this vertical map on the right-hand side is an isomorphism. As a result, we obtain that the horizontal map at the top is an isomorphism as well. $\square$

Now let

$$B \coloneqq \mathrm{Hom}(\mathbb{F}_Z[-n], (\iota \circ \Sigma^{-n})(t))\colon (\Sigma^{-n}(D))^\circ \to \mathrm{vect}_{\mathbb{F}},$$

then we have

$$B = (h_\gamma \circ \iota \circ \Sigma^{-n})(t)|_{\Sigma^{-n}(D)} \cong B_{\Sigma^{-n}(t)}|_{\Sigma^{-n}(D)}$$

by (9.4) and Corollary 8.3. We consider

$$\mathrm{Nat}(-, P) = \mathrm{Nat}(-, \mathrm{Hom}(\mathbb{F}_I[-n], (\iota \circ \Sigma^{-n})(t)))$$

as a functor from the category of presheaves $(\Sigma^{-n}(D))^\circ \to \mathrm{Vect}_{\mathbb{F}}$ vanishing on $\partial\mathbb{M} \cap \Sigma^{-n}(D)$ to the category of vector spaces over $\mathbb{F}$.

Lemma 9.23. *The natural transformation*

$$B = \mathrm{Hom}(\mathbb{F}_Z[-n], (\iota \circ \Sigma^{-n})(t)) \to \mathrm{Hom}(\mathbb{F}_I[-n], (\iota \circ \Sigma^{-n})(t)) = P$$

is a universal element of the functor $\mathrm{Nat}(-, P)$. *In other words, for any natural transformation $\eta\colon F \to P$ with F vanishing on $\partial\mathbb{M} \cap \Sigma^{-n}(D)$ there is a unique natural transformation $\eta'\colon F \to B$ such that the diagram*

$$\begin{tikzcd} & B \arrow[d] \\ F \arrow[ur, "\eta'"] \arrow[r, "\eta"'] & P \end{tikzcd}$$

commutes.

Proof. As $P \cong P'$ we have the short exact sequence

$$0 \to B \to P \to \mathrm{Ran}_{\partial\mathbb{M}} P|_{\partial\mathbb{M}} \to 0,$$

where $\mathrm{Ran}_{\partial\mathbb{M}} P|_{\partial\mathbb{M}}$ is the right Kan extension of the restriction of P to $\partial\mathbb{M} \cap \Sigma^{-n}(D)$ along the inclusion $\partial\mathbb{M} \cap \Sigma^{-n}(D) \hookrightarrow \Sigma^{-n}(D)$. As $\mathrm{Ran}_{\partial\mathbb{M}} F|_{\partial\mathbb{M}} \cong 0$ for any presheaf $F\colon (\Sigma^{-n}(D))^\circ \to \mathrm{Vect}_{\mathbb{F}}$ vanishing on $\partial\mathbb{M} \cap \Sigma^{-n}(D)$ the result follows. $\square$

Lemma 9.24. *For any presheaf* $F \colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ *the restriction map*

$$\mathrm{Nat}(F, B_{\Sigma^{-n}(t)}) \cong \mathrm{Nat}(F, (h_\gamma \circ \iota \circ \Sigma^{-n})(t)) \to \mathrm{Nat}(F|_{\Sigma^{-n}(D)}, B)$$

is an isomorphism.

Proposition 9.25. *For any object* F *of* $D^+(q_\gamma, \partial q)$ *the natural map*

$$\mathrm{Hom}_{D^+(q_\gamma)}(F, (\iota \circ \Sigma^{-n})(t)) \xrightarrow{h_\gamma} \mathrm{Nat}(h_\gamma(F), (h_\gamma \circ \iota \circ \Sigma^{-n})(t))$$

is an isomorphism.

Proof. We consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{D^+(q_\gamma)}(F, (\iota \circ \Sigma^{-n})(t)) \arrow[d, "h_\gamma"'] \arrow[dr, "\mathrm{Hom}(\mathbb{F}_I[-n],-)"] & \\
\mathrm{Nat}(h_\gamma(F), (h_\gamma \circ \iota \circ \Sigma^{-n})(t)) \arrow[d] & \mathrm{Nat}(\mathrm{Hom}(\mathbb{F}_I[-n], F), P) \arrow[d] \\
\mathrm{Nat}(\mathrm{Hom}(\mathbb{F}_Z[-n], F), B) \arrow[r] & \mathrm{Nat}(\mathrm{Hom}(\mathbb{F}_I[-n], F), P).
\end{tikzcd}$$

By Lemma 9.24 the vertical map on the lower left-hand side is an isomorphism. Moreover, by Lemma 9.23 the horizontal map at the bottom is an isomorphism. Furthermore, the diagonal map on the upper right-hand side induced by the functor $\mathrm{Hom}(\mathbb{F}_I[-n], -)$ is an isomorphism by Lemma 9.22. Thus, it suffices to show that the vertical map on the right-hand side

$$\mathrm{Nat}(\mathrm{Hom}(\mathbb{F}_I[-n], F), P) \to \mathrm{Nat}(\mathrm{Hom}(\mathbb{F}_I[-n], F), P)$$

is an isomorphism. To this end, we post-compose the pointwise distinguished triangle (9.5) of functors with the cohomological functor $\mathrm{Hom}_{D^+(q_\gamma)}(-, F)$ to obtain the exact sequence

$$\mathrm{Hom}(\mathbb{F}_C[1-n], F) \to \mathrm{Hom}(\mathbb{F}_Z[-n], F) \to \mathrm{Hom}(\mathbb{F}_I[-n], F) \to \mathrm{Hom}(\mathbb{F}_C[-n], F)$$

of presheaves $(\Sigma^{-n}(D))^\circ \to \mathrm{Vect}_\mathbb{F}$. Now we apply the exact functor $\mathrm{Nat}(-, P)$ to this exact sequence to obtain the exact sequence

$$\begin{tikzcd}
\mathrm{Nat}(\mathrm{Hom}(\mathbb{F}_C[-n], F), P) \arrow[d] \\
\mathrm{Nat}(\mathrm{Hom}(\mathbb{F}_I[-n], F), P) \arrow[d] \\
\mathrm{Nat}(\mathrm{Hom}(\mathbb{F}_Z[-n], F), P) \arrow[d] \\
\mathrm{Nat}(\mathrm{Hom}(\mathbb{F}_C[1-n], F), P)
\end{tikzcd} \tag{9.6}$$

in turn. By Corollary 9.21 we have

$$\mathrm{Nat}(\mathrm{Hom}(\mathbb{F}_C[-n], F), P) \cong \{0\} \cong \mathrm{Nat}(\mathrm{Hom}(\mathbb{F}_C[1-n], F), P)$$

and thus the vertical map in the center of (9.6) has to be an isomorphism. $\square$

9.2 Products Vanishing on ∂q

In the following we provide a proof of Proposition 9.13 by considering the induced maps on cohomology stalks of the direct sum and the coreflection of the derived product into $D^+(q_\gamma, \partial q)$. To this end, let $t \in q \setminus \partial q$ and let $n \in \mathbb{Z}$ as in Section 9.1.

Lemma 9.26. *For a family of objects $\{F_i \mid i \in I\}$ in $D^+(q_\gamma, \partial q)$ the naturally induced map*

$$\mathrm{Nat}\left(h_\gamma\left(\bigoplus_{i\in I} F_i\right), (h_\gamma \circ \iota \circ \Sigma^{-n})(t)\right) \longrightarrow \mathrm{Nat}\left(\bigoplus_{i\in I} h_\gamma(F_i), (h_\gamma \circ \iota \circ \Sigma^{-n})(t)\right)$$

is an isomorphism.

Proof. We consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{D^+(q_\gamma)}(\bigoplus_{i\in I} F_i, (\iota\circ\Sigma^{-n})(t)) \arrow[r, "h_\gamma"] \arrow[dd] & \mathrm{Nat}\left(h_\gamma\left(\bigoplus_{i\in I} F_i\right), (h_\gamma\circ\iota\circ\Sigma^{-n})(t)\right) \arrow[d] \\
 & \mathrm{Nat}\left(\bigoplus_{i\in I} h_\gamma(F_i), (h_\gamma\circ\iota\circ\Sigma^{-n})(t)\right) \arrow[d] \\
\prod_{i\in I} \mathrm{Hom}_{D^+(q_\gamma)}(F_i, (\iota\circ\Sigma^{-n})(t)) \arrow[r, "\prod_{i\in I} h_\gamma"'] & \prod_{i\in I} \mathrm{Nat}\left(h_\gamma(F_i), (h_\gamma\circ\iota\circ\Sigma^{-n})(t)\right).
\end{tikzcd}$$

By Proposition 9.25 the two horizontal maps are isomorphisms. Moreover, the vertical map on the left-hand side and the vertical map on the lower right-hand side are both isomorphisms by the universal property of the direct sum. Thus, the vertical map on the upper right-hand side is an isomorphism as well. $\square$

Now let $p\colon I \to \mathrm{int}\,\mathbb{M}$ be some map of sets such that the assignment

$$\mathrm{int}\,\mathbb{M} \to \mathbb{N}_0,\ u \mapsto \#p^{-1}(u)$$

is an admissible Betti function. Again we write $B_i \coloneqq B_{p(i)}\colon \mathbb{M}^\circ \to \mathrm{vect}_{\mathbb{F}}$ for any $i \in I$.

Proof of Proposition 9.13. Let

$$\kappa\colon \bigoplus_{i\in I}(\iota\circ p)(i) \to R\prod_{i\in I}(\iota\circ p)(i)$$

be the naturally induced homomorphism from the direct sum to the derived product. Then there is a unique homomorphism

$$\kappa^{\#}\colon \bigoplus_{i\in I}(\iota\circ p)(i) \to R\flat R\prod_{i\in I}(\iota\circ p)(i)$$

such that the diagram

$$\begin{tikzcd}
 & R\flat R\prod_{i\in I}(\iota\circ p)(i) \arrow[d, "\varepsilon_{R\prod_{i\in I}(\iota\circ p)(i)}"] \\
\bigoplus_{i\in I}(\iota\circ p)(i) \arrow[ur, "\kappa^{\#}"] \arrow[r, "\kappa"'] & R\prod_{i\in I}(\iota\circ p)(i)
\end{tikzcd} \tag{9.7}$$

commutes. We show that the induced map on cohomology stalks

$$\varinjlim_{t\in U} H^n(U;\kappa^{\#})\colon \varinjlim_{t\in U} H^n\left(U;\bigoplus_{i\in I}(\iota\circ p)(i)\right) \longrightarrow \varinjlim_{t\in U} H^n\left(U; R\flat R\prod_{i\in I}(\iota\circ p)(i)\right)$$

is an isomorphism. By Lemma D.5 we may as well show that the induced map

$$\begin{tikzcd}
\mathrm{Hom}_{D^+(X)}\left(R\flat R\prod_{i\in I}(\iota\circ p)(i), (\iota\circ\Sigma^{-n})(t)\right) \arrow[d, "\mathrm{Hom}_{D^+(X)}\left(\kappa^{\#},(\iota\circ\Sigma^{-n})(t)\right)"] \\
\mathrm{Hom}_{D^+(X)}\left(\bigoplus_{i\in I}(\iota\circ p)(i), (\iota\circ\Sigma^{-n})(t)\right)
\end{tikzcd}$$

is an isomorphism. Proposition 9.25 in turn implies that it suffices to show that the induced map

$$\begin{tikzcd}
\mathrm{Nat}\left(h_\gamma\left(R\flat R\prod_{i\in I}(\iota\circ p)(i)\right), (h_\gamma\circ\iota\circ\Sigma^{-n})(t)\right) \arrow[d, "\mathrm{Nat}\left(h_\gamma\left(\kappa^{\#}\right),(h_\gamma\circ\iota\circ\Sigma^{-n})(t)\right)"] \\
\mathrm{Nat}\left(h_\gamma\left(\bigoplus_{i\in I}(\iota\circ p)(i)\right), (h_\gamma\circ\iota\circ\Sigma^{-n})(t)\right)
\end{tikzcd}$$

is an isomorphism. To this end, we apply the contravariant functor

$$\mathrm{Nat}(h_\gamma(-), (h_\gamma\circ\iota\circ\Sigma^{-n})(t))\colon D^+(q_\gamma)^\circ \to \mathrm{Vect}_{\mathbb{F}}$$

to the commutative triangle (9.7) to obtain the commutative triangle

$$\begin{tikzcd}
\mathrm{Nat}\left(h_\gamma\left(R\prod_{i\in I}(\iota\circ p)(i)\right), (h_\gamma\circ\iota\circ\Sigma^{-n})(t)\right) \arrow[dr, "\mathrm{Nat}\left(h_\gamma\left(\varepsilon_{R\prod_{i\in I}(\iota\circ p)(i)}\right),(h_\gamma\circ\iota\circ\Sigma^{-n})(t)\right)"] \arrow[ddr, bend right, "\mathrm{Nat}\left(h_\gamma(\kappa),(h_\gamma\circ\iota\circ\Sigma^{-n})(t)\right)"'] & \\
& \mathrm{Nat}\left(h_\gamma\left(R\flat R\prod_{i\in I}(\iota\circ p)(i)\right), (h_\gamma\circ\iota\circ\Sigma^{-n})(t)\right) \arrow[d, "\mathrm{Nat}\left(h_\gamma\left(\kappa^{\#}\right),(h_\gamma\circ\iota\circ\Sigma^{-n})(t)\right)"] \\
& \mathrm{Nat}\left(h_\gamma\left(\bigoplus_{i\in I}(\iota\circ p)(i)\right), (h_\gamma\circ\iota\circ\Sigma^{-n})(t)\right).
\end{tikzcd}$$

By Lemma 8.5 the upper left diagonal map $\mathrm{Nat}\left(h_\gamma\left(\varepsilon_{R\prod_{i\in I}(\iota\circ p)(i)}\right), (h_\gamma\circ\iota\circ\Sigma^{-n})(t)\right)$ is an isomorphism. Thus, it suffices to show that the curved map on the left-hand side $\mathrm{Nat}\left(h_\gamma(\kappa), (h_\gamma\circ\iota\circ\Sigma^{-n})(t)\right)$ is an isomorphism. To this end, we consider the commutative diagram

$$\begin{tikzcd}
h_\gamma\left(\bigoplus_{i\in I}(\iota\circ p)(i)\right) \arrow[r, "h_\gamma(\kappa)"] & h_\gamma\left(R\prod_{i\in I}(\iota\circ p)(i)\right) \arrow[d] \\
\bigoplus_{i\in I}(h_\gamma\circ\iota\circ p)(i) \arrow[u] \arrow[r] & \prod_{i\in I}(h_\gamma\circ\iota\circ p)(i) \\
\bigoplus_{i\in I} B_{p(i)} \arrow[u] \arrow[r, equal] & \prod_{i\in I} B_{p(i)}. \arrow[u]
\end{tikzcd}$$

Here the equality at the bottom follows from Lemma 9.9. Moreover, the two vertical arrows in the second row are natural isomorphisms by Corollary 8.3. By the universal property of the derived product the vertical arrow on the upper right-hand side is a natural isomorphism as well. Furthermore, if we apply the functor

$$\mathrm{Nat}(-, (h_\gamma\circ\iota\circ\Sigma^{-n})(t)) \colon \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ} \to \mathrm{Vect}_{\mathbb{F}} \tag{9.8}$$

to the vertical map on the upper left-hand side, then we obtain an isomorphism by Lemma 9.26. Thus applying the functor (9.8) to the horizontal arrow at the top yields an isomorphism as well and this implies the result. $\square$

Part IV

Abelian Categorification of Persistence Diagrams

In algebraic topology the use of exact sequences, such as the Mayer–Vietoris sequence, or spectral sequences for the computation of invariants is ubiquitous, and it seems desirable to harness similar methods when computing invariants of topological data analysis. However, widely used invariants such as the *rank invariant* of *persistence modules* are not *additive*, which diminishes the utility of such methods. In this Part IV, which grew out of the preprint [BF22], we provide a *Mayer–Vietoris principle* for extended persistence diagrams; this is our final Theorem 11.22. In the beginning of Chapter 11 we provide a detailed explanation how an effective Mayer–Vietoris principle for extended persistence diagrams calls for an *abelian categorification* of extended persistence diagrams in the sense of [KMS09]. Essentially, such an abelian categorification is an abelian category whose *Grothendieck group* contains all extended persistence diagrams as elements. Now with RISC we already have a functorial invariant taking values in the abelian category of presheaves $\mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$. However, as we will see in Chapter 11 the category of all presheaves $\mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ is not suitable for at least two reasons (one of them being the Eilenberg swindle). So we consider an abelian subcategory of the category of presheaves $\mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ containing all relative interlevel set cohomologies (induced by singular cohomology with coefficients in $\mathbb{F}$) instead. We start with the construction of this subcategory of presheaves in Chapter 10 reusing our results from the previous Part III.

Related Local to Global Principles in TDA. The idea to port *local to global principles* from algebraic topology such as the Mayer–Vietoris principle to the realm of persistent homology is not new. In particular, [Car16] gave a talk considering Mayer–Vietoris (spectral) sequences and the Künneth formula in the context of Vietoris–Rips persistent homology of finite metric spaces. As we consider the persistence theory of functions in this exposition and as we segment their domains in our Mayer–Vietoris principle rather than the points of finite metric spaces, we see no direct relation between the Mayer–Vietoris principle provided here and the local to global principles by [Car16]. The results by [TC23] have a stronger connection to the present monograph. More specifically, [TC23] provides a Mayer–Vietoris spectral sequence for persistent homology. As far as we understand, one bottleneck of the approach by [TC23] is the resolution of *extension problems* to obtain the persistence barcode from the information provided by his spectral sequence. More concretely, while it is possible to obtain the pages of the Mayer–Vietoris spectral sequence itself by considering local information alone, in order to solve the associated extension problems it is necessary to inspect the filtered simplicial complex as a whole. In the present exposition we approach this problem (in the special case of ordinary Mayer–Vietoris sequences) from another angle. More specifically, we consider RISC as an invariant taking values in an abelian category other than 1-dimensional persistence modules (or sheaves on $\mathbb{R}$ for that matter) retaining more information on the level of homomorphisms. As we will see in the final Theorem 11.22, this additional information is sufficient to compute the extended persistence diagram of a function $f \colon X \to \mathbb{R}$ from local information associated to an open cover $A \cup B = X$ alone. We also note that [TC23] provided an implementation of his methods.

10 Abelianization of Tame Derived Sheaves

In this Chapter 10 we view the equivalence of RISC and derived level set persistence Corollary 9.4 from a more conceptual angle. One thing that is odd about Corollary 9.4 is that in spite the category of tame derived sheaves $D_t^+(\mathbb{R})$ and the category of pfd sequentially continuous cohomological functors $\mathcal{J}_\mathrm{f}$ being equivalent, they have very different constructions. More specifically, the category $D_t^+(\mathbb{R})$ is a triangulated subcategory of the derived category $D^+(\mathbb{R})$ of sheaves on $\mathbb{R}$ whereas $\mathcal{J}_\mathrm{f}$ is a subcategory of the abelian category of presheaves $\mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$. In this Chapter 10 we show that the smallest abelian subcategory of the category of presheaves $\mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$ containing $\mathcal{J}_\mathrm{f}$ is an *abelianization* of the triangulated category $D_t^+(\mathbb{R})$ and moreover, $\mathcal{J}_\mathrm{f}$ is the subcategory of projectives of this abelian category, which in general is the *idempotent completion* of the corresponding triangulated category by [BS01].

Suppose we are again in the generic situation of Chapter 7, where we consider sheaves on the topological space q_γ whose underlying set is the boundary of some closed proper downset of $\mathbb{M}$. As before we denote the full subcategory of pfd presheaves $\mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ that are cohomological, sequentially continuous, and have bounded above support by $\mathcal{J}$ and the full subcategory of $\mathcal{J}$-presentable presheaves in the sense of Definition E.6 by $\mathrm{pres}(\mathcal{J})$. Also note that an abelian category is *Frobenius* if it has enough projectives and enough injectives and both coincide. In this chapter we show the following.

Theorem 10.1. *Considering the sequence of $\mathbb{F}$-linear categories and functors*

$$D_t^+(q_\gamma, \partial q) \xrightarrow[h_{\gamma,0,t}]{\sim} \mathcal{J} \hookrightarrow \mathrm{pres}(\mathcal{J}) \hookrightarrow \mathrm{Vect}_\mathbb{F}^{\mathbb{M}^\circ}$$

we have:

(i) $\mathrm{pres}(\mathcal{J})$ *is an abelian subcategory of the category of presheaves* $\mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$*,*

(ii) the cohomological functor $D_t^+(q_\gamma, \partial q) \xrightarrow{h_\gamma} \mathcal{J} \hookrightarrow \mathrm{pres}(\mathcal{J})$ *exhibits* $\mathrm{pres}(\mathcal{J})$ *as an abelianization of* $D_t^+(q_\gamma, \partial q)$ *in the sense of Definition E.14,*

(iii) and $\mathrm{pres}(\mathcal{J})$ *is an abelian Frobenius category with* $\mathcal{J}$ *being the subcategory of projectives.*

We note that Theorem 10.1.(ii) is a reasonable statement as $D_t^+(q_\gamma, \partial q)$ is both essentially small by Theorem 9.1 and triangulated by Corollary 8.26.

Corollary 10.2. *If ∂q is closed in q_γ, then the cohomological functor* $D_t^+(\dot{q}) \xrightarrow{Ri_*} D_t^+(q_\gamma) \xrightarrow{h_\gamma} \mathcal{J} \hookrightarrow \mathrm{pres}(\mathcal{J})$ *exhibits* $\mathrm{pres}(\mathcal{J})$ *as an abelianization of* $D_t^+(\dot{q})$*.*

We note here as well, that Corollary 10.2 is a reasonable statement as $D_t^+(\dot{q})$ is both essentially small by Corollary 9.3 and triangulated by Corollary 8.27.

Proof. This statement follows from Theorem 10.1.(ii) in conjunction with Corollary 8.7 and Lemma 8.22. □

Corollary 10.3. *The cohomological functor* $D_t^+(\mathbb{R}) \xrightarrow{h_{\mathbb{R}}} \mathcal{J}_{\mathrm{f}} \hookrightarrow \mathrm{pres}(\mathcal{J}_{\mathrm{f}})$ *exhibits* $\mathrm{pres}(\mathcal{J}_{\mathrm{f}})$ *as an abelianization of* $D_t^+(\mathbb{R})$*, where* $h_{\mathbb{R}}\colon D^+(\mathbb{R}) \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}_{\mathrm{f}}^{\circ}}$ *is the restricted Yoneda embedding defined by* (8.14)*.*

The following proof of Corollary 10.3 is very similar to the proof of Corollary 9.4.

Proof. The map $\blacktriangle|_{\mathbb{R}}\colon \mathbb{R} \to \dot{q}$ is a homeomorphism. Moreover, by Lemma C.12 we have

$$Ri_* \circ D^+\left((\blacktriangle|_{\mathbb{R}})_*\right) \cong Ri_* \circ R(\blacktriangle|_{\mathbb{R}})_* \cong R(\blacktriangle|_{\mathbb{R}})_*,$$

where the symbol $\blacktriangle|_{\mathbb{R}}$ on the right-hand side denotes the embedding $\blacktriangle|_{\mathbb{R}}\colon \mathbb{R} \to q_\gamma$. With this the statement follows from Corollary 10.2 and Lemma 8.15. □

As an otherwise unrelated side note, we also mention the following corollary to Theorem 10.1.

Corollary 10.4. *The triangulated category* $D_t^+(q_\gamma, \partial q)$ *has* split idempotents.

Proof. By [Kra07, Section 4.3] and Theorem 10.1(ii) the subcategory of projectives in $\mathrm{pres}(\mathcal{J})$ has split idempotents, which is $\mathcal{J}$ by Theorem 10.1(iii). Furthermore, $\mathcal{J}$ and $D_t^+(q_\gamma, \partial q)$ are equivalent as $\mathbb{F}$-linear categories by Theorem 9.1. □

We also note that the proof of Corollary 10.4 is in some sense a specialization of just one implication of a more general result within the theory of triangulated categories, which states that any (essentially small) triangulated category has split idempotents iff it is additively equivalent to the subcategory of projectives of its abelianization, see for example [BS01].

10.1 Abelianization of $D_t^+(q_\gamma, \partial q)$

In this Section 10.1 we show Theorem 10.1.(ii), i.e. the category of $\mathcal{J}$-presentable presheaves $\mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ is an abelianization of $D_t^+(q_\gamma, \partial q)$. Now [Kra07, Section 4.2] has provided a construction of the abelianization of any triangulated category $\mathcal{T}$, namely the category of *coherent presheaves* $\mathcal{T}^\circ \to \mathrm{Ab}$ in the sense of Definition E.13, which is very similar to $\mathrm{pres}(\mathcal{J})$ when $\mathcal{T} = D_t^+(q_\gamma, \partial q)$. In order to harness this result by [Kra07, Section 4.2] we now construct an equivalence of categories

$$\mathrm{coh}\big(D_t^+(q_\gamma, \partial q)\big) \to \mathrm{pres}(\mathcal{J})$$

using the sheaf-theoretical Happel functor $\iota\colon \mathbb{M} \to D_t^+(q_\gamma, \partial q)$ as in Definition 7.17. To this end, let $\mathcal{P}$ be the category of $\mathbb{F}$-linear presheaves $D_t^+(q_\gamma, \partial q)^\circ \to \mathrm{Vect}_{\mathbb{F}}$ and let

$$\text{よ}_{\mathbb{F}}\colon D_t^+(q_\gamma, \partial q) \to \mathcal{P},\ F \mapsto \mathrm{Hom}_{D_t^+(q_\gamma, \partial q)}(-, F)$$

be the $\mathbb{F}$-linear Yoneda embedding as in Appendix E.2.3. Then the *pointwise forgetful functor* $\mathcal{P} \to \mathrm{Add}\big(D_t^+(q_\gamma, \partial q)^\circ, \mathrm{Ab}\big)$ restricts to an isomorphism of additive categories $\mathrm{pres}(\text{よ}_{\mathbb{F}}) \xrightarrow{\cong} \mathrm{pres}(\text{よ}) = \mathrm{coh}(\mathcal{A})$ by Lemma E.18. Thus, in order to obtain an equivalence $\mathrm{coh}\big(D_t^+(q_\gamma, \partial q)\big) \to \mathrm{pres}(\mathcal{J})$ it suffices to show that the pullback

$$\mathrm{pres}(\text{よ}_{\mathbb{F}}) \to \mathrm{pres}(\mathcal{J}),\ F \mapsto F \circ \iota$$

along the sheaf-theoretical Happel functor $\iota\colon \mathbb{M} \to D_t^+(q_\gamma, \partial q)$ is an equivalence.

Proposition 10.5. *The pullback*

$$\iota^*\colon \mathcal{P} \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ},\ F \mapsto \iota^* F := F \circ \iota$$

along the sheaf-theoretical Happel functor $\iota\colon \mathbb{M} \to D_t^+(q_\gamma, \partial q)$ restricts to an equivalence of categories

$$\mathrm{pres}(\text{よ}_{\mathbb{F}}) \xrightarrow{\iota^*} \mathrm{pres}(\mathcal{J}). \tag{10.1}$$

Proof. First we note that we have the commutative triangle

$$\begin{tikzcd} D_t^+(q_\gamma, \partial q) \arrow[d, "\text{よ}_{\mathbb{F}}"'] \arrow[dr, "h_\gamma"] & \\ \mathcal{P} \arrow[r, "\iota^*"'] & \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ} \end{tikzcd} \tag{10.2}$$

of additive categories and functors.

We start by showing the essential surjectivity of the restricted pullback (10.1). To this end, let $\varphi\colon F \to G$ be a natural transformation with $F, G\colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ presheaves in $\mathcal{J}$. We have to construct a $\text{よ}_{\mathbb{F}}$-presentable $\mathbb{F}$-linear presheaf whose pullback along $\iota\colon \mathbb{M} \to D_t^+(q_\gamma, \partial q)$ is a cokernel of $\varphi\colon F \to G$. By Theorem 10.1.(iii) we may choose a homomorphism of derived sheaves $\tilde{\varphi}\colon \tilde{F} \to \tilde{G}$ such that there is a commutative square of the form

$$\begin{tikzcd} h_\gamma(\tilde{F}) \arrow[r, "\cong"] \arrow[d, "h_\gamma(\tilde{\varphi})"'] & F \arrow[d, "\varphi"] \\ h_\gamma(\tilde{G}) \arrow[r, "\cong"] & G. \end{tikzcd} \tag{10.3}$$

By the commutativity of (10.2), the exactness of $\iota^*\colon \mathcal{P} \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ}$, and the commutativity of (10.3), the pullback $\iota^* \mathrm{coker} \text{よ}_{\mathbb{F}}(\tilde{\varphi})$ of the cokernel for $\text{よ}_{\mathbb{F}}(\tilde{\varphi})\colon \text{よ}_{\mathbb{F}}(\tilde{F}) \to \text{よ}_{\mathbb{F}}(\tilde{G})$ is a cokernel of $\varphi\colon F \to G$.

In order to show that the restricted pullback (10.1) is fully faithful we consider $\text{よ}_{\mathbb{F}}$-presentations

$$\text{よ}_{\mathbb{F}}(P_1) \to \text{よ}_{\mathbb{F}}(P_0) \to F \to 0 \tag{10.4}$$

$$\text{and} \quad \text{よ}_{\mathbb{F}}(Q_1) \xrightarrow{\text{よ}_{\mathbb{F}}(\delta)} \text{よ}_{\mathbb{F}}(Q_0) \to G \to 0. \tag{10.5}$$

Now suppose we have a natural transformation $\varphi\colon \iota^* F \to \iota^* G$ in $\mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ}$. In order to show that (10.1) is full we construct a preimage of φ under ι^*. By Proposition 8.17, Corollary 2.7, and the commutativity of (10.2) there are natural transformations $\varphi_i\colon h_\gamma(P_i) \to h_\gamma(Q_i)$, $i = 1, 2$ such that the diagram

$$\begin{tikzcd}
h_\gamma(P_1) \arrow[r, dashed, "\varphi_1"] \arrow[d] & h_\gamma(Q_1) \arrow[d, "h_\gamma(\delta)"] \\
h_\gamma(P_0) \arrow[r, dashed, "\varphi_0"] \arrow[d] & h_\gamma(Q_0) \arrow[d] \\
\iota^* F \arrow[r, "\varphi"'] & \iota^* G
\end{tikzcd} \tag{10.6}$$

commutes. By Theorem 10.1.(iii) there are homomorphisms of derived sheaves $\tilde{\varphi}_i\colon P_i \to Q_i$, $i = 1, 2$ such that $h_\gamma(\tilde{\varphi}_i) = \varphi_i$ for $i = 1, 2$. By the universal property of cokernels in $\mathcal{P}$ (and Theorem 10.1.(iii)) there is a homomorphism $\tilde{\varphi}\colon F \to G$ such that the diagram

$$\begin{tikzcd}
\text{よ}_{\mathbb{F}}(P_1) \arrow[r, "\varphi_1"] \arrow[d] & \text{よ}_{\mathbb{F}}(Q_1) \arrow[d, "\text{よ}_{\mathbb{F}}(\delta)"] \\
\text{よ}_{\mathbb{F}}(P_0) \arrow[r, "\varphi_0"] \arrow[d] & \text{よ}_{\mathbb{F}}(Q_0) \arrow[d] \\
F \arrow[r, dashed, "\tilde{\varphi}"'] & G
\end{tikzcd}$$

commutes. In conjunction with the uniqueness part of the universal property for cokernels in $\mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ}$ and the commutativity of (10.2) we obtain that $\iota^* \tilde{\varphi} = \varphi$.

Finally, we show that (10.1) is faithful. To this end, let $\varphi\colon F \to G$ be a natural transformation with $\iota^* \varphi = 0$. By the Yoneda lemma and the exactness of (10.5) at G there is a derived sheaf homomorphism $\tilde{\varphi}\colon P_0 \to Q_0$ such that the diagram

$$\begin{tikzcd}
\text{よ}_{\mathbb{F}}(P_0) \arrow[r, dashed, "\text{よ}_{\mathbb{F}}(\tilde{\varphi})"] \arrow[d] & \text{よ}_{\mathbb{F}}(Q_0) \arrow[d] \\
F \arrow[r, "\varphi"'] & G
\end{tikzcd}$$

commutes. In conjunction with the commutativity of (10.2) we obtain the commutative diagram

$$
\begin{tikzcd}
 & h_\gamma(Q_1) \arrow[d, "h_\gamma(\delta)"] \\
h_\gamma(P_0) \arrow[ur, dashed, "\sigma"] \arrow[r, "h_\gamma(\tilde{\varphi})"] \arrow[d] \arrow[dr, "0"] & h_\gamma(Q_0) \arrow[d] \\
\iota^* F \arrow[r, "\iota^*\varphi=0"'] & \iota^* G.
\end{tikzcd}
\tag{10.7}
$$

By Proposition 8.17 and Corollary 2.7 the presheaf $h_\gamma(P_0)$ is projective in the category of presheaves $\mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ vanishing on $\partial\mathbb{M}$. So by the exactness of ι^* and (10.5) at Q_0 there is a lift $\sigma\colon P_0 \to Q_1$ as indicated by the dashed arrow in (10.7). Moreover, by Theorem 10.1.(iii) there is a derived sheaf homomorphism $\tilde{\sigma}\colon P_0 \to Q_1$ with $h_\gamma(\tilde{\sigma}) = \sigma$. Using Theorem 10.1.(iii) once more we obtain the commutative triangle

$$
\begin{tikzcd}
 & Q_1 \arrow[d, "\delta"] \\
P_0 \arrow[ur, "\tilde{\sigma}"] \arrow[r, "\tilde{\varphi}"'] & Q_0.
\end{tikzcd}
$$

Furthermore, as (10.5) is a complex we have the commutative diagram

$$
\begin{tikzcd}
 & \text{よ}_\mathbb{F}(Q_1) \arrow[d, "\text{よ}_\mathbb{F}(\delta)"] \\
\text{よ}_\mathbb{F}(P_0) \arrow[ur, "\text{よ}_\mathbb{F}(\tilde{\sigma})"] \arrow[r, "\text{よ}_\mathbb{F}(\tilde{\varphi})"] \arrow[d] \arrow[dr, "0"] & \text{よ}_\mathbb{F}(Q_0) \arrow[d] \\
F \arrow[r, "\varphi"'] & G
\end{tikzcd}
$$

and hence $\varphi = 0$ by the exactness of (10.4) at F. □

Proof of Theorem 10.1.(i). This follows directly from the previous Proposition 10.5, Lemma E.19, and the exactness of the pullback $\iota^*\colon \mathcal{P} \to \mathrm{Vect}_\mathbb{F}^{\mathbb{M}^\circ}$. □

Proof of Theorem 10.1.(ii). By Lemma E.18 and Proposition 10.5 we have the commutative diagram

$$\begin{tikzcd}
 & D_t^+(q_\gamma, \partial q) \arrow[dl, "\bar{\text{よ}}"'] \arrow[d, "\text{よ}_{\mathbb{F}}"] \arrow[dr, "h_\gamma"] & \\
\operatorname{coh}\bigl(D_t^+(q_\gamma, \partial q)\bigr) & \operatorname{pres}(\text{よ}_{\mathbb{F}}) \arrow[l, "\cong"'] \arrow[r, "\sim", "\iota^*"'] & \operatorname{pres}(\mathcal{J})
\end{tikzcd}$$

with each horizontal arrow an additive (even exact) isomorphism respectively equivalence as indicated. Thus, the result follows in conjunction with Lemma E.16. □

10.2 Projective Covers of $\mathcal{J}$-Presentable Presheaves

In the previous Section 10.1 we provided a proof of Theorem 10.1.(ii), which states that the category of $\mathcal{J}$-presentable presheaves $\operatorname{pres}(\mathcal{J})$ is an abelianization of the triangulated category $D_t^+(q_\gamma, \partial q)$ of tame derived sheaves on q_γ vanishing on ∂q. Now Theorem 10.1.(ii) implies in particular, that $\operatorname{pres}(\mathcal{J})$ is an abelian Frobenius category by [Kra07, Section 4.2], which is almost the statement of Theorem 10.1.(iii). The only question that remains, is whether $\mathcal{J}$ is the subcategory of projectives of $\operatorname{pres}(\mathcal{J})$. Considering we already know from Corollaries 2.7 and E.9 (or more general facts on abelianizations of triangulated categories) that any presheaf in $\mathcal{J}$ is projective in $\operatorname{pres}(\mathcal{J})$, this question does not seem unreasonable at all. In this Section 10.2 we use Corollary E.11 to show that the converse is true as well.

Now in order to prove the Structure Theorem 2.6 in Chapter 2 we first constructed an approximation by reduction modulo radical in the sense of Definition 2.18 following [HL81, Theorem 1, Proposition 5] and then we showed this approximation is an isomorphism. Here we show that approximating $\mathcal{J}$-presentable presheaves by reduction modulo radical yields projective covers in the sense of Definition E.4. (Actually, the condition that presheaves in $\mathcal{J}$ have bounded above support is not needed for any of the results in this Section 10.2. However, in order to reduce the amount of terminology introduced we keep this assumption.)

10.2.1 Properties of Reducing Projections

We start with a discussion of reducing projections in the sense of Definition 2.16 in the context of $\mathcal{J}$-presentable presheaves on $\mathbb{M}$.

Lemma 10.6. *Suppose $F \colon \mathbb{M}^\circ \to \operatorname{vect}_{\mathbb{F}}$ is pfd and sequentially continuous and let $G \hookrightarrow F$ be a proper subpresheaf of F. Then, there is a point $w \in \mathbb{M}$ and a commutative triangle of the form*

$$\begin{tikzcd}
G \arrow[r, hook] \arrow[dr, "0"'] & F \arrow[d, two heads] \\
 & S_w,
\end{tikzcd}$$

where $S_w\colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ *is the simple presheaf associated to the point* $w \in \mathbb{M}$ *as defined by* (2.1)*.*

Corollary 10.7. *If* $F\colon \mathbb{M}^\circ \to \mathrm{vect}_{\mathbb{F}}$ *is pfd and sequentially continuous, then any proper subpresheaf of* F *is contained pointwise in a maximal subpresheaf of* F*.*

Proof of Lemma 10.6. Since the pointwise inclusion $G \hookrightarrow F$ is proper, there is a point $u := (x, y) \in \mathbb{M}$ such that $G(u)$ is a proper subspace of $F(u)$. Thus, there is a non-vanishing linear form $\alpha \in (F(u))^*$ with $\alpha|_{G(u)} = 0$. As $F\colon \mathbb{M}^\circ \to \mathrm{vect}_{\mathbb{F}}$ is pfd and sequentially continuous, the dual covariant functor $F^*\colon \mathbb{M} \to \mathrm{vect}_{\mathbb{F}}$ is sequentially co-continuous, i.e., for any increasing sequence $(v_k)_{k=1}^{\infty}$ in $\mathbb{M}$ converging to $v \in \mathbb{M}$ the natural map

$$\varinjlim_k F^*(v_k) \to F^*(v) \tag{10.8}$$

is an isomorphism. We consider the restriction of $F^*\colon \mathbb{M}^\circ \to \mathrm{vect}_{\mathbb{F}}$ to the maximal horizontal line segment $h \subset \mathbb{M}$ through u. Since F^* vanishes on l_0, the linear form α has to die at some point $v \in h$. Let $\beta := F^*(u \preceq v)(\alpha)$ be the pullback of α to $F(v)$. As $F^*\colon \mathbb{M}^\circ \to \mathrm{vect}_{\mathbb{F}}$ is sequentially cocontinuous we have $\beta \neq 0$ (or in other words α is a die-hard linear form). Now let $g \subset \mathbb{M}$ be the maximal vertical line segment through v. As F^* vanishes on l_1, the linear form β has to die at some point $w \in g$. Let $\gamma := F^*(v \preceq w)(\beta)$ be the pullback of β to $G(w)$. Again, $\gamma \neq 0$ as F^* is sequentially cocontinuous. Now $\gamma \in F^*(w)$ determines a pointwise surjective family of maps $\rho\colon F \to S_w$. As all internal pullbacks of γ are trivial, the family of maps $\rho\colon F \to S_w$ is a natural transformation. □

Lemma 10.8. *If* $F\colon \mathbb{M}^\circ \to \mathrm{vect}_{\mathbb{F}}$ *is pfd and sequentially continuous, then any reducing projection* $\theta\colon F \to \bigoplus_{i\in I} S_i$ *of a reduced dual frame for* F *in the sense of Definitions 2.15 and 2.16 is an essential epimorphism of presheaves in the sense of Definition E.1.*

Our proof of this lemma is inspired by the proof of a similar statement by [Kra15, Lemma 3.3.(2)].

Proof. In order to show that the reducing projection θ is essential, we use the characterization Lemma E.3. To this end, let $G \hookrightarrow F$ be a subpresheaf of $F\colon \mathbb{M}^\circ \to \mathrm{vect}_{\mathbb{F}}$ with $F = G + \ker\theta$ and assume for a contradiction $G \neq F$. Then there is a maximal subpresheaf $G' \hookrightarrow F$ containing $G\colon \mathbb{M}^\circ \to \mathrm{vect}_{\mathbb{F}}$ pointwise by Corollary 10.7. Thus, we have

$$F = G + \ker\theta = G + \operatorname{rad} F \hookrightarrow G'$$

by Proposition 2.17 and Corollary 2.13. This is a contradiction to G' being a proper subpresheaf of F. Therefore, we have $F = G$. Thus, the reducing projection $\theta\colon F \to \bigoplus_{i\in I} S_i$ is essential. □

By the following lemma the previous result is applicable in particular when F is $\mathcal{J}$-presentable.

Lemma 10.9. *If* $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ *is a* $\mathcal{J}$*-presentable presheaf, then* F *is pfd, has bounded above support, and is sequentially continuous.*

Proof. Let

$$Q \xrightarrow{\delta} P \to F \to 0$$

be a presentation of F by presheaves P and Q in $\mathcal{J}$. As P is pfd and has bounded above support, the same is true for F. It remains to show that F is sequentially continuous. To this end, let $(C^\bullet, \delta^\bullet)$ be the cochain complex of presheaves $\mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ with

$$C^n := \begin{cases} P & n = 0 \\ Q & n = -1 \\ 0 & \text{otherwise} \end{cases}$$

and $\delta^{-1} := \delta$. Now suppose $(u_k)_{k=1}^\infty$ is an increasing sequence in $\mathbb{M}$ converging to $u \in \mathbb{M}$. We consider the commutative diagram

$$\begin{tikzcd} H^0(C^\bullet(u)) \arrow[r] \arrow[d, "\cong"] & \varprojlim_k H^0(C^\bullet(u_k)) \arrow[d, "\cong"] \\ F(u) \arrow[r] & \varprojlim_k F(u_k). \end{tikzcd} \tag{10.9}$$

We have to show that the lower horizontal map of (10.9) is an isomorphism. To this end, it suffices to show that the upper horizontal map of (10.9) is an isomorphism. As P and Q are pfd, the projective systems

$$\left\{H^{-1}(C^\bullet(u_k))\right\}_{k=1}^\infty \quad \text{and} \quad \{C^n(u_k)\}_{k=1}^\infty, \quad n \in \mathbb{Z}$$

satisfy the Mittag-Leffler condition. Thus, the horizontal map at the top of (10.9) is an isomorphism by the sequential continuity of P and Q and by [KS90, Proposition 1.12.4]. $\square$

Corollary 10.10. *If $F\colon \mathbb{M}^\circ \to \mathrm{vect}_\mathbb{F}$ is $\mathcal{J}$-presentable, then any reducing projection $\theta\colon F \to \bigoplus_{i\in I} S_i$ of a reduced dual frame for F in the sense of Definitions 2.15 and 2.16 is an essential epimorphism of presheaves in the sense of Definition E.1.*

Proof. This follows directly from Lemmas 10.9 and 10.8. $\square$

10.2.2 Approximations by Reduction modulo Radical as Essential Epis

Now suppose that $F\colon \mathbb{M}^\circ \to \mathrm{vect}_\mathbb{F}$ is a $\mathcal{J}$-presentable presheaf in the sense of Definition E.6 and that we have a reduced dual frame for F in the sense of Definition 2.15 provided to us as a map $p\colon I \to \mathbb{M}$ together with an I-tuple

$$(\alpha_i)_{i\in I} \in \prod_{i\in I} \mathrm{Nat}\big(F, S_i\big),$$

where we write $S_i := S_{p(i)}\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ and $B_i := B_{p(i)}\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ for $i \in I$. Moreover, let $\theta\colon F \longrightarrow \prod_{i\in I} S_i$ be the reducing projection of this reduced dual frame in the sense of Definition 2.16 and let $\varphi\colon \bigoplus_{i\in I} B_i \to F$ be an approximation of F by reduction modulo radical lifting this reduced dual frame, i.e. φ is a lift as in the diagram

$$\begin{tikzcd} & F \arrow[d, two heads, "\theta"] \\ \bigoplus_{i\in I} B_i \arrow[ur, dashed, "\varphi"] \arrow[r, two heads, "\upsilon"'] & \bigoplus_{i\in I} S_i, \end{tikzcd} \tag{10.10}$$

where $\upsilon\colon \bigoplus_{i\in I} B_i \to \bigoplus_{i\in I} S_i$ is the canonical homomorphism, see also Definition 2.18.

Lemma 10.11. *The direct sum $\bigoplus_{i\in I} B_i\colon \mathbb{M}^\circ \to \mathrm{Vect}_\mathbb{F}$ is a presheaf in $\mathcal{J}$.*

Proof. By Definition 2.15 we have $\#p^{-1}(v) = \beta^0(F)(v)$ for any point $v \in \operatorname{int}\mathbb{M}$, hence the result follows from Lemmas 9.8 and 9.9. $\square$

Next we show that the approximation by reduction modulo radical $\varphi\colon \bigoplus_{i\in I} B_i \to F$ is an essential epimorphism of presheaves in the sense of Definition E.1.

Lemma 10.12. *Any approximation by reduction modulo radical φ of a $\mathcal{J}$-presentable presheaf F is a natural epimorphism.*

Proof. Considering the commutative triangle (10.10) and the fact that $\theta\colon F \to \bigoplus_{i\in I} S_i$ is an essential epimorphism by Corollary 10.10, we see that $\varphi\colon \bigoplus_{i\in I} B_i \to F$ is an epimorphism as well. $\square$

Lemma 10.13. *The natural epimorphism $\upsilon\colon \bigoplus_{i\in I} B_i \to \bigoplus_{i\in I} S_i$ is essential.*

Proof. Since we have

$$\dim_\mathbb{F} \mathrm{Nat}(B_u, S_v) = \dim_\mathbb{F} S_v(u) = \begin{cases} 1 & u = v \\ 0 & u \neq v \end{cases} \tag{10.11}$$

by the Yoneda Lemma 2.4 for any $u, v \in \operatorname{int}\mathbb{M}$, the I-tuple

$$(\mathrm{pr}_i \circ \upsilon_i)_{i\in I} \in \prod_{i\in I} \mathrm{Nat}\big(\textstyle\bigoplus_{i\in I} B_i, S_i\big)$$

is a reduced dual frame for the direct sum $\bigoplus_{i\in I} B_i$. Moreover, $\upsilon\colon \bigoplus_{i\in I} B_i \to \bigoplus_{i\in I} S_i$ is the reducing projection of this reduced dual frame. Thus, υ is an essential epimorphism by Corollaries 10.11 and 10.10. $\square$

Lemma 10.14. *Any approximation by reduction modulo radical φ of a $\mathcal{J}$-presentable presheaf F is an essential epimorphism of presheaves.*

Proof. We consider the commutative triangle (10.10). By Lemma 10.12 the natural transformation $\varphi\colon \bigoplus_{i\in I} B_i \to F$ is an epimorphism. Moreover, $\upsilon\colon \bigoplus_{i\in I} B_i \to \bigoplus_{i\in I} S_i$ is essential by Lemma 10.13, hence $\varphi\colon \bigoplus_{i\in I} B_i \to F$ is essential by Lemma E.2. $\square$

10.2.3 Projective Covers by Reduction modulo Radical

Now let $\mathcal{C}$ be the category of presheaves $\mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ vanishing on $\partial\mathbb{M}$. Then $\mathcal{J}$ is a full replete additive subcategory of projectives in $\mathcal{C}$ by Corollary 2.7. In particular, the categories $\mathcal{J}$ and $\mathcal{C}$ provide a context for the results provided in Appendix E.2.1.

Lemma 10.15. *Any approximation by reduction modulo radical φ of a $\mathcal{J}$-presentable presheaf $F\colon \mathbb{M}^\circ \to \mathrm{vect}_{\mathbb{F}}$ is a projective cover of F in $\mathcal{C}$.*

Proof. By the Yoneda Lemma 2.4 the direct sum $\bigoplus_{i\in I} B_i\colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ is projective in $\mathcal{C}$. Moreover, as $\mathcal{C}$ is an abelian subcategory of the category of presheaves $\mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ and as $\varphi\colon \bigoplus_{i\in I} B_i \to F$ is an essential epimorphism of presheaves by Lemma 10.14, φ is an essential epimorphism in $\mathcal{C}$ as well. □

Proposition 10.16. *Any approximation by reduction modulo radical φ of a $\mathcal{J}$-presentable presheaf $F\colon \mathbb{M}^\circ \to \mathrm{vect}_{\mathbb{F}}$ is a projective cover of F in the category of $\mathcal{J}$-presentable presheaves* $\mathrm{pres}(\mathcal{J})$.

Proof. Without loss of generality we assume that $\varphi\colon \bigoplus_{i\in I} B_i \to F$ is the approximation by reduction modulo radical constructed above. By Lemma 10.11 the direct sum $\bigoplus_{i\in I} B_i$ is a presheaf in $\mathcal{J}$. In conjunction with Lemma 10.15 we obtain that φ is a projective cover in $\mathcal{C}$ of F by a presheaf in $\mathcal{J}$. By Corollary E.10 the natural transformation $\varphi\colon \bigoplus_{i\in I} B_i \to F$ is a projective cover in $\mathrm{pres}(\mathcal{J})$ as well. □

Thus, approximations by reduction modulo radical in the sense of Definition 2.18 provide a way of constructing projective covers for $\mathcal{J}$-presentable presheaves.

Proposition 10.17. *The additive category $\mathcal{J}$ is the subcategory of projectives in* $\mathrm{pres}(\mathcal{J})$.

Proof. This follows from Lemma 10.15 and Corollary E.11. □

Corollary 10.18 (Acyclicity of $\mathrm{pres}(\mathcal{J})$)**.** *Any projective object P of* $\mathrm{pres}(\mathcal{J})$ *decomposes as*

$$P \cong \bigoplus_{v\in \operatorname{int}\mathbb{M}} B_v^{\oplus\mu(v)},$$

where $\mu \coloneqq \beta^0(P)\colon \operatorname{int}\mathbb{M} \to \mathbb{N}_0$.

Proof. This follows in conjunction with Theorem 2.6. □

This corollary essentially says that any projective of $\mathrm{pres}(\mathcal{J})$ is *free* in the sense of it being in the essential image of the left adjoint to the (right) exact forgetful monadic functor from $\mathrm{pres}(\mathcal{J})$ to the category of $\operatorname{int}\mathbb{M}$-indexed families of vector spaces over $\mathbb{F}$, in which all objects are projective; this is also related to [CJT21, Section 10.1]. In the terminology of [CGR+22, Paragraph 2.3] we may also say that $\mathrm{pres}(\mathcal{J})$ is *acyclic*.

Remark 10.19. In summary we used the reduction modulo radical by [HL81, Theorem 1, Proposition 5] in conjunction with the Structure Theorem 2.6 to provide a construction of projective covers for $\mathcal{J}$-presentable presheaves $\mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$, which entails the structure theorem Corollary 10.18 for projectives in $\mathrm{pres}(\mathcal{J})$. Moreover, our proof of the Structure Theorem 2.6 makes use of the reduction modulo radical by [HL81] itself. Now initially, one of the main applications of the reduction modulo radical is the result that projective functors on posets are free; this is [HL81, Proposition 5]. So even though, our initial motivation to use the reduction modulo radical is to show that certain cohomological presheaves rather than projectives are free, we used it for precisely that same purpose in the end. For a similar correlation between projective covers and decompositions of projectives consider [Kra15, Proposition 4.1] and the equivalence (1) $\Leftrightarrow$ (4) in particular, which also inspired the present exposition.

Proof of Theorem 10.1.(iii). By [Kra07, Section 4.2] the category of coherent presheaves $\mathrm{coh}\big(D_t^+(q_\gamma, \partial q)\big)$ is an abelian Frobenius category, hence $\mathrm{pres}(\mathcal{J})$ is Frobenius as well by Lemma E.18 and Proposition 10.5. Moreover, by Corollaries 2.7 and E.11 the subcategory $\mathcal{J} \subset \mathrm{pres}(\mathcal{J})$ is the category of projectives in $\mathrm{pres}(\mathcal{J})$. $\square$

11 RISC Categorifies Extended Persistence Diagrams

For this entire final Chapter 11 we assume that $\mathcal{H}^\bullet$ is a cohomology theory taking values in $\mathrm{Vect}_{\mathbb{F}}$, mapping weak equivalences to isomorphisms, and with $\mathcal{H}^n$ vanishing for $n \ll 0$.

Categorification in the Context of Algebraic Persistence Theory. The notion of *abelian categorification*, that the title of this Part IV refers to, has been introduced by [KMS09] and we provide the special case we use here with Definition F.8. Roughly speaking, a *categorification* of an abelian group A is a category $\mathcal{C}$ with some extra properties or structure, so there is some notion of *additivity* of invariants on $\mathcal{C}$ and A is the codomain of some *universal* additive invariant on $\mathcal{C}$. Here the adjective "abelian" to "categorification" calls for $\mathcal{C}$ to be an abelian category. We discuss some of these notions in Appendix F. In particular, the additive category of *definable* sheaves on $\mathbb{R}$ is an *additive categorification* (of the group completion) of level set barcodes by [Cur14, Section 15.3] and [Gui16, Corollary 7.3], but not an abelian categorification of level set barcodes in spite the fact that the category of sheaves on $\mathbb{R}$ is abelian, see also Definition F.1. The abelian (or derived) category of $\mathbb{R}$-constructible sheaves rather categorifies the abelian group of *constructible functions* by [KS90, Theorem 9.7.1], which retain less information than level set barcodes. An analogous result using *Waldhausen categories* has been provided by [GS23a, Section 5.1] and has then been generalized to multiple parameters in [GS23b]. The authors of [BCZ23] discuss similar notions of categorification in the context of persistence theory. Another form of categorification in the context of multi-parameter persistence theory has been provided by [BOO21, Section 4.2], which specializes to an additive categorification in the context of ordinary persistence theory. Similarly, the homotopy category of the category of $\mathbb{R}$-constructible or tame sheaves on $\mathbb{R}$ is a *triangulated categorification* of extended persistence diagrams, see also Remark F.7. Further invariants of multidimensional persistence modules defined in terms of Grothendieck groups of Quillen exact categories were introduced by [BBH21]. So considering there is a wealth of previous results on categorifications in the context of persistence theory, why should we need an *abelian* categorification?

Towards a Mayer–Vietoris Principle for Extended Persistence Diagrams. We now demonstrate how an adaptation of the *Mayer–Vietoris principle* from algebraic topology to the persistence theory of functions calls for an *abelian* categorification. To this end, we consider a topological space X with an open cover $A \cup B = X$ as well as the associated

Mayer–Vietoris sequence in singular homology:

$$\begin{tikzcd}
 & & \cdots \arrow[r] & H_2(X) \arrow[dlll, rounded corners] \\
 & H_1(A \cap B) \arrow[r, "j_1"'] & H_1(A) \oplus H_1(B) \arrow[r] & H_1(X) \arrow[dlll, rounded corners] \\
 & H_0(A \cap B) \arrow[r, "j_0"'] & H_0(A) \oplus H_0(B) \arrow[r] & H_0(X) \arrow[r] & 0.
\end{tikzcd} \tag{11.1}$$

Now suppose we intend to compute the first Betti number $\beta_1(X) = \operatorname{rank} H_1(X)$. Then we may use the exactness of the Mayer–Vietoris sequence (11.1) to obtain the short exact sequence

$$0 \to \operatorname{coker} j_1 \to H_1(X) \to \ker j_0 \to 0. \tag{11.2}$$

As the rank of abelian groups is an *additive invariant* we obtain

$$\beta_1(X) = \operatorname{rank} H_1(X) = \operatorname{rank}(\operatorname{coker} j_1) + \operatorname{rank}(\ker j_0).$$

Here the upshot is that in order to obtain the maps $j_0\colon H_0(A \cap B) \to H_0(A) \oplus H_0(B)$ and j_1 we only need to compute the singular homology of A, B, and $A \cap B$. Now suppose we also have a continuous function $f\colon X \to \mathbb{R}$. Then we have the long exact sequence of Leray sheaves

$$\begin{tikzcd}
0 \arrow[r] & R^0 f_* \mathbb{F}_X \arrow[r] & R^0 f_* \mathbb{F}_A \oplus R^0 f_* \mathbb{F}_B \arrow[r, "j^0"] & R^0 f_* \mathbb{F}_{A \cap B} \arrow[dll, rounded corners] \\
 & R^1 f_* \mathbb{F}_X \arrow[r] & R^1 f_* \mathbb{F}_A \oplus R^1 f_* \mathbb{F}_B \arrow[r, "j^1"] & R^1 f_* \mathbb{F}_{A \cap B} \arrow[dll, rounded corners] \\
 & R^2 f_* \mathbb{F}_X \arrow[r] & \cdots
\end{tikzcd}$$

and hence, a short exact sequence

$$0 \to \operatorname{coker} j^0 \to R^1 f_* \mathbb{F}_X \to \ker j^1 \to 0. \tag{11.3}$$

However, the short exact sequence (11.3) may not split even for the most down-to-earth finiteness assumptions on $f\colon X \to \mathbb{R}$ and its restrictions to A, B, and $A \cap B$. Thus, it is not possible to compute the level set barcode of $R^1 f_* \mathbb{F}_X$ in terms of the level set barcodes for $\operatorname{coker} j_0$ and $\ker j_1$. Or in other words, as an invariant on the abelian category of say *definable* sheaves [Cur14, Definition 15.3.1] the level set barcode is merely *split-additive* but not *additive*. These issues continue to persist (no pun intended), when considering the triangulated categories we discussed so far. While under mild finiteness assumptions there is a distinguished triangle

$$Rf_* \mathbb{F}_X \to Rf_* \mathbb{F}_A \oplus Rf_* \mathbb{F}_B \to Rf_* \mathbb{F}_{A \cap B} \to Rf_* \mathbb{F}_X[1] \tag{11.4}$$

in the bounded derived category $D^b_{\mathbb{R}c}(\mathbb{R})$ of $\mathbb{R}$-*constructible sheaves* [KS18, Section 1.1], the Grothendieck group of $D^b_{\mathbb{R}c}(\mathbb{R})$ retains too little information to compute the (graded) level set barcode or extended persistence diagram of $Rf_* \mathbb{F}_X$ in terms of an additive invariant on the triangulated category $D^b_{\mathbb{R}c}(\mathbb{R})$. Moreover, the additivity of invariants on $D^b_{\mathbb{R}c}(\mathbb{R})$ is in stark contrast to *stability* by [Ber23], whose results imply that any stable distance on an additive invariant of $D^b_{\mathbb{R}c}(\mathbb{R})$ necessarily assigns a distance of 0 to the derived pushforwards along any two (sufficiently tame) functions that are defined on the same space. Having made this observation, we may still hope for better results by considering the corresponding homotopy category as its Grothendieck group retains sufficient information by [Ros15], see also Remark F.7. However, in general there is no distinguished triangle as (11.4) in the homotopy category of the category of sheaves (let alone any issues related to tameness and functoriality in this context). So in order to adapt the Mayer–Vietoris principle to persistence theory, we need another approach. As a side note, we will get to a point, where we get a triangulated categorification of extended persistence diagrams from the bounded derived category $D^b(\mathrm{pres}(\mathcal{J}))$.

In Theorem 1.32 we provided a counterpart to the Mayer–Vietoris sequence for RISC, which we now use as a starting point for a Mayer–Vietoris principle for extended persistence diagrams in the sense of Definition 2.2. To this end, let $f \colon X \to \mathbb{R}$ be a continues function and let $A \cup B = X$ be an open cover with f, $f|_A$, $f|_B$ and $f|_{A \cap B}$ being $\mathcal{H}^\bullet$-tame. Then we have the exact sequence

$$h(f|_A) \circ \Sigma \oplus h(f|_B) \circ \Sigma \xrightarrow{\varphi \circ \Sigma} h(f|_{A\cap B}) \circ \Sigma \xrightarrow{d} h(f) \to h(f|_A) \oplus h(f|_B) \xrightarrow{\varphi} h(f|_{A\cap B}) \tag{1.10 revisited}$$

by Theorem 1.32. Thus, we may consider the short exact sequence

$$0 \to \operatorname{coker} \varphi \circ \Sigma \to h(f) \to \ker \varphi \to 0 \tag{11.5}$$

of presheaves $\mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$ a counterpart to (11.2) in the context of relative interlevel set cohomology. We note however, that the middle term of (11.5) is the entire RISC of $f \colon X \to \mathbb{R}$ whereas the middle term of (11.2) is the homology of X in a single degree. So strictly speaking, (11.5) is a counterpart the graded variant of (11.2). Now in Definition 2.2 we already defined the extended persistence diagram in terms of the 0-th Betti function of RISC, which can be associated to any other pfd presheaf $\mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$ as well. Unfortunately, 0-th Betti functions are not additive as an invariant of pfd presheaves $\mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$. While $\dim_\mathbb{F}$ is an additive invariant of the category of finite-dimensional vector spaces over $\mathbb{F}$, the functors

$$\mathrm{Nat}(-, S_u) \colon \left(\mathrm{Vect}_\mathbb{F}^{\mathbb{M}_\mathrm{f}^\circ}\right)^\circ \to \mathrm{Vect}_\mathbb{F},\ F \mapsto \mathrm{Nat}(F, S_u) \quad \text{for } u \in \operatorname{int} \mathbb{M}_\mathrm{f}$$

are merely left-exact. Now the usual approach to making up for this, is to consider the associated derived functors. So ignoring any issues related to infinite dimensionality and divergence for a minute, we may associate to any "sufficiently tame" presheaf $F \colon \mathbb{M}_\mathrm{f}^\circ \to \mathrm{Vect}_\mathbb{F}$ its *(higher) Betti functions*

$$\beta^n(F) \colon \operatorname{int} \mathbb{M}_\mathrm{f} \to \mathbb{N}_0,\ u \mapsto \dim_\mathbb{F} \mathrm{Ext}^n_\mathcal{C}(F, S_u) \quad \text{for } n \in \mathbb{N}_0 \tag{11.6}$$

as well as its *Euler function*

$$\chi(F) \coloneqq \sum_{n=0}^{\infty}(-1)^n \beta^n(F) \colon \operatorname{int}\mathbb{M}_{\mathrm{f}} \to \mathbb{Z},\ u \mapsto \sum_{n=0}^{\infty}(-1)^n \dim \operatorname{Ext}^n_{\mathcal{C}}(F, S_u) \tag{11.7}$$

mimicking the Euler characteristic from algebraic topology. Now even though the Euler function χ is a good candidate for an additive invariant, it is different from the 0-th Betti function, which is the invariant that we seek to compute. In order to adjust for this, we return to our earlier observation Corollary 2.7 that any sequentially continuous pfd cohomological presheaf $\mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ is projective in the full subcategory of presheaves vanishing on $\partial\mathbb{M}_{\mathrm{f}}$, which includes the relative interlevel set cohomology $h(f) \colon \mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ of any $\mathcal{H}^{\bullet}$-tame function $f \colon X \to \mathbb{R}$ in particular. So if we define $\mathcal{C}$ to be the full subcategory of presheaves $\mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ vanishing on $\partial\mathbb{M}_{\mathrm{f}}$, and derive the functors

$$\operatorname{Nat}(-, S_u) \colon \mathcal{C}^{\circ} \to \mathrm{Vect}_{\mathbb{F}},\ F \mapsto \operatorname{Nat}(F, S_u) \quad \text{for } u \in \operatorname{int}\mathbb{M}_{\mathrm{f}}$$

not as a functors on all presheaves $\mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ but as functors $\mathcal{C}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ and we adjust our definitions of (higher) Betti and Euler functions accordingly as indicated by the greyed out subscripts of $\mathcal{C}$ in (11.6) and (11.7), then we have $\beta^n(h(f)) = 0$ for $n \geq 1$ pointwise and hence $\beta^0(h(f)) = \chi(h(f)) \colon \operatorname{int}\mathbb{M}_{\mathrm{f}} \to \mathbb{N}_0$. However, some issues remain even for presheaves in $\mathcal{C}$. First, the (higher) Betti functions $\beta^n(F) \colon \operatorname{int}\mathbb{M}_{\mathrm{f}} \to \mathbb{N}_0$ may not be well-defined for an arbitrary presheaf $F \colon \mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ vanishing on $\partial\mathbb{M}_{\mathrm{f}}$ even if it is pfd and second, the Euler function $\chi(F) \coloneqq \sum_{n \in \mathbb{N}_0}(-1)^n \beta^n(F)$ is an infinite sum, so it may not be well-defined either. We will deal with these issues in Sections 11.1 and 11.2, respectively.

11.1 (Higher) Betti Functions

We now extend Definition 2.1 to the notion of an *n-th Betti function* for an arbitrary non-negative degree $n \in \mathbb{N}_0$. As mentioned in the previous paragraph, we seek a definition of β^n that ensures the pointwise equality $\beta^n(h(f)) = 0$ for $n \geq 1$ and any $\mathcal{H}^{\bullet}$-tame function $f \colon X \to \mathbb{R}$. So we consider the derived functors of the left-exact functors

$$\operatorname{Nat}(-, S_u) \colon \mathcal{C}^{\circ} \to \mathrm{Vect}_{\mathbb{F}},\ F \mapsto \operatorname{Nat}(F, S_u) \quad \text{for } u \in \operatorname{int}\mathbb{M},$$

where $\mathcal{C}$ is the category of presheaves $\mathbb{M}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ vanishing on $\partial\mathbb{M}$. Moreover, in order to ensure that we obtain integer-valued (higher) Betti functions using formula (11.6), we need to further constrain the subcategory of presheaves $\mathbb{M}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ for which (higher) Betti functions are defined. Now we recall from the previous paragraph, that we intend to define (higher) Betti functions for a full abelian subcategory of the category of presheaves $\mathbb{M}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ containing the relative interlevel set cohomology $h(f) \colon \mathbb{M}_{\mathrm{f}}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ of any $\mathcal{H}^{\bullet}$-tame function $f \colon X \to \mathbb{R}$, so we have well-defined higher Betti functions for all terms of the short exact sequence (11.5). Furthermore, in the previous Chapter 10 we constructed the category $\operatorname{pres}(\mathcal{J})$ as an abelianization of the triangulated category $D^+_t(q_\gamma, \partial q)$ and by our assumption that $\mathcal{H}^n = 0$ for $n \ll 0$ from the beginning of this

Chapter 11, the category $\mathcal{J}_{\mathrm{f}}$ contains the RISC of any $\mathcal{H}^{\bullet}$-tame function. Thus, it suffices to define (higher) Betti functions for any $\mathcal{J}$-presentable presheaf.

Now let $F\colon \mathbb{M}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ be a $\mathcal{J}$-presentable presheaf. By Theorem 10.1.(iii) there is a resolution $P_{\bullet} \xrightarrow{\sim} F$ by presheaves P_n, $n \in \mathbb{N}_0$ in $\mathcal{J}$. Moreover, as $P_n\colon \mathbb{M}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ is projective in $\mathcal{C}$ by Corollary 2.7 for any $n \in \mathbb{N}_0$, we have

$$\mathrm{Ext}^n_{\mathcal{C}}(F, S_u) \cong H^n(\mathrm{Nat}(P_{\bullet}, S_u)) \tag{11.8}$$

for any $u \in \operatorname{int} \mathbb{M}$ and any $n \in \mathbb{N}_0$. Furthermore, as $P_n\colon \mathbb{M}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ is pfd for any $n \in \mathbb{N}_0$, the right-hand side of (11.8) is finite-dimensional. Thus, the following makes for a sound definition of (higher) Betti functions.

Definition 11.1 (Betti Functions)**.** For a $\mathcal{J}$-presentable presheaf $F\colon \mathbb{M}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ and any $n \in \mathbb{N}_0$ we define the *n-th Betti function* of F by

$$\beta^n(F)\colon \operatorname{int} \mathbb{M} \to \mathbb{N}_0,\ u \mapsto \dim_{\mathbb{F}} \mathrm{Ext}^n_{\mathcal{C}}(F, S_u).$$

As $\mathrm{Ext}^0_{\mathcal{C}}(-,-) \cong \mathrm{Nat}(-,-)$ by the left-exactness of $\mathrm{Nat}(-,-)$, Definitions 2.1 and 11.1 are consistent. In Remark 11.5 below we will see that above notion of a Betti function is closely related to the notion of a *bigraded Betti number* by [LW19, Defintion 2.1]. (They are not the same though, because of our condition on all presheaves to vanish on $\partial\mathbb{M}$.) In the remainder of this Section 11.1 we show that any (higher) Betti function in the sense of Definition 11.1 is an admissible Betti function in the sense of Definition 9.7.

Lemma 11.2. *Let $\varphi\colon P \to F$ be a projective cover in* $\mathrm{pres}(\mathcal{J})$ *and let $\kappa\colon K \to P$ be its kernel, so we have the short exact sequence*

$$0 \to K \xrightarrow{\kappa} P \xrightarrow{\varphi} F \to 0.$$

Then $\beta^{n+1}(F) = \beta^n(K)$ for all $n \in \mathbb{N}_0$.

Proof. Let $u \in \operatorname{int} \mathbb{M}$ be a point inside the interior of $\mathbb{M}$. As $\mathrm{Nat}(-, S_u)$ is left-exact as a functor on $\mathcal{C}^{\circ}$ and since $P\colon \mathbb{M}^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ is projective in $\mathcal{C}$ by Theorem 10.1.(iii) and Corollary 2.7, we obtain the long exact sequence

$$\begin{array}{l} \longrightarrow \mathrm{Ext}^2_{\mathcal{C}}(F, S_u) \longrightarrow \{0\} \longrightarrow \cdots . \\ \\ \longrightarrow \mathrm{Ext}^1_{\mathcal{C}}(F, S_u) \longrightarrow \{0\} \longrightarrow \mathrm{Ext}^1_{\mathcal{C}}(K, S_u) \longrightarrow \\ \\ 0 \longrightarrow \mathrm{Nat}(F, S_u) \longrightarrow \mathrm{Nat}(P, S_u) \longrightarrow \mathrm{Nat}(K, S_u) \longrightarrow \end{array} \tag{11.9}$$

Moreover, by Proposition 10.16 we have

$$\dim \mathrm{Nat}(F, S_u) = \beta^0(F)(u) = \beta^0(P)(u) = \dim \mathrm{Nat}(P, S_u),$$

hence the naturally induced map $\mathrm{Nat}(F, S_u) \to \mathrm{Nat}(P, S_u)$ on the lower left-hand side of (11.9) is an isomorphism. In conjunction with (11.9) we obtain a natural isomorphism

$$\mathrm{Ext}^n_{\mathcal{C}}(K, S_u) \to \mathrm{Ext}^{n+1}_{\mathcal{C}}(F, S_u)$$

for each $n \in \mathbb{N}_0$. □

Proposition 11.3. *Any $\mathcal{J}$-presentable presheaf $F \colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ admits a (potentially infinite) minimal projective resolution*

$$\cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to F \to 0$$

by presheaves $P_n \colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ in $\mathcal{J}$ with $\beta^n(F) = \beta^0(P_n) \colon \operatorname{int} \mathbb{M} \to \mathbb{N}_0$ for all $n \in \mathbb{N}_0$.

Proof. By Proposition 10.16 and Theorem 10.1.(iii) any $\mathcal{J}$-presentable presheaf has a minimal projective resolution

$$\cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to F \to 0$$

by presheaves $P_n \colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ in $\mathcal{J}$. The second statement relating the higher Betti functions of $F \colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ to the 0-th Betti functions of the projectives $P_\bullet$ follows by induction from Lemma 11.2. □

Corollary 11.4. *Let $F \colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ be a $\mathcal{J}$-presentable presheaf. Then $\beta^n(F) \colon \operatorname{int} \mathbb{M} \to \mathbb{N}_0$ is an admissible Betti function for any $n \in \mathbb{N}_0$.*

Proof. This follows in conjunction with Corollary 2.10. □

Remark 11.5 (Relation to Bigraded Betti Numbers by [LW19]). Suppose we have a minimal projective resolution as in Proposition 11.3 and let $n \in \mathbb{N}_0$. Then we have

$$\beta^n(F) = \beta^0(P_n) = \mathrm{Hilb}(P_n / \operatorname{rad} P_n) \tag{11.10}$$

by Proposition 11.3 and Lemma 2.14. Moreover, the equation (11.10) is the direct translation of the notion of a *bigraded Betti number* by [LW19, Defintion 2.1] into the present setting.

11.2 Euler Functions

In the previous Section 11.1 we showed that any $\mathcal{J}$-presentable presheaf has well-defined higher Betti functions. However, as we will see that any non-projective $\mathcal{J}$-presentable presheaf has infinite projective dimension, it is a priori unclear, whether the *Euler function* of formula (11.7) is well-defined. In order to show the existence of well-defined Euler functions, we will use projective resolutions with special properties.

Definition 11.6 (Special Resolutions)**.** Let

$$\dots \xrightarrow{\delta_{n+1}} P_n \xrightarrow{\delta_n} \dots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\epsilon} F \to 0 \tag{11.11}$$

be a long exact sequence of presheaves $\mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$, henceforth also referred to as a *resolution* of $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$. We say the resolution (11.11) is

- *eventually twisted* 3*-periodic* if $P_{n+3} = P_n \circ \Sigma$ and $\delta_{n+3} = -\delta_n \circ \Sigma$ for $n \gg 0$, and
- *pointwise eventually vanishing* if for any $u \in \mathbb{M}$ and $n \gg 0$ we have $P_n(u) \cong \{0\}$.

The terminology of Definition 11.6 is adopted from a closely related predicate of a similar name for finite-dimensional algebras, see for example [Han20].

Lemma 11.7. *Any $\mathcal{J}$-presentable presheaf $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ has an eventually twisted* 3*-periodic resolution by presheaves in $\mathcal{J}$.*

Proof. By Definition E.6 and Theorem 9.1 we may choose a presentation

$$h_\gamma(Y) \xrightarrow{h_\gamma(\delta)} h_\gamma(X) \to F \to 0$$

with X and Y derived sheaves of $D^+_t(q_\gamma, \partial q)$. Then we may form homotopy pullback squares

$$\begin{tikzcd}
X[-1] \arrow[r, "\theta"] \arrow[d] & Z \arrow[r] \arrow[d, "\kappa"] & 0 \arrow[d] \\
0 \arrow[r] & Y \arrow[r, "\delta"'] & X.
\end{tikzcd}$$

As $h_\gamma(X[-1]) \cong h_\gamma(X) \circ \Sigma$ we may set

$$\begin{aligned}
&P_{3n} \coloneqq h_\gamma(X) \circ \Sigma^n, && P_{3n+1} \coloneqq h_\gamma(Y) \circ \Sigma^n, && P_{3n+2} \coloneqq h_\gamma(Z) \circ \Sigma^n,\\
&\delta_{3n+1} \coloneqq (-1)^n h_\gamma(\delta) \circ \Sigma^n, && \delta_{3n+2} \coloneqq (-1)^n h_\gamma(\kappa) \circ \Sigma^n, && \text{and } \delta_{3n+3} \coloneqq (-1)^n h_\gamma(\theta) \circ \Sigma^n
\end{aligned}$$

for all $n \in \mathbb{N}_0$. Moreover, as $h_\gamma\colon D^+(q_\gamma) \to \mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ}$ is cohomological and as $\mathrm{pres}(\mathcal{J})$ is an abelian subcategory of $\mathrm{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ}$ by Theorem 10.1.(i), the sequence (11.11) is indeed exact. Theorem 10.1 further implies that $P_n\colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ is projective for each $n \in \mathbb{N}_0$. $\square$

Lemma 11.7 has the following two corollaries.

Corollary 11.8. *Any $\mathcal{J}$-presentable presheaf has a pointwise eventually vanishing resolution by presheaves in $\mathcal{J}$.*

Corollary 11.9. *For a $\mathcal{J}$-presentable presheaf $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ we have $\beta^{n+3}(F) = \beta^n(F) \circ \Sigma$ for $n \gg 0$ (in fact $n \geq 1$).*

Corollary 11.10. *For a $\mathcal{J}$-presentable presheaf $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ the sequence $(\beta^n)_{n=0}^\infty$ of Betti functions is pointwise eventually vanishing.*

By Corollary 11.10 the following makes for a sound definition.

Definition 11.11 (Euler Function)**.** For a $\mathcal{J}$-presentable presheaf $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ and any $n \in \mathbb{N}_0$ we define the *n-th Betti function* of F by

$$\chi(F) \coloneqq \sum_{n=0}^{\infty} (-1)^n \beta^n(F)\colon \operatorname{int} \mathbb{M}_{\mathrm{f}} \to \mathbb{Z},\, u \mapsto \sum_{n=0}^{\infty} (-1)^n \beta^n(F)(u).$$

Admissible Euler Functions. In the same way that we characterized the set of functions $\operatorname{int}\mathbb{M} \to \mathbb{N}_0$ that may occur as a Betti function of a $\mathcal{J}$-presentable presheaf with Definition 9.7 and Lemma 9.9, we now characterize the set of functions $\operatorname{int}\mathbb{M} \to \mathbb{Z}$ that may occur as an Euler function of a $\mathcal{J}$-presentable presheaf. In Proposition 11.19 below we will see that the abelian group of these *admissible Euler functions* is isomorphic to the Grothendieck group of $\operatorname{pres}(\mathcal{J})$, see also Remark F.4.

Definition 11.12 (Admissible Euler Function)**.** We say that a function $\mu\colon \operatorname{int}\mathbb{M} \to \mathbb{Z}$ is an *admissible Euler function* if its pointwise absolute value

$$|\mu|\colon \operatorname{int}\mathbb{M} \to \mathbb{Z},\, u \mapsto |\mu(u)|$$

is an admissible Betti function in the sense of Definition 9.7, i.e. $\mu\colon \operatorname{int}\mathbb{M} \to \mathbb{Z}$ has bounded above support and satisfies the estimate

$$\sum_{v \in R_u} |\mu(v)| < \infty$$

for any $u \in \operatorname{int}\mathbb{M}$, where $R_u := (\uparrow u) \cap \operatorname{int}(\downarrow T(u)) \subset \operatorname{int}\mathbb{M}$. We denote the abelian group of admissible Euler functions by $\mathbb{E}$.

Lemma 11.13. *For a $\mathcal{J}$-presentable presheaf $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ the Euler function $\chi(F)\colon \operatorname{int}\mathbb{M} \to \mathbb{Z}$ is an admissible Euler function.*

Proof. Corollary 11.9 implies that for any bounded region $R \subset \mathbb{M}$ the restriction of $\beta^n(F)\colon \mathbb{M} \to \mathbb{N}_0$ to R vanishes for $n \gg 0$. Thus, the pointwise absolute value

$$|\chi(F)|\colon \operatorname{int}\mathbb{M} \to \mathbb{N}_0,\, u \mapsto |\chi(F)(u)|$$

is an admissible Betti function. □

Additivity of the Euler Function. We now show that the Euler function $\chi\colon \operatorname{Ob}(\operatorname{pres}(\mathcal{J})) \to \mathbb{E}$ is an additive invariant on $\operatorname{pres}(\mathcal{J})$ in the sense of Definition F.3. To this end, it will be useful to have the following counterpart to the Euler-Poincaré formula.

Lemma 11.14 (Euler-Poincaré Formula)**.** *Let*

$$\dots \xrightarrow{\delta_{n+1}} P_n \xrightarrow{\delta_n} \dots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\epsilon} F \to 0$$

be a pointwise eventually vanishing resolution by presheaves in $\mathcal{J}$. Then we have $\chi(F) = \sum_{n=0}^{\infty} (-1)^n \beta^0(P_n)$.

Proof. Let $u \in \operatorname{int}\mathbb{M}$. Then the cochain complex $\operatorname{Nat}(P_\bullet, S_u)$ is essentially bounded as $P_n(u) \cong \{0\}$ for $n \gg 0$. So in conjunction with the Euler-Poincaré formula [tom08, Proposition 12.4.1] for bounded (co)chain complexes of finite-dimensional vector spaces over $\mathbb{F}$, we obtain that

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n \beta^0(P_n)(u) &= \sum_{n=0}^{\infty}(-1)^n \dim \operatorname{Nat}(P_n, S_u)\\ &= \sum_{n=0}^{\infty}(-1)^n \dim H^n \operatorname{Nat}(P_\bullet, S_u)\\ &= \sum_{n=0}^{\infty}(-1)^n \dim \operatorname{Ext}^n_{\mathcal{C}}(F, S_u)\\ &= \sum_{n=0}^{\infty}(-1)^n \beta^n(F)(u)\\ &= \chi(F)(u). \qquad \square\end{aligned}$$

With the Euler-Poincaré formula at hand, we may reduce the additivity of $\chi\colon \operatorname{Ob}(\operatorname{pres}(\mathcal{J})) \to \mathbb{E}$ to the "additivity" of $\beta^0\colon \operatorname{Ob}(\mathcal{J}) \to \mathbb{E}$, $P \mapsto \beta^0(P)$. (We used scare quotes in the previous sentence as it can be easily misinterpreted when considering $\mathcal{J}$ a triangulated category via the equivalence of Theorem 9.1. Considering $\mathcal{J}$ a *Quillen exact category* via the inclusion into the category of presheaves $\mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ yields the right notion of additivity for the present context; also consider Remarks 11.18, F.4, and F.7 below.)

Lemma 11.15. *For a short exact sequence*

$$0 \to P \to Q \to R \to 0 \tag{11.12}$$

of presheaves in $\mathcal{J}$ we have $\beta^0(Q) = \beta^0(P) + \beta^0(R)$.

Proof. By Corollary 2.7 the short exact sequence (11.12) splits, hence the result follows directly from the additivity of the functor $\operatorname{Nat}(-, S_u)\colon \mathcal{J} \to \mathrm{Vect}_{\mathbb{F}}$ for any $u \in \operatorname{int}\mathbb{M}$. $\square$

Proposition 11.16 (Additivity of Euler Functions)**.** *For a short exact sequence*

$$0 \to F \to G \to H \to 0$$

of $\mathcal{J}$-presentable presheaves we have $\chi(G) = \chi(F) + \chi(H)$.

Proof. By Corollary 11.8 we may choose pointwise eventually vanishing resolutions

$$\begin{aligned}&\cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to F \to 0\\ \text{and}\quad &\cdots \to R_n \to \cdots \to R_2 \to R_1 \to R_0 \to H \to 0\end{aligned}$$

by presheaves in $\mathcal{J}$. By Theorem 10.1.(iii) and the horseshoe lemma [Gri07, Proposition XII.2.4] there is a resolution

$$\cdots \to Q_n \to \cdots \to Q_2 \to Q_1 \to Q_0 \to G \to 0$$

by presheaves in $\mathcal{J}$ as well as a short exact sequence of chain complexes

$$0 \to P_\bullet \to Q_\bullet \to R_\bullet \to 0 \tag{11.13}$$

such that the diagram

$$\begin{tikzcd} P_0 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & R_0 \arrow[d] \\ F \arrow[r] & G \arrow[r] & H \end{tikzcd}$$

commutes. Moreover, by the exactness of (11.13) the resolution (11.2) is pointwise eventually vanishing. Thus, the result follows in conjunction with Lemmas 11.14 and 11.15. □

11.3 Euler Functions as an Abelian Categorification

Proof of Universality. As with the proof of additivity, we will also reduce the *universality* of $\chi\colon \mathrm{Ob}(\mathrm{pres}(\mathcal{J})) \to \mathbb{E}$, $F \mapsto \chi(F)$ as an additive invariant to the universality of $\beta^0\colon \mathrm{Ob}(\mathcal{J}) \to \mathbb{E}$, $P \mapsto \beta^0(P)$ as a split-additive invariant.

Lemma 11.17. *The family of* 0*-th Betti functions* $\beta^0\colon \mathrm{Ob}(\mathcal{J}) \to \mathbb{E}$, $P \mapsto \beta^0(P)$ *provides a universal split-additive invariant on* $\mathcal{J}$ *in the sense of Definition F.1.*

Proof. By Lemma 9.8 the 0-th Betti function of any $\mathcal{J}$-presentable presheaf is indeed an admissible Euler function. The split-additivity of β^0 follows directly from the additivity of the functor $\mathrm{Nat}(-, S_u)\colon \mathcal{J} \to \mathrm{Vect}_{\mathbb{F}}$ for any $u \in \operatorname{int}\mathbb{M}$. Now suppose $\zeta\colon \mathrm{Ob}(\mathcal{J}) \to A$ is another split-additive invariant on $\mathcal{J}$. We have to show that there is a unique homomorphism $\varphi\colon \mathbb{E} \to A$ of abelian groups making the diagram

$$\begin{tikzcd} \mathrm{Ob}(\mathcal{J}) \arrow[d, "\beta^0"'] \arrow[dr, "\zeta"] & \\ \mathbb{E} \arrow[r, dashed, "\varphi"'] & A \end{tikzcd} \tag{11.14}$$

commute. In order to show the existence and uniqueness of such a homomorphism $\varphi\colon \mathbb{E} \to A$ we introduce the following notation. For an admissible Euler function $\mu\colon \operatorname{int}\mathbb{M} \to \mathbb{Z}$ let $\mu^+, \mu^-\colon \operatorname{int}\mathbb{M} \to \mathbb{N}_0$ be the unique functions such that we have

$$\mu = \mu^+ - \mu^- \quad \text{and} \quad \mu^+ \leq |\mu| \geq \mu^-$$

pointwise. Then μ^+ and μ^- are necessarily admissible Betti functions, so the direct sums $\bigoplus_{v \in \operatorname{int}\mathbb{M}} B_v^{\oplus \mu^+(v)}$ and $\bigoplus_{v \in \operatorname{int}\mathbb{M}} B_v^{\oplus \mu^-(v)}$ are presheaves in $\mathcal{J}$ by Lemma 9.9. We now show there is at most one homomorphism $\varphi\colon \mathbb{E} \to A$ making the diagram (11.14) commute. To this end, let $\mu\colon \operatorname{int}\mathbb{M} \to \mathbb{Z}$ be an admissible Euler function. Then we have

$$\begin{aligned}
\varphi(\mu) &= \varphi(\mu^+ - \mu^-) \\
&= \varphi(\mu^+) - \varphi(\mu^-) \\
&= \varphi\left(\beta^0\left(\bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus\mu^+(v)}\right)\right) - \varphi\left(\beta^0\left(\bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus\mu^-(v)}\right)\right) \\
&= \zeta\left(\bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus\mu^+(v)}\right) - \zeta\left(\bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus\mu^-(v)}\right)
\end{aligned}$$

proving uniqueness. Now let

$$\varphi\colon \mathbb{E} \to A,\ \mu \mapsto \zeta\left(\bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus\mu^+(v)}\right) - \zeta\left(\bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus\mu^-(v)}\right).$$

We have to show that the diagram (11.14) commutes and that $\varphi\colon \mathbb{E} \to A$ is a homomorphism of abelian groups. We start with the commutativity of (11.14). To this end, let $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ be a presheaf in $\mathcal{J}$. By the Structure Theorem 2.6 we have $F \cong \bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\mu}$, where $\mu \coloneqq \beta^0(F)\colon \operatorname{int}\mathbb{M} \to \mathbb{N}_0$, and hence

$$\varphi\big(\beta^0(F)\big) = \varphi(\mu) = \zeta\left(\bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus\mu(v)}\right) = \zeta(F)$$

as $\zeta\colon \mathrm{Ob}(\mathcal{J}) \to A$ is invariant under isomorphisms. Finally, we show that $\varphi\colon \mathbb{E} \to A$ is a homomorphism of abelian groups. To this end, let $\mu, \nu\colon \operatorname{int}\mathbb{M} \to \mathbb{Z}$ be admissible Euler functions. Then we have

$$\begin{aligned}
\varphi(\mu+\nu) &= \zeta\left(\bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus(\mu+\nu)^+(v)}\right) - \zeta\left(\bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus(\mu+\nu)^-(v)}\right)\\
&= \zeta\left(\bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus\mu^+(v)} \oplus \bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus\nu^+(v)}\right)\\
&\qquad - \zeta\left(\bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus\mu^-(v)} \oplus \bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus\nu^-(v)}\right)\\
&= \zeta\left(\bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus\mu^+(v)}\right) + \zeta\left(\bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus\nu^+(v)}\right)\\
&\qquad - \zeta\left(\bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus\mu^-(v)}\right) - \zeta\left(\bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus\nu^-(v)}\right)\\
&= \zeta\left(\bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus\mu^+(v)}\right) - \zeta\left(\bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus\mu^-(v)}\right)\\
&\qquad + \zeta\left(\bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus\nu^+(v)}\right) - \zeta\left(\bigoplus_{v\in\operatorname{int}\mathbb{M}} B_v^{\oplus\nu^-(v)}\right)\\
&= \varphi(\mu)+\varphi(\nu)
\end{aligned}$$

by the isomorphism invariance and split-additivity of $\zeta\colon \operatorname{Ob}(\mathcal{J}) \to A$. □

Remark 11.18. By Theorem 10.1.(ii) the additive category $\mathcal{J}$ is the subcategory of projectives of the abelian Frobenius category $\operatorname{pres}(\mathcal{J})$. So we may also say that $\beta^0\colon \operatorname{Ob}(\mathcal{J}) \to \mathbb{E}$ is a universal additive invariant with respect to the unique exact structure on $\mathcal{J}$ by Remark F.4. We have to be careful though, as β^0 is not additive with respect to the triangulated structure inherited from $D_t^+(q_\gamma, \partial q)$ for some boundary q of a closed proper downset of $\mathbb{M}$ via the $\mathbb{F}$-linear equivalence $h_{\gamma,0,t}\colon D_t^+(q_\gamma,\partial q) \to \mathcal{J}$ of Theorem 9.1, see also Remark F.7.

Proposition 11.19 (Universality of χ as an Additive Invariant)**.** *The family of Euler functions* $\chi\colon \operatorname{Ob}(\operatorname{pres}(\mathcal{J})) \to \mathbb{E}$, $F \mapsto \chi(F)$ *provides a universal additive invariant on* $\operatorname{pres}(\mathcal{J})$ *in the sense of Definition F.3.*

Proof. By Lemma 11.13 the Euler function of any $\mathcal{J}$-presentable presheaf is indeed an admissible Euler function. Moreover, $\chi\colon \operatorname{Ob}(\operatorname{pres}(\mathcal{J})) \to \mathbb{E}$ is additive by Proposition 11.16. Now suppose $\zeta\colon \operatorname{Ob}(\operatorname{pres}(\mathcal{J})) \to A$ is another additive invariant on $\operatorname{pres}(\mathcal{J})$. We consider the diagram

$$\begin{tikzcd}
\operatorname{Ob}(\mathcal{J}) \arrow[r, hook] \arrow[dr, "\beta^0"'] & \operatorname{Ob}(\operatorname{pres}(\mathcal{J})) \arrow[d, "\chi"] \arrow[dr, "\zeta"] & \\
& \mathbb{E} \arrow[r, dashed, "\varphi"'] & A.
\end{tikzcd} \tag{11.15}$$

First we note that the left triangle in (11.15) commutes, as $\beta^n(P) = 0$ pointwise for $n \geq 1$ and any presheaf P in $\mathcal{J}$ by Corollary 2.7. We have to show there is a unique homomorphism $\varphi\colon \mathbb{E} \to A$ of abelian groups making the right triangle of (11.15) commute. Now any such φ will also make the outer parallelogram commute. Moreover, by Theorem 10.1.(iii) and Lemma 11.17 such a homomorphism $\varphi\colon \mathbb{E} \to A$ uniquely exists. It remains to shown that such a φ obtained from the universality of $\beta^0\colon \mathrm{Ob}(\mathcal{J}) \to \mathbb{E}$ also makes the right triangle in (11.15) commute. To this end, let $F\colon \mathbb{M}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ be a $\mathcal{J}$-presentable presheaf. By Lemma 11.7 we may choose an eventually twisted 3-periodic resolution

$$\ldots \xrightarrow{\delta_{n+1}} P_n \xrightarrow{\delta_n} \ldots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\epsilon} F \to 0$$

by presheaves in $\mathcal{J}$. Then the infinite direct sum $\bigoplus_{n=0}^{\infty} P_n$ is pfd and has bounded above support, hence it is a presheaf in $\mathcal{J}$. In conjunction with Lemma F.5 we obtain

$$\begin{aligned}
\zeta(F) &= \zeta\left(\bigoplus_{n=0}^{\infty} P_{2n}\right) - \zeta\left(\bigoplus_{n=0}^{\infty} P_{2n+1}\right) \\
&= \beta^0\left(\bigoplus_{n=0}^{\infty} P_{2n}\right) - \beta^0\left(\bigoplus_{n=0}^{\infty} P_{2n+1}\right) \\
&= \chi\left(\bigoplus_{n=0}^{\infty} P_{2n}\right) - \chi\left(\bigoplus_{n=0}^{\infty} P_{2n+1}\right) \\
&= \chi(F).
\end{aligned}$$

□

Abelian Categorification of $\mathbb{E}$. Now for both $\mathcal{J}$-presentable presheaves as well as admissible Euler functions we may consider the pullbacks along $\Sigma\colon \mathbb{M} \to \mathbb{M}$:

$$\Sigma^*\colon \mathrm{pres}(\mathcal{J}) \to \mathrm{pres}(\mathcal{J}), \begin{cases} F \mapsto \Sigma^* F := F \circ \Sigma \\ \eta \mapsto \Sigma^* \eta := \eta \circ \Sigma \end{cases}$$
$$\text{and} \qquad \Sigma^*\colon \mathbb{E} \longrightarrow \mathbb{E}, \;\; \mu \mapsto \Sigma^* \mu := \mu \circ \Sigma.$$

Then we have commutative squares

$$\begin{tikzcd}
\mathrm{Ob}(\mathrm{pres}(\mathcal{J})) \arrow[r, "\Sigma^*"] \arrow[d, "\beta^n"'] & \mathrm{Ob}(\mathrm{pres}(\mathcal{J})) \arrow[d, "\beta^n"] \\
\mathbb{E} \arrow[r, "\Sigma^*"'] & \mathbb{E}
\end{tikzcd}$$

for any $n \in \mathbb{N}_0$ and thus also the commutative square

$$\begin{tikzcd}
\mathrm{Ob}(\mathrm{pres}(\mathcal{J})) \arrow[r, "\Sigma^*"] \arrow[d, "\chi"'] & \mathrm{Ob}(\mathrm{pres}(\mathcal{J})) \arrow[d, "\chi"] \\
\mathbb{E} \arrow[r, "\Sigma^*"'] & \mathbb{E}.
\end{tikzcd} \tag{11.16}$$

Moreover, as $\Sigma^* \colon \mathbb{E} \to \mathbb{E}$ is an automorphism of the abelian group of admissible Euler functions, it induces an action by the ring of integer Laurent polynomials $\mathbb{Z}[x, x^{-1}]$:

$$\begin{aligned} \mathbb{Z}[x, x^{-1}] \times \mathbb{E} &\to \mathbb{E}, \\ (x, \mu) &\mapsto x.\mu \coloneqq \Sigma^* \mu = \mu \circ \Sigma. \end{aligned} \tag{11.17}$$

Definition 11.20. We name the pullback functor $\Sigma^* \colon \mathrm{pres}(\mathcal{J}) \to \mathrm{pres}(\mathcal{J})$ the *suspension* of $\mathcal{J}$-presentable presheaves, so as to make $\mathrm{pres}(\mathcal{J})$ a strictly stable category with suspension Σ^* in the sense of Definition 1.8.

Theorem 11.21 (Abelian Categorification of $\mathbb{E}$)**.** *The strictly stable category of $\mathcal{J}$-presentable presheaves is an abelian categorification of the $\mathbb{Z}[x, x^{-1}]$-module $\mathbb{E}$ of admissible Euler functions provided by* (11.17) *in the sense of Definition F.8.*

Proof. This follows directly from Proposition 11.19 and the commutativity of (11.16). □

A Mayer–Vietoris Principle for Extended Persistence Diagrams. Finally, we return to the objective from the beginning of this Chapter 11.

Theorem 11.22 (Mayer–Vietoris Principle for Extended Persistence Diagrams)**.** *Suppose we have a continuous function $f \colon X \to \mathbb{R}$ as well as an open cover $A \cup B = X$ with f, $f|_A$, $f|_B$ and $f|_{A \cap B}$ being $\mathcal{H}^\bullet$-tame. Moreover, let $\varphi \colon h(f|_A) \oplus h(f|_B) \to h(f|_{A \cap B})$ be the natural transformation that is induced by the functoriality of RISC, see also Section 1.1.5. Then we have* $\mathrm{Dgm}(f; \mathcal{H}^\bullet) = \Sigma^* \chi(\operatorname{coker} \varphi) + \chi(\ker \varphi)$.

Proof. By Theorem 1.32 we have the exact sequence

$$h(f|_A) \circ \Sigma \oplus h(f|_B) \circ \Sigma \xrightarrow{\varphi \circ \Sigma} h(f|_{A \cap B}) \circ \Sigma \xrightarrow{d} h(f) \to h(f|_A) \oplus h(f|_B) \xrightarrow{\varphi} h(f|_{A \cap B}). \tag{1.10 revisited}$$

(Strictly speaking, we chose different signs for φ in Theorem 1.32, but this has no effect on the isomorphism classes of its kernel and cokernel.) Moreover, we obtain the short exact sequence

$$0 \to \Sigma^* \operatorname{coker} \varphi \to h(f) \to \ker \varphi \to 0$$

from (1.10). In conjunction with Corollary 2.7, Proposition 11.16, and the commutativity of (11.16) we obtain

$$\begin{aligned} \mathrm{Dgm}(f; \mathcal{H}^\bullet) = \beta^0(h(f)) = \chi(h(f)) &= \chi(\Sigma^* \operatorname{coker} \varphi) + \chi(\ker \varphi) \\ &= \Sigma^* \chi(\operatorname{coker} \varphi) + \chi(\ker \varphi). \quad \square \end{aligned}$$

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Appendices

A Some 2-Categorical Notions

A.1 Truncated Lax Limits in the Strict 2-Category of Categories

Let CAT be the strict 2-category of (potentially) locally large categories. In the following we define the notion of a 0-truncated lax limit for functors $\mathcal{B} \to \mathrm{CAT}_0$ from an ordinary category $\mathcal{B}$ to the underlying ordinary locally large category CAT_0 of CAT. The reason why we call it a 0-truncated lax limit rather than a lax limit is that we define it as a class rather than a category. This is only for simplicity. While it is possible to define lax limits as categories, we will only need to know their class of objects. So we only climb as high as we need and stop at level 0. We should also point out that this is a "brutal" or "stupid" form of truncation, which is not invariant under equivalence of categories and thus different from the usual 0-truncation.

Definition A.1 (0-Truncated Lax Limit)**.** For an ordinary category $\mathcal{B}$ and a functor $F\colon \mathcal{B} \to \mathrm{CAT}_0$ an *element* of the 0-*truncated lax limit* $\lim_0 F$ is an assigment G that assigns to any object A of $\mathcal{B}$ an object G_A of the category $F(A)$ and to any morphism $\varphi\colon A \to B$ of $\mathcal{B}$ a morphism

$$G_\varphi\colon F(\varphi)(G_A) \to G_B$$

of the category $F(B)$ such that

$$G_{\mathrm{id}_A} = \mathrm{id}_{G_A}$$

for all objects A of $\mathcal{B}$ and for all composable morphisms $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ of $\mathcal{B}$ the diagram

$$\begin{tikzcd} F(\psi\circ\varphi)(G_A) \arrow[d, "F(\psi)(G_\varphi)"'] \arrow[dr, "G_{\psi\circ\varphi}"] & \\ F(\psi)(G_B) \arrow[r, "G_\psi"'] & G_C \end{tikzcd} \tag{A.1}$$

commutes (in the category $F(C)$). The class of all of these elements then is the 0-*truncated lax limit* $\lim_0 F$ of $F\colon \mathcal{B} \to \mathrm{CAT}_0$. By extension this defines a functor from the locally large category of functors $\mathcal{B} \to \mathrm{CAT}_0$ to the locally large category of classes.

A.2 Initial Lax Cocones in Strict 2-Categories

In the following let $\mathbf{C}$ be a strict 2-category and let $\mathbf{C}_0$ denote the underlying ordinary category.

Definition A.2 (Lax Cocone)**.** For an ordinary category $\mathcal{B}$ and a (contravariant) functor $F\colon \mathcal{B}^\circ \to \mathbf{C}_0$ a *lax cocone* under F consists of

- an object $\mathbb{X}$ of $\mathbf{C}$ that we call the *vertex* of the cocone,
- for any object A of $\mathcal{B}$ a 1-morphism $G(A)\colon F(A) \to \mathbb{X}$, and
- for any morphism $\varphi\colon A \to B$ of $\mathcal{B}$ a 2-morphism $G(\varphi)\colon G(A) \circ F(\varphi) \Longrightarrow G(B)$

such that $G(\mathrm{id}_A) = \mathrm{id}_{G(A)}$ for all objects A of $\mathcal{B}$ and the diagram

$$\begin{tikzcd} G(A) \circ F(\psi \circ \varphi) \arrow[d, Rightarrow, "G(\psi)"'] \arrow[dr, Rightarrow, "G(\psi\circ\varphi)"] & \\ G(B) \circ F(\psi) \arrow[r, Rightarrow, "G(\varphi)"'] & G(C) \end{tikzcd} \tag{A.2}$$

commutes for all composable morphisms $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ of $\mathcal{B}$. We denote the class of lax cocones under F with vertex $\mathbb{X}$ by $\mathrm{LCocone}(F, \mathbb{X})$.

In Definition A.2 we describe the notion of lax cocones merely for functors from an ordinary category to $\mathbf{C}_0$ rather than a strict 2-functor to $\mathbf{C}$. We only make this restriction as we will not need the more general notion and ordinary functors are sufficient for our purposes. Also, considering contravariant functors in place of covariant functors is merely a notational convenience as we use this notion for contravariant functors only. Now suppose we have a (contravariant) functor $F\colon \mathcal{B}^\circ \to \mathbf{C}_0$ and a 1-morphism $\Phi\colon \mathbb{X} \to \mathbb{Y}$ of $\mathbf{C}$. Then post-composition and whiskering from the left yields a map

$$\mathrm{LCocone}(F, \Phi)\colon \mathrm{LCocone}(F, \mathbb{X}) \longrightarrow \mathrm{LCocone}(F, \mathbb{Y}).$$

This way we obtain a functor

$$\mathrm{LCocone}(F, -)\colon \mathbf{C}_0 \to \mathrm{Class}$$

from $\mathbf{C}_0$ to the category of classes. We denote the category of elements of this functor by $\mathrm{el}(\mathrm{LCocone}(F, -))$. Now the objects of $\mathrm{el}(\mathrm{LCocone}(F, -))$ are precisely the cocones under F in the sense of Definition A.2.

Definition A.3 (Initial Lax Cocone)**.** We say that cocone under a functor $F\colon \mathcal{B}^\circ \to \mathbf{C}_0$ is *initial* if it is initial as an object of the category of elements $\mathrm{el}(\mathrm{LCocone}(F, -))$.

Remark A.4. For two objects $\mathbb{X}$ and $\mathbb{Y}$ of $\mathbf{C}$, let $1\text{-}\mathrm{Hom}(\mathbb{X}, \mathbb{Y})$ denote the (potentially) locally large category of 1-morphisms $\mathbb{X} \to \mathbb{Y}$. Then we have the strict 2-bifunctor $1\text{-}\mathrm{Hom}(-, -)\colon \mathbf{C}^\circ \times \mathbf{C} \to \mathrm{CAT}$, where CAT denotes the strict 2-category of (potentially) locally large categories. Now suppose we have a (contravariant) functor $F\colon \mathcal{B}^\circ \to \mathbf{C}_0$ and an object $\mathbb{X}$ of $\mathbf{C}$. Then we have the functor

$$1\text{-}\mathrm{Hom}(F(-), \mathbb{X})\colon \mathcal{B} \to \mathrm{CAT}_0,\ B \mapsto 1\text{-}\mathrm{Hom}(F(B), \mathbb{X}).$$

So we may form the 0-truncated lax limit $\lim_0 1\text{-Hom}(F(-), \mathbb{X})$ of $1\text{-Hom}(F(-), \mathbb{X})$ in the sense of Definition A.1. Moreover, if we unravel Definition A.1 for the functor $1\text{-Hom}(F(-), \mathbb{X})$, then we see that the elements of $\lim_0 1\text{-Hom}(F(-), \mathbb{X})$ are precisely the lax cocones under F with vertex $\mathbb{X}$. This is the reason why it is called a "lax" cocone in Definition A.2. Furthermore, by the functoriality of $\lim_0$ we have the functor

$$\lim_0 1\text{-Hom}(F(-), ?) \colon \mathbf{C}_0 \to \mathrm{Class},$$

which is naturally isomorphic to $\mathrm{LCocone}(F, ?) \colon \mathbf{C}_0 \to \mathrm{Class}$.

Remark A.5. There is another property on lax cocones, which is that of being a *lax colimit*. This property is stronger than the property of being initial. Moreover, as initial objects are unique up to unique 1-isomorphism, initial lax cocones and lax colimits coincide as far as they exist. However, it may well happen that no lax colimit exists for a certain functor while a corresponding initial lax cocone does. For a simple example we may consider the dual of the example by [Kel05, near diagram (3.54)], where there is no coproduct for a certain pair objects in the stronger sense while there is an initial cocone under this pair. (Note that lax coproducts are just coproducts and vice versa.)

B Algebraic Preliminaries

B.1 Middle Exact Squares

Definition B.1 (Middle Exact Square)**.** We say that a commutative square

$$\begin{tikzcd} A \arrow[r, "f_{AB}"] \arrow[d, "f_{AC}"'] & B \arrow[d, "f_{BD}"] \\ C \arrow[r, "f_{CD}"'] & D \end{tikzcd}$$

of vector spaces (or modules) is *middle exact* if the sequence

$$A \xrightarrow{\begin{pmatrix} f_{AB} \\ f_{AC} \end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix} f_{BD} & -f_{CD} \end{pmatrix}} D$$

is exact (at the middle term $B \oplus C$).

Now suppose we have two adjacent middle exact squares

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F, \end{tikzcd}$$

with the maps denoted by f_{AB}, $f_{AC} = f_{BC} \circ f_{AB}$, and so forth.

Lemma B.2. *We have* $f_{EF}^{-1}(\operatorname{Im} f_{DF} + \operatorname{Im} f_{BF}) = \operatorname{Im} f_{DE} + \operatorname{Im} f_{BE}$.

Proof. It is clear that the right-hand side is a subspace of the left-hand side. The other inclusion follows from the following diagram chase. Suppose we have

$$u \in f_{EF}^{-1}(\operatorname{Im} f_{DF} + \operatorname{Im} f_{BF}),$$

then there are vectors $v \in B$ and $w \in D$ with

$$f_{EF}(u) = f_{DF}(w) + f_{BF}(v).$$

Moreover, we have that

$$f_{EF}(u - f_{DE}(w) - f_{BE}(v)) = 0,$$

hence the term

$$\begin{pmatrix} 0 \\ u - f_{DE}(w) - f_{BE}(v) \end{pmatrix} \in C \oplus E$$

is in the kernel of $\begin{pmatrix} f_{CF} & -f_{EF} \end{pmatrix}$. By the exactness of the sequence

$$B \xrightarrow{\begin{pmatrix} f_{BC} \\ f_{BE} \end{pmatrix}} C \oplus E \xrightarrow{\begin{pmatrix} f_{CF} & -f_{EF} \end{pmatrix}} F$$

there is a $v' \in B$ with

$$f_{BE}(v') = u - f_{DE}(w) - f_{BE}(v),$$

which is equivalent to $f_{DE}(w) + f_{BE}(v + v') = u$. □

Now we consider a diagram of four adjacent middle exact squares

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & F \arrow[d] \\ G \arrow[r] & H \arrow[r] & I. \end{tikzcd} \tag{B.1}$$

We note that middle exact squares "compose" to middle exact squares. If we compose the two squares in the first row of (B.1) as well as the two squares in the second row and then transpose the diagram, we obtain this diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & G \arrow[d] \\ C \arrow[r] & F \arrow[r] & I \end{tikzcd} \tag{B.2}$$

of two adjacent middle exact squares, which we will use at the end of the proof of the following proposition.

Proposition B.3. *The map f_{EI} induces a natural isomorphism*

$$\frac{E}{\operatorname{Im} f_{DE} + \operatorname{Im} f_{BE}} \cong \frac{\operatorname{Im} f_{EI} + \operatorname{Im} f_{CI}}{\operatorname{Im} f_{DI} + \operatorname{Im} f_{CI}}.$$

Proof. As the upper right square of (B.1) is middle exact we have

$$\operatorname{Im} f_{BF} = \operatorname{Im} f_{EF} \cap \operatorname{Im} f_{CF}. \tag{B.3}$$

With this we obtain the chain of three isomorphisms and two equalities

$$\begin{aligned}
\frac{E}{\operatorname{Im} f_{DE} + \operatorname{Im} f_{BE}} &\cong \frac{\operatorname{Im} f_{EF}}{\operatorname{Im} f_{DF} + \operatorname{Im} f_{BF}} \\
&= \frac{\operatorname{Im} f_{EF}}{\operatorname{Im} f_{DF} + \operatorname{Im} f_{EF} \cap \operatorname{Im} f_{CF}} \\
&= \frac{\operatorname{Im} f_{EF}}{\operatorname{Im} f_{EF} \cap (\operatorname{Im} f_{DF} + \operatorname{Im} f_{CF})} \\
&\cong \frac{\operatorname{Im} f_{EF} + \operatorname{Im} f_{CF}}{\operatorname{Im} f_{DF} + \operatorname{Im} f_{CF}} \\
&\cong \frac{\operatorname{Im} f_{EI} + \operatorname{Im} f_{CI}}{\operatorname{Im} f_{DI} + \operatorname{Im} f_{CI}}.
\end{aligned}$$

Here the first isomorphism follows from Lemma B.2 applied to the two squares at the top of (B.1) and the first isomorphism theorem. The first equality follows from (B.3). The second equality follows from the modular law for the lattice of subspaces. The second isomorphism follows from the second isomorphism theorem and the last isomorphism again from Lemma B.2 applied to (B.2) and the first isomorphism theorem. □

B.2 The Radical of a Presheaf on a Poset

In this appendix we adapt several notions from the theory of modules to presheaves following [CGR+22, Section 3.1] closely. Let P be a poset and let $F\colon P^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ be a presheaf on P. We define the *radical* of F to be the subpresheaf

$$\operatorname{rad} F\colon P^{\circ} \to \mathrm{Vect}_{\mathbb{F}},\, p \mapsto \sum_{q \succ p} \operatorname{Im} F(p \preceq q),$$

where q ranges over all $q \in P$ with $q \succ p$. In this appendix we show that $\operatorname{rad} F$ is indeed the intersection of all maximal (proper) subpresheaves of F. We say that a presheaf is *semisimple* if all internal maps between values at different elements of P are zero. Now a subpresheaf $G \subseteq F$ is maximal, iff the quotient F/G is *simple*, i.e. it has no non-zero proper subpresheaves. Moreover, a presheaf $S\colon P^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$ on a poset P is simple iff it is semisimple and S is supported on a single element $p \in P$ with $S(p) \cong K$. Thus, any simple presheaf on P is isomorphic to the *simple presheaf associated to some element $p \in P$:*

$$S_p\colon P^{\circ} \to \mathrm{Vect}_{\mathbb{F}},\, q \mapsto \begin{cases} \mathbb{F} & p = q \\ \{0\} & \text{otherwise.} \end{cases}$$

With this we may write the intersection of all maximal subpresheaves as

$$\widetilde{\operatorname{rad}}\, F := \bigcap_{\substack{p \in P \\ \varphi \in \operatorname{Nat}(F, S_p)}} \ker \varphi.$$

We now show that $\widetilde{\operatorname{rad}} F = \operatorname{rad} F$. To this end, we first consider the case where F is semisimple.

Lemma B.4. *If $F\colon P^\circ \to \mathrm{Vect}_{\mathbb{F}}$ is semisimple, then $\widetilde{\operatorname{rad}} F = 0$ or equivalently $\widetilde{\operatorname{rad}} F = \operatorname{rad} F$.*

Proof. Let $v \in F(p) \setminus \{0\}$. We now construct a natural transformation $\varphi\colon F \to S_p$ with $\varphi_p(v) \neq 0$ as a witness for $v \notin \widetilde{\operatorname{rad}}(F)(p)$. To this end, we choose a linear form $\alpha\colon F(p) \to K$ with $\alpha(v) \neq 0$. As F is semisimple, the evaluation at p yields an isomorphism

$$\mathrm{Nat}(F, S_p) \cong \mathrm{Hom}_K(F(p), K),$$

hence there is a (unique) natural transformation $\varphi\colon F \to S_p$ with $\varphi_p = \alpha$. $\square$

Corollary B.5. *We have $\widetilde{\operatorname{rad}}(F/\operatorname{rad} F) = 0$.*

Using the previous corollary we may reduce the general case to the semisimple one.

Lemma B.6. *Let $\varphi\colon F \to S_p$ be a homomorphism of presheaves on P for some $p \in P$. Then the triangle*

$$\begin{tikzcd} \operatorname{rad} F \arrow[r, hook] \arrow[dr, "0"'] & F \arrow[d, "\varphi"] \\ & S_p \end{tikzcd}$$

commutes.

Proposition B.7. *Let $F\colon P^\circ \to \mathrm{Vect}_{\mathbb{F}}$ be a presheaf on P. Then we have $\widetilde{\operatorname{rad}} F = \operatorname{rad} F$.*

Proof. Let $\mathrm{pr}\colon F \to F/\operatorname{rad} F$ be the pointwise projection onto the quotient space. By Lemma B.6 the precomposition by the projection pr yields an isomorphism

$$\mathrm{Nat}(F/\operatorname{rad} F, S_p) \cong \mathrm{Nat}(F, S_p)$$

for any $p \in P$. In conjunction with Corollary B.5 we obtain

$$\begin{aligned}
\widetilde{\operatorname{rad}} F &= \bigcap_{\substack{p \in P \\ \varphi \in \operatorname{Nat}(F, S_p)}} \ker \varphi \\
&= \bigcap_{\substack{p \in P \\ \varphi \in \operatorname{Nat}(F/\operatorname{rad} F, S_p)}} \ker(\varphi \circ \operatorname{pr}) \\
&= \bigcap_{\substack{p \in P \\ \varphi \in \operatorname{Nat}(F/\operatorname{rad} F, S_p)}} \operatorname{pr}^{-1}(\ker \varphi) \\
&= \operatorname{pr}^{-1} \left(\bigcap_{\substack{p \in P \\ \varphi \in \operatorname{Nat}(F/\operatorname{rad} F, S_p)}} \ker \varphi \right) \\
&= \operatorname{pr}^{-1} \left(\widetilde{\operatorname{rad}}(F/\operatorname{rad} F) \right) \\
&= \operatorname{pr}^{-1}(0) \\
&= \operatorname{rad} F.
\end{aligned}$$

□

Corollary B.8. *The radical* $\operatorname{rad} F$ *is the pointwise intersection of all maximal sub-presheaves of* $F \colon P^{\circ} \to \mathrm{Vect}_{\mathbb{F}}$.

C Some Properties of Adjunctions and Monads

C.1 The Beck–Chevalley Condition

Suppose we have a square

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, "G_1"] \arrow[d, "H_1"'] \arrow[dr, phantom, "\overset{\zeta}{\swarrow}"] & \mathcal{C}_2 \arrow[d, "H_2"] \\ \mathcal{C}_3 \arrow[r, "G_2"'] & \mathcal{C}_4 \end{tikzcd} \tag{C.1}$$

of categories and functors that commutes up to a natural isomorphism $\zeta\colon H_2 \circ G_1 \Rightarrow G_2 \circ H_1$. Moreover, suppose $G_1 : \mathcal{C}_1 \to \mathcal{C}_2$ and $G_2 : \mathcal{C}_3 \to \mathcal{C}_4$ have left adjoints $F_1 : \mathcal{C}_2 \to \mathcal{C}_1$ and $F_2 : \mathcal{C}_4 \to \mathcal{C}_3$, respectively. Then we also have the square diagram

$$\begin{tikzcd} \mathcal{C}_1 \arrow[d, "H_1"'] & \mathcal{C}_2 \arrow[l, "F_1"'] \arrow[d, "H_2"] \arrow[dl, phantom, "\overset{\xi}{\nwarrow}"] \\ \mathcal{C}_3 & \mathcal{C}_4, \arrow[l, "F_2"] \end{tikzcd}$$

where $\xi\colon F_2 \circ H_2 \Rightarrow H_1 \circ F_1$, defined as the composition

$$\begin{tikzcd} F_2 \circ H_2 \arrow[d, Rightarrow, "F_2 \circ H_2 \circ \eta^1"] \\ F_2 \circ H_2 \circ G_1 \circ F_1 \arrow[d, Rightarrow, "F_2 \circ \zeta \circ F_1"] \\ F_2 \circ G_2 \circ H_1 \circ F_1 \arrow[d, Rightarrow, "\varepsilon^2 \circ H_1 \circ F_1"] \\ H_1 \circ F_1, \end{tikzcd}$$

is the so called *mate* of $\zeta\colon H_2 \circ G_1 \Rightarrow G_2 \circ H_1$. Here $\eta^1 : \mathrm{id}_{\mathcal{C}_2} \Rightarrow G_1 \circ F_1$ denotes the unit of the adjunction $F_1 \dashv G_1$. Similarly, $\varepsilon^2\colon F_2 \circ G_2 \Rightarrow \mathrm{id}_{\mathcal{C}_3}$ denotes the counit of the adjunction $F_2 \dashv G_2$.

Definition C.1 (Beck–Chevalley Condition)**.** We say that the square diagram (C.1) satisfies the *Beck–Chevalley condition* if $\xi\colon F_2 \circ H_2 \Rightarrow H_1 \circ F_1$ is a natural isomorphism.

Morally, a commutative square of categories and functors satisfies the Beck–Chevalley condition, if the horizontal arrows have left adjoints and the corresponding square with left adjoints commutes as well, which is not precisely the same as Definition C.1, but it is close enough for intuition.

There is a dual version of the Beck–Chevalley condition as well, which involves right adjoints in place of left adjoints. To this end, we consider the square diagram

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, "F_1"] \arrow[d, "H_1"'] & \mathcal{C}_2 \arrow[d, "H_2"] \\ \mathcal{C}_3 \arrow[r, "F_2"'] \arrow[ur, phantom, "\overset{\zeta}{\Nearrow}"] & \mathcal{C}_4, \end{tikzcd} \tag{C.2}$$

with $\zeta \colon F_2 \circ H_1 \Rightarrow H_2 \circ F_1$ a natural isomorphism. Moreover, suppose $F_1 \colon \mathcal{C}_1 \to \mathcal{C}_2$ and $F_2 \colon \mathcal{C}_3 \to \mathcal{C}_4$ have right adjoints $G_1 \colon \mathcal{C}_2 \to \mathcal{C}_1$ and $G_2 \colon \mathcal{C}_4 \to \mathcal{C}_3$, respectively. Then we have the square diagram

$$\begin{tikzcd} \mathcal{C}_1 \arrow[d, "H_1"'] \arrow[dr, phantom, "\overset{\xi}{\Searrow}"] & \mathcal{C}_2 \arrow[l, "G_1"'] \arrow[d, "H_2"] \\ \mathcal{C}_3 & \mathcal{C}_4 \arrow[l, "G_2"] \end{tikzcd}$$

with $\xi \colon H_1 \circ G_1 \Rightarrow G_2 \circ H_2$ provided by the composition

$$\begin{aligned} & H_1 \circ G_1 \\ & \Downarrow \eta_2 \circ H_1 \circ G_1 \\ & G_2 \circ F_2 \circ H_1 \circ G_1 \\ & \Downarrow G_2 \circ \zeta \circ G_1 \\ & G_2 \circ H_2 \circ F_1 \circ G_1 \\ & \Downarrow G_2 \circ H_2 \circ \varepsilon_1 \\ & G_2 \circ H_2. \end{aligned}$$

Definition C.2 (Dual Beck–Chevalley Condition)**.** We say that the square diagram (C.2) satisfies the *dual Beck–Chevalley condition* if $\xi \colon H_1 \circ G_1 \Rightarrow G_2 \circ H_2$ is a natural isomorphism.

C.2 Composition of Monadic Functors

In general the composition of two monadic functors is not a monadic functor. In this Appendix C.2 we combine two results by [Bec69] to obtain a sufficient condition for the monadicity of a composition of monadic functors.

Definition C.3 (Distributive Adjoint Situation)**.** A *distributive adjoint situation over* a category $\mathcal{A}$ is a diagram of functors and adjunctions

$$
\begin{tikzcd}
& \tilde{\mathcal{B}} \arrow[dl, bend left, "\tilde{U}_1"] \arrow[dr, "\tilde{U}_0"] & \\
\mathcal{B}_0 \arrow[ur, bend left, "\tilde{F}_1"] \arrow[ur, phantom, "\dashv"] \arrow[dr, bend left, "U_0"] \arrow[dr, phantom, "\dashv"] & & \mathcal{B}_1 \arrow[dl, bend left, "U_1"] \\
& \mathcal{A} \arrow[ul, bend left, "F_0"] \arrow[ur, bend left, "F_1"] \arrow[ur, phantom, "\dashv"] &
\end{tikzcd}
\tag{C.3}
$$

such that the square

$$
\begin{tikzcd}
\tilde{\mathcal{B}} \arrow[r, "\tilde{U}_1"] \arrow[d, "\tilde{U}_0"'] & \mathcal{B}_0 \arrow[d, "U_0"] \\
\mathcal{B}_1 \arrow[r, "U_1"'] & \mathcal{A}
\end{tikzcd}
$$

is a commutative Beck–Chevalley square in the sense of Definition C.1.

Proposition C.4. *Suppose we have a distributive adjoint situation as in the diagram* (C.3)*. If U_0, U_1, and $\tilde{U}_1$ are monadic, then $U_0 \circ \tilde{U}_1$ is monadic as well.*

Proof. By [Bec69, Section 3] there is a reflective adjunction $\breve{\sigma} \dashv \sigma$ from distribute adjoint situations over $\mathcal{A}$ to so called *distributive laws in* $\mathcal{A}$, whose unit is composed of the corresponding *semantical comparison functors.* Suppose U_0, U_1, and $\tilde{U}_1$ are monadic. Then the corresponding semantical comparison functors are isomorphisms of categories. Thus, the distributive adjoint situation at hand is in the reflective subcategory of the adjunction $\breve{\sigma} \dashv \sigma$. Moreover, for any distributive adjoint situation (C.3) in the image of $\breve{\sigma}$, the composition $U_0 \circ \tilde{U}_1$ is monadic by [Bec69, Section 2]. □

C.3 Strong Monoidal Monads and Enrichment

Suppose $\mathcal{V}$ is a cartesian monoidal category and that $T\colon \mathcal{V} \to \mathcal{V}$ is a strong monoidal monad on $\mathcal{V}$ with unit $\eta\colon \mathrm{id}_{\mathcal{V}} \to T$ and multiplication $\mu\colon T \circ T \to T$. Then both the *Kleisli category* $\mathcal{V}_T$, which we define below, and the Eilenberg–Moore category $\mathcal{V}^T$ of T inherit monoidal structures from $\mathcal{V}$. In this Appendix C.3 we provide some properties of categories enriched in $\mathcal{V}$, in $\mathcal{V}_T$, or in $\mathcal{V}^T$. Let $U^T : \mathcal{V}^T \to \mathcal{V}$ be the forgetful functor from the Eilenberg–Moore category $\mathcal{V}^T$ to $\mathcal{V}$. Moreover, we denote the category of small $\mathcal{V}$-enriched categories by $\mathcal{V}$-cat with similar notation for small $\mathcal{V}^T$-categories.

Lemma C.5. *The base change*

$$U^T_\bullet \colon \mathcal{V}^T\text{-cat} \to \mathcal{V}\text{-cat}\, \mathcal{C} \mapsto U^T_\bullet \mathcal{C}$$

along the forgetful functor $U^T \colon \mathcal{V}^T \to \mathcal{V}$ is monadic.

Proof. Suppose $\mathcal{C}$ is a small $\mathcal{V}$-enriched category. Then the multiplication of T yields the structure of a T-algebra on each of the objects of morphisms of $T_\bullet \mathcal{C}$. Thus, there is a small $\mathcal{V}^T$-category $\tilde{\mathcal{C}}$, such that $U^T_\bullet \tilde{\mathcal{C}} = T_\bullet \mathcal{C}$, hence the base change

$$T_\bullet \colon \mathcal{V}\text{-cat} \to \mathcal{V}\text{-cat}$$

is the monad induced by $U^T_\bullet$. Moreover, if $\mathcal{C}$ is a $T_\bullet$-algebra with structure functor $\alpha \colon T_\bullet \mathcal{C} \to \mathcal{C}$, then α has to preserve the objects of $\mathcal{C}$ by the unit property of a monad algebra. Thus, small $\mathcal{V}^T$-categories are the same thing as algebras over the monad $T_\bullet$. □

Now suppose A is an object of $\mathcal{V}$ and that $\alpha \colon T(A) \to A$ is a T-algebra structure map, i.e. A and α provide an object of the Eilenberg–Moore category $\mathcal{V}^T$. Moreover, suppose we have a morphism $\varphi \colon B \to A$ of $\mathcal{V}$. Then we write

$$\varphi^+ \colon T(B) \xrightarrow{T(\varphi)} T(A) \xrightarrow{\alpha} A$$

for the composition of $T(\varphi)$ and α. This way we obtain the natural transformation

$$\mathrm{Hom}(-, A) \to \mathrm{Hom}(T(-), A),\ \varphi \mapsto \varphi^+ \coloneqq \alpha \circ T(\varphi),$$

which corresponds to the element $\alpha \in \mathrm{Hom}(T(A), A)$ under the bijection obtained from the Yoneda lemma. As a special case suppose that $A = T(\bar{A})$ is the *free* T-algebra generated by an object $\bar{A}$ of $\mathcal{V}$ with structure morphism $\alpha = \mu_{\bar{A}} \colon T(T(\bar{A})) \to T(\bar{A})$. Then we obtain the *Kleisli extension*

$$\mathrm{Hom}(B, T(\bar{A})) \to \mathrm{Hom}(T(B), T(\bar{A})),\ \varphi \mapsto \varphi^+.$$

We may use this notation to provide a lightweight definition of the Kleisli category $\mathcal{V}_T$.

Definition C.6 (Kleisli Category)**.** The objects of the *Kleisli category* $\mathcal{V}_T$ are the objects of $\mathcal{V}$. For two objects A and B of $\mathcal{V}$ the *morphisms* $A \to B$ of $\mathcal{V}_T$ are provided by the morphisms $A \to T(B)$ of $\mathcal{V}$. For an object A of $\mathcal{V}$ the *identity* at A of $\mathcal{V}_T$ is the unit $\eta_A \colon A \to T(A)$ and for two morphisms $\varphi \colon A \to T(B)$ and $\psi \colon B \to T(C)$ in $\mathcal{V}_T$ their *composition in* $\mathcal{V}_T$ is defined by the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi \circ \varphi"'] & T(B) \arrow[r, "\psi^+"] & T(C). \end{tikzcd}$$

With some abuse of notation we write

$$T\colon \mathcal{V}_T \to \mathcal{V}^T,\ \begin{cases} A \mapsto T(A) \\ \varphi \mapsto \varphi^+ \end{cases}$$

for the fully faithful strong monoidal functor sending each object of the Kleisli category to the corresponding free T-algebra. Now suppose $\mathcal{C}$ is a $\mathcal{V}_T$-enriched category. Recall, a morphism $X \to Y$ in the underlying ordinary category $\mathcal{C}_0$ is a morphism $* \to \mathrm{Hom}(X,Y)$ of $\mathcal{V}_T$, where $*$ is the terminal object of $\mathcal{V}$. Moreover, a morphism $* \to \mathrm{Hom}(X,Y)$ of $\mathcal{V}_T$ is just a morphism $* \to T(\mathrm{Hom}(X,Y))$ of $\mathcal{V}$ by Definition C.6.

Definition C.7. We say that a morphism $\varphi\colon X \to Y$ of the underlying ordinary category of a $\mathcal{V}_T$-enriched category $\mathcal{C}$ is *pure*, if it can be written as a composition:

$$\begin{tikzcd} * \arrow[r, "\overline{\varphi}"] \arrow[rr, bend right, "\varphi"'] & \mathrm{Hom}(X,Y) \arrow[r, "\eta_{\mathrm{Hom}(X,Y)}"] & T(\mathrm{Hom}(X,Y)). \end{tikzcd}$$

Moreover, we say that a $\mathcal{V}_T$-enriched category $\mathcal{C}$ has *pure identities*, if all identities of $\mathcal{C}_0$ are pure.

Now $\mathcal{V}_T$-categories with pure identities form a strict full sub-2-category $\mathcal{V}_T\text{-Cat}_p$ of the strict 2-category $\mathcal{V}_T$-Cat. Moreover, the base change along $T\colon \mathcal{V}_T \to \mathcal{V}^T$ yields an embedding of $\mathcal{V}_T\text{-Cat}_p$ into $\mathcal{V}^T$-Cat in a strict 2-categorical sense:

$$T_\bullet\colon \mathcal{V}_T\text{-Cat}_p \to \mathcal{V}^T\text{-Cat},\ \mathcal{C} \mapsto T_\bullet\mathcal{C}.$$

We will now characterize the base change $T_\bullet\mathcal{C}$ of a $\mathcal{V}_T$-enriched category $\mathcal{C}$ with pure identities by a universal property.

Definition C.8. For a $\mathcal{V}_T$-enriched category $\mathcal{C}$ with pure identities and a $\mathcal{V}^T$-enriched category $\mathcal{D}$ we define a locally large category $\mathrm{EFunc}'(\mathcal{C},\mathcal{D})$ as follows[1]. An *object* of $\mathrm{EFunc}'(\mathcal{C},\mathcal{D})$ consists of a map

$$G_0\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D}),\ X \mapsto G_0(X)$$

from the objects of $\mathcal{C}$ to the objects of $\mathcal{D}$ as well as an $\mathrm{Ob}(\mathcal{C})^2$-indexed family of morphisms

$$G_{1,X,Y}\colon \mathrm{Hom}(X,Y) \to \mathrm{Hom}(G_0(X),G_0(Y)), \quad X,Y \in \mathrm{Ob}(\mathcal{C})$$

such that the diagrams

$$\begin{tikzcd} * \arrow[r, "\overline{\mathrm{id}}_X"] \arrow[dr, "\mathrm{id}_{G_0(X)}"'] & \mathrm{Hom}(X,X) \arrow[d, "G_{1,X,X}"] \\ & \mathrm{Hom}(G_0(X),G_0(X)) \end{tikzcd} \tag{C.4}$$

[1] The "E" in EFunc$'$ stands for "equivariant" as $\mathcal{V}^T$-enriched functors are $\mathcal{V}$-enriched functors that are equivariant with respect to $T\colon \mathcal{V} \to \mathcal{V}$.

and

$$\begin{tikzcd}
\mathrm{Hom}(Y,Z)\times \mathrm{Hom}(X,Y) \arrow[r] \arrow[d, "G_{1,Y,Z}\times G_{1,X,Y}"'] & T(\mathrm{Hom}(X,Z)) \arrow[d, "G^{+}_{1,X,Z}"] \\
\mathrm{Hom}(G_0(Y),G_0(Z))\times \mathrm{Hom}(G_0(X),G_0(Y)) \arrow[r] & \mathrm{Hom}(G_0(X),G_0(Z))
\end{tikzcd} \tag{C.5}$$

commute for any three objects X, Y, and Z of $\mathcal{C}$.

For two such objects G and H, a *morphism* $G \to H$ of $\mathrm{EFunc}'(\mathcal{C},\mathcal{D})$ is an $\mathrm{Ob}(\mathcal{C})$-indexed family of morphisms

$$\alpha_X \colon * \to \mathrm{Hom}(G_0(X), H_0(X)), \quad X \in \mathrm{Ob}(\mathcal{C})$$

such that the diagram

$$\begin{tikzcd}
 & \mathrm{Hom}(X,Y) \arrow[dl] \arrow[dr] & \\
* \times \mathrm{Hom}(X,Y) \arrow[d, "\alpha_Y\times G_{1,X,Y}"] & & \mathrm{Hom}(X,Y)\times * \arrow[d, "H_{1,X,Y}\times\alpha_X"] \\
\mathrm{Hom}(G_0(Y),H_0(Y))\times \mathrm{Hom}(G_0(X),G_0(Y)) \arrow[dr] & & \mathrm{Hom}(H_0(X),H_0(Y))\times \mathrm{Hom}(G_0(X),H_0(X)) \arrow[dl] \\
 & \mathrm{Hom}(G_0(X),H_0(Y)) &
\end{tikzcd} \tag{C.6}$$

commutes for any two objects X and Y of $\mathcal{C}$. Identities and compositions in $\mathrm{EFunc}'(\mathcal{C},\mathcal{D})$ are defined analogous to identities and compositions for $\mathcal{V}$-natural transformations.

By extension this defines a strict 2-bifunctor

$$\mathrm{EFunc}' \colon (\mathcal{V}_T\text{-}\mathrm{Cat}_p)^\circ \times \mathcal{V}^T\text{-}\mathrm{Cat} \longrightarrow \mathrm{CAT},\ (\mathcal{C},\mathcal{D}) \mapsto \mathrm{EFunc}'(\mathcal{C},\mathcal{D})$$

taking values in the strict 2-category CAT of locally large categories.

Now for any $\mathcal{V}_T$-enriched category $\mathcal{C}$ with pure identities we may define an object $H'_{\mathcal{C}} \coloneqq H' \in \mathrm{EFunc}'(\mathcal{C}, T_\bullet\mathcal{C})$ by the equations

$$H'_0 \colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(T_\bullet\mathcal{C}),\ X \mapsto X$$
$$H'_{1,X,Y} \coloneqq \eta_{\mathrm{Hom}(X,Y)} \colon \mathrm{Hom}(X,Y) \to T(\mathrm{Hom}(X,Y)).$$

Definition C.9. We name $H'_{\mathcal{C}}$ the *universal object* of $\mathrm{EFunc}'(\mathcal{C}, T_\bullet\mathcal{C})$.

Now for two $\mathcal{V}^T$-enriched categories $\mathcal{D}$ and $\mathcal{E}$, let $\mathcal{V}^T\text{-Func}(\mathcal{D},\mathcal{E})$ be the locally large category of $\mathcal{V}^T$-enriched functors $\mathcal{D}\to\mathcal{E}$ and V-natural transformations. Then the following Lemma C.10 justifies the naming convention of Definition C.9.

Lemma C.10. *The strict 2-functor $\mathcal{V}^T\text{-Func}(\mathcal{C},-)\colon \mathcal{V}^T\text{-Cat}\to \mathrm{CAT}$ and the evaluation at the universal object $H'_{\mathcal{C}}$ yield an isomorphism of categories $\mathcal{V}^T\text{-Func}(\mathcal{C},\mathcal{D})\to \mathrm{EFunc}'(\mathcal{C},\mathcal{D})$ for any $\mathcal{V}^T$-enriched category $\mathcal{D}$.*

Proof. The result follows from the adjointness between the forgetful functor $U^T\colon \mathcal{V}^T\to\mathcal{V}$ from the Eilenberg–Moore category $\mathcal{V}^T$ to $\mathcal{V}$ and its left adjoint sending any object A of $\mathcal{V}$ to the free T-algebra with carrier $T(A)$ and structure morphism $\mu_A\colon T(T(A))\to T(A)$. □

C.4 Properties of Derived Adjunctions

Let

$$\begin{tikzcd} \mathcal{B} \arrow[r, bend right, "F"'] \arrow[r, phantom, "\top"] & \mathcal{C} \arrow[l, bend right, "G"'] \end{tikzcd}$$

be an adjunction of abelian categories. Moreover, we assume that the left adjoint F is exact (or equivalently left exact) and that $\mathcal{C}$ has enough injectives.

Lemma C.11. *For any object $X\in\mathcal{B}$ the derived unit*

$$\eta_X^{\mathrm{D}^+(\mathcal{B})}\colon X\xrightarrow{\eta_X}(G\circ F)(X)\to(RG\circ F)(X)$$

is a quasi-isomorphism, and hence an isomorphism of the derived category $\mathrm{D}^+(\mathcal{B})$, iff the ordinary unit η_X is an isomorphism and $F(X)$ is G-acyclic, i.e. $\big(R^kG\circ F\big)(X)=0$ for any $k\neq 0$.

Proof. This follows from 2-out-of-3 for quasi-isomorphisms and the fact that the coaugmentation $G(Y)\to RG(Y)$ for Y an object of $\mathcal{C}$ is a quasi-isomorphism iff Y is G-acyclic. □

Lemma C.12. *Let $H\colon\mathcal{B}\to\mathcal{A}$ be a left exact functor that has a right derived functor $RH\colon D^+(\mathcal{B})\to D^+(\mathcal{A})$. Then the derived functors RH and RG compose as $R(H\circ G)\cong RH\circ RG$.*

Lemma C.13. *If the adjunction $\mathrm{D}^+(F)=LF\dashv RG$ is coreflective, then the essential image of $\mathrm{D}^+(F)$ is the full subcategory $\mathrm{D}^+_{F(\mathcal{B})}(\mathcal{C})$ of complexes whose cohomology objects are in the essential image of F. In particular, $\mathrm{D}^+_{F(\mathcal{B})}(\mathcal{C})$ is a triangulated subcategory of $\mathrm{D}^+(\mathcal{C})$.*

D Properties of Sheaf Operations

Let X be a topological space and let $\mathrm{Sh}(X)$ be the category of sheaves on X with values in the category of abelian groups. We write $D^+(X) \coloneqq D^+(\mathrm{Sh}(X))$ for the bounded below derived category of $\mathrm{Sh}(X)$.

Lemma D.1. *Let $i\colon U \hookrightarrow X$ and $j\colon A \hookrightarrow X$ be inclusions with U open and $A \cup U = X$. If F is a sheaf on X, then the naturally induced map on kernels*

$$\begin{tikzcd}
0 \arrow[r] & (\ker \circ \eta^j)_F \arrow[r] \arrow[d, dashed] & F \arrow[r, "\eta^j_F"] \arrow[d, "\eta^i_F"'] & j_*j^{-1}F \arrow[d, "(j_*\circ j^{-1}\circ\eta^i)_F"] \\
0 \arrow[r] & (\ker \circ \eta^j \circ i_* \circ i^{-1})_F \arrow[r] & i_*i^{-1}F \arrow[r, "(\eta^j\circ i_*\circ i^{-1})_F"'] & j_*j^{-1}i_*i^{-1}F
\end{tikzcd}$$

is an isomorphism, where $\eta^i\colon \mathrm{id}_{\mathrm{Sh}(X)} \to i_*i^{-1}$ and η^j denote the units of the adjunctions $i^{-1} \dashv i_*$ and $j^{-1} \dashv j_*$ respectively.

Lemma D.2. *Let U be an open subset of X, let $C \coloneqq X \setminus U$, let $i\colon U \hookrightarrow X$ be the inclusion, and let $\mathrm{Sh}(X, C)$ be the full subcategory of sheaves vanishing on C. We write $(-)_U\colon \mathrm{Sh}(X) \to \mathrm{Sh}(X, C)$, $F \mapsto F_U$ for the corresponding functor defined in [KS90, Page 93]. Then we have $i_! = (-)_U \circ i_*$ and moreover, the composition of adjunctions*

$$\begin{tikzcd}
\mathrm{Sh}(X,C) \arrow[r, bend right] \arrow[r, phantom, "\top"] & \mathrm{Sh}(X) \arrow[l, bend right, "(-)_U"'] \arrow[r, bend right, "i^{-1}"'] \arrow[r, phantom, "\top"] & \mathrm{Sh}(U) \arrow[l, bend right, "i_*"']
\end{tikzcd}$$

yields an exact adjoint equivalence:

$$\begin{tikzcd}
\mathrm{Sh}(X,C) \arrow[r, bend right, "i^{-1}"'] \arrow[r, phantom, "\top"] & \mathrm{Sh}(U). \arrow[l, bend right, "i_!"']
\end{tikzcd}$$

Now let $U, V \subseteq X$ be open subsets, let $i\colon U \hookrightarrow X$ be the inclusion, let $A \subseteq X$ be a closed subset, let $Z \coloneqq V \cap A$, and suppose that $Z \subseteq U$.

Lemma D.3. *The commutative square*

$$\begin{tikzcd}
\mathrm{Sh}(U) \arrow[r, "i_*"] \arrow[d, "\Gamma_Z(U;-)"'] & \mathrm{Sh}(X) \arrow[d, "\Gamma_Z(X;-)"] \\
\mathrm{Ab} \arrow[r, equal] & \mathrm{Ab}
\end{tikzcd}$$

satisfies the Beck–Chevalley condition as defined in Definition C.1, where Ab *is the category of abelian groups.*

Proof. Let F be a sheaf on X. Then we have the composition of isomorphisms

$$\begin{aligned}\Gamma_Z(X;F) &\cong \Gamma_Z(V;F)\\ &\cong \Gamma_Z(U\cap V;F)\\ &= \Gamma_Z(U\cap V;F|_U)\\ &\cong \Gamma_Z(U\cap V;i^{-1}F)\\ &\cong \Gamma_Z(U;i^{-1}F),\end{aligned}$$

which is the mate of the identity natural transformation. □

This lemma has the following corollary.

Corollary D.4. *The square diagram*

$$\begin{tikzcd} D^+(U) \arrow[r, "Ri_*"] \arrow[d, "H_Z(U;-)"'] \arrow[dr, phantom, "\cong"] & D^+(X) \arrow[d, "H_Z(X;-)"] \\ \mathrm{Ab} \arrow[r, equal] & \mathrm{Ab} \end{tikzcd}$$

satisfies the Beck–Chevalley condition.

Now let $\mathrm{Sh}(X)$ be the category of $\mathbb{F}$-linear sheaves on X for some fixed field $\mathbb{F}$, let F be an object of $D^+(X) \coloneqq D^+(\mathrm{Sh}(X))$, let $n\in\mathbb{Z}$, let $x\in X$ be a point of X, and let S_x be the skyscraper sheaf at x. We consider the map

$$\mathrm{Hom}_{D^+(X)}(F, S_x[-n]) \to \left(\varinjlim_{U\ni x} H^n(U;F)\right)^* \tag{D.1}$$

which takes any homomorphism $\psi\colon F\to S_x[-n]$ to the family of maps $\{H^n(U;\psi)\colon H^n(U;F)\to\mathbb{F}\}_{U\ni x}$ and then to the naturally induced map of type $\varinjlim_{U\ni x} H^n(U;F)\to\mathbb{F}$.

Lemma D.5. *The map* (D.1) *is an isomorphism.*

Proof. As S_x is injective, as $H^n(U;S_x[-n])\cong\{0\}$ for all opens U excluding x, and by the exactness of the dual space functor (or the UCT for cohomology) the map (D.1) is an isomorphism. □

D.1 Mayer–Vietoris Sequences for Local Sheaf Cohomology

In addition to the original Mayer–Vietoris sequence computing the homology of a union of open subsets, there is a generalization computing the relative homology of a component-wise union of pairs of open subsets by [tom08, Theorem 10.7.7]. Here we provide a counterpart for local sheaf cohomology. To this end, let X be a topological space and let (X_i, A_i) for $i = 1, 2$ be pairs of open subsets of X in the sense that $A_i \subseteq X_i$ for $i = 1, 2$. Moreover, let

$$\begin{aligned} X_0 &\coloneqq X_1 \cap X_2, & A_0 &\coloneqq A_1 \cap A_2, \\ X_3 &\coloneqq X_1 \cup X_2, \quad \text{and} & A_3 &\coloneqq A_1 \cup A_2. \end{aligned}$$

Lemma D.6. *If F is a flabby sheaf on X, then the commutative diagram*

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \Gamma_{\overline{A}_3}(X_3; F) \arrow[r] \arrow[d] & \Gamma_{\overline{A}_1}(X_1; F) \oplus \Gamma_{\overline{A}_2}(X_2; F) \arrow[r, "(1\ -1)"] \arrow[d] & \Gamma_{\overline{A}_0}(X_0; F) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Gamma(X_3; F) \arrow[r] \arrow[d] & \Gamma(X_1; F) \oplus \Gamma(X_2; F) \arrow[r, "(1\ -1)"] \arrow[d] & \Gamma(X_0; F) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Gamma(A_3; F) \arrow[r] \arrow[d] & \Gamma(A_1; F) \oplus \Gamma(A_2; F) \arrow[r, "(1\ -1)"] \arrow[d] & \Gamma(A_0; F) \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

has exact rows and columns, where $\overline{A}_i$ is the complement $X \setminus A_i$ of A_i in X for $i = 0, 1, 2, 3$.

Proof. The columns are exact as F is a flabby sheaf and by the definition [KS90, (2.3.14)] of $\Gamma_{\overline{A}_i}(X_i; F)$ for $i = 0, 1, 2, 3$. The two bottom rows are exact by the sheaf condition and as F is a flabby sheaf. By the nine lemma the top row is exact as well. $\square$

Corollary D.7. *If G is a complex of flabby sheaves on X, then we have the short exact sequence*

$$0 \longrightarrow \Gamma_{\overline{A}_3}(X_3; G) \longrightarrow \Gamma_{\overline{A}_1}(X_1; G) \oplus \Gamma_{\overline{A}_2}(X_2; G) \xrightarrow{(1\ -1)} \Gamma_{\overline{A}_0}(X_0; G) \longrightarrow 0$$

of cochain complexes.

While Corollary D.7 is the result of this Appendix D.1 that we use in the main part (Section 8.2 specifically), we include the following Proposition D.8 to illustrate how Corollary D.7 relates to Mayer–Vietoris sequences for local sheaf cohomology.

Proposition D.8 (Mayer–Vietoris for Local Sheaf Cohomology)**.** *If F is an object of*

$D^+(X)$, then there is a long exact sequence

$$
\begin{aligned}
0 &\longrightarrow H^n_{\overline{A}_3}(X_3;F) \longrightarrow H^n_{\overline{A}_1}(X_1;F) \oplus H^n_{\overline{A}_2}(X_2;F) \longrightarrow H^n_{\overline{A}_0}(X_0;F) \longrightarrow \\
&\longrightarrow H^{n+1}_{\overline{A}_3}(X_3;F) \longrightarrow H^{n+1}_{\overline{A}_1}(X_1;F) \oplus H^{n+1}_{\overline{A}_2}(X_2;F) \longrightarrow H^{n+1}_{\overline{A}_0}(X_0;F) \longrightarrow \\
&\longrightarrow H^{n+2}_{\overline{A}_3}(X_3;F) \longrightarrow H^{n+2}_{\overline{A}_1}(X_1;F) \oplus H^{n+2}_{\overline{A}_2}(X_2;F) \longrightarrow H^{n+2}_{\overline{A}_0}(X_0;F) \longrightarrow \\
&\longrightarrow H^{n+3}_{\overline{A}_3}(X_3;F) \longrightarrow \cdots
\end{aligned}
$$

for some $n \in \mathbb{Z}$.

Proof. This follows directly from Corollary D.7 and the zig-zag lemma. □

E Projectives and Abelianization

E.1 Projective Covers

The following is taken from [Kra15, Section 3]. Let $\mathcal{C}$ be an abelian category.

Definition E.1 (Essential Epimorphism)**.** An epimorphism $\psi\colon Y \to Z$ is *essential*, if any morphism $\varphi\colon X \to Y$ is an epimorphism, provided that the composite $\psi \circ \varphi$ is an epimorphism.

Lemma E.2 ([Kra15, Lemma 3.1])**.** *Let $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ be epimorphisms. Then $\psi \circ \varphi\colon X \to Z$ is essential iff both φ and ψ are essential.*

Lemma E.3. *If $\mathcal{C}$ is a category of presheaves taking values in $\mathrm{Vect}_{\mathbb{F}}$, then an epimorphism of presheaves $\psi\colon Y \to Z$ is essential iff any subpresheaf $X \hookrightarrow Y$ with $X + \ker \psi = Y$ is equal to Y.*

Definition E.4 (Projective Cover)**.** An epimorphism $\varphi\colon P \to X$ is *a projective cover* of X if P is projective and φ is essential.

Lemma E.5. *Any projective cover $\varphi\colon P \to Q$ of a projective object Q is an isomorphism.*

Proof. The identity $\mathrm{id}_Q\colon Q \to Q$ is a projective cover as well and so φ is an isomorphism by [Kra15, Corollary 3.5]. □

E.2 Abelianization of Triangulated Categories

The *abelianization of a triangulated category* is a "universal construction" due to [Fre66, Ver96]. In this Appendix E.2 we describe a construction of the abelianization by [Kra07, Section 4.2] based on the theory of *coherent functors* by [Aus66]. To this end, we start by introducing a few notions and notations.

Definition E.6. For a full subcategory $\mathcal{J}$ of an abelian category $\mathcal{C}$ and an object X of $\mathcal{C}$ a *$\mathcal{J}$-presentation* of X is an exact sequence of the form

$$Q \to P \to X \to 0$$

with P and Q objects of $\mathcal{J}$. We say that X is *$\mathcal{J}$-presentable* if there is an $\mathcal{J}$-presentation. Moreover, we write $\mathrm{pres}(\mathcal{J})$ for the full subcategory of $\mathcal{J}$-presentable objects in $\mathcal{C}$. Furthermore, if $\mathcal{J}$ is the image of a fully faithful functor $F\colon \mathcal{A} \to \mathcal{C}$, then we also write *$F$-presentation*, *$F$-presentable*, and $\mathrm{pres}(F)$ in place of *$\mathcal{J}$-presentation*, *$\mathcal{J}$-presentable*, and $\mathrm{pres}(\mathcal{J})$ respectively.

The reason we define this notion of presentability in Definition E.6 for a full subcategory $\mathcal{J} \subseteq \mathcal{C}$ rather than a subclass of the objects of $\mathcal{C}$ is notational convenience. Similarly, this choice also entails the condition that $F\colon \mathcal{A} \to \mathcal{C}$ be fully faithful in Definition E.6, which is not essential.

E.2.1 Closure Under Cokernels

For this entire subsection, suppose that $\mathcal{J}$ is a full subcategory of projectives of an abelian category $\mathcal{C}$.

Lemma E.7. *The cokernel of a morphism between $\mathcal{J}$-presentable objects is again $\mathcal{J}$-presentable.*

Proof. Let

$$X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \to 0 \tag{E.1}$$

be an exact sequence with X and Y being $\mathcal{J}$-presentable. We choose presentations

$$\begin{aligned} &P_1 \to P_0 \to X \to 0 \\ \text{and}\quad &Q_1 \xrightarrow{\delta} Q_0 \xrightarrow{\epsilon} Y \to 0 \end{aligned}$$

as in Definition E.6. As P_0 is projective in $\mathcal{C}$, there is a morphism $\tilde{\varphi}\colon P_0 \to Q_0$ such that the square

$$\begin{tikzcd} P_0 \arrow[r, dashed, "\tilde{\varphi}"] \arrow[d] & Q_0 \arrow[d] \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

commutes. We aim to show that the sequence

$$P_0 \oplus Q_1 \xrightarrow{(\tilde{\varphi}\ \ \delta)} Q_0 \xrightarrow{\psi\circ\epsilon} Z \to 0$$

is exact. To this end, let $\xi\colon Q_0 \to W$ be a morphism such that

$$\xi \circ (\tilde{\varphi}\ \ \delta) = 0. \tag{E.2}$$

We consider the commutative diagram

$$\begin{tikzcd} & Q_1 \arrow[d, "\delta"] & P_0 \oplus Q_1 \arrow[d, "(\tilde{\varphi}\ \ \delta)"] & \\ P_0 \arrow[r, "\tilde{\varphi}"] \arrow[d, two heads] & Q_0 \arrow[r, equal] \arrow[d, two heads, "\epsilon"] & Q_0 \arrow[d, "\psi\circ\epsilon"] \arrow[ddr, bend left, "\xi"] & \\ X \arrow[r, "\varphi"'] \arrow[drrr, dashed, bend right, "0"'] & Y \arrow[r, "\psi"'] \arrow[drr, dashed, bend right, "\xi'"] & Z \arrow[dr, dashed, "\xi''"] & \\ & & & W. \end{tikzcd} \tag{E.3}$$

Now (E.2) implies in particular that $\xi \circ \delta = 0$, hence there is a unique natural transformation $\xi' \colon Y \to W$ such that
$$\xi' \circ \epsilon = \xi \tag{E.4}$$
as indicated in (E.3). Now
$$\xi' \circ \epsilon \circ \tilde{\varphi} = \xi \circ \tilde{\varphi} = 0$$
by (E.2). Moreover, as the vertical arrow on the left-hand side of (E.3) is an epimorphism $\xi' \circ \varphi = 0$ as well. By the exactness of (E.1) there is a unique natural transformation $\xi'' \colon Z \to W$ such that $\xi'' \circ \psi = \xi'$ as indicated in (E.3). In conjunction with (E.4) we obtain $\xi'' \circ \psi \circ \epsilon = \xi' \circ \epsilon = \xi$. ☐

Corollary E.8. *A morphism of $\mathcal{J}$-presentable objects is an epimorphism in* $\operatorname{pres}(\mathcal{J})$ *iff it is an epimorphism in $\mathcal{C}$.*

This corollary has yet another corollary.

Corollary E.9. *Any object of $\mathcal{J}$ is projective in* $\operatorname{pres}(\mathcal{J})$*.*

Proof. By our assumptions on $\mathcal{J}$, any object P of $\mathcal{J}$ is projective in $\mathcal{C}$. Moreover, as any epimorphism of $\operatorname{pres}(\mathcal{J})$ is an epimorphism of $\mathcal{C}$ by Corollary E.8, P has the lifting property of projective objects with respect to all epimorphisms of $\operatorname{pres}(\mathcal{J})$ in particular. ☐

Corollary E.10. *Any projective cover $\varphi \colon P \to X$ in $\mathcal{C}$ with X being $\mathcal{J}$-presentable and P an object of $\mathcal{J}$ is a projective cover in* $\operatorname{pres}(\mathcal{J})$*.*

Proof. By Corollary E.9 the object P is projective. Moreover, by Corollary E.8 the morphism $\varphi \colon P \to X$ is an essential epimorphism of $\operatorname{pres}(\mathcal{J})$ as well. ☐

Corollary E.11. *If any $\mathcal{J}$-presentable object X of $\mathcal{C}$ admits a projective cover $P \to X$ by an object P of $\mathcal{J}$, then $\mathcal{J}$ is the subcategory of projectives in* $\operatorname{pres}(\mathcal{J})$*.*

Proof. Suppose Q is projective in $\operatorname{pres}(\mathcal{J})$ and let $\varphi \colon P \to Q$ be a projective cover with P in $\mathcal{J}$. Then φ is a projective cover in $\operatorname{pres}(\mathcal{J})$ by Corollary E.10, hence φ is an isomorphism by Lemma E.5. ☐

Remark E.12. As it turns out, the inclusion of full categories $\mathcal{J} \hookrightarrow \operatorname{pres}(\mathcal{J})$ *exhibits* $\operatorname{pres}(\mathcal{J})$ as a cocompletion of $\mathcal{J}$ by finite colimits in an additively free way. We also note that it is possible to obtain the results of Section 10.1 using this idea in conjunction with the results by [Kra07, Section 4.2]. However, instead of making this idea precise and providing the corresponding proofs, we carry out a more elementary line of reasoning and we refer to [Fre66] for more details on this approach to abelianization.

E.2.2 Coherent Presheaves as an Abelianization

We denote the category of abelian groups by Ab and for two additive categories $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{A}$ essentially small we write $\mathrm{Add}(\mathcal{A}, \mathcal{B})$ for the category of additive functors $\mathcal{A} \to \mathcal{B}$. Moreover, we write

$$よ\colon \mathcal{A} \to \mathrm{Add}(\mathcal{A}^\circ, \mathrm{Ab}),\ A \mapsto \mathrm{Hom}_{\mathcal{A}}(-, A)$$

for the additive Yoneda embedding of an additive category $\mathcal{A}$ into the abelian category of additive presheaves $\mathcal{A}^\circ \to \mathrm{Ab}$. With this we may specialize Definition E.6 as follows.

Definition E.13. For an additive category $\mathcal{A}$ an additive presheaf $F\colon \mathcal{A}^\circ \to \mathrm{Ab}$ is *coherent* if it is よ-presentable in the sense of Definition E.6, i.e. there is an exact sequence of the form

$$\mathrm{Hom}_{\mathcal{A}}(-, B) \to \mathrm{Hom}_{\mathcal{A}}(-, A) \to F \to 0$$

for some objects A and B of $\mathcal{A}$. We denote the full subcategory of coherent presheaves $\mathcal{A}^\circ \to \mathrm{Ab}$ by $\mathrm{coh}(\mathcal{A})$ and the *suspended Yoneda embedding* by

$$\overline{よ}\colon \mathcal{A} \to \mathrm{coh}(\mathcal{A}),\ A \mapsto \mathrm{Hom}_{\mathcal{A}}(-, A).$$

Now for a triangulated category $\mathcal{T}$ and for abelian categories $\mathcal{C}$ and $\mathcal{D}$ with $\mathcal{T}$ and $\mathcal{C}$ essentially small we write $\mathrm{Cohom}(\mathcal{T}, \mathcal{D})$ and $\mathrm{Ex}(\mathcal{C}, \mathcal{D})$ for the categories of cohomological and exact functors, respectively.

Definition E.14 (Abelianization, [Fre66, Ver96])**.** Suppose $F\colon \mathcal{T} \to \mathcal{C}$ is a cohomological functor with $\mathcal{T}$ and $\mathcal{C}$ essentially small. Then F *exhibits* $\mathcal{C}$ *as an abelianization* of $\mathcal{T}$ if the *pullback functor*

$$F^!_{\mathcal{D}}\colon \mathrm{Ex}(\mathcal{C}, \mathcal{D}) \to \mathrm{Cohom}(\mathcal{T}, \mathcal{D}),\ G \mapsto G \circ F$$

is an equivalence of categories for any abelian category $\mathcal{D}$.

Theorem E.15 ([Kra07, Section 4.2])**.** *For an essentially small triangulated category* $\mathcal{T}$ *the category of coherent presheaves* $\mathrm{coh}(\mathcal{T})$ *is an abelian subcategory of the presheaf category* $\mathrm{Add}(\mathcal{T}^\circ, \mathrm{Ab})$ *and the suspended Yoneda embedding*

$$\overline{よ}\colon \mathcal{T} \to \mathrm{coh}(\mathcal{T}),\ X \mapsto \mathrm{Hom}_{\mathcal{T}}(-, X)$$

exhibits $\mathrm{coh}(\mathcal{T})$ *as an abelianization of* $\mathcal{T}$*.*

Lemma E.16. *Suppose we have a cohomological functor* $F\colon \mathcal{T} \to \mathcal{C}$ *as well as an exact equivalence* $G\colon \mathrm{coh}(\mathcal{T}) \to \mathcal{C}$ *with* $\mathcal{T}$ *and* $\mathcal{C}$ *essentially small and such that the diagram of additive functors and categories*

$$\begin{tikzcd} \mathcal{T} \arrow[d, "\overline{よ}"'] \arrow[dr, "F"] & \\ \mathrm{coh}(\mathcal{T}) \arrow[r, "G"'] & \mathcal{C} \end{tikzcd}$$

commutes. Then $F\colon \mathcal{T} \to \mathcal{C}$ *exhibits* $\mathcal{C}$ *as an abelianization of* $\mathcal{T}$*.*

Proof. Let $\mathcal{D}$ be an abelian category and let

$$G^!_{\mathcal{D}} \colon \mathrm{Ex}(\mathcal{C}, \mathcal{D}) \to \mathrm{Ex}(\mathrm{coh}(\mathcal{T}), \mathcal{D})$$

be the pullback of exact functors along the exact equivalence $G \colon \mathrm{coh}(\mathcal{T}) \to \mathcal{C}$. Then the pullback functor $G^!_{\mathcal{D}}$ itself is an equivalence of categories. Moreover, as the diagram of pullback functors

$$\begin{tikzcd}
\mathrm{Cohom}(\mathcal{T}, \mathcal{D}) & \\
\mathrm{Ex}(\mathrm{coh}(\mathcal{T}), \mathcal{D}) \arrow[u, "\overline{\text{よ}}^{!}_{\mathcal{D}}"] & \mathrm{Ex}(\mathcal{C}, \mathcal{D}) \arrow[l, "G^!_{\mathcal{D}}"] \arrow[ul, "F^!_{\mathcal{D}}"']
\end{tikzcd}$$

commutes and as $\overline{\text{よ}}^{!}_{\mathcal{D}} \colon \mathrm{Ex}(\mathcal{C}, \mathcal{D}) \to \mathrm{Cohom}(\mathcal{T}, \mathcal{D})$ is an equivalence by Theorem E.15 due to [Kra07, Section 4.2], the pullback functor $F^!_{\mathcal{D}} \colon \mathrm{Ex}(\mathcal{C}, \mathcal{D}) \to \mathrm{Cohom}(\mathcal{T}, \mathcal{D})$ is an equivalence as well. □

E.2.3 Coherent Linear Structure

In Definition E.13 we defined the additive category of coherent presheaves $\mathrm{coh}(\mathcal{A})$ on an essentially small additive category $\mathcal{A}$ as it can be used to model the abelianization of $\mathcal{A}$ when $\mathcal{A}$ is triangulated using the construction by [Kra07, Section 4.2]. Now in our application of the abelianization we work with an $\mathbb{F}$-linear triangulated category and also another subcategory of $\mathbb{F}$-linear presheaves as an abelianization, where $\mathbb{F}$ is a field although the content of this Appendix E.2.3 only requires that $\mathbb{F}$ is a ring. For this reason, it will be useful to describe $\mathrm{coh}(\mathcal{A})$ as a subcategory of $\mathbb{F}$-linear presheaves when $\mathcal{A}$ is $\mathbb{F}$-linear. To this end, let $\mathcal{A}$ be an essentially small $\mathbb{F}$-linear category and let $\mathcal{P}$ be the category of $\mathbb{F}$-linear presheaves $\mathcal{A}^\circ \to \mathrm{Vect}_{\mathbb{F}}$. Then the forgetful functor $\mathrm{Vect}_{\mathbb{F}} \to \mathrm{Ab}$ yields a *pointwise forgetful functor* $\mathcal{P} \to \mathrm{Add}(\mathcal{A}^\circ, \mathrm{Ab})$ on presheaf categories by post-composition.

Lemma E.17. *The pointwise forgetful functor* $\mathcal{P} \to \mathrm{Add}(\mathcal{A}^\circ, \mathrm{Ab})$ *strictly creates all limits and colimits in the sense of [ML98, Section V.1].*

Proof. It is well known that the forgetful functor $U \colon \mathrm{Vect}_{\mathbb{F}} \to \mathrm{Ab}$ is monadic, hence it strictly creates all limits by [ML98, Exercise VI.2.2]. As $U \colon \mathrm{Vect}_{\mathbb{F}} \to \mathrm{Ab}$ also has a right adjoint provided by the co-extension of scalars assigning to each abelian group A the vector space $\mathrm{Hom}(\mathbb{F}, A)$ of group homomorphisms $\mathbb{F} \to A$, the functor U is comonadic as well. So U strictly creates all colimits of $\mathrm{Vect}_{\mathbb{F}}$ by [ML98, Exercise VI.2.2] and duality. Moreover, as limits and colimits of presheaves are computed pointwise, the pointwise forgetful functor $\mathcal{P} \to \mathrm{Add}(\mathcal{A}^\circ, \mathrm{Ab})$ strictly creates all limits and colimits as well. □

Now let

$$\text{よ}_{\mathbb{F}} \colon \mathcal{A} \to \mathcal{P},\ F \mapsto \mathrm{Hom}_{\mathcal{A}}(-, F)$$

be the $\mathbb{F}$-linear Yoneda embedding. Then we may consider the full subcategory of $よ_{\mathbb{F}}$-presentable presheaves $\operatorname{pres}(よ_{\mathbb{F}}) \subset \mathcal{P}$. Clearly, the pointwise forgetful functor $\mathcal{P} \to \operatorname{Add}(\mathcal{A}^\circ, \mathrm{Ab})$ maps any $よ_{\mathbb{F}}$-presentable presheaf $\mathcal{A}^\circ \to \mathrm{Vect}_{\mathbb{F}}$ to a $よ$-presentable presheaf $\mathcal{A}^\circ \to \mathrm{Ab}$, i.e. a coherent presheaf in the sense of Definition E.13. As the following Lemma E.18 shows, this correspondence is one-to-one.

Lemma E.18. *The pointwise forgetful functor* $\mathcal{P} \to \operatorname{Add}(\mathcal{A}^\circ, \mathrm{Ab})$ *restricts to an additive isomorphism of categories* $\operatorname{pres}(よ_{\mathbb{F}}) \to \operatorname{pres}(よ) = \operatorname{coh}(\mathcal{A})$.

Proof. By Lemma E.17 the pointwise forgetful functor $\mathcal{P} \to \operatorname{Add}(\mathcal{A}^\circ, \mathrm{Ab})$ strictly creates all colimits in the sense of [ML98, Section V.1], hence its restriction $\hat{U} \colon \operatorname{pres}(よ_{\mathbb{F}}) \to \operatorname{pres}(よ)$ is a bijection on the corresponding classes of objects. Thus, it remains to be shown that $\hat{U}$ is fully faithful. Clearly, the pointwise forgetful functor $\mathcal{P} \to \operatorname{Add}(\mathcal{A}^\circ, \mathrm{Ab})$ is faithful, so $\hat{U} \colon \operatorname{pres}(よ_{\mathbb{F}}) \to \operatorname{pres}(よ)$ is faithful as well. Now suppose we have two $よ_{\mathbb{F}}$-presentations

$$\begin{aligned} &\operatorname{Hom}_{\mathcal{A}}(-, X_1) \to \operatorname{Hom}_{\mathcal{A}}(-, X_0) \to F \to 0 \\ \text{and} \quad &\operatorname{Hom}_{\mathcal{A}}(-, Y_1) \to \operatorname{Hom}_{\mathcal{A}}(-, Y_0) \to G \to 0 \end{aligned} \tag{E.5}$$

as well as a natural transformation $\eta \colon F \to G$ that is a group homomorphism in each component but a priori not $\mathbb{F}$-linear. By the exactness of (E.5) and the Yoneda lemma there are morphisms $\varphi_0 \colon X_0 \to Y_0$ and $\varphi_1 \colon X_1 \to Y_1$ such that the diagram

$$\begin{tikzcd} \operatorname{Hom}_{\mathcal{A}}(-, X_1) \arrow[r, dashed, "\operatorname{Hom}_{\mathcal{A}}(-,\varphi_1)"] \arrow[d] & \operatorname{Hom}_{\mathcal{A}}(-, Y_1) \arrow[d] \\ \operatorname{Hom}_{\mathcal{A}}(-, X_0) \arrow[r, dashed, "\operatorname{Hom}_{\mathcal{A}}(-,\varphi_0)"] \arrow[d] & \operatorname{Hom}_{\mathcal{A}}(-, Y_0) \arrow[d] \\ F \arrow[r, "\eta"'] & G \end{tikzcd}$$

commutes. Moreover, by the universal property of cokernels in $\mathcal{P}$ there is a pointwise $\mathbb{F}$-linear natural transformation $\tilde{\eta} \colon F \to G$ such that the diagram

$$\begin{tikzcd} \operatorname{Hom}_{\mathcal{A}}(-, X_1) \arrow[r, "\operatorname{Hom}_{\mathcal{A}}(-,\varphi_1)"] \arrow[d] & \operatorname{Hom}_{\mathcal{A}}(-, Y_1) \arrow[d] \\ \operatorname{Hom}_{\mathcal{A}}(-, X_0) \arrow[r, "\operatorname{Hom}_{\mathcal{A}}(-,\varphi_0)"] \arrow[d] & \operatorname{Hom}_{\mathcal{A}}(-, Y_0) \arrow[d] \\ F \arrow[r, dashed, "\tilde{\eta}"'] & G \end{tikzcd}$$

commutes. Furthermore, by the uniqueness part of the universal property of cokernels in $\operatorname{Add}(\mathcal{A}^\circ, \mathrm{Ab})$ we have $\eta = \tilde{\eta}$. $\square$

The following subtle Lemma E.19 and its proof are in some sense orthogonal to the previous Lemma E.18.

Lemma E.19. *If $\mathcal{A}$ is triangulated, then* $\operatorname{pres}(\text{よ}_{\mathbb{F}})$ *is an abelian subcategory of $\mathcal{P}$.*

Proof. As $\operatorname{pres}(\text{よ}_{\mathbb{F}})$ is a full subcategory of $\mathcal{P}$, it suffices to show that all kernels and cokernels of homomorphisms in $\operatorname{pres}(\text{よ}_{\mathbb{F}})$ are again $\text{よ}_{\mathbb{F}}$-presentable. Moreover, any cokernel of a homomorphism in $\operatorname{pres}(\text{よ}_{\mathbb{F}})$ is $\text{よ}_{\mathbb{F}}$-presentable by Lemma E.7 and the Yoneda lemma. Thus, merely the $\text{よ}_{\mathbb{F}}$-presentability of kernels remains to be shown. To this end, we consider an exact sequence

$$0 \to K \to F \to G$$

in $\mathcal{P}$ with F and G being $\text{よ}_{\mathbb{F}}$-presentable. We have to show that K is $\text{よ}_{\mathbb{F}}$-presentable as well. By Theorem E.15 the presheaf K is よ-presentable (i.e. coherent), hence there is よ-presentation

$$\operatorname{Hom}_{\mathcal{A}}(-,Q) \to \operatorname{Hom}_{\mathcal{A}}(-,P) \xrightarrow{\theta} K \to 0 \tag{E.6}$$

in $\operatorname{Add}(\mathcal{A}^\circ, \mathrm{Ab})$. By Lemma E.17 each component of $\theta\colon \operatorname{Hom}_{\mathcal{A}}(-,P) \to K$ is $\mathbb{F}$-linear and moreover, the sequence (E.6) is exact in $\mathcal{P}$ as well. $\square$

F Additive Invariants

Split-Additive Invariants. Before we get to *additive* invariants we consider the weaker notion of a *split-additive* invariant.

Definition F.1 (Split-Additive Invariant)**.** A *split-additive invariant* on an additive category $\mathcal{C}$ is an abelian group A together with a map $\zeta\colon \mathrm{Ob}(\mathcal{A}) \to A$ mapping isomorphic objects to identical elements of A and adhering to the equation

$$\zeta(X \oplus Y) = \zeta(X) + \zeta(Y)$$

for any objects X and Y of $\mathcal{C}$. We say that a split-additive invariant $\zeta\colon \mathrm{Ob}(\mathcal{A}) \to A$ is *universal* if for any other split-additive invariant $\zeta'\colon \mathrm{Ob}(\mathcal{A}) \to A'$ there is a unique homomorphism $\varphi\colon A \to A'$ making the diagram

$$\begin{tikzcd} \mathrm{Ob}(\mathcal{A}) \arrow[d, "\zeta"'] \arrow[dr, "\zeta'"] & \\ A \arrow[r, "\varphi"'] & A' \end{tikzcd}$$

commute.

Remark F.2. Universal split-additive invariants are unique up to unique isomorphism and a well known construction of a universal split-additive invariant on an essentially small additive category $\mathcal{C}$ is the *split Grothendieck group* denoted as $K_0^{\oplus}(\mathcal{A})$.

Additive Invariants on Abelian Categories. In the same way that a split-additive invariant applied to a direct sum is the sum of its parts, an additive invariant applied to an arbitrary extension can be computed as the sum of its parts as well.

Definition F.3 (Additive Invariant)**.** An *additive invariant* on an abelian category $\mathcal{C}$ is an abelian group A together with a map $\zeta\colon \mathrm{Ob}(\mathcal{C}) \to A$ such that

$$\zeta(Y) = \zeta(X) + \zeta(Z)$$

whenever there is a short exact sequence

$$0 \to X \to Y \to Z \to 0.$$

Universality of additive invariants is defined analogously to the split-additive case Definition F.3.

Considering isomorphisms of abelian categories have vanishing kernels and cokernels, additive invariants identify isomorphic objects.

Remark F.4. For an essentially small abelian category $\mathcal{C}$ the *Grothendieck group* denoted as $K_0(\mathcal{C})$ is a well known construction of a universal additive invariant. Moreover, the notions introduced by Definitions F.1 and F.3 are specializations of a more general notion, namely *additive invariants* on *Quillen exact categories*, these are additive categories endowed with an *exact structure*. In this terminology, an invariant on an additive category $\mathcal{C}$ is *split-additive* iff it is additive with respect to the smallest exact structure on $\mathcal{C}$, which is the *split-exact structure*. Similarly, an invariant on an abelian category $\mathcal{C}$ is additive iff it is additive with respect to the largest exact structure on $\mathcal{C}$. So we may view Definitions F.1 F.3 as two ends to a spectrum. We also note that an additive category, that is the subcategory of projectives of some abelian Frobenius category, has only one exact structure, the split-exact structure. So in that case there is no ambiguity between these two notions of additivity. However, ambiguities still arise from the notion of an additive invariant on a triangulated category as in Definition F.6.

Lemma F.5. *Let $\zeta\colon \mathrm{Ob}(\mathcal{C}) \to A$ be an additive invariant on an abelian category $\mathcal{C}$ and let*

$$\ldots \xrightarrow{\delta_{n+1}} P_n \xrightarrow{\delta_n} \ldots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \to X \to 0$$

be a resolution of an object X of $\mathcal{C}$ by some other objects P_n, $n \in \mathbb{N}_0$ of $\mathcal{C}$ such that the infinite direct sum $\bigoplus_{n=0}^{\infty} P_n$ exists in $\mathcal{C}$. Then we have the equation

$$\zeta\left(\bigoplus_{n=0}^{\infty} P_{2n}\right) = \zeta(X) + \zeta\left(\bigoplus_{n=0}^{\infty} P_{2n+1}\right).$$

Proof. We consider the exact sequence

$$\bigoplus_{n=0}^{\infty} P_{2n+3} \xrightarrow{\varphi_3} \bigoplus_{n=0}^{\infty} P_{2n+2} \xrightarrow{\varphi_2} \bigoplus_{n=0}^{\infty} P_{2n+1} \xrightarrow{\varphi_1} \bigoplus_{n=0}^{\infty} P_{2n}, \tag{F.1}$$

where

$$\varphi_3 \coloneqq \bigoplus_{n=0}^{\infty} \delta_{2n+3}, \quad \varphi_2 \coloneqq \bigoplus_{n=0}^{\infty} \delta_{2n+2}, \quad \text{and} \quad \varphi_1 \coloneqq \bigoplus_{n=0}^{\infty} \delta_{2n+1}.$$

First we note that

$$\operatorname{coker}\varphi_1 \cong \operatorname{coker}\delta_1 \oplus \operatorname{coker}\varphi_3 \cong X \oplus \operatorname{coker}\varphi_3. \tag{F.2}$$

Moreover, by the exactness of (F.1) we have $\operatorname{coker}\varphi_3 \cong \operatorname{Im}\varphi_2 = \ker\varphi_1$. In conjunction with (F.2) we obtain $\operatorname{coker}\varphi_1 \cong X \oplus \ker\varphi_1$ and hence

$$\zeta(\operatorname{coker}\varphi_1) = \zeta(X) + \zeta(\ker\varphi_1).$$

From this equation in turn we obtain

$$\begin{aligned}\zeta\left(\bigoplus_{n=0}^{\infty} P_{2n}\right) &= \zeta(\operatorname{Im}\varphi_1) + \zeta(\operatorname{coker}\varphi_1) \\ &= \zeta(\operatorname{Im}\varphi_1) + \zeta(X) + \zeta(\ker\varphi_1) \\ &= \zeta(X) + \zeta(\ker\varphi_1) + \zeta(\operatorname{Im}\varphi_1) \\ &= \zeta(X) + \zeta\left(\bigoplus_{n=0}^{\infty} P_{2n+1}\right). \end{aligned} \qquad \square$$

Additive Invariants on Triangulated Categories. In this exposition there is no use of additive invariants on triangulated categories. However, it does help us with putting the present work into perspective with the established theory on additive invariants and previous prior work in persistence theory.

Definition F.6. An *additive invariant* on a triangulated category $\mathcal{T}$ is an abelian group A together with a map $\zeta\colon \mathrm{Ob}(\mathcal{T}) \to A$ such that

$$\zeta(Y) = \zeta(X) + \zeta(Z)$$

whenever there is a distinguished triangle

$$X \to Y \to Z \to \Sigma(X).$$

Universality of additive invariants is defined analogously to the split-additive case Definition F.3.

Remark F.7. As in the case of abelian categories, the *Grothendieck group* $K_0(\mathcal{T})$ of an essentially small triangulated category $\mathcal{T}$ is a well known construction of a universal additive invariant. As it turns out, any universal additive invariant on a triangulated category is surjective. Moreover, any additive invariant on an abelian category $\mathcal{C}$ uniquely extends to an additive invariant on the bounded derived category $D^b(\mathcal{C})$ and conversely, any additive invariant on $D^b(\mathcal{C})$ restricts to an additive invariant on $\mathcal{C}$. Similarly, any split-additive invariant on $\mathcal{C}$ uniquely extends to the bounded homotopy category $K^b(\mathcal{C})$ by [Ros15]. So conceptually, passing from an essentially small abelian category $\mathcal{C}$ to its bounded derived category $D^b(\mathcal{C})$ (bounded homotopy category $K^b(\mathcal{C})$) we add more objects to $\mathcal{C}$ so we may represent any element of its (split) Grothendieck group by a complex. As a side effect, any morphism of $D^b(\mathcal{C})$, those of $\mathcal{C}$ in particular, is epic iff it is split. As a result, the only exact structure on $D^b(\mathcal{C})$ is the split-exact structure. Considering that $D^b(\mathcal{C})$ has an abelianization this observation is a special case of the case for projectives of an abelian Frobenius category described in Remark F.4. Thus, the different Grothendieck groups obtained in this way from $\mathcal{C}$ fit into the commutative diagram

$$\begin{tikzcd}
K_0^{\oplus}(\mathcal{C}) \arrow[r, tail] \arrow[d, equal] \arrow[rr, tail, bend left] & K_0^{\oplus}\big(K^b(\mathcal{C})\big) \arrow[r, two heads] \arrow[d, two heads] & K_0^{\oplus}\big(D^b(\mathcal{C})\big) \arrow[d, two heads] \\
K_0^{\oplus}(\mathcal{C}) \arrow[r, "\cong"] \arrow[d, two heads] & K_0\big(K^b(\mathcal{C})\big) \arrow[r, two heads] & K_0\big(D^b(\mathcal{C})\big) \arrow[d, equal] \\
K_0(\mathcal{C}) \arrow[rr, "\cong"] & & K_0\big(D^b(\mathcal{C})\big).
\end{tikzcd} \tag{F.3}$$

Additive Invariants on Strictly Stable Categories. Now suppose that we have a strictly stable abelian (or merely additive) category $\mathcal{C}$ with an exact (or additive) suspension functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$ in the sense of Definition 1.8 as well as a universal additive (or merely split-additive) invariant $\zeta\colon \mathrm{Ob}(\mathcal{C}) \to A$. Then

$$\zeta \circ \Sigma\colon \mathrm{Ob}(\mathcal{C}) \to A,\ X \mapsto \zeta(\Sigma(X))$$

is another additive (or split-additive) invariant on $\mathcal{C}$, hence we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Ob}(\mathcal{C}) \arrow[r, "\Sigma"] \arrow[d, "\zeta"'] & \mathrm{Ob}(\mathcal{C}) \arrow[d, "\zeta"] \\
A \arrow[r, dashed, "x._"'] & A
\end{tikzcd} \tag{F.4}$$

with the lower horizontal arrow an endomorphism of the abelian group A. Moreover, considering the universality of $\zeta\colon \mathrm{Ob}(\mathcal{C}) \to A$ it is easy to see that the lower horizontal arrow in (F.4) is an automorphism of A. Now an abelian group endowed with an automorphism is "the same thing" as a module over the ring of integer Laurent polynomials $\mathbb{Z}[x, x^{-1}]$:

$$\begin{aligned} \mathbb{Z}[x, x^{-1}] \times A &\to A, \\ (x, a) &\mapsto x.a. \end{aligned}$$

Thus, any universal (split-)additive invariant of a strictly stable abelian (additive) category inherits a unique additive $\mathbb{Z}[x, x^{-1}]$-action such that the diagram (F.4) commutes.

Abelian Categorification. In the previous paragraphs we considered additive invariants associated to (strictly stable) abelian or additive categories. Categorification reverses the order in this relation.

Definition F.8 (Abelian Categorification)**.** An *abelian categorification* of a module A over the ring of integer Laurent polynomials $\mathbb{Z}[x, x^{-1}]$ is a strictly stable abelian category $\mathcal{C}$ with exact suspension functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$ as well as a universal additive invariant $\zeta\colon \mathrm{Ob}(\mathcal{C}) \to A$ such that the inherited $\mathbb{Z}[x, x^{-1}]$-action coincides with the $\mathbb{Z}[x, x^{-1}]$-module structure.

This Definition F.8 is consistent with [KMS09, Definition 2.7] and [Maz12, Definition 1.12].