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CMC-Trinoids with Properly Embedded Annular Ends

Philipp Lang

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Vorsitzender: Univ.-Prof. Dr. J. Scheurle

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1. Hon.-Prof. Dr. J. Dorfmeister
2. Univ.-Prof. Dr. T. N. Hoffmann
3. Univ.-Prof. Dr. F. Pedit, Eberhard Karls Universität Tübingen

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Abstract

We consider CMC-trinoids in Euclidian three-space with properly embedded annular ends. Starting with a holomorphic potential $\tilde{\eta}$ and a special solution Ψ to the differential equation $\mathrm{d}\Psi = \Psi\tilde{\eta}$, we characterize all solutions to this differential equation which produce CMC-trinoids with properly embedded annular ends via the loop group method. Moreover, we give a classification of CMC-trinoids with properly embedded annular ends with respect to their symmetry properties in terms of the monodromy matrices of the solution Ψ associated with the trinoid ends.

Contents

1 Introduction

Among the surfaces of constant mean curvature $H \neq 0$, CMC-surfaces for short, only a few subclasses have been classified. The first ones have been the surfaces of revolution among the CMC-surfaces, the Delaunay surfaces. They were found almost 200 years ago [7], and are still of interest, since every properly embedded annular end of a CMC-surface is asymptotically a Delaunay surface [25].

More generally, CMC-immersions of round cylinders into $\mathbb{R}^3$ are fairly well understood. The class of CMC-tori has been investigated extensively using different mathematical techniques [31], [3], [23] and is clearly so far the best investigated one among all CMC-immersions.

All the surface classes mentioned so far have an abelian fundamental group. The simplest non-abelian groups are perhaps those which are free and have only two generators. Thus it seems to be particularly important to understand the *CMC-trinoids*, i.e. CMC-immersions of the thrice-punctured Riemannian sphere $\mathcal{T}_3$ into $\mathbb{R}^3$.

Among the CMC-trinoids $\mathcal{T}_3 \to \mathbb{R}^3$ clearly the embeddings are of particular interest. It seems to be difficult to classify this class of CMC-immersions. However, in a beautiful piece of work, Große-Brauckmann, Kusner and Sullivan have classified the Alexandrov embedded CMC-surfaces $\mathcal{T}_3 \to \mathbb{R}^3$ [21].

In [27] it was shown, however, that there are CMC-trinoids $\mathcal{T}_3 \to \mathbb{R}^3$, which have properly embedded annular ends but are not Alexandrov embedded. Further examples of such surfaces have been given in [17] using the loop group method [15] for a certain class of starting potentials. Naturally, the class of CMC-trinoids with properly embedded annular ends encompasses the class of the (globally) properly embedded trinoids. In this sense, the investigation of CMC-trinoids with properly embedded annular ends seems to be a natural next step for the understanding of all CMC-trinoids.

In [8] it is shown that all CMC-trinoids $\mathcal{T}_3 \to \mathbb{R}^3$ with properly embedded annular ends can be obtained via the loop group method from the potentials of [17]. Based on this result, this thesis provides a classification of all CMC-trinoids which can be obtained via the loop group method from the potentials of [17], and thus in particular a classification of all CMC-trinoids with properly embedded annular ends, in terms of the monodromy matrices associated with the trinoid ends.

We give all possible triples of monodromy matrices associated with the ends of a CMC-trinoid $\mathcal{T}_3 \to \mathbb{R}^3$ which can be obtained from the potentials of [17]. Moreover, we investigate the possible symmetries of a given CMC-trinoid with properly embedded annular ends under Euclidean motions in $\mathbb{R}^3$ and characterize these symmetries in terms of the corresponding monodromy matrices. I.e., we state necessary and sufficient conditions on the monodromy matrices of a given CMC-trinoid $\mathcal{T}_3 \to \mathbb{R}^3$ with properly embedded annular ends, such that the (image of the) given CMC-trinoid is invariant under a specific Euclidean motion in $\mathbb{R}^3$.

In section 2 we review the loop group method from [15] for constructing CMC-immersions from holomorphic potentials. A holomorphic potential $\tilde{\eta}$ is a $\mathrm{sl}(2,\mathbb{C})$-valued differential one-form, which is defined on the universal cover $\tilde{M}$ of a Riemann surface M. Furthermore, $\tilde{\eta}$ involves a loop parameter λ and depends holomorphically on both $z \in \tilde{M}$ and $\lambda \in \mathbb{C}^*$. Given a holomorphic potential $\tilde{\eta}$, the first step of the loop group method consists in solving the differential equation $\mathrm{d}\Psi = \Psi\tilde{\eta}$ for a $\mathrm{SL}(2,\mathbb{C})$-valued mapping Ψ on $\tilde{M}$, satisfying some initial condition $\Psi(z_*) = \Psi_0$. Ψ also depends on λ and the form of this dependence is determined by the initial condition Ψ_0. Assuming that Ψ_0 (and thus Ψ) is defined for all λ from some r-circle $C^{(r)}$, $0 < r \leq 1$, one can proceed with the second step of the loop group method. This involves (for each $z_0 \in \tilde{M}$) an r-Iwasawa decomposition of the λ-dependent loop $\Psi(z_0) : C^{(r)} \to \mathrm{SL}(2,\mathbb{C})$, i.e. a (pointwise) factorization of Ψ into a loop F on $C^{(r)}$, which can be extended holomorphically to the open annulus $r < |\lambda| < \frac{1}{r}$ and is unitary on the unit circle S^1, and a loop B_+ on $C^{(r)}$, which can be extended holomorphically to the disc $|\lambda| < r$, $\Psi = FB_+$. The factor F produces in the third and final step of the loop group method, by evaluating the so called Sym-Bobenko formula for $\lambda = 1$, a CMC-immersion ψ on $\tilde{M}$. ψ "descends" to a CMC-immersion ϕ on M if and only if the monodromy matrices $M(\tilde{\gamma}, \lambda)$ of Ψ associated with the covering transformations $\tilde{\gamma}$ of $\tilde{M}$ satisfy certain "closing conditions" (cf. theorem 2.11). In particular, all monodromy matrices $M(\tilde{\gamma}, \lambda)$ of Ψ need to be unitary for λ from S^1.[1]

A change in the initial condition Ψ_0 corresponds to modifying the solution Ψ by a dressing matrix $T = T(\lambda)$ to obtain a new solution $\hat{\Psi} = T\Psi$. This new solution $\hat{\Psi}$ produces a new CMC-immersion $\hat{\psi}$ on $\tilde{M}$ via the loop group method. Since there is (in general) no obvious relation between corresponding factors in the Iwasawa decompositions of Ψ and $\hat{\Psi}$, respectively, it is (in general) not possible to control the effect of dressing the solution Ψ by T into a new solution $\hat{\Psi}$ on the level of the corresponding CMC-immersions

[1]Actually, the Sym-Bobenko formula produces a CMC-immersion ψ_λ on $\tilde{M}$ for each $\lambda \in S^1$. Throughout this thesis however, if not stated otherwise, we consider the choice $\lambda = 1$.

ψ and $\hat{\psi}$. However, the change in the monodromy matrices is well understood: If the monodromy matrices of Ψ are given by $M(\tilde{\gamma}, \lambda)$, the monodromy matrices of $\hat{\Psi} = T\Psi$ are given by $\hat{M}(\tilde{\gamma}, \lambda) = TM(\tilde{\gamma}, \lambda)T^{-1}$. (Note that this only holds if the original CMC-immersion ψ associated with Ψ has an umbilic point [14]. However, for trinoids, this is always the case.) Based on this fact, one derives the following "recipe" for the construction of CMC-immersions on a Riemann surface M:

1. Given a holomorphic potential $\tilde{\eta}$, which is defined on the universal cover $\tilde{M}$ of M, solve the differential equation $\mathrm{d}\Psi = \Psi\tilde{\eta}$ for a solution Ψ with some initial value $\Psi(z_*) = \Psi_0$. Denote the monodromy matrices of Ψ under the covering transformations $\tilde{\gamma}$ of $\tilde{M}$ by $M(\tilde{\gamma}, \lambda)$.

2. Determine a dressing matrix $T = T(\lambda)$, such that the "dressed" monodromy matrices $\hat{M}(\tilde{\gamma}, \lambda) = TM(\tilde{\gamma}, \lambda)T^{-1}$ satisfy the conditions of theorem 2.11. In particular, one needs to ensure that $\hat{M}(\tilde{\gamma}, \lambda)$ is unitary for λ from S^1 and for all covering transformations $\tilde{\gamma}$. Then, the "dressed" solution $\hat{\Psi} = T\Psi$ produces via the loop group method for $\lambda = 1$ a CMC-immersion ψ on $\tilde{M}$, which descends to a CMC-immersion ϕ on M.

Section 2 is arranged as follows: In 2.1, we define the different loop groups which serve as domains of definition for the factors occuring in the Iwasawa decomposition, which is presented in 2.2. Section 2.3 introduces holomorphic potentials, which form the initial data for the loop group method explicated in 2.4. In 2.5, we introduce the monodromy matrices and cite from [11] our basic theorem 2.11 for the construction of CMC-immersions on a (not necessarily simply connected) Riemann surface M. In 2.6, we apply the loop group method and theorem 2.11 to explicitly construct the already mentioned Delaunay surfaces, CMC-surfaces of revolution in $\mathbb{R}^3$ parametrized by the punctured complex plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

In section 3, we apply the framework built up in section 2 to CMC-trinoids, i.e. to CMC-immersions in $\mathbb{R}^3$ parametrized by $\mathcal{T}_3$, the two-sphere $S^2 = \{x \in \mathbb{R}^3;\ |x| = 1\}$ with three points removed. As fit to our purposes, we identify $\mathcal{T}_3$ via stereographic projection with the twice-punctured complex plane (or, equivalently, the thrice-punctured extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$), $M = \mathbb{C} \setminus \{0, 1\} = \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$, and actually interpret a CMC-trinoid as a CMC-immersion $\phi : M \to \mathbb{R}^3$. The universal cover of M is given by $\tilde{M} = \mathbb{H}$, the upper half-plane in $\mathbb{C}$.

The three points $z_j = j$, $j = 0, 1, \infty$, removed from $\hat{C}$ are singularities of ϕ and thus induce three *(annular) ends* of the surface $\phi(M)$. While we allow arbitrary self-intersections of the surface away from its ends, we require the ends to be *properly embedded*. More precisely, we require that on a sufficiently small punctured neighborhood around each singularity the immersion ϕ is a proper CMC-embedding. Therefore, according to [25], the ends asymptotically show the behaviour of (unduloidal) Delaunay surfaces. Based on this fact and following [17], we introduce a family of holomorphic potentials η on M, which near each singularity z_j take the form of a "perturbed" Delaunay potential $\hat{\eta}_j$.

By [17], the corresponding holomorphic potential $\tilde{\eta} = \pi^*\eta$ on $\tilde{M}$, obtained from η via pullback by the universal covering map π, yields via the loop group method a CMC-trinoid with three properly embedded annular ends at z_j, $j = 0, 1, \infty$, showing the asymptotic behaviour of the respective Delaunay surface produced by (the pullback of) $\hat{\eta}_j$ via the loop group method. More precisely (cf. theorem 3.14), for a given solution Ψ to the differential equation $\mathrm{d}\Psi = \Psi\tilde{\eta}$, there exists a dressing matrix $T = T(\lambda)$ generating a new solution $T\Psi$, which produces via the loop group method a CMC-trinoid with three properly embedded annular ends. Note that it is claimed in [8] that all trinoids with properly embedded annular ends can be constructed from our potentials.

The main features of section 3 are the following: First, we explicitly compute a "starting" solution Φ to the differential equation $\mathrm{d}\Phi = \Phi\eta$. To achieve this, we use the well known fact that the given differential equation can be retraced to a (scalar) hypergeometric differential equation, whose solutions are well known and can be expressed in terms of hypergeometric functions. Moreover, we know by [17], that near each singularity z_j the differential equation $\mathrm{d}\hat{\Phi}_j = \hat{\Phi}_j\hat{\eta}_j$ possesses a solution $\hat{\Phi}_j$ of a special form, called an *EDP-solution*. Φ is locally around z_j related to $\hat{\Phi}_j$ by a gauge matrix $V_{+,j}$. Combining these two facts, Φ can be explicitly computed (cf. lemma 3.37). Moreover, the monodromy matrices $M(\tilde{\gamma}, \lambda)$ of the corresponding pullback $\Psi = \pi^*\Phi$ solving $\mathrm{d}\Psi = \Psi\tilde{\eta}$ are determined.

The second feature of section 3 consists in the characterization of all possible dressing matrices T rendering Ψ into a new solution $T\Psi$ to $\mathrm{d}\Psi = \Psi\tilde{\eta}$, which produces a CMC-trinoid via the loop group method. As indicated before, this is achieved by ensuring that the dressed monodromy matrices $\hat{M}(\tilde{\gamma}, \lambda) = TM(\tilde{\gamma}, \lambda)T^{-1}$ satisfy the conditions of theorem 2.11. In this context, we restrict our considerations to the three monodromy matrices $M_j(\lambda) := M(\tilde{\gamma}_j, \lambda)$, $j = 0, 1, \infty$, corresponding to the three covering transformations $\tilde{\gamma}_j$, $j = 0, 1, \infty$, of $\tilde{M}$, which represent three simple loops in M, surrounding exactly once the singularity z_j (counter-clockwise) without enclosing the other two singularities, respectively. Since

$\tilde{\gamma}_j$, $j = 0, 1, \infty$, generate the group of covering transformations $\tilde{\gamma}$ of $\tilde{M}$ (actually, even two of them do), a matrix T dresses the starting solution Ψ into a new solution $T\Psi$ to $\mathrm{d}\Psi = \Psi\tilde{\eta}$, which produces a CMC-trinoid via the loop group method, if and only if the three dressed monodromy matrices $\hat{M}_j = TM_jT^{-1}$ satisfy the conditions of theorem 2.11. Explicitly, $T\Psi$ produces a CMC-trinoid, if and only if the three monodromy matrices $\hat{M}_j$ are of the form

$$\hat{M}_j = -\left[\cos(2\pi\mu_j)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i\sin(2\pi\mu_j)\begin{pmatrix}p_j & \overline{q_j}\\ q_j & -p_j\end{pmatrix}\right]$$

with λ-dependent functions p_j, $\overline{p_j}$, q_j and $\overline{q_j}$ satisfying a number of conditions, which are summarized in theorem 3.59.

Section 3 is organized as follows: In section 3.1, we introduce CMC-trinoids as CMC-immersions of the twice-punctured complex plane $M = \mathbb{C}\setminus\{0,1\}$. Section 3.2 presents the universal cover $\tilde{M} = \mathbb{H}$ of M and defines the corresponding covering map $\pi : \tilde{M} \to M$. In section 3.3, we study the monodromy action of the fundamental group of M on $\tilde{M}$, which constitutes the basis for the definition of the monodromy matrices later. Since the loop group method actually produces CMC-immersions into su(2) rather than into $\mathbb{R}^3$, we need to identify su(2) and $\mathbb{R}^3$. This is done in section 3.4. Sections 3.5 and 3.6 are dedicated to the definition of the trinoid potentials η (on M) and $\tilde{\eta}$ (on $\tilde{M}$). Our starting solution Φ to the differential equation $\mathrm{d}\Phi = \Phi\eta$, along with the corresponding starting solution Ψ to $\mathrm{d}\Psi = \Psi\tilde{\eta}$, is explicitly computed in sections 3.7 and 3.8. Finally, section 3.9 deals with the possible dressing matrices $T = T(\lambda)$ transforming Ψ into a new solution $T\Psi$, which gives a CMC-trinoid via the loop group method.

Section 4 opens the second part of this thesis: Having so far determined all solutions Ψ to the differential equation $\mathrm{d}\Psi = \Psi\tilde{\eta}$, which generate CMC-trinoids via the loop group method, it seems natural to ask to what extent geometrical properties of a CMC-trinoid ϕ produced can be read off the respective generating solution Ψ. In particular, one can ask how symmetry properties of (the image $\phi(M)$ of) a given trinoid $\phi : M \to \mathbb{R}^3$ show in the corresponding generating solution Ψ. In this thesis, we give a comprehensive answer to this question in the case of CMC-trinoids with properly embedded annular ends.

Given a CMC-trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends, we define a *symmetry* of ϕ as an Euclidean motion $\mathcal{T}$, i.e. an orthogonal transformation followed by a translation on $\mathbb{R}^3$, which preserves the image of ϕ in $\mathbb{R}^3$: $\mathcal{T}(\phi(M)) = \phi(M)$. Denoting, as before, by π the universal covering $\tilde{M} \to M$, $\mathcal{T}$ also defines a symmetry of the CMC-immersion $\psi := \phi\circ\pi : \tilde{M} \to \mathbb{R}^3$, i.e. $\mathcal{T}(\psi(\tilde{M})) = \psi(\tilde{M})$.

A priori, due to the Iwasawa decomposition of Ψ in the second step of the loop group method, it is difficult to retrace any symmetry properties of ϕ (resp. ψ) back to the corresponding generating solution Ψ to the differential equation $\mathrm{d}\Psi = \Psi\tilde{\eta}$. Though it is possible to reconstruct the *extended frame* F from ψ, which returns ψ by insertion into the Sym-Bobenko formula for $\lambda = 1$ [10], the subsequent step of reconstructing Ψ from F (i.e., of reversing the Iwasawa decomposition $\Psi = FB_+$) is highly non-trivial. Nevertheless, we observe that the monodromy matrices M_j of Ψ with respect to the covering transformations $\tilde{\gamma}_j$ also occur as "monodromy matrices" of F:

$$F(\tilde{\gamma}_j(z), \lambda) = \pm\hat{M}_jF(z,\lambda)k_j(z),$$

where k_j depends directly on $\tilde{\gamma}_j$.

This observation forms the basis for our approach of retracing any symmetry properties of ϕ (resp. ψ) to the level of the generating solution Ψ: We translate the given symmetry properties of ϕ (resp. ψ) to the level of the extended frame F of ψ (section 4) and deduce certain "symmetry restrictions" on the "monodromy matrices" of F (sections 5 to 9). In this way, we actually obtain "symmetry restrictions" on the monodromy matrices of Ψ, which in turn determines the solutions to the differential equation $\mathrm{d}\Psi = \Psi\tilde{\eta}$ generating trinoids with properly embedded annular ends with the respective symmetry properties.

In detail, we proceed as follows: Let $\phi : M \to \mathbb{R}^3$ be a CMC-trinoid with properly embedded annular ends, which is symmetric with respect to the Euclidean motion $\mathcal{T}$. Denote by ψ the corresponding CMC-immersion $\tilde{M} \to \mathbb{R}^3$, which is also symmetric with respect to $\mathcal{T}$. Depending on whether $\mathcal{T}$ preserves orientation or reverses orientation on $\mathbb{R}^3$, it can be shown (cf. theorem 4.9, based on results from [12]) that there exist a pair of biholomorphic (resp. bi-antiholomorphic) mappings $\gamma : M \to M$ and $\tilde{\gamma} : \tilde{M} \to \tilde{M}$ translating the symmetry $\mathcal{T}$ to M and $\tilde{M}$, respectively:

$$\mathcal{T}\circ\phi = \phi\circ\gamma, \tag{1.0.1}$$

$$\mathcal{T}\circ\psi = \psi\circ\tilde{\gamma}. \tag{1.0.2}$$

Moreover, γ and $\tilde{\gamma}$ are linked by the relation $\pi\circ\tilde{\gamma} = \gamma\circ\pi$. (Note that, in order to obtain this result, the assumption that ϕ possesses properly embedded annular ends takes effect.) By the relations above, the symmetry properties of ϕ (resp. ψ) translate to the level of the associated extended frame as follows (cf. theorem 4.17): If $\mathcal{T}$ preserves orientation, then

$$F(\tilde{\gamma}(z), \lambda) = M_{\tilde{\gamma}}(\lambda) F(z, \lambda) k_{\mathcal{T},\tilde{\gamma}}(z), \tag{1.0.3}$$

where $k_{\mathcal{T},\tilde{\gamma}}$ depends on $\tilde{\gamma}$ and $M_{\tilde{\gamma}}(\lambda)$ denotes a λ-dependent matrix, which is unitary on the unit circle, $\lambda \in S^1$. If $\mathcal{T}$ reverses orientation, then

$$F(\tilde{\gamma}(z), \lambda^{-1}) = M_{\tilde{\gamma}}(\lambda) \overline{F(z, \lambda)} k_{\mathcal{T},\tilde{\gamma}}(z), \tag{1.0.4}$$

where $k_{\mathcal{T},\tilde{\gamma}}$, as above, depends on $\tilde{\gamma}$ and $M_{\tilde{\gamma}}(\lambda)$ denotes a λ-dependent matrix, which is unitary on the unit circle, $\lambda \in S^1$. In particular, in the case $\mathcal{T} = \mathcal{I}$, the identity mapping on $\mathbb{R}^3$ and naturally a symmetry of ϕ (resp. ψ), we obtain the "monodromy relations" for F given above, involving the covering transformations $\tilde{\gamma}_j$, $j = 0, 1, \infty$, associated with $\mathcal{T} = \mathcal{I}$ in the sense that $\mathcal{I} \circ \psi = \psi \circ \tilde{\gamma}_j$.

The biholomorphic (resp. bi-antiholomorphic) mapping $\gamma : M \to M$ associated with the trinoid symmetry $\mathcal{T}$ can be extended to a biholomorphic (resp. bi-antiholomorphic) mapping $\gamma_{\text{extd}} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, which necessarily permutes the three points $z_j = j \in \{0, 1, \infty\}$ according to a permutation σ of the set $\{0, 1, \infty\}$, i.e. $\gamma_{\text{extd}}(z_j) = z_{\sigma(j)}$. In this way, however, γ_{extd} (and thus γ) is completely determined by σ and can thus be explicitly computed. As there are six possibilities for σ, we obtain twelve possibilities for γ, six biholomorphic ones and six bi-antiholomorphic ones. One easily infers that there are only twelve possible trinoid symmetries $\mathcal{T}$, six orientation preserving ones and six orientation reversing ones. Moreover, these can be explicitly determined as well (cf. theorem 4.31) and characterized by their respective permutation behaviour concerning the trinoid ends: The six possible orientation preserving trinoid symmetries are the identity mapping $\mathcal{I}$, the rotation $\mathcal{R}$ by the angle $\pm\frac{2\pi}{3}$ rotating the trinoid ends into each other, its inverse $\mathcal{R}^{-1}$ and the three rotations $\mathcal{R}_j$ by the angle π, each preserving the trinoid end at the singularity z_j while interchanging the other two. The six possible orientation reversing trinoid symmetries are the reflection $\mathcal{S}$ in some plane, preserving each of the three trinoid ends, the three reflections $\mathcal{S}_j$, each preserving the trinoid end at the singularity z_j while interchanging the other two, the rotoreflection[2] $\hat{\mathcal{S}}$ composed of $\mathcal{R}$ and $\mathcal{S}$, and its inverse $\hat{\mathcal{S}}^{-1}$.

Once given the twelve possibilities for γ, one can compute the associated mappings $\tilde{\gamma} : \tilde{M} \to \tilde{M}$ from the relation $\pi \circ \tilde{\gamma} = \gamma \circ \pi$. (Note that $\tilde{\gamma}$ is only determined uniquely up to left composition with a covering transformation $\tilde{M} \to \tilde{M}$.) Thus, we explicitly obtain for each possible trinoid symmetry $\mathcal{T}$ a biholomorphic (resp. bi-antiholomorphic) mapping $\tilde{\gamma}_{\mathcal{T}} : \tilde{M} \to \tilde{M}$, which we put into relation with the covering transformations $\tilde{\gamma}_j$ corresponding to the "monodromy matrices" of F. This is done separately for each symmetry type in sections 5 to 9. Generally speaking, the relations between $\tilde{\gamma}_{\mathcal{T}}$ and the covering transformations $\tilde{\gamma}_j$ translate by use of the "monodromy relations" and the "symmetry relations" of F given above into relations on the "monodromy matrices" $\hat{M}_j$ of F, involving the respective "symmetry monodromy matrix" $M_{\tilde{\gamma}_{\mathcal{T}}}$. The latter relations translate directly into further constraints on the mappings p_j, q_j occurring in $\hat{M}_j$ and thus, by theorem 3.59 into constraints on the generating solution Ψ to the differential equation $\mathrm{d}\Psi = \Psi\tilde{\eta}$, which produces the respective trinoid with properly embedded annular ends, which is symmetric with respect to $\mathcal{T}$. The explicit results are given in theorems 5.9, 6.6, 7.5, 8.6 and 9.7, respectively.

In general, the further constraints on the mappings p_j, q_j occurring in $\hat{M}_j$, which are obtained by evaluating the deduced relations between the monodromy matrices $\hat{M}_j$ and the respective "symmetry monodromy matrix" $M_{\tilde{\gamma}_{\mathcal{T}}}$, also introduce more parameters, namely the (λ-dependent) entries of $M_{\tilde{\gamma}_{\mathcal{T}}}$. However, this additional freedom can be fixed by an appropriate "normalization" of the extended frame F: First, observe that during the reconstruction of the extended frame F from ψ, F is "normalized", such that $F(z_*, \lambda) = \mathrm{I}$ for all $\lambda \in S^1$ at an arbitrarily chosen base point $z_* \in \tilde{M}$. This normalization of F corresponds to a λ-dependent rotation and shift of the associated family ψ_λ of F, obtained from F via the loop group method. Now, choosing (if possible) for $z_* \in \tilde{M}$ a fixed point of the biholomorphic (resp. bi-antiholomorphic) mapping $\tilde{\gamma}_{\mathcal{T}} : \tilde{M} \to \tilde{M}$ associated with a considered trinoid symmetry $\mathcal{T}$, we infer from the "symmetry relation" of F given above that

$$M_{\tilde{\gamma}}(\lambda) = (k_{\mathcal{T},\tilde{\gamma}}(z_*))^{-1}, \tag{1.0.5}$$

i.e. we obtain a "symmetry monodromy matrix" $M_{\tilde{\gamma}_{\mathcal{T}}}$ which is actually independent of λ and explicitly known (since $k_{\mathcal{T},\tilde{\gamma}}(z_*)$ is). Thus, the additional parameters mentioned above are fixed, and the corresponding additional constraints on the mappings p_j, q_j occurring in $\hat{M}_j$ become by far more explicit.

[2] *A rotoreflection defines a Euclidean motion, i.e. an isometry of $\mathbb{R}^3$ (cf. section 4.1), which is composed of an arbitrary rotation $\mathcal{R}$ in $\mathbb{R}^3$ followed by a reflection $\mathcal{S}$ in $\mathbb{R}^3$, such that the reflection plane of $\mathcal{S}$ is orthogonal to the rotation axis of $\mathcal{R}$.*

(Note that, however, such an "appropriate normalization" of F is only possible, if the considered mapping $\tilde{\gamma}_{\mathcal{T}}$ possesses any fixed point in $\tilde{M}$. It turns out that this is the case for all possible trinoid symmetries except for the rotoreflections $\mathcal{T} = \hat{\mathcal{S}}$ and $\mathcal{T} = \hat{\mathcal{S}}^{-1}$.)

If the "appropriate normalization" of F as described above is possible, we can explicitly translate a given symmetry property of a trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends (resp. of the corresponding CMC-immersion $\psi : \tilde{M} \to \mathbb{R}^3$), into constraints on the mappings p_j, q_j occurring in the monodromy matrices $\hat{M}_j$ of the generating solution Ψ to the differential equation $\mathrm{d}\Psi = \Psi\tilde{\eta}$. The explicit results are given in theorems 5.13, 6.9, 7.8 and 8.9, respectively. Moreover, it turns out that the obtained constraints on the mappings p_j, q_j are not only necessary but also sufficient for Ψ generating a trinoid with properly embedded annular ends and the respective symmetry. This is proved in theorems 5.14, 6.10, 7.9 and 8.10, respectively.

Section 4 is organized as follows: First, we give the definition of a trinoid symmetry in section 4.1. The procedure of recovering the extended frame F from the CMC-immersion $\psi : \tilde{M} \to \mathbb{R}^3$ associated with a trinoid $\phi : M \to \mathbb{R}^3$ is studied in section 4.2. Actually, this is done in the generalized setting of an arbitrary Riemann surface M with universal cover $\tilde{M}$ and a pair of conformal CMC-immersions $\phi : M \to \mathbb{R}^3$ and $\psi : \tilde{M} \to \mathbb{R}^3$ linked via the universal covering $\pi : \tilde{M} \to M$, $\psi = \phi \circ \pi$. Sections 4.3 and 4.4 explicate the translation of a given symmetry property of a trinoid ϕ with properly embedded annular ends (resp. of the corresponding CMC-immersion ψ) to the level of the extended frame F of ψ. Section 4.5 applies this translation to the symmetry $\mathcal{T} = \mathcal{I}$ and the covering transformations $\tilde{\gamma}_j$, $j = 0, 1, \infty$, associated with the monodromy matrices of Ψ to obtain the "monodromy relations" for F mentioned earlier. Finally, section 4.6 provides the explicit forms of the biholomorphic (resp. bi-antiholomorphic) mappings $\gamma : M \to M$ asssociated with the possible trinoid symmetries, as well as the twelve possible trinoid symmetries themselves.

Sections 5 to 9 translate the "symmetry relation" of the extended frame F associated with a particular trinoid symmetry $\mathcal{T}$ of a trinoid ϕ with properly embedded annular ends into constraints on the monodromy matrices $\hat{M}_j$ of the solution Ψ to the differential equation $\mathrm{d}\Psi = \Psi\tilde{\eta}$, which produces ϕ (or, more precisely, the corresponding CMC-immersion ψ which "descends" to ϕ). Section 5 deals with the symmetries $\mathcal{R}$ and $\mathcal{R}^{-1}$, section 6 treats the symmetries $\mathcal{R}_j$, $j = 0, 1, \infty$, section 7 discusses $\mathcal{S}$, section 8 studies the symmetries $\mathcal{S}_j$, $j = 0, 1, \infty$, and section 9 is concerned with the symmetries $\hat{\mathcal{S}}$ and $\hat{\mathcal{S}}^{-1}$.

Throughout this thesis, we act on the assumption that the reader is familiar with the basic notions of differential geometry. In particular, this involves: (parametrized) surfaces in $\mathbb{R}^3$, the mean curvature of a surface, (differentiable) manifolds, differentiable mappings between manifolds, the differential of a differential mapping, Riemannian manifolds. A comprehensive introduction to differential geometry can be found in [6]. The following notions are of particular interest for our concerns:

A differentiable mapping $f : M \to \mathbb{R}^3$ on a Riemannian manifold M is called an *immersion*, if the corresponding differential at each point $p \in M$, $\mathrm{d}f : T_pM \to \mathbb{R}^3$, is injective. An immersion $f : M \to \mathbb{R}^3$ on a Riemannian manifold M is called an *embedding*, if it is a homeomorphism onto its image, i.e. if the mapping $f : M \to f(M)$ is continuous and injective with a continuous inverse mapping $f^{-1} : f(M) \to M$.

Let M be a Riemannian manifold with differentiable structure $\{U_\alpha, x_\alpha\}$. An immersion $f : M \to \mathbb{R}^3$ on M is called a *CMC-immersion* (or a *CMC-H-immersion*), if, for each α, the mapping $f \circ x_\alpha : U_\alpha \to \mathbb{R}^3$ is a (parametrized) surface of constant mean curvature H.

Furthermore, this thesis involves basic topological definitions and results, which have been assembled in appendix A. For a detailed introduction to algebraic topology, the reader is referred to [20].

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2 Outline of the loop group method

We begin by giving a brief review of the "loop group method" for the construction of constant mean curvature surfaces from holomorphic potentials as presented in [15]. (This method is often also referred to as the "DPW-method".) This review will involve introducing the basic concepts (loop groups, Iwasawa decomposition, holomorphic potentials) as well as giving an outline of the loop group method itself, illustrated by the simple example of the already mentioned Delaunay surfaces.

2.1 Loop Groups

Let $\mathrm{SL}(2,\mathbb{C})$ denote the special linear group of complex 2×2 matrices and $\sigma : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(2,\mathbb{C})$ denote the conjugation by the Pauli matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. For each $r \in (0,1]$, we define by

$$\Lambda_r \mathrm{SL}(2,\mathbb{C})_\sigma = \{\gamma : C^{(r)} \to \mathrm{SL}(2,\mathbb{C}) \text{ smooth } ; \gamma(-\lambda) = \sigma(\gamma(\lambda))\} \tag{2.1.1}$$

the *(twisted) loop group* of smooth maps from the r-cirle $C^{(r)} = \{\lambda \in \mathbb{C}; |\lambda| = r\}$ into $\mathrm{SL}(2,\mathbb{C})$. The Lie algebra $\Lambda_r \mathrm{sl}(2,\mathbb{C})_\sigma$ of $\Lambda_r \mathrm{SL}(2,\mathbb{C})_\sigma$ is given by

$$\Lambda_r \mathrm{sl}(2,\mathbb{C})_\sigma = \{x : C^{(r)} \to \mathrm{sl}(2,\mathbb{C}) \text{ smooth } ; x(-\lambda) = \sigma_3 x(\lambda)\sigma_3\}, \tag{2.1.2}$$

where $\mathrm{sl}(2,\mathbb{C})$ denotes the Lie algebra of $\mathrm{SL}(2,\mathbb{C})$. The conditions $\gamma(-\lambda) = \sigma(\gamma(\lambda))$ and $x(-\lambda) = \sigma_3 x(\lambda)\sigma_3$ will be referred to as the *twisting conditions* for $\gamma \in \Lambda_r \mathrm{SL}(2,\mathbb{C})_\sigma$ and $x \in \Lambda_r \mathrm{sl}(2,\mathbb{C})_\sigma$, respectively. Note that the twisting condition for a smooth matrix function $\gamma : C^{(r)} \to \mathrm{SL}(2,\mathbb{C})$ (resp. for a smooth matrix function $x : C^{(r)} \to \mathrm{sl}(2,\mathbb{C})$) is met if and only if the off-diagonal entries of γ (resp. of x) are odd functions, while the diagonal entries of γ (resp. of x) are even functions of $\lambda \in C^{(r)}$.

Furthermore, we denote by $\Lambda_r^+ \mathrm{SL}(2,\mathbb{C})_\sigma$ the subgroup of maps $\gamma \in \Lambda_r \mathrm{SL}(2,\mathbb{C})_\sigma$ that extend holomorphically to the open disc $I^{(r)} = \{\lambda \in \mathbb{C}; |\lambda| < r\}$, and - by abuse of notation - by $\Lambda_r \mathrm{SU}(2)_\sigma$ the subgroup of maps $\gamma \in \Lambda_r \mathrm{SL}(2,\mathbb{C})_\sigma$ that extend holomorphically to the open annulus $A^{(r)} = \{\lambda \in \mathbb{C}; r < |\lambda| < \frac{1}{r}\}$ and take values in the special unitary group $\mathrm{SU}(2)$ on the unit circle $S^1 = C^{(1)}$.

In the case $r = 1$, we omit any subscripts "r", simply denoting the groups $\Lambda_1 \mathrm{SL}(2,\mathbb{C})_\sigma$, $\Lambda_1^+ \mathrm{SL}(2,\mathbb{C})_\sigma$, $\Lambda_1 \mathrm{SU}(2)_\sigma$ by $\Lambda \mathrm{SL}(2,\mathbb{C})_\sigma$, $\Lambda^+ \mathrm{SL}(2,\mathbb{C})_\sigma$, $\Lambda \mathrm{SU}(2)_\sigma$, respectively. We deal analogously with the corresponding Lie algebras.

Remark 2.1. The topology introduced above for the loop groups and Lie algebras is a Frechet topology. Sometimes it is preferable to work with Banach structures instead of with Frechet structures. In this case one could require, e.g., that all matrix coefficients are contained in the Wiener algebra on the unit circle (cf., e.g., [9]). For the purposes of this work the topology of the groups will play a minor role.

2.2 Iwasawa decomposition

It is known from [32] that the multiplication map $\Lambda_r \mathrm{SU}(2)_\sigma \times \Lambda_r^+ \mathrm{SL}(2,\mathbb{C})_\sigma \to \Lambda_r \mathrm{SL}(2,\mathbb{C})_\sigma$ is surjective, that is, any $\gamma \in \Lambda_r \mathrm{SL}(2,\mathbb{C})_\sigma$ may be written as

$$\gamma = \gamma_u \gamma_+, \tag{2.2.1}$$

where $\gamma_u \in \Lambda_r \mathrm{SU}(2)_\sigma$ and $\gamma_+ \in \Lambda_r^+ \mathrm{SL}(2,\mathbb{C})_\sigma$. The splitting (2.2.1) is called an *r-Iwasawa decomposition* of $\gamma \in \Lambda_r \mathrm{SL}(2,\mathbb{C})_\sigma$, or, if $r = 1$, just *Iwasawa decomposition* of γ. By additionally requiring that $\gamma_+(0)$ is diagonal with positive real entries, the factors of the splitting (2.2.1) are uniquely determined. In this case the multiplication map is a real-analytic diffeomorphism, and we will therefore speak of the *unique r-Iwasawa decomposition* (resp. *unique Iwasawa decomposition*) of γ. For a proof of this, the reader is referred to [32] and [30].

2.3 Holomorphic potentials

Next we will outline how one obtains from an immersion $\psi : \tilde{M} \to \mathbb{R}^3$ of constant mean curvature $H \neq 0$ on a simply connected domain $\tilde{M} \subseteq \mathbb{C}$ an $\mathrm{sl}(2,\mathbb{C})$-valued holomorphic differential one-form on $\tilde{M}$ involving a loop parameter $\lambda \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, the so called holomorphic potential $\tilde{\eta}$.

Let $\psi : \tilde{M} \to \mathbb{R}^3$ be a CMC-immersion. Consider the *extended frame* $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ corresponding to ψ as defined in [15].[3] According to [15], there exists $B_+ : \tilde{M} \to \Lambda^+\mathrm{SL}(2,\mathbb{C})_\sigma$ such that

$$\Psi = FB_+ \text{ is holomorphic in both } z \in \tilde{M} \text{ and } \lambda \in \mathbb{C}^* \tag{2.3.1}$$

Ψ is called an *holomorphic frame* associated with ψ. The corresponding *Maurer-Cartan form*

$$\tilde{\eta} = \Psi^{-1}\mathrm{d}\Psi \tag{2.3.2}$$

is holomorphic in both $z \in \tilde{M}$ and $\lambda \in \mathbb{C}^*$ as well and is called the *holomorphic potential* associated with the immersion ψ.

Remark 2.2. The extended frame F associated with ψ is not determined uniquely, but only up to the choice of some initial value $F(z_*, \lambda) \in \Lambda\mathrm{SU}(2)_\sigma$ for some $z_* \in \tilde{M}$. It is in particular always possible to achieve $F(z_*, \lambda) = \mathrm{I}$ for a chosen base point $z_* \in \tilde{M}$ by replacing a given frame $F_0(z, \lambda)$ by $F(z, \lambda) := F_0(z_*, \lambda)^{-1}F_0(z, \lambda)$.

In the following sections we recapitulate the procedure of constructing CMC-immersions $\phi : M \to \mathbb{R}^3$ of a Riemann surface M into $\mathbb{R}^3$ from a given holomorphic potential $\tilde{\eta}$, which is defined on the universal cover $\tilde{M}$ of the (not necessarily simply connected) Riemann surface M. In order to construct ϕ, we proceed as follows: First, we apply the loop group method to the holomorphic potential $\tilde{\eta}$ to obtain a CMC-immersion $\psi : \tilde{M} \to \mathbb{R}^3$ (cf. section 2.4). More precisely, the loop group method will produce a whole family ψ_λ of CMC-immersions $\tilde{M} \to \mathbb{R}^3$, parametrized by a loop parameter $\lambda \in S^1$. Second, we turn to the question under which circumstances ψ_λ descends to a CMC-immersion $\phi_\lambda : M \to \mathbb{R}^3$, at least for a special choice of the loop parameter λ (see section 2.5).

2.4 The loop group method

As indicated above, we can construct immersions of constant mean curvature $H \neq 0$ defined on the universal cover $\tilde{M}$ of a Riemann surface M from holomorphic potentials introduced in section 2.3 by applying the "loop group method" presented in [15]. Carrying out this procedure involves the following three steps:

1. Given a holomorphic potential $\tilde{\eta}$, solve the differential equation

$$\mathrm{d}\Psi = \Psi\tilde{\eta}. \tag{2.4.1}$$

2. Perform (for each $z \in \tilde{M}$) an r-Iwasawa decomposition

$$\Psi = FB_+. \tag{2.4.2}$$

Note that, by construction, F involves the loop parameter $\lambda \in C^{(r)}$ and can be holomorphically extended (in λ) to the annulus $A^{(r)}$ containing the 1-sphere $S^1 := C^{(1)}$.

3. Interpreting F as an element of $\Lambda\mathrm{SU}(2)_\sigma$ and writing $\lambda = e^{i\theta}$ for the loop parameter $\lambda \in S^1$, evaluate the *Sym-Bobenko formula*

$$\mathrm{SymBob}(F) = -\frac{1}{2H}\left(\frac{\partial}{\partial\theta}F \cdot F^{-1} + \frac{i}{2}F\sigma_3F^{-1}\right) \tag{2.4.3}$$

for any $\lambda_0 \in S^1$ to obtain a CMC-immersion ψ_{λ_0} defined on $\tilde{M}$.

Remark 2.3. For our purposes, that is for the construction of trinoids with properly embedded annular ends from holomorphic potentials (cf. section 3), we can think of the starting potential $\tilde{\eta}$ on $\tilde{M}$ as the pullback of some potential η defined on $M = \tilde{M}/\Gamma$, where Γ denotes the fundamental group of M (cf. [9]). Thus we ensure that $\tilde{\eta}$ is an *invariant* holomorphic potential, i.e. invariant under the action of the fundamental group Γ on $\tilde{M}$. (Cf. section A.4 of appendix A for a detailed discussion of the mentioned action of the fundamental group Γ of M on $\tilde{M}$.)

More precisely, we note that the choice of a holomorphic potential associated with a given CMC-immersion $\psi : \tilde{M} \to \mathbb{R}^3$ is not unique (cf. [15]). By theorem 3.2 of [9], it is possible to associate with ψ an invariant holomorphic potential $\tilde{\eta}$. The fact that we can thus assume w.l.o.g. that $\tilde{\eta}$ is an invariant holomorphic potential will be useful in the following section, when we address ourselves to the question, whether a CMC-immersion $\tilde{M} \to \mathbb{R}^3$ produced from $\tilde{\eta}$ descends to a CMC-immersion $M \to \mathbb{R}^3$.

[3] We review the procedure of constructing the extended frame F from a CMC-immersion $\psi : \tilde{M} \to \mathbb{R}^3$ in section 4.4.

By the theory of ordinary differential equations, the solution to (2.4.1) is uniquely determined as soon as we prescribe an initial value condition

$$\Psi(z_*) = \Psi_0 \tag{2.4.4}$$

for an arbitrary base point $z_* \in \tilde{M}$ and some $\Psi_0 \in \Lambda_r\mathrm{SL}(2,\mathbb{C})_\sigma$ for some $r \in (0,1]$. Whereas any solution Ψ to (2.4.1) will be holomorphic in $z \in \tilde{M}$, Ψ will only be holomorphic in $\lambda \in \mathbb{C}^*$ if and only if Ψ_0 is holomorphic in $\lambda \in \mathbb{C}^*$.

Remark 2.4. In our discussions throughout this work we will deal with solutions to (2.4.1), whose initial values at z_* are not explicitly known and therefore might be *not* holomorphic in $\lambda \in \mathbb{C}^*$. In the case that a solution Ψ to (2.4.1) is actually singular for certain values of $\lambda \in \mathbb{C}^*$, we can only proceed with the second step of the loop group method, if Ψ is at least holomorphic (in λ) on some circle $C^{(r_0)}$, $r_0 \in (0,1]$. (In this case, resume the loop group method by performing an r-Iwasawa decomposition of Ψ with $r = r_0$.) Thus, when dealing with a solution Ψ to (2.4.1) at any time, we need to ensure that Ψ is holomorphic (in λ) on some circle $C^{(r_0)}$, $r_0 \in (0,1]$.

Given a solution $\tilde{\Psi}$ to (2.4.1) with initial value $\tilde{\Psi}(z_*)$, it is easy to verify that $\hat{\Psi}(z) := \Psi_0\tilde{\Psi}(z_*)^{-1}\tilde{\Psi}(z)$ also solves (2.4.1) and, moreover, meets the initial value condition (2.4.4). Consequently any solution Ψ to (2.4.1) can be modified (by a multiplication from the left independent of z) to meet a prescribed initial condition. In particular, a solution singular for certain values of $\lambda \in \mathbb{C}^*$, can in such a way be transformed into a new solution, which is holomorphic in $\lambda \in \mathbb{C}^*$ or, at least, holomorphic in $\lambda \in S^1$ (i.e. holomorphic in λ on an open neighborhood of S^1). Solutions to (2.4.1), which are holomorphic in $\lambda \in S^1$ are of special interest for this work (cf. theorem 2.11).

Definition 2.5. Let Ψ be a solution to (2.4.1). Then, the action of replacing Ψ by

$$\hat{\Psi} = T\Psi, \tag{2.4.5}$$

where T denotes some z-independent loop in $\Lambda_r\mathrm{SL}(2,\mathbb{C})_\sigma$, is referred to as *r-dressing* or simply *dressing* Ψ by T.

By dressing a solution Ψ of (2.4.1), we obtain a new solution $\hat{\Psi}$ to (2.4.1), "only" changing the initial condition. Such a change, however, has profound consequences in step two of the loop group method, as there is no trivial relation between the frames F and $\hat{F}$ involved in the Iwasawa decompositions of Ψ and $\hat{\Psi}$, respectively. This means, that dressing a solution Ψ to (2.4.1) will (in general) give rise to significant changes in the CMC-immersion $\psi = \psi_{\lambda_0}$ generated by step three of the loop group method. In fact, the manipulation of the initial value Ψ_0 given by (2.4.5) turns out to be crucial for our purposes, as it plays the decisive role when it comes to deciding whether ψ will descend to a CMC-immersion ϕ on M or not. This issue will be discussed further in the following section.

Remark 2.6. We would like to remark, that in other places the dressing action of T on a solution to (2.4.1) is sometimes only defined for $T \in \Lambda_r^+\mathrm{SL}(2,\mathbb{C})_\sigma$. This is motivated by the following considerations: Let Ψ be a solution to (2.4.1) and $T \in \Lambda_r\mathrm{SL}(2,\mathbb{C})_\sigma$ with Iwasawa decomposition $T = T_uT_+$. Denote by $\hat{\Psi}$ (resp. $\tilde{\Psi}$) the new solution to (2.4.1) obtained from dressing Ψ by T (resp. by T_+ only): $\hat{\Psi} = T\Psi$, $\tilde{\Psi} = T_+\Psi$. Moreover, let $\hat{\Psi} = \hat{F}\hat{B}_+$ be the Iwasawa decomposition of $\hat{\Psi}$. Then, the Iwasawa decomposition of $\tilde{\Psi} = T_u^{-1}\hat{\Psi}$ is obviously given by $\tilde{\Psi} = \tilde{F}\tilde{B}_+$ with $\tilde{F} = T_u^{-1}\hat{F}$ and $\tilde{B}_+ = \hat{B}_+$. Thus, the extended frames $\tilde{F}$ and $\hat{F}$ involved in the Iwasawa decompositions of $\tilde{\Psi}$ and $\hat{\Psi}$ differ only by T_u^{-1}, which is unitary for $\lambda \in S^1$. It is easy to verify, that, consequently, the families $\tilde{\psi}_\lambda$ and $\hat{\psi}_\lambda$, $\lambda \in S^1$, of CMC-immersions $\tilde{M} \to \mathbb{R}^3$ obtained from $\tilde{F}$ and $\hat{F}$, respectively, by the third step of the loop group method differ only by a (λ-dependent) rigid motion in $\mathbb{R}^3$. In this sense, dressing Ψ by T or only by T_+ yields new solutions $\hat{\Psi}$ and $\tilde{\Psi}$ to (2.4.1), which induce "essentially the same" CMC-immersions via the loop group method.

We end this section with the following observation: Given a solution Ψ to the differential equation (2.4.1) and a loop $g \in \Lambda_r^+\mathrm{SL}(2,\mathbb{C})_\sigma$, the mapping $\hat{\Psi} := \Psi g$ solves the equation

$$\mathrm{d}\hat{\Psi} = \hat{\Psi}\hat{\eta}, \tag{2.4.6}$$

where $\hat{\eta}$ is given by

$$\hat{\eta} = \tilde{\eta}\#g := g^{-1}\tilde{\eta}g + g^{-1}\mathrm{d}g. \tag{2.4.7}$$

Definition 2.7. Let Ψ be a solution to (2.4.1). Then, the action of replacing Ψ by

$$\hat{\Psi} = \Psi g, \tag{2.4.8}$$

where g denotes some z-independent loop in $\Lambda_r^+\mathrm{SL}(2,\mathbb{C})_\sigma$, is referred to as *$r$-gauging* or simply *gauging* Ψ by g. The potential $\hat{\eta}$ associated with the holomorphic potential $\tilde{\eta}$ by (2.4.7) is called the *gauged potential*.

The use of the above observation consists in the relation between the unitary factors in the Iwasawa decompositions of Ψ and $\hat{\Psi} = \Psi g$, respectively, during the course of the loop group method: As $g \in \Lambda_r^+\mathrm{SL}(2,\mathbb{C})_\sigma$, the r-Iwasawa decomposition of $\hat{\Psi}$ is given by

$$\hat{\Psi} = F(B_+ g), \tag{2.4.9}$$

which means that the unitary factors in the Iwasawa decompositions of Ψ and $\hat{\Psi}$ are actually the same. Thus, Ψ and $\hat{\Psi}$ produce the *same* CMC-immersions via the loop group method. Consequently, this allows for replacing a given holomorphic potential $\tilde{\eta}$ by a corresponding gauged potential $\hat{\eta}$ without changing the CMC-immersion produced by the loop group method.

2.5 Monodromy

Next we investigate under which circumstances a given immersion ψ on the universal cover $\tilde{M}$ of a Riemann surface M will descend to an immersion ϕ defined on M. The answer to this question is closely linked to the behaviour of the holomorphic frame Ψ associated with ψ (cf. section 2.3) under the covering transformations $\tilde{\gamma}$ corresponding to the elements $[\gamma]$ of the fundamental group Γ of M.[4] This transformation behaviour of Ψ is expressed by a z-independent matrix, the *monodromy matrix* $M(\gamma,\lambda)$. We will briefly state the results pertinent to this article, for more details see section 2.4 of [17].

Lemma 2.8. *Given a holomorphic potential $\tilde{\eta}$ on $\tilde{M}$ which is invariant under Γ in the sense of remark 2.3 and a class of loops $[\gamma] \in \Gamma$, any solution $\Psi : \tilde{M} \to \Lambda_r\mathrm{SL}(2,\mathbb{C})_\sigma$ to* (2.4.1) *will transform under the covering transformation $\tilde{\gamma} : z \mapsto [\gamma]\cdot z$ (cf. section A.4) according to*

$$\Psi(\tilde{\gamma}(z),\lambda) = M(\gamma,\lambda)\Psi(z,\lambda), \tag{2.5.1}$$

where $M(\gamma,\lambda)$ denotes some $\Lambda_r\mathrm{SL}(2,\mathbb{C})_\sigma$ matrix depending on $[\gamma]$, but independent of z. $M(\gamma,\lambda)$ is called the monodromy matrix of Ψ with respect to $[\gamma]$.

Remark 2.9. We would like to add some comments concerning the premises of the above lemma. Theorem A.14 of appendix A states how to construct the covering transformation $z \mapsto [\gamma]\cdot z$ from an element $[\gamma]$ of the fundamental group of M at a base point x. This construction involves the choice of a point $y \in \tilde{M}$, which is mapped to x by the universal covering $\pi : \tilde{M} \to M$. Thus, when speaking of "the" covering transformation on $\tilde{M}$ corresponding to an element $[\gamma] \in \Gamma$ (like in the above lemma), we tacitly assume that the necessary choices have already been made: First of all, we assume that we have chosen a base point $x \in M$, which allows for representing "the" fundamental group Γ of M by $\pi_1(M,x)$ and thus for working with loops γ based at x. Moreover, we assume that we have chosen a point $y \in \tilde{M}$, such that $\pi(y) = x$. In this framework we can apply theorem A.14 to obtain the covering transformation $\tilde{\gamma} : z \mapsto [\gamma]\cdot z$.

Remark 2.10. Carrying out an r-Iwasawa decomposition of a solution Ψ to (2.4.1), we obtain

$$\Psi = F B_+, \tag{2.5.2}$$

where $F \in \Lambda_r\mathrm{SU}(2)_\sigma$ and $B_+ \in \Lambda_r^+\mathrm{SL}(2,\mathbb{C})_\sigma$. Furthermore, the above lemma yields

$$\begin{aligned} F(\tilde{\gamma}(z),\lambda) = \Psi(\tilde{\gamma}(z),\lambda)B_+^{-1}(\tilde{\gamma}(z),\lambda) &= M(\gamma,\lambda)\Psi(z,\lambda)B_+^{-1}(z,\lambda)B_+(z,\lambda)B_+^{-1}(\tilde{\gamma}(z),\lambda) \\ &= M(\gamma,\lambda)F(z,\lambda)B_+(z,\lambda)B_+^{-1}(\tilde{\gamma}(z),\lambda), \end{aligned} \tag{2.5.3}$$

where $M(\gamma,\lambda)$ denotes the monodromy matrix of Ψ with respect to $[\gamma]$. Thus, in case that $M(\gamma,\lambda) \in \Lambda_r\mathrm{SU}(2)_\sigma$, we obtain

$$F(\tilde{\gamma}(z),\lambda) = M(\gamma,\lambda)F(z,\lambda)k(z,\tilde{\gamma},\lambda), \tag{2.5.4}$$

[4] Note that we assume throughout this work that M is path connected and that thus the fundamental groups of M at any two points in M are isomorphic to each other (cf. lemma A.4). We will therefore w.l.o.g. speak of "the" fundamental group of M, actually considering the fundamental group of M at an arbitrarily chosen point in M. Cf. section A.4 for more details on the action of Γ on $\tilde{M}$.

where $k(z, \tilde{\gamma}, \lambda) := B_+(z, \lambda) B_+^{-1}(\tilde{\gamma}(z), \lambda)$. Thereby, for fixed $z \in \tilde{M}$, we have $k \in \Lambda_r^+ \mathrm{SL}(2, \mathbb{C})_\sigma$ and at the same time $k \in \Lambda_r \mathrm{SU}(2)_\sigma$ (since $F \circ \gamma$, M and F are elements of $\Lambda_r \mathrm{SU}(2)_\sigma$). Thus, for fixed $z \in \tilde{M}$, k actually denotes a diagonal matrix in SU(2), which is independent of λ:

$$F(\tilde{\gamma}(z), \lambda) = M(\gamma, \lambda) F(z, \lambda) k(z, \tilde{\gamma}). \tag{2.5.5}$$

Under the above assumption, i.e. in the case $M(\gamma, \lambda) \in \Lambda_r \mathrm{SU}(2)_\sigma$, we will sometimes speak of the monodromy matrix $M(\gamma, \lambda)$ of F (with respect to $[\gamma]$), expressing in this way that $F(\tilde{\gamma}(z), \lambda)$ is linked to $F(z, \lambda)$ by $M(\gamma, \lambda)$ as in (2.5.5).

The basic theorem for all our considerations is obtained from theorem 2.7 of [11]:

Theorem 2.11. *Let M be a Riemann surface with universal cover $\tilde{M}$ and fundamental group Γ. Let $\tilde{\eta}$ be a holomorphic potential on $\tilde{M}$, which is invariant under Γ in the sense of remark 2.3. Furthermore, let Ψ be a solution to (2.4.1) and let $\psi : \tilde{M} \to \mathbb{R}^3$ be the CMC-immersion obtained from Ψ by the loop group method for $\lambda_0 = 1$. Then, ψ descends to a CMC-immersion ϕ on $M = \tilde{M}/\Gamma$ if and only if*

1. *$M(\gamma, \lambda)$ is unitary for all $[\gamma] \in \Gamma$, $\lambda \in S^1$* and
2. *$M(\gamma, \lambda = 1) = \pm \mathrm{I}$ for all $[\gamma] \in \Gamma$* and
3. *$\partial_\lambda M(\gamma, \lambda)|_{\lambda=1} = 0$ for all $[\gamma] \in \Gamma$.*

Theorem 2.11 provides the key for "tuning" the loop group method, such that it will generate a CMC-immersion $\psi = \psi_1$ on $\tilde{M}$ that descends to an immersion ϕ on M: Given a solution Ψ to (2.4.1), dressing it by $T = T(\lambda) \in \Lambda_r \mathrm{SL}(2, \mathbb{C})_\sigma$ will produce a new solution $\hat{\Psi} = T\Psi$. Denoting the monodromy matrices of Ψ by $M(\gamma, \lambda)$, $[\gamma] \in \Gamma$, the monodromy matrices of $\hat{\Psi}$ are then given by $\hat{M}(\gamma, \lambda) = T(\lambda) M(\gamma, \lambda) (T(\lambda))^{-1}$, $[\gamma] \in \Gamma$. (Again, note that this relation only holds in the setting of CMC-immersions with an umbilic point and thus in particular in the trinoid setting. [14])

Thus, to obtain a CMC-immersion $\phi : M \to \mathbb{R}^3$ from a given potential $\tilde{\eta}$, the strategy will be to find an appropriate dressing matrix T that will modify a given solution Ψ with monodromy matrices $M(\gamma, \lambda)$, $[\gamma] \in \Gamma$, such that the monodromy matrices $\hat{M}(\gamma, \lambda)$ of $\hat{\Psi} = T\Psi$ will meet the conditions given in theorem 2.11.

In particular, if a given solution Ψ to (2.4.1) is singular for certain values of $\lambda \in S^1$, also its monodromy matrices $M(\gamma, \lambda)$, $[\gamma] \in \Gamma$, will not even be defined for these values of λ. Thus, in this case, we need to find a dressing matrix T, which "removes" these singularities, such that $\hat{\Psi} = T\Psi$ is holomorphic in $\lambda \in S^1$.

2.6 Delaunay surfaces

As pointed out in the introduction, for the study of CMC-immersions with properly embedded annular ends *Delaunay surfaces* are of particular importance. These are CMC-surfaces of revolution around an axis in $\mathbb{R}^3$, the *Delaunay axis*, and parametrized by the punctured complex plane $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. For a detailed discussion of Delaunay surfaces, we refer to [16]. Here, we only summarize some basic results, which we will use in this work.

By section 3.2.1 of [16] all Delaunay surfaces (up to rigid motions) can be constructed from holomorphic potentials of the form

$$\tilde{\eta} = D\mathrm{d}z = \begin{pmatrix} 0 & X \\ \overline{X} & 0 \end{pmatrix} \mathrm{d}z, \tag{2.6.1}$$

where $X(\lambda) = s\lambda^{-1} + t\lambda$, $\overline{X}(\lambda) = s\lambda + t\lambda^{-1}$ and $s, t \in \mathbb{R}$ with $(s+t)^2 = \frac{1}{4}$. The matrix D is called a *Delaunay matrix*.

Remark 2.12. As for X and $\overline{X}$, we have adopted the notation of [17]. Note that, while $\overline{X}(\lambda)$ yields the usual complex conjugate of $X(\lambda)$ for $\lambda \in S^1$, we interpret $\overline{X}$ for values of $\lambda \in \mathbb{C}^* \setminus S^1$ as the unique holomorphic extension of the complex conjugate of $X|_{S^1}$ to $\mathbb{C}^*$. This motivates the definition of $\overline{X}$ above for all $\lambda \in \mathbb{C}^*$. Moreover, for $\lambda \in \mathbb{C}^*$, X and $\overline{X}$ are linked via the formula

$$\overline{X(\lambda)} = \overline{X}(\frac{1}{\overline{\lambda}}). \tag{2.6.2}$$

Remark 2.13. The choice of s and t will determine the special shape of the produced Delaunay surface (see [16] for details). We are especially interested in *embedded* Delaunay surfaces, which are also referred to as *unduloids*. These correspond to s and t such that $st > 0$, i.e. either $s, t > 0$ or $s, t < 0$. However, since the potentials $D\mathrm{d}z$ and $-D\mathrm{d}z$ are gauge equivalent (and thus produce the same surfaces, cf. section 2.4), all unduloids can be obtained from potentials of the form (2.6.1) with $s, t > 0$, $s + t = \frac{1}{2}$.

Given a potential $\tilde{\eta}$ of the form (2.6.1), it is easy to verify that

$$\Psi = e^{zD} \tag{2.6.3}$$

solves the differential equation (2.4.1). Around the point $z = 0$, Ψ picks up the *Delaunay monodromy matrix* $M(\gamma, \lambda)$:

$$\Psi(\tilde{\gamma}(z), \lambda) = M(\gamma, \lambda)\Psi(z, \lambda), \tag{2.6.4}$$

where $\tilde{\gamma} : z \mapsto z + 2\pi i$ denotes the covering transformation corresponding to the simply closed curve γ in $\mathbb{C}^*$, which encloses the point $z = 0$, w.l.o.g. defined by $\gamma : [0, 1] \to \mathbb{C}^*$, $t \mapsto e^{2\pi i t}$. Note that γ already generates the fundamental group Γ of $\mathbb{C}^*$. A simple computation yields

$$M(\gamma, \lambda) = e^{2\pi i D}. \tag{2.6.5}$$

Via the loop group method, Ψ gives rise to a CMC-immersion $\psi = \psi_1$ defined on the universal cover of $\mathbb{C} \setminus \{0\}$. Applying theorem 2.11, we prove that ψ descends to a CMC-surface ϕ on $\mathbb{C}^*$ by showing that the monodromy matrix $M(\gamma, \lambda) = e^{2\pi i D}$ of Ψ with respect to γ meets the conditions of theorem 2.11.

We restrict to the unduloid case $s, t > 0$, $s + t = \frac{1}{2}$ (cf. remark 2.13). Moreover, we assume $s \geq t$. Despite those restrictions, it turns out that the case $s = t$ needs to be treated separately.

Let first $s = t$. In particular, since $(s + t)^2 = \frac{1}{4}$, this implies $st > 0$. In view of remark 2.13, we can assume $s, t > 0$. Together, this implies $s = t = \frac{1}{4}$ and thus $X(\lambda) = \overline{X}(\lambda) = \frac{1}{4}(\lambda^{-1} + \lambda)$, which allows for writing

$$D = \frac{1}{4}(\lambda^{-1} + \lambda)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{4}(\lambda^{-1} + \lambda)\tilde{S}\sigma_3\tilde{S}^{-1}, \tag{2.6.6}$$

where $\tilde{S} := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Consequently, we derive that

$$e^{2\pi i D} = \tilde{S}e^{\frac{\pi i}{2}(\lambda^{-1}+\lambda)\sigma_3}\tilde{S}^{-1} = \tilde{S}\begin{pmatrix} e^{\frac{\pi i}{2}(\lambda^{-1}+\lambda)} & 0 \\ 0 & e^{-\frac{\pi i}{2}(\lambda^{-1}+\lambda)} \end{pmatrix}\tilde{S}^{-1}. \tag{2.6.7}$$

This shows that $M(\gamma, \lambda) = e^{2\pi i D}$ is unitary for all $\lambda \in S^1$ (as a product of unitary matrices), which means that the first condition of theorem 2.11 is met. Moreover,

$$e^{2\pi i D}|_{\lambda=1} = \tilde{S}\begin{pmatrix} e^{\pi i} & 0 \\ 0 & e^{-\pi i} \end{pmatrix}\tilde{S}^{-1} = -\mathrm{I}, \tag{2.6.8}$$

which proves that the second condition of theorem 2.11 is met. Finally, we have

$$(\partial_\lambda e^{2\pi i D})_{\lambda=1} = \tilde{S}\begin{pmatrix} \frac{\pi i}{2}e^{\frac{\pi i}{2}(\lambda^{-1}+\lambda)}(1 - \lambda^{-2}) & 0 \\ 0 & -\frac{\pi i}{2}e^{\frac{\pi i}{2}(\lambda^{-1}+\lambda)}(1 - \lambda^{-2}) \end{pmatrix}_{\lambda=1}\tilde{S}^{-1} = 0, \tag{2.6.9}$$

which means that also the third condition of theorem 2.11 is satisfied. Altogether, by theorem 2.11, Ψ induces in the case $s = t$ a CMC-immersion ϕ on $\mathbb{C}^*$.

Let now $s \neq t$, i.e. $s > t$. We consider the function

$$\mu(\lambda) = \sqrt{X(\lambda)\overline{X}(\lambda)} \tag{2.6.10}$$

from appendix B, which (in the present case $s \neq t$) is holomorphic and non-zero on a sufficiently small neighborhood of S^1 in $\mathbb{C}^*$ (cf. lemmaB.1 and remark B.2). In view of the relation

$$(\mu(\lambda))^2 = X(\lambda)\overline{X}(\lambda) = s^2(1 + \frac{t}{s}\lambda^2)(1 + \frac{t}{s}\lambda^{-2}), \tag{2.6.11}$$

we infer that

$$\mu(\lambda) = \epsilon s\mu_+(\lambda)\mu_-(\lambda), \tag{2.6.12}$$

where $\epsilon \in \{\pm 1\}$ and the mappings μ_+ and μ_-, respectively, are well defined (recall $s > t$) by

$$\mu_+(\lambda) = \sqrt{1 + \frac{t}{s}\lambda^2} := 1 + \frac{1}{2}\frac{t}{s}\lambda^2 - \frac{1}{8}\frac{t^2}{s^2}\lambda^4 \pm \dots, \tag{2.6.13}$$

$$\mu_-(\lambda) = \sqrt{1 + \frac{t}{s}\lambda^{-2}} := 1 + \frac{1}{2}\frac{t}{s}\lambda^{-2} - \frac{1}{8}\frac{t^2}{s^2}\lambda^{-4} \pm \dots. \tag{2.6.14}$$

Note that μ_+ is (at least) holomorphic and non-zero for $\lambda \in \{\lambda \in \mathbb{C}; |\lambda| < \sqrt{\frac{s}{t}}\}$, while μ_- is (at least) holomorphic and non-zero for $\lambda \in \{\lambda \in \mathbb{C}; |\lambda| > \sqrt{\frac{t}{s}}\}$. In particular, both μ_+ and μ_- are holomorphic and non-zero on a sufficiently small neighborhood of S^1 in $\mathbb{C}^*$. Moreover, μ_+ and μ_-, respectively, allow for explicity defining the complex square roots

$$\sqrt{\mu_+(\lambda)} := 1 + \frac{1}{4}\frac{t}{s}\lambda^2 - \frac{3}{32}\frac{t^2}{s^2}\lambda^4 \pm \dots, \tag{2.6.15}$$

$$\sqrt{\mu_-(\lambda)} := 1 + \frac{1}{4}\frac{t}{s}\lambda^{-2} - \frac{3}{32}\frac{t^2}{s^2}\lambda^{-4} \pm \dots, \tag{2.6.16}$$

which (by analogous arguments) are also holomorphic and non-zero on a sufficiently small neighborhood of S^1 in $\mathbb{C}^*$.

In view of the considerations above, we set for $\mu = \epsilon s \mu_+ \mu_-$ with $\epsilon \in \{\pm 1\}$ from now on

$$\sqrt{\mu(\lambda)} := \tilde{\epsilon}\sqrt{s}\sqrt{\mu_+(\lambda)}\sqrt{\mu_-(\lambda)}, \tag{2.6.17}$$

where $\tilde{\epsilon} = 1$ if $\epsilon = 1$ and $\tilde{\epsilon} = i$ if $\epsilon = -1$. Moreover, we define for the mappings $\lambda X(\lambda) = s(1 + \frac{t}{s}\lambda^2) = s(\mu_+(\lambda))^2$ and $\lambda^{-1}\overline{X}(\lambda) = s(1 + \frac{t}{s}\lambda^{-2}) = s(\mu_-(\lambda))^2$ the square roots

$$\sqrt{\lambda X(\lambda)} := \sqrt{s}\mu_+(\lambda), \tag{2.6.18}$$

$$\sqrt{\lambda^{-1}\overline{X}(\lambda)} := \epsilon\sqrt{s}\mu_-(\lambda). \tag{2.6.19}$$

Altogether, the mappings $\sqrt{\mu)}$, $\sqrt{\lambda X}$ and $\sqrt{\lambda^{-1}\overline{X}}$ defined as above are holomorphic and non-zero on a sufficiently small neighborhood of S^1 in $\mathbb{C}^*$. Consequently, we infer that the expressions

$$\frac{\sqrt{\lambda X}}{\sqrt{\mu}}, \frac{\sqrt{\lambda^{-1}\overline{X}}}{\sqrt{\mu}} \tag{2.6.20}$$

are well defined and holomorphic on a sufficiently small neighborhood U of S^1 in $\mathbb{C}^*$. Therefore we can proceed by writing for all $\lambda \in U$:

$$D = \begin{pmatrix} 0 & X \\ \overline{X} & 0 \end{pmatrix} = \mu R S \sigma_3 S^{-1} R^{-1}, \tag{2.6.21}$$

where

$$R = \begin{pmatrix} \frac{\sqrt{\lambda X}}{\sqrt{\mu}} & 0 \\ 0 & \frac{\sqrt{\lambda^{-1}\overline{X}}}{\sqrt{\mu}} \end{pmatrix}, \quad S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\lambda^{-1} \\ \lambda & 1 \end{pmatrix}. \tag{2.6.22}$$

(Note that, since $\sqrt{\lambda X}\sqrt{\lambda^{-1}\overline{X}} = \sqrt{s}\mu_+\epsilon\sqrt{s}\mu_- = \mu$, we have $\det(R) = 1$. Moreover, we obviously have $\det(S) = 1$.)

By (2.6.21) we infer that for all $\lambda \in U$

$$e^{2\pi i D} = R S e^{2\pi i \mu \sigma_3} S^{-1} R^{-1} = R S \begin{pmatrix} e^{2\pi i \mu} & 0 \\ 0 & e^{-2\pi i \mu} \end{pmatrix} S^{-1} R^{-1}. \tag{2.6.23}$$

Since D is Hermitian for $\lambda \in S^1$, $2\pi i D$ is skew-Hermitian for $\lambda \in S^1$. This implies directly that $M(\gamma, \lambda) = e^{2\pi i D}$ is unitary for $\lambda \in S^1$, which means that the first condition of theorem 2.11 is met. Moreover, using $\mu(\lambda = 1) = \frac{1}{2}$ from lemma B.3, we infer from (2.6.23) that

$$e^{2\pi i D}|_{\lambda=1} = R|_{\lambda=1} S|_{\lambda=1} \begin{pmatrix} e^{\pi i} & 0 \\ 0 & e^{-\pi i} \end{pmatrix} (S^{-1})|_{\lambda=1} (R^{-1})|_{\lambda=1} = -\mathrm{I}, \tag{2.6.24}$$

which proves that the second condition of theorem 2.11 is met. Finally, using the fact that the matrices R, R^{-1}, S and R^{-1} are holomorphic at $\lambda = 1$ together with $(\partial_\lambda \mu(\lambda))_{\lambda=1} = 0$ from lemma B.3, we have

$$
\begin{aligned}
(\partial_\lambda e^{2\pi i D})_{\lambda=1} = \Bigg[(\partial_\lambda R) S \begin{pmatrix} e^{2\pi i\mu} & 0 \\ 0 & e^{-2\pi i\mu} \end{pmatrix} S^{-1} R^{-1} + R(\partial_\lambda S) \begin{pmatrix} e^{2\pi i\mu} & 0 \\ 0 & e^{-2\pi i\mu} \end{pmatrix} S^{-1} R^{-1} \\
+ RS\partial_\lambda \begin{pmatrix} e^{2\pi i\mu} & 0 \\ 0 & e^{-2\pi i\mu} \end{pmatrix} S^{-1} R^{-1} - RS \begin{pmatrix} e^{2\pi i\mu} & 0 \\ 0 & e^{-2\pi i\mu} \end{pmatrix} S^{-1} (\partial_\lambda S) S^{-1} R^{-1} \\
- RS \begin{pmatrix} e^{2\pi i\mu} & 0 \\ 0 & e^{-2\pi i\mu} \end{pmatrix} S^{-1} R^{-1} (\partial_\lambda R) R^{-1} \Bigg]_{\lambda=1} \\
= \Bigg[-(\partial_\lambda R) R^{-1} - R(\partial_\lambda S) S^{-1} R^{-1} + RS \begin{pmatrix} 2\pi i e^{2\pi i\mu} \partial_\lambda \mu & 0 \\ 0 & -2\pi i e^{-2\pi i\mu} \partial_\lambda \mu \end{pmatrix} S^{-1} R^{-1} \\
+ R(\partial_\lambda S) S^{-1} R^{-1} + (\partial_\lambda R) R^{-1} \Big]_{\lambda=1} = 0, \quad (2.6.25)
\end{aligned}
$$

which means that also the third condition of theorem 2.11 is satisfied. Altogether, by theorem 2.11, Ψ induces also in the case $s > t$ a CMC-immersion ϕ on $\mathbb{C}^*$.

3 Trinoids

We have introduced trinoids as CMC-immersions $\mathcal{T}_3 \to \mathbb{R}^3$ of the thrice-punctured two-sphere into $\mathbb{R}^3$. Without loss of generality, we will assume that the three points removed are $S = (0,0,-1)^T$, $P = (1,0,0)^T$ and $N = (0,0,1)^T$ in $\mathbb{R}^3$, i.e. $\mathcal{T}_3 = S^2 \setminus \{S,P,N\}$. However, during the course of this thesis it will be sometimes convenient to consider alternative models for both the trinoid domain and the target space of the trinoid. We therefore identify the thrice-punctured two-sphere with the complex plane $\mathbb{C}$ with the two points 0 and 1 removed (or, equivalently, with the extended complex plane $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ with the three points 0, 1 and ∞ removed). The identifying map is a restriction of the well known stereographic projection $S^2 \to \hat{\mathbb{C}}$. We also identify the target space $\mathbb{R}^3$ of the trinoid with the matrix Lie algebra $\mathrm{su}(2) = \{\frac{1}{2}\begin{pmatrix} -iz & -ix-y \\ -ix+y & iz \end{pmatrix};\ x,y,z \in \mathbb{R}\}$. The identifying map $\mathbb{R}^3 \to \mathrm{su}(2)$ is denoted by J. Using the standard inner product on $\mathbb{R}^3$ and defining an inner product on su(2) by $< A,B > := -2\mathrm{trace}(AB)$ it is easy to verify that J is an isometry. Furthermore, J defines a Lie algebra homomorphism between $\mathbb{R}^3$ equipped with the cross product and su(2) equipped with the Lie bracket $[A,B] := AB - BA$. These alternative models of the trinoid domain and the target space of the trinoid are explained in detail in sections 3.1 and 3.4, respectively.

In sections 3.2 and 3.3, we study the universal cover of the trinoid domain as well as the monodromy action of its fundamental group on the universal cover. The reader who is not familiar with these notions is referred to appendix A, where we give a basic introduction to the underlying topological concepts, based on the book of Fulton [20].

In sections 3.5 to 3.9 we introduce the *trinoid potential* η, which produces trinoids via the DPW-method. Moreover, we explicitly compute a family of solutions to the differential equation (2.4.1), whose members have unitary monodromy matrices with respect to the elements of the fundamental group of M.

3.1 Trinoids on the domain $M = \mathbb{C} \setminus \{0,1\}$

Trinoids are CMC-immersions of the thrice-punctured two-sphere $\mathcal{T}_3$ into $\mathbb{R}^3$. As stated earlier, we can assume w.l.o.g. that the three points removed from the two-sphere are located at $S = (0,0,-1)^T$, $P = (1,0,0)^T$ and $N = (0,0,1)^T$ in $\mathbb{R}^3$. Thus a trinoid is originally defined as a CMC-immersion $\mathcal{T}_3 = S^2 \setminus \{S,P,N\} \to \mathbb{R}^3$. However, we find it more convenient to interpret a trinoid ϕ as a CMC-immersion of the twice-punctured complex plane (or, equivalently, the thrice-punctured extended complex plane), i.e. $\phi : M \to \mathbb{R}^3$, where

$$M = \mathbb{C} \setminus \{0,1\} = \hat{\mathbb{C}} \setminus \{0,1,\infty\}. \tag{3.1.1}$$

This is possible by identifying $\mathcal{T}_3$ and M by the well known *stereographic projection*

$$p : S^2 \to \hat{C}, \quad (x,y,z)^T \mapsto \begin{cases} \frac{x}{1-z} + i\frac{y}{1-z} & \text{for } z \neq 1 \\ \infty & \text{for } z = 1 \end{cases} \tag{3.1.2}$$

that defines a diffeomorphism $S^2 \to \hat{C}$, i.e. a differentiable bijection $S^2 \to \hat{C}$ with differentiable inverse mapping. As $p(S) = 0$, $p(P) = 1$ and $p(N) = \infty$, p in particular allows to identify $\mathcal{T}_3 = S^2 \setminus \{S,N,P\}$ and $\hat{\mathbb{C}} \setminus \{0,1,\infty\} = M$.

Thus, given a CMC-immersion $\phi_0 : \mathcal{T}_3 \to \mathbb{R}^3$, the mapping $\phi := \phi_0 \circ p^{-1}$ defines a CMC-immersion of M into $\mathbb{R}^3$ parametrizing the same surface, $\phi(M) = \phi_0(\mathcal{T}_3)$.

From now on, we interpret trinoids as CMC-immersions $M \to \mathbb{R}^3$. Furthermore, reparametrizing a given trinoid $\phi : M \to \mathbb{R}^3$ (or, more precisely, the associated surface $\phi(M)$) if needed, we will assume without loss of generality that ϕ is *conformal*, i.e. that the metric on $\phi(M)$ induced by ϕ is given by $\mathrm{d}s^2 = e^u(\mathrm{d}x^2 + \mathrm{d}y^2)$ for some real valued function $u : M \to \mathbb{R}$. (In other words, we assume without loss of generality that $\phi(M)$ is parametrized in conformal coordinates, which is always possible; cf., e.g., [1].) With respect to these considerations, we give the following (adjusted) definitions:

Definition 3.1. Let $M = \mathbb{C} \setminus \{0,1\}$ and, for $j = 0,1,\infty$, $z_j = j \in \hat{C}$.

1. A conformal CMC-immersion $\phi : M \to \mathbb{R}^3$ is called a *trinoid* (on M).
2. Let $\phi : M \to \mathbb{R}^3$ be a trinoid. A non-empty subset $B_j \subseteq \phi(M)$ is called an *annular end of ϕ (at z_j)*, if there exists a punctured neighborhood U_j of z_j in M, such that $B_j = \phi(U_j)$ and $\lim_{z \to z_j} \phi(z) = \infty$. Without loss of generality, if not stated otherwise, we will assume that U_j is open in M and that $U_j \cup \{z_j\}$ is simply connected in $\hat{\mathbb{C}}$.

3. Let $\phi : M \to \mathbb{R}^3$ be a trinoid and B_j be an annular end of ϕ at z_j with $B_j = \phi(U_j)$ for an appropriate punctured neighborhood U_j of z_j in M. B_j is called a *(properly) embedded annular end of ϕ (at z_j)*, if the mapping $\phi|_{U_j} : U_j \to \mathbb{R}^3$ is a (proper) embedding.[5]

We complete this section by recording the following result:

Lemma 3.2. *Let $M = \mathbb{C} \setminus \{0, 1\}$ and $\phi : M \to \mathbb{R}^3$ be a trinoid with three embedded annular ends. Then, M with the metric induced by ϕ is complete.*

Remark 3.3. The proof of lemma 3.2 involves basic concepts and results of analysis and differential geometry. We assume the reader is familiar with the notions of a *(complete) metric space* and the notion of a *(geodesically) complete Riemannian manifold.*

Proof of lemma 3.2. By a well known result of differential geometry ("Hopf-Rinow"), it is enough to show that M (interpreted as a Riemannian manifold) is geodesically complete, i.e. that at any point $p \in M$ and for each unit vector v from the tangent plane T_pM to M at p the geodesic γ at p in the direction of v, $\gamma(t) = \exp(tv)$, is defined for all $t \in \mathbb{R}$.

Let $p \in M$ and $v \in T_pM$ with $|v| = 1$. By theorem 1 of section 4-7 and lemma 1 of section 4-6 of [6], there exists $\epsilon > 0$ such that the curve $\gamma(t) = \exp(tv)$ defines a geodesic in M for all $t \in I = (-\epsilon, \epsilon)$. We extend the curve $\gamma : I \to M$ to the maximal interval $(s_0, t_0) \subseteq \mathbb{R}$ such that $\gamma : (s_0, t_0) \to M$, $\gamma(t) = \exp(tv)$, defines a geodesic in M.

Consider a sequence t_n, $n \in \mathbb{N}$, in (s_0, t_0) with $\lim_{n\to\infty} t_n = t_0$. The corresponding sequence $\gamma(t_n)$, $n \in \mathbb{N}$, in M possesses an accumulation point p_0 in the compact superset $\hat{\mathbb{C}}$ of M and thus a subsequence, which converges to p_0. Denoting this subsequence by abuse of notation again by $\gamma(t_n)$, $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \gamma(t_n) = p_0 \tag{3.1.3}$$

Assume $p_0 \in M$. Then, by theorem 1a of section 4-7 of [6], there exist $\epsilon_1, \epsilon_2 > 0$ such that at any point $q \in M$, which lies inside the sphere $B_{\epsilon_1}(p_0)$ of radius ϵ_1 around p_0, and for all $w \in T_qM$ with $|w| = 1$ the curve $\delta : (-\epsilon_2, \epsilon_2) \to M$, $\delta(t) = \exp(tw)$, defines a geodesic in M with $\delta(0) = q$. By (3.1.3) and since $\lim_{n\to\infty} t_n = t_0$, there exists $n_0 \in \mathbb{N}$ such that $\gamma(t_{n_0}) \in M \cap B_{\epsilon_1}(p_0)$ and $t_0 - t_{n_0} < \epsilon_2$. Thus, the curve $\delta_0 : (-\epsilon_2, \epsilon_2) \to M$, $\delta_0(t) = \exp(tw_0)$ with $w_0 = \gamma'(t_{n_0}) \in T_{\gamma(t_{n_0})}M$ defines a geodesic in M. Moreover, $\delta_0(0) = \gamma(t_{n_0})$ and $\delta_0(t) = \gamma(t + t_{n_0})$ for all $t \in (-\epsilon_2, \epsilon_2)$, i.e. δ_0 extends the geodesic $\gamma : (s_0, t_0) \to M$ beyond t_0 to a geodesic defined on the interval $(s_0, t_{n_0} + \epsilon_2) \supset (s_0, t_0)$, a contradiction to the assumption that (s_0, t_0) defines the maximal interval in $\mathbb{R}$, where γ is a geodesic. Thus it remains to consider $p_0 \in \hat{\mathbb{C}} \setminus M$, i.e. $p_0 \in \{0, 1, \infty\}$. Then, by (3.1.3) and by the fact that ϕ defines an embedding on a small enough punctured neighborhood of p_0 with $\lim_{z\to p_0} \phi(z) = \infty$, we infer that

$$\lim_{n\to\infty} \phi(\gamma(t_n)) = \infty. \tag{3.1.4}$$

This implies $\lim_{n\to\infty} d(\phi(p), \phi(\gamma(t_n))) = \infty$, where d denotes the metric on $\phi(M)$. Consequently, denoting by d_M the induced metric on M, we obtain

$$\infty = \lim_{n\to\infty} d(\phi(p), \phi(\gamma(t_n))) = \lim_{n\to\infty} d_M(p, \gamma(t_n)) = \lim_{n\to\infty} t_n = t_0, \tag{3.1.5}$$

which implies that γ, $\gamma(t) = \exp(tv)$, defines a geodesic on the interval (s_0, ∞).

Considering a sequence s_n, $n \in \mathbb{N}$, in (s_0, ∞) with $\lim_{n\to\infty} s_n = s_0$, we infer by the analogous argument as above that $s_0 = -\infty$ and conclude that γ, $\gamma(t) = \exp(tv)$, actually defines a geodesic in M on the interval $(-\infty, \infty) = \mathbb{R}$, which finishes the proof. □

3.2 The universal cover $\tilde{M}$ of M

It is well known that the universal cover $\tilde{M}$ of the twice-punctured complex plane M can be taken to be the upper half plane

$$\tilde{M} = \mathbb{H} := \{z = x + iy \in \mathbb{C}; x, y \in \mathbb{R}, y > 0\}. \tag{3.2.1}$$

[5] An embedding $U \to V$ is called *proper*, if inverse images of compact subsets of V are compact.

The corresponding covering map[6] is given by

$$\pi : \tilde{M} \to M, \quad \pi(z) := \frac{\wp(\frac{1}{2}; \mathbb{Z}z + \mathbb{Z}) - \wp(\frac{z+1}{2}; \mathbb{Z}z + \mathbb{Z})}{\wp(\frac{1}{2}; \mathbb{Z}z + \mathbb{Z}) - \wp(\frac{z}{2}; \mathbb{Z}z + \mathbb{Z})}, \tag{3.2.2}$$

where $\wp$ denotes the *Weierstrass function*

$$\wp(z; \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) := \frac{1}{z^2} + \sum_{0 \neq \omega \in \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right), \tag{3.2.3}$$

which is defined for $z \in \mathbb{C}$ and $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ with $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$. As a function of its first variable z, $\wp$ is a meromorphic function on $\mathbb{C}$ with second order poles at the points of the period lattice

$$\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2. \tag{3.2.4}$$

More precisely, $\wp$ is an elliptic (i.e. doubly-periodic) function with respect to Ω and thus satisfies

$$\wp(z + \omega; \Omega) = \wp(z; \Omega) \text{ for all } z \in \mathbb{C} \text{ and all } \omega \in \Omega. \tag{3.2.5}$$

Moreover, considering the defining equation (3.2.3), we obtain

$$\wp(-z; \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) = \wp(z; \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2), \tag{3.2.6}$$

$$\wp(\mu z; \mathbb{Z}\mu\omega_1 + \mathbb{Z}\mu\omega_2) = \mu^{-2}\wp(z; \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) \text{ for all } \mu \in \mathbb{C} \setminus \{0\} \tag{3.2.7}$$

$$\wp(\bar{z}; \mathbb{Z}\bar{\omega}_1 + \mathbb{Z}\bar{\omega}_2) = \overline{\wp(z; \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)}. \tag{3.2.8}$$

A detailed study of the function $\wp$ can be found, e.g., in [24]. In the special case $\omega_2 = 1$ we additionally have $\mathbb{Z}\tilde{\omega}_1 + \mathbb{Z} = \mathbb{Z}\omega_1 + \mathbb{Z}$ for all $\tilde{\omega}_1 \in \mathbb{C}$ with $\tilde{\omega}_1 - \omega_1 \in \mathbb{Z}$ and therefore

$$\wp(z; \mathbb{Z}\tilde{\omega}_1 + \mathbb{Z}) = \wp(z; \mathbb{Z}\omega_1 + \mathbb{Z}) \text{ for all } \tilde{\omega}_1 \in \mathbb{C} \text{ with } \tilde{\omega}_1 - \omega_1 \in \mathbb{Z}. \tag{3.2.9}$$

The following lemma records some useful properties of the covering map $\pi : \tilde{M} \to M$.

Lemma 3.4. *The covering map $\pi : \tilde{M} \to M$ as given in* (3.2.2) *satisfies*

$$\pi(z+1) = \frac{1}{\pi(z)}, \tag{3.2.10}$$

$$\pi(-\frac{1}{z}) = 1 - \pi(z), \tag{3.2.11}$$

$$\pi(-\bar{z}) = \overline{\pi(z)}. \tag{3.2.12}$$

Remark 3.5. Note that for $z \in \tilde{M}$ also $z+1$, $-\frac{1}{z}$ and $-\bar{z}$ are elements of $\tilde{M}$. Therefore the left-hand sides of the equations stated in the above lemma are well defined.

Proof of Lemma 3.4. We start with the proof of (3.2.10). Using equations (3.2.9) and (3.2.5) we obtain

$$\wp(\frac{1}{2}; \mathbb{Z}(z+1) + \mathbb{Z}) = \wp(\frac{1}{2}; \mathbb{Z}z + \mathbb{Z}), \tag{3.2.13}$$

$$\wp(\frac{(z+1)+1}{2}; \mathbb{Z}(z+1) + \mathbb{Z}) = \wp(\frac{z}{2} + 1; \mathbb{Z}z + \mathbb{Z}) = \wp(\frac{z}{2}; \mathbb{Z}z + \mathbb{Z}), \tag{3.2.14}$$

$$\wp(\frac{z+1}{2}; \mathbb{Z}(z+1) + \mathbb{Z}) = \wp(\frac{z+1}{2}; \mathbb{Z}z + \mathbb{Z}) \tag{3.2.15}$$

and thus

$$\pi(z+1) = \frac{\wp(\frac{1}{2}; \mathbb{Z}(z+1) + \mathbb{Z}) - \wp(\frac{(z+1)+1}{2}; \mathbb{Z}(z+1) + \mathbb{Z})}{\wp(\frac{1}{2}; \mathbb{Z}(z+1) + \mathbb{Z}) - \wp(\frac{z+1}{2}; \mathbb{Z}(z+1) + \mathbb{Z})} = \frac{\wp(\frac{1}{2}; \mathbb{Z}z + \mathbb{Z}) - \wp(\frac{z}{2}; \mathbb{Z}z + \mathbb{Z})}{\wp(\frac{1}{2}; \mathbb{Z}z + \mathbb{Z}) - \wp(\frac{z+1}{2}; \mathbb{Z}z + \mathbb{Z})} = \frac{1}{\pi(z)}. \tag{3.2.16}$$

[6]We have slightly modified the covering map $\tilde{M} \to M$, $z \mapsto \frac{\wp(\frac{1}{2}; \mathbb{Z}z+\mathbb{Z}) - \wp(\frac{z+1}{2}; \mathbb{Z}z+\mathbb{Z})}{\wp(\frac{z}{2}; \mathbb{Z}z+\mathbb{Z}) - \wp(\frac{z+1}{2}; \mathbb{Z}z+\mathbb{Z})}$ given in chapter I, §4 of [24] by composing it with the Moebius transformation $M \to M$, $z \mapsto \frac{z}{z-1}$. This way we ensure that the boundaries of the single sheets of our chosen tesselation of $\tilde{M}$ are mapped by the covering map π onto the "cuts" in $M = \mathbb{C}$ extending from 0 (resp. 1) to $-\infty$ (resp. $+\infty$) along the negative (resp. positive) real axis.

For the proof of (3.2.11) we apply (3.2.7), (3.2.5) and the identity of sets $\mathbb{Z} = -\mathbb{Z}$ to obtain

$$\wp(\frac{1}{2}; \mathbb{Z}(-\frac{1}{z}) + \mathbb{Z}) = z^2 \wp(\frac{z}{2}; \mathbb{Z}(-1) + \mathbb{Z}z) = z^2 \wp(\frac{z}{2}; \mathbb{Z}z + \mathbb{Z}), \tag{3.2.17}$$

$$\wp(\frac{-\frac{1}{z}+1}{2}; \mathbb{Z}(-\frac{1}{z}) + \mathbb{Z}) = z^2 \wp(\frac{-1+z}{2}; \mathbb{Z}(-1) + \mathbb{Z}z) = z^2 \wp(\frac{z+1}{2} - 1; \mathbb{Z}z + \mathbb{Z}) = z^2 \wp(\frac{z+1}{2}; \mathbb{Z}z + \mathbb{Z}), \tag{3.2.18}$$

$$\wp(\frac{-\frac{1}{z}}{2}; \mathbb{Z}(-\frac{1}{z}) + \mathbb{Z}) = z^2 \wp(\frac{-1}{2}; \mathbb{Z}(-1) + \mathbb{Z}z) = z^2 \wp(\frac{1}{2} - 1; \mathbb{Z}z + \mathbb{Z}) = z^2 \wp(\frac{1}{2}; \mathbb{Z}z + \mathbb{Z}). \tag{3.2.19}$$

This implies

$$\pi(-\frac{1}{z}) = \frac{\wp(\frac{1}{2}; \mathbb{Z}(-\frac{1}{z}) + \mathbb{Z}) - \wp(\frac{-\frac{1}{z}+1}{2}; \mathbb{Z}(-\frac{1}{z}) + \mathbb{Z})}{\wp(\frac{1}{2}; \mathbb{Z}(-\frac{1}{z}) + \mathbb{Z}) - \wp(\frac{-\frac{1}{z}}{2}; \mathbb{Z}(-\frac{1}{z}) + \mathbb{Z})} = \frac{\wp(\frac{z}{2}; \mathbb{Z}z + \mathbb{Z}) - \wp(\frac{z+1}{2}; \mathbb{Z}z + \mathbb{Z})}{\wp(\frac{z}{2}; \mathbb{Z}z + \mathbb{Z}) - \wp(\frac{1}{2}; \mathbb{Z}z + \mathbb{Z})} = 1 - \pi(z). \tag{3.2.20}$$

It remains to prove (3.2.12). Using equations (3.2.8) and (3.2.5) we obtain

$$\wp(\frac{1}{2}; \mathbb{Z}\bar{z} + \mathbb{Z}) = \overline{\wp(\frac{1}{2}; \mathbb{Z}z + \mathbb{Z})}, \tag{3.2.21}$$

$$\wp(\frac{1-\bar{z}}{2}; \mathbb{Z}\bar{z} + \mathbb{Z}) = \overline{\wp(\frac{1-z}{2}; \mathbb{Z}z + \mathbb{Z})} = \overline{\wp(\frac{1+z}{2} - z; \mathbb{Z}z + \mathbb{Z})} = \overline{\wp(\frac{1+z}{2}; \mathbb{Z}z + \mathbb{Z})}, \tag{3.2.22}$$

$$\wp(\frac{-\bar{z}}{2}; \mathbb{Z}\bar{z} + \mathbb{Z}) = \overline{\wp(\frac{-z}{2}; \mathbb{Z}z + \mathbb{Z})} = \overline{\wp(\frac{z}{2} - z; \mathbb{Z}z + \mathbb{Z})} = \overline{\wp(\frac{z}{2}; \mathbb{Z}z + \mathbb{Z})}. \tag{3.2.23}$$

Therefore,

$$\pi(-\bar{z}) = \frac{\wp(\frac{1}{2}; \mathbb{Z}\bar{z} + \mathbb{Z}) - \wp(\frac{1-\bar{z}}{2}; \mathbb{Z}\bar{z} + \mathbb{Z})}{\wp(\frac{1}{2}; \mathbb{Z}\bar{z} + \mathbb{Z}) - \wp(\frac{-\bar{z}}{2}; \mathbb{Z}\bar{z} + \mathbb{Z})} = \frac{\overline{\wp(\frac{1}{2}; \mathbb{Z}z + \mathbb{Z})} - \overline{\wp(\frac{1+z}{2}; \mathbb{Z}z + \mathbb{Z})}}{\overline{\wp(\frac{1}{2}; \mathbb{Z}z + \mathbb{Z})} - \overline{\wp(\frac{z}{2}; \mathbb{Z}z + \mathbb{Z})}} = \overline{\pi(z)}. \tag{3.2.24}$$

□

As a direct consequence of the above lemma we obtain

$$\pi(z+2) = \frac{1}{\pi(z+1)} = \pi(z), \tag{3.2.25}$$

$$\pi(\frac{z}{2z+1}) = 1 - \pi(\frac{-2z-1}{z}) = 1 - \pi(-2 - \frac{1}{z}) = 1 - \pi(-\frac{1}{z}) = \pi(z). \tag{3.2.26}$$

This shows that the two mappings

$$\tilde{U} : \tilde{M} \to \tilde{M}, \quad z \mapsto z + 2, \tag{3.2.27}$$

$$\tilde{S} : \tilde{M} \to \tilde{M}, \quad z \mapsto \frac{z}{2z+1} \tag{3.2.28}$$

satisfy

$$\pi \circ \tilde{U} = \pi \text{ and} \tag{3.2.29}$$

$$\pi \circ \tilde{S} = \pi, \tag{3.2.30}$$

respectively. As, moreover, $\tilde{U}$ and $\tilde{S}$ are homeomorphisms of $\tilde{M}$ (with inverse mappings $\tilde{U}^{-1} : z \mapsto z - 2$ and $\tilde{S}^{-1} : z \mapsto \frac{z}{-2z+1}$), $\tilde{U}$ and $\tilde{S}$ define two covering transformations on $\tilde{M}$. In fact, it turns out that the whole automorphism group of π (cf. appendix A, definition A.7) is generated by $\tilde{U}$ and $\tilde{S}$ (cf. chapter IV, 5 of [34]). Figure 3.1 below shows a tesselation of $\tilde{M}$ with respect to the sheet $\mathcal{F}$ given by

$$\mathcal{F} = \{z = x + iy \in \tilde{M}; -1 \le x < 1 \text{ and } |z + \frac{1}{2}| \ge \frac{1}{2} \text{ and } |z - \frac{1}{2}| > \frac{1}{2}\}. \tag{3.2.31}$$

Figure 3.2 shows in more detail where in M the different parts of $\mathcal{F}$ are mapped by π. (This can be validated by a direct computation.)

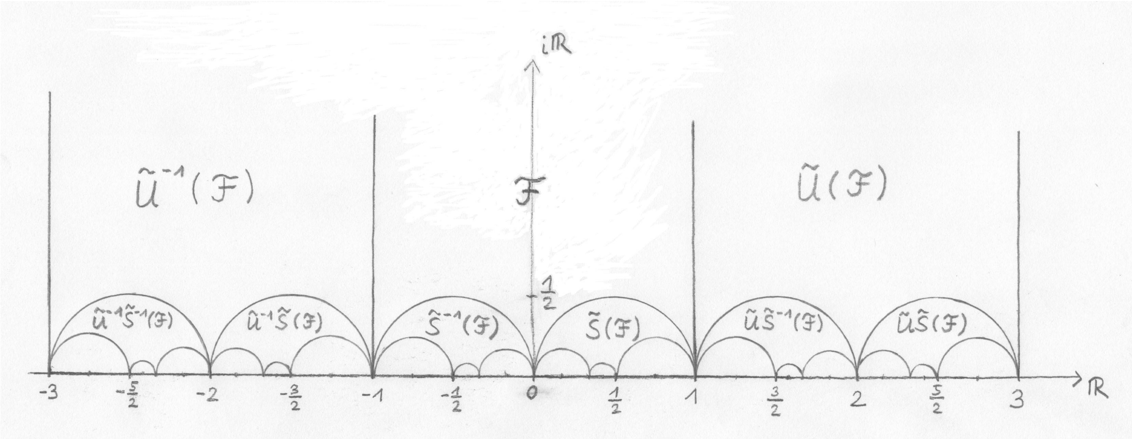

Figure 3.1: A tesselation of the upper half plane $\tilde{M} = \mathbb{H}$ with respect to $\mathcal{F}$.

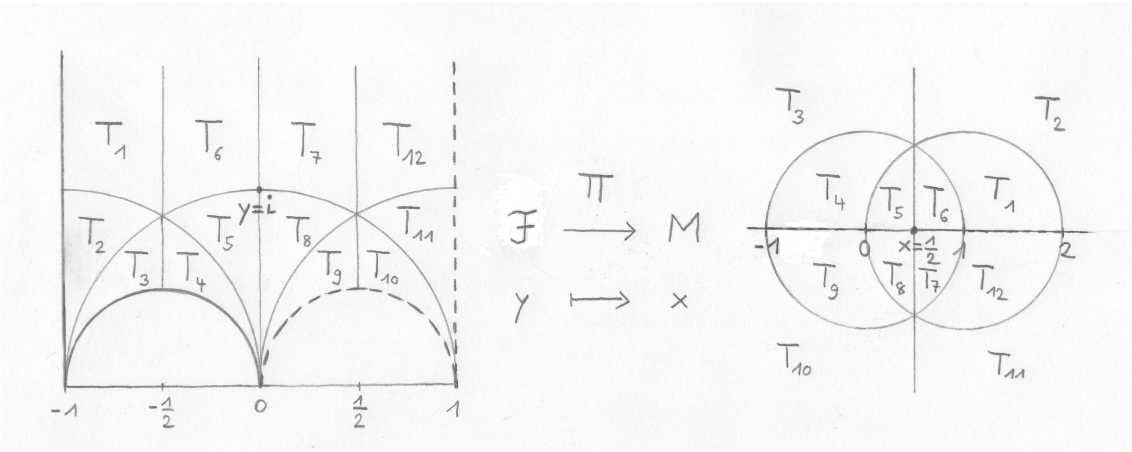

Figure 3.2: Correspondence between the main sheet $\mathcal{F}$ of $\tilde{M}$ and M via $\pi|_{\mathcal{F}}$

Remark 3.6. Naturally, the universal covering π defines a local homeomorphism $\tilde{M} \to M$. Note that, moreover, π is holomorphic on $\tilde{M}$ (cf. chapter I, §4 of [24]), which implies that π actually defines a *conformal* mapping $\tilde{M} \to M$ (cf. chapter VI, §1 of [4]). Consequently, to a given trinoid $\phi : M \to \mathbb{R}^3$, i.e. a CMC-immersion of M into $\mathbb{R}^3$, corresponds a CMC-immersion $\psi := \phi \circ \pi$ of $\tilde{M}$ into $\mathbb{R}^3$. Furthermore, ψ is conformal if and only if ϕ is conformal. I.e., to any conformal CMC-immersion $M \to \mathbb{R}^3$ corresponds a conformal CMC-immersion $\tilde{M} \to \mathbb{R}^3$.

Recall that, conversely, a given CMC-immersion $\psi : \tilde{M} \to \mathbb{R}^3$ "descends" only to a CMC-immersion $\phi : M \to \mathbb{R}^3$, if certain conditions are met (cf. theorem 2.11). However, if this is the case, and if, in addition, ψ is conformal, then ϕ will be conformal as well.

Altogether, we infer that the conformal CMC-immersions $M \to \mathbb{R}^3$ correspond via π to the "descending" conformal CMC-immersions $\tilde{M} \to \mathbb{R}^3$. As stated earlier, we can without loss of generality restrict our study of trinoids to the study of *conformal* CMC-immersions $M \to \mathbb{R}^3$. Consequently, we will also restrict ourselves to the study of *conformal* CMC-immersions $\tilde{M} \to \mathbb{R}^3$. Thus, we will tacitly assume from now on that any CMC-immersion $\psi : \tilde{M} \to \mathbb{R}^3$ produced by the loop group method has already been reparametrized into a conformal CMC-immersion of $\tilde{M}$ into $\mathbb{R}^3$.

3.3 The fundamental group Γ of M and its monodromy action on $\tilde{M}$

In this section, we introduce the fundamental group Γ of $M = \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$. (Recall that, as M is path-connected, Γ is up to isomorphisms independent of the choice of a base point in M.) Moreover, we explicitly construct the covering transformations on $\tilde{M}$ corresponding to the generating elements of Γ. The underlying ideas are provided in appendix A.

We first consider the fundamental group of the trinoid domain $\mathcal{T}_3 = S^2 \setminus \{S, P, N\}$. As $\mathcal{T}_3$ is path-connected, its fundamental group is up to isomorphisms independent of the choice of a base point in $\mathcal{T}_3$.

We choose the basepoint to be $(\frac{4}{5}, 0, -\frac{3}{5}) \in \mathcal{T}_3$. The fundamental group of $\mathcal{T}_3$ is then generated by the equivalence classes of two loops γ_S and γ_P based at $(\frac{4}{5}, 0, -\frac{3}{5})$, where γ_S (resp. γ_P) surrounds exactly once the point S (resp. P) without enclosing P and N (resp. S and N) at the same time. Thereby, we say that a loop γ in $\mathcal{T}_3$ *surrounds* or *encloses* a given point in S^2, if this point lies on the right hand side while "walking along" the loop γ from $\gamma(0)$ to $\gamma(1)$ on the "outside" of S^2. Note that any loop surrounding exactly once the point N (and neither S nor P) is homotopic to the loop product $\gamma_P^{-1} \cdot \gamma_S^{-1}$.

By applying the stereographic projection p defined in (3.1.2) we can translate the generating elements of the fundamental group of $S^2 \setminus \{S, P, N\}$ into the corresponding generating elements of the fundamental group of $M = \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$: The loop γ_S (resp. γ_P) is mapped by p onto a loop γ_0 (resp. γ_1) in M, which is based at $\frac{1}{2} = p((\frac{4}{5}, 0, -\frac{3}{5})) \in M$. Note that the stereographic projection "unfolds" the thrice-punctured two sphere onto the twice-punctured complex plane, relating the "outside" of S^2 to the "lower side" of $\hat{\mathbb{C}}$. Furthermore, p preserves orientation. Thus, as γ_S (resp. γ_P) keeps the point S (resp. P) on its right hand side when traced from $\gamma_S(0)$ to $\gamma_S(1)$ (resp. from $\gamma_P(0)$ to $\gamma_P(1)$) on the "outside" of S^2, γ_0 (resp. γ_1) keeps the point $0 = p(S)$ (resp. $1 = p(P)$) on its right hand side as well when traced from $\gamma_0(0)$ to $\gamma_0(1)$ (resp. from $\gamma_1(0)$ to $\gamma_1(1)$) on the "lower side" of $\hat{C}$. This means that - viewed "from underneath" $\hat{C}$ - γ_0 (resp. γ_1) encloses the point 0 (resp. 1) clockwise in M. Consequently, taking the more familiar point of view by looking at the extended complex plane $\hat{C}$ "from above", γ_0 (resp. γ_1) encloses the point 0 (resp. 1) *counter-clockwise* in M, keeping it on its *left* hand side while evolving from $\gamma_0(0)$ to $\gamma_0(1)$ (resp. from $\gamma_1(0)$ to $\gamma_1(1)$). From now on, we say that a loop γ in M *surrounds* or *encloses* a given point in $\hat{\mathbb{C}}$, if this point lies on the left hand side while "walking along" γ from $\gamma(0)$ to $\gamma(1)$ on the "upper side" of $\hat{\mathbb{C}}$. Naturally, as γ_S (resp. γ_P) encloses the point S (resp. P) in $\mathcal{T}_3$ exactly once without enclosing P and N (resp. S and N) at the same time, γ_0 (resp. γ_1) encloses the point 0 (resp. 1) in M exactly once without enclosing $1 = p(P)$ and $\infty = p(N)$ (resp. $0 = p(S)$ and $\infty = p(N)$) at the same time. Finally, $\gamma_\infty := \gamma_1^{-1} \cdot \gamma_0^{-1}$ defines a loop (based at $\frac{1}{2}$) surrounding the point ∞ exactly once without enclosing 0 and 1.

Altogether, the fundamental group Γ of M is generated by the homotopy equivalence classes of γ_0 and γ_1:

$$\Gamma =< [\gamma_0], [\gamma_1] > . \tag{3.3.1}$$

Next, by applying theorem A.14, we construct the covering transformations $\tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\gamma}_\infty : \tilde{M} \to \tilde{M}$ corresponding to the loops γ_0, γ_1 and γ_∞ in M. Note that, as any loop homotopic to γ_0 (resp. γ_1, resp. γ_∞) induces the same covering transformation $\tilde{\gamma}_0$ (resp. $\tilde{\gamma}_1$, resp. $\tilde{\gamma}_\infty$), it is enough to define γ_0, γ_1 and γ_∞ only "qualitatively" (as done above).

As Γ is based at the point $x = \frac{1}{2} \in M$, we need to choose a point $y \in \tilde{M}$, such that $\pi(y) = x$. Let

$$y = i. \tag{3.3.2}$$

Using equation (3.2.11), we observe that y satisfies

$$\pi(y) = \pi(-\frac{1}{y}) = 1 - \pi(y) \tag{3.3.3}$$

and thus $\pi(y) = \frac{1}{2} = x$, as desired. Now, following the procedure described in section A.4, we denote by $\tilde{\gamma}_j(y)$ the endpoint of the unique lift of the loop γ_j to a path in $\tilde{M}$ starting at y. Keeping in mind that the point $\tilde{\gamma}_j(y)$ will be the same for any loop homotopic to γ_j, and bringing in our detailed knowledge about corresponding domains in $\tilde{M}$ and M (i.e. domains homeomorphic with respect to π), we can forgo any further technical calculations and determine the values $\tilde{\gamma}_j(y)$ by just looking at the figures below:

As the loop γ_0 (as given in the figure 3.3) runs from $\gamma_0(0)$ to $\gamma_0(1)$, it takes course through the subdomains T_5, T_4, T_9 and T_8 in M. Consequently, its lift starting at y takes course through the corresponding subdomains in $\tilde{M}$, ending at another preimage of x, namely at the point $\tilde{S}^{-1}(y)$ (cf. figure 3.1). This implies $\tilde{\gamma}_0(y) = \tilde{S}^{-1}(y)$. By use of theorem A.8, we conclude that actually $\tilde{\gamma}_0 = \tilde{S}^{-1}$ on $\tilde{M}$. Analogously, by tracing the loops γ_1 (resp. γ_∞) in the figure 3.4 (resp. figure 3.5) above through T_7, T_{12}, T_1 and T_6 (resp. T_6, T_1, T_{12}, T_7, T_8, T_9, T_4 and T_5) we obtain the corresponding lifts starting at y and ending at $\tilde{U}((y))$ (resp. at $\tilde{U}^{-1}(\tilde{S}(y))$). From this, we conclude as above that $\tilde{\gamma}_1 = \tilde{U}$ and $\tilde{\gamma}_\infty = \tilde{U}^{-1}\tilde{S}$ on $\tilde{M}$.

Summarizing our previous considerations, the elements $[\gamma_0]$, $[\gamma_1]$ and $[\gamma_\infty]$ in Γ give rise to the following

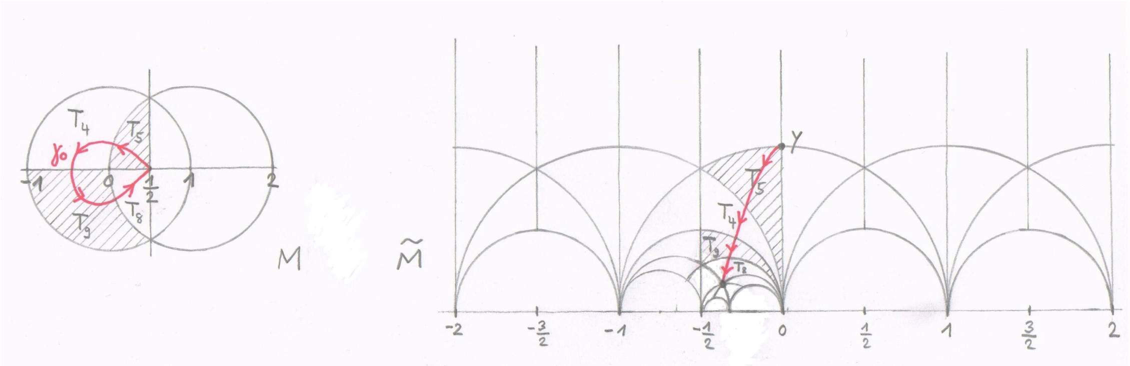

Figure 3.3: The loop γ_0 on M and its (qualitative) lift on $\tilde{M}$ connecting y and $\tilde{S}^{-1}(y) =: \tilde{\gamma}_0(y)$.

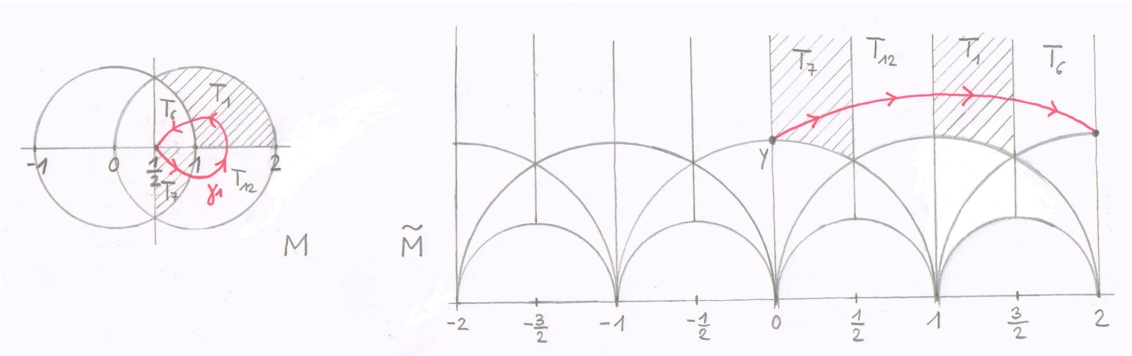

Figure 3.4: The loop γ_1 on M and its (qualitative) lift on $\tilde{M}$ connecting y and $\tilde{U}(y) =: \tilde{\gamma}_1(y)$.

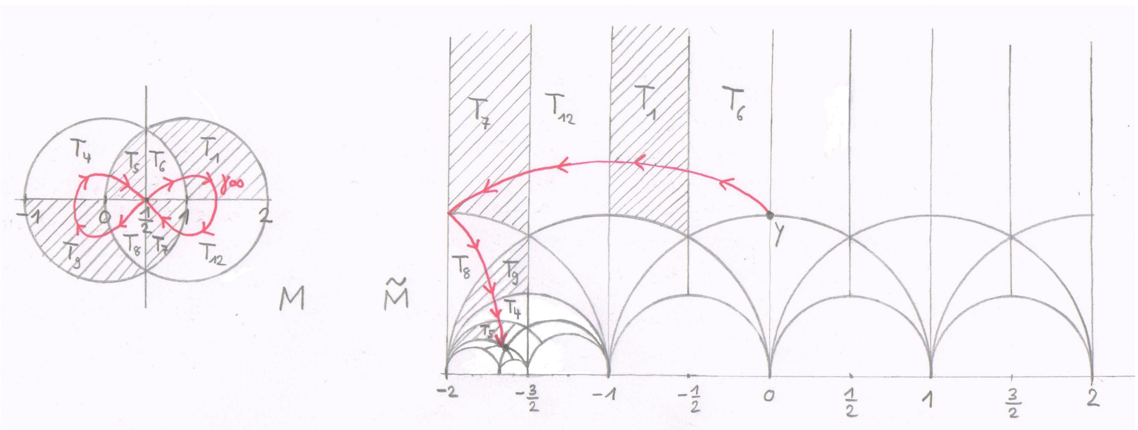

Figure 3.5: The loop γ_∞ on M and its (qualitative) lift on $\tilde{M}$ connecting y and $\tilde{U}^{-1}\tilde{S}(y) =: \tilde{\gamma}_\infty(y)$.

covering transformations $\tilde{\gamma}_0$, $\tilde{\gamma}_1$ and $\tilde{\gamma}_\infty$, respectively:

$$\tilde{\gamma}_0 : \tilde{M} \to \tilde{M}, \quad \tilde{\gamma}_0(z) = \tilde{S}^{-1}(z) = \frac{z}{-2z+1}, \tag{3.3.4}$$

$$\tilde{\gamma}_1 : \tilde{M} \to \tilde{M}, \quad \tilde{\gamma}_1(z) = \tilde{U}(z) = z + 2, \tag{3.3.5}$$

$$\tilde{\gamma}_\infty : \tilde{M} \to \tilde{M}, \quad \tilde{\gamma}_\infty(z) = \tilde{U}^{-1}\tilde{S}(z) = \frac{-3z-2}{2z+1}. \tag{3.3.6}$$

Note that, since $[\gamma_0]$ and $[\gamma_1]$ generate the fundamental group Γ of M, we infer by theorem A.14 that $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ generate the automorphism group $\mathrm{Aut}(\tilde{M}/M)$ of covering transformations $\tilde{M} \to \tilde{M}$. More precisely, we can state the following

Lemma 3.7.

$$\mathrm{Aut}(\tilde{M}/M) \;=\; <\tilde{\gamma}_0, \tilde{\gamma}_1> \;=\; \{\tilde{\gamma} : \tilde{M} \to \tilde{M}; \tilde{\gamma} \textit{ biholomorphic}, \pi \circ \tilde{\gamma} = \pi\} \tag{3.3.7}$$

Proof. As indicated before, the identity $\mathrm{Aut}(\tilde{M}/M) =< \tilde{\gamma}_0, \tilde{\gamma}_1 >$ is a direct consequence of (3.3.1) and theorem A.14. Moreover, in view of definition A.7 and using the fact that $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ define biholomorphic mappings $\tilde{M} \to \tilde{M}$, we infer that $\mathrm{Aut}(\tilde{M}/M) = \{\tilde{\gamma} : \tilde{M} \to \tilde{M}; \tilde{\gamma} \text{ biholomorphic}, \pi \circ \tilde{\gamma} = \pi\}$, which finishes the proof. □

Considering, again, a solution Ψ to the differential equation (2.4.1) as in section 2.5 and denoting the monodromy matrices of Ψ with respect to $[\gamma_j]$, $j = 0, 1, \infty$, by $M_j(\lambda) := M(\gamma_j, \lambda)$, $j = 0, 1, \infty$, we have by (2.5.1)

$$\Psi(\tilde{\gamma}_0(z), \lambda) = M_0(\lambda)\Psi(z, \lambda) \tag{3.3.8}$$

$$\Psi(\tilde{\gamma}_1(z), \lambda) = M_1(\lambda)\Psi(z, \lambda) \tag{3.3.9}$$

$$\Psi(\tilde{\gamma}_\infty(z), \lambda) = M_\infty(\lambda)\Psi(z, \lambda). \tag{3.3.10}$$

As $\tilde{\gamma}_0 \circ \tilde{\gamma}_1 \circ \tilde{\gamma}_\infty$ is the identity mapping on $\tilde{M}$, we have furthermore

$$\Psi(z, \lambda) = \Psi((\tilde{\gamma}_0 \circ \tilde{\gamma}_1 \circ \tilde{\gamma}_\infty)(z), \lambda) = M_0(\lambda)M_1(\lambda)M_\infty(\lambda)\Psi(z, \lambda), \tag{3.3.11}$$

which implies

$$M_0(\lambda)M_1(\lambda)M_\infty(\lambda) = \mathrm{I}. \tag{3.3.12}$$

3.4 The su(2) model of $\mathbb{R}^3$

After the study of the trinoid domain in the previous sections we now have a closer look at the target space of a trinoid, $\mathbb{R}^3$. Using the loop group method, we obtain for any $\lambda_0 \in S^1$ a CMC-immersion defined on the universal cover $\tilde{M}$ of the trinoid domain M by evaluating the Sym-Bobenko formula (2.4.3) at $\lambda = \lambda_0$. However, we observe that for a given extended frame $F \in \Lambda\mathrm{SU}(2)_\sigma$ the corresponding CMC-immersion

$$\mathrm{SymBob}(F)|_{\lambda=\lambda_0} = -\frac{1}{2H}(\frac{\partial}{\partial t}F \cdot F^{-1} + \frac{i}{2}F\sigma_3F^{-1})|_{\lambda=\lambda_0} \tag{3.4.1}$$

actually defines a mapping $\tilde{M} \to \mathrm{su}(2)$ from $\tilde{M}$ into the matrix Lie Algebra

$$\mathrm{su}(2) = \{\frac{-i}{2}\begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}; \quad (x, y, z) \in \mathbb{R}^3\}, \tag{3.4.2}$$

which is the Lie algebra of the matrix group SU(2). To obtain a mapping $\tilde{M} \to \mathbb{R}^3$ (as desired), we have to identify su(2) with the 3-dimensional Euclidean space $\mathbb{R}^3$. The corresponding identifying map is given by

$$J : \mathbb{R}^3 \to \mathrm{su}(2), \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} -iz & -ix - y \\ -ix + y & iz \end{pmatrix}. \tag{3.4.3}$$

Denoting by "$\times$" the cross product on $\mathbb{R}^3$ and by "$[\cdot, \cdot]$" the Lie bracket on su(2), we have for any two elements r_1, r_2 of $\mathbb{R}^3$ the identity

$$J(r_1 \times r_2) = [Jr_1, Jr_2]. \tag{3.4.4}$$

Due to this equation, J defines an isomorphism between the Lie Algebra $\mathbb{R}^3$ equipped with the cross product and the Lie Algebra su(2) equipped with the Lie bracket. Keeping this in mind, we can now state

$$\mathrm{SymBob}(F)|_{\lambda=\lambda_0} = -\frac{1}{2H}(\frac{\partial}{\partial t}F \cdot F^{-1} + \frac{i}{2}F\sigma_3F^{-1})|_{\lambda=\lambda_0} = J(\psi) \tag{3.4.5}$$

where ψ denotes the desired CMC-immersion $\tilde{M} \to \mathbb{R}^3$.

The automorphism group of the Lie Algebra $\mathbb{R}^3$ (equipped with the cross product) is given by the group SO(3) of all orthogonal 3×3-matrices with determinant $+1$ (cf. [10], section A.3). The isomorphism J provides a one-to-one correspondence between SO(3) and the automorphism group Aut(su(2)) of su(2):

$$\mathrm{Aut}(\mathrm{su}(2)) = J\mathrm{SO}(3)J^{-1}, \tag{3.4.6}$$

i.e. any automorphism $U \in \mathrm{SO}(3)$ of $\mathbb{R}^3$ induces an automorphism $J \circ U \circ J^{-1}$ of su(2) and vice versa. Moreover, as any automorphism of su(2) can be realized by conjugation with a unitary matrix $P \in \mathrm{SU}(2)$

of determinant 1, which is uniquely determined up to sign (again, c.f. [10], section A.3), we can write for $X \in \mathrm{su}(2)$

$$(J \circ U \circ J^{-1})(X) = PXP^{-1}. \tag{3.4.7}$$

Similarly, given an automorphism $V \in \mathrm{O}(3) \setminus \mathrm{SO}(3)$ of $\mathbb{R}^3$ along with the corresponding automorphism $J \circ V \circ J^{-1}$ of su(2), there exists a unitary matrix $P \in \mathrm{SU}(2)$ of determinant 1, which is uniquely determined up to sign, such that for $X \in \mathrm{su}(2)$

$$(J \circ V \circ J^{-1})(X) = -PXP^{-1}. \tag{3.4.8}$$

This is a consequence of the following: Consider the automorphism $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathrm{O}(3) \setminus \mathrm{SO}(3)$ of $\mathbb{R}^3$, which induces the automorphism $J \circ T \circ J^{-1}$ of su(2) given by

$$(J \circ T \circ J^{-1})(X) = -\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} X \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} =: Y. \tag{3.4.9}$$

As $V \circ T \in \mathrm{SO}(3)$, there exists $\hat{P} \in \mathrm{SU}(2)$, such that $(J \circ V \circ T \circ J^{-1})(X) = \hat{P}X\hat{P}^{-1}$ for $X \in \mathrm{su}(2)$. Setting $P := \hat{P}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \in \mathrm{SU}(2)$, this implies

$$(J \circ V \circ J^{-1})(Y) = (J \circ V \circ T \circ J^{-1})(X) = \hat{P}X\hat{P}^{-1} = -PYP^{-1}, \tag{3.4.10}$$

for all $Y \in \mathrm{su}(2)$, which proves (3.4.8).

3.5 The trinoid potential

In this section, we introduce a class of potentials, which will produce trinoids with properly embedded annular ends via the loop group method. Following [17], we define these potentials on the trinoid domain $M = \hat{\mathbb{C}} \setminus \{0, 1, \infty\} = \tilde{M}/\Gamma$ (rather than on the universal cover $\tilde{M} = \mathbb{H}$ of M). From each such potential η we can obtain a holomorphic potential $\tilde{\eta}$ on $\tilde{M}$ by carrying out the pullback construction induced by the covering $\pi : \tilde{M} \to M$, i.e.

$$\tilde{\eta} = \pi^* \eta. \tag{3.5.1}$$

In view of 2.3, $\tilde{\eta}$ is invariant under the action of the fundamental group Γ on $\tilde{M}$ and thus allows for the application of theorem 2.11.

While keeping in mind the necessity of the pullback construction to obtain the "true" trinoid potential $\tilde{\eta}$ (for use with the loop group method), we restrict our considerations from now on to the corresponding potential η on M. Accordingly, instead of solving equation (2.4.1), we turn to the corresponding differential equation on M:

$$\mathrm{d}\Phi = \Phi\eta. \tag{3.5.2}$$

Note that any solution Φ to (3.5.2) naturally induces the pullback solution $\Psi = \pi^*\Phi$ to (2.4.1), as

$$\mathrm{d}\Psi = \mathrm{d}(\pi^*\Phi) = \pi^*(\mathrm{d}\Phi) = (\pi^*\Phi)(\pi^*\eta) = \Psi\tilde{\eta}. \tag{3.5.3}$$

Remark 3.8. The potential η we consider comes along with three singularities at $z_0 = 0$, $z_1 = 1$ and $z_\infty = \infty$. These singularities carry over to the solution to the differential equation (3.5.2). Thus, there exists no holomorphic solution to (3.5.2), which is well defined globally on M. (Suppose there exists such a solution Φ. Tracing it along a closed curve $\gamma : [0, 1] \to M$, which is based at $x = \gamma(0) = \gamma(1)$ and surrounds one of the singularities of Φ, one obtains on return to x a different value $\Phi(\gamma(1)) \neq \Phi(\gamma(0))$ - a well known result of complex analysis. But, as $\gamma(0) = \gamma(1)$, this means that Φ is multiply defined at x, a contradiction.)
Consequently, in order to obtain a well defined holomorphic solution Φ to (3.5.2), we need to restrict to a simply connected subdomain of M. To this end we introduce certain "cuts", i.e. half-lines, which we exclude from the domain. More precisely, we cut M along the real axis from 0 to $-\infty$ and from 1 to $+\infty$. Thus, instead of M, we consider the simply connected domain

$$\mathcal{D} = \mathbb{C} \setminus \{x \in \mathbb{R}; x \leq 0 \text{ or } x \geq 1\}. \tag{3.5.4}$$

On $\mathcal{D}$, a holomorphic solution to equation (3.5.2) is well defined.
To a given solution Φ to (3.5.2) on $\mathcal{D}$ corresponds a solution Ψ to equation (2.4.1) via the pullback

construction: $\Psi = \pi^*\Phi$. By this construction, Ψ will be at first defined on a subset $\mathcal{F}_0$ of $\tilde{M}$, which is homeomorphic (via π) to $\mathcal{D}$. However, by continuing Φ holomorphically "across the cuts" in M, we can accordingly (again, via π) continue Ψ beyond $\mathcal{F}_0$ in $\tilde{M}$. (Each time we cross one of the cuts in M, we think of entering a "new copy" of $\mathcal{D}$ and thus circumvent the problem of Φ not being well defined.) By these means, we obtain a holomorphic solution Ψ to equation (2.4.1), which is defined globally on $\tilde{M}$.

We now explicitly introduce the potential η, which we will use throughout this work. As mentioned earlier, η comes along with three singularities at $z_0 = 0$, $z_1 = 1$ and $z_\infty = \infty$. These singularities will carry over to the solution to the differential equation (3.5.2) as well as to the induced immersion ϕ parametrizing the surface and thus will generate the three trinoid ends. Following section 3.1 of [17], we may restrict to the case where η is off-diagonal, that is of the form

$$\eta = \begin{pmatrix} 0 & \nu(z,\lambda) \\ \tau(z,\lambda) & 0 \end{pmatrix} \mathrm{d}z, \tag{3.5.5}$$

where, for now, ν and τ denote some holomorphic functions in $z \in M = \hat{\mathbb{C}} \setminus \{z_0, z_1, z_\infty\}$ which also depend on $\lambda \in \mathbb{C}^*$.

We would like to construct trinoids with properly embedded annular ends. According to [25], these ends asymptotically show the behaviour of unduloidal Delaunay surfaces, which have been studied in section 2.6. Therefore, we further assume that the potential η near each singularity z_j adopts some of the properties of the corresponding (unduloidal) Delaunay potential

$$\frac{1}{z - z_j} D_j \mathrm{d}z \tag{3.5.6}$$

involving the off-diagonal Delaunay matrix

$$D_j = \begin{pmatrix} 0 & X_j \\ \overline{X_j} & 0 \end{pmatrix}, \tag{3.5.7}$$

where

$$X_j = s_j\lambda^{-1} + t_j\lambda, \;\; \overline{X_j} = s_j\lambda + t_j\lambda^{-1}, \tag{3.5.8}$$

$$s_j \in [\frac{1}{4}, \frac{1}{2}), \;\; s_j + t_j = \frac{1}{2}. \tag{3.5.9}$$

Remark 3.9. Note that the Delaunay potential given above defines the translation of the Delaunay potential given in section 2.6 (defined on the universal cover of $\mathbb{C} \setminus \{z_j\}$) to the space $\mathbb{C} \setminus \{z_j\}$ itself. Therefore, this alternate version of the Delaunay potential relates to the potential η, which is defined on M, not on $\tilde{M}$.

Remark 3.10. When dealing with the singularity $z_\infty = \infty$, we introduce the coordinate transformation $u = \frac{1}{z}$ and consider the potential $\eta(u, \lambda)$ near the singularity $u_\infty = 0$ and, accordingly, the (unduloidal) Delaunay potential

$$\frac{1}{u} D_\infty \mathrm{d}u \tag{3.5.10}$$

involving the off-diagonal Delaunay matrix D_∞ given in (3.5.7).

Remark 3.11. In general, unduloidal Delaunay surfaces are obtained via the loop groop method from a holomorphic potential of the form (3.5.6) with parameters $s_j, t_j > 0$ satisfying $s_j + t_j = \frac{1}{2}$ (cf. remarks 2.13 and 3.9). However, it turns out that (for each $j \in \{0, 1, \infty\}$) our potential η introduced in (3.5.5) is gauge equivalent to a "perturbed" Delaunay potential of the form (3.5.6) with parameters $s_j, t_j > 0$ satisfying $s_j + t_j = \frac{1}{2}$ *and* the further restriction $s_j \geq t_j$ (cf. remark 3.31). Consequently, following [17], all CMC-surfaces with properly embedded annular ends which can be obtained from holomorphic potentials of the form (3.5.5) show the asymptotic behaviour of unduloidal Delaunay surfaces generated from holomorphic potentials of the form (3.5.6) with

$$s_j, t_j > 0, \;\; s_j + t_j = \frac{1}{2}, \;\; s_j \geq t_j, \tag{3.5.11}$$

or, equivalently,

$$s_j \in [\frac{1}{4}, \frac{1}{2}), \;\; s_j + t_j = \frac{1}{2}. \tag{3.5.12}$$

Therefore, we consider right away only Delaunay potentials associated with parameters s_j, t_j satisfying (3.5.12), i.e. in particular $s_j \geq t_j$.

First of all, as the singularities of Delaunay potentials are regular, we require all singularities z_j of η to be regular singularities. More precisely, they will be regular singular points of a second order scalar ODE associated with the differential equation (3.5.2) in the sense of the following straightforward lemma.

Lemma 3.12. *Every solution Φ of the differential equation* (3.5.2) *can be written in the form*

$$\Phi = \begin{pmatrix} \frac{y_1'}{\nu} & y_1 \\ \frac{y_2'}{\nu} & y_2 \end{pmatrix}, \tag{3.5.13}$$

where y_1, y_2 is a fundamental system of the differential equation

$$y'' - \frac{\nu'}{\nu} y' - \nu\tau y = 0. \tag{3.5.14}$$

We require that z_0, z_1, z_∞ are regular singular points of (3.5.14), i.e. we require that equation (3.5.14) is of *Fuchsian type* (cf., e.g., chapter 7 of [2]) with three singular points. In general, a Fuchsian equation can have a singularity which, however, does not show up in the solutions. Such a singularity is called *apparent singularity.* In our case we do not want any apparent singularities, since otherwise we would have fewer than three properly embedded annular ends. From [2] and sections 3.3 and 3.5 of [17] we obtain that the three ends at 0, 1 and ∞ are non-apparent regular singular points if and only if

$$\nu(z,\lambda) = \lambda^{-1} z^{-a_0}(z-1)^{-a_1}, \tag{3.5.15}$$

$$\tau(z,\lambda) = -\lambda z^{a_0}(z-1)^{a_1}\left[\frac{b_0(\lambda)}{z^2} + \frac{b_1(\lambda)}{(z-1)^2} + \frac{c_0(\lambda)}{z} + \frac{c_1(\lambda)}{z-1}\right], \tag{3.5.16}$$

for some integers a_0, a_1, a_∞ and some even functions b_0, b_1, b_∞, c_0, c_1 in $\lambda \in \mathbb{C}^*$ satisfying

$$a_0 + a_1 + a_\infty = 2, \tag{3.5.17}$$

$$b_0(\lambda) + b_1(\lambda) + 0 \cdot c_0(\lambda) + 1 \cdot c_1(\lambda) = b_\infty(\lambda), \tag{3.5.18}$$

$$c_0(\lambda) + c_1(\lambda) = 0. \tag{3.5.19}$$

The potential η is defined on M. Therefore, its pullback $\tilde{\eta}$ to the universal cover $\tilde{M}$ is invariant under the covering transformations $\tilde{\gamma}_0$, $\tilde{\gamma}_1$ and $\tilde{\gamma}_\infty$, which correspond to surrounding the singularities z_0, z_1 and z_∞, respectively, in M. According to section 3.3, the pulled back solution $\Psi := \pi^*\Phi$ to (2.4.1) picks up a monodromy matrix $M_j(\lambda)$ under the covering transformation $\tilde{\gamma}_j$. Following [17], these monodromy matrices can also be computed directly on M: "Cutting" $M = \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ along the real axis from 0 to $-\infty$ and from 1 to $+\infty$, we obtain a simply connected subdomain

$$\mathcal{D} = \mathbb{C} \setminus \{x \in \mathbb{R}; x \leq 0 \;\text{ or }\; x \geq 1\} \tag{3.5.20}$$

of M, on which we can globally solve the differential equation (3.5.2) (cf. remark 3.8). However, in order to extend a given solution Φ to (3.5.2), which is defined on $\mathcal{D}$, to M, one has to "cross the cuts", which results in a change of the starting solution. More precisely, one observes the following: When extending a starting solution Φ to (3.5.2), which is defined on a neighborhood of the point $x = \frac{1}{2}$ in $\mathcal{D}$, holomorphically along the loop γ_j based at x, which encloses the singularity z_j, one obtains "on return" to x another solution Φ_{new}, which differs from Φ by a z-independent matrix. This change in Φ when surrounding z_j corresponds exactly to the change in $\Psi = \pi^*\Phi$ under the covering tranformation $\tilde{\gamma}_j$ induced by $[\gamma_j]$, i.e. the matrix representing the change in Φ when surrounding z_j coincides with the monodromy matrix $M_j(\lambda)$ picked up by Ψ under $\tilde{\gamma}_j$. So, according to

$$\Psi(\tilde{\gamma}_j(z), \lambda) = M_j(\lambda)\Psi(z,\lambda) \tag{3.5.21}$$

we write

$$\Phi(z, \gamma_j, \lambda) = M_j(\lambda)\Phi(z,\lambda), \tag{3.5.22}$$

where $\Phi(z, \gamma_j, \lambda)$ denotes the value $\Phi_{\text{new}}(z,\lambda)$ of the modified solution Φ_{new} obtained by extending Φ holomorphically along the loop γ_j. In view of (3.5.22), we will sometimes refer to the monodromy matrix $M_j(\lambda)$ of Ψ with respect to $[\gamma_j]$ also as monodromy matrix of Φ with respect to the loop γ_j in M.

By use of lemma 3.12, one can explicitly compute *up to conjugation* the monodromy matrices of a solution Φ to (3.5.2) by studying the behaviour of (i.e. the change in) the fundamental systems of the differential equation (3.5.14) when surrounding the singularities z_0, z_1 and z_∞. In particular, the eigenvalues of the monodromy matrices are known. This is done in section 3.4 of [17], and we refer there for more details.

One should expect that the monodromy matrix M_j of Φ (and Ψ) corresponding to the singularity z_j is somehow related to the monodromy matrix of the Delaunay surface which is the asymptotic shape of the end of the trinoid we aim for at z_j . It is therefore natural to assume that M_j is conjugate to the monodromy matrix of the corresponding Delaunay surface (and thus possesses the same eigenvalues). Actually, one can prove that this is necessarily the case [8]. Referring to section 3.6 of [17], this is equivalent to requiring for each $j \in \{0, 1, \infty\}$

$$b_j(\lambda) = \frac{1}{4}(1 - a_j)^2 - \mu_j^2, \tag{3.5.23}$$

where

$$\mu_j = \sqrt{X_j \overline{X_j}} = \sqrt{\frac{1}{4} + w_j(\lambda - \lambda^{-1})^2}, \quad w_j = s_j t_j \tag{3.5.24}$$

and $\pm\mu_j$ are the eigenvalues of D_j. The relation between μ_j and w_j in (3.5.24) is proved, along with other useful properties of μ_j, in appendix B.

Remark 3.13. If $s_j \neq t_j$, μ_j defines by lemma B.1 a holomorphic function of λ on the cut plane $\mathbb{C}^* \setminus W_{1,j}$, where

$$W_{1,j} = \{\lambda \in \mathbb{C}^*; \Re(\lambda) = 0 \text{ and } \Im(\lambda) \in (-\infty, -\sqrt{\frac{s_j}{t_j}}] \cup [-\sqrt{\frac{t_j}{s_j}}, \sqrt{\frac{t_j}{s_j}}] \cup [\sqrt{\frac{s_j}{t_j}}, +\infty)\}. \tag{3.5.25}$$

Moreover, μ_j is well defined and continuous on the slightly larger set $\mathbb{C}^* \setminus \tilde{W}_{1,j}$, where

$$\tilde{W}_{1,j} = W_{1,j} \setminus \{\pm i\sqrt{\frac{t_j}{s_j}}, \pm i\sqrt{\frac{s_j}{t_j}}\}. \tag{3.5.26}$$

In particular, μ_j defines a continuous and holomorphic mapping on (a sufficiently small open neighborhood of) the unit circle S^1 in $\mathbb{C}^*$. If $s_j = t_j = \frac{1}{4}$, μ_j defines by lemma B.1 a holomorphic function of $\lambda \in \mathbb{C}^*$.

In any case, the mapping μ_j^2 can be holomorphically extended to $\mathbb{C}^*$:

$$(\mu_j(\lambda))^2 = X_j(\lambda)\overline{X_j}(\lambda) = \frac{1}{4} + w_j(\lambda - \lambda^{-1})^2. \tag{3.5.27}$$

Consequently, also the functions b_j defined in (3.5.23) are holomorphic for $\lambda \in \mathbb{C}^*$.

By the choice of D_0, D_1, D_∞ and some integers a_0, a_1, a_∞ satisfying (3.5.17) the functions b_j and c_j are given by equations (3.5.23), (3.5.18) and (3.5.19) explicitly, whereby η is determined completely. While we can assume w.l.o.g. $a_0 = 0$, $a_1 = 0$ and $a_\infty = 2$ (we carry this out explicitly in section 3.6), the choice of the D_j will determine whether the associated potential η will give rise to a "descending" CMC-immersion ψ in the sense of theorem 2.11. In order to ensure this, we further need to require for $\lambda \in S^1$

$$0 \leq \frac{cos(\pi(\mu_0 - \mu_1 - \mu_\infty))cos(\pi(\mu_0 - \mu_1 + \mu_\infty))}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)} \leq 1. \tag{3.5.28}$$

Equation (3.5.28) will be referred to as the *unitarizability condition*, as it is equivalent with the existence of a solution Ψ to (2.4.1), which has *unitary* monodromy matrices at the singularities at z_0, z_1, and z_∞. In other words, (3.5.28) holds if and only if any solution of (2.4.1) can be "dressed" (cf. section 2.4) into a new solution with unitary monodromy matrices.

Altogether, by [17], theorem 5.4.1 and corollary 5.4.2, and [26], theorems 3.5 and 5.9, we have in fact

Theorem 3.14. *Let D_0, D_1, D_∞ be Delaunay matrices satisfying* (3.5.28) *for all $\lambda \in S^1$. Let η be of the form* (3.5.5) *associated with the given Delaunay matrices. Assume that η satisfies equations* (3.5.15) *to* (3.5.19) *and* (3.5.23)*. Then, η yields for $\lambda = 1$ a trinoid with properly embedded annular ends after some appropriate r-dressing.*

Remark 3.15. Actually, theorem 5.4.1 and corollary 5.4.2 of [17] only ensure that the potential η yields for $\lambda = 1$ a trinoid with *embedded* annular ends after some appropriate r-dressing. However, by theorems 3.5 and 5.9 of [26], embeddedness of the annular ends implies proper embeddedness.

Remark 3.16. We note that the unitarizability condition (3.5.28) does not hold (on S^1) if at least two of the three Delaunay matrices D_0, D_1, D_∞ are associated with parameters s_j, t_j satisfying $s_j = t_j = \frac{1}{4}$: Suppose $s_j = t_j = \frac{1}{4}$ for at least two $j \in \{0, 1, \infty\}$. The corresponding mappings μ_j then satisfy

$$\mu_j(i) = \frac{1}{4}(i - i) = 0. \tag{3.5.29}$$

Now, consider the expression

$$\frac{cos(\pi(\mu_0 - \mu_1 - \mu_\infty))cos(\pi(\mu_0 - \mu_1 + \mu_\infty))}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)} \tag{3.5.30}$$

from (3.5.28). Since $\mu_j(i) = 0$ for at least two $j \in \{0, 1, \infty\}$, the numerator of (3.5.30) simplifies into $\cos^2(\pi\mu_k(i))$, where $k \in \{0, 1, \infty\}$ denotes the index for which *not* necessarily $s_k = t_k = \frac{1}{4}$ holds. Nevertheless, we have

$$\mu_k(i) = \sqrt{\frac{1}{4} - 4w_k} \quad \in [0, \frac{1}{2}), \tag{3.5.31}$$

where we have used that $w_k \in (0, \frac{1}{16}]$ (cf. lemma B.4). This implies $\cos^2(\pi\mu_k(i)) \in (0, 1]$, i.e. the numerator of (3.5.30) takes a positive real value at $\lambda = i$. In contrast, since $\mu_j(i) = 0$ for at least two $j \in \{0, 1, \infty\}$, the denominator of (3.5.30) equals 0 at $\lambda = i$. Consequently, (3.5.28) does not hold at $\lambda = i$.

Altogether, we record for ther following considerations that in order to construct a (conformal) CMC-immersion $M \to \mathbb{R}^3$ on $M = \mathbb{C} \setminus \{0, 1\}$, we need to start with a potential η of the form (3.5.5), such that $s_j \neq t_j$ for at least two $j \in \{0, 1, \infty\}$, where s_j, t_j denote the parameters of the Delaunay matrices D_0, D_1, D_∞ associated with η. Otherwise, the unitarizability condition (3.5.28) is not satisfiable for all $\lambda \in S^1$. Re-indexing[7] the Delaunay matrices D_0, D_1, D_∞ if necessary, we assume from now on without loss of generality that

$$s_0 \neq t_0, \quad s_1 \neq t_1. \tag{3.5.32}$$

Definition 3.17. Let $M = \mathbb{C} \setminus \{0, 1\}$, $\tilde{M} = \mathbb{H}$ and $\pi : \tilde{M} \to M$ be the universal covering defined in (3.2.2). Let D_0, D_1 and D_∞ be Delaunay matrices with eigenvalues $\pm\mu_0$, $\pm\mu_1$ and $\pm\mu_\infty$, respectively, which satisfy (3.5.28) for all $\lambda \in S^1$. A potential η of the form

$$\eta = \begin{pmatrix} 0 & \nu(z, \lambda) \\ \tau(z, \lambda) & 0 \end{pmatrix} \mathrm{d}z, \tag{3.5.33}$$

where

$$\nu(z, \lambda) = \lambda^{-1} z^{-a_0} (z - 1)^{-a_1}, \tag{3.5.34}$$

$$\tau(z, \lambda) = -\lambda z^{a_0} (z - 1)^{a_1} \left[\frac{b_0(\lambda)}{z^2} + \frac{b_1(\lambda)}{(z - 1)^2} + \frac{c_0(\lambda)}{z} + \frac{c_1(\lambda)}{z - 1} \right] \tag{3.5.35}$$

for some integers a_0, a_1, a_∞ and some even functions b_0, b_1, b_∞, c_0, c_1 of $\lambda \in \mathbb{C}^*$ satisfying

$$a_0 + a_1 + a_\infty = 2, \tag{3.5.36}$$

$$b_j(\lambda) = \frac{1}{4}(1 - a_j)^2 - \mu_j^2, \tag{3.5.37}$$

$$b_0(\lambda) + b_1(\lambda) + c_1(\lambda) = b_\infty(\lambda), \tag{3.5.38}$$

$$c_0(\lambda) + c_1(\lambda) = 0 \tag{3.5.39}$$

will be called a *trinoid potential (on M)*. Note that such potentials are holomorphic for $z \in M$ and for $\lambda \in \mathbb{C}^*$.

Given a trinoid potential η on M, we obtain a potential $\tilde{\eta}$ on $\tilde{M}$ by the pullback construction induced by π,

$$\tilde{\eta} = \pi^*\eta, \tag{3.5.40}$$

which is holomorphic for $z \in \tilde{M}$ and for $\lambda \in \mathbb{C}^*$ and will be called a *trinoid potential (on $\tilde{M}$)*.

[7] Actually, re-indexing the Delaunay matrices D_0, D_1, D_∞ corresponds to replacing the potential η by $\gamma^*\eta$, where γ denotes some Moebius transformation on $\hat{\mathbb{C}}$.

Remark 3.18. It is claimed in [8] that all trinoids with properly embedded annular ends can be constructed via the loop group method from potentials of the form (3.5.33) (or, more precisely, of the form (3.5.40)).

Starting with a trinoid potential η and the associated pullback potential $\tilde{\eta}$, any solution Ψ to (2.4.1), by theorem 3.14, can be dressed by some appropriately chosen matrix $T = T(\lambda)$ into a solution $\hat{\Psi} = T\Psi$, which in turn will produce a trinoid with properly embedded annular ends, defined on M. The same matrix T tranforms the corresponding solution Φ to the equation (3.5.2) into another solution $\hat{\Phi} = T\Phi$. If Ψ (resp. Φ) picks up the monodromy matrix $M_j(\lambda)$ around the singularity z_j, $\hat{\Psi}$ (resp. $\hat{\Phi}$) has the monodromy matrix $\hat{M}_j(\lambda) = T(\lambda)M_j(\lambda)(T(\lambda))^{-1}$ at z_j (cf. section 2.5). These monodromy matrices necessarily satisfy the conditions of theorem 2.11. In particular, they are unitary on S^1. Thus, in order to find a solution $\hat{\Psi}$ yielding a trinoid with properly embedded annular ends on M in the sense of theorem 2.11, we will perform the following two steps:

1. We compute a solution Φ to (3.5.2) with monodromy matrices M_j at z_j, $j = 0, 1, \infty$ (see sections 3.7 and 3.8). Note that $\Psi = \pi^*\Phi$ defines a solution to (2.4.1) and possesses the same monodromy matrices as Φ.

2. We determine all possible dressing matrices T such that the "dressed monodromy matrices" $\hat{M}_j = T(\lambda)M_j(\lambda)(T(\lambda))^{-1}$ satisfy the conditions of theorem 2.11. Then, the corresponding new solution $\hat{\Psi} = T\Psi$ to 2.4.1 produces via the loop group method a (conformal) CMC-immersion $\tilde{M} \to \mathbb{R}^3$, which, by theorem 2.11, descends to a (conformal) CMC-immersion $M \to \mathbb{R}^3$.

Remark 3.19. Note that, since the elements $[\gamma_0], [\gamma_1] \in \Gamma$ corresponding to the monopdromy matrices $\hat{M}_0$ and $\hat{M}_1$, respectively, generate the fundamental group Γ, it is enough to verify the three conditions of theorem 2.11 only for $\hat{M}_0$ and $\hat{M}_1$.

Remark 3.20. Starting with a trinoid potential η, the associated pullback potential $\tilde{\eta}$ and a solution Ψ to the differential equation (2.4.1), we compute all possible dressing matrices T yielding a new solution $\hat{\Psi} = T\Psi$ to (2.4.1), which produces (via the loop group method) a CMC-immersion ψ on $\tilde{M}$ that, by theorem 2.11, descends to a CMC-immersion ϕ on M. These possible dressing matrices in particular encompass those dressing matrices, which actually induce trinoids with properly embedded annular ends.

3.6 The standardized trinoid potential

Recall the trinoid potential η as introduced in section 3.5:

$$\eta = \begin{pmatrix} 0 & \nu(z,\lambda) \\ \tau(z,\lambda) & 0 \end{pmatrix} \mathrm{d}z. \tag{3.6.1}$$

As stated before, the choice of three off-diagonal Delaunay matrices D_0, D_1, D_∞ with eigenvalues $\pm\mu_0$, $\pm\mu_1$, $\pm\mu_\infty$, respectively, together with the choice of three integers a_0, a_1, a_∞ satisfying $a_0 + a_1 + a_\infty = 2$ will determine the potential η completely. The functions ν and τ are given by

$$\nu(z,\lambda) = \lambda^{-1} z^{-a_0}(z-1)^{-a_1}, \tag{3.6.2}$$

$$\tau(z,\lambda) = -\lambda z^{a_0}(z-1)^{a_1} \left[\frac{b_0(\lambda)}{z^2} + \frac{b_1(\lambda)}{(z-1)^2} + \frac{c_0(\lambda)}{z} + \frac{c_1(\lambda)}{z-1} \right], \tag{3.6.3}$$

where b_0, b_1, b_∞, c_0, c_1 are obtained from

$$b_j(\lambda) = \frac{1}{4}(1-a_j)^2 - \mu_j^2 \quad \text{for } j = 0, 1, \infty, \tag{3.6.4}$$

$$b_0(\lambda) + b_1(\lambda) + 0 \cdot c_0(\lambda) + 1 \cdot c_1(\lambda) = b_\infty(\lambda), \tag{3.6.5}$$

$$c_0(\lambda) + c_1(\lambda) = 0. \tag{3.6.6}$$

The purpose of this section is to show that - for a fixed choice of Delaunay matrices D_0, D_1, D_∞ - the different trinoid potentials corresponding to different possible choices of integers a_0, a_1, a_∞ are *gauge-equivalent*, i.e. related by certain gauge transformations (cf. section 2.4), and therefore will produce the same surface via the DPW-method. This implies that we can assume w.l.o.g. $a_0 = 0$, $a_1 = 0$, $a_\infty = 2$. The corresponding trinoid potential will be called the *standardized trinoid potential.*

Theorem 3.21. *Let D_0, D_1, D_∞ be Delaunay matrices with eigenvalues $\pm\mu_0$, $\pm\mu_1$, $\pm\mu_\infty$, respectively. Let a_0, a_1, a_∞ be some integers satisfying $a_0 + a_1 + a_\infty = 2$ and η denote the trinoid potential determined by μ_0, μ_1, μ_∞ and a_0, a_1, a_∞ as above. Furthermore, let $\hat{a}_0 = 0$, $\hat{a}_1 = 0$, $\hat{a}_\infty = 2$ and $\hat{\eta}$ denote the trinoid potential corresponding to μ_0, μ_1, μ_∞ and $\hat{a}_0, \hat{a}_1, \hat{a}_\infty$. Then we have*

$$\eta\#l = \hat{\eta}, \tag{3.6.7}$$

where $l = g_0 g_1 h_0 h_1$ and

$$g_0 = \begin{pmatrix} z^{-\frac{a_0}{2}} & 0 \\ 0 & z^{\frac{a_0}{2}} \end{pmatrix}, \; g_1 = \begin{pmatrix} (z-1)^{-\frac{a_1}{2}} & 0 \\ 0 & (z-1)^{\frac{a_1}{2}} \end{pmatrix}, \; h_0 = \begin{pmatrix} 1 & 0 \\ \frac{a_0\lambda}{2z} & 1 \end{pmatrix}, \; h_1 = \begin{pmatrix} 1 & 0 \\ \frac{a_1\lambda}{2(z-1)} & 1 \end{pmatrix}. \tag{3.6.8}$$

Remark 3.22. Note that g_0, g_1, h_0 and h_1 are well defined only on a simply connected subdomain of M, e.g. on the cut domain $\mathcal{D}$ introduced in remark 3.8. Keeping this in mind, the following proof of the above theorem at first only holds for $z \in \mathcal{D}$, i.e. we prove $\eta\#l = \hat{\eta}$ for all $z \in \mathcal{D}$. However, by continuity arguments, we can afterwards extend this result to M and thus obtain, as claimed, $\eta\#l = \hat{\eta}$ for all $z \in M$.

Proof of theorem 3.21. Abbreviating

$$Q = Q(z,\lambda) = \frac{b_0(\lambda)}{z^2} + \frac{b_1(\lambda)}{(z-1)^2} + \frac{c_0(\lambda)}{z} + \frac{c_1(\lambda)}{z-1} \tag{3.6.9}$$

for the moment and recalling the gauge equation

$$\eta\#g = g^{-1}\eta g + g^{-1}\mathrm{d}g, \tag{3.6.10}$$

we have

$$\begin{aligned} \eta\#(g_0g_1h_0h_1) = (\eta\#g_0)\#(g_1h_0h_1) &= \begin{pmatrix} -\frac{a_0}{2}z^{-1} & \lambda^{-1}(z-1)^{-a_1} \\ -\lambda(z-1)^{a_1}Q & \frac{a_0}{2}z^{-1} \end{pmatrix} \mathrm{d}z\#(g_1h_0h_1) \\ &= \begin{pmatrix} -\frac{a_0}{2}z^{-1} - \frac{a_1}{2}(z-1)^{-1} & \lambda^{-1} \\ -\lambda Q & \frac{a_0}{2}z^{-1} + \frac{a_1}{2}(z-1)^{-1} \end{pmatrix} \mathrm{d}z\#(h_0h_1) \\ &= \begin{pmatrix} -\frac{a_1}{2}(z-1)^{-1} & \lambda^{-1} \\ -\lambda Q + \lambda\frac{a_0a_1}{2z(z-1)} + \lambda\frac{a_0^2}{4z^2} - \lambda\frac{a_0}{2z^2} & \frac{a_1}{2}(z-1)^{-1} \end{pmatrix} \mathrm{d}z\#h_1 = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda\hat{Q} & 0 \end{pmatrix} \mathrm{d}z, \end{aligned} \tag{3.6.11}$$

where

$$\begin{aligned} \hat{Q} = Q - \frac{a_0a_1}{2z(z-1)} - \frac{a_0^2}{4z^2} + \frac{a_0}{2z^2} - \frac{a_1^2}{4(z-1)^2} + \frac{a_1}{2(z-1)^2} \\ = \frac{4b_0 - a_0^2 + 2a_0}{4z^2} + \frac{4b_1 - a_1^2 + 2a_1}{4(z-1)^2} + \frac{c_1 - \frac{1}{2}a_0a_1}{z(z-1)}. \end{aligned} \tag{3.6.12}$$

As

$$\frac{1}{4}(4b_j - a_j^2 + 2a_j) = \hat{b}_j \tag{3.6.13}$$

and (using $a_0 + a_1 + a_\infty = 2$)

$$\begin{aligned} c_1 - \frac{1}{2}a_0a_1 = b_\infty - b_0 - b_1 - \frac{1}{2}a_0a_1 = -\frac{1}{4} + \mu_0^2 + \mu_1^2 - \mu_\infty^2 - \frac{1}{2}a_\infty + \frac{1}{2}(a_0 + a_1) + \frac{1}{4}a_\infty^2 \\ - \frac{1}{4}(a_0^2 + a_1^2 + 2a_0a_1) = -\frac{1}{4} + \mu_0^2 + \mu_1^2 - \mu_\infty^2 = \hat{b}_\infty - \hat{b}_0 - \hat{b}_1 = \hat{c}_1, \end{aligned} \tag{3.6.14}$$

we infer that

$$\hat{Q} = \frac{\hat{b}_0}{z^2} + \frac{\hat{b}_1}{(z-1)^2} + \frac{\hat{c}_1}{z(z-1)} = \frac{\hat{b}_0}{z^2} + \frac{\hat{b}_1}{(z-1)^2} + \frac{-\hat{c}_1}{z} + \frac{\hat{c}_1}{z-1} \tag{3.6.15}$$

and thus, as claimed, $\eta\#l = \hat{\eta}$. □

In view of theorem (3.21), we will from now on restrict our study without loss of generality to trinoid potentials with $a_0 = a_1 = 0$ and $a_\infty = 2$, explicitly introduced in the following definition.

Definition 3.23. Let $M = \mathbb{C} \setminus \{0, 1\}$, $\tilde{M} = \mathbb{H}$ and $\pi : \tilde{M} \to M$ be the universal covering defined in (3.2.2). Let D_0, D_1 and D_∞ be Delaunay matrices with eigenvalues $\pm\mu_0$, $\pm\mu_1$ and $\pm\mu_\infty$, respectively, which satisfy (3.5.28) for all $\lambda \in S^1$. A potential η of the form

$$\eta = \begin{pmatrix} 0 & \nu(z,\lambda) \\ \tau(z,\lambda) & 0 \end{pmatrix} \mathrm{d}z, \tag{3.6.16}$$

where

$$\nu(z,\lambda) = \lambda^{-1}, \tag{3.6.17}$$

$$\tau(z,\lambda) = -\lambda \left[\frac{b_0(\lambda)}{z^2} + \frac{b_1(\lambda)}{(z-1)^2} + \frac{c_0(\lambda)}{z} + \frac{c_1(\lambda)}{z-1} \right] \tag{3.6.18}$$

for some even functions b_0, b_1, b_∞, c_0, c_1 of $\lambda \in \mathbb{C}^*$ satisfying

$$b_j(\lambda) = \frac{1}{4} - \mu_j^2, \tag{3.6.19}$$

$$b_0(\lambda) + b_1(\lambda) + c_1(\lambda) = b_\infty(\lambda), \tag{3.6.20}$$

$$c_0(\lambda) + c_1(\lambda) = 0 \tag{3.6.21}$$

will be called a *standardized trinoid potential (on M)*. Note that such potentials are holomorphic for $z \in M$ and for $\lambda \in \mathbb{C}^*$.

Given a standardized trinoid potential η on M, we obtain a potential $\tilde{\eta}$ on $\tilde{M}$ by the pullback construction induced by π,

$$\tilde{\eta} = \pi^* \eta, \tag{3.6.22}$$

which is holomorphic for $z \in \tilde{M}$ and for $\lambda \in \mathbb{C}^*$ and will be called a *standardized trinoid potential (on $\tilde{M}$)*.

3.7 The Fuchsian ODE

We consider a standardized trinoid potential η of the form (3.6.16). In order to solve (3.5.2) we take a closer look at the Fuchsian differential equation (3.5.14), which reads more explicitly as

$$y'' + \left(\frac{b_0}{z^2} + \frac{b_1}{(z-1)^2} + \frac{c_0}{z} + \frac{c_1}{z-1} \right) y = 0. \tag{3.7.1}$$

The corresponding indicial equations around the singularities $z_0 = 0$, $z_1 = 1$ and $z_\infty = \infty$ are given by $w(w-1) + b_j$ for $j = 0, 1, \infty$, respectively (cf. section 7.2 of [2]), and possess the roots

$$r_{j,\pm} = \frac{1}{2}\left(1 \pm \sqrt{1 - 4b_j}\right) = \frac{1}{2}\left(1 \pm 2\mu_j\right) \text{ for } j = 0, 1, \tag{3.7.2}$$

$$r_{\infty,\pm} = \frac{1}{2}\left(-1 \pm \sqrt{1 - 4b_j}\right) = \frac{1}{2}\left(-1 \pm 2\mu_j\right), \tag{3.7.3}$$

where we have simplified by using (3.6.19).

Remark 3.24. In view of remark 3.13, μ_j defines a holomorphic mapping for $\lambda \in \mathbb{C}^* \setminus W_{1,j}$, where

$$W_{1,j} = \begin{cases} \{\lambda \in \mathbb{C}^*; \Re(\lambda) = 0 \text{ and } \Im(\lambda) \in (-\infty, -\sqrt{\frac{s_j}{t_j}}] \cup [-\sqrt{\frac{t_j}{s_j}}, \sqrt{\frac{t_j}{s_j}}] \cup [\sqrt{\frac{s_j}{t_j}}, +\infty)\} & \text{if } s_j \neq t_j \\ \emptyset & \text{if } s_j = t_j \end{cases} \tag{3.7.4}$$

Consequently, the functions $r_{j,\pm}$, $j = 0, 1, \infty$, depend holomorphically on $\lambda \in \mathbb{C}^* \setminus W_{1,j}$. From now on, we restrict our considerations to the domain $\mathbb{C}^* \setminus W_1$, where

$$W_1 = W_{1,0} \cup W_{1,1} \cup W_{1,\infty}. \tag{3.7.5}$$

On $\mathbb{C}^* \setminus W_1$, all the mappings μ_j and $r_{j,\pm}$ are holomorphic in λ.

Defining $r_j := r_{j,+}$ and substituting

$$y = z^{r_0}(z-1)^{r_1} w, \tag{3.7.6}$$

equation (3.7.1) translates into the hypergeometric differential equation

$$w'' + \frac{-\gamma + (1+\alpha+\beta)z}{z(z-1)} w' + \frac{\alpha\beta}{z(z-1)} w = 0, \tag{3.7.7}$$

where

$$\alpha = r_{0,+} + r_{1,+} + r_{\infty,+} = \frac{1}{2} + \mu_0 + \mu_1 + \mu_\infty, \tag{3.7.8}$$

$$\beta = r_{0,+} + r_{1,+} + r_{\infty,-} = \frac{1}{2} + \mu_0 + \mu_1 - \mu_\infty, \tag{3.7.9}$$

$$\gamma = 1 + r_{0,+} - r_{0,-} = 1 + 2\mu_0. \tag{3.7.10}$$

The theory of hypergeometric functions requires a special discussion for the cases where γ or $\alpha+\beta-\gamma$ are integers. In view of equations (3.7.8), (3.7.9) and (3.7.10), these cases can be avoided by excluding all values of $\lambda \in \mathbb{C}^*$ from our considerations, for which either $\mu_0(\lambda)$ or $\mu_1(\lambda)$ is a half-integer. We denote the subset of $\lambda \in \mathbb{C}^*$, for which either $\mu_0(\lambda)$ or $\mu_1(\lambda)$ is a half-integer, by W_2:

$$W_2 = \{\lambda \in \mathbb{C}^*; \mu_0(\lambda) \in \frac{1}{2}\mathbb{Z} \text{ or } \mu_1(\lambda) \in \frac{1}{2}\mathbb{Z}\}. \tag{3.7.11}$$

In particular, by lemma B.3, we have $1 \in W_2$. Standard analysis of the functions μ_0 and μ_1 yields that W_2 forms a discrete subset of $\mathbb{C}^*$, which does not possess any accumulation points on S^1. Taking into account remark 3.24, we restrict our further calculations to the λ-domain $\mathbb{C}^* \setminus (W_1 \cup W_2)$, i.e. in what follows we will only use $\lambda \in \mathbb{C}^* \setminus (W_1 \cup W_2)$.

Assuming $\gamma \notin \mathbb{Z}$ and $\alpha + \beta - \gamma \notin \mathbb{Z}$, which is the case for all $\lambda \in \mathbb{C}^* \setminus (W_1 \cup W_2)$, there are the following natural fundamental systems w_{j1}, w_{j2} of (3.7.7) at z_j, $j = 0, 1$ (cf. chapter 8 of [2]):

$$w_{01} = F(\alpha, \beta, \gamma; z), \tag{3.7.12}$$

$$w_{02} = z^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; z), \tag{3.7.13}$$

$$w_{11} = F(\alpha, \beta, \alpha + \beta - \gamma + 1; 1 - z), \tag{3.7.14}$$

$$w_{12} = (1-z)^{\gamma-\alpha-\beta} F(\gamma - \beta, \gamma - \alpha, \gamma - \alpha - \beta + 1; 1 - z). \tag{3.7.15}$$

where F denotes the hypergeometric series

$$F(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)} \frac{z^n}{n!}. \tag{3.7.16}$$

Remark 3.25. Recall that we are interested in finding a solution Φ to the differential equation (3.5.2). In particular, we want to study the monodromy matrices associated with Φ. In view of remark 3.19 it suffices to restrict our considerations to the monodromy matrices corresponding to the singularities $z_0 = 0$ and $z_1 = 1$, since they generate the monodromy group. Consequently, our discussion of a solution Φ to (3.5.2) can be restricted to the analysis of its behaviour near z_0 and z_1. It therefore suffices to consider in this work only fundamental systems to the equation (3.7.7) defined near z_0 and z_1, respectively.

By definition of the hypergeometric series, the solutions w_{01}, w_{02} (resp. w_{11}, w_{12}) to (3.7.7) given in (3.7.12) and (3.7.13) (resp. (3.7.14) and (3.7.15)) are, at first, defined (with respect to z) on the open disc of radius 1 around $z_0 = 0$ (resp. $z_1 = 1$). However, by [2], w_{01} and w_{02} (resp. w_{11} and w_{12}) can be extended holomorphically to the single cut complex plane $\mathbb{C} \setminus \{x \in \mathbb{R}; x \geq 1\}$ (resp. to the single cut complex plane $\mathbb{C} \setminus \{x \in \mathbb{R}; x \leq 0\}$). Consequently, all w_{jk} are well defined on the double cut complex plane $\mathcal{D} = \mathbb{C} \setminus \{x \in \mathbb{R}; x \leq 0 \text{ or } x \geq 1\}$. Moreover, according to [2], p. 235, the following relations hold on $\mathcal{D}$:

$$w_{01} = \kappa_{11}^{01} w_{11} + \kappa_{12}^{01} w_{12}, \tag{3.7.17}$$

$$w_{02} = \kappa_{11}^{02} w_{11} + \kappa_{12}^{02} w_{12}, \tag{3.7.18}$$

where

$$\kappa_{11}^{01} = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \tag{3.7.19}$$

$$\kappa_{12}^{01} = \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}, \tag{3.7.20}$$

$$\kappa_{11}^{02} = \frac{\Gamma(\gamma-\alpha-\beta)\Gamma(2-\gamma)}{\Gamma(1-\alpha)\Gamma(1-\beta)}, \tag{3.7.21}$$

$$\kappa_{12}^{02} = \frac{\Gamma(\alpha+\beta-\gamma)\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)}, \tag{3.7.22}$$

and Γ denotes the Gamma function $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$.

Remark 3.26. The Gamma function Γ is originally defined on the complex half plane $\{\lambda \in \mathbb{C}; \Re(\lambda) > 0\}$. However, Γ can be holomorphically extended to the complex plane $\mathbb{C}$ excluding the non-positive integers $\mathbb{Z}_0^-$. Throughout this work, we interpret Γ as being defined on $\mathbb{C} \setminus \mathbb{Z}_0^-$. In order to ensure that the connection coefficients κ_{1j}^{0i}, $i, j \in \{0, 1\}$, are well defined and holomorphic on their domain of definition, we eliminate any $\lambda \in \mathbb{C}^*$ from our considerations, for which the argument of any of the Gamma functions occurring in the κ_{1j}^{0i} takes values in $\mathbb{Z}_0^-$. Denoting the set of these λ-values by W_3, we restrict our study to the λ-domain $\mathbb{C}^* \setminus (W_1 \cup W_2 \cup W_3)$. As W_2 before, W_3 is also a discrete subset of $\mathbb{C}^*$, which does not possess any accumulation points on S^1. All λ-dependent functions introduced up to this point, including the mappings κ_{1j}^{0i}, $i, j \in \{0, 1\}$, are holomorpic for $\lambda \in \mathbb{C}^* \setminus (W_1 \cup W_2 \cup W_3)$.

From (3.7.12) to (3.7.15) together with (3.7.6) we obtain fundamental systems y_{j1}, y_{j2} around z_j solving the Fuchsian equation (3.7.1):

$$y_{01} = z^{r_0}(1-z)^{r_1}F(\alpha, \beta, \gamma; z), \tag{3.7.23}$$

$$y_{02} = z^{r_0+1-\gamma}(1-z)^{r_1}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z), \tag{3.7.24}$$

$$y_{11} = z^{r_0}(1-z)^{r_1}F(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z), \tag{3.7.25}$$

$$y_{12} = z^{r_0}(1-z)^{r_1+\gamma-\alpha-\beta}F(\gamma-\beta, \gamma-\alpha, \gamma-\alpha-\beta+1; 1-z). \tag{3.7.26}$$

Note that relations (3.7.17) and (3.7.18) relating the fundamental system w_{01}, w_{02} to w_{11}, w_{12} are equivalent to the relations

$$y_{01} = \kappa_{11}^{01}y_{11} + \kappa_{12}^{01}y_{12}, \tag{3.7.27}$$

$$y_{02} = \kappa_{11}^{02}y_{11} + \kappa_{12}^{02}y_{12}, \tag{3.7.28}$$

for the fundamental systems y_{01}, y_{02} and y_{11}, y_{12}.

Like the solutions w_{jk} to (3.7.7), the solutions y_{jk} to (3.7.1) are well defined and holomorphic (with respect to $z \in M$) on the double cut complex plane $\mathcal{D} = \mathbb{C}\setminus\{x \in \mathbb{R}; x \leq 0 \text{ or } x \geq 1\}$.

3.8 Solving $\mathrm{d}\Phi = \Phi\eta$

The following considerations involve logarithms and square roots of the complex variable z. It is well known, that these functions are singular at $z = 0$ and can thus be defined holomorphically only for values of z from the complex plane cut from $z = 0$ to ∞ along an arbitrary half-line. For our purposes, it is convenient to consider both the complex plane cut from 0 to ∞ along the negative real axis,

$$\mathbb{C}_{>0} = \mathbb{C} \setminus \{x \in \mathbb{R};\ x \leq 0\} = \{z = re^{i\varphi} \in \mathbb{C}; r \in \mathbb{R}^+, \varphi \in (-\pi, \pi)\}, \tag{3.8.1}$$

and the complex plane cut from 0 to ∞ along the positive real axis,

$$\mathbb{C}_{<0} = \mathbb{C} \setminus \{x \in \mathbb{R};\ x \geq 0\} = \{z = re^{i\varphi} \in \mathbb{C}; r \in \mathbb{R}^+, \varphi \in (0, 2\pi)\}. \tag{3.8.2}$$

We denote the holomorphic natural logarithm on $\mathbb{C}_{>0}$ by ln and the holomorphic square root on $\mathbb{C}_{>0}$ by $\sqrt{\cdot}$. By contrast, we denote the holomorphic natural logarithm on $\mathbb{C}_{<0}$ by $\ln^*$ and the holomorphic square root on $\mathbb{C}_{<0}$ by $\sqrt[*]{\cdot}$. The definition below lists the explicit mappings, which we use throughout this work, involving the real natural logarithm $\ln_{\mathbb{R}} : \mathbb{R}^+ \to \mathbb{R}$.

Definition 3.27. The holomorphic logarithm ln on $\mathbb{C}_{>0}$ is defined by

$$\ln : \mathbb{C}_{>0} \to \mathbb{C}_{>0}, \quad z \mapsto \ln(z) = \ln_{\mathbb{R}}(r) + i\varphi, \tag{3.8.3}$$

where $z = re^{i\varphi}$ with $r \in \mathbb{R}^+$ and $\varphi \in (-\pi, \pi)$. The holomorphic square root $\sqrt{}$ on $\mathbb{C}_{>0}$ is defined by

$$\sqrt{} : \mathbb{C}_{>0} \to \mathbb{C}_{>0}, \quad z \mapsto \sqrt{z} = e^{\frac{1}{2}\ln(z)}. \tag{3.8.4}$$

The holomorphic logarithm $\ln^*$ on $\mathbb{C}_{<0}$ is defined by

$$\ln^* : \mathbb{C}_{<0} \to \mathbb{C}_{<0}, \quad z \mapsto \ln^*(z) = \ln_{\mathbb{R}}(r) + i\varphi, \tag{3.8.5}$$

where $z = re^{i\varphi}$ with $r \in \mathbb{R}^+$ and $\varphi \in (0, 2\pi)$. The holomorphic square root $\sqrt[*]{}$ on $\mathbb{C}_{<0}$ is defined by

$$\sqrt[*]{} : \mathbb{C}_{<0} \to \mathbb{C}_{<0}, \quad z \mapsto \sqrt[*]{z} = e^{\frac{1}{2}\ln^*(z)}. \tag{3.8.6}$$

Lemma 3.28. *Let $z = re^{i\varphi} \in \mathbb{C}_{<0}$, with $r \in \mathbb{R}^+$ and $\varphi \in (0, 2\pi)$. Then, we have*

$$\sqrt[*]{z} = i\sqrt{-z}. \tag{3.8.7}$$

Proof. Since $z = re^{i\varphi} \in \mathbb{C}_{<0}$ with $r \in \mathbb{R}^+$ and $\varphi \in (0, 2\pi)$, we infer that $-z = -re^{i\varphi} = re^{i(\varphi-\pi)} \in \mathbb{C}_{>0}$. Using this, the claim is a direct consequence of the above definition:

$$\sqrt[*]{z} = e^{\frac{1}{2}\ln^*(z)} = e^{\frac{1}{2}(\ln_{\mathbb{R}}(r)+i\varphi)} = e^{\frac{1}{2}i\pi} \cdot e^{\frac{1}{2}(\ln_{\mathbb{R}}(r)+i(\varphi-\pi))} = ie^{\frac{1}{2}\ln(re^{i(\varphi-\pi)})} = ie^{\frac{1}{2}\ln(-z)} = i\sqrt{-z}. \tag{3.8.8}$$

□

With these preparations made, we resume our study of the differential equation

$$\mathrm{d}\Phi = \Phi\eta. \tag{3.8.9}$$

Recalling the result of section 3.6, by setting $a_0 = a_1 = 0$ and $a_\infty = 2$ and choosing three off-diagonal Delaunay matrices D_0, D_1, D_∞ with eigenvalues $\pm\mu_0$, $\pm\mu_1$, $\pm\mu_\infty$, respectively, we consider w.l.o.g. the standardized trinoid potential

$$\eta = \begin{pmatrix} 0 & \nu(z,\lambda) \\ \tau(z,\lambda) & 0 \end{pmatrix} \mathrm{d}z, \tag{3.8.10}$$

where

$$\nu(z,\lambda) = \lambda^{-1}, \tag{3.8.11}$$

$$\tau(z,\lambda) = -\lambda\left[\frac{b_0(\lambda)}{z^2} + \frac{b_1(\lambda)}{(z-1)^2} + \frac{c_0(\lambda)}{z} + \frac{c_1(\lambda)}{z-1}\right], \tag{3.8.12}$$

and b_0, b_1, b_∞, c_0, c_1 are obtained from

$$b_j(\lambda) = \frac{1}{4} - \mu_j^2 \quad \text{for } j = 0, 1, \infty, \tag{3.8.13}$$

$$b_0(\lambda) + b_1(\lambda) + 0 \cdot c_0(\lambda) + 1 \cdot c_1(\lambda) = b_\infty(\lambda), \tag{3.8.14}$$

$$c_0(\lambda) + c_1(\lambda) = 0. \tag{3.8.15}$$

In order to solve the differential equation (3.8.9) we want to understand its solutions near the singularities of η. For this it will turn out to be helpful to use special forms of the potential and the solutions. We consider, for $j = 0, 1$, the gauged potential (cf. section 2.4)

$$\hat{\eta}_j = V_{+,j}^{-1}\eta V_{+,j} + V_{+,j}^{-1}\mathrm{d}V_{+,j}, \tag{3.8.16}$$

where

$$V_{+,0} = \begin{pmatrix} \sqrt{z} \cdot \sqrt{\lambda X_0}^{-1} & 0 \\ -\frac{\lambda}{2}\sqrt{z}^{-1} \cdot \sqrt{\lambda X_0}^{-1} & \sqrt{z}^{-1} \cdot \sqrt{\lambda X_0} \end{pmatrix}, \tag{3.8.17}$$

$$V_{+,1} = \begin{pmatrix} i\sqrt{1-z} \cdot \sqrt{\lambda X_1}^{-1} & 0 \\ \frac{\lambda}{2}i\sqrt{1-z}^{-1} \cdot \sqrt{\lambda X_1}^{-1} & -i\sqrt{1-z}^{-1} \cdot \sqrt{\lambda X_1} \end{pmatrix}. \tag{3.8.18}$$

Remark 3.29. The gauge matrix $V_{+,j}$, $j = 0, 1$, is given in section 5.2 of [17] as

$$V_{+,j} = \begin{pmatrix} \sqrt{z - z_j} \cdot \sqrt{\lambda X_j}^{-1} & 0 \\ -\frac{\lambda}{2}\sqrt{z - z_j}^{-1} \cdot \sqrt{\lambda X_j}^{-1} & \sqrt{z - z_j}^{-1} \cdot \sqrt{\lambda X_j} \end{pmatrix}, \tag{3.8.19}$$

where the square roots involving z, i.e. $\sqrt{z - z_j}$ and $\sqrt{z - z_j}^{-1}$, are not explicitly defined until we choose a half-line $\mathcal{L}_j$ in $\mathbb{C}$, extending from z_j to ∞, and restrict the mentioned square roots to $\mathbb{C} \setminus \mathcal{L}_j$. For our purposes, it is convenient to think of $V_{+,0}$ as being defined (in z) on $\mathbb{C} \setminus \{x \in \mathbb{R}; x \leq 0\}$ and of $V_{+,1}$ as being defined (in z) on $\mathbb{C} \setminus \{x \in \mathbb{R}; x \geq 1\}$. Thus, we interpret the undetermined square roots $\sqrt{z - z_j}$ and $\sqrt{z - z_j}^{-1}$ of [17] as $\sqrt{z - z_0}$ and $\sqrt{z - z_0}^{-1}$ in the sense of definition 3.27 in the case $j = 0$, but as $\sqrt[*]{z - z_1}$ and $\sqrt[*]{z - z_1}^{-1}$ in the sense of definition 3.27 in the case $j = 1$. Thus, we obtain equation (3.8.17) and, in view of lemma 3.28, equation (3.8.18), where - *now* - the occurring square roots involving z are defined according to definition 3.27.

Remark 3.30. Recall that the λ-dependent functions X_j and $\overline{X}_j$ are defined by $X_j(\lambda) = s_j\lambda^{-1} + t_j\lambda$ and $\overline{X}_j(\lambda) = s_j\lambda + t_j\lambda^{-1}$, respectively with positive real parameters s_j, t_j satisfying $s_j \geq t_j$ and $s_j + t_j = \frac{1}{2}$. Moreover, we assume for $j = 0, 1$ that $s_j \neq t_j$, i.e. $s_j > t_j$ (cf. remark 3.16).

We note that under this assumption the square roots $\sqrt{\lambda X_j}$ and $\sqrt{\lambda X_j}^{-1}$, $j = 0, 1$, occurring in $V_{+,0}$ and $V_{+,1}$ are holomorphic at $\lambda = 0$. (See section 5.2 of [17] for details.) Therefore, the matrices $V_{+,0}$ and $V_{+,1}$ are elements of the loop group $\Lambda_r^+\mathrm{SL}(2,\mathbb{C})_\sigma$ for some $r \in (0, 1]$, thus ensuring that the potential $\hat{\eta}_j$ defined in equation (3.8.16) actually defines a gauged version of the starting potential η.

Remark 3.31. By section 5.2 of [17], the gauged potentials $\hat{\eta}_j$, $j = 0, 1$, defined in (3.8.16) are perturbed versions of the corresponding (unduloidal) Delaunay potentials $\frac{1}{z - z_j} D_j \mathrm{d}z$ defined in (3.5.6), where the off-diagonal Delaunay matrices D_j given in (3.5.7) involve parameters s_j and t_j which in particular satisfy $s_j \geq t_j$.

We note without proof, that there exists also for $j = \infty$ a matrix $V_{+,\infty} \in \Lambda_r^+\mathrm{SL}(2,\mathbb{C})_\sigma$ which gauges the potential $\eta(u, \lambda)$ (obtained from the starting potential $\eta(z, \lambda)$ by applying the coordinate transformation $u = \frac{1}{z}$, cf. remark 3.10) into a perturbed version of the (unduloidal) Delaunay potential $\frac{1}{u} D_\infty \mathrm{d}u$, where the Delaunay matrix D_∞ is given in (3.5.7) and involves parameters s_∞ and t_∞, which in particular satisfy $s_\infty \geq t_\infty$.

By sections 4.2 and 4.3 of [17], there exists for the gauged potential $\hat{\eta}_j$ an *EDP-representation*[8] $\hat{\Phi}_j^*$, which is holomorphic (in z) on a cut disc $\mathcal{D}_j^*$ around z_j and satisfies $\mathrm{d}\hat{\Phi}_j^* = \hat{\Phi}_j^* \hat{\eta}_j$ there. We will call $\hat{\Phi}_j^*$ an *EDP-solution* for short.

$$\hat{\Phi}_j^* = e^{\ln(z - z_j) D_j} \cdot P_j, \tag{3.8.20}$$

where $P_j = \mathrm{I} + (z - z_j)P_{j,1} + (z - z_j)^2 P_{j,2} + \ldots$ is holomorphic at $z = z_j$, $P(z = z_j) = \mathrm{I}$ and P_j is uniquely determined by these properties and the fact that (3.8.20) solves (3.8.9) (cf. [5]). The cut discs $\mathcal{D}_j^*$, where the respective solution $\hat{\Phi}_j^*$ is defined, are given by

$$\mathcal{D}_0^* = \{z \in \mathbb{C}; |z| < \epsilon_0^*\} \setminus \{x \in \mathbb{R}; x \leq 0\}, \tag{3.8.21}$$

$$\mathcal{D}_1^* = \{z \in \mathbb{C}; |z - 1| < \epsilon_1^*\} \setminus \{x \in \mathbb{R}; x \leq 1\}, \tag{3.8.22}$$

where ϵ_0^* and ϵ_1^* denote sufficiently small positive real numbers.

For our concerns, it will be more convenient to work with the following local solutions $\hat{\Phi}_j$ to $\mathrm{d}\hat{\Phi}_j = \hat{\Phi}_j \hat{\eta}_j$ around z_j, slightly modifying $\hat{\Phi}_j^*$ in the case $j = 1$:

$$\hat{\Phi}_0 = e^{\ln(z) D_0} \cdot P_0, \tag{3.8.23}$$

$$\hat{\Phi}_1 = e^{\ln(1 - z) D_1} \cdot P_1. \tag{3.8.24}$$

We prove the following lemma:

Lemma 3.32. *The mappings $\hat{\Phi}_0$ and $\hat{\Phi}_1$ as defined above solve*

$$\mathrm{d}\hat{\Phi}_j = \hat{\Phi}_j \hat{\eta}_j \tag{3.8.25}$$

[8]The expression EDP-representation is an abbreviation for exponential-Delaunay-powerseries-representation.

holomorphically on a cut disc $\mathcal{D}_0$ around $z_0 = 0$ and on a cut disc $\mathcal{D}_1$ around $z_1 = 1$, respectively, where

$$\mathcal{D}_0 = \{z \in \mathbb{C}; |z| < \epsilon_0\} \setminus \{x \in \mathbb{R}; x \leq 0\}, \tag{3.8.26}$$

$$\mathcal{D}_1 = \{z \in \mathbb{C}; |z-1| < \epsilon_1\} \setminus \{x \in \mathbb{R}; x \geq 1\}, \tag{3.8.27}$$

for sufficiently small $\epsilon_j > 0$.

Proof. We start with the case $j = 0$. For some $\epsilon_0 > 0$, P_0 is defined (and holomorphic) on $\{z \in \mathbb{C}; |z| < \epsilon_0\}$. By definition of the complex natural logarithm, $e^{\ln(z)D_0}$ is holomorphic on $\mathbb{C}^* \setminus \{x \in \mathbb{R}; x \leq 0\}$. Together, $\hat{\Phi}_0$ is holomorphic on $\mathcal{D}_0$. Moreover, as $\hat{\Phi}_0 = \hat{\Phi}_0^*$, it is clear that $\hat{\Phi}_0$ solves (3.8.25) on $\mathcal{D}_0$.

We turn to the case $j = 1$. Similar as before, P_1 is defined (and holomorphic) on $\{z \in \mathbb{C}; |z-1| < \epsilon_1\}$ for some $\epsilon_1 > 0$, while $e^{\ln(1-z)D_1}$ is holomorphic on $\mathbb{C}^* \setminus \{x \in \mathbb{R}; x \geq 1\}$. Thus, $\hat{\Phi}_1$ is holomorphic on $\mathcal{D}_1$. Moreover, since

$$\ln(1-z) - \ln(z-1) = \begin{cases} +i\pi & \text{for } z \in \mathbb{C} \text{ with } \Im(z) < 0 \\ -i\pi & \text{for } z \in \mathbb{C} \text{ with } \Im(z) > 0 \end{cases} \tag{3.8.28}$$

we infer that for $z \in \mathbb{C}^* \setminus \mathbb{R}$

$$e^{\ln(1-z)D_1} e^{-\ln(z-1)D_1} = e^{(\ln(1-z)-\ln(z-1))D_1} = e^{\alpha(z)i\pi D_1}, \tag{3.8.29}$$

where $\alpha(z) = +1$ if $\Im(z) < 0$ and $\alpha(z) = -1$ if $\Im(z) > 0$. Thus, for $z \in \mathcal{D}_1 \cap (\mathbb{C}^* \setminus \mathbb{R})$, we obtain

$$\hat{\Phi}_1 = e^{\alpha(z)i\pi D_1} e^{\ln(z-1)D_1} P_1 = \begin{cases} e^{i\pi D_1}\hat{\Phi}_1^* & \text{if } \Im(z) < 0 \\ e^{-i\pi D_1}\hat{\Phi}_1^* & \text{if } \Im(z) > 0 \end{cases} \tag{3.8.30}$$

Consequently, using the fact that $\hat{\Phi}_1^*$ solves $\mathrm{d}\hat{\Phi}_1^* = \hat{\Phi}_1^*\hat{\eta}_1$ on $\mathcal{D}_1^*$, we have for $z \in \mathcal{D}_1$ with $\Im(z) < 0$ that

$$\mathrm{d}\hat{\Phi}_1 = e^{i\pi D_1}\mathrm{d}\hat{\Phi}_1^* = e^{i\pi D_1}\hat{\Phi}_1^*\hat{\eta}_1 = \hat{\Phi}_1\hat{\eta}_1 \tag{3.8.31}$$

and, analogously, for $z \in \mathcal{D}_1$ with $\Im(z) > 0$ that

$$\mathrm{d}\hat{\Phi}_1 = e^{-i\pi D_1}\mathrm{d}\hat{\Phi}_1^* = e^{-i\pi D_1}\hat{\Phi}_1^*\hat{\eta}_1 = \hat{\Phi}_1\hat{\eta}_1. \tag{3.8.32}$$

Together, $\hat{\Phi}_1$ solves (3.8.25) for $z \in \mathcal{D}_1 \cap (\mathbb{C}^* \setminus \mathbb{R})$. By continuity, $\hat{\Phi}_1$ solves (3.8.25) for $z \in \mathcal{D}_1$, which finishes the proof. □

By $\hat{\Phi}_0$ and $\hat{\Phi}_1$, we have found two local solutions to equation (3.8.25) on a cut disc $\mathcal{D}_0$ around $z_0 = 0$ and on a cut disc $\mathcal{D}_1$ around $z_1 = 1$, respectively. From these, we obtain local solutions Φ_j, $j = 0, 1$, to the original differential equation (3.8.9) by setting

$$\Phi_0 = \hat{\Phi}_0 V_{+,0}^{-1} = e^{\ln(z)D_0} P_0 V_{+,0}^{-1}, \tag{3.8.33}$$

$$\Phi_1 = \hat{\Phi}_1 V_{+,1}^{-1} = e^{\ln(1-z)D_1} P_1 V_{+,1}^{-1}, \tag{3.8.34}$$

Remark 3.33. By definition, $V_{+,0}$ (resp. $V_{+,1}$) is holomorphic (in z) on $\mathbb{C} \setminus \{x \in \mathbb{R}; x \leq 0\}$ (resp. on $\mathbb{C} \setminus \{x \in \mathbb{R}; x \geq 1\}$). Moreover, $\hat{\Phi}_j$ is holomorphic (in z) on $\mathcal{D}_j$. Consequently, also Φ_j is holomorphic (in z) on $\mathcal{D}_j$.

The following lemma provides the monodromy matrix of Φ_j with respect to the loop γ_j in M around z_j, which is close enough to z_j, such that γ_j only encloses the singularity z_j. W.l.o.g., we set

$$\gamma_0(t) = \frac{1}{2}e^{it}, \quad -\pi < t < \pi, \tag{3.8.35}$$

$$\gamma_1(t) = 1 + \frac{1}{2}e^{it}, \quad 0 < t < 2\pi. \tag{3.8.36}$$

Remark 3.34. Note that the loops γ_0 and γ_1 as defined above pass through the point $\frac{1}{2} \in M$, as presumed in our considerations of the monodromy matrices of Φ_j in section 3.5.

Lemma 3.35. *With γ_0 and γ_1 defined by* (3.8.35) *and* (3.8.36)*, respectively, the solutions Φ_j to* (3.8.9) *given in* (3.8.33) *and* (3.8.34) *satisfy*

$$\Phi_j(z, \gamma_j, \lambda) = M_j^*(\lambda)\Phi_j(z, \lambda), \tag{3.8.37}$$

where

$$M_j^*(\lambda) = -e^{2\pi i D_j}. \tag{3.8.38}$$

Remark 3.36. Note that in (3.5.22) the monodromy matrix of a solution Φ to (3.8.9) with respect to the loop γ_j in M around z_j is denoted by $M_j(\lambda)$. To avoid inconsistency with our notation, we thus denote the monodromy matrix of the solution Φ_j to (3.8.9) with respect to the loop γ_j in M around z_j by $M_j^*(\lambda)$.)

Proof of lemma 3.35. As stated before, extending the solution Φ_j to (3.8.9), which is defined on the cut domain $\mathcal{D}_j$ around the singularity z_j of η, holomorphically along a closed loop γ_j, which encloses z_j, across the cut results in a change of the starting solution. This change is expressed in form of the monodromy matrix $M_j^*(\lambda)$ of Φ_j with respect to the loop γ_j (cf. (3.5.22)):

$$\Phi_j(z, \gamma_j, \lambda) = M_j^*(\lambda)\Phi_j(z, \lambda). \tag{3.8.39}$$

In order to compute the monodromy matrix $M_j^*(\lambda)$, we investigate the behaviour of $\Phi_j|_{\gamma_j}$ near the cut in $\mathcal{D}_j$, once approaching the cut from below and once from above. We start with the case $j = 0$. Since the cut in $\mathcal{D}_0$ is given by $(-\epsilon_0, 0] \subseteq \mathbb{R}$ and $\gamma_0(t) = \frac{1}{2}e^{it}$ with $t \in (-\pi, \pi)$, $M_0^*(\lambda)$ represents the transition from $\lim_{t\to-\pi} \Phi_0(\gamma_0(t))$ to $\lim_{t\to\pi} \Phi_0(\gamma_0(t))$. Thus, consider the following relations.

First, we have

$$\lim_{t\to\pi} e^{\ln(\gamma_0(t))D_0} = e^{(\ln_{\mathbb{R}}(\frac{1}{2})+\pi i)D_0} = e^{2\pi i D_0} e^{(\ln_{\mathbb{R}}(\frac{1}{2})-\pi i)D_0} = e^{2\pi i D_0} \cdot \lim_{t\to-\pi} e^{\ln(\gamma_0(t))D_0}. \tag{3.8.40}$$

Furthermore, as

$$\begin{aligned}\lim_{t\to\pi} \sqrt{\gamma_0(t)}^{\pm 1} &= \lim_{t\to\pi} e^{\pm\frac{1}{2}\ln(\frac{1}{2}e^{it})} = e^{\pm\frac{1}{2}(\ln_{\mathbb{R}}(\frac{1}{2})+\pi i)} = e^{\pm\pi i} e^{\pm\frac{1}{2}(\ln_{\mathbb{R}}(\frac{1}{2})-\pi i)} \\ &= (-1) \cdot \lim_{t\to-\pi} e^{\pm\frac{1}{2}\ln(\frac{1}{2}e^{it})} = (-1) \cdot \lim_{t\to-\pi} \sqrt{\gamma_0(t)}^{\pm 1},\end{aligned} \tag{3.8.41}$$

we infer that

$$\lim_{t\to\pi} V_{+,0}^{-1}(\gamma_0(t), \lambda) = (-1) \cdot \lim_{t\to-\pi} V_{+,0}^{-1}(\gamma_0(t), \lambda). \tag{3.8.42}$$

Finally, since P_0 is holomorphic (in z) around z_0, we have

$$\lim_{t\to\pi} P_0(\gamma_0(t), \lambda) = \lim_{t\to-\pi} P_0(\gamma_0(t), \lambda). \tag{3.8.43}$$

Altogether, we conclude that

$$\lim_{t\to\pi} \Phi_0(\gamma_0(t), \lambda) = \lim_{t\to\pi} \left(e^{\ln(\gamma_0(t))D_0} P_0(\gamma_0(t), \lambda) V_{+,0}^{-1}(\gamma_0(t), \lambda)\right) = -e^{2\pi i D_0} \cdot \lim_{t\to-\pi} \Phi_0(\gamma_0(t), \lambda) \tag{3.8.44}$$

and, consequently,

$$M_0^*(\lambda) = -e^{2\pi i D_0}. \tag{3.8.45}$$

We turn to the case $j = 1$. Since the cut in $\mathcal{D}_1$ is given by $[1, 1 + \epsilon_1) \subseteq \mathbb{R}$ and $\gamma_1(t) = 1 + \frac{1}{2}e^{it}$ with $t \in (0, 2\pi)$, $M_1^*(\lambda)$ represents the transition from $\lim_{t\to0} \Phi_1(\gamma_1(t))$ to $\lim_{t\to2\pi} \Phi_1(\gamma_1(t))$. Since $\ln(1 - \gamma_1(t)) = \ln(-\frac{1}{2}e^{it}) = \ln(\frac{1}{2}e^{i(t-\pi)}) = \ln_{\mathbb{R}}(\frac{1}{2}) + i(t - \pi)$, we have

$$\lim_{t\to2\pi} e^{\ln(1-\gamma_1(t))D_1} = e^{(\ln_{\mathbb{R}}(\frac{1}{2})+\pi i)D_1} = e^{2\pi i D_1} e^{(\ln_{\mathbb{R}}(\frac{1}{2})-\pi i)D_1} = e^{2\pi i D_1} \cdot \lim_{t\to0} e^{\ln(1-\gamma_1(t))D_1}. \tag{3.8.46}$$

Moreover, as

$$\begin{aligned}\lim_{t\to2\pi} \left(\sqrt{1-\gamma_1(t)}\right)^{\pm 1} &= \lim_{t\to2\pi} \left(\sqrt{-\frac{1}{2}e^{it}}\right)^{\pm 1} = \lim_{t\to2\pi} \left(\sqrt{\frac{1}{2}e^{i(t-\pi)}}\right)^{\pm 1} \\ &= \lim_{t\to2\pi} e^{\pm\frac{1}{2}\ln(\frac{1}{2}e^{i(t-\pi)})} = e^{\pm\frac{1}{2}(\ln_{\mathbb{R}}(\frac{1}{2})+\pi i)} = e^{\pm\pi i} e^{\pm\frac{1}{2}(\ln_{\mathbb{R}}(\frac{1}{2})-\pi i)} \\ &= (-1) \cdot \lim_{t\to0} e^{\pm\frac{1}{2}\ln(\frac{1}{2}e^{i(t-\pi)})} = (-1) \cdot \lim_{t\to0} \left(\sqrt{1-\gamma_1(t)}\right)^{\pm 1},\end{aligned} \tag{3.8.47}$$

we infer that

$$\lim_{t\to2\pi} V_{+,1}^{-1}(\gamma_1(t), \lambda) = (-1) \cdot \lim_{t\to0} V_{+,1}^{-1}(\gamma_1(t), \lambda). \tag{3.8.48}$$

Finally, as P_1 is holomorphic (in z) around z_1, we have

$$\lim_{t\to2\pi} P_1(\gamma_1(t), \lambda) = \lim_{t\to0} P_1(\gamma_1(t), \lambda). \tag{3.8.49}$$

Altogether, we conclude that

$$\lim_{t\to 2\pi}\Phi_1(\gamma_1(t),\lambda)=\lim_{t\to 2\pi}\left(e^{\ln(1-\gamma_1(t))D_1}P_1(\gamma_1(t),\lambda)V_{+,1}^{-1}(\gamma_1(t),\lambda)\right)=-e^{2\pi iD_1}\cdot\lim_{t\to 0}\Phi_1(\gamma_1(t),\lambda) \quad (3.8.50)$$

and, consequently,

$$M_1^*(\lambda)=-e^{2\pi iD_1}. \quad (3.8.51)$$

□

By lemma 3.12, Φ_j may be described in terms of an appropriate fundamental system solving (3.5.14) around z_j, which itself may be expressed in terms of the fundamental system y_{j1}, y_{j2} given in equations (3.7.23) to (3.7.26). That is, we may write for $j=0,1$

$$\Phi_j=\begin{pmatrix}\frac{\alpha_j y'_{j1}+\beta_j y'_{j2}}{\nu} & \alpha_j y_{j1}+\beta_j y_{j2}\\ \frac{\delta_j y'_{j1}+\epsilon_j y'_{j2}}{\nu} & \delta_j y_{j1}+\epsilon_j y_{j2}\end{pmatrix}, \quad (3.8.52)$$

where α_j, β_j, δ_j, ϵ_j denote z-independent functions of λ. It turns out that, by evaluating in (3.8.33) (resp. in (3.8.34)) the properties of both P_j and the fundamental system y_{j1}, y_{j2} at z_j, especially the holomorphicity on a cut disc around z_j, the connection coefficients α_j, β_j, δ_j, ϵ_j can be computed explicitly:

Lemma 3.37. *Let $j\in\{0,1\}$. The connection coefficients α_j, β_j, δ_j, ϵ_j occuring in* (3.8.52) *are given by*

$$\alpha_j=-\beta_j=(i)^j\frac{X_j}{2\mu_j\sqrt{\lambda X_j}}, \quad (3.8.53)$$

$$\delta_j=\epsilon_j=(i)^j\frac{1}{2\sqrt{\lambda X_j}}. \quad (3.8.54)$$

Proof. The proof of this lemma is quite technical and therefore postponed until appendix C. □

Recall that $y_{01}, y_{02}, y_{11}, y_{12}$ may be extended holomorphically (in z) to $\mathcal{D}$, the complex plane excluding two "cuts" from 0 to $-\infty$ and from 1 to $+\infty$, as introduced in (3.5.20). By (3.8.52), also Φ_0 and Φ_1 can be extended holomorphically (in z) to $\mathcal{D}$. Denoting the extensions again by Φ_0 and Φ_1, respectively, we obtain two solutions to (3.8.9) - now defined globally for z from the simply connected, "double-cut" complex plane $\mathcal{D}$ - which will only differ by a matrix $A=A(\lambda)$, which is independent of z. That is, we have

$$\Phi_0=A(\lambda)\Phi_1. \quad (3.8.55)$$

The matrix A can be explicitly computed:

Lemma 3.38.

$$A=-i\frac{\sqrt{\mu_1}}{\sqrt{\mu_0}}R_0S\begin{pmatrix}\kappa_{11}^{01} & \lambda^{-1}\kappa_{12}^{01}\\ \lambda\kappa_{11}^{02} & \kappa_{12}^{02}\end{pmatrix}S^{-1}R_1^{-1}, \quad (3.8.56)$$

where $R_j=\begin{pmatrix}\frac{\sqrt{\lambda X_j}}{\sqrt{\mu_j}} & 0\\ 0 & \frac{\sqrt{\lambda^{-1}X_j}}{\sqrt{\mu_j}}\end{pmatrix}$ *and* $S=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -\lambda^{-1}\\ \lambda & 1\end{pmatrix}$.

Proof. First, using equations (3.8.52), (3.7.27) and (3.7.28), we have

$$\Phi_0=\begin{pmatrix}\alpha_0 & \beta_0\\ \delta_0 & \epsilon_0\end{pmatrix}\begin{pmatrix}\kappa_{11}^{01} & \kappa_{12}^{01}\\ \kappa_{11}^{02} & \kappa_{12}^{02}\end{pmatrix}\begin{pmatrix}\frac{y'_{11}}{\nu} & y_{11}\\ \frac{y'_{12}}{\nu} & y_{12}\end{pmatrix}, \quad (3.8.57)$$

$$\Phi_1=\begin{pmatrix}\alpha_1 & \beta_1\\ \delta_1 & \epsilon_1\end{pmatrix}\begin{pmatrix}\frac{y'_{11}}{\nu} & y_{11}\\ \frac{y'_{12}}{\nu} & y_{12}\end{pmatrix}. \quad (3.8.58)$$

This yields

$$A=\Phi_0\Phi_1^{-1}=\begin{pmatrix}\alpha_0 & \beta_0\\ \delta_0 & \epsilon_0\end{pmatrix}\begin{pmatrix}\kappa_{11}^{01} & \kappa_{12}^{01}\\ \kappa_{11}^{02} & \kappa_{12}^{02}\end{pmatrix}\begin{pmatrix}\alpha_1 & \beta_1\\ \delta_1 & \epsilon_1\end{pmatrix}^{-1}. \quad (3.8.59)$$

Since, moreover,

$$\begin{pmatrix}\alpha_j & \beta_j\\ \delta_j & \epsilon_j\end{pmatrix}=\frac{(i)^j}{2\sqrt{\lambda X_j}}\begin{pmatrix}\frac{X_j}{\mu_j} & -\frac{X_j}{\mu_j}\\ 1 & 1\end{pmatrix}=\frac{(i)^j}{\sqrt{2}\sqrt{\mu_j}}R_jS\begin{pmatrix}\lambda^{-1} & 0\\ 0 & 1\end{pmatrix}, \quad (3.8.60)$$

and, consequently,

$$\begin{pmatrix} \alpha_j & \beta_j \\ \delta_j & \epsilon_j \end{pmatrix}^{-1} = (-i)^j \sqrt{2}\sqrt{\mu_j} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} S^{-1} R_j^{-1}, \tag{3.8.61}$$

we end up with

$$A = -i\frac{\sqrt{\mu_1}}{\sqrt{\mu_0}} R_0 S \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \kappa_{11}^{01} & \kappa_{12}^{01} \\ \kappa_{11}^{02} & \kappa_{12}^{02} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} S^{-1} R_j^{-1} = -i\frac{\sqrt{\mu_1}}{\sqrt{\mu_0}} R_0 S \begin{pmatrix} \kappa_{11}^{01} & \lambda^{-1}\kappa_{12}^{01} \\ \lambda\kappa_{11}^{02} & \kappa_{12}^{02} \end{pmatrix} S^{-1} R_1^{-1}. \tag{3.8.62}$$

□

Moreover, we have the following result:

Lemma 3.39. *The matrix A defined in* (3.8.56) *satisfies*

$$\det(A) = 1. \tag{3.8.63}$$

In particular,

$$\kappa_{11}^{01}\kappa_{12}^{02} - \kappa_{12}^{01}\kappa_{11}^{02} = -\frac{\mu_0}{\mu_1}. \tag{3.8.64}$$

Proof. Since Φ_0 and Φ_1 take values in $\Lambda_r\mathrm{SL}(2,\mathbb{C})_\sigma$, the identity $\det(A) = 1$ follows directly from (3.8.55). Moreover, as $\det(R_j) = \det(R_1^{-1}) = \det(S) = \det(S^{-1}) = 1$, we obtain in view of (3.8.56)

$$1 = \det(A) = \det(-i\frac{\sqrt{\mu_1}}{\sqrt{\mu_0}} \begin{pmatrix} \kappa_{11}^{01} & \lambda^{-1}\kappa_{12}^{01} \\ \lambda\kappa_{11}^{02} & \kappa_{12}^{02} \end{pmatrix}) = -\frac{\mu_1}{\mu_0}(\kappa_{11}^{01}\kappa_{12}^{02} - \kappa_{12}^{01}\kappa_{11}^{02}), \tag{3.8.65}$$

which also proves the relation (3.8.64). □

Taking into account (3.8.55), we are now able to explicitly compute the monodromy matrices of Φ_0 at z_0 and z_1, as well as the monodromy matrices of Φ_1 at z_0 and z_1. Since Φ_0 and Φ_1 are linked by (3.8.55), it suffices to consider the solution $\Phi = \Phi_0$.

Theorem 3.40. *The solution*

$$\Phi = \Phi_0 = A\Phi_1 \tag{3.8.66}$$

to the differential equation (3.8.9) *satisfies*

$$\Phi(z, \gamma_j, \lambda) = M_j(\lambda)\Phi(z, \lambda), \tag{3.8.67}$$

where

$$M_0(\lambda) = -e^{2\pi i D_0} = -\left[\cos(2\pi\mu_0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\sin(2\pi\mu_0)\begin{pmatrix} 0 & \frac{X_0}{\mu_0} \\ \frac{X_0}{\mu_0} & 0 \end{pmatrix}\right], \tag{3.8.68}$$

$$M_1(\lambda) = -Ae^{2\pi i D_1}A^{-1} = -\left[\cos(2\pi\mu_1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\sin(2\pi\mu_1)A\begin{pmatrix} 0 & \frac{X_1}{\mu_1} \\ \frac{X_1}{\mu_1} & 0 \end{pmatrix}A^{-1}\right]. \tag{3.8.69}$$

Proof. By (3.8.37), we know that $\Phi_j(z, \gamma_j, \lambda) = M_j^*(\lambda)\Phi_j(z, \lambda)$, where $M_j^*(\lambda) = -e^{2\pi i D_j}$. Consequently, using (3.8.55),

$$\Phi(z, \gamma_0, \lambda) = \Phi_0(z, \gamma_0, \lambda) = M_0^*(\lambda)\Phi_0(z, \lambda) = M_0^*(\lambda)\Phi(z, \lambda), \tag{3.8.70}$$

$$\Phi(z, \gamma_1, \lambda) = A\Phi_1(z, \gamma_1, \lambda) = AM_1^*(\lambda)A^{-1}A\Phi_1(z, \lambda) = AM_1^*(\lambda)A^{-1}\Phi(z, \lambda). \tag{3.8.71}$$

This proves

$$M_0(\lambda) = M_0^*(\lambda) = -e^{2\pi i D_0}, \tag{3.8.72}$$

$$M_1(\lambda) = AM_1^*(\lambda)A^{-1} = -Ae^{2\pi i D_1}A^{-1}. \tag{3.8.73}$$

Finally, referring to (2.6.23) (note that by assumption $s_j > t_j$ for $j = 0, 1$), we have

$$e^{2\pi i D_j} = R_j S \begin{pmatrix} e^{2\pi i\mu_j} & 0 \\ 0 & e^{-2\pi i\mu_j} \end{pmatrix} S^{-1} R_j^{-1}. \tag{3.8.74}$$

Using $e^{\pm 2\pi i\mu_j} = \cos(2\pi\mu_j) \pm i\sin(2\pi\mu_j)$, the above equation yields

$$e^{2\pi i D_j} = \cos(2\pi\mu_j)\mathrm{I} + i\sin(2\pi\mu_j)R_j S\sigma_3 S^{-1}R_j^{-1} = \cos(2\pi\mu_j)\mathrm{I} + i\sin(2\pi\mu_j)\begin{pmatrix} 0 & \frac{X_j}{\mu_j} \\ \frac{X_j}{\mu_j} & 0 \end{pmatrix}. \tag{3.8.75}$$

Altogether, this finishes the proof. □

Remark 3.41. In view of remark 3.26 and the definitions of Φ_0, Φ_1 and A, our solution $\Phi = \Phi_0 = A\Phi_1$ to the differential equation (3.8.9) is (at least) well defined and holomorphic in λ for $\lambda \in \mathbb{C}^* \setminus (W_1 \cup W_2 \cup W_3)$. The same holds for the monodromy matrices M_0 and M_1 of Φ.

Recalling the structure of the sets W_1, W_2 and W_3 (in particular the fact that none of these sets possesses any accumulation points on the unit circle S^1), we note that the set $\mathbb{C}^* \setminus (W_1 \cup W_2 \cup W_3)$ contains for at least some $r \in (0,1)$ an open annulus containing the r-circle $C^{(r)}$. (More precisely, any $r \in (0,1)$ "close enough" to 1 will do.)

3.9 Simultaneous unitarization of the monodromy matrices

In the previous section, we have determined a solution Φ to (3.8.9), given explicitly by

$$\Phi = \Phi_0 = e^{\ln(z)D_0}P_0V_{+,0}^{-1} = \begin{pmatrix} \frac{\alpha_0 y_{01}' + \beta_0 y_{02}'}{\nu} & \alpha_0 y_{01} + \beta_0 y_{02} \\ \frac{\delta_0 y_{01}' + \epsilon_0 y_{02}'}{\nu} & \delta_0 y_{01} + \epsilon_0 y_{02} \end{pmatrix}, \tag{3.9.1}$$

where the fundamental system y_{01}, y_{02} is given by (3.7.23) and (3.7.24) and the connection coefficients $\alpha_0, \beta_0, \delta_0, \epsilon_0$ are defined in (3.8.53) and (3.8.54). Φ is defined (in z) on the double cut complex plane $\mathcal{D} \subseteq M$ and thus induces by remark 3.8 a solution $\Psi = \Phi \circ \pi$ to the differential equation

$$\mathrm{d}\Psi = \Psi\tilde{\eta}, \tag{3.9.2}$$

which is defined for $z \in \tilde{M}$.

By surrounding the singularities z_0 and z_1 in M, and thus "crossing the cuts", which have been excluded from the z-domain of definition of Φ, Φ picks up the monodromy matrices $M_0(\lambda)$ and $M_1(\lambda)$, respectively, which we have explicitly computed in theorem 3.40:

$$M_0(\lambda) = -e^{2\pi i D_0} = -\left[\cos(2\pi\mu_0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\sin(2\pi\mu_0)\begin{pmatrix} 0 & \frac{X_0}{\mu_0} \\ \frac{X_0}{\mu_0} & 0 \end{pmatrix}\right], \tag{3.9.3}$$

$$M_1(\lambda) = -Ae^{2\pi i D_1}A^{-1} = -\left[\cos(2\pi\mu_1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\sin(2\pi\mu_1)A\begin{pmatrix} 0 & \frac{X_1}{\mu_1} \\ \frac{X_1}{\mu_1} & 0 \end{pmatrix}A^{-1}\right]. \tag{3.9.4}$$

As explained earlier, $M_0(\lambda)$ and $M_1(\lambda)$ are also the monodromy matrices of Ψ with respect to the covering transformations $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$, respectively, which correspond to the loops γ_0 and γ_1 in M enclosing the singularities z_0 and z_1, respectively:

$$\Psi(\tilde{\gamma}_j(z), \lambda) = M_j(\lambda)\Psi(z, \lambda) \quad j = 0, 1. \tag{3.9.5}$$

By remark 3.41, Φ (and thus Ψ) is well defined and holomorphic in λ for $\lambda \in \mathbb{C}^* \setminus (W_1 \cup W_2 \cup W_3)$. Recalling the definitions of the sets W_1, W_2 and W_3, respectively, we observe that $\mathbb{C}^* \setminus (W_1 \cup W_2 \cup W_3)$ contains (a sufficiently small open neighborhood of) an r-circle $C^{(r)}$ for some $0 < r < 1$, which is close enough to 1. (Since, as mentioned earlier, $1 \in W_2$, the unit circle S^1 is *not* contained in $\mathbb{C}^* \setminus (W_1 \cup W_2 \cup W_3)$.) This implies that Φ and Ψ are in particular holomorphic (in λ) on (a sufficiently small open neighborhood of) an r-circle $C^{(r)}$. Thus, carrying out an r-Iwasawa decomposition in the second step of the loop group method, Ψ produces by evaluating the associated extended frame (which is holomorphic for λ in the annulus $A^{(r)}$) at $\lambda = 1$ a CMC-immersion $\psi : \tilde{M} \to \mathbb{R}^3$. Referring to theorem 2.11, ψ yields a CMC-immersion $\phi : M \to \mathbb{R}^3$ if and only if the monodromy matrices of Ψ meet the conditions of theorem 2.11, or, by remark 3.19 equivalently, if and only if the "generating" monodromy matrices M_0 and M_1 of Ψ meet the conditions of theorem 2.11. However, in general, this is *not* the case: While M_0 is unitary for $\lambda \in S^1$, M_1 is in general *not* unitary for $\lambda \in S^1$ (cf. appendix D).

Therefore, in the following, we modify Ψ to obtain another solution $\hat{\Psi}$ to the differential equation (3.9.2) with (generating) monodromy matrices $\hat{M}_j$ ($j = 0, 1$), which actually meet the conditions of theorem 2.11. More precisely, we modify Ψ by an appropriate λ-dependent dressing matrix $T = T(\lambda)$, such that the monodromy matrices $\hat{M}_j = TM_jT^{-1}$ of the dressed solution $\hat{\Psi} = T\Psi$ satisfy the conditions of theorem 2.11. (Note that the existence of such a matrix T is provided by theorem 3.14.)

Note that, by section 2.2 of [17], any CMC-immersion $\psi : \tilde{M} \to \mathbb{R}^3$ can be obtained from a solution $\tilde{\Psi}$ to the differential equation (3.9.2), which is holomorphic in $\lambda \in \mathbb{C}^*$ (cf. section 2.2 of [17]) and thus comes along with monodromy matrices, which are holomorphic in $\lambda \in \mathbb{C}^*$ as well. Therefore, we additionally assume that the monodromy matrices $\hat{M}_j = TM_jT^{-1}$ $(j = 0, 1)$ of the dressed solution $\hat{\Psi} = T\Psi$ are holomorphic in $\lambda \in \mathbb{C}^*$.

Altogether, our next goal will be to compute explicitly a dressing matrix $T = T(\lambda)$, such that the monodromy matrices $\hat{M}_j = TM_jT^{-1}$ $(j = 0, 1)$ of the dressed solution $\hat{\Psi} = T\Psi$ are holomorphic in $\lambda \in \mathbb{C}^*$ and satisfy the conditions of theorem 2.11, i.e. such that for $j = 0, 1$

$$\hat{M}_j(\lambda) \text{ is holomorphic in } \lambda \in \mathbb{C}^*, \tag{3.9.6}$$

$$\hat{M}_j(\lambda) \text{ is unitary for all } \lambda \in S^1, \tag{3.9.7}$$

$$\hat{M}_j(\lambda = 1) = \pm \mathrm{I} \textit{ and} \tag{3.9.8}$$

$$\partial_\lambda \hat{M}_j(\lambda)|_{\lambda=1} = 0. \tag{3.9.9}$$

Remark 3.42. Among the conditions (3.9.6) to (3.9.9), satisfying the condition (3.9.7) poses the main difficulty. This is emphasized by the name of the current section.

Remark 3.43. Our starting solution Ψ to the differential equation (3.9.2), as well as the corresponding monodromy matrices M_j are holomorphic in λ on the domain $\mathbb{C}^* \setminus (W_1 \cup W_2 \cup W_3)$. In order to obtain via dressing by T from Ψ a new solution $\hat{\Psi} = T\Psi$ with monodromy matrices $\hat{M}_j = TM_jT^{-1}$, which are holomorphic in $\lambda \in \mathbb{C}^*$, T necessarily needs to "cancel" the existing singularities from the monodromy matrices M_j of Ψ. Consequently, it is not only possible but even probable, that T itself possesses singularities in $\mathbb{C}^*$ and thus will not be well defined on the whole (punctured) λ-plane $\mathbb{C}^*$.

In the following, we will compute the dressing matrix T purely formally. In particular, we will for now ignore the λ-domain of definition of T, postponing this issue to remark 3.57.

Let Φ be as in (3.9.1), $\Psi = \Phi \circ \pi$ and the (common) monodromy matrices M_j $(j = 0, 1)$ of Φ and Ψ given by (3.9.3) and (3.9.4). We want to determine a dressing matrix $T = T(\lambda)$, such that the monodromy matrices $\hat{M}_j = TM_jT^{-1}$ $(j = 0, 1)$ of the dressed solution $\hat{\Psi} = T\Psi$ satisfy the conditions (3.9.6) to (3.9.9). First, we compute the general form of $\hat{M}_j$: Observing that any conjugate of the matrix $\begin{pmatrix} 0 & \frac{X_j}{\mu_j} \\ \frac{X_j}{\mu_j} & 0 \end{pmatrix}$ by a λ-dependent matrix will be of the form $\begin{pmatrix} p_j & r_j \\ q_j & -p_j \end{pmatrix}$ for some λ-dependent functions p_j, q_j, r_j, the dressed monodromy matrices $\hat{M}_j$ are in view of (3.8.68) and (3.8.69) of the general form

$$\hat{M}_j = TM_jT^{-1} = -\left[\cos(2\pi\mu_j)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\sin(2\pi\mu_j)\begin{pmatrix} p_j & r_j \\ q_j & -p_j \end{pmatrix}\right], \tag{3.9.10}$$

where

$$\begin{pmatrix} p_0 & r_0 \\ q_0 & -p_0 \end{pmatrix} = T\begin{pmatrix} 0 & \frac{X_0}{\mu_0} \\ \frac{X_0}{\mu_0} & 0 \end{pmatrix}T^{-1}, \tag{3.9.11}$$

$$\begin{pmatrix} p_1 & r_1 \\ q_1 & -p_1 \end{pmatrix} = TA\begin{pmatrix} 0 & \frac{X_1}{\mu_1} \\ \frac{X_1}{\mu_1} & 0 \end{pmatrix}A^{-1}T^{-1}. \tag{3.9.12}$$

Since $\det(M_j) = 1$ and conjugating a matrix does not affect its determinant, we moreover infer that the functions p_j, q_j, r_j satisfy

$$p_j^2 + q_jr_j = 1. \tag{3.9.13}$$

Remark 3.44. In order to ensure that the monodromy matrices $\hat{M}_j$, $j = 0, 1$ of the form (3.9.10) are elements of the twisted loop group $\Lambda_r\mathrm{SU}(2)_\sigma$ we furthermore require for $j = 0, 1$ that

$$\begin{aligned} &p_j \text{ is an even function of } \lambda, \\ &q_j \text{ is an odd function of } \lambda, \\ &r_j \text{ is an odd function of } \lambda. \end{aligned} \tag{3.9.14}$$

In view of (3.9.10), we reformulate the conditions (3.9.6) to (3.9.9) in terms of the functions p_j, q_j and r_j occurring in the dressed monodromy matrices $\hat{M}_j$. The condition (3.9.6) translates into the following constraints on the functions p_j, q_j and r_j $(j = 0, 1)$:

$$\begin{aligned} \sin(2\pi\mu_j)p_j &\text{ is holomorphic for } \lambda \in \mathbb{C}^*, \\ \sin(2\pi\mu_j)q_j &\text{ is holomorphic for } \lambda \in \mathbb{C}^*, \\ \sin(2\pi\mu_j)r_j &\text{ is holomorphic for } \lambda \in \mathbb{C}^*. \end{aligned} \tag{3.9.15}$$

(Writing $\cos(2\pi\mu_j)$ in its power series representation,

$$\cos(2\pi\mu_j) = \sum_{k=0}^{\infty}(-1)^k \frac{(2\pi\mu_j)^{2k}}{(2k)!}, \tag{3.9.16}$$

we see that $\cos(2\pi\mu_j)$ only involves even powers of μ_j. Since, by remark 3.13, μ_j^2 is defined as a holomorphic function on $\mathbb{C}^*$, $\cos(2\pi\mu_j)$ is holomorphic for $\lambda \in \mathbb{C}^*$.)

In order to express (3.9.7) in a different way, we note that a matrix U is in SU(2) if and only if U is of the form $U = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$ with $u\bar{u} + v\bar{v} = 1$. Applying this to (3.9.10), we see that $\hat{M}_j(\lambda)$ is in SU(2) for all $\lambda \in S^1$ if and only if the functions p_j, q_j and r_j $(j = 0, 1)$ satisfy (in addition to already existing conditions)

$$\begin{aligned} p_j(\lambda) &= \overline{p_j(\lambda)} \text{ for all } \lambda \in \tilde{S}^1, \\ r_j(\lambda) &= \overline{q_j(\lambda)} \text{ for all } \lambda \in \tilde{S}^1, \end{aligned} \tag{3.9.17}$$

where $\tilde{S}^1$ denotes the unit sphere S^1 excluding the discrete subset of values of $\lambda \in S^1$, for which $\sin(2\pi\mu_j(\lambda)) = 0$. (In particular, the functions p_j, q_j and r_j are necessarily well defined on $\tilde{S}^1$. Note that at $\lambda \in S^1 \setminus \tilde{S}^1$, this is not necessarily the case: here, p_j, q_j and r_j might possess simple poles, which cancel with the (simple) zeros of $\sin(2\pi\mu_j(\lambda))$.)

Remark 3.45. In view of (3.9.15), the functions p_j, q_j and r_j, and thus also the functions $\overline{p_j}$ and $\overline{q_j}$ defined by

$$\overline{p_j}(\lambda) := \overline{p_j(\frac{1}{\bar{\lambda}})}, \tag{3.9.18}$$

$$\overline{q_j}(\lambda) := \overline{q_j(\frac{1}{\bar{\lambda}})}, \tag{3.9.19}$$

respectively, are at least holomorphic on the (common) subdomain of $\mathbb{C}^*$, where (for all $j \in \{0, 1, \infty\}$) $\mu_j(\lambda)$ is holomorphic and $\sin(2\pi\mu_j(\lambda)) \neq 0$, i.e. on $\mathbb{C}^* \setminus (W_1 \cup W_4)$, where W_1 is given in (3.7.5) and

$$W_4 = \{\lambda \in \mathbb{C}^*; \sin(2\pi\mu_j(\lambda)) = 0\}. \tag{3.9.20}$$

(As stated earlier, the mappings μ_j, $j = 0, 1, \infty$, are holomorphic on $\mathbb{C}^* \setminus W_1$ and thus in particular on $\mathbb{C}^* \setminus (W_1 \cup W_4)$. Moreover, the sets W_1 and W_4 are "symmetric" with respect to the unit circlce S^1 in the sense that (for $k = 1, 4$) $\lambda \in W_k$ if and only if $\frac{1}{\bar{\lambda}} \in W_k$. This ensures that, together with the functions p_j, q_j and r_j, also the functions $\overline{p_j}$ and $\overline{q_j}$ are well defined (and holomorphic) on $\mathbb{C}^* \setminus (W_1 \cup W_4)$.)

Note for later use (cf. remark 3.57) that, as W_1 before, W_4 does not possess any accumulation points on the unit circle S^1. Consequently, the set $\mathbb{C}^* \setminus (W_1 \cup W_4)$ contains for at least some $r \in (0, 1)$ an open annulus containing the r-circle $C^{(r)}$. (More precisely, any $r \in (0, 1)$ "close enough" to 1 will do.)

Combining the result above with (3.9.17), we conclude that p_j and $\overline{p_j}$ (resp. r_j and $\overline{q_j}$) define holomorphic functions on $\mathbb{C}^* \setminus (W_1 \cup W_4)$, which coincide on $\tilde{S}^1$:

$$p_j(\lambda) = \overline{p_j(\lambda)} = \overline{p_j(\frac{1}{\bar{\lambda}})} = \overline{p_j}(\lambda), \tag{3.9.21}$$

$$r_j(\lambda) = \overline{q_j(\lambda)} = \overline{q_j(\frac{1}{\bar{\lambda}})} = \overline{q_j}(\lambda). \tag{3.9.22}$$

Consequently, p_j and $\overline{p_j}$ (resp. r_j and $\overline{q_j}$) coincide everywhere on $\mathbb{C}^* \setminus (W_1 \cup W_4)$, i.e. we can actually replace (3.9.17) by

$$\begin{aligned} p_j &= \overline{p_j}, \\ r_j &= \overline{q_j}. \end{aligned} \tag{3.9.23}$$

Finally, by two simple calculations based on the relations $\mu_j(\lambda = 1) = \frac{1}{2}$ and $(\partial_\lambda \mu_j)(\lambda = 1) = 0$ from lemma B.3, the conditions (3.9.8) and (3.9.9) translate into the following further constraints on the functions p_j, q_j and r_j $(j = 0, 1)$:

$$p_j, q_j, r_j \text{ take finite values in } \mathbb{C} \text{ for } \lambda = 1, \tag{3.9.24}$$

$$p_j, q_j, r_j \text{ are holomorphic (in } \lambda) \text{ at } \lambda = 1. \tag{3.9.25}$$

In the following, we focus on (3.9.7), the first condition of theorem 2.11. Thus, our next goal will be to compute explicitly a dressing matrix $T = T(\lambda)$, such that $\hat{\Psi} = T\Psi$ has *unitary* monodromy matrices $\hat{M}_0$ and $\hat{M}_1$ for all $\lambda \in S^1$. In view of our considerations above, we see that $\hat{M}_j$ $(j = 0, 1)$ is unitary and of determinant 1 on S^1 if an only if it is of the form

$$\hat{M}_j = -\left[\cos(2\pi\mu_j)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i\sin(2\pi\mu_j)\begin{pmatrix}p_j & \overline{q_j}\\ q_j & -p_j\end{pmatrix}\right] \tag{3.9.26}$$

with λ-dependent functions p_j, $\overline{p_j}$ (defined in (3.9.18)), q_j and $\overline{q_j}$ (defined in (3.9.19)) satisfying

$$p_j^2 + q_j\overline{q_j} = 1 \text{ and } p_j = \overline{p_j} \tag{3.9.27}$$

and

$$\begin{pmatrix}p_0 & \overline{q_0}\\ q_0 & -p_0\end{pmatrix} = T\begin{pmatrix}0 & \frac{X_0}{\mu_0}\\ \frac{\overline{X_0}}{\mu_0} & 0\end{pmatrix}T^{-1}, \tag{3.9.28}$$

$$\begin{pmatrix}p_1 & \overline{q_1}\\ q_1 & -p_1\end{pmatrix} = TA\begin{pmatrix}0 & \frac{X_1}{\mu_1}\\ \frac{\overline{X_1}}{\mu_1} & 0\end{pmatrix}A^{-1}T^{-1}. \tag{3.9.29}$$

Remark 3.46. We can assume that $q_0, q_1 \not\equiv 0$ on S^1: If, by accident, some T should give $q_j \equiv 0$ for some j, it is easy to show that we can modify T by multiplying from the left by a unitary matrix $U = U(\lambda)$ such that $\hat{T} = UT$ will yield $q_0, q_1 \not\equiv 0$ on S^1, while unitarity of the dressed monodromy matrices on S^1 will be maintained. That is, if simultaneously unitarizing M_0, M_1 is possible at all, it is also possible in a way such that $q_0, q_1 \not\equiv 0$. Note that modifying T by U will result (only) in a rotation and/or translation of the generated CMC-surface $M \to \mathbb{R}^3$.

Remark 3.47. Note that (3.9.26) holds also for $j = \infty$, as the monodromy matrix $\hat{M}_\infty$ is a unitary conjugate of the Delaunay monodromy matrix $-e^{2\pi i D_\infty}$. Note that, however, in the special case that $s_\infty = t_\infty = \frac{1}{4}$, we have $\mu_j(\lambda) = \frac{1}{4}(\lambda + \lambda^{-1})$, i.e. μ_j defines (in contrast to the general case $s_\infty \neq t_\infty$) an odd function of λ. Thus in the case $s_\infty = t_\infty = \frac{1}{4}$, the condition that the monodromy matrix $\hat{M}_\infty$ is an element of the twisted loop group $\Lambda_r\mathrm{SU}(2)_\sigma$ is equivalent to requiring

$$\begin{aligned}&p_j \text{ is an odd function of } \lambda,\\ &q_j \text{ is an even function of } \lambda,\end{aligned} \tag{3.9.30}$$

replacing the relations (3.9.14), which apply for $j = 0, 1$ and for $j = \infty$ in the case that $s_\infty \neq t_\infty$.

Remark 3.48. As a consequence of equation (3.3.12), the unitarized monodromy matrices $\hat{M}_j$ satisfy

$$\hat{M}_0(\lambda)\hat{M}_1(\lambda)\hat{M}_\infty(\lambda) = TM_0(\lambda)T^{-1}TM_1(\lambda)T^{-1}TM_\infty(\lambda)T^{-1} = TM_0(\lambda)M_1(\lambda)M_\infty(\lambda)T^{-1} = \mathrm{I}, \tag{3.9.31}$$

i.e.

$$\hat{M}_0(\lambda)\hat{M}_1(\lambda)\hat{M}_\infty(\lambda) = \mathrm{I}. \tag{3.9.32}$$

Rewriting this as $\hat{M}_\infty = \hat{M}_1^{-1}\hat{M}_0^{-1}$, $\hat{M}_1 = \hat{M}_0^{-1}\hat{M}_\infty^{-1}$ or $\hat{M}_0 = \hat{M}_\infty^{-1}\hat{M}_1^{-1}$, respectively, and applying (3.9.26), we obtain the following three pairs of scalar equations, where each pair is equivalent to (3.9.32). The first pair of equations reads

$$\begin{aligned}\cos(2\pi\mu_\infty) + i\sin(2\pi\mu_\infty)p_\infty = &-\cos(2\pi\mu_0)\cos(2\pi\mu_1) + i\cos(2\pi\mu_0)\sin(2\pi\mu_1)p_1\\ &+ i\sin(2\pi\mu_0)\cos(2\pi\mu_1)p_0 + \sin(2\pi\mu_0)\sin(2\pi\mu_1)(p_0p_1 + q_0\overline{q_1}),\end{aligned} \tag{3.9.33}$$

$$\begin{aligned}i\sin(2\pi\mu_\infty)q_\infty = &\, i\cos(2\pi\mu_0)\sin(2\pi\mu_1)q_1 + i\sin(2\pi\mu_0)\cos(2\pi\mu_1)q_0\\ &+ \sin(2\pi\mu_0)\sin(2\pi\mu_1)(p_0q_1 - p_1q_0).\end{aligned} \tag{3.9.34}$$

The second pair of equations reads

$$\begin{aligned}\cos(2\pi\mu_1) + i\sin(2\pi\mu_1)p_1 = &-\cos(2\pi\mu_\infty)\cos(2\pi\mu_0) + i\cos(2\pi\mu_\infty)\sin(2\pi\mu_0)p_0 \\ &+ i\sin(2\pi\mu_\infty)\cos(2\pi\mu_0)p_\infty + \sin(2\pi\mu_\infty)\sin(2\pi\mu_0)(p_\infty p_0 + q_\infty\overline{q_0}),\end{aligned} \tag{3.9.35}$$

$$\begin{aligned}i\sin(2\pi\mu_1)q_1 = &\ i\cos(2\pi\mu_\infty)\sin(2\pi\mu_0)q_0 + i\sin(2\pi\mu_\infty)\cos(2\pi\mu_0)q_\infty \\ &+ \sin(2\pi\mu_\infty)\sin(2\pi\mu_0)(p_\infty q_0 - p_0 q_\infty).\end{aligned} \tag{3.9.36}$$

The third pair of equations reads

$$\begin{aligned}\cos(2\pi\mu_0) + i\sin(2\pi\mu_0)p_0 = &-\cos(2\pi\mu_1)\cos(2\pi\mu_\infty) + i\cos(2\pi\mu_1)\sin(2\pi\mu_\infty)p_\infty \\ &+ i\sin(2\pi\mu_1)\cos(2\pi\mu_\infty)p_1 + \sin(2\pi\mu_1)\sin(2\pi\mu_\infty)(p_1 p_\infty + q_1\overline{q_\infty}),\end{aligned} \tag{3.9.37}$$

$$\begin{aligned}i\sin(2\pi\mu_0)q_0 = &\ i\cos(2\pi\mu_1)\sin(2\pi\mu_\infty)q_\infty + i\sin(2\pi\mu_1)\cos(2\pi\mu_\infty)q_1 \\ &+ \sin(2\pi\mu_1)\sin(2\pi\mu_\infty)(p_1 q_\infty - p_\infty q_1).\end{aligned} \tag{3.9.38}$$

Equations (3.9.26) to (3.9.29) give necessary and sufficient conditions for T to unitarize both M_0 and M_1. More precisely, T will render M_0 and M_1 unitary on S^1 if and only if there exist functions p_0, q_0, p_1, q_1 depending on λ such that the equations (3.9.27), (3.9.28) and (3.9.29) hold. In this case the unitarized monodromy matrices $\hat{M}_j$ are given by (3.9.26). In the following, we will discuss the unitarizing conditions (3.9.28) and (3.9.29) in more detail.

Lemma 3.49. *The unitarizing conditions* (3.9.28) *and* (3.9.29) *hold if and only if*

$$(\Delta_0 S)\sigma_3(\Delta_0 S)^{-1} = (TR_0S)\sigma_3(TR_0S)^{-1}, \tag{3.9.39}$$

$$(\Delta_1 S)\sigma_3(\Delta_1 S)^{-1} = (TAR_1S)\sigma_3(TAR_1S)^{-1}, \tag{3.9.40}$$

where

$$R_j = \begin{pmatrix} \frac{\sqrt{\lambda X_j}}{\sqrt{\mu_j}} & 0 \\ 0 & \frac{\sqrt{\lambda^{-1}\overline{X_j}}}{\sqrt{\mu_j}} \end{pmatrix}, \quad S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\lambda^{-1} \\ \lambda & 1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Delta_j = \frac{1}{\sqrt{\lambda^{-1}q_j}}\begin{pmatrix} 1 & \lambda^{-1}p_j \\ 0 & \lambda^{-1}q_j \end{pmatrix}. \tag{3.9.41}$$

Remark 3.50. In order to ensure that the (λ-dependent) matrices Δ_j are well defined, we have to exclude any values of $\lambda \in \mathbb{C}^*$ from our considerations, for which $q_j(\lambda) = 0$. These values of λ form a discrete subset of $\mathbb{C}^*$, which does not possess any accumulation points on the unit circle S^1. In order to ensure that the matrices Δ_j are holomorphic in λ, we exclude for each $\lambda \in \mathbb{C}^*$ with $q_j(\lambda) = 0$ the radial cut from λ to 0 (if $|\lambda| < 1$) or from λ to ∞ (if $|\lambda| \geq 1$), respectively, from our considerations.

Denoting the union of all these cuts by W_5, we note for later use (cf. remark 3.57) that the subset $\mathbb{C}^* \setminus W_5$ of $\mathbb{C}^*$ by construction still contains for some $r \in (0,1)$ an open annulus containing the r-circle $C^{(r)}$. (More precisely, any $r \in (0,1)$ "close enough" to 1 will do.)

Proof of lemma 3.49. First, recalling (2.6.21), we have

$$\begin{pmatrix} 0 & \frac{X_j}{\mu_j} \\ \frac{\overline{X_j}}{\mu_j} & 0 \end{pmatrix} = (R_jS)\sigma_3(R_jS)^{-1}. \tag{3.9.42}$$

Moreover, a straightforward computation yields

$$\begin{pmatrix} p_j & \overline{q_j} \\ q_j & -p_j \end{pmatrix} = (\Delta_j S)\sigma_3(\Delta_j S)^{-1}. \tag{3.9.43}$$

Inserting these two relations into (3.9.28) and (3.9.29), these equations read as

$$(\Delta_0 S)\sigma_3(\Delta_0 S)^{-1} = (TR_0S)\sigma_3(TR_0S)^{-1}, \tag{3.9.44}$$

$$(\Delta_1 S)\sigma_3(\Delta_1 S)^{-1} = (TAR_1S)\sigma_3(TAR_1S)^{-1}, \tag{3.9.45}$$

which proves the claim. □

Remark 3.51. The decompositon $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = S\sigma_3S^{-1}$ is convenient for computational purposes. Otherwise one would read (3.9.39) and (3.9.40) as equations between conjugates of σ_1. In this case, however, the matrices L_j defined in (3.9.48) below would take a more complicated form.

Lemma 3.52. *Let $A, B, C \in \mathrm{Gl}(2,\mathbb{C})$. Then $BAB^{-1} = CAC^{-1}$ if and only if there exists some $L \in \mathrm{Gl}(2,\mathbb{C})$ satisfying $C = BL$ and $LA = AL$.*

Proof. First, let $A, B, C \in \mathrm{Gl}(2,\mathbb{C})$, such that $BAB^{-1} = CAC^{-1}$. Setting $L := B^{-1}C \in \mathrm{Gl}(2,\mathbb{C})$, we obviously have $C = BL$. Moreover, using the assumption, $LA = B^{-1}CA = AB^{-1}C = AL$, which proves one direction of the claim.

For the other direction, let $A, B, C, L \in \mathrm{Gl}(2,\mathbb{C})$, such that $C = BL$ and $LA = AL$. Then, $BAB^{-1} = CL^{-1}ALC^{-1} = CL^{-1}LAC^{-1} = CAC^{-1}$. $\square$

By the above lemmas 3.49 and 3.52, equations (3.9.39) and (3.9.40) are equivalent to

$$TR_0S = \Delta_0 SL_0, \tag{3.9.46}$$
$$TAR_1S = \Delta_1 SL_1 \tag{3.9.47}$$

for some matrices L_j which commute with σ_3. This implies in particular that the matrices L_j are diagonal. Moreover, since the matrices T, R_j, S, Δ_j and A have determinant 1 (for T, this follows from the relation $\hat{\Psi} = T\Psi$), equations (3.9.46) and (3.9.47) imply $\det(L_j) = 1$ as well. Hence we obtain

$$L_j = \begin{pmatrix} \omega_j & 0 \\ 0 & \omega_j^{-1} \end{pmatrix}, \tag{3.9.48}$$

where $\omega_j = \omega_j(\lambda)$ denotes some λ-dependent function.

With these preparations made, we can prove the following theorem.

Theorem 3.53. *Let M_0 and M_1 be given by (3.9.3) and (3.9.4), respectively. Then, a matrix T unitarizes M_0 and M_1 simultaneously on S^1 if and only if it is of the form*

$$T = \frac{1}{2\sqrt{\mu_0}\sqrt{\lambda^{-1}q_0}} \begin{pmatrix} \sqrt{\lambda^{-1}\overline{X_0}}\left[(\omega_0+\omega_0^{-1}) + p_0(\omega_0-\omega_0^{-1})\right] & \lambda^{-1}\sqrt{\lambda X_0}\left[(\omega_0-\omega_0^{-1}) + p_0(\omega_0+\omega_0^{-1})\right] \\ \sqrt{\lambda^{-1}\overline{X_0}}q_0(\omega_0-\omega_0^{-1}) & \lambda^{-1}\sqrt{\lambda X_0}q_0(\omega_0+\omega_0^{-1}) \end{pmatrix}, \tag{3.9.49}$$

for functions p_0, q_0, p_1, q_1, ω_0 and ω_1 of λ, which satisfy

$$p_j^2 + q_j\overline{q_j} = 1 \quad \textit{and} \quad p_j = \overline{p_j} \quad \textit{for} \quad j = 0,1, \tag{3.9.50}$$
$$p_0p_1 + \frac{q_0\overline{q_1} + \overline{q_0}q_1}{2} = \frac{\cos(2\pi\mu_0)\cos(2\pi\mu_1) + \cos(2\pi\mu_\infty)}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)}, \tag{3.9.51}$$
$$\omega_0 = \delta\frac{\sqrt{\kappa_{12}^{02}}}{\sqrt{\kappa_{12}^{01}}}\frac{\sqrt{-q_0+q_1-p_0q_1+p_1q_0}}{\sqrt{q_0+q_1+p_0q_1-p_1q_0}}, \tag{3.9.52}$$
$$\omega_1 = \tilde{\delta}\frac{\sqrt{\kappa_{11}^{01}}}{\sqrt{\kappa_{12}^{01}}}\frac{\sqrt{-q_0+q_1-p_0q_1+p_1q_0}}{\sqrt{q_0+q_1-p_0q_1+p_1q_0}}, \tag{3.9.53}$$

where $\delta, \tilde{\delta} \in \{\pm 1\}$, such that

$$\delta\tilde{\delta} = \frac{\sqrt{\mu_0}}{\sqrt{\mu_1}}\frac{\sqrt{q_0+q_1+p_0q_1-p_1q_0}\sqrt{q_0+q_1-p_0q_1+p_1q_0}}{-2i\lambda\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}\lambda\sqrt{\kappa_{11}^{01}}\sqrt{\kappa_{12}^{02}}}. \tag{3.9.54}$$

Moreover, if T is of the form (3.9.49), the unitarized monodromy matrices $\hat{M}_j = TM_jT^{-1}$ are given by (3.9.26).

Proof. As explicated earlier, a matrix T unitarizes M_0 and M_1 simultaneously (for $\lambda \in S^1$) if and only if T satisfies the equations (3.9.39) and (3.9.40) for λ-dependent functions p_0, q_0, p_1 and q_1 satisfying $p_j^2 + q_j\overline{q_j} = 1$ and $p_j = \overline{p_j}$. Moreover, presuming T simultaneously unitarizes M_0 and M_1, the unitarized monodromy matrices $\hat{M}_j = TM_jT^{-1}$ are then of the form (3.9.26). Thus, it remains to prove, that T satisfies (3.9.39) and (3.9.40) if and only if it is of the form (3.9.49).

As stated before, (3.9.39) and (3.9.40) are equivalent to the equations (3.9.46) and (3.9.47). We can further transform these equations equivalently into

$$T = \Delta_0 SL_0S^{-1}R_0^{-1}, \tag{3.9.55}$$
$$S^{-1}R_0^{-1}AR_1S = L_0^{-1}S^{-1}\Delta_0^{-1}\Delta_1 SL_1. \tag{3.9.56}$$

Thus, T satisfies (3.9.39) and (3.9.40) if and only if there exist λ-dependent functions p_j, q_j and ω_j, such that (3.9.56) holds. (Of course, we still need to additionally ensure that the conditions given in (3.9.27) are met.) Once we have found functions ω_j, p_j, q_j satisfying (3.9.56) and (3.9.27), we are able to compute T from (3.9.55), that is

$$T = \frac{1}{2\sqrt{\mu_0}\sqrt{\lambda^{-1}q_0}} \begin{pmatrix} \sqrt{\lambda^{-1}\overline{X_0}}\left[(\omega_0+\omega_0^{-1})+p_0(\omega_0-\omega_0^{-1})\right] & \lambda^{-1}\sqrt{\lambda X_0}\left[(\omega_0-\omega_0^{-1})+p_0(\omega_0+\omega_0^{-1})\right] \\ \sqrt{\lambda^{-1}\overline{X_0}}q_0(\omega_0-\omega_0^{-1}) & \lambda^{-1}\sqrt{\lambda X_0}q_0(\omega_0+\omega_0^{-1}) \end{pmatrix}, \tag{3.9.57}$$

as claimed.

To finish the proof, we now focus on equation (3.9.56), which reads more explicitly (cf. (3.8.56) for the matrix A)

$$\begin{aligned} &-i\frac{\sqrt{\mu_1}}{\sqrt{\mu_0}}\begin{pmatrix} \kappa_{11}^{01} & \lambda^{-1}\kappa_{12}^{01} \\ \lambda\kappa_{11}^{02} & \kappa_{12}^{02} \end{pmatrix} \\ &= \frac{1}{2\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}}\begin{pmatrix} \omega_0^{-1}\omega_1\lambda^{-1}(q_0+q_1-p_0q_1+p_1q_0) & \omega_0^{-1}\omega_1^{-1}\lambda^{-1}(-q_0+q_1-p_0q_1+p_1q_0) \\ \omega_0\omega_1(-q_0+q_1+p_0q_1-p_1q_0) & \omega_0\omega_1^{-1}\lambda^{-1}(q_0+q_1+p_0q_1-p_1q_0) \end{pmatrix}. \end{aligned} \tag{3.9.58}$$

Naturally, this matrix equation gives rise to four scalar equations. By taking into account (3.7.19) to (3.7.22) and (3.9.27), these may be equivalently transformed into the following three equations. The computations necessary to carry out this transformation involve the use of some identities for Gamma functions, but are straight forward apart from that and given explicitly in appendix E. We end up with

$$\omega_0 = \delta\frac{\sqrt{\kappa_{12}^{02}}}{\sqrt{\kappa_{12}^{01}}}\frac{\sqrt{-q_0+q_1-p_0q_1+p_1q_0}}{\sqrt{q_0+q_1+p_0q_1-p_1q_0}}, \tag{3.9.59}$$

$$\omega_1 = \tilde{\delta}\frac{\sqrt{\kappa_{11}^{01}}}{\sqrt{\kappa_{12}^{01}}}\frac{\sqrt{-q_0+q_1-p_0q_1+p_1q_0}}{\sqrt{q_0+q_1-p_0q_1+p_1q_0}}, \tag{3.9.60}$$

$$p_0p_1 + \frac{q_0\overline{q_1}+\overline{q_0}q_1}{2} = \frac{\cos(2\pi\mu_0)\cos(2\pi\mu_1)+\cos(2\pi\mu_\infty)}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)}, \tag{3.9.61}$$

where $\delta, \tilde{\delta} \in \{\pm 1\}$, such that

$$\delta\tilde{\delta} = \frac{\sqrt{\mu_0}}{\sqrt{\mu_1}}\frac{\sqrt{q_0+q_1+p_0q_1-p_1q_0}\sqrt{q_0+q_1-p_0q_1+p_1q_0}}{-2i\lambda\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}\sqrt{\kappa_{11}^{01}}\sqrt{\kappa_{12}^{02}}}. \tag{3.9.62}$$

Thus, equation (3.9.56) holds if and only if the involved functions p_j, q_j and ω_j satisfy the three equations above. (Again, note that we additionally assume all the time that the conditions given in (3.9.27) are met.) Altogether, the claim is proved. □

Corollary 3.54. *Let M_0 and M_1 be given by* (3.9.3) *and* (3.9.4)*, respectively. Then, to find a matrix T, which simultaneously unitarizes M_0 and M_1 on S^1, one can proceed as follows:*

1. *Solve* (3.9.51) *for functions p_0, q_0, p_1, q_1 satisfying* (3.9.50).
2. *Compute ω_0 from* (3.9.52).
3. *Compute T from* (3.9.49).

Remark 3.55. We would like to remark that equation (3.9.51) is solvable for functions p_0, q_0, p_1, q_1 in $\lambda \in S^1$ satisfying (3.9.50) if and only if the eigenvalues μ_j of the Delaunay matrices D_j inducing the potential η as explicated in section 3.5 meet the unitarizability condition (3.5.28) for all $\lambda \in S^1$. This is proved in appendix F.

Remark 3.56. Given functions p_0, q_0, p_1, q_1 satisfying (3.9.27) and (3.9.51), the corresponding functions p_∞ and q_∞ (with $p_\infty^2 + q_\infty\overline{q_\infty} = 1$) occurring in the monodromy matrix $\hat{M}_\infty$ can be explicitly computed from equations (3.9.33) and (3.9.34) representing the matrix identity $\hat{M}_0\hat{M}_1\hat{M}_\infty(\lambda) = \mathrm{I}$. By a straightforward computation, which is given in appendix G, we can prove the following statement: For

$j = 0, 1, \infty$, let p_j, q_j be the functions occurring in the unitary monodromy matrix $\hat{M}_j$ as in (3.9.26), satisfying (3.9.27). In view of the identity (3.9.32), we have

$$\begin{aligned} & p_0, q_0, p_1, q_1 \ \text{ solve } \ p_0 p_1 + \frac{q_0\overline{q_1} + \overline{q_0} q_1}{2} = \frac{\cos(2\pi\mu_0)\cos(2\pi\mu_1) + \cos(2\pi\mu_\infty)}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)} \\ & \qquad \text{and } \ p_\infty, q_\infty \ \text{ are given by (3.9.33) and (3.9.34)} \\ \Longleftrightarrow \quad & p_0, q_0, p_\infty, q_\infty \ \text{ solve } \ p_0 p_\infty + \frac{q_0\overline{q_\infty} + \overline{q_0} q_\infty}{2} = \frac{\cos(2\pi\mu_0)\cos(2\pi\mu_\infty) + \cos(2\pi\mu_1)}{\sin(2\pi\mu_0)\sin(2\pi\mu_\infty)} \\ & \qquad \text{and } \ p_1, q_1 \ \text{ are given by (3.9.35) and (3.9.36)} \\ \Longleftrightarrow \quad & p_1, q_1, p_\infty, q_\infty \ \text{ solve } \ p_1 p_\infty + \frac{q_1\overline{q_\infty} + \overline{q_1} q_\infty}{2} = \frac{\cos(2\pi\mu_1)\cos(2\pi\mu_\infty) + \cos(2\pi\mu_0)}{\sin(2\pi\mu_1)\sin(2\pi\mu_\infty)} \\ & \qquad \text{and } \ p_0, q_0 \ \text{ are given by (3.9.37) and (3.9.38).} \end{aligned} \tag{3.9.63}$$

Remark 3.57. Returning to remark 3.43, we now turn to the question, for which values of λ our preceeding considerations are valid. First, recall from remark 3.41, that – for the time being – we have restricted λ to the domain $\mathbb{C}^* \setminus (W_1 \cup W_2 \cup W_3)$, where everything is holomorphic in λ. Recapitulating now what we have done in this section, we observe that, in order to keep all expressions well defined and holomorphic in λ, we need to exclude further points from the λ-domain. Namely, this is necessary when dealing with the functions p_j, q_j and $\overline{q_j}$ occurring in the unitarized monodromy matrices $\hat{M}_j$ (excluding the subsets W_1 and W_4 of $\mathbb{C}^*$ from our considerations, cf. remark 3.45) and when introducing the matrices Δ_j (excluding the subset W_5 of $\mathbb{C}^*$, cf. remark 3.50).

Consequently, our computations are valid for all λ from the λ-domain

$$\mathbb{C}^* \setminus (W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5). \tag{3.9.64}$$

In particular, the matrix T is well defined and holomorphic for $\lambda \in \mathbb{C}^* \setminus (W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5)$.

For our study, it is crucial to observe that the set $\mathbb{C}^* \setminus (W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5)$ contains an r-circle $C^{(r)}$ for some $r \in (0,1)$. More precisely, $\mathbb{C}^* \setminus (W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5)$ contains for each $r \in (0,1)$ "close enough" to 1 an open annulus containing the r-circle $C^{(r)}$. (Cf. remarks 3.41, 3.45 and 3.50 for this.) Altogether, the matrix T is in particular well defined (and holomorphic) for λ from some r-circle $C^{(r)}$ and thus actually defines an r-dressing matrix (as highly desired).

Remark 3.58. By a slight modification of the computations carried out in appendix E, one can prove the following, generalized result, which does not require explicit knowledge of the connection coefficients κ_{11}^{01}, κ_{12}^{01}, κ_{11}^{02} and κ_{12}^{02}:

Let M_0 and M_1 be given by (3.9.3) and (3.9.4), respectively. Then, a matrix T unitarizes M_0 and M_1 simultaneously on S^1 if and only if it is of the form

$$T = \frac{1}{2\sqrt{\mu_0}\sqrt{\lambda^{-1}q_0}} \begin{pmatrix} \sqrt{\lambda^{-1}\overline{X_0}}\left[(\omega_0 + \omega_0^{-1}) + p_0(\omega_0 - \omega_0^{-1})\right] & \lambda^{-1}\sqrt{\lambda X_0}\left[(\omega_0 - \omega_0^{-1}) + p_0(\omega_0 + \omega_0^{-1})\right] \\ \sqrt{\lambda^{-1}\overline{X_0}} q_0(\omega_0 - \omega_0^{-1}) & \lambda^{-1}\sqrt{\lambda X_0} q_0(\omega_0 + \omega_0^{-1}) \end{pmatrix}, \tag{3.9.65}$$

for functions p_0, q_0, p_1, q_1, ω_0 and ω_1 of λ, which satisfy

$$p_j^2 + q_j\overline{q_j} = 1 \ \text{ and } \ p_j = \overline{p_j}, \tag{3.9.66}$$

$$-4q_0q_1\frac{\mu_1}{\mu_0}\kappa_{11}^{01}\kappa_{12}^{02} = (q_0+q_1)^2 - (p_0q_1 - p_1q_0)^2, \tag{3.9.67}$$

$$-4q_0q_1\frac{\mu_1}{\mu_0}\kappa_{11}^{02}\kappa_{12}^{01} = (q_0-q_1)^2 - (p_0q_1 - p_1q_0)^2, \tag{3.9.68}$$

$$\omega_0 = \delta\frac{\sqrt{\kappa_{12}^{02}}}{\sqrt{\kappa_{12}^{01}}}\frac{\sqrt{-q_0+q_1-p_0q_1+p_1q_0}}{\sqrt{q_0+q_1+p_0q_1-p_1q_0}}, \tag{3.9.69}$$

$$\omega_1 = \tilde{\delta}\frac{\sqrt{\kappa_{11}^{01}}}{\sqrt{\kappa_{12}^{01}}}\frac{\sqrt{-q_0+q_1-p_0q_1+p_1q_0}}{\sqrt{q_0+q_1-p_0q_1+p_1q_0}}, \tag{3.9.70}$$

where $\delta, \tilde{\delta} \in \{\pm 1\}$, such that

$$\delta\tilde{\delta} = \frac{\sqrt{\mu_0}}{\sqrt{\mu_1}}\frac{\sqrt{q_0+q_1+p_0q_1-p_1q_0}\sqrt{q_0+q_1-p_0q_1+p_1q_0}}{-2i\lambda\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}\sqrt{\kappa_{11}^{01}}\sqrt{\kappa_{12}^{02}}}. \tag{3.9.71}$$

Moreover, if T is of the form (3.9.65), the unitarized monodromy matrices $\hat{M}_j = TM_jT^{-1}$ are given by (3.9.26).

Inserting the connection coefficients κ_{11}^{01}, κ_{12}^{01}, κ_{11}^{02} and κ_{12}^{02} from (3.7.19) to (3.7.22) restores the statement of theorem 3.53. (Actually, this is proved in lemma E.1.)

We can now state the following result:

Theorem 3.59. *Let $M = \mathbb{C} \setminus \{0, 1\}$, $\tilde{M} = \mathbb{H}$ and $\pi : \tilde{M} \to M$ the universal covering map as defined in* (3.2.2)*. Let η be a standardized trinoid potential of the form* (3.6.16) *on M and $\tilde{\eta} = \pi^*\eta$ the corresponding standardized trinoid potential on $\tilde{M}$. Moreover, let Φ be the solution to the differential equation $\mathrm{d}\Phi = \Phi\eta$ given in* (3.9.1) *and $\Psi = \pi^*\Phi$ the corresponding solution to the differential equation $\mathrm{d}\Psi = \Psi\tilde{\eta}$. Then, to find a dressing matrix $T = T(\lambda)$, which dresses the solution Ψ into a new solution $\hat{\Psi}$, which possesses holomorphic monodromy matrices in $\lambda \in \mathbb{C}^*$ and will produce a descending CMC-immersion $\tilde{M} \to \mathbb{R}^3$ via the loop group method, one can proceed as follows:*

1. *Find λ-dependent functions p_0, q_0, p_1, q_1 satisfying*
   - *(a)* $\sin(2\pi\mu_j)p_j$ *and* $\sin(2\pi\mu_j)q_j$ *are holomorphic for $\lambda \in \mathbb{C}^*$ (holomorphicity condition),*
   - *(b) p_j is an even function of λ, q_j is an odd function of λ (twisting condition),*
   - *(c) $p_j^2 + q_j\overline{q_j} = 1$ (determinant 1 condition),*
   - *(d) $p_j = \overline{p_j}$ (unitarity condition),*
   - *(e) $p_0p_1 + \frac{q_0\overline{q_1}+\overline{q_0}q_1}{2} = \frac{\cos(2\pi\mu_0)\cos(2\pi\mu_1)+\cos(2\pi\mu_\infty)}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)}$ (simultaneous unitarizability condition),*
   - *(f) p_j and q_j take finite values in $\mathbb{C}$ and are holomorphic at $\lambda = 1$ (closing condition) .*
2. *Compute ω_0 from* (3.9.52)*.*
3. *Compute T from* (3.9.49)*.*

Remark 3.60. Theorem 3.59 allows for the construction of a family of dressing matrices $T = T(\lambda)$, which dress a special starting solution Ψ to the differential equation $\mathrm{d}\Psi = \Psi\tilde{\eta}$ into new solutions $\hat{\Psi} = T\Psi$, which will (via the loop group method) produce a *descending* CMC-immersion $\tilde{M} \to \mathbb{R}^3$ and thus a trinoid $\phi : M \to \mathbb{R}^3$ in the sense of definition 3.1. By theorem 3.14, the choice of an appropriate dressing matrix T will ensure that the corresponding trinoid ϕ has properly embedded annular ends. In view of remark 3.18, *all* trinoids with properly embedded annular ends can be obtained (by different choices of T). Nevertheless, we conjecture that the given family of dressing matrices is even "larger" in the sense that, for appropriate choices of T, even trinoids $M \to \mathbb{R}^3$ with "non-properly embedded" or "non-embedded" ends can be produced. It is not clear how the corresponding surfaces in $\mathbb{R}^3$ would look like. In particular, such trinoids would not necessarily show the asymptotic behaviour of Delaunay surfaces. For the rest of this thesis, we restrict our considerations to trinoids with properly embedded annular ends.

Remark 3.61. Each dressing matrix $T = T(\lambda)$ provided by theorem 3.59 produces a solution $\hat{\Psi} = T\Psi$ to the differential equation

$$\mathrm{d}\Psi = \Psi\tilde{\eta} \tag{3.9.72}$$

with monodromy matrices $\hat{M}_j$, which are holomorphic in $\lambda \in \mathbb{C}^*$. However, $\hat{\Psi}$ itself is not necessarily holomorphic (in λ) on $\mathbb{C}^*$. Nevertheless, based on remarks 3.21 and 3.39 of [18], we have the following result:

Assume T is as in theorem 3.59. Denote by $\hat{\Psi} = T\Psi$ the corresponding solution to (3.9.72), which has holomorphic monodromy matrices in $\lambda \in \mathbb{C}^*$ and generates via the loop group method a descending CMC-immersion $\tilde{M} \to \mathbb{R}^3$. Assume $\tilde{T}$ is another dressing matrix provided by theorem 3.59, which produces a solution $\tilde{\Psi} = \tilde{T}\Psi$ to (3.9.72), which has holomorphic monodromy matrices in $\lambda \in \mathbb{C}^*$, generates via the loop group method a descending CMC-immersion $\tilde{M} \to \mathbb{R}^3$ *and* is holomorphic in $\lambda \in \mathbb{C}^*$. In particular, both T and $\tilde{T}$ simultaneously unitarize the monodromy matrices of the starting solution Ψ. Thus, by remarks 3.21 and 3.39 of [18], T and $\tilde{T}$ only differ by an element $U = U(\lambda)$ of $\Lambda_r\mathrm{SU}(2)_\sigma$ (for an appropriate $r \in (0, 1)$): $\tilde{T} = UT$.

Altogether, any dressed solution $\hat{\Psi} = T\Psi$ to (3.9.72) obtained by theorem 3.59 is related to another solution $\tilde{\Psi} = \tilde{T}\Psi$ to (3.9.72), which is holomorphic in $\lambda \in \mathbb{C}^*$ by some $U \in \Lambda_r\mathrm{SU}(2)_\sigma$: $\tilde{\Psi} = U\hat{\Psi}$. (Both $\hat{\Psi}$ and $\tilde{\Psi}$ have holomorphic monodromy matrices in λ on $\mathbb{C}^*$ and produce via the loop group method descending CMC-immersions $\tilde{M} \to \mathbb{R}^3$.) The additional dressing by U transforming $\hat{\Psi}$ into $\tilde{\Psi}$ corresponds on the level of the associated CMC-immersions to a λ-dependent rotation and/or translation of the members of the related associated families.

4 Trinoid symmetries

So far, we have classified trinoids $\phi : M \to \mathbb{R}^3$ produced by the (standardized) trinoid potential η in terms of functions p_0, p_1, q_0, q_1 solving (3.9.51). In the following, we are going to refine this classification by translating possible symmetry properties of the image of the trinoid $\phi(M)$ into further constraints on the functions p_0, p_1, q_0, q_1. From now on, we restrict our considerations to trinoids with properly embedded annular ends.

As p_0, p_1, q_0, q_1 appear in the monodromy matrices of the holomorphic frame Ψ associated with the mapping $\psi := \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$ (and thus, by remark 2.10, in the monodromy matrices of the extended frame F associated with ψ), we translate any symmetry properties of ϕ into symmetry properties of ψ and transfer these to the level of the extended frame $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ afterwards.

Remark 4.1. Both the definitions and the results of the sections 4.1 and 4.2 are valid and thus formulated in the generalized setting of an arbitrary Riemann surface M with universal cover $\tilde{M}$ and a pair of conformal CMC-immersions $\phi : M \to \mathbb{R}^3$ and $\psi : \tilde{M} \to \mathbb{R}^3$ linked via the universal covering $\pi : \tilde{M} \to M$, $\psi = \phi \circ \pi$. We return to the trinoid setting with $M = \mathbb{C} \setminus \{0, 1\}$ and $\tilde{M} = \mathbb{H}$ in section 4.3.

4.1 Definitions

Let $\mathrm{Iso}(\mathbb{R}^3)$ denote the *isometry group* of $\mathbb{R}^3$, i.e. the group of all distance preserving affine isomorphisms on $\mathbb{R}^3$ with respect to the Euclidean metric

$$d : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}, \quad d(x, y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_2 - x_2)^2}. \tag{4.1.1}$$

The elements of $\mathrm{Iso}(\mathbb{R}^3)$, often referred to as *Euclidean motions* on $\mathbb{R}^3$, are of the form

$$\mathcal{T} : \mathbb{R}^3 \to \mathbb{R}^3, \quad x \mapsto \mathcal{T}(x) := \mathcal{A}_\mathcal{T} x + t_\mathcal{T}, \tag{4.1.2}$$

where $\mathcal{A}_\mathcal{T}$ denotes a (real) orthogonal 3×3-matrix, and $t_\mathcal{T}$ denotes a translation vector in $\mathbb{R}^3$. Thus, any element of $\mathrm{Iso}(\mathbb{R}^3)$ is composed of an orthogonal transformation and a translation. Moreover, $\mathcal{T} \in \mathrm{Iso}(\mathbb{R}^3)$ of the form (4.1.2) preserves orientation on $\mathbb{R}^3$ if and only if the associated matrix $\mathcal{A}_\mathcal{T}$ satisfies $\det(\mathcal{A}_\mathcal{T}) = +1$, i.e. if and only if $\mathcal{A}_\mathcal{T} \in \mathrm{SO}(3)$. Accordingly, $\mathcal{T} \in \mathrm{Iso}(\mathbb{R}^3)$ of the form (4.1.2) reverses orientation on $\mathbb{R}^3$ if and only if the associated matrix $\mathcal{A}_\mathcal{T}$ satisfies $\det(\mathcal{A}_\mathcal{T}) = -1$, i.e. if and only if $\mathcal{A}_\mathcal{T} \in \mathrm{O}(3) \setminus \mathrm{SO}(3)$.

Given a conformal CMC-immersion $\phi : M \to \mathbb{R}^3$ of a Riemann surface M into $\mathbb{R}^3$, we define the *symmetry group* of $\phi(M)$ by

$$\mathrm{Sym}(\phi(M)) := \{\mathcal{T} \in \mathrm{Iso}(\mathbb{R}^3) \mid \mathcal{T}(\phi(M)) = \phi(M)\}. \tag{4.1.3}$$

For the corresponding conformal CMC-immersion $\psi := \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$, we define analogously

$$\mathrm{Sym}(\psi(\tilde{M})) := \{\mathcal{T} \in \mathrm{Iso}(\mathbb{R}^3) \mid \mathcal{T}(\psi(\tilde{M})) = \psi(\tilde{M})\}. \tag{4.1.4}$$

By construction, we have $\psi(\tilde{M}) = \phi(M)$ and therefore obviously $\mathrm{Sym}(\phi(M)) = \mathrm{Sym}(\psi(\tilde{M}))$.

Definition 4.2. Let $\phi : M \to \mathbb{R}^3$ be a conformal CMC-immersion of a Riemann surface M into $\mathbb{R}^3$. Let $\pi : \tilde{M} \to M$ be the universal covering of M and $\psi := \phi \circ \pi$. The mapping ϕ (or ψ) is called *symmetric* with respect to some Euclidean motion $\mathcal{T} \in \mathrm{Iso}(\mathbb{R}^3)$ if and only if $\mathcal{T} \in \mathrm{Sym}(\phi(M)) = \mathrm{Sym}(\psi(\tilde{M}))$, i.e. if and only if

$$\mathcal{T}(\phi(M)) = \phi(M) \;\text{ or, equivalently, }\; \mathcal{T}(\psi(\tilde{M})) = \psi(\tilde{M}). \tag{4.1.5}$$

In this case, $\mathcal{T}$ is called a *symmetry* of ϕ (or ψ).

4.2 The extended frame

Throughout this section, let ϕ be a conformal CMC-immersion $M \to \mathbb{R}^3$ of a Riemann surface M into $\mathbb{R}^3$, and $\psi := \phi \circ \pi$ the corresponding conformal CMC-immersion $\tilde{M} \to \mathbb{R}^3$, where $\tilde{M}$ denotes the universal cover of M and $\pi : \tilde{M} \to M$ the associated covering map.

In the following, we review the procedure of constructing the corresponding extended frame F from ψ as presented in [10], slightly modifying it at the same time to make it fit our needs. As we will only give an outline of the basic procedure, the reader is referred to the appendix of [10] for more details.

Since the CMC-immersion $\psi : \tilde{M} \to \mathbb{R}^3$ is conformal, the metric on $\psi(\tilde{M})$ induced by ψ is given by $\mathrm{d}s^2 = e^u(\mathrm{d}x^2 + \mathrm{d}y^2)$ for some real valued function $u : \tilde{M} \to \mathbb{R}$. More explicitly, we have

$$< \psi_x(x,y), \psi_x(x,y) > = < \psi_y(x,y), \psi_y(x,y) > = e^{u(x,y)}, \tag{4.2.1}$$
$$< \psi_x(x,y), \psi_y(x,y) > = 0, \tag{4.2.2}$$

where $< \cdot, \cdot >$ denotes the standard inner product of $\mathbb{R}^3$ and we interpret ψ as a mapping of two real variables x and y, writing $z \in \tilde{M}$ as $z = x + iy$.[9] Defining

$$N(x,y) = \frac{\psi_x(x,y) \times \psi_y(x,y)}{|\ \psi_x(x,y) \times \psi_y(x,y)\ |}, \tag{4.2.3}$$

we obtain an orthogonal matrix (depending on $z = x + iy \in \tilde{M}$)

$$\mathcal{U} = (e^{-\frac{u(x,y)}{2}}\psi_x(x,y), e^{-\frac{u(x,y)}{2}}\psi_y(x,y), N(x,y)) \in \mathrm{SO}(3) \tag{4.2.4}$$

representing the natural orthonormal frame corresponding to ψ. By possibly rotating the surface $\psi(\tilde{M})$ and shifting it afterwards, we can assume $\mathcal{U}(z_*) = \mathcal{G}(1)$ and $\psi(z_*) = \frac{1}{2H}e_3$ for any preassigned $z_* \in \tilde{M} = \mathbb{H}$, where $e_3 = (0,0,1) \in \mathbb{R}^3$ and

$$\mathcal{G}(\lambda) = \mathcal{G}(\lambda)^{-1} = \begin{pmatrix} \cos(t) & \sin(t) & 0 \\ \sin(t) & -\cos(t) & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathrm{SO}(3) \quad \text{for all } \lambda = e^{it} \in S^1. \tag{4.2.5}$$

Remark 4.3. The normalization of ψ given above is different from the one of [10], where $\mathcal{U}(z_*) = \mathrm{I}$. The reasons for our normalization will become apparent by the following: Instead of translating the moving frame $\mathcal{U}$ of ψ directly into an extended frame $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ by the procedure given in [10], we will consider the moving frame $\tilde{\mathcal{U}}$ of $\tilde{\psi} := \mathcal{G}(1)\psi$ which satisfies $\tilde{\mathcal{U}}(z_*) = \mathrm{I}$. By applying the method of [10], we obtain $\tilde{F}$ which exactly produces the original immersion ψ via the Sym-Bobenko formula (rather than some rotated and translated version of ψ as in [10]).

For ψ as above and $\mathcal{A} \in \mathrm{O}(3)$, consider the (also conformal) CMC-immersion $\tilde{\psi} := \mathcal{A}\psi : \tilde{M} \to \mathbb{R}^3$. The corresponding orthonormal frame of $\tilde{\psi}$ is represented by

$$\tilde{\mathcal{U}} = (e^{-\frac{\tilde{u}(x,y)}{2}}\tilde{\psi}_x(x,y), e^{-\frac{\tilde{u}(x,y)}{2}}\tilde{\psi}_y(x,y), \tilde{N}(x,y)) \in \mathrm{SO}(3), \tag{4.2.6}$$

where

$$\tilde{N}(x,y) = \frac{\tilde{\psi}_x(x,y) \times \tilde{\psi}_y(x,y)}{|\ \tilde{\psi}_x(x,y) \times \tilde{\psi}_y(x,y)\ |}. \tag{4.2.7}$$

The relation between $\tilde{\mathcal{U}}$ and $\mathcal{U}$ is stated in the following lemma.

Lemma 4.4. *Let $\psi : \tilde{M} \to \mathbb{R}^3$ be a conformal CMC-immersion, $\mathcal{A} \in \mathrm{O}(3)$ and $\tilde{\psi} := \mathcal{A}\psi$. Then the corresponding orthogonal matrices $\mathcal{U}$ and $\tilde{\mathcal{U}}$ given in* (4.2.4) *and* (4.2.6)*, respectively, satisfy*

$$\tilde{\mathcal{U}} = \mathcal{A}\mathcal{U} \qquad \text{if } \mathcal{A} \in \mathrm{SO}(3), \tag{4.2.8}$$
$$\tilde{\mathcal{U}} = \mathcal{A}\mathcal{U}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \text{if } \mathcal{A} \in \mathrm{O}(3) \setminus \mathrm{SO}(3). \tag{4.2.9}$$

Proof. As indicated above, $\tilde{\psi}$ is also a conformal CMC-immersion. More precisely, we have

$$\tilde{\psi}_x = \mathcal{A}\psi_x, \quad \tilde{\psi}_y = \mathcal{A}\psi_y \tag{4.2.10}$$

and thus $e^{\tilde{u}} = \langle \tilde{\psi}_x, \tilde{\psi}_x \rangle = \langle \psi_x, \psi_x \rangle = e^u$, which implies

$$\tilde{u} = u. \tag{4.2.11}$$

Because of $\mathcal{A} \in \mathrm{O}(3)$, we have for any two vectors $v, w \in \mathbb{R}^3$ the relation $(\mathcal{A}v) \times (\mathcal{A}w) = \det(\mathcal{A})\mathcal{A}(v \times w)$ and therefore

$$\tilde{N} = \frac{(\mathcal{A}\psi_x) \times (\mathcal{A}\psi_y)}{|\ (\mathcal{A}\psi_x) \times (\mathcal{A}\psi_y)\ |} = \frac{\det(\mathcal{A})\mathcal{A}(\psi_x \times \psi_y)}{|\ \det(\mathcal{A})\mathcal{A}(\psi_x \times \psi_y)\ |} = \det(\mathcal{A})\mathcal{A}\frac{\psi_x \times \psi_y}{|\ \psi_x \times \psi_y\ |} = \det(\mathcal{A})\mathcal{A}N. \tag{4.2.12}$$

Altogether, the claim follows. ∎

[9] It is well known that the universal cover $\tilde{M}$ of a Riemann surface M is (up to conformal transformations) given by either $\tilde{M} = \hat{\mathbb{C}}$ or $\tilde{M} = \mathbb{C}$ or $\tilde{M} = \mathbb{H}$. In the case under consideration, we can interpret $\tilde{M}$ as a subset of $\hat{\mathbb{C}}$.

From now on, let $\mathcal{A} := \mathcal{G}(1) \in \mathrm{SO}(3)$, where $\mathcal{G}(1)$ is defined by (4.2.5). Hence, we consider from now on the conformal CMC-immersion

$$\tilde{\psi} = \mathcal{G}(1)\psi : \tilde{M} \to \mathbb{R}^3. \tag{4.2.13}$$

For later use we note $\tilde{\psi}(z_*) = -\frac{1}{2H}e_3$. Considering the corresponding orthonormal frame $\tilde{U}$ as given in (4.2.6), we derive by the above lemma that $\tilde{\mathcal{U}} = \mathcal{G}(1)\mathcal{U}$ and thus $\tilde{\mathcal{U}}(z_*) = \mathrm{I}$. For now, we consider $\tilde{\mathcal{U}}$ instead of $\mathcal{U}$ and therefore find ourselves back in the setting of [10].

Interpreting $\tilde{\mathcal{U}}$ as an automorphism of $\mathbb{R}^3$, the mapping $J \circ \tilde{\mathcal{U}} \circ J^{-1}$, where $J : \mathbb{R}^3 \to \mathrm{su}(2)$ as in (3.4.3), defines an automorphism of su(2). Furthermore (cf. equation (3.4.7)), there exists a (up to sign) unique $\tilde{P} \in \mathrm{SU}(2)$ such that

$$(J \circ \tilde{\mathcal{U}} \circ J^{-1})(X) = \tilde{P}X\tilde{P}^{-1} \quad \text{for all } X \in \mathrm{su}(2). \tag{4.2.14}$$

Additionally, as $\tilde{\mathcal{U}}(z_*) = \mathrm{I}$, we have $\tilde{P}(z_*) = \pm\mathrm{I}$. W.l.o.g., we assume $\tilde{P}(z_*) = +\mathrm{I}$, thus defining $\tilde{P}$ uniquely.

Following A.4 of [10], $\tilde{P}$ satisfies

$$\tilde{P}^{-1}\tilde{P}_z = \begin{pmatrix} -\frac{1}{4}u_z & Qe^{-\frac{u}{2}} \\ -\frac{1}{2}e^{\frac{u}{2}}H & \frac{1}{4}u_z \end{pmatrix}, \quad \tilde{P}^{-1}\tilde{P}_{\bar{z}} = \begin{pmatrix} \frac{1}{4}u_{\bar{z}} & \frac{1}{2}e^{\frac{u}{2}}H \\ -\bar{Q}e^{-\frac{u}{2}} & -\frac{1}{4}u_{\bar{z}} \end{pmatrix}. \tag{4.2.15}$$

Introducing the loop parameter $\lambda \in S^1$, we define the mapping $\tilde{P}_\lambda : \tilde{M} \to \Lambda\mathrm{SU}(2)$ into the (untwisted) loop group $\Lambda\mathrm{SU}(2) = \{g : S^1 \to \mathrm{SU}(2) \text{ smooth}\}$ by

$$\tilde{P}_\lambda^{-1}(\tilde{P}_\lambda)_z = \begin{pmatrix} -\frac{1}{4}u_z & \lambda^{-2}Qe^{-\frac{u}{2}} \\ -\frac{1}{2}e^{\frac{u}{2}}H & \frac{1}{4}u_z \end{pmatrix}, \quad \tilde{P}_\lambda^{-1}(\tilde{P}_\lambda)_{\bar{z}} = \begin{pmatrix} \frac{1}{4}u_{\bar{z}} & \frac{1}{2}e^{\frac{u}{2}}H \\ -\lambda^2\bar{Q}e^{-\frac{u}{2}} & -\frac{1}{4}u_{\bar{z}} \end{pmatrix} \tag{4.2.16}$$

and $\tilde{P}_\lambda(z_*) = \mathrm{I}$. After conjugating $\tilde{P}_\lambda$ by $G(\lambda)^{-1}$, where

$$G(\lambda) = \begin{pmatrix} 0 & i\lambda^{-\frac{1}{2}} \\ i\lambda^{\frac{1}{2}} & 0 \end{pmatrix}, \tag{4.2.17}$$

we finally obtain

$$\tilde{F}_\lambda = G(\lambda)^{-1}\tilde{P}_\lambda G(\lambda) : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma \tag{4.2.18}$$

with $\tilde{F}_\lambda(z_*) = \mathrm{I}$.

By section A.7 of [10], the Sym-Bobenko formula (evaluated at $\lambda = 1$) allows to recover $\tilde{\psi}$ from $\tilde{F}_\lambda$ up to an Euclidian motion:

$$[-\frac{1}{2H}(\frac{\partial}{\partial\theta}\tilde{F}_\lambda \cdot \tilde{F}_\lambda^{-1} + \frac{i}{2}\tilde{F}_\lambda\sigma_3\tilde{F}_\lambda^{-1})]|_{\lambda=1} = G(1)^{-1}J(\tilde{\psi})G(1) + C(1), \tag{4.2.19}$$

where $\lambda = e^{i\theta}$ and $C = C(\lambda)$ denotes a z-independent translation matrix in su(2).

Keeping in mind that $\tilde{F}_\lambda(z_*) = \mathrm{I}$ for all $\lambda \in S^1$, by evaluating the left hand side of (4.2.19) at $z = z_*$ we obtain $-\frac{1}{2H}\frac{i}{2}\sigma_3$. Since $\tilde{\psi}(z_*) = -\frac{1}{2H}e_3$ as stated earlier and therefore $J(\tilde{\psi}(z_*)) = \frac{1}{2H}\frac{i}{2}\sigma_3$, the right hand side of (4.2.19) for $z = z_*$ reads as $G(1)^{-1}J(\tilde{\psi}(z_*))G(1) + C(1) = -\frac{1}{2H}\frac{i}{2}\sigma_3 + C(1)$. Comparing both expressions for the left and for the right hand side, we obtain $C(1) = 0$. Therefore, the extended frame $\tilde{F}$ produces exactly the (su(2) version of) the original immersion ψ via the Sym-Bobenko formula evaluated at $\lambda = 1$:

$$[-\frac{1}{2H}(\frac{\partial}{\partial\theta}\tilde{F}_\lambda \cdot \tilde{F}_\lambda^{-1} + \frac{i}{2}\tilde{F}_\lambda\sigma_3\tilde{F}_\lambda^{-1})]|_{\lambda=1} = G(1)^{-1}J(\tilde{\psi})G(1) = J(\psi). \tag{4.2.20}$$

To understand the last step in the equation above, note that we have

$$(J \circ \mathcal{G}(\lambda) \circ J^{-1})(X) = G(\lambda)XG(\lambda)^{-1} \text{ for all } X \in \mathrm{su}(2) \tag{4.2.21}$$

and thus $J(\tilde{\psi}) = (J \circ \mathcal{G}(1))(\psi) = G(1)J(\psi)G(1)^{-1}$.

From now on, by abuse of the notation of [10], we denote the frame $\tilde{F}_\lambda$ as constructed above by F to match our notation of the previous sections. Altogether, starting with a (normalized) CMC-immersion ψ, we have reversed the last step of the DPW-method by recovering the extended frame F which exactly produces ψ by evaluating the Sym-Bobenko formula at $\lambda = 1$. This result is summarized in the following theorem.

Theorem 4.5. *Let $\psi : \tilde{M} \to \mathbb{R}^3$ be a conformal CMC-immersion and $\mathcal{U}$, $\mathcal{G}(1)$ defined by (4.2.4), (4.2.5), respectively. Moreover, let $\mathcal{U}(z_*) = \mathcal{G}(1)$ and $\psi(z_*) = \frac{1}{2H}e_3$ at some $z_* \in \tilde{M}$. Then, the natural orthonormal moving frame of $\mathcal{G}(1)\psi$ translates by the procedure presented in [10] into an extended frame $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ satisfying*

$$[-\frac{1}{2H}(\frac{\partial}{\partial\theta}F \cdot F^{-1} + \frac{i}{2}F\sigma_3F^{-1})]|_{\lambda=1} = J(\psi). \tag{4.2.22}$$

4.3 Trinoids with properly embedded annular ends

We return to the trinoid setting. I.e., let $M = \mathbb{C} \setminus \{0,1\}$, $\tilde{M} = \mathbb{H}$ and $\pi : \tilde{M} \to M$ be the universal covering as defined in (3.2.2). From now on, we will restrict our considerations to trinoids $M \to \mathbb{R}^3$ with properly embedded annular ends.

Hence, from now on, let $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends B_j, $j = 0, 1, \infty$, at $z_j = j \in \hat{\mathbb{C}}$,$j = 0, 1, \infty$, respectively (cf. 3.1). Moreover, assume ϕ is derived via the loop group method from a standardized trinoid potential η (on M) of the form (3.6.16). Recall that η is determined by a triple of Delaunay matrices D_0, D_1 and D_∞ of the form

$$D_j = \begin{pmatrix} 0 & X_j \\ \overline{X_j} & 0 \end{pmatrix}, \tag{4.3.1}$$

where

$$X_j = s_j\lambda^{-1} + t_j\lambda, \;\; \overline{X_j} = s_j\lambda + t_j\lambda^{-1}, \tag{4.3.2}$$

$$s_j \in [\frac{1}{4}, \frac{1}{2}), \;\; s_j + t_j = \frac{1}{2}. \tag{4.3.3}$$

As stated earlier, for each $j \in \{0, 1, \infty\}$, the properly embedded annular end B_j of ϕ asymptotically shows the behaviour of the unduloidal Delaunay surface produced from the Delaunay potential $\frac{1}{z-z_j}D_j\mathrm{d}z$. We define the *trinoid axis of ϕ (at z_j)* as the axis of revolution of the asymptotic Delaunay surface of the properly embedded annular end B_j. The trinoid axis of ϕ (at z_j) is denoted by $A_j = C_j + \mathbb{R}v_j$, involving a base point $C_j \in \mathbb{R}^3$ and a unit direction vector $v_j \in \mathbb{R}^3$, pointing towards the trinoid end B_j. It is well known that the direction vectors of the trinoid axes are subject to the *balancing formula*[10]

$$w_0v_0 + w_1v_1 + w_\infty v_\infty = 0, \tag{4.3.4}$$

where, as before, $w_j = s_jt_j$.

We note the following result:

Lemma 4.6. *Let $M = \mathbb{C} \setminus \{0, 1\}$ and, for $j = 0, 1, \infty$, $z_j = j \in \hat{\mathbb{C}}$. Moreover, let $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends $B_j = \phi(U_j)$ at z_j for $j = 0, 1, \infty$. Then, for some $j \in \{0, 1, \infty\}$, there exists an open subset $\hat{U}_j \subseteq U_j$ in M, such that $\hat{B}_j := \phi(\hat{U}_j)$ is a properly embedded annular end of ϕ at z_j which satifies*

$$\hat{B}_j \cap \phi(M \setminus \hat{U}_j) = \emptyset. \tag{4.3.5}$$

Proof. We assume without loss of generality that the punctured neighborhoods U_j of z_j, $j = 0, 1, \infty$, are open in M and pairwise disjoint, i.e.

$$U_0 \cap U_1 = U_0 \cap U_\infty = U_1 \cap U_\infty = \emptyset. \tag{4.3.6}$$

In addition, we assume without loss of generality that, for each $j \in \{0, 1, \infty\}$, the set $U_j \cup \{z_j\}$ is simply connected in $\hat{\mathbb{C}}$.

As before, denote for each $j \in \{0, 1, \infty\}$ by v_j the unit direction vector of the trinoid axis corresponding to the trinoid end B_j at z_j (and pointing towards the end). The vectors v_0, v_1 and v_∞ are subject to the balancing formula (4.3.4),

$$w_0v_0 + w_1v_1 + w_\infty v_\infty = 0, \tag{4.3.7}$$

[10]By chapter 7 of [29], the direction vectors v_0, v_1 and v_∞ of the trinoid axes satisfy the equation

$$m_0v_0 + m_1v_1 + m_\infty v_\infty = 0,$$

for some real constants m_j called the *trinoid weights*, which are associated with the corresponding asymptotic Delaunay surfaces of the trinoid ends B_j, respectively. By section 5.4 of [19], these weights are up to a common factor κ identical with the parameters $w_j = s_jt_j$ of the respective Delaunay surfaces: $m_j = \kappa w_j$. Consequently, the formula above can be reformulated as (4.3.4).

where $w_j > 0$. This implies that there exists a plane in $\mathbb{R}^3$, which separates one of the v_j's, say v_r, from the other two, say v_k and v_l. Consequently, after shrinking the ends if necessary, we have

$$B_r \cap B_k = \emptyset = B_r \cap B_l. \tag{4.3.8}$$

Given the compact subset

$$\hat{M} = M \setminus (U_0 \cup U_1 \cup U_\infty) \tag{4.3.9}$$

of M, we consider the open set

$$V_r = \phi^{-1}(B_r \setminus \phi(\hat{M})) \subseteq M. \tag{4.3.10}$$

We will prove that

$$V_r \subseteq U_r. \tag{4.3.11}$$

To this end, let $z \in V_r$. Assume $z \notin U_r$. In view of (4.3.9), we have $z \in \hat{M}$ or $z \in U_k$ or $z \in U_l$. Consider the first case, $z \in \hat{M}$. Then, $\phi(z) \in \phi(\hat{M})$ and thus $\phi(z) \notin \phi(V_r)$, which implies by (4.3.10) that $z \notin V_r$, a contradiction. We turn to the second case, $z \in U_k$. Then, $\phi(z) \in \phi(U_k) = B_k$ and thus by (4.3.8) $\phi(z) \notin B_r$. This yields again $\phi(z) \notin \phi(V_r)$ and therefore $z \notin V_r$, a contradiction. Analogous to the second case, also the third case, $z \in U_l$, is led to a contradiction. Altogether, we infer that necessarily $z \in U_r$, which proves (4.3.11).

Since $\hat{M}$ is compact (in M), $\phi(\hat{M})$ is compact and in particular bounded (in $\mathbb{R}^3$). Consequently, since $\lim_{z\to z_r} \phi(z) = \infty$ by assumption, V_r contains a punctured neighborhood $\hat{U}_r$ of z_r. Without loss of generality, we can assume that $\hat{U}_r$ is open in M and that the set $\hat{U}_r \cup \{z_r\}$ is simply connected in $\hat{\mathbb{C}}$. Set

$$\hat{B}_r = \phi(\hat{U}_r). \tag{4.3.12}$$

As $\hat{U}_r \subseteq U_r$ and, by assumption, ϕ defines a proper embedding of U_r, also $\phi|_{\hat{U}_r}$ is a proper embedding. We infer that $\hat{B}_r$ defines a properly embedded annular end of ϕ at z_r.

It remains to show

$$\hat{B}_r \cap \phi(M \setminus \hat{U}_r) = \emptyset. \tag{4.3.13}$$

For a start we observe by (4.3.6) and (4.3.9) that

$$M \setminus \hat{U}_r = \hat{M} \cup U_k \cup U_l \cup (U_r \setminus \hat{U}_r) \tag{4.3.14}$$

and thus

$$\phi(M \setminus \hat{U}_r) = \phi(\hat{M}) \cup B_k \cup B_l \cup \phi(U_r \setminus \hat{U}_r). \tag{4.3.15}$$

As $\hat{B}_r = \phi(\hat{U}_r) \subseteq \phi(V_r) = B_r \setminus \phi(\hat{M})$, we have $\hat{B}_r \cap \phi(\hat{M}) = \emptyset$. Moreover, $\hat{B}_r = \phi(\hat{U}_r) \subseteq \phi(U_r) = B_r$ and thus, by (4.3.8), $\hat{B}_r \cap B_k = \emptyset = \hat{B}_r \cap B_l$. Finally, since $\phi|_{U_r}$ is an embedding, we have $\hat{B}_r \cap \phi(U_r \setminus \hat{U}_r) = \emptyset$. Altogether, we obtain

$$\hat{B}_r \cap \phi(M \setminus \hat{U}_r) = (\hat{B}_r \cap \phi(\hat{M})) \cup (\hat{B}_r \cap B_k) \cup (\hat{B}_r \cap B_l) \cup (\hat{B}_r \cap \phi(U_r \setminus \hat{U}_r)) = \emptyset, \tag{4.3.16}$$

as claimed. □

4.4 The extended frame symmetry transformations

We are in the trinoid setting presented in section 4.3. I.e., let $M = \mathbb{C} \setminus \{0, 1\}$, $\tilde{M} = \mathbb{H}$ and $\pi : \tilde{M} \to M$ be the universal covering as defined in (3.2.2). Moreover, let $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and $\psi = \phi \circ \pi$ the corresponding conformal CMC-immersion $\tilde{M} \to \mathbb{R}^3$. Finally, let F denote the extended frame corresponding to ψ in the sense of theorem 4.5.

We are interested in translating any symmetry properties of ϕ to the level of the extended frame F. Thus, we assume that ϕ is symmetric with respect to $\mathcal{T} \in \mathrm{Sym}(\phi(M))$, i.e.

$$\mathcal{T}(\phi(M)) = \phi(M). \tag{4.4.1}$$

Recalling from section 4.1 that $\mathrm{Sym}(\phi(M)) = \mathrm{Sym}(\psi(\tilde{M}))$, we infer that $\mathcal{T}$ is also a symmetry of the conformal CMC-immersion ψ, i.e.

$$\mathcal{T}(\psi(M)) = \psi(M). \tag{4.4.2}$$

In order to translate the symmetry property of ψ to the level of the extended frame F, we will trace the symmetry property of ψ through the process generating F from ψ, which has been presented in section 4.2. Theorem 4.9, based on results of [12], associates with the given symmetry $\mathcal{T}$ of ψ a pair of bijections, $\gamma : M \to M$ and $\tilde{\gamma} : \tilde{M} \to \tilde{M}$, and is crucial for our purposes. We make the necessary preparations for this theorem in the following.

Let U and V denote open subsets of $\mathbb{C}$. A function $f : U \to V$ is called *biholomorphic*, if f is a holomorphic bijection with holomorphic inverse function. f is called *bi-antiholomorphic*, if f is an antiholomorphic bijection with antiholomorphic inverse function. We define the following sets:

$$\mathrm{Aut}(\tilde{M}) = \{\tilde{\gamma} : \tilde{M} \to \tilde{M}; \tilde{\gamma} \text{ biholomorphic}\}, \tag{4.4.3}$$

$$\mathrm{Aut}^*(\tilde{M}) = \{\tilde{\gamma} : \tilde{M} \to \tilde{M}; \tilde{\gamma} \text{ bi-antiholomorphic}\}, \tag{4.4.4}$$

$$\mathrm{Aut}(M) = \{\gamma : M \to M; \gamma \text{ biholomorphic}\}, \tag{4.4.5}$$

$$\mathrm{Aut}^*(M) = \{\gamma : M \to M; \gamma \text{ bi-antiholomorphic}\}. \tag{4.4.6}$$

Lemma 4.7. *Let $M = \mathbb{C} \setminus \{0, 1\}$, $\tilde{M} = \mathbb{H}$ and $\pi : \tilde{M} \to M$ be the universal covering as defined in* (3.2.2)*. Moreover, let $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and $\psi = \phi \circ \pi$ the corresponding conformal CMC-immersion $\tilde{M} \to \mathbb{R}^3$. Then, we have*

$$\{\tilde{\gamma} \in \mathrm{Aut}(\tilde{M}); \pi \circ \tilde{\gamma} = \pi\} = \{\tilde{\gamma} \in \mathrm{Aut}(\tilde{M}); \psi \circ \tilde{\gamma} = \psi\} \tag{4.4.7}$$

Proof. First, let $\tilde{\gamma} \in \mathrm{Aut}(\tilde{M})$ with $\pi \circ \tilde{\gamma} = \pi$. Since $\psi = \phi \circ \pi$, we infer that

$$\psi \circ \tilde{\gamma} = \phi \circ \pi \circ \tilde{\gamma} = \phi \circ \pi = \psi, \tag{4.4.8}$$

which already proves the relation "$\subseteq$".

It remains to prove the relation "$\supseteq$". To this end, let $\tilde{\gamma} \in \mathrm{Aut}(\tilde{M})$ satisfying $\psi \circ \tilde{\gamma} = \psi$. Since $\phi : M \to \mathbb{R}^3$ is a trinoid with properly embedded annular ends, there exists by lemma 4.6 for some $j \in \{0, 1, \infty\}$ a punctured neighborhood U_j of z_j in M, such that $B_j = \phi(U_j)$ is a properly embedded annular end of ϕ at z_j satisfying $B_j \cap \phi(M \setminus U_j) = \emptyset$. As stated in the proof of lemma 4.6, we can assume without loss of generality that U_j is open in M and that $U_j \cup \{z_j\}$ is simply connected in $\hat{\mathbb{C}}$.

We consider the tesselation of $\tilde{M}$ induced by the sheet $\mathcal{F} \subseteq \tilde{M}$ defined in (3.2.31) and the group $\mathrm{Aut}(\tilde{M}/M)$ of covering transformations on $\tilde{M}$ given in (3.3.7). Via the universal covering π, U_j corresponds to an open subset $\tilde{U}_j$ of $\mathcal{F}$. In particular, the mapping $\pi|_{\tilde{U}_j} : tildeU_j \to U_j$ is bijective. Consequently, we have $\psi(\tilde{U}_j) \cap \psi(\mathcal{F} \setminus \tilde{U}_j) = \emptyset$.

Naturally, we have $\tilde{\gamma}(\tilde{U}_j) \subseteq \tilde{M}$. However, by restricting $\tilde{\gamma}$ to an appropriate open subset of $\tilde{U}_j$, the resulting restricted map takes values in only one sheet $\tilde{\mathcal{F}}$ of our tesselation of $\tilde{M}$. Therefore, we assume without loss of generality that $\tilde{\gamma}$ maps $\tilde{U}_j$ into some sheet $\tilde{\mathcal{F}}$, i.e. $\tilde{\gamma}|_{\tilde{U}_j} : \tilde{U}_j \to \tilde{\mathcal{F}}$.

Recalling that the group $\mathrm{Aut}(\tilde{M}/M)$ acts transitively on the set of the sheets of our tesselation of $\tilde{M}$, there exists $\tilde{\delta} \in \mathrm{Aut}(\tilde{M}/M)$, such that $\tilde{\delta}(\tilde{\mathcal{F}}) = \mathcal{F}$. Considering the mapping $\tilde{\delta} \circ \tilde{\gamma} : \tilde{M} \to \tilde{M}$, we observe by using $\pi \circ \tilde{\delta} = \pi$ and $\psi \circ \tilde{\gamma} = \psi$ that

$$\psi \circ \tilde{\delta} \circ \tilde{\gamma} = \phi \circ \pi \circ \tilde{\delta} \circ \tilde{\gamma} = \phi \circ \pi \circ \tilde{\gamma} = \psi \circ \tilde{\gamma} = \psi. \tag{4.4.9}$$

Let now $z \in \tilde{U}_j$. In this case, we have $\tilde{\gamma}(z) \in \tilde{\mathcal{F}}$ and thus $\tilde{\delta}(\tilde{\gamma}(z)) \in \mathcal{F}$. By (4.4.9), we have

$$\psi(z) = \psi(\tilde{\delta}(\tilde{\gamma}(z))). \tag{4.4.10}$$

Since $\psi(\tilde{U}_j) \cap \psi(\mathcal{F} \setminus \tilde{U}_j) = \emptyset$, this implies $\tilde{\delta}(\tilde{\gamma}(z)) \in \tilde{U}_j$. However, since $\psi = \phi \circ \pi$ is injective on $\tilde{U}_j$, we infer from (4.4.10) that actually $\tilde{\delta}(\tilde{\gamma}(z)) = z$.

Altogether, we have proved $\tilde{\delta} \circ \tilde{\gamma}|_{\tilde{U}_j} = \mathrm{id}$. Since $\tilde{\delta}$ and $\tilde{\gamma}$ are holomorphic on $\tilde{M}$, this relation carries over to $\tilde{M}$: $\tilde{\delta} \circ \tilde{\gamma} = \mathrm{id}$. But this implies $\tilde{\gamma} = \tilde{\delta}^{-1} \in \mathrm{Aut}(\tilde{M}/M)$ and in particular $\pi \circ \tilde{\gamma} = \pi$, which finishes the proof. □

Based on theorem 2.7 of [12], we can state the following result:

Lemma 4.8. *Let $\phi : M \to \mathbb{R}^3$ be a conformal CMC-immersion of a Riemann surface M into $\mathbb{R}^3$. Let $\pi : \tilde{M} \to M$ be the universal covering of M and $\psi := \phi \circ \pi$ a conformal CMC-immersion of $\tilde{M}$ into $\mathbb{R}^3$. Finally, let ϕ - or, equivalently, ψ - be symmetric with respect to an orientation preserving (resp. orientation reversing) Euclidean motion $\mathcal{T} \in \mathrm{Sym}(\phi(M)) = \mathrm{Sym}(\psi(\tilde{M}))$. Then, if M with the metric induced by ϕ is complete, there exists a mapping $\tilde{\gamma} \in \mathrm{Aut}(\tilde{M})$ (resp. $\tilde{\gamma} \in \mathrm{Aut}^*(\tilde{M})$), such that*

$$\mathcal{T} \circ \psi = \psi \circ \tilde{\gamma}. \tag{4.4.11}$$

The mapping $\tilde{\gamma}$ is unique up to composition from the left with an element of $\mathrm{Aut}(\tilde{M}/M)$.

Proof. In the case of an orientation preserving Euclidean motion $\mathcal{T}$ this is proved in theorem 2.7 of [12]. The proof for orientation reversing $\mathcal{T}$ is completely analogous. □

With these preparations made we can now turn to the announced

Theorem 4.9. *Let $M = \mathbb{C} \setminus \{0,1\}$ with universal cover $\tilde{M} = \mathbb{H}$ and covering map $\pi : \tilde{M} \to M$ as defined in (3.2.2). Let $\mathrm{Aut}(\tilde{M}/M)$ denote the automorphism group of π. Moreover, let $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and $\psi = \phi \circ \pi$ the corresponding conformal CMC-immersion $\tilde{M} \to \mathbb{R}^3$. Finally, let ϕ - or, equivalently, ψ - be symmetric with respect to an orientation preserving (resp. orientation reversing) Euclidean motion $\mathcal{T} \in \mathrm{Sym}(\phi(M)) = \mathrm{Sym}(\psi(\tilde{M}))$. Then there exist $\gamma \in \mathrm{Aut}(M)$ and $\tilde{\gamma} \in \mathrm{Aut}(\tilde{M})$ (resp. $\gamma \in \mathrm{Aut}^*(M)$ and $\tilde{\gamma} \in \mathrm{Aut}^*(\tilde{M})$) satisfying*

$$\mathcal{T} \circ \phi = \phi \circ \gamma, \tag{4.4.12}$$
$$\mathcal{T} \circ \psi = \psi \circ \tilde{\gamma}. \tag{4.4.13}$$

While γ is unique, $\tilde{\gamma}$ is unique up to composition from the left with an element of $\mathrm{Aut}(\tilde{M}/M)$. Furthermore, γ and $\tilde{\gamma}$ are related by

$$\pi \circ \tilde{\gamma} = \gamma \circ \pi. \tag{4.4.14}$$

Proof. First, recall that (by lemma 3.2) M with the metric induced by ϕ is complete. Thus, we can apply lemma 4.8 to relate to the orientation preserving (resp. orientation reversing) Euclidean motion $\mathcal{T}$ a mapping $\tilde{\gamma} \in \mathrm{Aut}(\tilde{M})$ (resp. $\tilde{\gamma} \in \mathrm{Aut}^*(\tilde{M})$) satisfying

$$\mathcal{T} \circ \psi = \psi \circ \tilde{\gamma}. \tag{4.4.15}$$

Moreover, $\tilde{\gamma}$ is unique up to composition from the left with an element of $\mathrm{Aut}(\tilde{M}/M)$.

Next, we prove the set identity

$$\tilde{\gamma}\mathrm{Aut}(\tilde{M}/M)\tilde{\gamma}^{-1} = \mathrm{Aut}(\tilde{M}/M). \tag{4.4.16}$$

Let $\tilde{\delta} \in \mathrm{Aut}(\tilde{M}/M)$. By lemma 4.7 we have $\psi \circ \tilde{\delta} = \psi$. Consequently,

$$\psi \circ \tilde{\gamma} \circ \tilde{\delta} \circ \tilde{\gamma}^{-1} = \mathcal{T} \circ \psi \circ \tilde{\delta} \circ \tilde{\gamma}^{-1} = \mathcal{T} \circ \psi \circ \tilde{\gamma}^{-1} = \psi, \tag{4.4.17}$$

which, noting that $\tilde{\gamma} \circ \tilde{\delta} \circ \tilde{\gamma}^{-1}$ is biholomorphic (even if $\tilde{\gamma}$ is bi-antihlomorphic) and applying again lemma 4.7, shows $\tilde{\gamma} \circ \tilde{\delta} \circ \tilde{\gamma}^{-1} \in \mathrm{Aut}(\tilde{M}/M)$ and thus proves the inclusion $\tilde{\gamma}\mathrm{Aut}(\tilde{M}/M)\tilde{\gamma}^{-1} \subseteq \mathrm{Aut}(\tilde{M}/M)$. Replacing $\tilde{\gamma}$ by $\tilde{\gamma}^{-1}$ in (4.4.17), we obtain analogously $\tilde{\gamma}^{-1}\mathrm{Aut}(\tilde{M}/M)\tilde{\gamma} \subseteq \mathrm{Aut}(\tilde{M}/M)$, or, equivalently, $\mathrm{Aut}(\tilde{M}/M) \subseteq \tilde{\gamma}\mathrm{Aut}(\tilde{M}/M)\tilde{\gamma}^{-1}$. Altogether, (4.4.16) follows.

We identify M with $\tilde{M}/\mathrm{Aut}(\tilde{M}/M)$ by the mapping $z \mapsto [w]$, where $w \in \tilde{M}$ with $\pi(w) = z$. (Cf. remark A.15 of appendix A.1 for more details.) Consider the mapping

$$\gamma : M \to M, \quad [w] \mapsto \gamma([w]) := [\tilde{\gamma}(w)]. \tag{4.4.18}$$

(Since $[w] = \{\tilde{\delta}(w); \tilde{\delta} \in \mathrm{Aut}(\tilde{M}/M)\}$ and, by (4.4.16), $[\tilde{\gamma}(\tilde{\delta}(w))] = [\tilde{\sigma}(\tilde{\gamma}(w))] = [\tilde{\gamma}(w)]$ for all $\tilde{\delta} \in \mathrm{Aut}(\tilde{M}/M)$ and appropriate $\tilde{\sigma} \in \mathrm{Aut}(\tilde{M}/M)$, γ is well defined.) With the identification of M and $\tilde{M}/\mathrm{Aut}(\tilde{M}/M)$ given above, we can write γ as

$$\gamma : M \to M, \quad z \mapsto \gamma(z) := \pi(\tilde{\gamma}(w)), \tag{4.4.19}$$

where $w \in \pi^{-1}(z)$ and the definition of γ is independent of the choice of w. By definition of γ, we have on $\tilde{M}$

$$\gamma \circ \pi = \pi \circ \tilde{\gamma}. \tag{4.4.20}$$

Moreover, since π is conformal, γ is biholomorphic (resp. bi-antiholomorphic) if $\tilde{\gamma}$ is biholomorphic (resp. bi-antiholomorphic).

Let now $z \in M$ and $w \in \pi^{-1}(z)$. Then,

$$\phi \circ \gamma(z) = \phi \circ \pi \circ \tilde{\gamma}(w) = \psi \circ \tilde{\gamma}(w) = \mathcal{T} \circ \psi(w) = \mathcal{T} \circ \phi \circ \pi(w) = \mathcal{T} \circ \phi(z). \tag{4.4.21}$$

This proves the relation

$$\mathcal{T} \circ \phi = \phi \circ \gamma, \tag{4.4.22}$$

which means that we have constructed γ with the claimed properties.

It remains to prove the uniqueness of γ. This can be seen as follows: Assume $\gamma_1, \gamma_2 \in \mathrm{Aut}(M)$ (resp. $\gamma_1, \gamma_2 \in \mathrm{Aut}^*(M)$) both satisfy the claimed relations. Then, in particular $\phi \circ \gamma_1 = \mathcal{T} \circ \phi = \phi \circ \gamma_2$, which implies

$$\phi \circ \gamma_1 \circ \gamma_2^{-1} = \phi. \tag{4.4.23}$$

Since ϕ is a trinoid with properly embedded annular ends there exists by lemma 4.6 an open subset U of M, such that $\phi|_U$ is an embedding with

$$\phi(U) \cap \phi(M \setminus U) = \emptyset. \tag{4.4.24}$$

This implies $\gamma_1 \circ \gamma_2^{-1}|_U = \mathrm{id}|_U$. As $\gamma_1 \circ \gamma_2^{-1}$ is biholomorphic (resp. bi-antiholomorphic) on M, this implies that actually $\gamma_1 \circ \gamma_2^{-1}|_M = \mathrm{id}|_M$ and therefore $\gamma_1 = \gamma_2$. □

By theorem 4.9, we can investigate the behaviour of ψ under the symmetry $\mathcal{T}$ by considering the composition $\psi \circ \tilde{\gamma}$. In the following, we explain how this relation carries over to the corresponding extended frame F from theorem 4.5.

According to the previous section, we consider $\tilde{\psi} = \mathcal{G}(1)\psi$ together with the associated orthogonal matrix $\tilde{\mathcal{U}} = \mathcal{G}(1)\mathcal{U}$ given in (4.2.6). We would like to compute

$$\tilde{\mathcal{U}} \circ \tilde{\gamma} = \mathcal{G}(1)(e^{-\frac{u\circ\tilde{\gamma}}{2}}\psi_x \circ \tilde{\gamma}, e^{-\frac{u\circ\tilde{\gamma}}{2}}\psi_y \circ \tilde{\gamma}, N \circ \tilde{\gamma}). \tag{4.4.25}$$

For this, we need to collect some technical results.

Lemma 4.10. *Interpreting $\tilde{\gamma} : \tilde{M} \to \tilde{M}$ as a mapping $(x, y) \mapsto (\tilde{\gamma}_1(x + iy), \tilde{\gamma}_2(x + iy)) \in \mathbb{R}^2$ of real variables x and y, where $\tilde{\gamma}_1 := \Re(\tilde{\gamma})$ and $\tilde{\gamma}_2 := \Im(\tilde{\gamma})$, we have*

$$e^{-\frac{u\circ\tilde{\gamma}}{2}} = e^{-\frac{u}{2}}\sqrt{\left(\frac{\partial\tilde{\gamma}_1}{\partial x}\right)^2 + \left(\frac{\partial\tilde{\gamma}_2}{\partial x}\right)^2}. \tag{4.4.26}$$

Proof. Because of $\mathcal{T} \in \mathrm{Iso}(\mathbb{R}^3)$ and $\mathcal{T} \circ \psi = \psi \circ \tilde{\gamma}$, $\tilde{\gamma}$ forms an isometry of $\tilde{M}$. This implies

$$\begin{aligned}
&e^u(\mathrm{d}x^2 + \mathrm{d}y^2) = e^{u\circ\tilde{\gamma}}(\mathrm{d}(\tilde{\gamma}_1)^2 + \mathrm{d}(\tilde{\gamma}_2)^2)\\
&= e^{u\circ\tilde{\gamma}}\left(\left(\frac{\partial\tilde{\gamma}_1}{\partial x}\right)^2\mathrm{d}x^2 + 2\frac{\partial\tilde{\gamma}_1}{\partial x}\frac{\partial\tilde{\gamma}_1}{\partial y}\mathrm{d}x\mathrm{d}y + \left(\frac{\partial\tilde{\gamma}_1}{\partial y}\right)^2\mathrm{d}y^2 + \left(\frac{\partial\tilde{\gamma}_2}{\partial x}\right)^2\mathrm{d}x^2 + 2\frac{\partial\tilde{\gamma}_2}{\partial x}\frac{\partial\tilde{\gamma}_2}{\partial y}\mathrm{d}x\mathrm{d}y + \left(\frac{\partial\tilde{\gamma}_2}{\partial y}\right)^2\mathrm{d}y^2\right)\\
&= e^{u\circ\tilde{\gamma}}\left(\left(\frac{\partial\tilde{\gamma}_1}{\partial x}\right)^2 + \left(\frac{\partial\tilde{\gamma}_2}{\partial x}\right)^2\right)\left(\mathrm{d}x^2 + \mathrm{d}y^2\right).
\end{aligned}$$

The last step follows from the (anti-)holomorphicity of $\tilde{\gamma}$: We either have $\frac{\partial\tilde{\gamma}_1}{\partial x} = \frac{\partial\tilde{\gamma}_2}{\partial y}$, $\frac{\partial\tilde{\gamma}_1}{\partial y} = -\frac{\partial\tilde{\gamma}_2}{\partial x}$ or $\frac{\partial\tilde{\gamma}_1}{\partial x} = -\frac{\partial\tilde{\gamma}_2}{\partial y}$, $\frac{\partial\tilde{\gamma}_1}{\partial y} = \frac{\partial\tilde{\gamma}_2}{\partial x}$. Altogether, the claim follows. □

Lemma 4.11. *Decomposing the symmetry $\mathcal{T} \in \mathrm{Sym}(\psi(\tilde{M}))$ into an orthogonal part $\mathcal{A}_\mathcal{T} \in \mathrm{O}(3)$ and an translational part $t_\mathcal{T} \in \mathbb{R}^3$,*

$$\mathcal{T} : \mathbb{R}^3 \to \mathbb{R}^3, x \mapsto \mathcal{A}_\mathcal{T}x + t_\mathcal{T}, \tag{4.4.27}$$

the following equations hold:

$$\psi_x \circ \tilde{\gamma} = \frac{1}{\left(\frac{\partial\tilde{\gamma}_1}{\partial x}\frac{\partial\tilde{\gamma}_2}{\partial y} - \frac{\partial\tilde{\gamma}_2}{\partial x}\frac{\partial\tilde{\gamma}_1}{\partial y}\right)}\mathcal{A}_\mathcal{T}\left(\frac{\partial\tilde{\gamma}_2}{\partial y}\psi_x - \frac{\partial\tilde{\gamma}_2}{\partial x}\psi_y\right), \tag{4.4.28}$$

$$\psi_y \circ \tilde{\gamma} = \frac{1}{\left(\frac{\partial\tilde{\gamma}_1}{\partial x}\frac{\partial\tilde{\gamma}_2}{\partial y} - \frac{\partial\tilde{\gamma}_2}{\partial x}\frac{\partial\tilde{\gamma}_1}{\partial y}\right)}\mathcal{A}_\mathcal{T}\left(-\frac{\partial\tilde{\gamma}_1}{\partial y}\psi_x + \frac{\partial\tilde{\gamma}_1}{\partial x}\psi_y\right), \tag{4.4.29}$$

$$N \circ \tilde{\gamma} = \mathcal{A}_\mathcal{T}N. \tag{4.4.30}$$

Proof. Defining the differential matrix

$$\mathrm{D}\tilde{\gamma}(x + iy) := \begin{pmatrix} \frac{\partial\tilde{\gamma}_1}{\partial x}(x + iy) & \frac{\partial\tilde{\gamma}_1}{\partial y}(x + iy) \\ \frac{\partial\tilde{\gamma}_2}{\partial x}(x + iy) & \frac{\partial\tilde{\gamma}_2}{\partial y}(x + iy) \end{pmatrix} \tag{4.4.31}$$

and analogously the differential matrices $\mathrm{D}\psi$ and $\mathrm{D}(\psi \circ \tilde{\gamma})$, we derive from the relation $\mathcal{T} \circ \psi = \psi \circ \tilde{\gamma}$ (provided by theorem 4.9)

$$(\psi_x \circ \tilde{\gamma}, \psi_y \circ \tilde{\gamma})\mathrm{D}\tilde{\gamma} = ((\mathrm{D}\psi) \circ \tilde{\gamma})\, \mathrm{D}\tilde{\gamma} = \mathrm{D}(\psi \circ \tilde{\gamma}) = \mathrm{D}(\mathcal{T} \circ \psi) = \mathcal{A}_{\mathcal{T}}\mathrm{D}\psi, \tag{4.4.32}$$

whence

$$(\psi_x \circ \tilde{\gamma}, \psi_y \circ \tilde{\gamma}) = \mathcal{A}_{\mathcal{T}}\mathrm{D}\psi(\mathrm{D}\tilde{\gamma})^{-1}. \tag{4.4.33}$$

Taking into account

$$(\mathrm{D}\tilde{\gamma})^{-1} = \frac{1}{\left(\frac{\partial\tilde{\gamma}_1}{\partial x}\frac{\partial\tilde{\gamma}_2}{\partial y} - \frac{\partial\tilde{\gamma}_2}{\partial x}\frac{\partial\tilde{\gamma}_1}{\partial y}\right)} \begin{pmatrix} \frac{\partial\tilde{\gamma}_2}{\partial y} & -\frac{\partial\tilde{\gamma}_1}{\partial y} \\ -\frac{\partial\tilde{\gamma}_2}{\partial x} & \frac{\partial\tilde{\gamma}_1}{\partial x} \end{pmatrix} \tag{4.4.34}$$

we obtain (4.4.28) and (4.4.29).

Recalling $\mathcal{A}_{\mathcal{T}} \in \mathrm{O}(3)$ and applying the same argument as in (4.2.12), we infer

$$N \circ \tilde{\gamma} = \frac{(\psi_x \circ \tilde{\gamma}) \times (\psi_y \circ \tilde{\gamma})}{|\,(\psi_x \circ \tilde{\gamma}) \times (\psi_y \circ \tilde{\gamma})\,|} = \det(\mathcal{A}_{\mathcal{T}})\mathcal{A}_{\mathcal{T}} \frac{\left(\frac{\partial\tilde{\gamma}_2}{\partial y}\psi_x - \frac{\partial\tilde{\gamma}_2}{\partial x}\psi_y\right) \times \left(-\frac{\partial\tilde{\gamma}_1}{\partial y}\psi_x + \frac{\partial\tilde{\gamma}_1}{\partial x}\psi_y\right)}{\left|\left(\frac{\partial\tilde{\gamma}_2}{\partial y}\psi_x - \frac{\partial\tilde{\gamma}_2}{\partial x}\psi_y\right) \times \left(-\frac{\partial\tilde{\gamma}_1}{\partial y}\psi_x + \frac{\partial\tilde{\gamma}_1}{\partial x}\psi_y\right)\right|}. \tag{4.4.35}$$

Continuing the calculation by using the (anti-)holomorphicity of $\tilde{\gamma}$ we obtain

$$N \circ \tilde{\gamma} = \det(\mathcal{A}_{\mathcal{T}})\mathcal{A}_{\mathcal{T}} \frac{\epsilon\left(\left(\frac{\partial\tilde{\gamma}_1}{\partial x}\right)^2 + \left(\frac{\partial\tilde{\gamma}_2}{\partial x}\right)^2\right)(\psi_x \times \psi_y)}{\left|\epsilon\left(\left(\frac{\partial\tilde{\gamma}_1}{\partial x}\right)^2 + \left(\frac{\partial\tilde{\gamma}_2}{\partial x}\right)^2\right)(\psi_x \times \psi_y)\right|} = \epsilon\det(\mathcal{A}_{\mathcal{T}})\mathcal{A}_{\mathcal{T}}N, \tag{4.4.36}$$

where $\epsilon \in \{\pm 1\}$ takes the value "+1" (resp. "−1") in the case of holomorphic (resp. antiholomorphic) $\tilde{\gamma}$. As (by theorem 4.9) $\tilde{\gamma}$ is holomorphic for orientation preserving $\mathcal{T}$, i.e. in the case $\det(\mathcal{A}_{\mathcal{T}}) = 1$, and antiholomorphic for orientation reversing $\mathcal{T}$, i.e. in the case $\det(\mathcal{A}_{\mathcal{T}}) = -1$, we obtain (merging both cases) $\epsilon\det(\mathcal{A}_{\mathcal{T}}) = +1$. Therefore $N \circ \tilde{\gamma} = \mathcal{A}_{\mathcal{T}}N$, which is (4.4.30). ☐

Combining the results of the two proceeding lemmas, we can write out how $\tilde{\mathcal{U}}$ transforms under the biholomorphic (resp. bi-antiholomorphic) mapping $\tilde{\gamma}$. We record this in the following theorem.

Theorem 4.12. *Let $\psi : \tilde{M} \to \mathbb{R}^3$ be a conformal CMC-immersion, which is symmetric with respect to $\mathcal{T} : x \mapsto \mathcal{A}_{\mathcal{T}}x + t_{\mathcal{T}}$. Assume ψ corresponds to a trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends via the universal covering $\pi : \tilde{M} \to M$, $\psi = \phi \circ \pi$. Let $\tilde{\gamma}$ be a biholomorphic (resp. bi-antiholomorphic) mapping $\tilde{M} \to \tilde{M}$ associated with $\mathcal{T}$ by theorem 4.9. Furthermore, let $\tilde{\psi} = \mathcal{G}(1)\psi$, where $\mathcal{G}(1)$ is given by* (4.2.5)*. Then the orthogonal matrix $\tilde{\mathcal{U}}$ corresponding to $\tilde{\psi}$ as defined in* (4.2.6) *satisfies*

$$\tilde{\mathcal{U}} \circ \tilde{\gamma} = \mathcal{G}(1)\mathcal{A}_{\mathcal{T}}(\mathcal{G}(1))^{-1}\tilde{\mathcal{U}}\mathcal{K}_{\mathcal{T},\tilde{\gamma}}, \tag{4.4.37}$$

where

$$\mathcal{K}_{\mathcal{T},\tilde{\gamma}} = \begin{pmatrix} A & B & 0 \\ -B & A & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{if } \mathcal{T} \text{ preserves orientation,} \tag{4.4.38}$$

$$\mathcal{K}_{\mathcal{T},\tilde{\gamma}} = \begin{pmatrix} A & B & 0 \\ B & -A & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{if } \mathcal{T} \text{ reverses orientation,} \tag{4.4.39}$$

and $A, B : \tilde{M} \to \mathbb{R}$ are defined by

$$A(x+iy) = \frac{\frac{\partial\tilde{\gamma}_1}{\partial x}(x+iy)}{\sqrt{\left(\frac{\partial\tilde{\gamma}_1}{\partial x}(x+iy)\right)^2 + \left(\frac{\partial\tilde{\gamma}_2}{\partial x}(x+iy)\right)^2}}, \tag{4.4.40}$$

$$B(x+iy) = \frac{\frac{\partial\tilde{\gamma}_2}{\partial x}(x+iy)}{\sqrt{\left(\frac{\partial\tilde{\gamma}_1}{\partial x}(x+iy)\right)^2 + \left(\frac{\partial\tilde{\gamma}_2}{\partial x}(x+iy)\right)^2}}. \tag{4.4.41}$$

Proof. We distinguish two cases. If $\mathcal{T}$ preserves orientation, the associated mapping $\tilde{\gamma}$ of theorem 4.9 is holomorphic. Otherwise, if $\mathcal{T}$ reverses orientation, $\tilde{\gamma}$ is antiholomorphic. Thus, setting

$$\epsilon := \epsilon(\mathcal{T}) := \begin{cases} +1 & \text{if } \mathcal{T} \text{ preserves orientation} \\ -1 & \text{if } \mathcal{T} \text{ reverses orientation} \end{cases} \tag{4.4.42}$$

we have $\frac{\partial \tilde{\gamma}_1}{\partial x} = \epsilon \frac{\partial \tilde{\gamma}_2}{\partial y}$, $\frac{\partial \tilde{\gamma}_1}{\partial y} = -\epsilon \frac{\partial \tilde{\gamma}_2}{\partial x}$. Applying this to equations (4.4.28) and (4.4.29) of the above lemma, we obtain

$$\psi_x \circ \tilde{\gamma} = \frac{1}{\left(\frac{\partial \tilde{\gamma}_1}{\partial x}\right)^2 + \left(\frac{\partial \tilde{\gamma}_2}{\partial x}\right)^2} \mathcal{A}_{\mathcal{T}} \left(\frac{\partial \tilde{\gamma}_1}{\partial x} \psi_x - \epsilon \frac{\partial \tilde{\gamma}_2}{\partial x} \psi_y \right), \tag{4.4.43}$$

$$\psi_y \circ \tilde{\gamma} = \frac{1}{\left(\frac{\partial \tilde{\gamma}_1}{\partial x}\right)^2 + \left(\frac{\partial \tilde{\gamma}_2}{\partial x}\right)^2} \mathcal{A}_{\mathcal{T}} \left(\frac{\partial \tilde{\gamma}_2}{\partial x} \psi_x + \epsilon \frac{\partial \tilde{\gamma}_1}{\partial x} \psi_y \right). \tag{4.4.44}$$

Altogether, taking into account (4.4.26), (4.4.30), (4.4.43) and (4.4.44), we obtain from (4.4.25):

$$\tilde{\mathcal{U}} \circ \tilde{\gamma} = \mathcal{G}(1)(e^{-\frac{u}{2}} \mathcal{A}_{\mathcal{T}} \left(A\psi_x - \epsilon B \psi_y\right), e^{-\frac{u}{2}} \mathcal{A}_{\mathcal{T}} \left(B\psi_x + \epsilon A \psi_y\right), \mathcal{A}_{\mathcal{T}} N), \tag{4.4.45}$$

where A, B are as in (4.4.40), (4.4.41), respectively. This transforms further into

$$\tilde{\mathcal{U}} \circ \tilde{\gamma} = \mathcal{G}(1) \mathcal{A}_{\mathcal{T}} (e^{-\frac{u}{2}} \psi_x, e^{-\frac{u}{2}} \psi_y, N) \begin{pmatrix} A & B & 0 \\ -\epsilon B & +\epsilon A & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathcal{G}(1) \mathcal{A}_{\mathcal{T}} \mathcal{U} \mathcal{K}_{\mathcal{T},\tilde{\gamma}}, \tag{4.4.46}$$

with $\mathcal{K}_{\mathcal{T},\tilde{\gamma}}$ as in (4.4.38), (4.4.39), respectively. As $\tilde{\mathcal{U}} = \mathcal{G}(1)\mathcal{U}$, the claim follows. □

As a consequence, the conjugation matrix $\tilde{P}$ corresponding to $\tilde{\mathcal{U}}$ by (4.2.14) transforms as follows:

Corollary 4.13. *We retain the notation and the assumptions of theorem 4.12. The conjugation matrix* $\tilde{P}$ *realizing the orthogonal matrix* $\tilde{\mathcal{U}}$ *in the* su(2) *model transforms under* $\tilde{\gamma}$ *as*

$$\tilde{P} \circ \tilde{\gamma} = \pm G(1) A_{\mathcal{T}} G(1)^{-1} \tilde{P} \hat{k}_{\mathcal{T},\tilde{\gamma}}, \tag{4.4.47}$$

where $G(1), A_{\mathcal{T}}, \hat{k}_{\mathcal{T},\tilde{\gamma}} \in \mathrm{SU}(2)$ *are the corresponding conjugation matrices realizing* $\mathcal{G}(1), \mathcal{A}_{\mathcal{T}}, \mathcal{K}_{\mathcal{T},\tilde{\gamma}} \in \mathrm{O}(3)$*, respectively, in the* su(2) *model, and the remaining freedom in the sign is caused by the fact that we work in* su(2) *and not in* O(3)*.*

Proof. As a first step, we interpret the O(3) matrices appearing in (4.4.37) in the su(2) model. To this end, recall

$$\mathcal{G}(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{4.4.48}$$

$$\mathcal{K}_{\mathcal{T},\tilde{\gamma}} = \begin{pmatrix} A & B & 0 \\ -B & A & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SO}(3) \quad \text{if } \mathcal{T} \text{ preserves orientation}, \tag{4.4.49}$$

$$\mathcal{K}_{\mathcal{T},\tilde{\gamma}} = \begin{pmatrix} A & B & 0 \\ B & -A & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{O}(3) \setminus \mathrm{SO}(3) \quad \text{if } \mathcal{T} \text{ reverses orientation}. \tag{4.4.50}$$

From (4.2.21) we already know

$$(J \circ \mathcal{G}(1) \circ J^{-1})(X) = G(1) X G(1)^{-1} \text{ for all } X \in \mathrm{su}(2), \tag{4.4.51}$$

where $G(1)$ is given by (4.2.17). The corresponding equations for $\mathcal{K}_{\mathcal{T},\tilde{\gamma}}$ read

$$(J \circ \mathcal{K}_{\mathcal{T},\tilde{\gamma}} \circ J^{-1})(X) = \hat{k}_{\mathcal{T},\tilde{\gamma}} X \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1} \text{ for all } X \in \mathrm{su}(2) \quad \text{if } \mathcal{T} \text{ preserves orientation}, \tag{4.4.52}$$

$$(J \circ \mathcal{K}_{\mathcal{T},\tilde{\gamma}} \circ J^{-1})(X) = -\hat{k}_{\mathcal{T},\tilde{\gamma}} X \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1} \text{ for all } X \in \mathrm{su}(2) \quad \text{if } \mathcal{T} \text{ reverses orientation}, \tag{4.4.53}$$

where

$$\hat{k}_{\mathcal{T},\tilde{\gamma}} = \begin{pmatrix} \sqrt{A+iB} & 0 \\ 0 & \overline{\sqrt{A+iB}} \end{pmatrix} \quad \text{if } \mathcal{T} \text{ preserves orientation,} \tag{4.4.54}$$

$$\hat{k}_{\mathcal{T},\tilde{\gamma}} = \begin{pmatrix} 0 & -\overline{\sqrt{A+iB}} \\ \sqrt{A+iB} & 0 \end{pmatrix} \quad \text{if } \mathcal{T} \text{ reverses orientation.} \tag{4.4.55}$$

The identities (4.4.52) and (4.4.53) are verified as follows: Using the fact that $|A+iB|^2 = (A+iB)\overline{(A+iB)} = (A+iB)(A-iB) = 1$ and thus $|A+iB| = 1$, we conclude that $|\sqrt{A+iB}|^2 = \sqrt{A+iB}\,\overline{\sqrt{A+iB}} = 1$. In view of this, equations (4.4.52) and (4.4.53) are obtained by a direct computation.[11]

Concerning the orthogonal part $\mathcal{A}_{\mathcal{T}}$ of the symmetry $\mathcal{T}$, we know that $J \circ \mathcal{A}_{\mathcal{T}} \circ J^{-1}$ defines an automorphism of su(2) which (recalling equations (3.4.7) and (3.4.8), respectively) is realized by

$$(J \circ \mathcal{A}_{\mathcal{T}} \circ J^{-1})(X) = A_{\mathcal{T}} X A_{\mathcal{T}}^{-1} \text{ for all } X \in \mathrm{su}(2) \quad \text{if } \mathcal{T} \text{ preserves orientation,} \tag{4.4.56}$$

$$(J \circ \mathcal{A}_{\mathcal{T}} \circ J^{-1})(X) = -A_{\mathcal{T}} X A_{\mathcal{T}}^{-1} \text{ for all } X \in \mathrm{su}(2) \quad \text{if } \mathcal{T} \text{ reverses orientation,} \tag{4.4.57}$$

where $A_{\mathcal{T}}$ defines a $\mathcal{T}$-dependent element of SU(2). Finally, we recall equation (4.2.14), which reads

$$(J \circ \tilde{\mathcal{U}} \circ J^{-1})(X) = \tilde{P} X \tilde{P}^{-1} \quad \text{for all } X \in \mathrm{su}(2). \tag{4.4.58}$$

Altogether, by theorem 4.12, we obtain for all $X \in \mathrm{su}(2)$

$$\begin{aligned}(\tilde{P} \circ \tilde{\gamma}) X (\tilde{P} \circ \tilde{\gamma})^{-1} &= (J \circ (\tilde{\mathcal{U}} \circ \tilde{\gamma}) \circ J^{-1})(X) = (J \circ (\mathcal{G}(1)\mathcal{A}_{\mathcal{T}}(\mathcal{G}(1))^{-1}\tilde{\mathcal{U}}\mathcal{K}_{\mathcal{T},\tilde{\gamma}}) \circ J^{-1})(X) \\ &= (J \circ \mathcal{G}(1) \circ J^{-1}) \circ (J \circ \mathcal{A}_{\mathcal{T}} \circ J^{-1}) \circ (J \circ \mathcal{G}(1))^{-1} \circ J^{-1}) \circ (J \circ \tilde{\mathcal{U}} \circ J^{-1}) \circ (J \circ \mathcal{K}_{\mathcal{T},\tilde{\gamma}} \circ J^{-1})(X) \\ &= G(1) A_{\mathcal{T}} G(1)^{-1} \tilde{P} \hat{k}_{\mathcal{T},\tilde{\gamma}} X \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1} \tilde{P}^{-1} G(1) A_{\mathcal{T}}^{-1} G(1)^{-1}. \end{aligned} \tag{4.4.59}$$

Note that the two minus signs occuring in the case of an orientation reversing symmetry $\mathcal{T}$ cancel.

So far, we have seen that $\tilde{P} \circ \tilde{\gamma}$ and $G(1) A_{\mathcal{T}} G(1)^{-1} \tilde{P} \hat{k}_{\mathcal{T},\tilde{\gamma}}$ conjugate $X \in \mathrm{su}(2)$ into the same element of su(2). But since this is true for all $X \in \mathrm{su}(2)$, we necessarily have

$$\tilde{P} \circ \tilde{\gamma} = \pm G(1) A_{\mathcal{T}} G(1)^{-1} \tilde{P} \hat{k}_{\mathcal{T},\tilde{\gamma}}, \tag{4.4.60}$$

which proves the claim. □

Remark 4.14. We define the complex square root $\sqrt{\ }$ occurring in $\hat{k}_{\mathcal{T},\tilde{\gamma}}$ (cf. (4.4.54) and (4.4.55)) on the z-plane $\mathbb{C}^*$ by

$$\sqrt{\ } : \mathbb{C}^* \to \mathbb{C}^*, \ z = re^{i\theta} \mapsto \sqrt{z} := \sqrt{r} e^{i\frac{\theta}{2}}, \tag{4.4.61}$$

where we write $z \in \mathbb{C}^*$ in the form $z = re^{i\theta}$ with $r \in \mathbb{R}^+$ and $\theta \in (-\pi, \pi]$, and $\sqrt{r}$ defines the value of the usual (real) square root of r. For future calculations, we state the following identities involving the complex square root as defined above. Note that these identities are, as is well-known for complex square roots in general, only determined up to sign. For all $z, z_1, z_2 \in C^*$ we have

$$\sqrt{z_1}\sqrt{z_2} = \pm\sqrt{z_1 z_2}, \tag{4.4.62}$$

$$\sqrt{\bar{z}} = \pm\overline{\sqrt{z}}, \tag{4.4.63}$$

$$\sqrt{z^{-1}} = \pm(\sqrt{z})^{-1}. \tag{4.4.64}$$

In order to translate the transformation property of $\tilde{P}$ under $\tilde{\gamma}$ stated in the proceeding corollary into a corresponding relation for $\tilde{P}_\lambda$ as introduced in the previous section, we need to make some further preparations. We start by defining the differential form

$$\zeta := \tilde{P}^{-1} \mathrm{d}\tilde{P}. \tag{4.4.65}$$

[11] Note that the stated identity $\sqrt{A+iB}\,\overline{\sqrt{A+iB}} = 1$, which suffices to prove equations (4.4.52) and (4.4.53), is obtained for any complex square root $\mathbb{C}^* \to \mathbb{C}^*$. We explicitly define the complex square root $\sqrt{\ }$ used in this work in remark 4.14.

In view of the corollary above, ζ transforms under $\tilde{\gamma}$ as

$$\tilde{\gamma}^*\zeta = (\tilde{P}\circ\tilde{\gamma})^{-1}\mathrm{d}(\tilde{P}\circ\tilde{\gamma}) = \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\tilde{P}^{-1}G(1)A_{\mathcal{T}}^{-1}G(1)^{-1}\mathrm{d}(G(1)A_{\mathcal{T}}G(1)^{-1}\tilde{P}\hat{k}_{\mathcal{T},\tilde{\gamma}})$$
$$= \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\tilde{P}^{-1}(\mathrm{d}\tilde{P}\hat{k}_{\mathcal{T},\tilde{\gamma}} + \tilde{P}\mathrm{d}\hat{k}_{\mathcal{T},\tilde{\gamma}}) = \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta\hat{k}_{\mathcal{T},\tilde{\gamma}} + \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\mathrm{d}\hat{k}_{\mathcal{T},\tilde{\gamma}}. \quad (4.4.66)$$

Furthermore, taking into account equations (4.2.15) we have

$$\zeta = \tilde{P}^{-1}\tilde{P}_z\mathrm{d}z + \tilde{P}^{-1}\tilde{P}_{\bar{z}}\mathrm{d}\bar{z} = \begin{pmatrix} -\frac{1}{4}u_z & Qe^{-\frac{u}{2}} \\ -\frac{1}{2}e^{\frac{u}{2}}H & \frac{1}{4}u_z \end{pmatrix}\mathrm{d}z + \begin{pmatrix} \frac{1}{4}u_{\bar{z}} & \frac{1}{2}e^{\frac{u}{2}}H \\ -\bar{Q}e^{-\frac{u}{2}} & -\frac{1}{4}u_{\bar{z}} \end{pmatrix}\mathrm{d}\bar{z}. \quad (4.4.67)$$

Performing the splitting

$$\zeta = \zeta_k + \zeta_p'\mathrm{d}z + \zeta_p''\mathrm{d}\bar{z}, \quad (4.4.68)$$

where

$$\zeta_k = \begin{pmatrix} -\frac{1}{4}u_z & 0 \\ 0 & \frac{1}{4}u_z \end{pmatrix}\mathrm{d}z + \begin{pmatrix} \frac{1}{4}u_{\bar{z}} & 0 \\ 0 & -\frac{1}{4}u_{\bar{z}} \end{pmatrix}\mathrm{d}\bar{z}, \quad (4.4.69)$$
$$\zeta_p' = \begin{pmatrix} 0 & Qe^{-\frac{u}{2}} \\ -\frac{1}{2}e^{\frac{u}{2}}H & 0 \end{pmatrix}, \quad (4.4.70)$$
$$\zeta_p'' = \begin{pmatrix} 0 & \frac{1}{2}e^{\frac{u}{2}}H \\ -\bar{Q}e^{-\frac{u}{2}} & 0 \end{pmatrix}, \quad (4.4.71)$$

equation (4.4.66) implies

$$\tilde{\gamma}^*\zeta = \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_k\hat{k}_{\mathcal{T},\tilde{\gamma}} + \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_p'\hat{k}_{\mathcal{T},\tilde{\gamma}}\mathrm{d}z + \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_p''\hat{k}_{\mathcal{T},\tilde{\gamma}}\mathrm{d}\bar{z} + \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\mathrm{d}\hat{k}_{\mathcal{T},\tilde{\gamma}}. \quad (4.4.72)$$

At the same time we have

$$\tilde{\gamma}^*\zeta = \tilde{\gamma}^*\zeta_k + \tilde{\gamma}^*(\zeta_p'\mathrm{d}z) + \tilde{\gamma}^*(\zeta_p''\mathrm{d}\bar{z}) = \tilde{\gamma}^*\zeta_k + (\zeta_p'\circ\tilde{\gamma})\mathrm{d}\tilde{\gamma}(z) + (\zeta_p''\circ\tilde{\gamma})\mathrm{d}\overline{\tilde{\gamma}(z)}. \quad (4.4.73)$$

Comparing equations (4.4.72) and (4.4.73), we can state the following lemma:

Lemma 4.15. *Let $\psi : \tilde{M} \to \mathbb{R}^3$ be a conformal CMC-immersion, which corresponds to a trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends via the universal covering $\pi : \tilde{M} \to M$, $\psi = \phi\circ\pi$. Moreover, let $\zeta = \tilde{P}^{-1}\mathrm{d}\tilde{P} = \zeta_k + \zeta_p'\mathrm{d}z + \zeta_p''\mathrm{d}\bar{z}$ as above, where $\tilde{P}$ corresponds, as in (4.2.14), to the orthogonal frame $\tilde{U}$ associated with ψ, which is given in (4.2.6). Furthermore, let $\mathcal{T}$ denote a symmetry of ψ, $\mathcal{T} : \mathbb{R}^3 \to \mathbb{R}^3$, $\mathcal{T}(x) = \mathcal{A}_{\mathcal{T}}x + t_{\mathcal{T}}$ with $\mathcal{A}_{\mathcal{T}} \in \mathrm{O}(3)$ and $t_{\mathcal{T}} \in \mathbb{R}^3$, and let $\tilde{\gamma}$ denote a biholomorphic (resp. bi-antiholomorphic) mapping $\tilde{M} \to \tilde{M}$ associated with $\mathcal{T}$ by theorem 4.9.*

1. *If $\mathcal{T}$ preserves orientation, the following holds:*

$$\tilde{\gamma}^*\zeta_k = \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_k\hat{k}_{\mathcal{T},\tilde{\gamma}} + \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\mathrm{d}\hat{k}_{\mathcal{T},\tilde{\gamma}} \quad (4.4.74)$$
$$(\zeta_p'\circ\tilde{\gamma})\partial_z\tilde{\gamma} = \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_p'\hat{k}_{\mathcal{T},\tilde{\gamma}} \quad (4.4.75)$$
$$(\zeta_p''\circ\tilde{\gamma})\partial_{\bar{z}}\overline{\tilde{\gamma}} = \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_p''\hat{k}_{\mathcal{T},\tilde{\gamma}}. \quad (4.4.76)$$

2. *If $\mathcal{T}$ reverses orientation, the following holds:*

$$\tilde{\gamma}^*\zeta_k = \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_k\hat{k}_{\mathcal{T},\tilde{\gamma}} + \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\mathrm{d}\hat{k}_{\mathcal{T},\tilde{\gamma}} \quad (4.4.77)$$
$$(\zeta_p'\circ\tilde{\gamma})\partial_{\bar{z}}\tilde{\gamma} = \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_p''\hat{k}_{\mathcal{T},\tilde{\gamma}} \quad (4.4.78)$$
$$(\zeta_p''\circ\tilde{\gamma})\partial_z\overline{\tilde{\gamma}} = \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_p'\hat{k}_{\mathcal{T},\tilde{\gamma}}. \quad (4.4.79)$$

Proof. As stated before, the claims follow from comparing equations (4.4.72) and (4.4.73).
We start with the first case, i.e. let $\mathcal{T}$ preserve orientation. Therefore, by theorem 4.9, $\tilde{\gamma}$ is holomorphic, and equation (4.4.73) reads as

$$\tilde{\gamma}^*\zeta = \tilde{\gamma}^*\zeta_k + (\zeta_p'\circ\tilde{\gamma})\partial_z\tilde{\gamma}\mathrm{d}z + (\zeta_p''\circ\tilde{\gamma})\partial_{\bar{z}}\overline{\tilde{\gamma}}\mathrm{d}\bar{z}. \quad (4.4.80)$$

We observe that $\tilde{\gamma}^*\zeta_k$ is of the form

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\mathrm{d}z + \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\mathrm{d}\bar{z}, \quad (4.4.81)$$

while $(\zeta_p' \circ \tilde{\gamma})\partial_z \tilde{\gamma} \mathrm{d}z$ is of the form

$$\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \mathrm{d}z, \tag{4.4.82}$$

and $(\zeta_p'' \circ \tilde{\gamma})\partial_{\bar{z}} \overline{\tilde{\gamma}} \mathrm{d}\bar{z}$ is of the form

$$\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \mathrm{d}\bar{z}. \tag{4.4.83}$$

As in the present case $\hat{k}_{\mathcal{T},\tilde{\gamma}}$ is explicitly given by (4.4.54), it is now easy to verify that the summands of $\tilde{\gamma}^*\zeta$ occuring in equation (4.4.72) are of the following forms, respectively: $\hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_k \hat{k}_{\mathcal{T},\tilde{\gamma}} + \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\mathrm{d}\hat{k}_{\mathcal{T},\tilde{\gamma}}$ is of the form (4.4.81), $\hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_p' \hat{k}_{\mathcal{T},\tilde{\gamma}}\mathrm{d}z$ is of the form (4.4.82), and $\hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_p'' \hat{k}_{\mathcal{T},\tilde{\gamma}}\mathrm{d}\bar{z}$ is of the form (4.4.83). Therefore, comparing equation (4.4.72) to (4.4.73) proves the claim in the first case.
We turn to the second case, i.e. let $\mathcal{T}$ reverse orientation. We proceed as in the first case. However, note that now, by theorem 4.9, $\tilde{\gamma}$ is antiholomorphic and thus equation (4.4.73) becomes

$$\tilde{\gamma}^*\zeta = \tilde{\gamma}^*\zeta_k + (\zeta_p' \circ \tilde{\gamma})\partial_{\bar{z}}\tilde{\gamma}(\bar{z})\mathrm{d}\bar{z} + (\zeta_p'' \circ \tilde{\gamma})\partial_z \overline{\tilde{\gamma}}\mathrm{d}z. \tag{4.4.84}$$

In this case, $\tilde{\gamma}^*\zeta_k$ is of the form (4.4.81), $(\zeta_p' \circ \tilde{\gamma})\partial_{\bar{z}}\tilde{\gamma}\mathrm{d}\bar{z}$ is of the form (4.4.83), and $(\zeta_p'' \circ \tilde{\gamma})\partial_z \overline{\tilde{\gamma}}\mathrm{d}z$ is of the form (4.4.82). In view of $\hat{k}_{\mathcal{T},\tilde{\gamma}}$, now explicitly given by (4.4.55), we investigate the summands of $\tilde{\gamma}^*\zeta$ in (4.4.72) and observe that, exactly as in the first case, $\hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_k \hat{k}_{\mathcal{T},\tilde{\gamma}} + \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\mathrm{d}\hat{k}_{\mathcal{T},\tilde{\gamma}}$ is of the form (4.4.81), $\hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_p' \hat{k}_{\mathcal{T},\tilde{\gamma}}\mathrm{d}z$ is of the form (4.4.82), and $\hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_p'' \hat{k}_{\mathcal{T},\tilde{\gamma}}\mathrm{d}\bar{z}$ is of the form (4.4.83). Combining (4.4.72) and (4.4.73) proves the claim in the second case. □

The technical result just established yields the following transformation property of $\tilde{P}_\lambda$ under $\tilde{\gamma}$:

Lemma 4.16. *We retain the notations and the assumptions of the previous lemma. Furthermore let $\tilde{P}_\lambda$ be the mapping $\tilde{M} \to \Lambda\mathrm{SU}(2)$ as defined in section 4.2. Then, the following statements hold:*

1. *If $\mathcal{T}$ is orientation preserving, $\tilde{P}_\lambda$ transforms under $\tilde{\gamma}$ as:*

$$\tilde{P}_\lambda \circ \tilde{\gamma} = \tilde{M}_{\tilde{\gamma}}(\lambda)\tilde{P}_\lambda \hat{k}_{\mathcal{T},\tilde{\gamma}}, \tag{4.4.85}$$

where $\hat{k}_{\mathcal{T},\tilde{\gamma}}$ is defined in (4.4.54) and $\tilde{M}_{\tilde{\gamma}}(\lambda)$ is independent of z. Moreover, for $\lambda = 1$, we have

$$\tilde{M}_{\tilde{\gamma}}(1) = \pm G(1) A_{\mathcal{T}} G(1)^{-1} \tag{4.4.86}$$

where, as before, $G(1), A_{\mathcal{T}}, \in \mathrm{SU}(2)$ are the corresponding conjugation matrices realizing $\mathcal{G}(1), \mathcal{A}_{\mathcal{T}} \in \mathrm{O}(3)$, respectively, in the $\mathrm{su}(2)$-model.

2. *If $\mathcal{T}$ is orientation reversing, $\tilde{P}_\lambda$ transforms under $\tilde{\gamma}$ as:*

$$\tilde{P}_{\lambda^{-1}} \circ \tilde{\gamma} = \tilde{M}_{\tilde{\gamma}}(\lambda)\tilde{P}_\lambda \hat{k}_{\mathcal{T},\tilde{\gamma}}, \tag{4.4.87}$$

where $\hat{k}_{\mathcal{T},\tilde{\gamma}}$ is defined in (4.4.55) and $\tilde{M}_{\tilde{\gamma}}(\lambda)$ is independent of z. Moreover, for $\lambda = 1$, we have

$$\tilde{M}_{\tilde{\gamma}}(1) = \pm G(1) A_{\mathcal{T}} G(1)^{-1} \tag{4.4.88}$$

where, as before, $G(1), A_{\mathcal{T}}, \in \mathrm{SU}(2)$ are the corresponding conjugation matrices realizing $\mathcal{G}(1), \mathcal{A}_{\mathcal{T}} \in \mathrm{O}(3)$, respectively, in the $\mathrm{su}(2)$-model.

Proof. Define the differential form

$$\zeta_\lambda := \tilde{P}_\lambda^{-1}\mathrm{d}\tilde{P}_\lambda. \tag{4.4.89}$$

By taking into account equations (4.2.16), we can write

$$\zeta_\lambda = \tilde{P}_\lambda^{-1}(\tilde{P}_\lambda)_z \mathrm{d}z + \tilde{P}_\lambda^{-1}(\tilde{P}_\lambda)_{\bar{z}}\mathrm{d}\bar{z} = \begin{pmatrix} -\frac{1}{4}u_z & \lambda^{-2}Qe^{-\frac{u}{2}} \\ -\frac{1}{2}e^{\frac{u}{2}}H & \frac{1}{4}u_z \end{pmatrix} \mathrm{d}z + \begin{pmatrix} \frac{1}{4}u_{\bar{z}} & \frac{1}{2}e^{\frac{u}{2}}H \\ -\lambda^2 \bar{Q}e^{-\frac{u}{2}} & -\frac{1}{4}u_{\bar{z}} \end{pmatrix} \mathrm{d}\bar{z}. \tag{4.4.90}$$

Recalling the splitting $\zeta = \zeta_k + \zeta_p'\mathrm{d}z + \zeta_p''\mathrm{d}\bar{z}$ of $\zeta = \tilde{P}^{-1}\mathrm{d}\tilde{P}$, where ζ_k, ζ_p' and ζ_p'' are given by equations (4.4.69) to (4.4.71), an easy computation allows to relate ζ_λ to ζ:

$$\zeta_\lambda = \zeta_k + \lambda^{-1}\begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \zeta_p' \begin{pmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & \lambda^{-\frac{1}{2}} \end{pmatrix} \mathrm{d}z + \lambda \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \zeta_p'' \begin{pmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & \lambda^{-\frac{1}{2}} \end{pmatrix} \mathrm{d}\bar{z}. \tag{4.4.91}$$

Applying $\tilde{\gamma}$ to this equation, we obtain

$$\tilde{\gamma}^*\zeta_\lambda = \tilde{\gamma}^*\zeta_k + \lambda^{-1}\begin{pmatrix}\lambda^{-\frac{1}{2}} & 0\\ 0 & \lambda^{\frac{1}{2}}\end{pmatrix}(\zeta_p'\circ\tilde{\gamma})\begin{pmatrix}\lambda^{\frac{1}{2}} & 0\\ 0 & \lambda^{-\frac{1}{2}}\end{pmatrix}\mathrm{d}\tilde{\gamma}(z) + \lambda\begin{pmatrix}\lambda^{-\frac{1}{2}} & 0\\ 0 & \lambda^{\frac{1}{2}}\end{pmatrix}(\zeta_p''\circ\tilde{\gamma})\begin{pmatrix}\lambda^{\frac{1}{2}} & 0\\ 0 & \lambda^{-\frac{1}{2}}\end{pmatrix}\mathrm{d}\overline{\tilde{\gamma}(z)}. \tag{4.4.92}$$

Now we distinguish between the two cases of orientation preserving and orientation reversing $\mathcal{T}$: If $\mathcal{T}$ preserves orientation, $\tilde{\gamma}$ is holomorphic, and we obtain

$$\tilde{\gamma}^*\zeta_\lambda = \tilde{\gamma}^*\zeta_k + \lambda^{-1}\begin{pmatrix}\lambda^{-\frac{1}{2}} & 0\\ 0 & \lambda^{\frac{1}{2}}\end{pmatrix}(\zeta_p'\circ\tilde{\gamma})\begin{pmatrix}\lambda^{\frac{1}{2}} & 0\\ 0 & \lambda^{-\frac{1}{2}}\end{pmatrix}\partial_z\tilde{\gamma}\mathrm{d}z + \lambda\begin{pmatrix}\lambda^{-\frac{1}{2}} & 0\\ 0 & \lambda^{\frac{1}{2}}\end{pmatrix}(\zeta_p''\circ\tilde{\gamma})\begin{pmatrix}\lambda^{\frac{1}{2}} & 0\\ 0 & \lambda^{-\frac{1}{2}}\end{pmatrix}\partial_{\bar{z}}\overline{\tilde{\gamma}}\mathrm{d}\bar{z}. \tag{4.4.93}$$

As a consequence of the previous lemma, this is equivalent to

$$\begin{aligned}\tilde{\gamma}^*\zeta_\lambda &= \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_k\hat{k}_{\mathcal{T},\tilde{\gamma}} + \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\mathrm{d}\hat{k}_{\mathcal{T},\tilde{\gamma}} + \lambda^{-1}\begin{pmatrix}\lambda^{-\frac{1}{2}} & 0\\ 0 & \lambda^{\frac{1}{2}}\end{pmatrix}\hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_p'\hat{k}_{\mathcal{T},\tilde{\gamma}}\begin{pmatrix}\lambda^{\frac{1}{2}} & 0\\ 0 & \lambda^{-\frac{1}{2}}\end{pmatrix}\mathrm{d}z\\
&\quad + \lambda\begin{pmatrix}\lambda^{-\frac{1}{2}} & 0\\ 0 & \lambda^{\frac{1}{2}}\end{pmatrix}\hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_p''\hat{k}_{\mathcal{T},\tilde{\gamma}}\begin{pmatrix}\lambda^{\frac{1}{2}} & 0\\ 0 & \lambda^{-\frac{1}{2}}\end{pmatrix}\mathrm{d}\bar{z}\\
&= \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_k\hat{k}_{\mathcal{T},\tilde{\gamma}} + \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\mathrm{d}\hat{k}_{\mathcal{T},\tilde{\gamma}} + \lambda^{-1}\hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\begin{pmatrix}\lambda^{-\frac{1}{2}} & 0\\ 0 & \lambda^{\frac{1}{2}}\end{pmatrix}\zeta_p'\begin{pmatrix}\lambda^{\frac{1}{2}} & 0\\ 0 & \lambda^{-\frac{1}{2}}\end{pmatrix}\hat{k}_{\mathcal{T},\tilde{\gamma}}\mathrm{d}z\\
&\quad + \lambda\hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\begin{pmatrix}\lambda^{-\frac{1}{2}} & 0\\ 0 & \lambda^{\frac{1}{2}}\end{pmatrix}\zeta_p''\begin{pmatrix}\lambda^{\frac{1}{2}} & 0\\ 0 & \lambda^{-\frac{1}{2}}\end{pmatrix}\hat{k}_{\mathcal{T},\tilde{\gamma}}\mathrm{d}\bar{z} = \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_\lambda\hat{k}_{\mathcal{T},\tilde{\gamma}} + \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\mathrm{d}\hat{k}_{\mathcal{T},\tilde{\gamma}},\end{aligned} \tag{4.4.94}$$

where we have used that the occurring diagonal matrices commute. But this implies

$$\begin{aligned}(\tilde{P}_\lambda\hat{k}_{\mathcal{T},\tilde{\gamma}})^{-1}\mathrm{d}(\tilde{P}_\lambda\hat{k}_{\mathcal{T},\tilde{\gamma}}) &= \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\tilde{P}_\lambda^{-1}\mathrm{d}\tilde{P}_\lambda\hat{k}_{\mathcal{T},\tilde{\gamma}} + \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\mathrm{d}\hat{k}_{\mathcal{T},\tilde{\gamma}}\\
&= \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_\lambda\hat{k}_{\mathcal{T},\tilde{\gamma}} + \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\mathrm{d}\hat{k}_{\mathcal{T},\tilde{\gamma}} = \tilde{\gamma}^*\zeta_\lambda = (\tilde{P}_\lambda\circ\tilde{\gamma})^{-1}\mathrm{d}(\tilde{P}_\lambda\circ\tilde{\gamma}),\end{aligned} \tag{4.4.95}$$

which means that $\tilde{P}_\lambda\circ\tilde{\gamma}$ and $\tilde{P}_\lambda\hat{k}_{\mathcal{T},\tilde{\gamma}}$ solve the same differential equation and therefore only differ by a matrix $\tilde{M}_{\tilde{\gamma}}(\lambda)$ independent of z:

$$\tilde{P}_\lambda\circ\tilde{\gamma} = \tilde{M}_{\tilde{\gamma}}(\lambda)\tilde{P}_\lambda\hat{k}_{\mathcal{T},\tilde{\gamma}}, \tag{4.4.96}$$

which is the equation claimed in the first case. For $\lambda = 1$, we have $\tilde{P}_{\lambda=1} = \tilde{P}$ and thus obtain

$$\tilde{P}\circ\tilde{\gamma} = \tilde{M}_{\tilde{\gamma}}(1)\tilde{P}\hat{k}_{\mathcal{T},\tilde{\gamma}}. \tag{4.4.97}$$

Comparing this to (4.4.47), we infer that $\tilde{M}_{\tilde{\gamma}}(1) = \pm G(1)A_{\mathcal{T}}G(1)^{-1}$.

Let now $\mathcal{T}$ be orientation reversing. This implies that $\tilde{\gamma}$ is antiholomorphic and equation (4.4.92) reads as

$$\tilde{\gamma}^*\zeta_\lambda = \tilde{\gamma}^*\zeta_k + \lambda^{-1}\begin{pmatrix}\lambda^{-\frac{1}{2}} & 0\\ 0 & \lambda^{\frac{1}{2}}\end{pmatrix}(\zeta_p'\circ\tilde{\gamma})\begin{pmatrix}\lambda^{\frac{1}{2}} & 0\\ 0 & \lambda^{-\frac{1}{2}}\end{pmatrix}\partial_{\bar{z}}\tilde{\gamma}\mathrm{d}\bar{z} + \lambda\begin{pmatrix}\lambda^{-\frac{1}{2}} & 0\\ 0 & \lambda^{\frac{1}{2}}\end{pmatrix}(\zeta_p''\circ\tilde{\gamma})\begin{pmatrix}\lambda^{\frac{1}{2}} & 0\\ 0 & \lambda^{-\frac{1}{2}}\end{pmatrix}\partial_z\overline{\tilde{\gamma}}\mathrm{d}z. \tag{4.4.98}$$

Applying lemma 4.15 again, using the second part this time, we obtain

$$\begin{aligned}\tilde{\gamma}^*\zeta_\lambda &= \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_k\hat{k}_{\mathcal{T},\tilde{\gamma}} + \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\mathrm{d}\hat{k}_{\mathcal{T},\tilde{\gamma}} + \lambda^{-1}\begin{pmatrix}\lambda^{-\frac{1}{2}} & 0\\ 0 & \lambda^{\frac{1}{2}}\end{pmatrix}\hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_p''\hat{k}_{\mathcal{T},\tilde{\gamma}}\begin{pmatrix}\lambda^{\frac{1}{2}} & 0\\ 0 & \lambda^{-\frac{1}{2}}\end{pmatrix}\mathrm{d}\bar{z}\\
&\quad + \lambda\begin{pmatrix}\lambda^{-\frac{1}{2}} & 0\\ 0 & \lambda^{\frac{1}{2}}\end{pmatrix}\hat{k}_{\mathcal{T}}^{-1}\zeta_p'\hat{k}_{\mathcal{T},\tilde{\gamma}}\begin{pmatrix}\lambda^{\frac{1}{2}} & 0\\ 0 & \lambda^{-\frac{1}{2}}\end{pmatrix}\mathrm{d}z.\end{aligned} \tag{4.4.99}$$

Recalling that $\hat{k}_{\mathcal{T},\tilde{\gamma}}$ is, in the present case, given by (4.4.55), we verify by a direct computation the identity

$$\hat{k}_{\mathcal{T},\tilde{\gamma}}\begin{pmatrix}\lambda^{\frac{1}{2}} & 0\\ 0 & \lambda^{-\frac{1}{2}}\end{pmatrix} = \begin{pmatrix}\lambda^{-\frac{1}{2}} & 0\\ 0 & \lambda^{\frac{1}{2}}\end{pmatrix}\hat{k}_{\mathcal{T},\tilde{\gamma}}. \tag{4.4.100}$$

Consequently, we obtain

$$\begin{aligned}\tilde{\gamma}^*\zeta_\lambda &= \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_k\hat{k}_{\mathcal{T},\tilde{\gamma}} + \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\mathrm{d}\hat{k}_{\mathcal{T},\tilde{\gamma}} + \lambda^{-1}\hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\begin{pmatrix}\lambda^{\frac{1}{2}} & 0\\ 0 & \lambda^{-\frac{1}{2}}\end{pmatrix}\zeta_p''\begin{pmatrix}\lambda^{-\frac{1}{2}} & 0\\ 0 & \lambda^{\frac{1}{2}}\end{pmatrix}\hat{k}_{\mathcal{T},\tilde{\gamma}}\mathrm{d}\bar{z}\\
&\quad + \lambda\hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\begin{pmatrix}\lambda^{\frac{1}{2}} & 0\\ 0 & \lambda^{-\frac{1}{2}}\end{pmatrix}\zeta_p'\begin{pmatrix}\lambda^{-\frac{1}{2}} & 0\\ 0 & \lambda^{\frac{1}{2}}\end{pmatrix}\hat{k}_{\mathcal{T},\tilde{\gamma}}\mathrm{d}z = \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_{\lambda^{-1}}\hat{k}_{\mathcal{T},\tilde{\gamma}} + \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\mathrm{d}\hat{k}_{\mathcal{T},\tilde{\gamma}},\end{aligned} \tag{4.4.101}$$

and similarly as before

$$(\tilde{P}_\lambda \hat{k}_{\mathcal{T},\tilde{\gamma}})^{-1}\mathrm{d}(\tilde{P}_\lambda \hat{k}_{\mathcal{T},\tilde{\gamma}}) = \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\tilde{P}_\lambda^{-1}\mathrm{d}\tilde{P}_\lambda \hat{k}_{\mathcal{T},\tilde{\gamma}} + \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\mathrm{d}\hat{k}_{\mathcal{T},\tilde{\gamma}}$$
$$= \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\zeta_\lambda \hat{k}_{\mathcal{T},\tilde{\gamma}} + \hat{k}_{\mathcal{T},\tilde{\gamma}}^{-1}\mathrm{d}\hat{k}_{\mathcal{T},\tilde{\gamma}} = \tilde{\gamma}^*\zeta_{\lambda^{-1}} = (\tilde{P}_{\lambda^{-1}} \circ \tilde{\gamma})^{-1}\mathrm{d}(\tilde{P}_{\lambda^{-1}} \circ \tilde{\gamma}). \quad (4.4.102)$$

So by following verbatim the argument of the first case, we derive

$$\tilde{P}_{\lambda^{-1}} \circ \tilde{\gamma} = \tilde{M}_{\tilde{\gamma}}(\lambda)\tilde{P}_\lambda \hat{k}_{\mathcal{T},\tilde{\gamma}}, \quad (4.4.103)$$

where $\tilde{M}_{\tilde{\gamma}}(\lambda)$ is independent of z. Moreover, for $\lambda = 1$, we have $\tilde{P}_{\lambda^{-1}} = \tilde{P}_\lambda = \tilde{P}$ and thus obtain

$$\tilde{P} \circ \tilde{\gamma} = \tilde{M}_{\tilde{\gamma}}(1)\tilde{P}\hat{k}_{\mathcal{T},\tilde{\gamma}}. \quad (4.4.104)$$

Comparing this to (4.4.47), we infer that $\tilde{M}_{\tilde{\gamma}}(1) = \pm G(1)A_{\mathcal{T}}G(1)^{-1}$. □

Finally, we state in theorem 4.17 the transformation behaviour of the extended frame $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ with respect to the biholomorphic (resp. bi-antiholomorphic) mapping $\tilde{\gamma} : \tilde{M} \to \tilde{M}$ associated with the symmetry $\mathcal{T}$ of a given trinoid with properly embedded annular ends. In preparation of this, we define the matrix $k_{\mathcal{T},\tilde{\gamma}} \in \mathrm{SU}(2)$, which is independent of λ, by

$$k_{\mathcal{T},\tilde{\gamma}} := \begin{pmatrix} \overline{\sqrt{A+iB}} & 0 \\ 0 & \sqrt{A+iB} \end{pmatrix}, \quad (4.4.105)$$

where $A, B : \tilde{M} \to \mathbb{R}$ depend on $\tilde{\gamma}$ and are explicitly given by the equations (4.4.40) and (4.4.41). Moreover, the occurring complex square roots are defined as in remark 4.14.

Theorem 4.17. *Let $\psi : \tilde{M} \to \mathbb{R}^3$ be a conformal CMC-immersion, which is symmetric with respect to $\mathcal{T} : x \mapsto \mathcal{A}_{\mathcal{T}}x + t_{\mathcal{T}}$. Assume ψ corresponds to a trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends via the universal covering $\pi : \tilde{M} \to M$, $\psi = \phi \circ \pi$. Let $\tilde{\gamma}$ denote a biholomorphic (resp. bi-antiholomorphic) mapping $\tilde{M} \to \tilde{M}$ associated with $\mathcal{T}$ by theorem 4.9. Then the extended frame $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ corresponding to ψ by theorem 4.5 transforms under $\tilde{\gamma}$ as follows.*

1. *If $\mathcal{T}$ preserves orientation, then*

$$F(\tilde{\gamma}(z), \lambda) = M_{\tilde{\gamma}}(\lambda)F(z, \lambda)k_{\mathcal{T},\tilde{\gamma}}(z), \quad (4.4.106)$$

where $k_{\mathcal{T},\tilde{\gamma}}$ is given in (4.4.105) and $M_{\tilde{\gamma}}(\lambda)$ denotes an element of $\Lambda\mathrm{SU}(2)_\sigma$, which is independent of z. In particular, for $\lambda = 1$, we have

$$M_{\tilde{\gamma}}(1) = \pm A_{\mathcal{T}}, \quad (4.4.107)$$

where $A_{\mathcal{T}} \in \mathrm{SU}(2)$ denotes the conjugation matrix realizing $\mathcal{A}_{\mathcal{T}} \in \mathrm{O}(3)$ in the $\mathrm{su}(2)$-model.

2. *If $\mathcal{T}$ reverses orientation, then*

$$F(\tilde{\gamma}(z), \lambda^{-1}) = M_{\tilde{\gamma}}(\lambda)\overline{F(z, \lambda)}k_{\mathcal{T},\tilde{\gamma}}(z), \quad (4.4.108)$$

where $k_{\mathcal{T},\tilde{\gamma}}$ is given in (4.4.105) and $M_{\tilde{\gamma}}(\lambda)$ denotes an element of $\Lambda\mathrm{SU}(2)_\sigma$, which is independent of z. In particular, for $\lambda = 1$, we have

$$M_{\tilde{\gamma}}(1) = \pm A_{\mathcal{T}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (4.4.109)$$

where $A_{\mathcal{T}} \in \mathrm{SU}(2)$ denotes the conjugation matrix realizing $\mathcal{A}_{\mathcal{T}} \in \mathrm{O}(3)$ in the $\mathrm{su}(2)$-model.

Proof. By construction of F, we have $F = G(\lambda)^{-1}\tilde{P}_\lambda G(\lambda)$. There are two cases to consider. In the case that $\mathcal{T}$ preserves orientation, we use the above lemma and obtain

$$F(\tilde{\gamma}(z), \lambda) = G(\lambda)^{-1}((\tilde{P}_\lambda \circ \tilde{\gamma})(z))G(\lambda) = G(\lambda)^{-1}\tilde{M}_{\tilde{\gamma}}(\lambda)\tilde{P}_\lambda(z)\hat{k}_{\mathcal{T},\tilde{\gamma}}(z)G(\lambda)$$
$$= G(\lambda)^{-1}\tilde{M}_{\tilde{\gamma}}(\lambda)G(\lambda)F(z, \lambda)G(\lambda)^{-1}\hat{k}_{\mathcal{T},\tilde{\gamma}}(z)G(\lambda). \quad (4.4.110)$$

As $\mathcal{T}$ is orientation preserving, we have by (4.2.17) and (4.4.54)

$$G(\lambda)^{-1}\hat{k}_{\mathcal{T},\tilde{\gamma}}G(\lambda) = \begin{pmatrix} \overline{\sqrt{A+iB}} & 0 \\ 0 & \sqrt{A+iB} \end{pmatrix} = k_{\mathcal{T},\tilde{\gamma}}, \tag{4.4.111}$$

and by defining $M_{\tilde{\gamma}}(\lambda) := G(\lambda)^{-1}\tilde{M}_{\tilde{\gamma}}(\lambda)G(\lambda)$, we altogether obtain $F(\tilde{\gamma}(z),\lambda) = M_{\tilde{\gamma}}(\lambda)F(z,\lambda)k_{\mathcal{T},\tilde{\gamma}}(z)$, which proves (4.4.106). Clearly, $M_{\tilde{\gamma}}(\lambda)$ is independent of z, and furthermore, as $F(\tilde{\gamma}(z),\lambda)$, $F(z,\lambda)$ and $k_{\mathcal{T},\tilde{\gamma}}(z)$ are elements of $\Lambda\mathrm{SU}(2)_\sigma$, so is $M_{\tilde{\gamma}}(\lambda)$. Finally, for $\lambda = 1$, we infer from (4.4.86) that

$$M_{\tilde{\gamma}}(1) = G(1)^{-1}\tilde{M}_{\tilde{\gamma}}(1)G(1) = \pm G(1)^{-1}G(1)\mathcal{A}_{\mathcal{T}}G(1)^{-1}G(1) = \pm\mathcal{A}_{\mathcal{T}}. \tag{4.4.112}$$

The second case to consider is the case of orientation reversing $\mathcal{T}$. Using the above lemma, we obtain analogously

$$\begin{aligned} F(\tilde{\gamma}(z),\lambda^{-1}) &= G(\lambda^{-1})^{-1}((\tilde{P}_{\lambda^{-1}}\circ\tilde{\gamma})(z))G(\lambda^{-1}) \\ &= G(\lambda^{-1})^{-1}\tilde{M}_{\tilde{\gamma}}(\lambda)\tilde{P}_\lambda(z)\hat{k}_{\mathcal{T},\tilde{\gamma}}(z)G(\lambda^{-1}) = G(\lambda^{-1})^{-1}\tilde{M}_{\tilde{\gamma}}(\lambda)G(\lambda)F(z,\lambda)G(\lambda)^{-1}\hat{k}_{\mathcal{T},\tilde{\gamma}}(z)G(\lambda^{-1}) \\ &= G(\lambda^{-1})^{-1}\tilde{M}_{\tilde{\gamma}}(\lambda)G(\lambda)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}F(z,\lambda)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}G(\lambda)^{-1}\hat{k}_{\mathcal{T},\tilde{\gamma}}(z)G(\lambda^{-1}). \end{aligned} \tag{4.4.113}$$

This time we consider equations (4.2.17) and (4.4.55) to obtain

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}G(\lambda)^{-1}\hat{k}_{\mathcal{T},\tilde{\gamma}}G(\lambda^{-1}) = \begin{pmatrix} \overline{\sqrt{A+iB}} & 0 \\ 0 & \sqrt{A+iB} \end{pmatrix} = k_{\mathcal{T},\tilde{\gamma}}. \tag{4.4.114}$$

Moreover, we have

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}F(z,\lambda)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \overline{F(z,\lambda)}, \tag{4.4.115}$$

and by defining $M_{\tilde{\gamma}}(\lambda) := G(\lambda^{-1})^{-1}\tilde{M}_{\tilde{\gamma}}(\lambda)G(\lambda)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we arrive at equation (4.4.108). $M_{\tilde{\gamma}}(\lambda)$ is independent of z and, as $F(\tilde{\gamma}(z),\lambda^{-1})$, $\overline{F(z,\lambda)}$ and $k_{\mathcal{T},\tilde{\gamma}}$ are elements of $\Lambda\mathrm{SU}(2)_\sigma$, so is $M_{\tilde{\gamma}}(\lambda)$. Finally, for $\lambda = 1$, we infer from (4.4.88) that

$$M_{\tilde{\gamma}}(1) = G(1)^{-1}\tilde{M}_{\tilde{\gamma}}(1)G(1)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \pm G(1)^{-1}G(1)\mathcal{A}_{\mathcal{T}}G(1)^{-1}G(1)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \pm\mathcal{A}_{\mathcal{T}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.4.116}$$

□

As we are interested in explicitly computing the matrix $k_{\mathcal{T},\tilde{\gamma}}$, we state it in a more convenient form, which involves more directly the mapping $\tilde{\gamma}$ associated with the symmetry $\mathcal{T}$ by theorem 4.9.

Lemma 4.18. *Let $\psi : \tilde{M} \to \mathbb{R}^3$ be a conformal CMC-immersion, which corresponds to a trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends via the universal covering $\pi : \tilde{M} \to M$, $\psi = \phi\circ\pi$. Moreover, let $\mathcal{T} \in \mathrm{Sym}(\psi(\tilde{M}))$ be given by $\mathcal{T} : x \mapsto \mathcal{A}_{\mathcal{T}}x + t_{\mathcal{T}}$. Furthermore, let $\tilde{\gamma}$ denote a biholomorphic (resp. bi-antiholomorphic) mapping $\tilde{M} \to \tilde{M}$ associated with $\mathcal{T}$ by theorem 4.9. Then, the (λ-independent) matrix $k_{\mathcal{T},\tilde{\gamma}}$ defined in* (4.4.105) *satisfies*

$$k_{\mathcal{T},\tilde{\gamma}} = \begin{pmatrix} \overline{\sqrt{\frac{\partial_z\tilde{\gamma}}{|\partial_z\tilde{\gamma}|}}} & 0 \\ 0 & \sqrt{\frac{\partial_z\tilde{\gamma}}{|\partial_z\tilde{\gamma}|}} \end{pmatrix} \quad \text{if } \mathcal{T} \text{ preserves orientation}, \tag{4.4.117}$$

$$k_{\mathcal{T},\tilde{\gamma}} = \begin{pmatrix} \overline{\sqrt{\frac{\partial_{\bar{z}}\tilde{\gamma}}{|\partial_{\bar{z}}\tilde{\gamma}|}}} & 0 \\ 0 & \sqrt{\frac{\partial_{\bar{z}}\tilde{\gamma}}{|\partial_{\bar{z}}\tilde{\gamma}|}} \end{pmatrix} \quad \text{if } \mathcal{T} \text{ reverses orientation}, \tag{4.4.118}$$

where we write for ease of notation $\partial_z\tilde{\gamma}$ (resp. $\partial_{\bar{z}}\tilde{\gamma}$) for $\frac{\partial\tilde{\gamma}}{\partial z}$ (resp. $\frac{\partial\tilde{\gamma}}{\partial\bar{z}}$).

Proof. We know from (4.4.105) that

$$k_{\mathcal{T},\tilde{\gamma}} = \begin{pmatrix} \sqrt{A+iB} & 0 \\ 0 & \sqrt{A+iB} \end{pmatrix}, \tag{4.4.119}$$

where A and B are given in (4.4.40) and (4.4.41). Recalling $\tilde{\gamma} = \tilde{\gamma}_1 + i\tilde{\gamma}_2$, we infer

$$A + iB = \frac{\frac{\partial \tilde{\gamma}_1}{\partial x}(x+iy) + i\frac{\partial \tilde{\gamma}_2}{\partial x}(x+iy)}{\sqrt{\left(\frac{\partial \tilde{\gamma}_1}{\partial x}(x+iy)\right)^2 + \left(\frac{\partial \tilde{\gamma}_2}{\partial x}(x+iy)\right)^2}} = \frac{\frac{\partial \tilde{\gamma}}{\partial x}}{\mid \frac{\partial \tilde{\gamma}}{\partial x} \mid}. \tag{4.4.120}$$

As $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$ and, moreover, $\frac{\partial \tilde{\gamma}}{\partial \bar{z}} = 0$ (resp. $\frac{\partial \tilde{\gamma}}{\partial z} = 0$) for holomorphic (resp. antiholomorphic) $\tilde{\gamma}$, i.e. in the case of an orientation preserving (resp. reversing) symmetry $\mathcal{T}$, we conclude that

$$A + iB = \frac{\partial_z \tilde{\gamma}}{\mid \partial_z \tilde{\gamma} \mid} \quad \text{if } \mathcal{T} \text{ preserves orientation,} \tag{4.4.121}$$

$$A + iB = \frac{\partial_{\bar{z}} \tilde{\gamma}}{\mid \partial_{\bar{z}} \tilde{\gamma} \mid} \quad \text{if } \mathcal{T} \text{ reverses orientation.} \tag{4.4.122}$$

By (4.4.119), the claim follows. □

As a special case of theorem 4.17 we formulate the following corollary, which states how the extended frame F transforms under a covering transformation $\tilde{\gamma}$ on $\tilde{M}$. This result is obtained by setting $\mathcal{T} = \mathcal{I} : \mathbb{R}^3 \to \mathbb{R}^3$, $x \mapsto \mathcal{I}(x) = x$ in theorem 4.17 and interpreting $\tilde{\gamma}$ as a (biholomorphic) mapping linked to $\mathcal{T}$ by theorem 4.9. (As, by definition, $\pi \circ \tilde{\gamma} = \pi$, we have obviously $\mathcal{T} \circ \psi = \psi = \phi \circ \pi = \phi \circ \pi \circ \tilde{\gamma} = \psi \circ \tilde{\gamma}$.)

Corollary 4.19. *Let $\psi : \tilde{M} \to \mathbb{R}^3$ be a conformal CMC-immersion, which corresponds to a trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends via the universal covering $\pi : \tilde{M} \to M$, $\psi = \phi \circ \pi$. Let $\tilde{\gamma}$ denote a covering transformation on $\tilde{M}$, i.e. a biholomorphic mapping $\tilde{M} \to \tilde{M}$ associated by theorem 4.9 with the identity mapping $\mathcal{I} : \mathbb{R}^3 \to \mathbb{R}^3$ (interpreted as a symmetry of ψ). Then the extended frame $F : \tilde{M} \to \Lambda \mathrm{SU}(2)_\sigma$ corresponding to ψ by theorem 4.5 transforms under $\tilde{\gamma}$ as follows:*

$$F(\tilde{\gamma}(z), \lambda) = M_{\tilde{\gamma}}(\lambda) F(z, \lambda) k_{\mathcal{I},\tilde{\gamma}}(z), \tag{4.4.123}$$

where $k_{\mathcal{I},\tilde{\gamma}}$ is given in (4.4.105) *with $\mathcal{T} = \mathcal{I}$ and $M_{\tilde{\gamma}}(\lambda)$ denotes an element of $\Lambda \mathrm{SU}(2)_\sigma$, which is independent of z.*

4.5 The extended frame monodromy relations

We apply corollary 4.19 to the covering transformations $\tilde{\gamma}_j$, $j = 0, 1, \infty$, on $\tilde{M}$ from section 3.3. Denoting the corresponding matrices $k_{\mathcal{I},\tilde{\gamma}_0}$, $k_{\mathcal{I},\tilde{\gamma}_1}$ and $k_{\mathcal{I},\tilde{\gamma}_\infty}$ in equation (4.4.123) of corollary 4.19 by k_0, k_1 and k_∞, respectively, we obtain for $j = 0, 1, \infty$:

$$F(\tilde{\gamma}_j(z), \lambda) = M_{\tilde{\gamma}_j}(\lambda) F(z, \lambda) k_j(z), \tag{4.5.1}$$

where $M_{\tilde{\gamma}_j}(\lambda)$ denotes an element of $\Lambda \mathrm{SU}(2)_\sigma$, which is independent of z.

The matrices $k_j = k_{\mathcal{I},\tilde{\gamma}_j}$, $j = 0, 1, \infty$, are given in (4.4.105) with $\mathcal{T} = \mathcal{I}$ and $\tilde{\gamma} = \tilde{\gamma}_j$. We compute k_0, k_1 and k_∞ explicitly in the following. First, recall from (3.3.4), (3.3.5) and (3.3.6) that

$$\tilde{\gamma}_0(z) = \frac{z}{-2z+1}, \tag{4.5.2}$$

$$\tilde{\gamma}_1(z) = z + 2, \tag{4.5.3}$$

$$\tilde{\gamma}_\infty(z) = \frac{-3z-2}{2z+1}. \tag{4.5.4}$$

Thus, we have

$$\partial_z \tilde{\gamma}_0 = \frac{1}{(1-2z)^2}, \quad \partial_z \tilde{\gamma}_1 = 1, \quad \partial_z \tilde{\gamma}_\infty = \frac{1}{(1+2z)^2}, \tag{4.5.5}$$

and, consequently,

$$\mid \partial_z \tilde{\gamma}_0 \mid = \frac{1}{(1-2z)(1-2\bar{z})}, \quad \mid \partial_z \tilde{\gamma}_1 \mid = 1, \quad \mid \partial_z \tilde{\gamma}_\infty \mid = \frac{1}{(1+2z)(1+2\bar{z})}. \tag{4.5.6}$$

Using lemma 4.18, we can easily compute k_0, k_1 and k_∞ from equation (4.4.117):

$$k_0(z) = \begin{pmatrix} \sqrt{\overline{\sqrt{\frac{1-2\bar{z}}{1-2z}}}} & 0 \\ 0 & \sqrt{\frac{1-2\bar{z}}{1-2z}} \end{pmatrix}, \tag{4.5.7}$$

$$k_1(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.5.8}$$

$$k_\infty(z) = \begin{pmatrix} \sqrt{\overline{\sqrt{\frac{1+2\bar{z}}{1+2z}}}} & 0 \\ 0 & \sqrt{\frac{1+2\bar{z}}{1+2z}} \end{pmatrix}. \tag{4.5.9}$$

Equation (4.5.1) states the transformation behaviour of F under the covering transformation $\tilde{\gamma}_j$ on $\tilde{M}$. Recall that, at the same time, we have for our dressed solution $\hat{\Psi} = T\Psi$ to equation (2.4.1) the monodromy relation

$$\hat{\Psi}(\tilde{\gamma}_j(z), \lambda) = \hat{M}_j(\lambda)\hat{\Psi}(z, \lambda), \tag{4.5.10}$$

where $\hat{M}_j(\lambda)$ denotes an element of $\Lambda\mathrm{SU}(2)_\sigma$, which is independent of z, and which is of the form (3.9.26). As explicated in remark 2.10, this relation carries over to the extended frame F corresponding to $\hat{\Psi}$:

$$F(\tilde{\gamma}_j(z), \lambda) = \hat{M}_j(\lambda)F(z, \lambda)k(z, \tilde{\gamma}_j), \tag{4.5.11}$$

where k denotes a diagonal matrix in SU(2), which is independent of λ. Like equation (4.5.1) before, equation (4.5.11) states *as well* the transformation behaviour of F under the covering transformation $\tilde{\gamma}_j$ on $\tilde{M}$.

Thus, in the terminology of [14], both $(\tilde{\gamma}_j, M_{\tilde{\gamma}_j})$ and $(\tilde{\gamma}_j, \hat{M}_j)$ define symmetries of the extended frame F. However, by theorem 2.1 of [14], this implies that $M_{\tilde{\gamma}_j}$ and $\hat{M}_j$ differ at most by a sign:

$$M_{\tilde{\gamma}_j}(\lambda) = \alpha_j \hat{M}_j(\lambda), \tag{4.5.12}$$

where $\alpha_j \in \{\pm 1\}$. Inserting this relation into equation (4.5.1), we obtain the following result:

Theorem 4.20. *Let $\tilde{M} = \mathbb{H}$ and $\psi : \tilde{M} \to \mathbb{R}^3$ be a conformal CMC-immersion, which corresponds to a trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends via the universal covering $\pi : \tilde{M} \to M$, $\psi = \phi \circ \pi$. Let $\tilde{\gamma}_j$, $j = 0, 1, \infty$, denote the covering transformations on $\tilde{M}$ from section 3.3. Then, the extended frame $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ corresponding to ψ by theorem 4.5 transforms under $\tilde{\gamma}_j$ as follows:*

$$F(\tilde{\gamma}_j(z), \lambda) = \alpha_j \hat{M}_j(\lambda)F(z, \lambda)k_j(z), \tag{4.5.13}$$

where $\alpha_j \in \{\pm 1\}$, the matrices $\hat{M}_j(\lambda)$ are of the form (3.9.26) *and the matrices $k_j(z)$ are given by equations* (4.5.7) *to* (4.5.9).

4.6 Trinoid symmetries

In this section, we investigate in detail the possible symmetries of a trinoid with properly embedded annular ends.

In the following, let $M = \mathbb{C} \setminus \{0, 1\}$ and $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends, which is symmetric with respect to an Euclidean motion $\mathcal{T} \in \mathrm{Sym}(\phi(M))$, i.e.

$$\mathcal{T}(\phi(M)) = \phi(M). \tag{4.6.1}$$

By theorem 4.9, there exists a unique biholomorhic (resp. bi-antiholomorhic) mapping $\gamma = \gamma(\mathcal{T}) : M \to M$ satisfying $\mathcal{T} \circ \phi = \phi \circ \gamma$. In fact, we observe that the correspondence between $\mathcal{T}$ and γ is one-to-one: For each $\tilde{\mathcal{T}} \in \mathrm{Sym}(\phi(M))$, which also satisfies $\tilde{\mathcal{T}} \circ \phi = \phi \circ \gamma$ (for the same γ), we have necessarily $\tilde{\mathcal{T}}|_{\phi(M)} \equiv \mathcal{T}|_{\phi(M)}$ and thus $\tilde{\mathcal{T}} \equiv \mathcal{T}$ (on $\mathbb{R}^3$).

The following lemma explicitly lists all biholomorhic (resp. bi-antiholomorhic) mappings $\gamma : M \to M$.

Lemma 4.21. *Let $M = \mathbb{C} \setminus \{0, 1\}$ and* $\mathrm{Aut}(M) = \{\gamma : M \to M; \gamma$ *biholomorphic*$\}$, $\mathrm{Aut}^*(M) = \{\gamma : M \to M; \gamma$ *bi-antiholomorphic*$\}$. *Then, the following holds:*

1.

$$\mathrm{Aut}(M) = \{\gamma_{(\ )}, \gamma_{(1\ \infty)}, \gamma_{(0\ \infty)}, \gamma_{(0\ 1)}, \gamma_{(0\ 1\ \infty)}, \gamma_{(0\ \infty\ 1)}\}, \tag{4.6.2}$$

where

$$\gamma_{(\ )} : M \to M, \quad \gamma_{(\ )}(z) = z, \tag{4.6.3}$$

$$\gamma_{(1\ \infty)} : M \to M, \quad \gamma_{(1\ \infty)}(z) = \frac{z}{z-1}, \tag{4.6.4}$$

$$\gamma_{(0\ \infty)} : M \to M, \quad \gamma_{(0\ \infty)}(z) = \frac{1}{z}, \tag{4.6.5}$$

$$\gamma_{(0\ 1)} : M \to M, \quad \gamma_{(0\ 1)}(z) = 1 - z, \tag{4.6.6}$$

$$\gamma_{(0\ 1\ \infty)} : M \to M, \quad \gamma_{(0\ 1\ \infty)}(z) = \frac{1}{1-z}, \tag{4.6.7}$$

$$\gamma_{(0\ \infty\ 1)} : M \to M, \quad \gamma_{(0\ \infty\ 1)}(z) = \frac{z-1}{z}. \tag{4.6.8}$$

2.

$$\mathrm{Aut}^*(M) = \{\gamma_{(\ )}, \gamma_{(1\ \infty)}, \gamma_{(0\ \infty)}, \gamma_{(0\ 1)}, \gamma_{(0\ 1\ \infty)}, \gamma_{(0\ \infty\ 1)}\}, \tag{4.6.9}$$

where

$$\gamma^*_{(\ )} : M \to M, \quad \gamma^*_{(\ )}(z) = \bar{z}, \tag{4.6.10}$$

$$\gamma^*_{(1\ \infty)} : M \to M, \quad \gamma^*_{(1\ \infty)}(z) = \frac{\bar{z}}{\bar{z}-1}, \tag{4.6.11}$$

$$\gamma^*_{(0\ \infty)} : M \to M, \quad \gamma^*_{(0\ \infty)}(z) = \frac{1}{\bar{z}}, \tag{4.6.12}$$

$$\gamma^*_{(0\ 1)} : M \to M, \quad \gamma^*_{(0\ 1)}(z) = 1 - \bar{z}, \tag{4.6.13}$$

$$\gamma^*_{(0\ 1\ \infty)} : M \to M, \quad \gamma^*_{(0\ 1\ \infty)}(z) = \frac{1}{1-\bar{z}}, \tag{4.6.14}$$

$$\gamma^*_{(0\ \infty\ 1)} : M \to M, \quad \gamma^*_{(0\ \infty\ 1)}(z) = \frac{\bar{z}-1}{\bar{z}}. \tag{4.6.15}$$

Remark 4.22. The notation introduced in the lemma above for the different biholomorhic (resp. bi-antiholomorhic) mappings $\gamma : M \to M$ is motivated by the way each γ (extended to a biholomorhic or bi-antiholomorhic mapping $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$) permutes the set $\{0, 1, \infty\}$. This is explained in more detail in the following proof.

Proof of lemma 4.21. Let $\gamma \in \mathrm{Aut}(M)$ (resp. $\gamma \in \mathrm{Aut}^*(M)$). Since γ defines a biholomorphic (resp. bi-antiholomorphic) mapping $M \to M$, it can be uniquely extended to a biholomorphic (resp. bi-antiholomorphic) mapping $\gamma_{\mathrm{extd}} : \hat{C} \to \hat{C}$ such that

$$\gamma_{\mathrm{extd}}|_M = \gamma. \tag{4.6.16}$$

Since γ_{extd} is bijective, we necessarily have

$$\gamma_{\mathrm{extd}}(\{0, 1, \infty\}) = \{0, 1, \infty\}, \tag{4.6.17}$$

i.e. γ_{extd} permutes the set $\{0, 1, \infty\}$ according to an appropriate permutation σ of $\{0, 1, \infty\}$:

$$\gamma_{\mathrm{extd}}(z_j) = z_{\sigma(j)} \tag{4.6.18}$$

for all $z_j = j \in \{0, 1, \infty\}$.

It is a well known result of complex analysis that γ_{extd} is of the form

$$\gamma_{\mathrm{extd}} : z \mapsto \frac{az+b}{cz+d} \tag{4.6.19}$$

with complex parameters a, b, c, d satisfying $ad - bc \neq 0$ in the case that γ (and thus γ_{extd}) is biholomorphic, and of the form

$$\gamma_{\mathrm{extd}} : z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d} \tag{4.6.20}$$

with complex parameters a, b, c, d satisfying $ad - bc \neq 0$ in the case that γ (and thus γ_{extd}) is bi-antiholomorphic. Evaluating the relation (4.6.18) for $j = 0, 1, \infty$ in (4.6.19) and (4.6.20) (of course, with respect to σ), the parameters a, b, c, d can be explicitly computed (up to a common complex scale, which cancels in (4.6.19) and (4.6.20), respectively). In view of (4.6.16), we obtain the possible explicit forms of $\gamma = \gamma_\sigma$ (resp. $\gamma = \gamma^*_\sigma$) given in equations (4.6.3) to (4.6.8) and equations (4.6.10) to (4.6.15), respectively.

This proves the relations "$\subseteq$" in the two claimed identities above. The relations "$\supseteq$" are trivial. □

Corresponding to the elements $\gamma \in \mathrm{Aut}(M)$ (resp. $\gamma \in \mathrm{Aut}^*(M)$), we define the following auxiliary functions $h : M \to \mathbb{C} \setminus \{0\}$, which are holomorphic (resp. antiholomorphic) in M:

Definition 4.23.

$$h_{(\ )} : M \to \mathbb{C} \setminus \{0\}, \quad h_{(\ )}(z) = 1, \tag{4.6.21}$$
$$h_{(1\ \infty)} : M \to \mathbb{C} \setminus \{0\}, \quad h_{(1\ \infty)}(z) = -i(z-1), \tag{4.6.22}$$
$$h_{(0\ \infty)} : M \to \mathbb{C} \setminus \{0\}, \quad h_{(0\ \infty)}(z) = -iz, \tag{4.6.23}$$
$$h_{(0\ 1)} : M \to \mathbb{C} \setminus \{0\}, \quad h_{(0\ 1)}(z) = -i, \tag{4.6.24}$$
$$h_{(0\ 1\ \infty)} : M \to \mathbb{C} \setminus \{0\}, \quad h_{(0\ 1\ \infty)}(z) = 1 - z, \tag{4.6.25}$$
$$h_{(0\ \infty\ 1)} : M \to \mathbb{C} \setminus \{0\}, \quad h_{(0\ \infty\ 1)}(z) = z, \tag{4.6.26}$$

$$h^*_{(\ )} : M \to \mathbb{C} \setminus \{0\}, \quad h^*_{(\ )}(z) = 1, \tag{4.6.27}$$
$$h^*_{(1\ \infty)} : M \to \mathbb{C} \setminus \{0\}, \quad h^*_{(1\ \infty)}(z) = -i(\bar{z}-1), \tag{4.6.28}$$
$$h^*_{(0\ \infty)} : M \to \mathbb{C} \setminus \{0\}, \quad h^*_{(0\ \infty)}(z) = -i\bar{z}, \tag{4.6.29}$$
$$h^*_{(0\ 1)} : M \to \mathbb{C} \setminus \{0\}, \quad h^*_{(0\ 1)}(z) = -i, \tag{4.6.30}$$
$$h^*_{(0\ 1\ \infty)} : M \to \mathbb{C} \setminus \{0\}, \quad h^*_{(0\ 1\ \infty)}(z) = 1 - \bar{z}, \tag{4.6.31}$$
$$h^*_{(0\ \infty\ 1)} : M \to \mathbb{C} \setminus \{0\}, \quad h^*_{(0\ \infty\ 1)}(z) = \bar{z}. \tag{4.6.32}$$

Lemma 4.24. *Let σ be a permutation of the set $\{0, 1, \infty\}$. Denote by γ_σ (resp. γ^*_σ) the element of* $\mathrm{Aut}(M)$ *(resp. of* $\mathrm{Aut}^*(M)$*) corresponding to σ as in lemma 4.21. Moreover, denote by h_σ (resp. h^*_σ) the auxiliary function corresponding to σ as defined in definition 4.23. Then, for all $z \in M$, the following holds:*

1.

$$\partial_{zz} h_s(z) = 0, \quad \partial_{\bar{z}\bar{z}} h^*_s(z) = 0. \tag{4.6.33}$$

2.

$$(h_s(z))^2 = \frac{1}{\partial_z \gamma_\sigma(z)}, \quad (h^*_s(z))^2 = \frac{1}{\partial_{\bar{z}} \gamma^*_\sigma(z)}. \tag{4.6.34}$$

Proof. This is proved by direct computation. □

Lemma 4.25. *Let $\eta = \eta(z, \lambda)$ be a standardized trinoid potential on $M = \mathbb{C} \setminus \{0, 1, \infty\}$ associated with three off-diagonal Delaunay matrices D_0, D_1, D_∞ possessing the eigenvalues $\pm\mu_j(\lambda)$, respectively,*

$$\eta = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q(z,\lambda) & 0 \end{pmatrix} \mathrm{d}z, \tag{4.6.35}$$

where

$$Q(z,\lambda) = \frac{b_0(\lambda)}{z^2} + \frac{b_1(\lambda)}{(z-1)^2} + \frac{c_0(\lambda)}{z} + \frac{c_1(\lambda)}{z-1}, \tag{4.6.36}$$

and the functions b_j, c_j satisfy equations (3.6.4), (3.6.5) *and* (3.6.6).

1. *Let σ be a permutation of the set $\{0, 1, \infty\}$. Denote by $\gamma := \gamma_\sigma$ the element of* $\mathrm{Aut}(M)$ *corresponding to σ as in lemma 4.21. Moreover, denote by $h := h_\sigma$ the auxiliary function corresponding to σ as given in definition 4.23 and define $W_+ : M \to \Lambda^+\mathrm{SL}(2,\mathbb{C})_\sigma$ by*

$$W_+(z,\lambda) = \begin{pmatrix} h(z) & 0 \\ -\lambda \partial_z h(z) & (h(z))^{-1} \end{pmatrix}. \tag{4.6.37}$$

Then, we have

$$\gamma^*\eta = \eta\#W_+ \quad \Longleftrightarrow \quad Q(\gamma(z),\lambda) = (h(z))^4 Q(z,\lambda), \tag{4.6.38}$$

where $\gamma^*\eta$ *denotes the transform of* η *under* γ *and* $\eta\#W_+$ *denotes the gauged potential* $W_+^{-1}\eta W_+ + W_+^{-1}\mathrm{d}W_+$*.*

2. *Let* σ *be a permutation of the set* $\{0,1,\infty\}$*. Denote by* $\gamma := \gamma_\sigma^*$ *the element of* $\mathrm{Aut}^*(M)$ *corresponding to* σ *as in lemma 4.21. Moreover, denote by* $h := h_\sigma^*$ *the auxiliary function corresponding to* σ *as given in definition 4.23 and define* $W_+ : M \to \Lambda^+\mathrm{SL}(2,\mathbb{C})_\sigma$ *by*

$$W_+(z,\lambda) = \begin{pmatrix} h(z) & 0 \\ -\lambda\partial_{\bar z}h(z) & (h(z))^{-1}\end{pmatrix}. \tag{4.6.39}$$

Then, we have

$$\gamma^*\eta(z,\lambda) = \overline{\eta(z,\lambda^{-1})}\#W_+ \quad \Longleftrightarrow \quad Q(\gamma(z),\lambda) = (h(z))^4\overline{Q(z,\lambda^{-1})}, \tag{4.6.40}$$

where $\gamma^*\eta$ *denotes the transform of* η *under* γ *and* $\overline{\eta(z,\lambda^{-1})}\#W_+$ *denotes the gauged potential* $W_+^{-1}\overline{\eta(z,\lambda^{-1})}W_+ + W_+^{-1}\mathrm{d}W_+$*.*

Proof. We start with the proof of the first case. Using (4.6.34) from lemma 4.24, we have

$$\gamma^*\eta = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q(\gamma(z),\lambda) & 0\end{pmatrix}\partial_z\gamma\mathrm{d}z = \begin{pmatrix} 0 & \lambda^{-1}h^{-2} \\ -\lambda h^{-2}Q(\gamma(z),\lambda) & 0\end{pmatrix}\mathrm{d}z. \tag{4.6.41}$$

Furthermore, using (4.6.33) from lemma 4.24, we compute

$$\begin{aligned}\eta\#W_+ &= W_+^{-1}\eta W_+ + W_+^{-1}\mathrm{d}W_+ \\ &= \begin{pmatrix} h^{-1} & 0 \\ \lambda\partial_z h & h\end{pmatrix}\begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q(z,\lambda) & 0\end{pmatrix}\begin{pmatrix} h & 0 \\ -\lambda\partial_z h & h^{-1}\end{pmatrix}\mathrm{d}z + \begin{pmatrix} h^{-1} & 0 \\ \lambda\partial_z h & h\end{pmatrix}\begin{pmatrix}\partial_z h & 0 \\ 0 & -h^{-2}\partial_z h\end{pmatrix}\mathrm{d}z \\ &= \begin{pmatrix} -h^{-1}\partial_z h & \lambda^{-1}h^{-2} \\ -\lambda h^2 Q - \lambda(\partial_z h)^2 & h^{-1}\partial_z h\end{pmatrix}\mathrm{d}z + \begin{pmatrix} h^{-1}\partial_z h & 0 \\ \lambda(\partial_z h)^2 & -h^{-1}\partial_z h\end{pmatrix}\mathrm{d}z = \begin{pmatrix} 0 & \lambda^{-1}h^{-2} \\ -\lambda h^2 Q & 0\end{pmatrix}\mathrm{d}z.\end{aligned} \tag{4.6.42}$$

Together, this proves (4.6.38).

We now turn to the second case. Using again (4.6.34) from lemma 4.24, we have

$$\gamma^*\eta = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q(\gamma(z),\lambda) & 0\end{pmatrix}\partial_{\bar z}\gamma(z)\mathrm{d}\bar z = \begin{pmatrix} 0 & \lambda^{-1}h^{-2} \\ -\lambda h^{-2}Q(\gamma(z),\lambda) & 0\end{pmatrix}\mathrm{d}\bar z. \tag{4.6.43}$$

Furthermore, using again (4.6.33) from lemma 4.24, we compute ($\lambda \in S^1$)

$$\begin{aligned}\overline{\eta(z,\lambda^{-1})}\#W_+ &= W_+^{-1}\overline{\eta(z,\lambda^{-1})}W_+ + W_+^{-1}\mathrm{d}W_+ \\ &= \begin{pmatrix} h^{-1} & 0 \\ \lambda\partial_{\bar z} h & h\end{pmatrix}\begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda\overline{Q(z,\lambda^{-1})} & 0\end{pmatrix}\begin{pmatrix} h & 0 \\ -\lambda\partial_{\bar z} h & h^{-1}\end{pmatrix}\mathrm{d}\bar z + \begin{pmatrix} h^{-1} & 0 \\ \lambda\partial_{\bar z} h & h\end{pmatrix}\begin{pmatrix}\partial_{\bar z} h & 0 \\ 0 & -h^{-2}\partial_{\bar z} h\end{pmatrix}\mathrm{d}\bar z \\ &= \begin{pmatrix} -h^{-1}\partial_{\bar z} h & \lambda^{-1}h^{-2} \\ -\lambda h^2\overline{Q(z,\lambda^{-1})} - \lambda(\partial_{\bar z} h)^2 & h^{-1}\partial_{\bar z} h\end{pmatrix}\mathrm{d}\bar z + \begin{pmatrix} h^{-1}\partial_{\bar z} h & 0 \\ \lambda(\partial_{\bar z} h)^2 & -h^{-1}\partial_{\bar z} h\end{pmatrix}\mathrm{d}\bar z \\ &= \begin{pmatrix} 0 & \lambda^{-1}h^{-2} \\ -\lambda h^2\overline{Q(z,\lambda^{-1})} & 0\end{pmatrix}\mathrm{d}z.\end{aligned} \tag{4.6.44}$$

Together, this proves (4.6.40). ☐

In view of the one-to-one correspondence explicated earlier, between possible orientation preserving (resp. orientation reversing) symmetries of a given trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends on the one hand and biholomorphic (resp. bi-antiholomorphic) mappings $\gamma : M \to M$ on the other hand, it is a direct consequence of lemma 4.21 that a given trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends allows for at most twelve symmetries, namely six orientation preserving ones and six orientation reversing ones.

As seen before, a biholomorphic (resp. bi-antiholomorphic) mapping $M \to M$ is entirely characterized by the way it (or, more precisely, its biholomorhic or bi-antiholomorhic extension $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$) permutes the set $\{0,1,\infty\}$. Recall that we write γ_σ (resp. γ_σ^*) for the unique biholomorphic (resp. bi-antiholomorphic) mapping $M \to M$, which permutes the set $\{0,1,\infty\}$ according to the permutation σ.

In view of the discussed correspondence, we adopt this notation for the twelve possible symmetries $\mathcal{T}$ of a given trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends, writing $\mathcal{T} = \mathcal{T}_\sigma$, σ permutation of $\{0,1,\infty\}$, for the orientation preserving ones and $\mathcal{T} = \mathcal{T}_\sigma^*$, σ permutation of $\{0,1,\infty\}$, for the orientation reversing ones.

In fact, denoting the twelve possible symmetries of a trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends by $\mathcal{T}_\sigma$ and $\mathcal{T}_\sigma^*$, where σ represents the six possible permutations of the set $\{0,1,\infty\}$, can be motivated more directly, as shown in the following lemma.

Lemma 4.26. *Let $M = \mathbb{C} \setminus \{0,1\}$ and $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends $B_j = \phi(U_j)$, which is symmetric with respect to the Euclidean motion $\mathcal{T} \in \mathrm{Sym}(\phi(M))$. Moreover, let γ denote the biholomorphic (resp. bi-antiholomorphic) mapping $M \to M$ associated with $\mathcal{T}$ by theorem 4.9 and characterized by the permutation σ of the set $\{0,1,\infty\}$, i.e. $\gamma = \gamma_\sigma$, $\mathcal{T} = \mathcal{T}_\sigma$ in the case that $\mathcal{T}$ preserves orientation and $\gamma = \gamma_\sigma^*$, $\mathcal{T} = \mathcal{T}_\sigma^*$ in the case that $\mathcal{T}$ reverses orientation. Then, the following holds:*

1. *For each $j \in \{0,1,\infty\}$, there exists an open, non-empty punctured neighborhood $\hat{U}_j \subseteq U_j$ of z_j such that*
$$\mathcal{T}(\phi(\hat{U}_j)) \subseteq B_{\sigma(j)}. \tag{4.6.45}$$

2. *For each $j \in \{0,1,\infty\}$, let $A_j \subseteq \mathbb{R}^3$ denote the trinoid axis of ϕ at $z_j = j$. Then,*
$$\mathcal{T}(A_j) = A_{\sigma(j)}. \tag{4.6.46}$$

Proof. We begin with the proof of the first claim. As before, denote by γ_{extd} the unique biholomorphic (resp. bi-antiholomorphic) mapping $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with $\gamma_{\mathrm{extd}}|M \equiv \gamma$. Then, by definition of $\gamma = \gamma_\sigma$ (resp. $\gamma = \gamma_\sigma^*$), γ_{extd} permutes the set $\{0,1,\infty\}$ according to σ, i.e. $\gamma_{\mathrm{extd}}(z_j) = z_{\sigma(j)}$ for all $z_j = j \in \{0,1,\infty\}$. Consequently, since γ is continuous on M, there exists for each $j \in \{0,1,\infty\}$ an open, non-empty punctured neighborhood $\hat{U}_j$ of z_j in M, such that
$$\gamma(\hat{U}_j) \subseteq U_{\sigma(j)}. \tag{4.6.47}$$
W.l.o.g., we can assume $\hat{U}_j \subseteq U_j$.

Since, by theorem 4.9, γ satisfies $\mathcal{T} \circ \phi = \phi \circ \gamma$ (on M), we conclude that, for each $j \in \{0,1,\infty\}$, there exists an open, non-empty punctured neighborhood $\hat{U}_j \subseteq U_j$ of z_j such that
$$\mathcal{T}(\phi(\hat{U}_j)) = \phi(\gamma(\hat{U}_j)) \subseteq \phi(U_{\sigma(j)}) = B_{\sigma(j)}, \tag{4.6.48}$$
as claimed.

The second claim is a direct consequence of the first one: By (4.6.45), there exists for each $j \in \{0,1,\infty\}$ an open, non-empty punctured neighborhood $\hat{U}_j \subseteq U_j$ of z_j, such that $\mathcal{T}$ maps $\hat{B}_j := \phi(\hat{U}_j)$ (which forms a "sub-end" of B_j) to $B_{\sigma(j)}$, i.e. $\mathcal{T}(\hat{B}_j)$ forms a "sub-end" of $B_{\sigma(j)}$. Recalling that, for each $j \in \{0,1,\infty\}$, the properly embedded annular end B_j of ϕ asymptotically shows the behaviour of an unduloidal Delaunay surface ϕ_j, we proceed as follows: Since $\mathcal{T}(\hat{B}_j) \subseteq B_{\sigma(j)}$, we infer that for each $j \in \{0,1,\infty\}$, $\mathcal{T}(\hat{B}_j)$ (and thus also the "super-end" $\mathcal{T}(B_j)$) asymptotically shows the behaviour of $\phi_{\sigma(j)}$. Moreover, since $\mathcal{T}$ is continuous, it necessarily maps the (images of the) corresponding Delaunay surfaces (as subsets of $\mathbb{R}^3$) onto each other, i.e. $\mathcal{T}(\mathrm{im}(\phi_j)) = \mathrm{im}(\phi_{\sigma(j)})$. Consequently, also the related revolution axes of the Delaunay surfaces, i.e. the trinoid axes A_j, $j \in \{0,1,\infty\}$, of ϕ are mapped by $\mathcal{T}$ onto each other:
$$\mathcal{T}(A_j) = A_{\sigma(j)}, \tag{4.6.49}$$
as claimed. □

Remark 4.27. Let $M = \mathbb{C} \setminus \{0,1\}$ and $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends $B_j = \phi(U_j)$, which is symmetric with respect to the Euclidean motion $\mathcal{T} \in \mathrm{Sym}(\phi(M))$. By lemma 4.26, $\mathcal{T}$ maps for each $j \in \{0,1,\infty\}$ at least some "outer part" $\phi(\hat{U}_j)$ of the trinoid end $B_j = \phi(U_j)$ to the trinoid end $B_{\sigma(j)}$. In this sense, $\mathcal{T}$ permutes the trinoid ends according to the permutation σ.

In addition to the result above, we have the following:

Lemma 4.28. *Let $M = \mathbb{C} \setminus \{0,1\}$ and $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends $B_j = \phi(U_j)$ and corresponding trinoid axes $A_j = C_j + \mathbb{R}v_j \subseteq \mathbb{R}^3$, involving a base point $C_j \in \mathbb{R}^3$ and a unit direction vector $v_j \in \mathbb{R}^3$, pointing towards the trinoid end B_j. Moreover, let ϕ be symmetric with respect to the Euclidean motion $\mathcal{T} \in \mathrm{Sym}(\phi(M))$ associated with the permutation σ of the set $\{0,1,\infty\}$, i.e. $\mathcal{T} = \mathcal{T}_\sigma$ (resp. $\mathcal{T} = \mathcal{T}^*_\sigma$) for orientation preserving (resp. orientation reversing) $\mathcal{T}$. Denote by $\mathcal{A} \in \mathrm{O}(3)$ (resp. by $t \in \mathbb{R}^3$) the orthogonal part (resp. the translational part) of $\mathcal{T}$, i.e. $\mathcal{T}(x) = \mathcal{A}x + t$ (on $\mathbb{R}^3$). Then, the following holds:*

1. *For all $j \in \{0,1,\infty\}$, we have*
$$\mathcal{A}v_j = v_{\sigma(j)}. \tag{4.6.50}$$

2. *Moreover, we can assume without loss of generality for all $j \in \{0,1,\infty\}$ that*
$$\mathcal{T}(C_j) = C_{\sigma(j)}. \tag{4.6.51}$$

Proof. The first claim is proved as follows: Let $j \in \{0,1,\infty\}$. Since, by lemma 4.26, $\mathcal{T}(A_j) = \mathcal{T}(A_{\sigma(j)})$, there exists for each $\lambda \in \mathbb{R}$ a real number $\mu_{\sigma(j)} = \mu_{\sigma(j)}(\lambda) \in \mathbb{R}$, such that
$$\mathcal{T}(C_j + \lambda v_j) = C_{\sigma(j)} + \mu_{\sigma(j)}(\lambda) v_{\sigma(j)}. \tag{4.6.52}$$
In particular, for $\lambda = 0$, we have
$$\mathcal{T}(C_j) = C_{\sigma(j)} + \mu_{\sigma(j)}(0) v_{\sigma(j)}. \tag{4.6.53}$$
Note that, as $\mathcal{T}$ defines a continuous bijection $\mathbb{R}^3 \to \mathbb{R}^3$, μ defines a continuous bijecton $\mathbb{R} \to \mathbb{R}$. Moreover, since $\mathcal{T}$ permutes the trinoid ends in the sense of lemma 4.26, it preserves the orientation of the also permuted, corresponding axes: "outer" points on the axis A_j (i.e. points on A_j relatively "close" to the end B_j) are mapped by $\mathcal{T}$ to "outer" points on the axis $A_{\sigma(j)}$. In other words, $\mu_{\sigma(j)} : \mathbb{R} \to \mathbb{R}$ defines a strictly increasing bijection.

In view of $\mathcal{T}(x) = \mathcal{A}x + t$ (on $\mathbb{R}^3$), we obtain by substracting (4.6.53) from (4.6.52) that
$$\lambda \mathcal{A} v_j = (\mu_{\sigma(j)}(\lambda) - \mu_{\sigma(j)}(0)) v_{\sigma(j)}. \tag{4.6.54}$$
As (4.6.54) holds for all $\lambda \in \mathbb{R}$, and since $v_j \neq 0 \neq v_{\sigma(j)}$ and μ_j is strictly increasing, $\mathcal{A}v_j$ necessarily equals a positive multiple of $v_{\sigma(j)}$. Since v_j and $v_{\sigma(j)}$ are unit vectors in $\mathbb{R}^3$, and $\mathcal{A}$ is orthogonal, we conclude that
$$\mathcal{A}v_j = v_{\sigma(j)}, \tag{4.6.55}$$
which finishes the proof of the first claim.

We now turn to the proof of the second claim, i.e. we show that we can always choose the base points C_j of the trinoid axes A_j, such that (4.6.51) holds for all $j \in \{0,1,\infty\}$.

For a start, we infer from (4.6.54) and (4.6.55) that $\mu_{\sigma(j)}(\lambda) = \mu_{\sigma(j)}(0) + \lambda$ and thus, by (4.6.52),
$$\mathcal{T}(C_j + \lambda v_j) = C_{\sigma(j)} + (\mu_{\sigma(j)}(0) + \lambda) v_{\sigma(j)}. \tag{4.6.56}$$

Denote as before by γ the biholomorphic (resp. bi-antiholomorphic) mapping $M \to M$ associated with $\mathcal{T}$ by theorem 4.9, $\mathcal{T} \circ \phi = \phi \circ \gamma$. In view of lemma 4.21, we observe that either $\gamma^2 \equiv \mathrm{id}$ (in case that $\sigma \in \{(\ ), (1\ \infty), (0\ \infty), (0\ 1)\}$) or at least $\gamma^6 \equiv \mathrm{id}$ (in case that $\sigma \in \{(0\ 1\ \infty), (0\ \infty\ 1)\}$) holds.

We consider the first case: $\gamma^2 \equiv \mathrm{id}$ and $\sigma \in \{(\ ), (1\ \infty), (0\ \infty), (0\ 1)\}$. Combining the identities $\mathcal{T} \circ \phi = \phi \circ \gamma$ and $\gamma^2 \equiv \mathrm{id}$ yields $\mathcal{T}^2|_{\phi(M)} \equiv \mathrm{id}$ and thus $\mathcal{T}^2|_{\mathbb{R}_3} \equiv \mathrm{id}$. Consequently, we have for all $x \in \mathbb{R}$ that $\mathcal{A}^2 x + \mathcal{A}t + t = \mathcal{T}^2(x) = x$, which directly implies (setting $x = 0$) $\mathcal{A}t + t = 0$ and thus $\mathcal{A}^2 = \mathrm{I}$. It follows that $\mathcal{A}$ has only eigenvalues ± 1 and therefore induces an eigenspace decomposition of $\mathbb{R}^3$ into U_+ and U_-. Note that $\mathcal{A}t + t = 0$ implies that $t \in U_-$.

The permutation σ either keeps all three points 0, 1 and ∞ fixed or keeps one point fixed while interchanging the other two. In the first case, we choose for each $j \in \{0,1,\infty\}$ an arbitrary $C_j \in A_j$ as base point of A_j. By (4.6.56), we have
$$\mathcal{A}C_j + t = \mathcal{T}(C_j) = C_j + \mu_j(0) v_j. \tag{4.6.57}$$
As $\sigma(j) = j$, we have $\mathcal{A}v_j = v_j$ and thus $v_j \in U_+$. Moreover, writing $C_j = C_j^+ + C_j^-$ with $C_j^+ \in U_+$, $C_j^- \in U_-$ and recalling $t \in U_-$, comparison of the U_+-parts of both sides in (4.6.57) yields $C_j^+ = C_j^+ + \mu_j(0) v_j$

and therefore $\mu_j(0) = 0$, which by (4.6.57) implies $\mathcal{T}(C_j) = C_j = C_{\sigma(j)}$. This finishes the proof for $\mathcal{T}$ associated with $\sigma = (\ )$.

In the case that σ keeps one point fixed (say, j), while interchanging the other two (say, k and l), we proceed as follows: Choose $C_j \in A_j$, then by the arguments given above $\mathcal{T}(C_j) = C_j = C_{\sigma(j)}$. Moreover, choose $C_k \in A_k$ and set $C_l := \mathcal{T}(C_k) \in A_l$. It remains to show $\mathcal{T}(C_l) = C_k$. But this follows directly from $\mathcal{T}^2 = \mathrm{id}$: $\mathcal{T}(C_l) = \mathcal{T}^2(C_k) = C_k$. This finishes the proof for $\mathcal{T}$ associated with $\sigma \in \{(1\ \infty), (0\ \infty), (0\ 1)\}$.

It remains to consider the second case: $\gamma^6 \equiv \mathrm{id}$ and $\sigma \in \{(0\ 1\ \infty), (0\ \infty\ 1)\}$. Combining the identities $\mathcal{T} \circ \phi = \phi \circ \gamma$ and $\gamma^6 \equiv \mathrm{id}$ yields $\mathcal{T}^6|_{\phi(M)} \equiv \mathrm{id}$ and thus $\mathcal{T}^6|_{\mathbb{R}_3} \equiv \mathrm{id}$. Consequently, we have for all $x \in \mathbb{R}$ that $\mathcal{A}^6x + \mathcal{A}^5t + \mathcal{A}^4t + \mathcal{A}^3t + \mathcal{A}^2t + \mathcal{A}t + t = \mathcal{T}^6(x) = x$, which directly implies (setting $x = 0$) $\mathcal{A}^5t+\mathcal{A}^4t+\mathcal{A}^3t+\mathcal{A}^2t+\mathcal{A}t+t = 0$ and thus $\mathcal{A}^6 = \mathrm{I}$. It follows that $\mathcal{A}^3$ has only eigenvalues ±1 and therefore induces an eigenspace decomposition of $\mathbb{R}^3$ into U_+ and U_-. Note that $\mathcal{A}^5t+\mathcal{A}^4t+\mathcal{A}^3t+\mathcal{A}^2t+\mathcal{A}t+t = 0$ implies that $\mathcal{A}^2t + \mathcal{A}t + t \in U_-$.

We consider w.l.o.g. only the permutation $\sigma = (0\ 1\ \infty)$. ($\sigma = (0\ \infty\ 1)$ is treated completely analogously.) Choose $C_0 \in A_0$ and set $C_1 := \mathcal{T}(C_0) \in A_1$, $C_\infty := \mathcal{T}(C_1) \in A_\infty$. It remains to show $\mathcal{T}(C_\infty) = C_0$, i.e. $\mathcal{T}^3(C_0) = C_0$. Repeated application of (4.6.56) yields

$$\mathcal{A}^3C_j + \mathcal{A}^2t + \mathcal{A}t + t = \mathcal{T}^3(C_j) = C_j + (\mu_j(0) + \mu_{\sigma(j)}(0) + \mu_{\sigma^2(j)}(0))v_j. \qquad (4.6.58)$$

As $\sigma^3(j) = j$, we have $\mathcal{A}^3v_j = v_j$ and thus $v_j \in U_+$. Moreover, writing $C_j = C_j^+ + C_j^-$ with $C_j^+ \in U_+$, $C_j^- \in U_-$ and recalling $\mathcal{A}^2t + \mathcal{A}t + t \in U_-$, comparison of the U_+-parts of both sides in (4.6.58) yields $C_j^+ = C_j^+ + (\mu_j(0) + \mu_{\sigma(j)}(0) + \mu_{\sigma^2(j)}(0))v_j$ and therefore $(\mu_j(0) + \mu_{\sigma(j)}(0) + \mu_{\sigma^2(j)}(0)) = 0$, which by (4.6.58) implies $\mathcal{T}^3(C_j) = C_j$, in particular $\mathcal{T}^3(C_0) = C_0$. This finishes the proof for $\mathcal{T}$ associated with $\sigma = (0\ 1\ \infty)$ (and $\sigma = (0\ \infty\ 1)$). □

Remark 4.29. Note that, by [28], in addition to (4.3.4) there holds another "balancing formula" involving the torques of the trinoid axes A_j, which implies (cf. [28]) that the three trinoid axes are coplanar in $\mathbb{R}^3$ *and* either are all parallel or meet in a common point. In fact, this holds more generally for all CMC-immersions with three annular ends which are asymptotic to (not necessarily unduloidal) Delaunay surfaces. Recent communication with R. Kusner and N. Schmitt suggests that in the case of trinoids with properly embedded (i.e. asymptotic unduloidal) annular ends parallel axes can not occur and thus the three trinoid axes necessarily meet in one point. In view of this, one could assume w.l.o.g. $C_0 = C_1 = C_\infty$ for the three base points for the trinoid axes A_0, A_1 and A_∞. In this setting, the claims of lemma 4.28 are almost trivial. Moreover, the proof of theorem 4.31 simplifies significantly. However, mainly to preserve the adaptability of lemma 4.28 and theorem 4.31 to possible future work (e.g., the study of CMC-immersions with three not necessarily properly embedded annular ends), we retain the more general (and more complicated) proofs here.

The following theorem now lists the twelve possible trinoid symmetries explicitly. Firstly, however we introduce some more notions.

Definition 4.30. Let $M = \mathbb{C} \setminus \{0, 1\}$ and $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends $B_j = \phi(U_j)$ and corresponding trinoid axes $A_j = C_j + \mathbb{R}v_j \subseteq \mathbb{R}^3$, $j = 0, 1, \infty$. Then:

1. The point $C := \frac{1}{3}(C_0 + C_1 + C_\infty) \in \mathbb{R}^3$ will be called the *trinoid center.*
2. Let C denote the trinoid center. Any plane in R^3 containing the affine subspace $C+\mathbb{R}v_0+\mathbb{R}v_1+\mathbb{R}v_\infty$ is called a *trinoid plane* and will often be denoted by E. Moreover, denoting the (up to sign unique) unit normal vector of a trinoid plane E by n, the line $C+\mathbb{R}n$ is called a *trinoid normal* and will often be denoted by A_n. Finally, given a trinoid plane E with normal vector n, for each $j \in \{0, 1, \infty\}$ the plane $C_j + \mathbb{R}v_j + \mathbb{R}n$ is called a *trinoid normal plane (along the trinoid axis A_j).* This plane will often be denoted by E_j.

Theorem 4.31. *Let $M = \mathbb{C} \setminus \{0, 1\}$ and $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends $B_j = \phi(U_j)$ and corresponding trinoid axes $A_j = C_j + \mathbb{R}v_j \subseteq \mathbb{R}^3$, $j = 0, 1, \infty$. Denote by C the trinoid center $\frac{1}{3}(C_0+C_1+C_\infty) \in \mathbb{R}^3$. Moreover, let $\mathcal{T} \in \mathrm{Sym}(\phi(M))$ and denote by σ the permutation of $\{0, 1, \infty\}$ representing the transformation behaviour of the trinoid ends B_j under $\mathcal{T}$. Then, the following holds:*

1. If $\mathcal{T}$ preserves orientation, i.e. $\mathcal{T} = \mathcal{T}_\sigma$, we have:

(a)

$$\mathcal{T}_\sigma = \mathcal{I} \quad \textit{for } \sigma = (\ ), \tag{4.6.59}$$

where $\mathcal{I}$ *denotes the identity mapping on* $\mathbb{R}^3$*.*

(b)

$$\mathcal{T}_\sigma = \mathcal{R}_0 \quad \textit{for } \sigma = (1\ \infty), \tag{4.6.60}$$

where $\mathcal{R}_0$ *denotes the rotation on* $\mathbb{R}^3$ *by the angle* π *around the trinoid axis* A_0*.*

(c)

$$\mathcal{T}_\sigma = \mathcal{R}_1 \quad \textit{for } \sigma = (0\ \infty), \tag{4.6.61}$$

where $\mathcal{R}_1$ *denotes the rotation on* $\mathbb{R}^3$ *by the angle* π *around the trinoid axis* A_1*.*

(d)

$$\mathcal{T}_\sigma = \mathcal{R}_\infty \quad \textit{for } \sigma = (0\ 1), \tag{4.6.62}$$

where $\mathcal{R}_\infty$ *denotes the rotation on* $\mathbb{R}^3$ *by the angle* π *around the trinoid axis* A_∞*.*

(e)

$$\mathcal{T}_\sigma = \mathcal{R} \quad \textit{for } \sigma = (0\ 1\ \infty), \tag{4.6.63}$$

where $\mathcal{R}$ *denotes the rotation on* $\mathbb{R}^3$ *by the angle*[12] $\pm\frac{2\pi}{3}$ *around the trinoid normal* $A_n = \{C + \lambda n; \lambda \in \mathbb{R}\}$*, where* $n \in S^2 \subseteq \mathbb{R}^3$ *satisfies* $n \perp v_j$ *for all* $j \in \{0, 1, \infty\}$*. (Note that, in this case, the vectors* v_j*,* $j \in \{0, 1, \infty\}$*, span a plane in* $\mathbb{R}^3$*, whence* n *is uniquely determined up to sign and we can speak of* the *trinoid normal* A_n *of* ϕ*.)*

(f)

$$\mathcal{T}_\sigma = \mathcal{R}^{-1} \quad \textit{for } \sigma = (0\ \infty\ 1), \tag{4.6.64}$$

where $\mathcal{R}$ *is given above.*

2. *If* $\mathcal{T}$ *reverses orientation, i.e.* $\mathcal{T} = \mathcal{T}_\sigma^*$*, we have:*

(a)

$$\mathcal{T}_\sigma^* = \mathcal{S} \quad \textit{for } \sigma = (\ ), \tag{4.6.65}$$

where $\mathcal{S}$ *denotes the reflection on* $\mathbb{R}^3$ *in some trinoid plane* $E = C + (\mathbb{R}n)^\perp$*, where* $n \in S^2 \subseteq \mathbb{R}^3$ *satisfies* $n \perp v_j$ *for all* $j \in \{0, 1, \infty\}$*. (Moreover,* E *is uniquely determined by the relation* $\mathcal{T} \circ \phi = \phi \circ \gamma$*.)*

(b)

$$\mathcal{T}_\sigma^* = \mathcal{S}_0 \quad \textit{for } \sigma = (1\ \infty), \tag{4.6.66}$$

where $\mathcal{S}_0$ *denotes the reflection on* $\mathbb{R}^3$ *in some trinoid normal plane* $E_0 = C_0 + \mathbb{R}v_0 + \mathbb{R}n$*, where* $n \in S^2 \subseteq \mathbb{R}^3$ *satisfies* $n \perp v_j$ *for all* $j \in \{0, 1, \infty\}$*. (Moreover,* E_0 *is uniquely determined by the relation* $\mathcal{T} \circ \phi = \phi \circ \gamma$*.)*

(c)

$$\mathcal{T}_\sigma^* = \mathcal{S}_1 \quad \textit{for } \sigma = (0\ \infty), \tag{4.6.67}$$

where $\mathcal{S}_1$ *denotes the reflection on* $\mathbb{R}^3$ *in some trinoid normal plane* $E_1 = C_1 + \mathbb{R}v_1 + \mathbb{R}n$*, where* $n \in S^2 \subseteq \mathbb{R}^3$ *satisfies* $n \perp v_j$ *for all* $j \in \{0, 1, \infty\}$*. (Moreover,* E_1 *is uniquely determined by the relation* $\mathcal{T} \circ \phi = \phi \circ \gamma$*.)*

(d)

$$\mathcal{T}_\sigma^* = \mathcal{S}_\infty \quad \textit{for } \sigma = (0\ 1), \tag{4.6.68}$$

where $\mathcal{S}_\infty$ *denotes the reflection on* $\mathbb{R}^3$ *in some trinoid normal plane* $E_\infty = C_\infty + \mathbb{R}v_\infty + \mathbb{R}n$*, where* $n \in S^2 \subseteq \mathbb{R}^3$ *satisfies* $n \perp v_j$ *for all* $j \in \{0, 1, \infty\}$*. (Moreover,* E_∞ *is uniquely determined by the relation* $\mathcal{T} \circ \phi = \phi \circ \gamma$*.)*

(e)

$$\mathcal{T}_\sigma^* = \hat{\mathcal{S}} \quad \textit{for } \sigma = (0\ 1\ \infty), \tag{4.6.69}$$

where $\hat{\mathcal{S}}$ *denotes the rotoreflection on* $\mathbb{R}^3$ *composed of the rotation by the angle* $\pm\frac{2\pi}{3}$ *around the trinoid normal* $A_n = \{C + \lambda n; \lambda \in \mathbb{R}\}$*, where* $n \in S^2 \subseteq \mathbb{R}^3$ *satisfies* $n \perp v_j$ *for all* $j \in \{0, 1, \infty\}$*, and the reflection in the trinoid plane* $E = C + (\mathbb{R}n)^\perp$*. (Note that, in this case, the vectors* v_j*,* $j \in \{0, 1, \infty\}$*, span a plane in* $\mathbb{R}^3$*, whence* n *is uniquely determined up to sign and we can speak of* the *trinoid normal* A_n *and* the *trinoid plane of* ϕ*, respectively.)*

[12]Note that the sign of the rotation angle is defined with respect to the orientation of the rotation axis. This is further discussed in section 5.

(f)

$$\mathcal{T}_\sigma^* = \hat{\mathcal{S}}^{-1} \quad \textit{for } \sigma = (0\ \infty\ 1), \tag{4.6.70}$$

where $\hat{\mathcal{S}}$ is given above.

Proof. Let $\mathcal{T} \in \mathrm{Sym}(\phi(M)$ and $\gamma : M \to M$ be the bi-(anti)holomorphic mapping associated with $\mathcal{T}$ by theorem 4.9 and explicitly given in lemma 4.21, satifying $\mathcal{T} \circ \phi = \phi \circ \gamma$. Let σ denote the permutation of the set $\{0, 1, \infty\}$ representing the transformation behaviour of the trinoid ends under $\mathcal{T}$. We start with the case that $\mathcal{T}$ preserves orientation, i.e. $\mathcal{T} = \mathcal{T}_\sigma$.

If $\sigma = (\ )$, we infer by lemma 4.21 that $\gamma(z) = z$. Thus, $\mathcal{T} \circ \phi = \phi$, which implies that $\mathcal{T}|_{\phi(M)} = \mathrm{id}$. Consequently, $\mathcal{T}|_{\mathbb{R}^3} = \mathcal{I}$, where $\mathcal{I}$ denotes the identity mapping on $\mathbb{R}^3$.

If $\sigma = (1\ \infty)$, we infer by lemma 4.21 that $\gamma(z) = \frac{z}{z-1}$ and thus $\gamma^2 = \mathrm{id}$. This implies $\mathcal{T}^2 \circ \phi = \phi$, i.e. $\mathcal{T}^2|_{\phi(M)} = \mathrm{id}$, and consequently $\mathcal{T}^2|_{\mathbb{R}^3} = \mathrm{id}$. Writing $\mathcal{T}$ explicitly as $\mathcal{T}(x) = \mathcal{A}x + t$ with $\mathcal{A} \in \mathrm{SO}(3)$ and $t \in \mathbb{R}^3$, we obtain $\mathcal{A}^2 x + \mathcal{A}t + t = x$ for all $x \in \mathbb{R}^3$ and therefore immediately (for $x = 0$) $\mathcal{A}t + t = 0$, which yields $\mathcal{A}^2 = \mathrm{I}$. Since $\mathcal{A} \in \mathrm{SO}(3)$, this implies that either $\mathcal{A} = \mathrm{I}$ or that $\mathcal{A}$ is a rotation by the angle π around some axis in $\mathbb{R}^3$ containing $0 \in \mathbb{R}^3$. The case $\mathcal{A} = \mathrm{I}$ yields $t = 0$ and thus $T = \mathrm{id}$, a contradiction to the fact that, according to σ, $\mathcal{T}$ swaps the trinoid ends B_1 and B_∞. Therefore, $\mathcal{A}$ is a rotation by the angle π around some axis in $\mathbb{R}^3$ containing $0 \in \mathbb{R}^3$. Since, by lemma 4.28, $\mathcal{A}$ preserves the direction vector v_0 of the trinoid axis A_0, i.e $\mathcal{A}v_0 = v_0$, $\mathcal{A}$ actually defines the rotation by the angle π around the axis $\mathbb{R}v_0$. Moreover, the same lemma allows for assuming without loss of generality that $\mathcal{T}$ keeps the base point C_0 of the trinoid axis A_0 fixed, i.e. $\mathcal{T}(C_0) = C_0$. Consequently, we have $t = C_0 - \mathcal{A}C_0$ and thus

$$\mathcal{T}(x) = \mathcal{A}x + C_0 - \mathcal{A}C_0 = \mathcal{A}(x - C_0) + C_0, \tag{4.6.71}$$

which means that $\mathcal{T}$ defines the rotation by the angle π around the rotation axis of $\mathcal{A}$ translated by C_0, i.e. around the trinoid axis $A_0 = C_0 + \mathbb{R}v_0$. The cases $\sigma = (0\ \infty)$ and $\sigma = (0\ 1)$ are treated completely analogously.

If $\sigma = (0\ 1\ \infty)$, we infer by lemma 4.21 that $\gamma(z) = \frac{1}{1-z}$ and thus $\gamma^3 = \mathrm{id}$. This implies $\mathcal{T}^3 \circ \phi = \phi$, i.e. $\mathcal{T}^3|_{\phi(M)} = \mathrm{id}$, and consequently $\mathcal{T}^3|_{\mathbb{R}^3} = \mathrm{id}$. Writing $\mathcal{T}$ explicitly as $\mathcal{T}(x) = \mathcal{A}x + t$ with $\mathcal{A} \in \mathrm{SO}(3)$ and $t \in \mathbb{R}^3$, we obtain $\mathcal{A}^3 x + \mathcal{A}^2 t + \mathcal{A}t + t = x$ for all $x \in \mathbb{R}^3$ and therefore immediately (for $x = 0$) $\mathcal{A}^2 t + \mathcal{A}t + t = 0$, which yields $\mathcal{A}^3 = \mathrm{I}$. Since $\mathcal{A} \in \mathrm{SO}(3)$, this implies that either $\mathcal{A} = \mathrm{I}$ or that $\mathcal{A}$ is a rotation by the angle $\pm\frac{2\pi}{3}$ around some axis in $\mathbb{R}^3$ containing $0 \in \mathbb{R}^3$. The case $\mathcal{A} = \mathrm{I}$ yields $t = 0$ and thus $T = \mathrm{id}$, a contradiction to the fact that, according to σ, $\mathcal{T}$ doesn't preserve the trinoid ends. Therefore, $\mathcal{A}$ is a rotation by the angle $\pm\frac{2\pi}{3}$ around some axis in $\mathbb{R}^3$ containing $0 \in \mathbb{R}^3$.

Assume now that two of the direction vectors v_0, v_1 and v_∞ of the trinoid axes are collinear, e.g. $v_0 = \pm v_1$. Applying this together with the relation $\mathcal{A}v_j = v_{\sigma(j)}$ from lemma 4.28 several times, we obtain $v_\infty = \mathcal{A}v_1 = \pm\mathcal{A}v_0 = \pm v_1 = v_0$ and thus $v_1 = \mathcal{A}v_0 = \mathcal{A}v_\infty = v_0$, i.e. $v_0 = v_1 = v_\infty$, a contradiction to the the balancing formula (4.3.4). (Similarly, the assumptions $v_0 = \pm v_\infty$ and $v_1 = \pm v_\infty$, respectively, yield contradictions.) Consequently, no two of the v_j, $j \in \{0, 1, \infty\}$, are collinear. By (4.3.4), however, these three vectors are coplanar and thus now necessarily span a plane E in $\mathbb{R}^3$. The normal vector $n \in S^2$ of E is determined up to sign. As $\mathcal{A}$ preserves E and thus also the line $\mathbb{R}n$, $\mathcal{A}$ actually defines the rotation by the angle $\pm\frac{2\pi}{3}$ around the axis $\mathbb{R}n$. Moreover, since (again by lemma 4.28) $\mathcal{T}$ permutes without loss of generality the base points C_j, $j \in \{0, 1, \infty\}$ of the trinoid axes according to σ, i.e. $\mathcal{T}(C_j) = \mathbb{C}_{\sigma(j)}$, we conclude that

$$t = C_1 - \mathcal{A}C_0 = C_\infty - \mathcal{A}C_1 = C_0 - \mathcal{A}C_\infty \tag{4.6.72}$$

and thus

$$\mathcal{T}(x) = \mathcal{A}x + \frac{1}{3}(C_1 - \mathcal{A}C_0 + C_\infty - \mathcal{A}C_1 + C_0 - \mathcal{A}C_\infty) = \mathcal{A}(x - C) + C, \tag{4.6.73}$$

which means that $\mathcal{T}$ defines the rotation by the angle $\pm\frac{2\pi}{3}$ around the rotation axis of $\mathcal{A}$ translated by C, i.e. around the trinoid normal $A_n = C + \mathbb{R}n$.

If $\sigma = (0\ \infty\ 1)$, we observe that $\mathcal{T}^{-1}$ corresponds to $\sigma^{-1} = (0\ 1\ \infty)$ and is also a symmetry of ϕ: $\mathcal{T}^{-1}(\phi(M)) = \phi(M)$. Thus (as shown above), $\mathcal{T}^{-1} = \mathcal{R}$, where $\mathcal{R}$ denotes the rotation on $\mathbb{R}^3$ by the angle $\pm\frac{2\pi}{3}$ around the axis $A_n = C + \mathbb{R}n$, where n denotes the (up to sign unique) normal vector of the plane E spanned by the direction vectors v_0, v_1 and v_∞ of the trinoid axes. Consequently, we have $\mathcal{T} = \mathcal{R}^{-1}$.

We now turn to the case that $\mathcal{T}$ reverses orientation, i.e. $\mathcal{T} = \mathcal{T}_\sigma^*$.

If $\sigma = (\ )$, we infer by lemma 4.21 that $\gamma(z) = \bar{z}$ and thus $\gamma^2 = \mathrm{id}$. This implies $\mathcal{T}^2 \circ \phi = \phi$, i.e. $\mathcal{T}^2|_{\phi(M)} = \mathrm{id}$, and consequently $\mathcal{T}^2|_{\mathbb{R}^3} = \mathrm{id}$. Writing $\mathcal{T}$ explicitly as $\mathcal{T}(x) = \mathcal{A}x + t$ with $\mathcal{A} \in \mathrm{O}(3) \setminus \mathrm{SO}(3)$

and $t \in \mathbb{R}^3$, we obtain $\mathcal{A}^2 x + \mathcal{A}t + t = x$ for all $x \in \mathbb{R}^3$ and therefore immediately (for $x = 0$) $\mathcal{A}t + t = 0$, which implies $\mathcal{A}^2 = \mathrm{I}$. Since $\mathcal{A} \in \mathrm{O}(3) \setminus \mathrm{SO}(3)$, we have $\tilde{\mathcal{A}} := -\mathcal{A} \in \mathrm{SO}(3)$ with $\tilde{\mathcal{A}}^2 = I$. This implies that either $\tilde{\mathcal{A}} = \mathrm{I}$ or that $\tilde{\mathcal{A}}$ is a rotation by the angle π around some axis in $\mathbb{R}^3$ containing $0 \in \mathbb{R}^3$. The case $\tilde{\mathcal{A}} = \mathrm{I}$ yields $t = 0$ (set $x = 0$ in $\mathcal{T}(x) = -x + t$) and thus $\mathcal{T} = -\mathrm{id}$, a contradiction to the fact that, according to σ, $\mathcal{T}$ preserves the trinoid ends. Therefore, $\tilde{\mathcal{A}}$ is a rotation by the angle π around some axis in $\mathbb{R}^3$ containing $0 \in \mathbb{R}^3$. Since, by lemma 4.28, $\mathcal{A}$ preserves the direction vectors v_j, $j \in \{0, 1, \infty\}$ of the trinoid axes, i.e. $\mathcal{A}v_j = v_j$, we infer that $\tilde{\mathcal{A}}v_j = -v_j$. Thus, $\tilde{\mathcal{A}}$ defines the rotation by the angle π around some axis $\mathbb{R}n$, where $n \in \mathbb{R}^3$ satifies $n \perp v_j$ for all $j \in \{0, 1, \infty\}$. (Since v_0, v_1 and v_∞ are coplanar by the balancing formula (4.3.4), such an n exists.) It follows by a direct computation that $\mathcal{A} = -\tilde{\mathcal{A}}$ defines the reflection in the plane $(\mathbb{R}n)^\perp$ at $0 \in \mathbb{R}^3$. Finally, as $\mathcal{T}$ by lemma 4.21 without loss of generality preserves the base points C_j, $j \in \{0, 1, \infty\}$, of the trinoid axes, i.e. $\mathcal{T}(C_j) = C_j$, we conclude that $t = C_j - \mathcal{A}C_j$ for all j and obtain

$$\mathcal{T}(x) = \mathcal{A}x + \frac{1}{3}(C_0 - \mathcal{A}C_0 + C_1 - \mathcal{A}C_1 + C_\infty - \mathcal{A}C_\infty) = \mathcal{A}(x - C) + C. \tag{4.6.74}$$

This means that $\mathcal{T}$ defines the reflection in the reflection plane of $\mathcal{A}$ translated by C, i.e. in the trinoid plane $E = C + (\mathbb{R}n)^\perp$. Note that the symmetry relation $\mathcal{T} \circ \phi = \phi \circ \gamma$ determines E completely.

If $\sigma = (1\ \infty)$, we infer by lemma 4.21 that $\gamma(z) = \frac{\bar{z}}{\bar{z}-1}$ and thus $\gamma^2 = \mathrm{id}$. This implies $\mathcal{T}^2 \circ \phi = \phi$, i.e. $\mathcal{T}^2|_{\phi(M)} = \mathrm{id}$, and consequently $\mathcal{T}^2|_{\mathbb{R}^3} = \mathrm{id}$. Writing $\mathcal{T}$ explicitly as $\mathcal{T}(x) = \mathcal{A}x + t$ with $\mathcal{A} \in \mathrm{O}(3) \setminus \mathrm{SO}(3)$ and $t \in \mathbb{R}^3$, we obtain $\mathcal{A}^2 x + \mathcal{A}t + t = x$ for all $x \in \mathbb{R}^3$ and therefore immediately (for $x = 0$) $\mathcal{A}t + t = 0$, which in turn yields $\mathcal{A}^2 = \mathrm{I}$. Since $\mathcal{A} \in \mathrm{O}(3) \setminus \mathrm{SO}(3)$, we have $\tilde{\mathcal{A}} := -\mathcal{A} \in \mathrm{SO}(3)$ with $\tilde{\mathcal{A}}^2 = \mathrm{I}$. This implies that either $\tilde{\mathcal{A}} = \mathrm{I}$ or that $\tilde{\mathcal{A}}$ is a rotation by the angle π around some axis in $\mathbb{R}^3$ containing $0 \in \mathbb{R}^3$. The case $\tilde{\mathcal{A}} = \mathrm{I}$ yields $t = 0$ (set $x = 0$ in $\mathcal{T}(x) = -x + t$) and thus $\mathcal{T} = -\mathrm{id}$, a contradiction to the fact that, according to σ, $\mathcal{T}$ preserves the trinoid end B_0. Therefore, $\tilde{\mathcal{A}}$ is a rotation by the angle π around some axis in $\mathbb{R}^3$ containing $0 \in \mathbb{R}^3$. Since, by lemma 4.28, $\mathcal{A}$ satisfies $\mathcal{A}v_0 = v_0$, we infer that $\tilde{\mathcal{A}}v_0 = -v_0$. Thus, $\tilde{\mathcal{A}}$ defines the rotation by the angle π around some axis $\mathbb{R}\tilde{n}$, where $\tilde{n} \in \mathbb{R}^3$ satifies $\tilde{n} \perp v_0$. It follows by a direct computation that $\mathcal{A} = -\tilde{\mathcal{A}}$ defines the reflection in the plane spanned by v_0 and $n := v_0 \times \tilde{n}$, where "$\times$" denotes the usual cross product on $\mathbb{R}^3$. By definition, $n \perp v_0$. Actually, we necessarily have $\tilde{n} \perp v_j$ for all $j \in \{0, 1, \infty\}$, which can be seen as follows: In case that any two of v_0, v_1 and v_∞ are collinear, all three are collinear by the balancing formula (4.3.4), whence $n \perp v_0$ implies that $\tilde{n} \perp v_j$ for all $j \in \{0, 1, \infty\}$. Otherwise, i.e. in the case that no two of v_0, v_1 and v_∞ are collinear, these three span a plane in $\mathbb{R}^3$. (Recall that they are coplanar by the balancing formula.) Since, by lemma 4.28, $\mathcal{A}v_j = v_{\sigma(j)}$ for all j, this plane is preserved under $\mathcal{A}$, i.e. it is the reflection plane of $\mathcal{A}$ itself or, otherwise, orthogonal to the reflection plane of $\mathcal{A}$. In the first case, we infer that $\mathcal{A}x = x$ for all points x of the reflection plane, in particular $\mathcal{A}v_1 = v_1$, which together with the relation $\mathcal{A}v_1 = v_{\sigma(1)} = v_\infty$ yields $v_1 = v_\infty$, a contradiction to the assumption that no two of the v_j's are collinear. Thus we are necessarily in the second case, i.e. the plane spanned by v_0, v_1 and v_∞ is orthogonal to the reflection plane of $\mathcal{A}$ (spanned by v_0 and n). Since v_0 is contained in both planes, we conclude that n is orthogonal to the plane spanned by v_0, v_1 and v_∞, i.e. $n \perp v_j$ for all $j \in \{0, 1, \infty\}$.

Altogether, $\mathcal{A}$ defines the reflection in the plane spanned by v_0 and n, where $n \perp v_j$ for all $j \in \{0, 1, \infty\}$. As $\mathcal{T}$ by lemma 4.28 preserves the base point C_0 of the trinoid axis A_0, i.e. $\mathcal{T}(C_0) = C_0$, we conclude that $t = C_0 - \mathcal{A}C_0$ for all j and obtain

$$\mathcal{T}(x) = \mathcal{A}(x - C_0) + C_0. \tag{4.6.75}$$

This means that $\mathcal{T}$ defines the reflection in the reflection plane of $\mathcal{A}$ translated by C_0, i.e. in the trinoid normal plane $E_0 = C_0 + \mathbb{R}v_0 + \mathbb{R}n$ along the trinoid axis A_0. Note that, by the symmetry relation $\mathcal{T} \circ \phi = \phi \circ \gamma$, E_0 and thus (up to sign) also n are determined completely. The cases $\sigma = (0\ \infty)$ and $\sigma = (0\ 1)$ are treated analogously.

If $\sigma = (0\ 1\ \infty)$, we infer by lemma 4.21 that $\gamma(z) = \frac{1}{1-\bar{z}}$ and thus $\gamma^6 = \mathrm{id}$. This implies $\mathcal{T}^6 \circ \phi = \phi$, i.e. $\mathcal{T}^6|_{\phi(M)} = \mathrm{id}$, and consequently $\mathcal{T}^6|_{\mathbb{R}^3} = \mathrm{id}$. Writing $\mathcal{T}$ explicitly as $\mathcal{T}(x) = \mathcal{A}x + t$ with $\mathcal{A} \in \mathrm{O}(3) \setminus \mathrm{SO}(3)$ and $t \in \mathbb{R}^3$, we obtain $\mathcal{A}^6 x + \mathcal{A}^5 t + \mathcal{A}^4 t + \mathcal{A}^3 t + \mathcal{A}^2 t + \mathcal{A}t + t = x$ for all $x \in \mathbb{R}^3$ and therefore immediately (for $x = 0$) $\mathcal{A}^5 t + \mathcal{A}^4 t + \mathcal{A}^3 t + \mathcal{A}^2 t + \mathcal{A}t + t = 0$, which in turn yields $\mathcal{A}^6 = \mathrm{I}$. Since $\mathcal{A} \in \mathrm{O}(3) \setminus \mathrm{SO}(3)$, we have $\tilde{\mathcal{A}} := -\mathcal{A} \in \mathrm{SO}(3)$ with $\tilde{\mathcal{A}}^6 = I$. There are four possible cases:

1. $\tilde{\mathcal{A}}$ defines the identity mapping on $\mathbb{R}^3$, *or*

2. $\tilde{\mathcal{A}}$ defines a rotation by the angle π around some axis containing $0 \in \mathbb{R}^3$, *or*

3. $\tilde{\mathcal{A}}$ defines a rotation by the angle $\mp\frac{2\pi}{3}$ around some axis containing $0 \in \mathbb{R}^3$, *or*

4. $\tilde{\mathcal{A}}$ defines a rotation by the angle $\mp\frac{\pi}{3}$ around some axis containing $0 \in \mathbb{R}^3$.

We lead each of the first three possible cases to a contradiction, using repeatedly the relation $\mathcal{A}v_j = v_{\sigma(j)}$ from lemma 4.28, i.e. here

$$\mathcal{A}v_0 = v_1 \quad \mathcal{A}v_1 = v_\infty \quad \mathcal{A}v_\infty = v_0. \tag{4.6.76}$$

First, assume that $\tilde{\mathcal{A}} = \mathrm{I}$, i.e. $\mathcal{A} = -\mathrm{I}$. Then, $-v_0 = \mathcal{A}v_0 = v_1 = -\mathcal{A}v_1 = -v_\infty = \mathcal{A}v_\infty = v_0$, which yields $v_0 = 0$, a contradiction. Second, assume that $\tilde{\mathcal{A}}$ defines a rotation by the angle π around some axis containing $0 \in \mathbb{R}^3$. In particular, $\tilde{\mathcal{A}}^2 = \mathrm{I}$ and thus also $\mathcal{A}^2 = \mathrm{I}$. Then, $v_0 = \mathcal{A}^2 v_0 = v_\infty = \mathcal{A}^2 v_\infty = v_1$, i.e. $v_0 = v_1 = v_\infty$, a contradiction to the balancing formula (4.3.4). Third, assume that $\tilde{\mathcal{A}}$ defines a rotation by the angle $\mp\frac{2\pi}{3}$ around some axis containing $0 \in \mathbb{R}^3$. In particular, we have $\tilde{\mathcal{A}}^3 = \mathrm{I}$ and thus $\mathcal{A}^3 = -\mathrm{I}$. This yields $-v_0 = \mathcal{A}^3 v_0 = v_0$, i.e. $v_0 = 0$, another contradiction.

As we have lead each of the first three cases to a contradiction, we are necessarily in the fourth case, i.e. $\tilde{\mathcal{A}}$ defines a rotation by the angle $\mp\frac{\pi}{3}$ around some axis containing $0 \in \mathbb{R}^3$. By a direct computation, this implies that $\mathcal{A} = -\tilde{\mathcal{A}}$ defines a rotoreflection on $\mathbb{R}^3$ composed of a rotation by the angle $\pm\frac{2\pi}{3}$ around some axis $\mathbb{R}n$ containing $0 \in \mathbb{R}^3$ and the reflection in the plane $(\mathbb{R}n)^\perp$.

Assume now that two of the direction vectors v_0, v_1 and v_∞ of the trinoid axes are collinear, e.g. $v_0 = \pm v_1$. Applying this together with the relation $\mathcal{A}v_j = v_{\sigma(j)}$ from lemma 4.28 several times, we obtain $v_\infty = \mathcal{A}v_1 = \pm\mathcal{A}v_0 = \pm v_1 = v_0$ and thus $v_1 = \mathcal{A}v_0 = \mathcal{A}v_\infty = v_0$, i.e. $v_0 = v_1 = v_\infty$, a contradiction to the the balancing formula (4.3.4). (Similarly, the assumptions $v_0 = \pm v_\infty$ and $v_1 = \pm v_\infty$, respectively, yield contradictions.)

Consequently, no two of the v_j, $j \in \{0, 1, \infty\}$ are collinear, which implies that these vectors (which are coplanar by (4.3.4)) actually span a plane in $\mathbb{R}^3$. Note that this plane is preserved by $\mathcal{A}$ and thus necessarily coincides with the plane $(\mathbb{R}n)^\perp$ introduced above, i.e. $(\mathbb{R}n)^\perp = \mathbb{R}v_0 + \mathbb{R}v_1 + \mathbb{R}v_\infty$. Therefore, n satifies $n \perp v_j$ for all $j \in \{0, 1, \infty\}$ and is determined up to sign. As $\mathcal{T}$ by lemma 4.28 without loss of generality permutes the base points C_j, $j \in \{0, 1, \infty\}$, of the trinoid axes, i.e. $\mathcal{T}(C_j) = C_{\sigma(j)}$, we conclude that $t = C_{\sigma(j)} - \mathcal{A}C_j$ for all j and obtain

$$\mathcal{T}(x) = \mathcal{A}x + \frac{1}{3}(C_1 - \mathcal{A}C_0 + C_\infty - \mathcal{A}C_1 + C_0 - \mathcal{A}C_\infty) = \mathcal{A}(x - C) + C. \tag{4.6.77}$$

This means that $\mathcal{T}$ defines the rotoreflection on $\mathbb{R}^3$ composed of the rotation by the angle $\pm\frac{2\pi}{3}$ around the trinoid normal $A_n = C + \mathbb{R}n$ and the reflection in the trinoid plane $E = C + (\mathbb{R}n)^\perp$.

If $\sigma = (0\ \infty\ 1)$, we observe that $\mathcal{T}^{-1}$ corresponds to $\sigma^{-1} = (0\ 1\ \infty)$ and is also a symmetry of ϕ: $\mathcal{T}^{-1}(\phi(M)) = \phi(M)$. Thus (as shown above), $\mathcal{T}^{-1} = \hat{\mathcal{S}}$ with $\hat{\mathcal{S}}$ as given above. Consequently, we have $\mathcal{T} = \hat{\mathcal{S}}^{-1}$. □

Theorem 4.31 explicitly lists the twelve Euclidean motions on $\mathbb{R}^3$, which qualify as possible symmetries of the given trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends. In the case that ϕ is actually symmetric with respect to one of these, say $\mathcal{T}$, lemma 4.21 provides the associated biholomorphic (resp. bi-antiholomorphic) mapping $\gamma : M \to M$, which allows for translating the symmetry property to the level of the trinoid domain M: $\mathcal{T} \circ \phi = \phi \circ \gamma$. This enables us, based on the results of section 4.4, to study in detail the impact of the possible symmetries of ϕ on the monodromy matrices of the extended frame F associated with the conformal CMC-immersion $\psi := \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where $\pi : \tilde{M} \to M$ denotes the universal covering defined in (3.2.2). This is done in the sections 5 to 9.

5 Rotational symmetry with respect to the trinoid normal

5.1 Definition

In this section we discuss trinoids $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends on $M = \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ which are symmetric in the sense of definition 4.2 with respect to the rotation $\mathcal{R}$ by the angle $\pm\frac{2\pi}{3}$ around the trinoid normal $A_n = \{C + \lambda n; \lambda \in \mathbb{R}\}$, where C denotes the trinoid center, and n denotes a normal vector of the trinoid plane E. (Recall from theorem 4.31 that, in the case that a trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends is symmetric with respect to the given Euclidean motion $\mathcal{R}$, there exists a unique trinoid plane and a unique trinoid normal of ϕ, which enables us to speak of *the* trinoid plane and *the* trinoid normal of ϕ, respectively.) $\mathcal{R}$ is uniquely determined by additionally prescribing that $\mathcal{R}$ permutes the trinoid ends according to the permutation $\sigma = (0\ 1\ \infty)$ of the set $\{0, 1, \infty\}$. Since we have

$$\mathcal{R}(\phi(M)) = \phi(M) \quad \Longleftrightarrow \quad \mathcal{R}^{-1}(\phi(M)) = \phi(M), \tag{5.1.1}$$

it is clear that a given trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends is symmetric with respect to $\mathcal{R}$, if and only if it is symmetric with respect $\mathcal{R}^{-1}$, defining the rotation around the trinoid normal A_n by the inverse angle $\mp\frac{2\pi}{3}$, permuting the trinoid ends according to the permutation $\sigma^{-1} = (0\ \infty\ 1)$.

Remark 5.1. Note that, in order to define (the sign of) the angle of rotation for $\mathcal{R}$, one first needs to determine an orientation of the axis of rotation of $\mathcal{R}$ itself. Depending on which choice we make for the orientation of the axis of rotation of $\mathcal{R}$, the angle of rotation of $\mathcal{R}$ will be either $+\frac{2\pi}{3}$ or $-\frac{2\pi}{3}$. Accordingly, the angls of rotation of $\mathcal{R}^{-1}$ will be either $-\frac{2\pi}{3}$ or $+\frac{2\pi}{3}$. However, for our purposes it suffices to characterize $\mathcal{R}$ (resp. $\mathcal{R}^{-1}$) by the property that it permutes the trinoid ends according to the permutation $\sigma = (0\ 1\ \infty)$ (resp. $\sigma^{-1} = (0\ \infty\ 1)$).

Definition 5.2. Let $M = \mathbb{C} \setminus \{0, 1\}$ and $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends. Let $\tilde{M} = \mathbb{H}$ and $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$ the conformal CMC-immersion associated with ϕ via the universal covering $\pi : \tilde{M} \to M$ given in (3.2.2). Let $A_n = \{C + \lambda n; \lambda \in \mathbb{R}\}$, where C denotes the trinoid center and n a normal vector of the trinoid plane E, be the trinoid normal. Then, if ϕ (or, equivalently, ψ) is symmetric with respect to the rotation $\mathcal{R}$ by the angle $\pm\frac{2\pi}{3}$ around A_n, which permutes the trinoid ends according to the permutation $\sigma = (0\ 1\ \infty)$ of the set $\{0, 1, \infty\}$,

$$\mathcal{R}(\phi(M)) = \phi(M), \quad \mathcal{R}(\psi(\tilde{M})) = \psi(\tilde{M}), \tag{5.1.2}$$

or, equivalently, if ϕ (or, equivalently, ψ) is symmetric with respect to the inverse rotation $\mathcal{R}^{-1}$,

$$\mathcal{R}^{-1}(\phi(M)) = \phi(M), \quad \mathcal{R}^{-1}(\psi(\tilde{M})) = \psi(\tilde{M}), \tag{5.1.3}$$

ϕ (or ψ) is called *rotationally symmetric with respect to the trinoid normal.*

We are now going to apply the results of the previous sections in order to translate the rotational symmetry of ϕ into further constraints on the functions p_0, p_1, q_0, q_1. Recall that p_0, p_1, q_0, q_1 occur in the monodromy matrices of the extended frame F of the conformal CMC-immersion $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$ associated with ϕ via the universal covering $\pi : \tilde{M} \to M$. So the question we now actually turn to is: Which monodromy matrices are possible for rotationally symmetric trinoids with properly embedded annular ends?

5.2 Implications of rotational symmetry with respect to the trinoid normal

The following result is an immediate consequence of definition 5.2:

Lemma 5.3. *Let $M = \mathbb{C} \setminus \{0, 1\}$ and $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends produced from a trinoid potential η as in theorem 3.14. Denote by D_0, D_1, D_∞ the corresponding Delaunay matrices with eigenvalues $\pm\mu_0$, $\pm\mu_1$, $\pm\mu_\infty$, respectively, where, for $j \in \{0, 1, \infty\}$, $\mu_j = \sqrt{X_j \overline{X_j}} = \sqrt{\frac{1}{4} + w_j(\lambda - \lambda^{-1})^2}$ and $w_j = s_j t_j$ as in section 3.5. Then, if ϕ is rotationally symmetric with respect to the trinoid normal, we have*

$$\mu := \mu_0 = \mu_1 = \mu_\infty = \sqrt{\frac{1}{4} + w(\lambda - \lambda^{-1})^2}, \tag{5.2.1}$$

where

$$w := w_0 = w_1 = w_\infty. \tag{5.2.2}$$

Proof. By definition 5.2, the ends of a trinoid with properly embedded annular ends, which is rotationally symmetric with respect to the trinoid normal, are rotated by the corresponding symmetry $\mathcal{R}$ (resp. $\mathcal{R}^{-1}$) into each other according to the permutation $\sigma = (0\ 1\ \infty)$ (resp. $\sigma^{-1} = (0\ \infty\ 1)$). This means that the asymptotic Delaunay surfaces associated with the ends are rotated into each other as well. Hence, these Delaunay surfaces only differ by a rigid motion on $\mathbb{R}^3$. In particular, this implies that the corresponding Delaunay matrices D_j, $j = 0, 1, \infty$, (see section 3.5 for more details) all possess the same eigenvalues. This yields $\mu_0 = \mu_1 = \mu_\infty$ and allows for defining $\mu := \mu_0 = \mu_1 = \mu_\infty$. Using lemma B.6, we infer that $w_0 = w_1 = w_\infty$, whence w given in (5.2.2) is well defined. Consequently, $\mu = \sqrt{\frac{1}{4} + w(\lambda - \lambda^{-1})^2}$ holds, which finishes the proof. □

Let $M = \mathbb{C} \setminus \{0, 1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ given in (3.2.2). Suppose ϕ (or, equivalently, ψ) is rotationally symmetric with respect to the trinoid normal, and denote the corresponding symmetry by $\mathcal{R}$. Since $\mathcal{R}$ preserves orientation on $\mathbb{R}^3$, we obtain by theorem 4.9 a pair of biholomorphic mappings, $\gamma_{\mathcal{R}} : M \to M$ and $\tilde{\gamma}_{\mathcal{R}} : \tilde{M} \to \tilde{M}$ satisfying

$$\mathcal{R} \circ \phi = \phi \circ \gamma_{\mathcal{R}}, \tag{5.2.3}$$
$$\mathcal{R} \circ \psi = \psi \circ \tilde{\gamma}_{\mathcal{R}}, \tag{5.2.4}$$
$$\pi \circ \tilde{\gamma}_{\mathcal{R}} = \gamma_{\mathcal{R}} \circ \pi. \tag{5.2.5}$$

Analogously, we obtain for $\mathcal{R}^{-1}$ a pair of biholomorphic mappings, $\gamma_{\mathcal{R}^{-1}} : M \to M$ and $\tilde{\gamma}_{\mathcal{R}^{-1}} : \tilde{M} \to \tilde{M}$ satisfying

$$\mathcal{R}^{-1} \circ \phi = \phi \circ \gamma_{\mathcal{R}^{-1}}, \tag{5.2.6}$$
$$\mathcal{R}^{-1} \circ \psi = \psi \circ \tilde{\gamma}_{\mathcal{R}^{-1}}, \tag{5.2.7}$$
$$\pi \circ \tilde{\gamma}_{\mathcal{R}^{-1}} = \gamma_{\mathcal{R}^{-1}} \circ \pi. \tag{5.2.8}$$

The mappings $\gamma_{\mathcal{R}}$ and $\gamma_{\mathcal{R}^{-1}}$ are uniquely determined and explicitly given by lemma 4.21:

$$\gamma_{\mathcal{R}}(z) = \frac{1}{1 - z}, \tag{5.2.9}$$
$$\gamma_{\mathcal{R}^{-1}}(z) = \frac{z - 1}{z}. \tag{5.2.10}$$

The mappings $\tilde{\gamma}_{\mathcal{R}}$ and $\tilde{\gamma}_{\mathcal{R}^{-1}}$ are uniquely determined up to composition from the left with an element of the automorphism group $\mathrm{Aut}(\tilde{M}/M)$ of π. The following lemma explicitly states a pair of valid choices for $\tilde{\gamma}_{\mathcal{R}}$ and $\tilde{\gamma}_{\mathcal{R}^{-1}}$:

Lemma 5.4. *Let $M = \mathbb{C} \setminus \{0, 1\}$, $\tilde{M} = \mathbb{H}$ and $\pi : \tilde{M} \to M$ be the universal covering as given in* (3.2.2)*. Let $\gamma_{\mathcal{R}} : M \to M$ and $\gamma_{\mathcal{R}^{-1}} : M \to M$ be given by* (5.2.9) *and* (5.2.10)*, respectively. Then, the following holds:*

1. *The mapping $\tilde{\gamma}_{\mathcal{R}} : \tilde{M} \to \tilde{M}$,*

$$\tilde{\gamma}_{\mathcal{R}}(z) = \frac{-z - 1}{z}, \tag{5.2.11}$$

is biholomorphic and satisfies

$$\pi \circ \tilde{\gamma}_{\mathcal{R}} = \gamma_{\mathcal{R}} \circ \pi, \tag{5.2.12}$$
$$\mathcal{R} \circ \psi = \psi \circ \tilde{\gamma}_{\mathcal{R}}. \tag{5.2.13}$$

2. *The mapping $\tilde{\gamma}_{\mathcal{R}^{-1}} : \tilde{M} \to \tilde{M}$,*

$$\tilde{\gamma}_{\mathcal{R}^{-1}}(z) = \frac{1}{-z - 1}, \tag{5.2.14}$$

is biholomorphic and satisfies

$$\pi \circ \tilde{\gamma}_{\mathcal{R}^{-1}} = \gamma_{\mathcal{R}^{-1}} \circ \pi, \tag{5.2.15}$$
$$\mathcal{R}^{-1} \circ \psi = \psi \circ \tilde{\gamma}_{\mathcal{R}^{-1}}. \tag{5.2.16}$$

Proof. We start with proving the first claim. Clearly, $\tilde{\gamma}_{\mathcal{R}}$ defines a Moebius transformation and thus a biholomorphic mapping $\tilde{M} \to \tilde{M}$. Moreover, by applying the relations (3.2.10) and (3.2.11) of lemma 3.4, we obtain for all $z \in \tilde{M}$

$$\pi \circ \tilde{\gamma}_{\mathcal{R}}(z) = \pi\left(\frac{-z-1}{z}\right) = \pi\left(-1 - \frac{1}{z}\right) = \frac{1}{\pi\left(-\frac{1}{z}\right)} = \frac{1}{1-\pi(z)} = \gamma_{\mathcal{R}} \circ \pi(z), \tag{5.2.17}$$

i.e. $\pi \circ \tilde{\gamma}_{\mathcal{R}} = \gamma_{\mathcal{R}} \circ \pi$. Finally,

$$\mathcal{R} \circ \psi = \mathcal{R} \circ \phi \circ \pi = \phi \circ \gamma_{\mathcal{R}} \circ \pi = \phi \circ \pi \circ \tilde{\gamma}_{\mathcal{R}} = \psi \circ \tilde{\gamma}_{\mathcal{R}}, \tag{5.2.18}$$

i.e. $\mathcal{R} \circ \psi = \psi \circ \tilde{\gamma}_{\mathcal{R}}$.

Now we turn to the second claim. Clearly, $\tilde{\gamma}_{\mathcal{R}^{-1}}$ defines a Moebius transformation and thus a biholomorphic mapping $\tilde{M} \to \tilde{M}$. Moreover, by applying the relations (3.2.11) and (3.2.10) of lemma 3.4, we obtain for all $z \in \tilde{M}$

$$\pi \circ \tilde{\gamma}_{\mathcal{R}^{-1}}(z) = \pi\left(\frac{1}{-z-1}\right) = 1 - \pi(z+1) = 1 - \frac{1}{\pi(z)} = \frac{\pi(z)-1}{\pi(z)} = \gamma_{\mathcal{R}^{-1}} \circ \pi(z), \tag{5.2.19}$$

i.e. $\pi \circ \tilde{\gamma}_{\mathcal{R}^{-1}} = \gamma_{\mathcal{R}^{-1}} \circ \pi$. Finally,

$$\mathcal{R}^{-1} \circ \psi = \mathcal{R}^{-1} \circ \phi \circ \pi = \phi \circ \gamma_{\mathcal{R}^{-1}} \circ \pi = \phi \circ \pi \circ \tilde{\gamma}_{\mathcal{R}^{-1}} = \psi \circ \tilde{\gamma}_{\mathcal{R}^{-1}}, \tag{5.2.20}$$

i.e. $\mathcal{R}^{-1} \circ \psi = \psi \circ \tilde{\gamma}_{\mathcal{R}^{-1}}$. □

Remark 5.5. Note that, since $\tilde{\gamma}_{\mathcal{R}} \circ \tilde{\gamma}_{\mathcal{R}^{-1}} = \tilde{\gamma}_{\mathcal{R}^{-1}} \circ \tilde{\gamma}_{\mathcal{R}} = \mathrm{id}$ for the mappings $\tilde{\gamma}_{\mathcal{R}}$ and $\tilde{\gamma}_{\mathcal{R}^{-1}}$ defined in (5.2.11) and (5.2.14), respectively, we have

$$\tilde{\gamma}_{\mathcal{R}^{-1}} = \tilde{\gamma}_{\mathcal{R}}^{-1}. \tag{5.2.21}$$

By the above lemma, we have explicitly determined mappings $\tilde{\gamma}_{\mathcal{R}}$ and $\tilde{\gamma}_{\mathcal{R}^{-1}}$ corresponding to the trinoid symmetries $\mathcal{R}$ and $\mathcal{R}^{-1}$, respectively, in the sense of theorem 4.9. Thus, we can apply theorem 4.17 to obtain

Theorem 5.6. *Let $M = \mathbb{C} \setminus \{0,1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion of $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ as defined in (3.2.2). Let ϕ be rotationally symmetric with respect to the trinoid normal. Denote by $\mathcal{R}$ and $\mathcal{R}^{-1}$ the corresponding symmetries permuting the trinoid ends according to the permutations $\sigma = (0\ 1\ \infty)$ and $\sigma^{-1} = (0\ \infty\ 1)$, respectively. Moreover, denote by $\tilde{\gamma}_{\mathcal{R}}$ and by $\tilde{\gamma}_{\mathcal{R}^{-1}}$ the biholomorphic mappings $\tilde{M} \to \tilde{M}$ associated with $\mathcal{R}$ and $\mathcal{R}^{-1}$, respectively, as in theorem 4.9 and explicitly given in lemma 5.4. Then, the following holds:*

1. *The extended frame $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ corresponding to ψ by theorem 4.5 transforms under $\tilde{\gamma}_{\mathcal{R}}$ as*

$$F(\tilde{\gamma}_{\mathcal{R}}(z), \lambda) = M_{\mathcal{R}}(\lambda) F(z, \lambda) k_{\mathcal{R}, \tilde{\gamma}_{\mathcal{R}}}(z), \tag{5.2.22}$$

where

$$k_{\mathcal{R}, \tilde{\gamma}_{\mathcal{R}}}(z) = \begin{pmatrix} \sqrt{\frac{\bar{z}}{z}} & 0 \\ 0 & \sqrt{\frac{\bar{z}}{z}} \end{pmatrix}, \tag{5.2.23}$$

and $M_{\mathcal{R}}$ denotes an element of $\Lambda\mathrm{SU}(2)_\sigma$, which is independent of z.

2. *The extended frame $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ corresponding to ψ by theorem 4.5 transforms under $\tilde{\gamma}_{\mathcal{R}^{-1}}$ as*

$$F(\tilde{\gamma}_{\mathcal{R}^{-1}}(z), \lambda) = M_{\mathcal{R}^{-1}}(\lambda) F(z, \lambda) k_{\mathcal{R}^{-1}, \tilde{\gamma}_{\mathcal{R}^{-1}}}(z), \tag{5.2.24}$$

where

$$k_{\mathcal{R}^{-1}, \tilde{\gamma}_{\mathcal{R}^{-1}}}(z) = \begin{pmatrix} \sqrt{\frac{\bar{z}+1}{z+1}} & 0 \\ 0 & \sqrt{\frac{\bar{z}+1}{z+1}} \end{pmatrix} \tag{5.2.25}$$

and $M_{\mathcal{R}^{-1}}$ denotes an element of $\Lambda\mathrm{SU}(2)_\sigma$, which is independent of z.

Proof. We start with the proof of the first part. Let $\tilde{\gamma}(z) = \tilde{\gamma}_{\mathcal{R}}(z) = \frac{-z-1}{z}$ for all $z \in \tilde{M} = \mathbb{H}$. (For convenience we omit the index $\mathcal{R}$ throughout this proof.) As $\mathcal{R}$ preserves orientation on $\mathbb{R}^3$, we apply the first part of theorem 4.17 to obtain

$$F(\tilde{\gamma}(z), \lambda) = M_{\tilde{\gamma}}(\lambda) F(z, \lambda) k_{\mathcal{R},\tilde{\gamma}}(z), \tag{5.2.26}$$

where $F : \tilde{M} \to \Lambda \mathrm{SU}(2)_\sigma$ denotes the extended frame corresponding to ψ by theorem 4.5 and $M_{\tilde{\gamma}}$ denotes an element of $\Lambda \mathrm{SU}(2)_\sigma$, which is independent of z. $k_{\mathcal{R},\tilde{\gamma}}(z)$ is given by equation (4.4.117) from lemma 4.18. By computing

$$\partial_z \tilde{\gamma}(z) = \frac{1}{z^2} \tag{5.2.27}$$

we infer that

$$\frac{\partial_z \tilde{\gamma}(z)}{\mid \partial_z \tilde{\gamma}(z) \mid} = \frac{\mid z \mid^2}{z^2} = \frac{\bar{z}}{z} \tag{5.2.28}$$

and thus obtain from (4.4.117)

$$k_{\mathcal{R},\tilde{\gamma}}(z) = \begin{pmatrix} \sqrt{\frac{\bar{z}}{z}} & 0 \\ 0 & \sqrt{\frac{\bar{z}}{z}} \end{pmatrix}. \tag{5.2.29}$$

As $\tilde{\gamma} = \tilde{\gamma}_{\mathcal{R}}$, we denote $M_{\tilde{\gamma}}$ by $M_{\mathcal{R}}$. This finishes the proof of equation (5.2.22).

To prove the second part of the theorem, we define $\tilde{\gamma}(z) = \tilde{\gamma}_{\mathcal{R}^{-1}}(z) = \frac{1}{-z-1}$ on $\tilde{M} = \mathbb{H}$. Everything is then done analogously. We have

$$\partial_z \tilde{\gamma}(z) = \frac{1}{(z+1)^2} \tag{5.2.30}$$

and thus

$$\frac{\partial_z \tilde{\gamma}(z)}{\mid \partial_z \tilde{\gamma}(z) \mid} = \frac{\mid z+1 \mid^2}{(z+1)^2} = \frac{\bar{z}+1}{z+1}. \tag{5.2.31}$$

Formula (4.4.117) from lemma 4.18 then yields

$$k_{\mathcal{R}^{-1},\tilde{\gamma}}(z) = \begin{pmatrix} \sqrt{\frac{\bar{z}+1}{z+1}} & 0 \\ 0 & \sqrt{\frac{\bar{z}+1}{z+1}} \end{pmatrix}, \tag{5.2.32}$$

and by setting $M_{\mathcal{R}^{-1}}(\lambda) := M_{\tilde{\gamma}}(\lambda)$, the first part of theorem 4.17 implies (5.2.24). □

Remark 5.7. The monodromy matrices $M_{\mathcal{R}}$ and $M_{\mathcal{R}^{-1}}$ of F under the biholomorphic mappings $\tilde{\gamma}_{\mathcal{R}}$ and $\tilde{\gamma}_{\mathcal{R}^{-1}}$ are linked as follows: As $\tilde{\gamma}_{\mathcal{R}^{-1}} = \tilde{\gamma}_{\mathcal{R}}^{-1}$, we have

$$F(z, \lambda) = F((\tilde{\gamma}_{\mathcal{R}} \circ \tilde{\gamma}_{\mathcal{R}^{-1}})(z), \lambda) = M_{\mathcal{R}}(\lambda) F(\tilde{\gamma}_{\mathcal{R}^{-1}}(z), \lambda) k_{\mathcal{R},\tilde{\gamma}_{\mathcal{R}}}(\tilde{\gamma}_{\mathcal{R}^{-1}}(z)) \tag{5.2.33}$$

and thus

$$M_{\mathcal{R}^{-1}}(\lambda) F(z, \lambda) k_{\mathcal{R}^{-1},\tilde{\gamma}_{\mathcal{R}^{-1}}}(z) = F(\tilde{\gamma}_{\mathcal{R}^{-1}}(z), \lambda) = (M_{\mathcal{R}}(\lambda))^{-1} F(z, \lambda) (k_{\mathcal{R},\tilde{\gamma}_{\mathcal{R}}}(\tilde{\gamma}_{\mathcal{R}^{-1}}(z)))^{-1}. \tag{5.2.34}$$

A direct computation yields $k_{\mathcal{R}^{-1},\tilde{\gamma}_{\mathcal{R}^{-1}}}(z) = \pm((k_{\mathcal{R},\tilde{\gamma}_{\mathcal{R}}}(\tilde{\gamma}_{\mathcal{R}^{-1}}(z)))^{-1}$, which implies

$$M_{\mathcal{R}^{-1}}(\lambda) = \pm((M_{\mathcal{R}}(\lambda))^{-1}. \tag{5.2.35}$$

5.3 Monodromy matrices of trinoids with properly embedded annular ends, which are rotationally symmetric with respect to the trinoid normal

Using the results of the previous section we are now able to describe the (unitary) monodromy matrices $\hat{M}_0, \hat{M}_1, \hat{M}_\infty$ associated with a trinoid with properly embedded annular ends, which is rotationally symmetric with respect to the trinoid normal. As a start, recall from section 3.3 the covering transformations $\tilde{\gamma}_j$, $j = 0, 1, \infty$, on $\tilde{M}$ generating the monodromy matrices $\hat{M}_j$, $j = 0, 1, \infty$:

$$\tilde{\gamma}_0(z) = \frac{z}{-2z+1} \tag{5.3.1}$$

$$\tilde{\gamma}_1(z) = z + 2 \tag{5.3.2}$$

$$\tilde{\gamma}_\infty(z) = \frac{-3z-2}{2z+1}. \tag{5.3.3}$$

Lemma 5.8. *Let $\tilde{M} = \mathbb{H}$ and $\tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\gamma}_\infty : \tilde{M} \to \tilde{M}$ be given as above.*

1. *For $\tilde{\gamma}_\mathcal{R} : \tilde{M} \to \tilde{M}, \tilde{\gamma}_\mathcal{R}(z) = \frac{-z-1}{z}$, the following identities hold:*

$$\tilde{\gamma}_\mathcal{R} \circ \tilde{\gamma}_0 = \tilde{\gamma}_1 \circ \tilde{\gamma}_\mathcal{R}, \quad \tilde{\gamma}_\mathcal{R} \circ \tilde{\gamma}_1 = \tilde{\gamma}_\infty \circ \tilde{\gamma}_\mathcal{R}, \quad \tilde{\gamma}_\mathcal{R} \circ \tilde{\gamma}_\infty = \tilde{\gamma}_0 \circ \tilde{\gamma}_\mathcal{R}. \tag{5.3.4}$$

2. *For $\tilde{\gamma}_{\mathcal{R}^{-1}} : \tilde{M} \to \tilde{M}, \tilde{\gamma}_{\mathcal{R}^{-1}}(z) = \frac{1}{-z-1}$, the following identities hold:*

$$\tilde{\gamma}_0 \circ \tilde{\gamma}_{\mathcal{R}^{-1}} = \tilde{\gamma}_{\mathcal{R}^{-1}} \circ \tilde{\gamma}_1, \quad \tilde{\gamma}_1 \circ \tilde{\gamma}_{\mathcal{R}^{-1}} = \tilde{\gamma}_{\mathcal{R}^{-1}} \circ \tilde{\gamma}_\infty, \quad \tilde{\gamma}_\infty \circ \tilde{\gamma}_{\mathcal{R}^{-1}} = \tilde{\gamma}_{\mathcal{R}^{-1}} \circ \tilde{\gamma}_0. \tag{5.3.5}$$

Proof. We start with the first part, i.e. $\tilde{\gamma}_\mathcal{R}(z) = \frac{-z-1}{z}$. The claim is proved by straightforward computation: For $z \in \tilde{M}$ we have

$$\tilde{\gamma}_\mathcal{R} \circ \tilde{\gamma}_0(z) = \tilde{\gamma}_\mathcal{R}\left(\frac{z}{-2z+1}\right) = \frac{z-1}{z} = \tilde{\gamma}_1\left(\frac{-z-1}{z}\right) = \tilde{\gamma}_1 \circ \tilde{\gamma}_\mathcal{R}(z) \tag{5.3.6}$$

$$\tilde{\gamma}_\mathcal{R} \circ \tilde{\gamma}_1(z) = \tilde{\gamma}_\mathcal{R}(z+2) = -\frac{z+3}{z+2} = \tilde{\gamma}_\infty\left(\frac{-z-1}{z}\right) = \tilde{\gamma}_1 \circ \tilde{\gamma}_\mathcal{R}(z) \tag{5.3.7}$$

$$\tilde{\gamma}_\mathcal{R} \circ \tilde{\gamma}_\infty(z) = \tilde{\gamma}_\mathcal{R}\left(\frac{-3z-2}{2z+1}\right) = -\frac{z+1}{3z+2} = \tilde{\gamma}_0\left(\frac{-z-1}{z}\right) = \tilde{\gamma}_0 \circ \tilde{\gamma}_\mathcal{R}(z). \tag{5.3.8}$$

Now considering the second part of the lemma, we have $\tilde{\gamma}_{\mathcal{R}^{-1}}(z) = \frac{1}{-z-1}$. Observe that this mapping is the inverse function of $\tilde{\gamma}_\mathcal{R}$ given in the first part. So the identities (5.3.5) follow directly from the first part by applying the automorphism $\tilde{\gamma}_{\mathcal{R}^{-1}}$ from both the left hand side and the right hand side to the identities (5.3.4). □

The above lemma is needed to prove the following theorem, which states further necessary conditions on the monodromy matrices of the extended frame F associated with a trinoid with properly embedded annular ends, which is rotationally symmetric with respect to the trinoid normal.

Theorem 5.9. *Let $M = \mathbb{C} \setminus \{0,1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ as defined in* (3.2.2)*. Let ϕ be rotationally symmetric with respect to the trinoid normal. Denote by $\mathcal{R}$ and $\mathcal{R}^{-1}$ the corresponding symmetries permuting the trinoid ends according to the permutations $\sigma = (0\ 1\ \infty)$ and $\sigma^{-1} = (0\ \infty\ 1)$, respectively. Furthermore, let $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ be the extended frame associated with ψ by theorem 4.5. Denote by $\hat{M}_0, \hat{M}_1, \hat{M}_\infty \in \Lambda\mathrm{SU}(2,\mathbb{C})_\sigma$ the unitary monodromy matrices*

$$\hat{M}_j = -\left[\cos(2\pi\mu_j)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i\sin(2\pi\mu_j)\begin{pmatrix}p_j & \overline{q_j}\\ q_j & -p_j\end{pmatrix}\right] \tag{5.3.9}$$

associated with F as in (4.5.13) *by*

$$F(\tilde{\gamma}_j(z), \lambda) = \alpha_j \hat{M}_j(\lambda) F(z,\lambda) k_j(z), \quad j = 0, 1, \infty, \tag{5.3.10}$$

where $\alpha_j \in \{\pm 1\}$ and $\tilde{\gamma}_j$ denote the covering transformations on $\tilde{M}$ from section 3.3. Finally, let $\tilde{\gamma}_\mathcal{R}$, $\tilde{\gamma}_{\mathcal{R}^{-1}}$ be the biholomorphic mappings $\tilde{M} \to \tilde{M}$ associated with $\mathcal{R}$ and $\mathcal{R}^{-1}$, respectively, as in theorem 4.9 and explicitly given in lemma 5.4, and let $M_\mathcal{R}(\lambda)$, $M_{\mathcal{R}^{-1}}(\lambda)$ be the corresponding monodromy matrices of F as given in equations (5.2.22) *and* (5.2.24)*. In view of remark 5.7, we set*

$$M_\mathcal{R}(\lambda) = \pm(M_{\mathcal{R}^{-1}}(\lambda))^{-1} =: \begin{pmatrix}a_\mathcal{R} & b_\mathcal{R}\\ -\overline{b_\mathcal{R}} & \overline{a_\mathcal{R}}\end{pmatrix}. \tag{5.3.11}$$

Then, the monodromy matrices satisfy

$$\hat{M}_1(\lambda) = M_\mathcal{R}(\lambda)\hat{M}_0(\lambda)M_\mathcal{R}(\lambda)^{-1}, \tag{5.3.12}$$

$$\hat{M}_\infty(\lambda) = M_\mathcal{R}(\lambda)\hat{M}_1(\lambda)M_\mathcal{R}(\lambda)^{-1}, \tag{5.3.13}$$

$$\hat{M}_0(\lambda) = M_\mathcal{R}(\lambda)\hat{M}_\infty(\lambda)M_\mathcal{R}(\lambda)^{-1}. \tag{5.3.14}$$

In terms of the functions p_j and q_j occurring in $\hat{M}_j$, equations (5.3.12) *to* (5.3.14) *read as*

$$p_1 = a_\mathcal{R}\overline{a_\mathcal{R}}p_0 + \overline{a_\mathcal{R}}b_\mathcal{R}q_0 + a_\mathcal{R}\overline{b_\mathcal{R}}\overline{q_0} - b_\mathcal{R}\overline{b_\mathcal{R}}p_0, \tag{5.3.15}$$

$$q_1 = -2\overline{a_\mathcal{R}}\overline{b_\mathcal{R}}p_0 + \overline{a_\mathcal{R}}^2 q_0 - \overline{b_\mathcal{R}}^2\overline{q_0}, \tag{5.3.16}$$

$$p_\infty = a_\mathcal{R}\overline{a_\mathcal{R}}p_1 + \overline{a_\mathcal{R}}b_\mathcal{R}q_1 + a_\mathcal{R}\overline{b_\mathcal{R}}\overline{q_1} - b_\mathcal{R}\overline{b_\mathcal{R}}p_1, \tag{5.3.17}$$
$$q_\infty = -2\overline{a_\mathcal{R}}\overline{b_\mathcal{R}}p_1 + \overline{a_\mathcal{R}}^2 q_1 - \overline{b_\mathcal{R}}^2\overline{q_1}, \tag{5.3.18}$$

$$p_0 = a_\mathcal{R}\overline{a_\mathcal{R}}p_\infty + \overline{a_\mathcal{R}}b_\mathcal{R}q_\infty + a_\mathcal{R}\overline{b_\mathcal{R}}\overline{q_\infty} - b_\mathcal{R}\overline{b_\mathcal{R}}p_\infty, \tag{5.3.19}$$
$$q_0 = -2\overline{a_\mathcal{R}}\overline{b_\mathcal{R}}p_\infty + \overline{a_\mathcal{R}}^2 q_\infty - \overline{b_\mathcal{R}}^2\overline{q_\infty}. \tag{5.3.20}$$

Proof. Consider the biholomorphic mapping $\tilde{\gamma}_\mathcal{R} : \tilde{M} \to \tilde{M}$ given in (5.2.11): $\tilde{\gamma}_\mathcal{R}(z) = \frac{-z-1}{z}$. Applying (the first part of) theorem 5.6 we obtain

$$F(\tilde{\gamma}_\mathcal{R}(z), \lambda) = M_\mathcal{R}(\lambda)F(z, \lambda)k_{\mathcal{R},\tilde{\gamma}_\mathcal{R}}(z), \tag{5.3.21}$$

where

$$k_{\mathcal{R},\tilde{\gamma}_\mathcal{R}}(z) = \begin{pmatrix} \sqrt{\frac{\bar{z}}{z}} & 0 \\ 0 & \sqrt{\frac{\bar{z}}{z}} \end{pmatrix}. \tag{5.3.22}$$

Combining this with the monodromy equations (5.3.10), and applying the identities (5.3.4) from the above lemma, we compute:

$$\begin{aligned} M_\mathcal{R}(\lambda)\hat{M}_0(\lambda)F(z,\lambda)k_0(z)k_{\mathcal{R},\tilde{\gamma}_\mathcal{R}}(\tilde{\gamma}_0(z)) &= \alpha_0 M_\mathcal{R}(\lambda)F(\tilde{\gamma}_0(z),\lambda)k_{\mathcal{R},\tilde{\gamma}_\mathcal{R}}(\tilde{\gamma}_0(z)) = \alpha_0 F(\tilde{\gamma}_\mathcal{R}\circ\tilde{\gamma}_0(z),\lambda) \\ &= \alpha_0 F(\tilde{\gamma}_1 \circ \tilde{\gamma}_\mathcal{R}(z),\lambda) = \alpha_1\alpha_0\hat{M}_1(\lambda)F(\tilde{\gamma}_\mathcal{R}(z),\lambda)k_1(\tilde{\gamma}_\mathcal{R}(z)) \\ &= \alpha_1\alpha_0\hat{M}_1(\lambda)M_\mathcal{R}(\lambda)F(z,\lambda)k_{\mathcal{R},\tilde{\gamma}_\mathcal{R}}(z)k_1(\tilde{\gamma}_\mathcal{R}(z)). \end{aligned} \tag{5.3.23}$$

Analogously, we obtain

$$M_\mathcal{R}(\lambda)\hat{M}_1(\lambda)F(z,\lambda)k_1(z)k_{\mathcal{R},\tilde{\gamma}_\mathcal{R}}(\tilde{\gamma}_1(z)) = \alpha_\infty\alpha_1\hat{M}_\infty(\lambda)M_\mathcal{R}(\lambda)F(z,\lambda)k_{\mathcal{R},\tilde{\gamma}_\mathcal{R}}(z)k_\infty(\tilde{\gamma}_\mathcal{R}(z)), \tag{5.3.24}$$
$$M_\mathcal{R},1(\lambda)\hat{M}_\infty(\lambda)F(z,\lambda)k_\infty(z)k_{\mathcal{R},\tilde{\gamma}_\mathcal{R}}(\tilde{\gamma}_\infty(z)) = \alpha_0\alpha_\infty\hat{M}_0(\lambda)M_\mathcal{R}(\lambda)F(z,\lambda)k_{\mathcal{R},\tilde{\gamma}_\mathcal{R}}(z)k_0(\tilde{\gamma}_\mathcal{R}(z)). \tag{5.3.25}$$

As

$$\begin{aligned} k_0(z)k_{\mathcal{R},\tilde{\gamma}_\mathcal{R}}(\tilde{\gamma}_0(z)) &= \begin{pmatrix} \sqrt{\frac{1-2\bar{z}}{1-2z}} & 0 \\ 0 & \sqrt{\frac{1-2\bar{z}}{1-2z}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{\frac{\bar{z}}{-2\bar{z}+1}}{\frac{z}{-2z+1}}} & 0 \\ 0 & \sqrt{\frac{\frac{\bar{z}}{-2\bar{z}+1}}{\frac{z}{-2z+1}}} \end{pmatrix} = \pm \begin{pmatrix} \sqrt{\frac{\bar{z}}{z}} & 0 \\ 0 & \sqrt{\frac{\bar{z}}{z}} \end{pmatrix} \\ &= \pm \begin{pmatrix} \sqrt{\frac{\bar{z}}{z}} & 0 \\ 0 & \sqrt{\frac{\bar{z}}{z}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \pm k_{\mathcal{R},\tilde{\gamma}_\mathcal{R}}(z)k_1(\tilde{\gamma}_\mathcal{R}(z)), \end{aligned} \tag{5.3.26}$$

where changes in sign may occur due to the power rules for complex numbers, equation (5.3.23) implies

$$M_\mathcal{R}(\lambda)\hat{M}_0(\lambda) = \epsilon\hat{M}_1(\lambda)M_\mathcal{R}(\lambda) \tag{5.3.27}$$

with $\epsilon \in \{\pm\alpha_1\alpha_0\}$, and therefore

$$\hat{M}_1(\lambda) = \epsilon M_\mathcal{R}(\lambda)\hat{M}_0(\lambda)M_\mathcal{R}(\lambda)^{-1}. \tag{5.3.28}$$

Taking into account equation (5.3.9), we compare the upper left and the lower right entries of $\hat{M}_1(\lambda)$ and $\epsilon M_\mathcal{R}(\lambda)\hat{M}_0(\lambda)M_\mathcal{R}(\lambda)^{-1}$. This yields

$$\begin{aligned} &-\cos(2\pi\mu) - i\sin(2\pi\mu)p_1 = \\ &\qquad -\epsilon\cos(2\pi\mu) - i\epsilon\sin(2\pi\mu)\left[a_\mathcal{R}\overline{a_\mathcal{R}}p_0 + \overline{a_\mathcal{R}}b_\mathcal{R}q_0 + a_\mathcal{R}\overline{b_\mathcal{R}}\overline{q_0} - b_\mathcal{R}\overline{b_\mathcal{R}}p_0\right], \end{aligned} \tag{5.3.29}$$

$$\begin{aligned} &-\cos(2\pi\mu) + i\sin(2\pi\mu)p_1 = \\ &\qquad -\epsilon\cos(2\pi\mu) + i\epsilon\sin(2\pi\mu)\left[a_\mathcal{R}\overline{a_\mathcal{R}}p_0 + \overline{a_\mathcal{R}}b_\mathcal{R}q_0 + a_\mathcal{R}\overline{b_\mathcal{R}}\overline{q_0} - b_\mathcal{R}\overline{b_\mathcal{R}}p_0\right]. \end{aligned} \tag{5.3.30}$$

Adding up these equations, we obtain

$$-2\cos(2\pi\mu) = -2\epsilon\cos(2\pi\mu), \tag{5.3.31}$$

which, as $\cos(2\pi\mu)$ doesn't vanish identically, implies $\epsilon = 1$ and thus as claimed

$$\hat{M}_1(\lambda) = M_{\mathcal{R}}(\lambda)\hat{M}_0(\lambda)M_{\mathcal{R}}(\lambda)^{-1}. \tag{5.3.32}$$

Moreover, this equation translates equivalently into the scalar equations (omitting redundant ones)

$$p_1 = a_{\mathcal{R}}\overline{a_{\mathcal{R}}}p_0 + \overline{a_{\mathcal{R}}}b_{\mathcal{R}}q_0 + a_{\mathcal{R}}\overline{b_{\mathcal{R}}}\,\overline{q_0} - b_{\mathcal{R}}\overline{b_{\mathcal{R}}}p_0, \tag{5.3.33}$$

$$q_1 = -2\overline{a_{\mathcal{R}}}\,\overline{b_{\mathcal{R}}}p_0 + \overline{a_{\mathcal{R}}}^2 q_0 - \overline{b_{\mathcal{R}}}^2\overline{q_0}. \tag{5.3.34}$$

Similar to the argument given above, we have

$$\begin{aligned} k_1(z)k_{\mathcal{R},\tilde{\gamma}_{\mathcal{R}}}(\tilde{\gamma}_1(z)) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{\bar{z}+2}{z+2}} & 0 \\ 0 & \sqrt{\frac{\bar{z}+2}{z+2}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\bar{z}+2}{z+2}} & 0 \\ 0 & \sqrt{\frac{\bar{z}+2}{z+2}} \end{pmatrix} \\ &= \pm \begin{pmatrix} \sqrt{\frac{\bar{z}}{z}} & 0 \\ 0 & \sqrt{\frac{\bar{z}}{z}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1+2\frac{-\bar{z}-1}{\bar{z}}}{1+2\frac{-z-1}{z}}} & 0 \\ 0 & \sqrt{\frac{1+2\frac{-\bar{z}-1}{\bar{z}}}{1+2\frac{-z-1}{z}}} \end{pmatrix} = \pm k_{\mathcal{R},\tilde{\gamma}_{\mathcal{R}}}(z)k_\infty(\tilde{\gamma}_{\mathcal{R}}(z)) \end{aligned} \tag{5.3.35}$$

and

$$\begin{aligned} k_\infty(z)k_{\mathcal{R},\tilde{\gamma}_{\mathcal{R}}}(\tilde{\gamma}_\infty(z)) &= \begin{pmatrix} \sqrt{\frac{1+2\bar{z}}{1+2z}} & 0 \\ 0 & \sqrt{\frac{1+2\bar{z}}{1+2z}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{\frac{-3\bar{z}-2}{2\bar{z}+1}}{\frac{-3z-2}{2z+1}}} & 0 \\ 0 & \sqrt{\frac{\frac{-3\bar{z}-2}{2\bar{z}+1}}{\frac{-3z-2}{2z+1}}} \end{pmatrix} \\ = \pm \begin{pmatrix} \sqrt{\frac{3\bar{z}+2}{3z+2}} & 0 \\ 0 & \sqrt{\frac{3\bar{z}+2}{3z+2}} \end{pmatrix} &= \pm \begin{pmatrix} \sqrt{\frac{\bar{z}}{z}} & 0 \\ 0 & \sqrt{\frac{\bar{z}}{z}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1-2\frac{-\bar{z}-1}{\bar{z}}}{1-2\frac{-z-1}{z}}} & 0 \\ 0 & \sqrt{\frac{1-2\frac{-\bar{z}-1}{\bar{z}}}{1-2\frac{-z-1}{z}}} \end{pmatrix} = \pm k_{\mathcal{R},\tilde{\gamma}_{\mathcal{R}}}(z)k_0(\tilde{\gamma}_{\mathcal{R}}(z)), \end{aligned} \tag{5.3.36}$$

which leads by (5.3.24) and (5.3.25) to $M_{\mathcal{R}}(\lambda)\hat{M}_1(\lambda) = \pm\alpha_\infty\alpha_1\hat{M}_\infty(\lambda)M_{\mathcal{R}}(\lambda)$ and $M_{\mathcal{R}}(\lambda)\hat{M}_\infty(\lambda) = \pm\alpha_0\alpha_\infty\hat{M}_0(\lambda)M_{\mathcal{R}}(\lambda)$, respectively. From this we obtain

$$\hat{M}_\infty(\lambda) = \pm\alpha_\infty\alpha_1 M_{\mathcal{R}}(\lambda)\hat{M}_1(\lambda)M_{\mathcal{R}}(\lambda)^{-1}, \tag{5.3.37}$$

$$\hat{M}_0(\lambda) = \pm\alpha_0\alpha_\infty M_{\mathcal{R}}(\lambda)\hat{M}_\infty(\lambda)M_{\mathcal{R}}(\lambda)^{-1}. \tag{5.3.38}$$

Replacing in the argument above $(\hat{M}_1, \hat{M}_0)$ by $(\hat{M}_\infty, \hat{M}_1)$ and by $(\hat{M}_0, \hat{M}_\infty)$, respectively, we obtain $\pm\alpha_\infty\alpha_1 = \pm\alpha_0\alpha_\infty = 1$, which yields

$$\hat{M}_\infty(\lambda) = M_{\mathcal{R}}(\lambda)\hat{M}_1(\lambda)M_{\mathcal{R}}(\lambda)^{-1}, \tag{5.3.39}$$

$$\hat{M}_0(\lambda) = M_{\mathcal{R}}(\lambda)\hat{M}_\infty(\lambda)M_{\mathcal{R}}(\lambda)^{-1}, \tag{5.3.40}$$

and in scalar form

$$p_\infty = a_{\mathcal{R}}\overline{a_{\mathcal{R}}}p_1 + \overline{a_{\mathcal{R}}}b_{\mathcal{R}}q_1 + a_{\mathcal{R}}\overline{b_{\mathcal{R}}}\,\overline{q_1} - b_{\mathcal{R}}\overline{b_{\mathcal{R}}}p_1, \tag{5.3.41}$$

$$q_\infty = -2\overline{a_{\mathcal{R}}}\,\overline{b_{\mathcal{R}}}p_1 + \overline{a_{\mathcal{R}}}^2 q_1 - \overline{b_{\mathcal{R}}}^2\overline{q_1}, \tag{5.3.42}$$

$$p_0 = a_{\mathcal{R}}\overline{a_{\mathcal{R}}}p_\infty + \overline{a_{\mathcal{R}}}b_{\mathcal{R}}q_\infty + a_{\mathcal{R}}\overline{b_{\mathcal{R}}}\,\overline{q_\infty} - b_{\mathcal{R}}\overline{b_{\mathcal{R}}}p_\infty, \tag{5.3.43}$$

$$q_0 = -2\overline{a_{\mathcal{R}}}\,\overline{b_{\mathcal{R}}}p_\infty + \overline{a_{\mathcal{R}}}^2 q_\infty - \overline{b_{\mathcal{R}}}^2\overline{q_\infty}. \tag{5.3.44}$$

This finishes the proof. □

5.4 Normalized trinoids with properly embedded annular ends, which are rotationally symmetric with respect to the trinoid normal

Let $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends, which is rotationally symmetric with respect to the trinoid normal, and let $\psi = \phi \circ \pi$ be the associated CMC-immersion $\tilde{M} \to \mathbb{R}^3$. Denote by $\mathcal{R}$ and $\mathcal{R}^{-1}$ the corresponding symmetries of ϕ (and ψ), i.e. the rotations by the angles $\pm\frac{2\pi}{3}$ around the trinoid normal.

We specialize the results of section 5.3 to the case that the extended frame $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ associated with ψ as in section 4.2 is "normalized" at

$$z_* = \frac{-1 + i\sqrt{3}}{2} \in \tilde{M}, \tag{5.4.1}$$

i.e.,

$$F(z_*, \lambda) = \mathrm{I}, \tag{5.4.2}$$

for all $\lambda \in S^1$. The special choice of z_* results in more explicit requirements on the functions p_0, p_1, q_0, q_1 occuring in the monodromy matrices of F.

Recall from section 4.2, that the normalization $F(z_*, \lambda) = \mathrm{I}$ of the extended frame F at some point $z_* \in \tilde{M}$, or, more precisely, the underlying normalization of the (conformal) CMC-immersion ψ,

$$\psi(z_*) = \frac{1}{2H} e_3, \quad \mathcal{U}(z_*) = \mathcal{G}(1), \tag{5.4.3}$$

where $\mathcal{U} \in \mathrm{SO}(3)$ represents the natural orthonormal frame corresponding to ψ, and $\mathcal{G}(1)$ is given in (4.2.5), corresponds to rotating and shifting the (image of the) trinoid in $\mathbb{R}^3$, such that the conditions (5.4.3) are met. It turns out (cf. corollary 5.12), that the choice of z_* as in (5.4.1) corresponds to arranging the (image of the) trinoid in $\mathbb{R}^3$, such that the trinoid plane of ϕ is parallel to the x-y-plane in $\mathbb{R}^3$, and that the rotation axis of $\mathcal{R}$ (and of $\mathcal{R}^{-1}$) is the z-axis in $\mathbb{R}^3$.

A trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends, which is rotationally symmetric with respect to the trinoid normal and, in addition, is "well positioned" in $\mathbb{R}^3$ in the sense that the associated conformal CMC-immersion $\psi : \tilde{M} \to M$ meets the normalization conditions (5.4.3), is called a *normalized* trinoid with properly embedded annular ends, which is rotationally symmetric with respect to the trinoid normal.

We now formulate a more explicit version of theorem 5.6:

Theorem 5.10. *Let $M = \mathbb{C} \setminus \{0, 1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ as defined in* (3.2.2)*. Assume that ψ has been normalized at z_* given in* (5.4.1)*, such that $\psi(z_*) = \frac{1}{2H}e_3$ and $F(z_*, \lambda) = \mathrm{I}$, where $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ denotes the extended frame corresponding to ψ by theorem 4.5. Moreover, let ϕ be rotationally symmetric with respect to the trinoid normal. Denote by $\mathcal{R}$ and $\mathcal{R}^{-1}$ the corresponding symmetries permuting the trinoid ends according to the permutations $\sigma = (0\ 1\ \infty)$ and $\sigma^{-1} = (0\ \infty\ 1)$, respectively. Moreover, denote by $\tilde{\gamma}_{\mathcal{R}}$ and by $\tilde{\gamma}_{\mathcal{R}^{-1}}$ the biholomorphic mappings $\tilde{M} \to \tilde{M}$ associated with $\mathcal{R}$ and $\mathcal{R}^{-1}$, respectively, as in theorem 4.9 and explicitly given in* (5.2.11) *and* (5.2.14)*, respectively:*

$$\tilde{\gamma}_{\mathcal{R}}(z) = \frac{-z-1}{z}, \quad \tilde{\gamma}_{\mathcal{R}^{-1}}(z) = \frac{1}{-z-1}. \tag{5.4.4}$$

Then, the following holds:

1. *The extended frame F transforms under $\tilde{\gamma}_{\mathcal{R}}$ as*

$$F(\tilde{\gamma}_{\mathcal{R}}(z), \lambda) = M_{\mathcal{R}}(\lambda) F(z, \lambda) \begin{pmatrix} \sqrt{\frac{\bar{z}}{z}} & 0 \\ 0 & \sqrt{\frac{\bar{z}}{z}} \end{pmatrix}, \tag{5.4.5}$$

where

$$M_{\mathcal{R}}(\lambda) = \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix}. \tag{5.4.6}$$

In particular, $M_{\mathcal{R}}$ is actually independent of λ.

2. The extended frame F transforms under $\tilde{\gamma}_{\mathcal{R}^{-1}}$ as

$$F(\tilde{\gamma}_{\mathcal{R}^{-1}}(z),\lambda) = M_{\mathcal{R}^{-1}}(\lambda)F(z,\lambda)\begin{pmatrix}\sqrt{\frac{\bar{z}+1}{z+1}} & 0\\ 0 & \sqrt{\frac{\bar{z}+1}{z+1}}\end{pmatrix}, \tag{5.4.7}$$

where

$$M_{\mathcal{R}^{-1}}(\lambda) = \begin{pmatrix} e^{-\frac{\pi i}{3}} & 0\\ 0 & e^{\frac{\pi i}{3}}\end{pmatrix}. \tag{5.4.8}$$

In particular, $M_{\mathcal{R}^{-1}}$ is actually independent of λ.

Proof. In view of theorem 5.6, we only have to prove equations (5.4.6) and (5.4.8).

In the first case, a direct computation shows $\tilde{\gamma}_{\mathcal{R}}(z_*) = \frac{-z_*-1}{z_*} = z_*$. Furthermore, by assumption, $F(z_*,\lambda) = \mathrm{I}$. Keeping this in mind, we evaluate equation (5.4.5) at $z = z_*$ to obtain

$$\mathrm{I} = F(z_*,\lambda) = F(\tilde{\gamma}_{\mathcal{R}}(z_*),\lambda) = M_{\mathcal{R}}(\lambda)F(z_*,\lambda)\begin{pmatrix}\sqrt{\frac{\bar{z}_*}{z_*}} & 0\\ 0 & \sqrt{\frac{\bar{z}_*}{z_*}}\end{pmatrix} = M_{\mathcal{R}}(\lambda)\mathrm{I}\begin{pmatrix} e^{-\frac{\pi i}{3}} & 0\\ 0 & e^{\frac{\pi i}{3}}\end{pmatrix}, \tag{5.4.9}$$

where we have explicitly computed the occurring complex square roots according to remark 4.14. This yields (5.4.6).

In the second case, we have by a direct computation $\tilde{\gamma}_{\mathcal{R}^{-1}}(z_*) = \frac{1}{-z_*-1} = z_*$ as well as, by assumption, $F(z_*,\lambda) = \mathrm{I}$. Evaluating equation (5.4.7) at $z = z_*$, we infer that

$$\mathrm{I} = F(z_*,\lambda) = F(\tilde{\gamma}_{\mathcal{R}^{-1}}(z_*),\lambda) = M_{\mathcal{R}^{-1}}(\lambda)F(z_*,\lambda)\begin{pmatrix}\sqrt{\frac{\bar{z}_*+1}{z_*+1}} & 0\\ 0 & \sqrt{\frac{\bar{z}_*+1}{z_*+1}}\end{pmatrix} = M_{\mathcal{R}^{-1}}(\lambda)\mathrm{I}\begin{pmatrix} e^{\frac{\pi i}{3}} & 0\\ 0 & e^{-\frac{\pi i}{3}}\end{pmatrix}, \tag{5.4.10}$$

and thus (5.4.8). □

Remark 5.11. In the previous sections, we started with a trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends, which admits the rotational symmetry $\mathcal{R}$ around the trinoid normal. Note that in this general case, the extended frame $F(z,\lambda)$ associated with the conformal CMC-immersion $\psi = \phi \circ \pi$ produces by the Sym-Bobenko formula an associated family $\psi_\lambda : \tilde{M} \to \mathbb{R}^3$, $\lambda \in S^1$, of CMC-immersions, which are invariant under some Euclidean motion $\mathcal{R}_\lambda$, respectively, induced by the monodromy matrix $M_{\mathcal{R}}(\lambda)$ of F under the biholomorphic mapping $\tilde{\gamma}_{\mathcal{R}}$ associated with $\mathcal{R} = \mathcal{R}_{\lambda=1}$. (Note that, since for $\lambda \neq 1$ the conditions of theorem 2.11 are in general not met, ψ_λ will for $\lambda \neq 1$ in general *not* descend to a CMC-immersion $\phi_\lambda : M \to \mathbb{R}^3$. However, ψ_λ will be symmetric with respect to $\mathcal{R}_\lambda$.) Note that it is not clear, a priori, whether $\mathcal{R}_\lambda$ is a rotation (in $\mathbb{R}^3$).

In the special case considered in this section, i.e. in the case that the CMC-immersion ψ as well as the extended frame F associated with a given trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends, which admits the rotational symmetry $\mathcal{R}$ around the trinoid normal, have been normalized in the sense of (5.4.3) and (5.4.2), we observe that the monodromy matrix $M_{\mathcal{R}}(\lambda)$ of F under the biholomorphic mapping $\tilde{\gamma}_{\mathcal{R}}$ associated with $\mathcal{R}$ is actually independent of λ. Thus, each element of the associated family ψ_λ, $\lambda \in S^1$, of $\psi = \psi_{\lambda=1}$ generated by F via the Sym-Bobenko formula, admits the same rotational symmetry $\mathcal{R}$ in $\mathbb{R}^3$. (However, still, ψ_λ will for $\lambda \neq 1$ in general *not* descend to a CMC-immersion $\phi_\lambda : M \to \mathbb{R}^3$.)

Corollary 5.12. *We retain the notation and the assumptions of theorem 5.10. The axis of rotation of the symmetries $\mathcal{R}$ and $\mathcal{R}^{-1}$ of the normalized trinoid ϕ is the z-axis in $\mathbb{R}^3$. The trinoid plane of ϕ is parallel to the x-y-plane in $\mathbb{R}^3$.*

Proof. Applying (the first part of) theorem 4.17, we know that the monodromy matrices $M_{\mathcal{R}}(\lambda)$ and $M_{\mathcal{R}^{-1}}(\lambda)$ explicitly given in theorem 5.10 satisfy at $\lambda = 1$

$$M_{\mathcal{R}}(1) = \pm A_{\mathcal{R}}, \tag{5.4.11}$$

$$M_{\mathcal{R}^{-1}}(1) = \pm A_{\mathcal{R}^{-1}}, \tag{5.4.12}$$

where $A_{\mathcal{R}} \in \mathrm{SU}(2)$ (resp. $A_{\mathcal{R}^{-1}} \in \mathrm{SU}(2)$) denotes the conjugation matrix realizing the orthogonal part $\mathcal{A}_{\mathcal{R}}$ of the symmetry $\mathcal{R}$ (resp. the orthogonal part $\mathcal{A}_{\mathcal{R}^{-1}}$ of the symmetry $\mathcal{R}^{-1}$) in the su(2)-model. In view of equations (5.4.6) and (5.4.8), this yields

$$A_{\mathcal{R}} = \pm \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix}, \tag{5.4.13}$$

$$A_{\mathcal{R}^{-1}} = \pm \begin{pmatrix} e^{-\frac{\pi i}{3}} & 0 \\ 0 & e^{\frac{\pi i}{3}} \end{pmatrix}. \tag{5.4.14}$$

Recalling that $\mathcal{A}_{\mathcal{R}}$ and $A_{\mathcal{R}}$ (resp. $\mathcal{A}_{\mathcal{R}^{-1}}$ and $A_{\mathcal{R}^{-1}}$) are linked via the Lie Algebra isomorphism $J : \mathbb{R}^3 \to$ su(2) defined in (3.4.3) as in (3.4.7), i.e.

$$(J \circ \mathcal{A}_{\mathcal{R}} \circ J^{-1})(X) = A_{\mathcal{R}} X A_{\mathcal{R}}^{-1} \qquad \text{for all } X \in \mathrm{su}(2), \tag{5.4.15}$$

$$(J \circ \mathcal{A}_{\mathcal{R}^{-1}} \circ J^{-1})(X) = A_{\mathcal{R}^{-1}} X A_{\mathcal{R}^{-1}}^{-1} \qquad \text{for all } X \in \mathrm{su}(2), \tag{5.4.16}$$

we obtain by a direct computation that

$$\mathcal{A}_{\mathcal{R}} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{A}_{\mathcal{R}^{-1}} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.4.17}$$

Thus, $\mathcal{A}_{\mathcal{R}}$ and $\mathcal{A}_{\mathcal{R}^{-1}}$ define rotations (in $\mathbb{R}^3$) by the angles $\pm\frac{2\pi}{3}$ around the z-axis in $\mathbb{R}^3$, $\mathbb{R}e_3$. Consequently, the symmetries $\mathcal{R}$ and $\mathcal{R}^{-1}$ of the normalized trinoid ϕ are rotations by the angles $\pm\frac{2\pi}{3}$ around an axis in $\mathbb{R}^3$, which is parallel to the z-axis. In particular, the trinoid plane of ϕ, which is orthogonal to this axis of rotation, is parallel to the x-y-plane in $\mathbb{R}^3$. As the point $\psi(z_*) \in \mathbb{R}^3$ with z_* given in (5.4.1) satisfies

$$\mathcal{R}(\psi(z_*)) = \psi(\tilde{\gamma}_{\mathcal{R}}(z_*)) = \psi(z_*), \tag{5.4.18}$$

$$\mathcal{R}^{-1}(\psi(z_*)) = \psi(\tilde{\gamma}_{\mathcal{R}^{-1}}(z_*)) = \psi(z_*), \tag{5.4.19}$$

it lies on the common axis of rotation of $\mathcal{R}$ and $\mathcal{R}^{-1}$. Since by assumption we have $\psi(z_*) = \frac{1}{2H}e_3$, we infer that the axis of rotation of $\mathcal{R}$ and $\mathcal{R}^{-1}$ is actually the z-axis in $\mathbb{R}^3$. □

Applying theorems 5.9 and 5.10, we obtain the following result:

Theorem 5.13. *Let $M = \mathbb{C} \setminus \{0,1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ as defined in (3.2.2). Assume that ψ has been normalized at z_* given in (5.4.1), such that $\psi(z_*) = \frac{1}{2H}e_3$ and $F(z_*, \lambda) = \mathrm{I}$, where $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ denotes the extended frame corresponding to ψ by theorem 4.5. Let ϕ be rotationally symmetric with respect to the trinoid normal. Then, the unitary monodromy matrices $\hat{M}_0, \hat{M}_1, \hat{M}_\infty \in \Lambda\mathrm{SU}(2,\mathbb{C})_\sigma$ associated with F as in (5.3.10) are of the form*

$$\hat{M}_0 = -\cos(2\pi\mu)\mathrm{I} - \frac{2i}{\sqrt{3}}\cos(\pi\mu)\begin{pmatrix} \cos(\pi\mu) & \overline{\zeta_0} \\ \zeta_0 & -\cos(\pi\mu) \end{pmatrix}, \tag{5.4.20}$$

$$\hat{M}_1 = -\cos(2\pi\mu)\mathrm{I} - \frac{2i}{\sqrt{3}}\cos(\pi\mu)\begin{pmatrix} \cos(\pi\mu) & e^{\frac{2\pi i}{3}}\overline{\zeta_0} \\ e^{-\frac{2\pi i}{3}}\zeta_0 & -\cos(\pi\mu) \end{pmatrix}, \tag{5.4.21}$$

$$\hat{M}_\infty = -\cos(2\pi\mu)\mathrm{I} - \frac{2i}{\sqrt{3}}\cos(\pi\mu)\begin{pmatrix} \cos(\pi\mu) & e^{-\frac{2\pi i}{3}}\overline{\zeta_0} \\ e^{\frac{2\pi i}{3}}\zeta_0 & -\cos(\pi\mu) \end{pmatrix}, \tag{5.4.22}$$

where ζ_0 is an odd function in λ and a solution to

$$\zeta_0\overline{\zeta_0} = 4\sin^2(\pi\mu) - 1. \tag{5.4.23}$$

Proof. As before, we denote the symmetries of ϕ by $\mathcal{R}$ and $\mathcal{R}^{-1}$ and by $\tilde{\gamma}_{\mathcal{R}}$, $\tilde{\gamma}_{\mathcal{R}^{-1}}$ the biholomorphic mappings $\tilde{M} \to \tilde{M}$ associated with $\mathcal{R}$ and $\mathcal{R}^{-1}$, respectively, as in theorem 4.9 and explicitly given in lemma 5.4. Moreover, let $M_{\mathcal{R}}(\lambda)$, $M_{\mathcal{R}^{-1}}(\lambda)$ be the corresponding monodromy matrices of F as introduced in equations (5.2.22) and (5.2.24). In view of theorem 5.10, we have

$$M_{\mathcal{R}}(\lambda) = (M_{\mathcal{R}^{-1}}(\lambda))^{-1} = \begin{pmatrix} a_{\mathcal{R}} & b_{\mathcal{R}} \\ -\overline{b_{\mathcal{R}}} & \overline{a_{\mathcal{R}}} \end{pmatrix} \tag{5.4.24}$$

where

$$a_{\mathcal{R}} = e^{\frac{\pi i}{3}} \text{ and } b_{\mathcal{R}} = 0. \tag{5.4.25}$$

Moreover, by theorem 5.9, we obtain the following relations between the unitary monodromy matrices $\hat{M}_j$ of the extended frame F:

$$\hat{M}_1(\lambda) = M_{\mathcal{R}}(\lambda)\hat{M}_0(\lambda)M_{\mathcal{R}}(\lambda)^{-1}, \tag{5.4.26}$$

$$\hat{M}_\infty(\lambda) = M_{\mathcal{R}}(\lambda)\hat{M}_1(\lambda)M_{\mathcal{R}}(\lambda)^{-1}, \tag{5.4.27}$$

$$\hat{M}_0(\lambda) = M_{\mathcal{R}}(\lambda)\hat{M}_\infty(\lambda)M_{\mathcal{R}}(\lambda)^{-1}, \tag{5.4.28}$$

which translate into the following scalar equations involving the functions p_j and q_j occurring in $\hat{M}_j$ (cf. (3.9.26)):

$$p_\infty = p_1 = p_0, \quad e^{\frac{2\pi i}{3}} q_\infty = q_1 = e^{-\frac{2\pi i}{3}} q_0. \tag{5.4.29}$$

Thus, in the case of a normalized trinoid, which is rotationally symmetric with respect to the trinoid normal, we obtain the following equivalent reformulations of (3.9.50) and (3.9.51), characterizing the monodromy matrices $\hat{M}_j$:

$$p_0 = \overline{p_0} \text{ and } p_0^2 + q_0\overline{q_0} = 1, \tag{5.4.30}$$

$$p_0^2 - \frac{q_0\overline{q_0}}{2} = \frac{\cos^2(2\pi\mu) + \cos(2\pi\mu)}{\sin^2(2\pi\mu)}. \tag{5.4.31}$$

Here, the second equation follows in view of $q_0\overline{q_1} + \overline{q_0}q_1 = 2q_0\overline{q_0}\cos\frac{2\pi}{3} = -q_0\overline{q_0}$.

We derive directly from (5.4.30) that

$$q_0\overline{q_0} = 1 - p_0^2. \tag{5.4.32}$$

Inserting this into the second equation, we obtain

$$\frac{3}{2}p_0^2 - \frac{1}{2} = \frac{\cos^2(2\pi\mu) + \cos(2\pi\mu)}{\sin^2(2\pi\mu)}, \tag{5.4.33}$$

or, equivalently,

$$p_0^2 = \frac{1}{3} + \frac{\cos(2\pi\mu)(\cos(2\pi\mu)+1)}{6\sin^2(\pi\mu)\cos^2(\pi\mu)} = \frac{1}{3} + \frac{\cos^2(\pi\mu) - \sin^2(\pi\mu)}{3\sin^2(\pi\mu)} = \frac{\cos^2(\pi\mu)}{3\sin^2(\pi\mu)}. \tag{5.4.34}$$

This in turn implies

$$q_0\overline{q_0} = 1 - \frac{\cos^2(\pi\mu)}{3\sin^2(\pi\mu)} = \frac{4\sin^2(\pi\mu) - 1}{3\sin^2(\pi\mu)}. \tag{5.4.35}$$

Next, recall that the monodromy matrices $\hat{M}_j$ satisfy (3.9.32), i.e. $\hat{M}_0\hat{M}_1\hat{M}_\infty = \mathrm{I}$, which reads in scalar form as (3.9.33) and (3.9.34). Inserting the previous results together with the identity $\mu := \mu_0 = \mu_1 = \mu_\infty$ from lemma 5.3 into (3.9.33), we obtain

$$\cos(2\pi\mu)+i\sin(2\pi\mu)p_0 = -\cos^2(2\pi\mu)+2i\cos(2\pi\mu)\sin(2\pi\mu)p_0+\sin^2(2\pi\mu)(p_0^2-\frac{q_0\overline{q_0}}{2}+i\frac{\sqrt{3}}{2}q_0\overline{q_0}), \tag{5.4.36}$$

which in view of (5.4.31) and (5.4.35) transforms into

$$\begin{aligned}\cos(2\pi\mu) + i\sin(2\pi\mu)p_0 = &-\cos^2(2\pi\mu) + 2i\cos(2\pi\mu)\sin(2\pi\mu)p_0 \\ &+ \cos^2(2\pi\mu) + \cos(2\pi\mu) + \frac{2i}{\sqrt{3}}cos^2(\pi\mu)(4\sin^2(\pi\mu) - 1),\end{aligned} \tag{5.4.37}$$

or, equivalently,

$$\sin(2\pi\mu)p_0(1 - 2\cos(2\pi\mu)) = \frac{2}{\sqrt{3}}cos^2(\pi\mu)(4\sin^2(\pi\mu) - 1). \tag{5.4.38}$$

Since $\sin(2\pi\mu) = 2\sin(\pi\mu)\cos(\pi\mu)$ and $\cos(2\pi\mu) = 1 - 2\sin^2(\pi\mu)$, this implies

$$p_0 = \frac{2cos^2(\pi\mu)}{\sqrt{3}\sin(2\pi\mu)} = \frac{cos(\pi\mu)}{\sqrt{3}\sin(\pi\mu)}, \tag{5.4.39}$$

determining p_0 completely. By a direct computation, we check that (3.9.34) yields no further conditions.

Altogether, we conclude that the functions p_j and q_j occurring in the unitary monodromy matrices $\hat{M}_j$ of the extended frame F of a normalized trinoid, which is rotationally symmetric with respect to the trinoid normal satisfy

$$p_0 = p_1 = p_\infty = \frac{\cos(\pi\mu)}{\sqrt{3}\sin(\pi\mu)}, \tag{5.4.40}$$

$$q_0 = e^{\frac{2\pi i}{3}} q_1 = e^{-\frac{2\pi i}{3}} q_\infty = \frac{\zeta_0}{\sqrt{3}\sin(\pi\mu)}, \tag{5.4.41}$$

where ζ_0 is obtained by solving

$$\zeta_0 \overline{\zeta_0} = 4\sin^2(\pi\mu) - 1. \tag{5.4.42}$$

Moreover, in view of remark 3.44, q_0 and thus also ζ_0 are necessarily odd functions in λ.

Applying our results to (3.9.26), we obtain the claimed forms for the monodromy matrices $\hat{M}_j$. $\square$

Theorem 5.13 describes the (unitary) monodromy matrices associated with the extended frame $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ of a trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends, which is rotationally symmetric with respect to the trinoid normal, and which has been normalized such that $F(z_*) = \mathrm{I}$ and $\psi(z_*) = \frac{1}{2H} e_3$, where $z_* \in \tilde{M}$ is given in (5.4.1) and ψ denotes the conformal CMC-immersion $\tilde{M} \to \mathbb{R}^3$ corresponding to ϕ. It turns out that, in this setting, we can also prove the converse result: A trinoid ϕ with properly embedded annular ends and with extended frame F satisfying $F(z_*) = \mathrm{I}$ at $z_* \in \tilde{M}$ from (5.4.1) and corresponding monodromy matrices of the form given in theorem 5.13 is necessarily rotationally symmetric with respect to the trinoid normal. This result is formulated in the following theorem.

Theorem 5.14. *Let η be a (standardized) trinoid potential associated with three off-diagonal Delaunay matrices D_0, D_1, D_∞ possessing the same eigenvalues $\pm\mu$. Denote by $\phi : M \to \mathbb{R}^3$ a trinoid with properly embedded annular ends on $M = \mathbb{C} \setminus \{0, 1\}$ generated by η via the loop group method. Moreover, let $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ be the extended frame associated with the mapping $\psi = \phi \circ \pi$ by theorem 4.5. Furthermore, let $F(z_*) = \mathrm{I}$ at $z_* \in \tilde{M}$ as given in (5.4.1). Assume the unitary monodromy matrices $\hat{M}_0, \hat{M}_1, \hat{M}_\infty \in \Lambda\mathrm{SU}(2, \mathbb{C})_\sigma$ associated with F are given by*

$$\hat{M}_0 = -\cos(2\pi\mu)\mathrm{I} - \frac{2i}{\sqrt{3}}\cos(\pi\mu)\begin{pmatrix} \cos(\pi\mu) & \overline{\zeta_0} \\ \zeta_0 & -\cos(\pi\mu) \end{pmatrix}, \tag{5.4.43}$$

$$\hat{M}_1 = -\cos(2\pi\mu)\mathrm{I} - \frac{2i}{\sqrt{3}}\cos(\pi\mu)\begin{pmatrix} \cos(\pi\mu) & e^{\frac{2\pi i}{3}}\overline{\zeta_0} \\ e^{-\frac{2\pi i}{3}}\zeta_0 & -\cos(\pi\mu) \end{pmatrix}, \tag{5.4.44}$$

$$\hat{M}_\infty = -\cos(2\pi\mu)\mathrm{I} - \frac{2i}{\sqrt{3}}\cos(\pi\mu)\begin{pmatrix} \cos(\pi\mu) & e^{-\frac{2\pi i}{3}}\overline{\zeta_0} \\ e^{\frac{2\pi i}{3}}\zeta_0 & -\cos(\pi\mu) \end{pmatrix}, \tag{5.4.45}$$

where ζ_0 is an odd function in λ and a solution to

$$\zeta_0 \overline{\zeta_0} = 4\sin^2(\pi\mu) - 1. \tag{5.4.46}$$

Then, ϕ is rotationally symmetric with respect to the trinoid normal.

Proof. Since the underlying Delaunay matrices D_0, D_1, D_∞ of the standardized trinoid potential η possess the same eigenvalues $\pm\mu$, we can write η explicitly as (cf. section 3.6)

$$\eta = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q(z, \lambda) & 0 \end{pmatrix} \mathrm{d}z, \tag{5.4.47}$$

where

$$Q(z, \lambda) = b(\lambda)\left[\frac{1}{z^2} + \frac{1}{(z-1)^2} + \frac{1}{z} + \frac{-1}{z-1}\right] = b(\lambda)\frac{z^2 - z + 1}{z^2(z-1)^2} \tag{5.4.48}$$

and $b(\lambda) = \frac{1}{4} - (\mu(\lambda))^2$. Considering the biholomorphic mapping $\gamma = \gamma_{\mathcal{R}} : M \to M$ defined by $z \mapsto \gamma(z) := \frac{1}{1-z}$ and the function $h : M \to \mathbb{C} \setminus \{0\}$, $z \mapsto h(z) = 1 - z$, we compute

$$Q(\gamma(z), \lambda) = b(\lambda)\frac{(\frac{1}{1-z})^2 - \frac{1}{1-z} + 1}{(\frac{1}{1-z})^2(\frac{1}{1-z} - 1)^2} = (z-1)^2 b(\lambda)\frac{1 - 1 + z + 1 - 2z + z^2}{z^2} = (h(z))^4 Q(z, \lambda). \tag{5.4.49}$$

Recalling from lemma 4.21 that $\gamma_{\mathcal{R}}$ corresponds to the permutation $\sigma = (0\ 1\ \infty)$ of the set $\{0, 1, \infty\}$, we apply lemma 4.25 to infer that η transforms under $\gamma_{\mathcal{R}}$ as

$$\gamma^*\eta = \eta \# W_+, \tag{5.4.50}$$

where

$$W_+ = W_+(z, \lambda) = \begin{pmatrix} h(z) & 0 \\ -\lambda \partial_z h(z) & (h(z))^{-1} \end{pmatrix}. \tag{5.4.51}$$

Applying the pullback construction with respect to the covering mapping $\pi : \tilde{M} \to M$ to (5.4.50), we obtain

$$\pi^*(\gamma^*\eta) = \pi^*(\eta \# W_+) = \tilde{\eta} \# \tilde{W}_+, \tag{5.4.52}$$

where $\tilde{\eta} = \pi^*\eta$ denotes the pullback potential of the trinoid potential η (cf. section 2.3) and $\tilde{W}_+ = W_+ \circ \pi$. Moreover, recall that the biholomorphic mapping $\tilde{\gamma} = \tilde{\gamma}_{\mathcal{R}} : \tilde{M} \to \tilde{M}$, $z \mapsto \frac{-z-1}{z}$ from lemma 5.4 satisfies $\gamma \circ \pi = \pi \circ \tilde{\gamma}$. Thus, the left hand side of (5.4.52) can be transformed as follows:

$$\begin{aligned} \pi^*(\gamma^*\eta) = \pi^* \left[\begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q(\gamma(z), \lambda) & 0 \end{pmatrix} \mathrm{d}\gamma(z) \right] = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q((\gamma \circ \pi)(z), \lambda) & 0 \end{pmatrix} \mathrm{d}(\gamma \circ \pi)(z) \\ = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q((\pi \circ \tilde{\gamma})(z), \lambda) & 0 \end{pmatrix} \mathrm{d}(\pi \circ \tilde{\gamma})(z) = \tilde{\gamma}^* \left[\begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q(\pi(z), \lambda) & 0 \end{pmatrix} \mathrm{d}\pi(z) \right] = \tilde{\gamma}^*(\pi^*\eta) = \tilde{\gamma}^*\tilde{\eta}. \end{aligned} \tag{5.4.53}$$

Altogether, (5.4.52) yields

$$\tilde{\gamma}^*\tilde{\eta} = \tilde{\eta} \# \tilde{W}_+. \tag{5.4.54}$$

Considering the extended frame F associated with the trinoid ϕ, we obtain a solution $\Psi = FB_+$ to the differential equation $\mathrm{d}\Psi = \Psi\tilde{\eta}$. Note that Ψ possesses the same (unitary) monodromy matrices as F at the singularities of the potential η, namely $\hat{M}_0$, $\hat{M}_1$ and $\hat{M}_\infty$.

Naturally, the mapping $\tilde{\gamma}^*\Psi = \Psi \circ \tilde{\gamma}$ defines a solution to the differential equation $\mathrm{d}(\tilde{\gamma}^*\Psi) = (\tilde{\gamma}^*\Psi)(\tilde{\gamma}^*\tilde{\eta})$, which in view of (5.4.54) reads as

$$\mathrm{d}(\tilde{\gamma}^*\Psi) = (\tilde{\gamma}^*\Psi)(\tilde{\eta} \# \tilde{W}_+). \tag{5.4.55}$$

Since this differential equation is also solved by the mapping $\Psi\tilde{W}_+$, i.e.

$$\mathrm{d}(\Psi\tilde{W}_+) = (\Psi\tilde{W}_+)(\tilde{\eta} \# \tilde{W}_+), \tag{5.4.56}$$

the mappings $\tilde{\gamma}^*\Psi$ and $\Psi\tilde{W}_+$ only differ by a λ-dependent matrix $\rho = \rho(\lambda)$:

$$\tilde{\gamma}^*\Psi = \rho\Psi\tilde{W}_+. \tag{5.4.57}$$

Now applying the relation (5.3.4), $\tilde{\gamma} \circ \tilde{\gamma}_0 = \tilde{\gamma}_1 \circ \tilde{\gamma}$, involving the covering transformations $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ on $\tilde{M}$ as given in section 3.3, we compute

$$\begin{aligned} \hat{M}_1(\lambda)\rho(\lambda)\Psi(z, \lambda)\tilde{W}_+(z, \lambda) = \hat{M}_1(\lambda)(\tilde{\gamma}^*\Psi(z, \lambda)) = \hat{M}_1(\lambda)\Psi(\tilde{\gamma}(z), \lambda) = \Psi((\tilde{\gamma}_1 \circ \tilde{\gamma})(z), \lambda) \\ = \Psi((\tilde{\gamma} \circ \tilde{\gamma}_0)(z), \lambda) = \tilde{\gamma}^*\Psi(\tilde{\gamma}_0(z), \lambda) = \rho(\lambda)\Psi(\tilde{\gamma}_0(z), \lambda)\tilde{W}_+(\tilde{\gamma}_0(z), \lambda) = \rho(\lambda)\hat{M}_0(\lambda)\Psi(z, \lambda)\tilde{W}_+(\tilde{\gamma}_0(z), \lambda). \end{aligned} \tag{5.4.58}$$

As $\tilde{W}_+$ defines the pullback of the mapping W_+, which is holomorphic on M (with respect to z), $\tilde{W}_+$ is holomorphic on $\tilde{M}$ and therefore does not pick up any monodromy under $\tilde{\gamma}_0$, i.e. $\tilde{W}_+(\tilde{\gamma}_0(z), \lambda) = \tilde{W}_+(z, \lambda)$. Thus, we conclude that

$$\hat{M}_1(\lambda)\rho(\lambda) = \rho(\lambda)\hat{M}_0(\lambda). \tag{5.4.59}$$

Analogously, applying $\tilde{\gamma} \circ \tilde{\gamma}_\infty = \tilde{\gamma}_0 \circ \tilde{\gamma}$ from (5.3.4), we have

$$\begin{aligned} \hat{M}_0(\lambda)\rho(\lambda)\Psi(z, \lambda)\tilde{W}_+(z, \lambda) = \hat{M}_0(\lambda)(\tilde{\gamma}^*\Psi(z, \lambda)) = \hat{M}_0(\lambda)\Psi(\tilde{\gamma}(z), \lambda) = \Psi((\tilde{\gamma}_0 \circ \tilde{\gamma})(z), \lambda) \\ = \Psi((\tilde{\gamma} \circ \tilde{\gamma}_\infty)(z), \lambda) = \tilde{\gamma}^*\Psi(\tilde{\gamma}_\infty(z), \lambda) = \rho(\lambda)\Psi(\tilde{\gamma}_\infty(z), \lambda)\tilde{W}_+(\tilde{\gamma}_\infty(z), \lambda) \\ = \rho(\lambda)\hat{M}_\infty(\lambda)\Psi(z, \lambda)\tilde{W}_+(\tilde{\gamma}_0(z), \lambda). \end{aligned} \tag{5.4.60}$$

Using the holomorphicity of $\tilde{W}_+$ on $\tilde{M}$, we know that $\tilde{W}_+(\tilde{\gamma}_\infty(z), \lambda) = \tilde{W}_+(z, \lambda)$, which yields

$$\hat{M}_0(\lambda)\rho(\lambda) = \rho(\lambda)\hat{M}_\infty(\lambda). \tag{5.4.61}$$

We set

$$\rho(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}, \tag{5.4.62}$$

where a, b, c and d define complex valued functions of λ satisfying $a(\lambda)d(\lambda) - b(\lambda)c(\lambda) = 1$. Then, by comparing the upper left entries of $\hat{M}_1(\lambda)\rho(\lambda)$ and $\rho(\lambda)\hat{M}_0(\lambda)$ (resp. of $\hat{M}_0(\lambda)\rho(\lambda)$ and $\rho(\lambda)\hat{M}_\infty(\lambda)$), we obtain $c(\lambda)e^{\frac{2\pi i}{3}}\overline{\zeta_0} = b(\lambda)\zeta_0$ (resp. $c(\lambda)\overline{\zeta_0} = b(\lambda)e^{\frac{2\pi i}{3}}\zeta_0$). Combining these two equations yields $b(\lambda) \equiv c(\lambda) \equiv 0$. Comparing now the upper right entries of $\hat{M}_1(\lambda)\rho(\lambda)$ and $\rho(\lambda)\hat{M}_0(\lambda)$ (resp. of $\hat{M}_0(\lambda)\rho(\lambda)$ and $\rho(\lambda)\hat{M}_\infty(\lambda)$), we obtain $e^{\frac{2\pi i}{3}}d(\lambda) = a(\lambda)$ (resp. $d(\lambda) = e^{-\frac{2\pi i}{3}}a(\lambda)$). Together with $a(\lambda)d(\lambda) - 0 = 1$, we conclude that $a(\lambda) = (d(\lambda))^{-1} = \pm e^{\frac{\pi i}{3}}$ and thus

$$\rho(\lambda) = \pm \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix}, \tag{5.4.63}$$

in particular $\rho(\lambda) \in \Lambda\mathrm{SU}(2)_\sigma$. Thus, $(\rho F \rho^{-1})(\rho B_+ \tilde{W}_+)$ defines an Iwasawa-decomposition of $\rho\Psi\tilde{W}_+$ (pointwise for all $z \in \tilde{M}$) with $\rho F \rho^{-1} \in \Lambda\mathrm{SU}(2)_\sigma$, $\rho B_+ \tilde{W}_+ \in \Lambda^+\mathrm{SL}(2,\mathbb{C})_\sigma$ and $(\rho F \rho^{-1})(z_*) = \mathrm{I}$. Therefore, we can write

$$(F \circ \tilde{\gamma})(B_+ \circ \tilde{\gamma}) = \tilde{\gamma}^*\Psi = \rho\Psi\tilde{W}_+ = (\rho F \rho^{-1})(\rho B_+ \tilde{W}_+). \tag{5.4.64}$$

This implies that, using the loop group method, $\tilde{\gamma}^*\Psi$ produces on the one hand the trinoid $J(\psi \circ \tilde{\gamma}) = \mathrm{SymBob}(F \circ \gamma)|_{\lambda=1}$ and on the other hand the rotated trinoid $\rho J(\psi)\rho^{-1} = \mathrm{SymBob}(\rho F \rho^{-1})|_{\lambda=1}$. Consequently, these two surfaces coincide, i.e.

$$J(\psi \circ \gamma)(\tilde{M}) = (\rho J(\psi)\rho^{-1})(\tilde{M}). \tag{5.4.65}$$

Using the identity $\rho J(\psi)\rho^{-1} = J \circ \mathcal{A}_\mathcal{R} \circ \psi$, where

$$\mathcal{A}_\mathcal{R} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.4.66}$$

from the proof of corollary 5.12, we switch into the $\mathbb{R}^3$ model and obtain $\psi \circ \tilde{\gamma} = \mathcal{A}_\mathcal{R} \circ \psi$. As $\tilde{\gamma}(\tilde{M}) = \tilde{M}$, this yields

$$\psi(\tilde{M}) = \mathcal{A}_\mathcal{R}(\psi(\tilde{M})). \tag{5.4.67}$$

This means that ψ (and thus also ϕ) is symmetric with respect to the Euclidean motion $\mathcal{A}_\mathcal{R} \in \mathrm{Iso}(\mathbb{R}^3)$ defining the rotation by the angle $\pm\frac{2\pi}{3}$ around the z-axis in $\mathbb{R}^3$. Thus, ϕ is necessarily rotationally symmetric with respect to the trinoid normal. (In view of theorem 4.31, which lists all possible trinoid symmetries, only the rotation by the angle $\pm\frac{2\pi}{3}$ around the trinoid normal shows the behaviour of $\mathcal{A}_\mathcal{R}$. Thus, we infer that the z-axis in $\mathbb{R}^3$ coincides with the trinoid normal, and that ϕ is rotationally symmetric with respect to the trinoid normal, coinciding with the z-axis in $\mathbb{R}^3$.) □

5.5 Solving $\zeta_0\overline{\zeta_0} = 4\sin^2(\pi\mu) - 1$

In order to describe the solutions ζ_0 to (5.4.42), we investigate the right hand side of (5.4.42). To this end, recall from lemma 5.3, that

$$\mu(\lambda) = \sqrt{\frac{1}{4} + w(\lambda - \lambda^{-1})^2}, \tag{5.5.1}$$

where $w = s_0t_0 = s_1t_1 = s_\infty t_\infty$ and s_j, t_j denote the parameters occurring in the Delaunay matrices D_j defined in (3.5.7).

Until now, we have treated μ as a holomorphic function of λ in the domain $\mathbb{C}^* \setminus W_1$ (cf. remark 3.24). For the following considerations, we extend μ to $\mathbb{C}^*$ by explicitly defining the occurring complex square root on $\mathbb{C}^*$. (Of course, this breaks the holomorphicity of μ. However, by our definition of the square root below, the restriction of μ to $\mathbb{C}^* \setminus W_1$ will still be holomorphic.)

We define the complex square root on the λ-plane $\mathbb{C}^*$ in analogy to the complex square root on the z-plane $\mathbb{C}^*$ given in remark 4.14 by

$$\sqrt{\ }: \mathbb{C}^* \to \mathbb{C}^*,\ \lambda = re^{i\theta} \mapsto \sqrt{\lambda} := \sqrt{r}e^{i\frac{\theta}{2}}, \tag{5.5.2}$$

where we write $\lambda \in \mathbb{C}^*$ in the form $\lambda = re^{i\theta}$ with $r \in \mathbb{R}^+$ and $\theta \in (-\pi, \pi]$, and $\sqrt{r}$ defines the value of the usual (real) square root of r. In particular, $\sqrt{\ }$ maps the negative real axis onto the positive imaginary axis. With this definition, we interpret μ from now on as being defined for $\lambda \in \mathbb{C}^*$.

The following lemma states the roots of the expression $4\sin^2(\pi\mu(\lambda)) - 1$.

Lemma 5.15.

$$4\sin^2(\pi\mu(\lambda))-1 = 0 \iff (\mu(\lambda))^2 = (\frac{1}{6}+k)^2 \text{ for } k \in \mathbb{Z} \iff \lambda \in \mathcal{I}_k := \{\pm\lambda_k, \pm\lambda_k^{-1}\} \text{ for } k \in \mathbb{Z}, \tag{5.5.3}$$

where, for $k \in \mathbb{Z}$,

$$\lambda_k := \sqrt{\frac{1}{2w}\left[d_k + \sqrt{d_k^2 - 4w^2}\right]} \tag{5.5.4}$$

and

$$d_k := (\frac{1}{6} + k)^2 - \frac{1}{4} + 2w. \tag{5.5.5}$$

Furthermore,

$$\lambda_k, -\lambda_k^{-1} \in i\mathbb{R}^+ \quad \text{for } k = 0, \tag{5.5.6}$$

$$\lambda_k, \lambda_k^{-1} \in \mathbb{R}^+ \quad \text{for } k \in \mathbb{Z} \setminus \{0\}. \tag{5.5.7}$$

Proof. We start the proof with the following observation:

$$4\sin^2(\pi\mu(\lambda)) - 1 = 0 \iff (2\sin(\pi\mu(\lambda)) - 1)(2\sin(\pi\mu(\lambda)) + 1) = 0 \iff \sin(\pi\mu(\lambda)) = \pm\frac{1}{2}$$

$$\iff \mu(\lambda) = \pm(\frac{1}{6} + k) \text{ for } k \in \mathbb{Z} \iff (\mu(\lambda))^2 = (\frac{1}{6} + k)^2 \text{ for } k \in \mathbb{Z}. \tag{5.5.8}$$

This already proves the first part of (5.5.3). Furthermore, as $\mu(\lambda) = \sqrt{\frac{1}{4} + w(\lambda - \lambda^{-1})^2}$, we have for all $k \in \mathbb{Z}$ and all $\lambda \in \mathbb{C}^*$

$$(\mu(\lambda))^2 = (\frac{1}{6} + k)^2 \iff \frac{1}{4} + w(\lambda - \lambda^{-1})^2 = (\frac{1}{6} + k)^2$$

$$\iff w\lambda^4 + (-(\frac{1}{6} + k)^2 + \frac{1}{4} - 2w)\lambda^2 + w = 0$$

$$\iff \lambda = \pm\sqrt{\frac{1}{2w}\left[(\frac{1}{6} + k)^2 - \frac{1}{4} + 2w \pm \sqrt{((\frac{1}{6} + k)^2 - \frac{1}{4} + 2w)^2 - 4w^2}\right]}. \tag{5.5.9}$$

Defining d_k as in (5.5.5) and for all $k \in \mathbb{Z}$

$$\lambda_{k,1} := -\lambda_{k,3} := \sqrt{\frac{1}{2w}\left[d_k + \sqrt{d_k^2 - 4w^2}\right]} \tag{5.5.10}$$

$$\lambda_{k,2} := -\lambda_{k,4} := \sqrt{\frac{1}{2w}\left[d_k - \sqrt{d_k^2 - 4w^2}\right]} \tag{5.5.11}$$

$$\tag{5.5.12}$$

we obtain

$$(\mu(\lambda))^2 = (\frac{1}{6} + k)^2 \iff \lambda \in \{\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}, \lambda_{k,4}\}. \tag{5.5.13}$$

Next we show for all $k \in \mathbb{Z}$ that

$$\lambda_{k,1}, \lambda_{k,2} \in i\mathbb{R}^+ \text{ and } \lambda_{k,3}, \lambda_{k,4} \in i\mathbb{R}^- \text{ for } k = 0 \tag{5.5.14}$$

$$\lambda_{k,1}, \lambda_{k,2} \in \mathbb{R}^+ \text{ and } \lambda_{k,3}, \lambda_{k,4} \in \mathbb{R}^- \text{ for } k \neq 0 \tag{5.5.15}$$

We first look at the case $k = 0$. We have $d_0^2 - 4w^2 = (-\frac{2}{9} + 2w)^2 - 4w^2 = \frac{4}{81} - \frac{8}{9}w$, whence, as $w \in (0, \frac{1}{18}]$ by lemma B.6 of appendix B, we infer that

$$d_0^2 - 4w^2 \geq 0. \tag{5.5.16}$$

As a consequence, $\sqrt{d_0^2 - 4w^2}$ lies on the positive real axis. Moreover, we have $d_0^2 > d_0^2 - 4w^2 \geq 0$ and thus $|d_0| > \sqrt{d_0^2 - 4w^2}$. Because of $w \in (0, \frac{1}{18}]$ we have $d_0 < 0$, which implies $-d_0 = |d_0| > \sqrt{d_0^2 - 4w^2}$. Altogether, we obtain

$$d_0 \pm \sqrt{d_0^2 - 4w^2} \leq d_0 + \sqrt{d_0^2 - 4w^2} < d_0 - d_0 = 0 \tag{5.5.17}$$

and conclude that

$$\sqrt{\frac{1}{2w}[d_0 \pm \sqrt{d_0^2 - 4w^2}]} = i\sqrt{\frac{1}{2w}[-d_0 \mp \sqrt{d_0^2 - 4w^2}]} \in i\mathbb{R}^+, \tag{5.5.18}$$

which proves (5.5.14).

Now we turn to the case $k \neq 0$. Here, $(\frac{1}{6} + k)^2 - \frac{1}{4} > 0$, and hence, using $w > 0$,

$$d_k^2 - 4w^2 = ((\frac{1}{6} + k)^2 - \frac{1}{4} + 2w)^2 - 4w^2 > (2w)^2 - 4w^2 = 0. \tag{5.5.19}$$

This implies that $\sqrt{d_k^2 - 4w^2}$ lies on the positive real axis. Furthermore, as a consequence of $(\frac{1}{6}+k)^2 - \frac{1}{4} > 0$ and $w > 0$, we have $d_k > 0$. Hence,

$$d_k + \sqrt{d_k^2 - 4w^2} > 0. \tag{5.5.20}$$

Moreover, as a consequence of $d_k^2 > d_k^2 - 4w^2 > 0$, we obtain $d_k = |d_k| > \sqrt{d_k^2 - 4w^2} > 0$ and thus

$$d_k \pm \sqrt{d_k^2 - 4w^2} \geq d_k - \sqrt{d_k^2 - 4w^2} > d_k - d_k = 0. \tag{5.5.21}$$

We infer that

$$\sqrt{\frac{1}{2w}[d_k \pm \sqrt{d_k^2 - 4w^2}]} \in \mathbb{R}^+, \tag{5.5.22}$$

which proves (5.5.15).

Finally, observing that

$$\lambda_{0,1}\lambda_{0,4} = -i^2\sqrt{\frac{1}{2w}[d_0 + \sqrt{d_0^2 - 4w^2}]}\sqrt{\frac{1}{2w}[d_0 - \sqrt{d_0^2 - 4w^2}]} = \frac{1}{2w}\sqrt{d_0^2 - (d_0^2 - 4w^2)} = 1, \tag{5.5.23}$$

we obtain

$$\lambda_{0,2} = -\lambda_{0,1}^{-1}, \quad \lambda_{0,3} = -\lambda_{0,1}, \quad \lambda_{0,4} = \lambda_{0,1}^{-1}. \tag{5.5.24}$$

Hence, as by definition $\lambda_0 = \lambda_{0,1}$, we have

$$\{\lambda_{0,1}, \lambda_{0,2}, \lambda_{0,3}, \lambda_{0,4}\} = \{\pm\lambda_0, \pm\lambda_0^{-1}\}, \tag{5.5.25}$$

where in view of (5.5.24)

$$\lambda_0, -\lambda_0^{-1} \in i\mathbb{R}^+, \tag{5.5.26}$$

which proves (5.5.6).

Analogously, for $k \neq 0$, we compute

$$\lambda_{k,1}\lambda_{k,2} = \sqrt{\frac{1}{2w}[d_k + \sqrt{d_k^2 - 4w^2}]}\sqrt{\frac{1}{2w}[d_k - \sqrt{d_k^2 - 4w^2}]} = \frac{1}{2w}\sqrt{d_k^2 - (d_k^2 - 4w^2)} = 1, \tag{5.5.27}$$

which implies (for $k \neq 0$)

$$\lambda_{k,2} = \lambda_{k,1}^{-1}, \quad \lambda_{k,3} = -\lambda_{k,1}, \quad \lambda_{k,4} = -\lambda_{k,1}^{-1}. \tag{5.5.28}$$

In view of $\lambda_k = \lambda_{k,1}$ we infer

$$\{\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}, \lambda_{k,4}\} = \{\pm\lambda_k, \pm\lambda_k^{-1}\} \text{ for } k \neq 0, \tag{5.5.29}$$

where, using (5.5.28),

$$\lambda_k, \lambda_k^{-1} \in \mathbb{R}^+ \text{ for } k \neq 0, \tag{5.5.30}$$

which proves (5.5.7).

Combining the relations (5.5.8) and (5.5.13) with the equations (5.5.25) and (5.5.29), the (second part of the) claimed relation (5.5.3) follows. □

Next, we state the following theorem.

Theorem 5.16. *For all $\lambda \in \mathbb{C}^*$ the following holds:*

$$4\sin^2(\pi\mu(\lambda)) - 1 = 4\pi^2 \prod_{k=-\infty}^{\infty} C_k (1 - \frac{\lambda}{\lambda_k})(1 + \frac{\lambda}{\lambda_k})(1 - \frac{\lambda^{-1}}{\lambda_k})(1 + \frac{\lambda^{-1}}{\lambda_k}), \tag{5.5.31}$$

where

$$C_k := \begin{cases} -\lambda_k^2 w & \text{for } k = 0 \\ \lambda_k^2 w & \text{for } k = -1 \\ \frac{\lambda_k^2 w}{k^2 (1+\frac{1}{k})^{\frac{1}{3}}} & \text{for } k \in \mathbb{Z} \setminus \{-1, 0\} \end{cases} \tag{5.5.32}$$

and λ_k is given in (5.5.4).

Proof. The proof of this theorem is quite technical and therefore given in the appendix H. □

We now return to equation (5.4.42):

$$\zeta_0 \overline{\zeta_0} = 4\sin^2(\pi\mu) - 1. \tag{5.5.33}$$

By theorem 5.16, we have

$$4\sin^2(\pi\mu(\lambda)) - 1 = 4\pi^2 \prod_{k=-\infty}^{\infty} C_k (1 - \frac{\lambda}{\lambda_k})(1 + \frac{\lambda}{\lambda_k})(1 - \frac{\lambda^{-1}}{\lambda_k})(1 + \frac{\lambda^{-1}}{\lambda_k}), \tag{5.5.34}$$

where C_k defines a positive real number for all $k \in \mathbb{Z}$. Thus, in order to solve (5.5.33) for ζ_0, we need to split the infinite product given in (5.5.34), i.e. we need to distribute the factors of this product among ζ_0 and $\overline{\zeta_0}$. In view of the defining relation

$$\overline{\zeta_0}(\lambda) = \overline{\zeta_0(\frac{1}{\overline{\lambda}})} \tag{5.5.35}$$

defining $\overline{\zeta_0}$ and the relations (5.5.6) and (5.5.7) from lemma 5.15, we observe that we necessarily have

$$\zeta_0(\pm\lambda_0) = 0 \iff \overline{\zeta_0}(\pm\overline{\lambda_0}^{-1}) = \overline{\zeta_0}(\mp\lambda_0^{-1}) = 0, \tag{5.5.36}$$

$$\zeta_0(\pm\lambda_0^{-1}) = 0 \iff \overline{\zeta_0}(\pm\overline{\lambda_0}) = \overline{\zeta_0}(\mp\lambda_0) = 0 \tag{5.5.37}$$

and, for all $k \in \mathbb{Z} \setminus \{0\}$,

$$\zeta_0(\pm\lambda_k) = 0 \iff \overline{\zeta_0}(\pm\overline{\lambda_k}^{-1}) = \overline{\zeta_0}(\pm\lambda_k^{-1}) = 0, \tag{5.5.38}$$

$$\zeta_0(\pm\lambda_k^{-1}) = 0 \iff \overline{\zeta_0}(\pm\overline{\lambda_k}) = \overline{\zeta_0}(\pm\lambda_k) = 0. \tag{5.5.39}$$

Thus, when distributing the factors of the infinite product given in (5.5.34) among ζ_0 and $\overline{\zeta_0}$, we need to respect the following relations:

$$(1 \pm \frac{\lambda}{\lambda_0}) \text{ contributes to } \zeta_0 \iff (1 \mp \frac{\lambda^{-1}}{\lambda_0}) \text{ contributes to } \overline{\zeta_0}, \tag{5.5.40}$$

$$(1 \pm \frac{\lambda^{-1}}{\lambda_0}) \text{ contributes to } \zeta_0 \iff (1 \mp \frac{\lambda}{\lambda_0}) \text{ contributes to } \overline{\zeta_0} \tag{5.5.41}$$

and, for all $k \in \mathbb{Z} \setminus \{0\}$,

$$(1 \pm \frac{\lambda}{\lambda_k}) \text{ contributes to } \zeta_0 \iff (1 \pm \frac{\lambda^{-1}}{\lambda_k}) \text{ contributes to } \overline{\zeta_0}, \tag{5.5.42}$$

$$(1 \pm \frac{\lambda^{-1}}{\lambda_k}) \text{ contributes to } \zeta_0 \iff (1 \pm \frac{\lambda}{\lambda_k}) \text{ contributes to } \overline{\zeta_0}. \tag{5.5.43}$$

Consequently, for each $k \in \mathbb{Z}$, ζ_0 necessarily contains exactly two of the four factors $(1 - \frac{\lambda}{\lambda_k})$, $(1 + \frac{\lambda}{\lambda_k})$, $(1 - \frac{\lambda^{-1}}{\lambda_k})$ and $(1 + \frac{\lambda^{-1}}{\lambda_k})$ in one of the following four possible combinations:

$$\zeta_0 \text{ contains } p_k^{(1)}(\lambda) := (1 - \frac{\lambda}{\lambda_k})(1 + \frac{\lambda}{\lambda_k}) \tag{5.5.44}$$

$$\textit{or } \zeta_0 \text{ contains } p_k^{(2)}(\lambda) := (1 - \frac{\lambda^{-1}}{\lambda_k})(1 + \frac{\lambda^{-1}}{\lambda_k}) \tag{5.5.45}$$

$$\textit{or } \zeta_0 \text{ contains } p_k^{(3)}(\lambda) := \begin{cases} (1 - \frac{\lambda}{\lambda_0})(1 - \frac{\lambda^{-1}}{\lambda_0}) & \text{for } k = 0 \\ (1 - \frac{\lambda}{\lambda_k})(1 + \frac{\lambda^{-1}}{\lambda_k}) & \text{for } k \in \mathbb{Z} \setminus \{0\} \end{cases} \tag{5.5.46}$$

$$\textit{or } \zeta_0 \text{ contains } p_k^{(4)}(\lambda) := \begin{cases} (1 + \frac{\lambda}{\lambda_0})(1 + \frac{\lambda^{-1}}{\lambda_0}) & \text{for } k = 0 \\ (1 + \frac{\lambda}{\lambda_k})(1 - \frac{\lambda^{-1}}{\lambda_k}) & \text{for } k \in \mathbb{Z} \setminus \{0\} \end{cases} \tag{5.5.47}$$

So far, we have investigated the question how the λ-dependent factors of the infinite product given in (5.5.34) can be distributed among ζ_0 and $\overline{\zeta_0}$ in order to solve (5.5.33). Also the constant factors $4\pi^2$ and C_k, $k \in \mathbb{Z}$, occurring in the representation (5.5.34) of the expression $4\sin^2(\pi\mu(\lambda)) - 1$ as an infinite product, need to be distributed among ζ_0 and $\overline{\zeta_0}$. Since $4\pi^2$ and C_k (for all $k \in \mathbb{Z}$) are positive real numbers, these factors are (in order to be in line with the definition of $\overline{\zeta_0}$) necessarily distributed "equally" among ζ_0 and $\overline{\zeta_0}$ in the sense that the respective square roots, i.e. 2π and (for all $k \in \mathbb{Z}$) $\sqrt{C_k}$, contribute to both ζ_0 and $\overline{\zeta_0}$.

Altogether, we set the following "basic" form of ζ_0:

$$\zeta_0(\lambda) = 2\pi \left(\prod_{k=-\infty}^{\infty} \sqrt{C_k} p_k^{(\nu_k)}(\lambda) \right), \tag{5.5.48}$$

where, for all $k \in \mathbb{Z}$, $\nu_k \in \{1, 2, 3, 4\}$.

Remark 5.17. Note that, in general, ζ_0 can still be modified by any λ-dependent function g satisfying $g\overline{g} = 1$. In particular, $g(\lambda) = \lambda$ is a valid choice, producing

$$\zeta_0(\lambda) = 2\pi\lambda \left(\prod_{k=-\infty}^{\infty} \sqrt{C_k} p_k^{(\nu_k)}(\lambda) \right). \tag{5.5.49}$$

Since we are interested in a function ζ_0 solving (5.5.33), which is odd in λ, we keep this possibility in mind for later use: If ζ_0 of the form (5.5.48) solves (5.5.33) and is even in λ, the function $\lambda\zeta_0$ is odd in λ while it as well solves (5.5.33).

The following observation is crucial for our further considerations: The infinite product

$$4\pi^2 \prod_{k=-\infty}^{\infty} C_k (1 - \frac{\lambda}{\lambda_k})(1 + \frac{\lambda}{\lambda_k})(1 - \frac{\lambda^{-1}}{\lambda_k})(1 + \frac{\lambda^{-1}}{\lambda_k}), \tag{5.5.50}$$

occurring in (5.5.34) is well defined for all $\lambda \in \mathbb{C}^*$, since this is the case for the expression $4\sin^2(\pi\mu(\lambda)) - 1$. (Writing $\sin(\pi\mu(\lambda))$ in its power series representation, we observe that $\sin^2(\pi\mu(\lambda))$ involves only even powers of $\mu(\lambda)$. Since, by remark 3.13, μ^2 is defined as a holomorphic function on $\mathbb{C}^*$, $\sin^2(\pi\mu(\lambda))$ and thus also $4\sin^2(\pi\mu(\lambda)) - 1$ are holomorphic and well defined for $\lambda \in \mathbb{C}^*$.) When splitting (5.5.50) in two infinite products representing ζ_0 and $\overline{\zeta_0}$, we still need to ensure that these two products are well defined on $\mathbb{C}^*$, i.e. that they take finite values for all $\lambda \in \mathbb{C}^*$. This leads to the notion of *(normal) convergence* of an infinite product (cf., e.g. [33], chapter 1).

Definition 5.18. 1. Let $\nu \in \mathbb{N}_0$ and $(a_n)_{n \geq \nu}$ be a sequence of complex numbers. Then, the sequence

$$\prod_{n=\nu}^{\infty} a_n := \left(\prod_{n=\nu}^{m} a_n \right)_{m \geq \nu} \tag{5.5.51}$$

is called an *infinite product*. $\prod_{n=\nu}^{\infty} a_n$ is said to be *convergent* if and only if $a_n = 0$ for only finitely many $n \geq \nu$, i.e. $a_n \neq 0$ for all $n \geq n_0$, and if the limit

$$\lim_{l \to \infty} \prod_{n=n_0}^{l} a_n \tag{5.5.52}$$

exists. In this case, the value of $\prod_{n=\nu}^{\infty} a_n$ is defined to be

$$a := a_\nu \cdot a_{\nu+1} \cdots a_{n_0-1} \cdot \lim_{l\to\infty} \prod_{n=n_0}^{l} a_n. \tag{5.5.53}$$

A finite product, which does not converge, is called *divergent*.

2. Let X be a locally-compact metric space X, e.g. a subset of $\mathbb{C}$. Let $\nu \in \mathbb{N}_0$ and $(f_n)_{n\geq\nu}$ be a sequence of continuous functions $X \to \mathbb{C}$. Moreover, define for all $n \geq \nu$ the continuous function $g_n : X \to \mathbb{C}$ by $g_n = f_n - 1$. Then, the sequence of functions

$$\prod_{n=\nu}^{\infty} f_n := \left(\prod_{n=\nu}^{m} f_n\right)_{m\geq\nu} \tag{5.5.54}$$

is called an *infinite product (of functions)*. $\prod_{n=\nu}^{\infty} f_n$ is called *normally convergent (on X)*, if the series $\sum_{n=\nu}^{\infty} g_n$ converges normally on X, i.e., if, for any compact subset K of X, $\sum_{n=\nu}^{\infty} |g_n|_K < \infty$, where $|g_n|_K := \sup_{z\in K}|g_n(z)|$. A finite product, which is not normally convergent on X, is called *divergent on X*.

Remark 5.19. The notions of convergence of an infinite product and of normal convergence of an infinite product of functions given in definition 5.18 are transferred to infinite products of the form $\prod_{n=-\infty}^{\infty} a_n$ and infinite products of functions of the form $\prod_{n=-\infty}^{\infty} f_n$, respectively, as follows: For complex numbers a_n, $n \in \mathbb{Z}$, the infinite product

$$\prod_{n=-\infty}^{\infty} a_n \tag{5.5.55}$$

is said to be convergent, if and only if the infinite products $\prod_{n=0}^{\infty} a_n$ and $\prod_{n=-\infty}^{-1} a_n := \prod_{n=1}^{\infty} a_{-n}$ converge. Otherwise, it is called divergent. For continuous functions $f_n : X \to \mathbb{C}$, $n \in \mathbb{Z}$, on a locally-compact metric space X, the infinite product of functions

$$\prod_{n=-\infty}^{\infty} f_n \tag{5.5.56}$$

is called normally convergent (on X), if and only if the infinite products $\prod_{n=0}^{\infty} f_n$ and $\prod_{n=-\infty}^{-1} f_n := \prod_{n=1}^{\infty} f_{-n}$ are normally convergent (on X). Otherwise, it is called divergent (on X).

We note the following useful result:

Lemma 5.20. *Let $\nu \in \mathbb{N}_0$ and $(a_n)_{n\geq\nu}$ be a sequence of real numbers $a_n > 0$. Assume the series $\sum_{n=\nu}^{\infty}(a_n - 1)$ converges to some limit $a \in \mathbb{R}$. Then, the infinite product $\prod_{n=\nu}^{\infty} a_n$ converges.*

Proof. Since $a_n \neq 0$ for all $n \geq \nu$, it is enough to show that the limit $\lim_{l\to\infty} \prod_{n=\nu}^{l} a_n$ exists. To this end, consider for all $l \geq \nu$ the estimate

$$0 \leq \prod_{n=\nu}^{l} a_n \leq \prod_{n=\nu}^{l} e^{a_n-1} = e^{\sum_{n=\nu}^{l}(a_n-1)}, \tag{5.5.57}$$

a direct consequence of the relation $x \leq e^{x-1}$ for all $x \in \mathbb{R}$. (5.5.57) implies

$$0 \leq \lim_{l\to\infty} \prod_{n=\nu}^{l} a_n \leq \lim_{l\to\infty} e^{\sum_{n=\nu}^{l}(a_n-1)} = e^a \in \mathbb{R}, \tag{5.5.58}$$

which proves the claim. □

Let now ζ_0 be of the form (5.5.48). In the following, we study the question, if, or, more precisely, for which "configurations" of the factors $p_k^{(\nu_k)}(\lambda)$, ζ_0 is well defined, i.e. normally convergent, on $\mathbb{C}^*$. As a start, we have the following result:

Lemma 5.21. *For all $k \in \mathbb{Z}$, let λ_k and C_k be given by* (5.5.4) *and* (5.5.32)*, respectively. Moreover, let the λ-dependent functions $p_k^{(\nu)}(\lambda)$, $\nu \in \{1, 2, 3, 4\}$ be defined by* (5.5.44), (5.5.45), (5.5.46) *and* (5.5.47)*, respectively. Then, we have:*

1. *The infinite product* $\prod_{k=-\infty}^{\infty} \sqrt{C_k}$ *converges.*

2. *The infinite product* $\prod_{k=-\infty}^{\infty} p_k^{(1)}$ *is normally convergent on* $\mathbb{C}^*$*.*

3. *The infinite product* $\prod_{k=-\infty}^{\infty} p_k^{(2)}$ *is normally convergent on* $\mathbb{C}^*$*.*

4. *The infinite product* $\prod_{k=-\infty}^{\infty} p_k^{(3)}$ *is divergent on* $\mathbb{C}^*$*.*

5. *The infinite product* $\prod_{k=-\infty}^{\infty} p_k^{(4)}$ *is divergent on* $\mathbb{C}^*$*.*

Proof. This is proved in appendix I. □

We return to ζ_0 of the form (5.5.48), i.e.

$$\zeta_0(\lambda) = 2\pi \left(\prod_{k=-\infty}^{\infty} \sqrt{C_k} p_k^{(\nu_k)}(\lambda) \right), \tag{5.5.59}$$

where, for all $k \in \mathbb{Z}$, $\nu_k \in \{1,2,3,4\}$. We want to ensure that ζ_0 is well defined and thus normally convergent on $\mathbb{C}^*$. Moreover, ζ_0 is meant to be an odd function of λ or an even function of λ (cf. remark 5.17).

Applying basic results concerning infinite products (cf., e.g., [33], chapter 1), the above lemma 5.21 immediately allows for the following conclusions:

1. ζ_0 is divergent (on $\mathbb{C}^*$) if and only if $\nu_k \in \{3,4\}$ for infinitely many $k \in \mathbb{Z}$. Consequently, presuming ζ_0 is well defined and thus normally convergent on $\mathbb{C}^*$, we infer that $\nu_k \in \{3,4\}$ for at most finitely many $k \in \mathbb{Z}$ and therefore (naturally) $\nu_k \in \{1,2\}$ for infinitely many $k \in \mathbb{Z}$.

2. By equations (5.5.44) to (5.5.47), we observe that, for all $k \in \mathbb{Z}$, $p_k^{(1)}$ and $p_k^{(2)}$ define even functions of λ, while $p_k^{(3)}$ and $p_k^{(4)}$ define functions of λ, which are neither even nor odd (in λ). Thus, any infinite product involving only factors of the form $p_k^{(1)}$ and/or $p_k^{(2)}$ will be even in λ, while any finite product of factors of the form $p_k^{(3)}$ and/or $p_k^{(4)}$ will be neither even nor odd in λ. Consequently, for ζ_0 of the form (5.5.48) with $\nu_k \in \{1,2\}$ for infinitely many $k \in \mathbb{Z}$ and $\nu_k \in \{3,4\}$ for at most finitely many $k \in \mathbb{Z}$, we infer that ζ_0 is even or odd in λ if and only if *all* factors $p_k^{(\nu_k)}$ occurring in ζ_0 are of the form $p_k^{(1)}$ or of the form $p_k^{(2)}$. (Actually, the case that ζ_0 is odd in λ does not occur.)

We summarize our considerations above in the following lemma:

Lemma 5.22. *Let* ζ_0 *be of the form*

$$\zeta_0(\lambda) = 2\pi \left(\prod_{k=-\infty}^{\infty} \sqrt{C_k} p_k^{(\nu_k)}(\lambda) \right), \tag{5.5.60}$$

where, for all $k \in \mathbb{Z}$, $\nu_k \in \{1,2,3,4\}$*,* C_k *is given in* (5.5.32)*, and the functions* $p_k^{(\nu_k)}(\lambda)$ *are defined by (depending on* ν_k*)* (5.5.44), (5.5.45), (5.5.46) *or* (5.5.47)*, respectively. Then, the following holds:*

ζ_0 *is well defined for all* $\lambda \in \mathbb{C}^*$ *and an even or an odd function of* λ*, if and only if* $\nu_k \in \{1,2\}$ *for all* $k \in \mathbb{Z}$*. Actually, if* $\nu_k \in \{1,2\}$ *for all* $k \in \mathbb{Z}$*,* ζ_0 *is well defined for all* $\lambda \in \mathbb{C}^*$ *and an even function of* λ*.*

In view of lemma 5.22 and remark 5.17, we can give the following basic form of a solution ζ_0 to equation (5.5.33), which is well defined for all $\lambda \in \mathbb{C}^*$ and an odd function in λ:

$$\zeta_0(\lambda) = 2\pi\lambda \left(\prod_{k=-\infty}^{\infty} \sqrt{C_k} p_k^{(\nu_k)}(\lambda) \right), \tag{5.5.61}$$

where, for all $k \in \mathbb{Z}$, $\nu_k \in \{1,2\}$, C_k is given in (5.5.32), and the functions $p_k^{(\nu_k)}(\lambda)$ are defined by (depending on ν_k) (5.5.44) or (5.5.45), respectively.

By remark 5.17, ζ_0 can still be modified by any λ-dependent function g, which, in order to preserve the properties of ζ_0, necessarily satisfies $g(-\lambda) = g(\lambda)$ and $g\bar{g} = 1$. Since g might possess singularities in

$\mathbb{C}^*$, after modifying ζ_0 of the form (5.5.61) by g, the new solution $g\zeta_0$ might not be well defined on $\mathbb{C}^*$. However, this is not necessary: returning to theorem 3.59, we only want to achieve

$$\sin(2\pi\mu)q_0 \text{ is holomorphic for } \lambda \in \mathbb{C}^*, \tag{5.5.62}$$
$$q_0 \text{ takes a finite value in } \mathbb{C} \text{ at } \lambda = 1 \text{ and is holomorphic at } \lambda = 1, \tag{5.5.63}$$

which in view of (cf. section 5.4)

$$q_0 = \frac{\zeta_0}{\sqrt{3}\sin(\pi\mu)} \tag{5.5.64}$$

translates into

$$\cos(\pi\mu)\zeta_0 \text{ is holomorphic for } \lambda \in \mathbb{C}^*, \tag{5.5.65}$$
$$\zeta_0 \text{ takes a finite value in } \mathbb{C} \text{ at } \lambda = 1 \text{ and is holomorphic at } \lambda = 1. \tag{5.5.66}$$

Altogether, the general form of a solution ζ_0 to equation (5.5.33) is given by:

$$\zeta_0(\lambda) = 2\pi\lambda g(\lambda) \left(\prod_{k=-\infty}^{\infty} \sqrt{C_k} p_k^{(\nu_k)}(\lambda) \right), \tag{5.5.67}$$

such that ζ_0 satisfies (5.5.65) and (5.5.66), the function g satisfies

$$g(-\lambda) = g(\lambda), \tag{5.5.68}$$
$$g\overline{g} = 1, \tag{5.5.69}$$

and where, for all $k \in \mathbb{Z}$, $\nu_k \in \{1, 2\}$, C_k is given in (5.5.32), and the functions $p_k^{(\nu_k)}(\lambda)$ are defined by (depending on ν_k) (5.5.44) or (5.5.45), respectively.

In following, we translate the conditions (5.5.65) and (5.5.66) into further constraints on the function g, which will lead to a general form of g. We observe the following: In order to satisfy (5.5.65), equation (5.5.67) implies that g may only have singularities at values of $\lambda \in \mathbb{C}^*$, where $\cos(\pi\mu(\lambda)) = 0$. Moreover, to fulfill (5.5.66), the value $\lambda = 1$ is excluded from this, i.e. g needs to be well defined at $\lambda = 1$ (although $\mu(1) = \frac{1}{2}$ and thus $\cos(\pi\mu(1)) = 0$). Since by (5.5.68) $g(-1) = g(1)$, g is also necessarily well defined at $\lambda = -1$. The zeros of the expression $\cos(\pi\mu(\lambda))$ are given in the following lemma.

Lemma 5.23.

$$\cos(\pi\mu(\lambda)) = 0 \iff (\mu(\lambda))^2 = (\frac{1}{2} + j)^2 \textit{ for } j \in \mathbb{Z} \iff \lambda \in \mathcal{J}_j := \{\pm\lambda_j, \pm\lambda_j^{-1}\} \textit{ for } j \in \mathbb{Z}, \tag{5.5.70}$$

where, for $j \in \mathbb{Z}$,

$$\lambda_j := \sqrt{\frac{1}{2w}\left[d_j + \sqrt{d_j^2 - 4w^2}\right]} \tag{5.5.71}$$

and

$$d_j := (\frac{1}{2} + j)^2 - \frac{1}{4} + 2w. \tag{5.5.72}$$

Furthermore, we have for all $j \in \mathbb{Z}$

$$\lambda_j, \lambda_j^{-1} \ \in \mathbb{R}^+. \tag{5.5.73}$$

Proof. We start the proof with the following observation:

$$\cos(\pi\mu(\lambda)) = 0 \iff \mu(\lambda) = \frac{1}{2} + j \text{ for } j \in \mathbb{Z} \iff (\mu(\lambda))^2 = (\frac{1}{2} + j)^2 \text{ for } j \in \mathbb{Z}. \tag{5.5.74}$$

This already proves the first part of (5.5.70). Furthermore, as $\mu(\lambda) = \sqrt{\frac{1}{4} + w(\lambda - \lambda^{-1})^2}$, we have for all $j \in \mathbb{Z}$ and all $\lambda \in \mathbb{C}^*$

$$\begin{aligned}
(\mu(\lambda))^2 = (\frac{1}{2} + j)^2 \iff& \frac{1}{4} + w(\lambda - \lambda^{-1})^2 = (\frac{1}{2} + k)^2 \\
\iff& w\lambda^4 + (-(\frac{1}{2} + j)^2 + \frac{1}{4} - 2w)\lambda^2 + w = 0 \\
\iff& \lambda = \pm\sqrt{\frac{1}{2w}\left[(\frac{1}{2} + j)^2 - \frac{1}{4} + 2w \pm \sqrt{((\frac{1}{2} + j)^2 - \frac{1}{4} + 2w)^2 - 4w^2}\right]}.
\end{aligned} \tag{5.5.75}$$

Defining d_j as in (5.5.72) and for all $j \in \mathbb{Z}$

$$\lambda_{j,1} := -\lambda_{j,3} := \sqrt{\frac{1}{2w}\left[d_j + \sqrt{d_j^2 - 4w^2}\right]} \tag{5.5.76}$$

$$\lambda_{j,2} := -\lambda_{j,4} := \sqrt{\frac{1}{2w}\left[d_j - \sqrt{d_j^2 - 4w^2}\right]} \tag{5.5.77}$$

$$\tag{5.5.78}$$

we obtain

$$(\mu(\lambda))^2 = (\frac{1}{2} + j)^2 \iff \lambda \in \{\lambda_{j,1}, \lambda_{j,2}, \lambda_{j,3}, \lambda_{j,4}\}. \tag{5.5.79}$$

Next we show for all $j \in \mathbb{Z}$ that

$$\lambda_{j,1}, \lambda_{j,2} \in \mathbb{R}^+ \text{ and } \lambda_{j,3}, \lambda_{j,4} \in \mathbb{R}^-. \tag{5.5.80}$$

Since, for all $j \in \mathbb{Z}$, $(\frac{1}{2} + j)^2 - \frac{1}{4} \geq 0$, and hence, using $w > 0$,

$$d_j^2 - 4w^2 = ((\frac{1}{2} + j)^2 - \frac{1}{4} + 2w)^2 - 4w^2 \geq (2w)^2 - 4w^2 = 0. \tag{5.5.81}$$

This implies that $\sqrt{d_j^2 - 4w^2}$ lies on the non-negative real axis. Furthermore, as a consequence of $(\frac{1}{2} + j)^2 - \frac{1}{4} \geq 0$ and $w > 0$, we have $d_j > 0$. Hence,

$$d_j + \sqrt{d_j^2 - 4w^2} > 0. \tag{5.5.82}$$

Moreover, as a consequence of $d_j^2 > d_j^2 - 4w^2 \geq 0$, we obtain $d_j = |d_j| > \sqrt{d_j^2 - 4w^2} \geq 0$ and thus

$$d_j \pm \sqrt{d_j^2 - 4w^2} \geq d_j - \sqrt{d_j^2 - 4w^2} > d_j - d_j = 0. \tag{5.5.83}$$

We infer that

$$\sqrt{\frac{1}{2w}[d_j \pm \sqrt{d_k^2 - 4w^2}]} \in \mathbb{R}^+, \tag{5.5.84}$$

which proves (5.5.80).

Finally, observing that for all $j \in \mathbb{Z}$

$$\lambda_{j,1}\lambda_{j,2} = \sqrt{\frac{1}{2w}[d_j + \sqrt{d_j^2 - 4w^2}]}\sqrt{\frac{1}{2w}[d_j - \sqrt{d_j^2 - 4w^2}]} = \frac{1}{2w}\sqrt{d_j^2 - (d_j^2 - 4w^2)} = 1, \tag{5.5.85}$$

we obtain

$$\lambda_{j,2} = \lambda_{j,1}^{-1}, \quad \lambda_{j,3} = -\lambda_{j,1}, \quad \lambda_{j,4} = -\lambda_{j,1}^{-1}. \tag{5.5.86}$$

In view of $\lambda_j = \lambda_{j,1}$ we infer that for all $j \in \mathbb{Z}$

$$\{\lambda_{j,1}, \lambda_{j,2}, \lambda_{j,3}, \lambda_{j,4}\} = \{\pm\lambda_j, \pm\lambda_j^{-1}\}, \tag{5.5.87}$$

where, using (5.5.86), for all $j \in \mathbb{Z}$

$$\lambda_j, \lambda_j^{-1} \in \mathbb{R}^+, \tag{5.5.88}$$

which proves (5.5.73).

Combining the relations (5.5.74) and (5.5.79) with the equation (5.5.87), the (second part of the) claimed relation (5.5.70) follows. ☐

By lemma 5.23, the set of zeros of the expression $\cos(\pi\mu(\lambda))$ (in $\mathbb{C}^*$) is given by

$$\bigcup_{j\in\mathbb{Z}} \mathcal{J}_j = \bigcup_{j\in\mathbb{Z}} \{\pm\lambda_j, \pm\lambda_j^{-1}\}. \tag{5.5.89}$$

Observing that, for all $j \in \mathbb{Z}$, $\lambda_j = \lambda_{-1-j}$ and thus

$$\mathcal{J}_j = \mathcal{J}_{-1-j}, \tag{5.5.90}$$

and that, furthermore, $\mathcal{J}_j$ contains ± 1 if and only if $j = 0$ (actually, $\mathcal{J}_0 = \{\pm 1\}$), we conclude that

$$\{\lambda \in \mathbb{C}^* \setminus \{\pm 1\}; \cos(\pi\mu(\lambda)) = 0\} = \mathcal{J} := \bigcup_{j \in \mathbb{N}} \mathcal{J}_j = \bigcup_{j \in \mathbb{N}} \{\pm\lambda_j, \pm\lambda_j^{-1}\}. \tag{5.5.91}$$

As indicated before, $\mathcal{J} = \{\lambda \in \mathbb{C}^* \setminus \{\pm 1\}; \cos(\pi\mu(\lambda)) = 0\}$ defines the set of λ-values in $\mathbb{C}^*$, where the function g may be singular. More precisely, since each $\lambda \in \mathcal{J}$ is a simple root of $\cos(\pi\mu)$ (proved by direct computation), g may at most have a simple pole at $\lambda \in \mathcal{J}$.

For each $j \in \mathbb{N}$, there are four points in $\mathbb{C}^*$, at which g may have simple poles: $\pm\lambda_j$ and $\pm\lambda_j^{-1}$, where λ_j is given by (5.5.71). In view of (5.5.68), we immediately infer that all poles of g come in pairs, i.e.

$$\lambda_j \text{ is a (simple) pole of } g \quad \Longleftrightarrow \quad -\lambda_j \text{ is a (simple) pole of } g, \tag{5.5.92}$$
$$\lambda_j^{-1} \text{ is a (simple) pole of } g \quad \Longleftrightarrow \quad -\lambda_j^{-1} \text{ is a (simple) pole of } g. \tag{5.5.93}$$

Assume now that, for some $j \in \mathbb{N}$, g possesses a pair of simple poles at $\pm\lambda_j$. Thus, written in product representation, g contains the factor $(1 - \frac{\lambda}{\lambda_j})^{-1}(1 + \frac{\lambda}{\lambda_j})^{-1}$. By (5.5.69), $\overline{g}$ consequently contains the inverse factor $(1 - \frac{\lambda}{\lambda_j})(1 + \frac{\lambda}{\lambda_j})$. In particular, $\overline{g}(\pm\lambda_j) = 0$. Since, by definition, $\overline{g}(\lambda) = \overline{g(\overline{\lambda}^{-1})}$, this implies directly (using $\lambda_j \in \mathbb{R}$), that $g(\pm\overline{\lambda_j}^{-1}) = g(\pm\lambda_j^{-1}) = 0$, which means that g also contains the factor $(1 - \frac{\lambda^{-1}}{\lambda_j})(1 + \frac{\lambda^{-1}}{\lambda_j})$. Altogether we conclude that, if, for some $j \in \mathbb{N}$, g possesses a pair of simple poles at $\pm\lambda_j$, then g contains in its product representation the factor

$$g_{j,1}(\lambda) := \frac{(1 - \frac{\lambda^{-1}}{\lambda_j})(1 + \frac{\lambda^{-1}}{\lambda_j})}{(1 - \frac{\lambda}{\lambda_j})(1 + \frac{\lambda}{\lambda_j})}. \tag{5.5.94}$$

Analogously, we infer that, if, for some $j \in \mathbb{N}$, g possesses a pair of simple poles at $\pm\lambda_j^{-1}$, then g contains in its product representation the factor

$$g_{j,2}(\lambda) := \frac{(1 - \frac{\lambda}{\lambda_j})(1 + \frac{\lambda}{\lambda_j})}{(1 - \frac{\lambda^{-1}}{\lambda_j})(1 + \frac{\lambda^{-1}}{\lambda_j})}. \tag{5.5.95}$$

(Obviously, for each $j \in \mathbb{N}$, g can only possess a pair of simple poles at $\pm\lambda_j$ or at $\pm\lambda_j^{-1}$, since the respective other pair will automatically become a pair of zeros of g.) We note that the factors $g_{j,1}$ and $g_{j,2}$ are even in λ and satisfy the relation (5.5.69). Thus, they are valid components of g.

So far, theoretically, g may contain an infinite product of factors of the form (5.5.94) and/or an infinite product of factors of the form (5.5.95). To prove that this is actually possible, we show that the following holds:

Lemma 5.24. *The infinite products (of functions)*

$$\prod_{j \in \mathbb{N}} g_{j,1}(\lambda) \quad \textit{and} \quad \prod_{j \in \mathbb{N}} g_{j,2}(\lambda), \tag{5.5.96}$$

where, for all $j \in \mathbb{N}$, $g_{j,1}$ is given in (5.5.94) *and $g_{j,2}$ is given in* (5.5.95), *are normally convergent on $\mathbb{C}^* \setminus \bigcup_{j \in \mathbb{N}} \{\pm\lambda_j\}$ and $\mathbb{C}^* \setminus \bigcup_{j \in \mathbb{N}} \{\pm\lambda_j^{-1}\}$, respectively.*

Proof. This is proved in appendix I. □

Finally, we observe that g can be completed by an additional factor, which is well defined and non-zero for all $\lambda \in \mathbb{C}^*$ and even in λ. By definition of $\overline{g}$ and equation (5.5.69), this factor can be written in the form

$$e^{ih(\lambda)}, \tag{5.5.97}$$

where h denotes a real valued even function of λ.

Altogether, we conclude that the function g can be written in the general form

$$g(\lambda) = e^{ih(\lambda)} \left(\prod_{j \in \mathbb{N}_1} g_{j,1}(\lambda) \right) \left(\prod_{j \in \mathbb{N}_2} g_{j,2}(\lambda) \right) \tag{5.5.98}$$

where h denotes a real valued even function of λ, the functions $g_{j,1}$ and $g_{j,2}$ are for all $j \in \mathbb{N}$ given in (5.5.94) and (5.5.95), respectively, and $\mathbb{N}_1$, $\mathbb{N}_2$ denote subsets of the natural numbers $\mathbb{N}$ satisfying $\mathbb{N}_1 \cap \mathbb{N}_2 = \emptyset$. In the case $\mathbb{N}_1 = \emptyset$ or $\mathbb{N}_2 = \emptyset$, the corresponding product over j is set to be 1.

We summarize our results:

Theorem 5.25. *The general solution ζ_0 to the equation*

$$\zeta_0\overline{\zeta_0} = 4\sin^2(\pi\mu) - 1 \tag{5.5.99}$$

satisfying

$$\zeta_0 \text{ is an odd function of } \lambda, \tag{5.5.100}$$
$$\cos(\pi\mu)\zeta_0 \text{ is holomorphic for } \lambda \in \mathbb{C}^*, \tag{5.5.101}$$
$$\zeta_0 \text{ takes a finite value in } \mathbb{C} \text{ at } \lambda = 1 \text{ and is holomorphic at } \lambda = 1 \tag{5.5.102}$$

is of the form

$$\zeta_0(\lambda) = 2\pi\lambda e^{ih(\lambda)} \left(\prod_{j\in\mathbb{N}_1} g_{j,1}(\lambda) \right) \left(\prod_{j\in\mathbb{N}_2} g_{j,2}(\lambda) \right) \left(\prod_{k=-\infty}^{\infty} \sqrt{C_k} p_k^{(\nu_k)}(\lambda) \right), \tag{5.5.103}$$

where h denotes a real valued even function of $\lambda \in \mathbb{C}^*$, the functions $g_{j,1}$ and $g_{j,2}$ are for all $j \in \mathbb{N}$ given by

$$g_{j,1}(\lambda) = \frac{(1-\frac{\lambda^{-1}}{\lambda_j})(1+\frac{\lambda^{-1}}{\lambda_j})}{(1-\frac{\lambda}{\lambda_j})(1+\frac{\lambda}{\lambda_j})}, \tag{5.5.104}$$

$$g_{j,2}(\lambda) = \frac{(1-\frac{\lambda}{\lambda_j})(1+\frac{\lambda}{\lambda_j})}{(1-\frac{\lambda^{-1}}{\lambda_j})(1+\frac{\lambda^{-1}}{\lambda_j})} \tag{5.5.105}$$

with λ_j given in (5.5.71), *$\mathbb{N}_1$, $\mathbb{N}_2$ denote subsets of the natural numbers $\mathbb{N}$ satisfying $\mathbb{N}_1 \cap \mathbb{N}_2 = \emptyset$, and where, moreover, $\nu_k \in \{1,2\}$ for all $k \in \mathbb{Z}$, C_k is for all $k \in \mathbb{Z}$ given in* (5.5.32), *and the functions $p_k^{(\nu_k)}(\lambda)$ are defined by (depending on ν_k)*

$$p_k^{(1)}(\lambda) = (1-\frac{\lambda}{\lambda_k})(1+\frac{\lambda}{\lambda_k}), \tag{5.5.106}$$

$$p_k^{(2)}(\lambda) = (1-\frac{\lambda^{-1}}{\lambda_k})(1+\frac{\lambda^{-1}}{\lambda_k}) \tag{5.5.107}$$

with λ_k given in (5.5.4). *(In the case $\mathbb{N}_1 = \emptyset$ or $\mathbb{N}_2 = \emptyset$, the corresponding product over j is set to be* 1.*)*

Remark 5.26. It is suspected (based on computer experiments) that the special solution ζ_0 to (5.5.99) associated with $h(\lambda) \equiv 0$, $\mathbb{N}_1 = \mathbb{N}_2 = \emptyset$ and $\nu_k = 1$ for $k < 0$, $\nu_k = 2$ for $k \geq 0$, i.e.

$$\begin{aligned}\zeta_0(\lambda) &= 2\pi\lambda \left(\prod_{k=-\infty}^{-1} \sqrt{C_k} p_k^{(1)}(\lambda) \right) \left(\prod_{k=0}^{\infty} \sqrt{C_k} p_k^{(2)}(\lambda) \right) \\ &= 2\pi\lambda \left(\prod_{k=-\infty}^{-1} \sqrt{C_k}(1-\frac{\lambda^2}{\lambda_k^2}) \right) \left(\prod_{k=0}^{\infty} \sqrt{C_k}(1-\frac{\lambda^{-2}}{\lambda_k^2}) \right),\end{aligned} \tag{5.5.108}$$

corresponds to the triple of unitarized monodromy matrices $\hat{M}_j$, $j = 0, 1, \infty$, associated with a (completely) properly embedded trinoid $M \to \mathbb{R}^3$. Other solutions $\tilde{\zeta_0}$ to (5.5.99), which differ from ζ_0 given in (5.5.108) only "on finitely many positions" in the sense that the corresponding subsets $\tilde{\mathbb{N}}_1, \tilde{\mathbb{N}}_2$ of $\mathbb{N}$ are finite and $\tilde{\nu}_k \neq \nu_k$ only for finitely many $k \in \mathbb{Z}$, seem to induce trinoids with finitely many "bubbles" (in the sense of [27]) but still with properly embedded annular ends. This seems to be perfectly consistent with the following observation: While ζ_0 given in (5.5.108) induces a dressing matrix T which determines a solution $\hat{\Psi} = T\Psi$ to the differential equation $\mathrm{d}\Psi = \Psi\tilde{\eta}$ (cf. section 3.9), $\tilde{\zeta_0}$ induces a dressing matrix $\tilde{T}$ which determines another solution $\tilde{\Psi} = \tilde{T}\Psi$ to the same differential equation with unitary monodromy matrices $\tilde{M}_j$, $j = 0, 1, \infty$, which differ from the unitary monodromy matrices $\hat{M}_j$, $j = 0, 1, \infty$ of $\hat{\Psi}$ only by conjugation with a matrix S, which is given as a finite product of conjugation matrices which "bring in" the factors $g_{j,1}(\lambda)$ for $j \in \tilde{\mathbb{N}}_1$ and $g_{j,2}(\lambda)$ for $j \in \tilde{\mathbb{N}}_2$, respectively, and replace for $k \in \mathbb{Z}$ with $\tilde{\nu}_k \neq \nu_k$ the factors $p_k^{(1)}(\lambda)$ by $p_k^{(2)}(\lambda)$ (and vice versa). The single conjugation matrices contributing to S seem to play the role of simple factor dressing matrices in the sense of [26]. Consequently, the solutions $\hat{\Psi}$ and $\tilde{\Psi} = S\hat{\Psi}$ would be related by a finite product of simple factors, which, by [26], would imply that $\tilde{\Psi}$ produces a trinoid with properly embedded annular ends.

According to these considerations, solutions to (5.5.99), which differ from ζ_0 given in (5.5.108) "on infinitely many positions", would produce trinoids which do not possess properly embedded annular ends (but are still rotationally symmetric with respect to the trinoid normal).

6 Rotational symmetry with respect to a trinoid axis

6.1 Definition

The second possible trinoid symmetry type we are going to study encompasses the symmetries with respect to the rotations $\mathcal{R}_0$, $\mathcal{R}_1$ and $\mathcal{R}_\infty$ by the angle π around the trinoid axes A_0, A_1 and A_∞, respectively. Recall that, by theorem 4.31, $\mathcal{R}_0$, $\mathcal{R}_1$ and $\mathcal{R}_\infty$ preserve orientation on $\mathbb{R}^3$ and permute the trinoid ends according to the permutations $(1\ \infty)$, $(0\ \infty)$ and $(0\ 1)$ of the set $\{0, 1, \infty\}$, respectively.

Definition 6.1. Let $M = \mathbb{C} \setminus \{0, 1\}$ and $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends. Let $\tilde{M} = \mathbb{H}$ and $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$ the conformal CMC-immersion associated with ϕ via the universal covering $\pi : \tilde{M} \to M$ given in (3.2.2). Then, if ϕ (or, equivalently, ψ) is symmetric with respect to the rotation $\mathcal{R}_l$ by the angle π around the trinoid axis A_l,

$$\mathcal{R}_l(\phi(M)) = \phi(M), \quad \mathcal{R}_l(\psi(\tilde{M})) = \psi(\tilde{M}), \tag{6.1.1}$$

ϕ (or ψ) is called *rotationally symmetric with respect to the trinoid axis A_l.*

In analogy to the previous section, we translate the symmetry property (6.1.1) into further constraints on the monodromy matrices associated with the extended frame F of ψ.

6.2 Implications of rotational symmetry with respect to a trinoid axis

As a direct consequence of definition 6.1, we state the following lemma:

Lemma 6.2. *Let $M = \mathbb{C} \setminus \{0, 1\}$ and $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends produced from a trinoid potential η as in theorem 3.14. Denote by D_0, D_1, D_∞ the corresponding Delaunay matrices with eigenvalues $\pm\mu_0$, $\pm\mu_1$, $\pm\mu_\infty$, respectively, where, for $j \in \{0, 1, \infty\}$, $\mu_j = \sqrt{X_j \overline{X_j}} = \sqrt{\frac{1}{4} + w_j(\lambda - \lambda^{-1})^2}$ and $w_j = s_j t_j$ as in section 3.5. Moreover, denote by B_0, B_1 and B_∞ the trinoid ends and by A_0, A_1 and A_∞ the trinoid axes. Then, the following holds:*

1. *If ϕ is rotationally symmetric with respect to the trinoid axis A_0, we have*

$$\mu_1 = \mu_\infty. \tag{6.2.1}$$

2. *If ϕ is rotationally symmetric with respect to the trinoid axis A_1, we have*

$$\mu_0 = \mu_\infty. \tag{6.2.2}$$

3. *If ϕ is rotationally symmetric with respect to the trinoid axis A_∞, we have*

$$\mu_0 = \mu_1. \tag{6.2.3}$$

Proof. We carry out the proof for the first case, i.e. suppose ϕ is rotationally symmetric with respect to the trinoid axis A_0. By theorem 4.31, the corresponding symmetry $\mathcal{R}_0$ preserves the trinoid end B_0, while it rotates the trinoid ends B_1 and B_∞ into each other. This means that the asymptotic Delaunay surfaces associated with the ends at B_1 and B_∞ are rotated into each other as well. Hence, these Delaunay surfaces only differ by a rigid motion on $\mathbb{R}^3$. In particular, this implies that the corresponding Delaunay matrices D_1 and D_∞ possess the same eigenvalues, i.e. $\mu_1 = \mu_\infty$.

The other two cases are proved analogously. □

Let $M = \mathbb{C} \setminus \{0, 1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ given in (3.2.2). Suppose ϕ (or, equivalently, ψ) is rotationally symmetric with respect to the trinoid axis A_l, and denote the corresponding symmetry by $\mathcal{R}_l$. Since $\mathcal{R}_l$ preserves orientation on $\mathbb{R}^3$, we obtain by theorem 4.9 a pair of biholomorphic mappings, $\gamma_{\mathcal{R}_l} : M \to M$ and $\tilde{\gamma}_{\mathcal{R}_l} : \tilde{M} \to \tilde{M}$ satisfying

$$\mathcal{R}_l \circ \phi = \phi \circ \gamma_{\mathcal{R}_l}, \tag{6.2.4}$$

$$\mathcal{R}_l \circ \psi = \psi \circ \tilde{\gamma}_{\mathcal{R}_l}, \tag{6.2.5}$$

$$\pi \circ \tilde{\gamma}_{\mathcal{R}_l} = \gamma_{\mathcal{R}_l} \circ \pi. \tag{6.2.6}$$

The mappings $\gamma_{\mathcal{R}_l}$, $l = 0, 1, \infty$, are uniquely determined and explicitly given by lemma 4.21:

$$\gamma_{\mathcal{R}_0}(z) = \frac{z}{z-1}, \tag{6.2.7}$$

$$\gamma_{\mathcal{R}_1}(z) = \frac{1}{z}, \tag{6.2.8}$$

$$\gamma_{\mathcal{R}_\infty}(z) = 1 - z. \tag{6.2.9}$$

The mappings $\tilde{\gamma}_{\mathcal{R}_l}$, $l = 0, 1, \infty$, are uniquely determined up to composition from the left with elements of the automorphism group $\mathrm{Aut}(\tilde{M}/M)$ of π. The following lemma explicitly states valid choices for $\tilde{\gamma}_{\mathcal{R}_l}$, $l = 0, 1, \infty$:

Lemma 6.3. *Let $M = \mathbb{C} \setminus \{0, 1\}$, $\tilde{M} = \mathbb{H}$ and $\pi : \tilde{M} \to M$ be the universal covering as given in* (3.2.2). *Let $\gamma_{\mathcal{R}_l} : M \to M$, $l = 0, 1, \infty$, be given by* (6.2.7), (6.2.8) *and* (6.2.9), *respectively. Then, the following holds:*

1. *The mapping $\tilde{\gamma}_{\mathcal{R}_0} : \tilde{M} \to \tilde{M}$,*

$$\tilde{\gamma}_{\mathcal{R}_0}(z) = \frac{-z-2}{z+1}, \tag{6.2.10}$$

is biholomorphic and satisfies

$$\pi \circ \tilde{\gamma}_{\mathcal{R}_0} = \gamma_{\mathcal{R}_0} \circ \pi, \tag{6.2.11}$$

$$\mathcal{R}_0 \circ \psi = \psi \circ \tilde{\gamma}_{\mathcal{R}_0}. \tag{6.2.12}$$

2. *The mapping $\tilde{\gamma}_{\mathcal{R}_1} : \tilde{M} \to \tilde{M}$,*

$$\tilde{\gamma}_{\mathcal{R}_1}(z) = \frac{-z-1}{2z+1}, \tag{6.2.13}$$

is biholomorphic and satisfies

$$\pi \circ \tilde{\gamma}_{\mathcal{R}_1} = \gamma_{\mathcal{R}_1} \circ \pi, \tag{6.2.14}$$

$$\mathcal{R}_1 \circ \psi = \psi \circ \tilde{\gamma}_{\mathcal{R}_1}. \tag{6.2.15}$$

3. *The mapping $\tilde{\gamma}_{\mathcal{R}_\infty} : \tilde{M} \to \tilde{M}$,*

$$\tilde{\gamma}_{\mathcal{R}_\infty}(z) = -\frac{1}{z}, \tag{6.2.16}$$

is biholomorphic and satisfies

$$\pi \circ \tilde{\gamma}_{\mathcal{R}_\infty} = \gamma_{\mathcal{R}_\infty} \circ \pi, \tag{6.2.17}$$

$$\mathcal{R}_\infty \circ \psi = \psi \circ \tilde{\gamma}_{\mathcal{R}_\infty}. \tag{6.2.18}$$

Proof. Direct computations show that $\tilde{\gamma}_{\mathcal{R}_l}$, $l = 0, 1, \infty$ define biholomorphic mappings $\tilde{M} \to \tilde{M}$. Moreover, by applying the relations (3.2.10) and (3.2.11) of lemma 3.4, we obtain for all $z \in \tilde{M}$

$$\begin{aligned}\pi \circ \tilde{\gamma}_{\mathcal{R}_0}(z) &= \pi\left(\frac{-z-2}{z+1}\right) = \pi\left(-1 - \frac{1}{z+1}\right) = \frac{1}{\pi\left(-\frac{1}{z+1}\right)} \\ &= \frac{1}{1-\pi(z+1)} = \frac{1}{1-\frac{1}{\pi(z)}} = \frac{\pi(z)}{\pi(z)-1} = \gamma_{\mathcal{R}_0} \circ \pi(z), \end{aligned} \tag{6.2.19}$$

$$\begin{aligned}\pi \circ \tilde{\gamma}_{\mathcal{R}_1}(z) &= \pi\left(\frac{-z-1}{2z+1}\right) = \pi\left(-1 + \frac{z}{2z+1}\right) = \frac{1}{\pi\left(\frac{z}{2z+1}\right)} \\ &= \frac{1}{1-\pi\left(-\frac{2z+1}{z}\right)} = \frac{1}{1-\pi\left(-2-\frac{1}{z}\right)} = \frac{1}{1-\pi\left(-\frac{1}{z}\right)} = \frac{1}{\pi(z)} = \gamma_{\mathcal{R}_1} \circ \pi(z), \end{aligned} \tag{6.2.20}$$

$$\pi \circ \tilde{\gamma}_{\mathcal{R}_\infty}(z) = \pi\left(-\frac{1}{z}\right) = 1 - \pi(z) = \gamma_{\mathcal{R}_\infty} \circ \pi(z), \tag{6.2.21}$$

i.e. $\pi \circ \tilde{\gamma}_{\mathcal{R}_l} = \gamma_{\mathcal{R}_l} \circ \pi$ for $l = 0, 1, \infty$. Consequently,

$$\mathcal{R}_l \circ \psi = \mathcal{R}_l \circ \phi \circ \pi = \phi \circ \gamma_{\mathcal{R}_l} \circ \pi = \phi \circ \pi \circ \tilde{\gamma}_{\mathcal{R}_l} = \psi \circ \tilde{\gamma}_{\mathcal{R}_l}, \tag{6.2.22}$$

i.e. $\mathcal{R}_l \circ \psi = \psi \circ \tilde{\gamma}_{\mathcal{R}_l}$ for $l = 0, 1, \infty$. □

By the above lemma, we have explicitly determined mappings $\tilde{\gamma}_{\mathcal{R}_l}$, $l = 0, 1, \infty$, corresponding to the trinoid symmetries $\mathcal{R}_l$, $l = 0, 1, \infty$, respectively, in the sense of theorem 4.9. Thus, we can apply theorem 4.17 to obtain

Theorem 6.4. *Let $M = \mathbb{C} \setminus \{0, 1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ as defined in (3.2.2). Let ϕ be rotationally symmetric with respect to the trinoid axis A_l. Denote the corresponding symmetry by $\mathcal{R}_l$ and by $\tilde{\gamma}_{\mathcal{R}_l}$ the biholomorphic mapping $\tilde{M} \to \tilde{M}$ associated with $\mathcal{R}_l$ as in theorem 4.9 and explicitly defined in lemma 6.3. Then, the extended frame $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ corresponding to ψ by theorem 4.5 transforms under $\tilde{\gamma}_{\mathcal{R}_l}$ as*

$$F(\tilde{\gamma}_{\mathcal{R}_l}(z), \lambda) = M_{\mathcal{R}_l}(\lambda) F(z, \lambda) k_{\mathcal{R}_l, \gamma_{\tilde{\mathcal{R}}_l}}(z), \tag{6.2.23}$$

where $M_{\mathcal{R}_l}(\lambda)$ denotes an element of $\Lambda\mathrm{SU}(2)_\sigma$, which is independent of z, and

$$k_{\mathcal{R}_0, \gamma_{\tilde{\mathcal{R}}_0}}(z) = \begin{pmatrix} \sqrt{\frac{\bar{z}+1}{z+1}} & 0 \\ 0 & \sqrt{\frac{\bar{z}+1}{z+1}} \end{pmatrix} \quad \text{in the case } l = 0, \tag{6.2.24}$$

$$k_{\mathcal{R}_1, \gamma_{\tilde{\mathcal{R}}_1}}(z) = \begin{pmatrix} \sqrt{\frac{2\bar{z}+1}{2z+1}} & 0 \\ 0 & \sqrt{\frac{2\bar{z}+1}{2z+1}} \end{pmatrix} \quad \text{in the case } l = 1, \tag{6.2.25}$$

$$k_{\mathcal{R}_\infty, \gamma_{\tilde{\mathcal{R}}_\infty}}(z) = \begin{pmatrix} \sqrt{\frac{\bar{z}}{z}} & 0 \\ 0 & \sqrt{\frac{\bar{z}}{z}} \end{pmatrix} \quad \text{in the case } l = \infty. \tag{6.2.26}$$

Proof. As $\mathcal{R}_l$ preserves orientation, we apply the first part of theorem 4.17 to obtain

$$F(\tilde{\gamma}_{\mathcal{R}_l}(z), \lambda) = M_{\mathcal{R}_l}(\lambda) F(z, \lambda) k_{\mathcal{R}_l, \tilde{\gamma}_{\mathcal{R}_l}}(z), \tag{6.2.27}$$

where $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ denotes the extended frame corresponding to ψ by theorem 4.5 and $M_{\mathcal{R}_l} := M_{\tilde{\gamma}_{\mathcal{R}_l}}$ denotes an element of $\Lambda\mathrm{SU}(2)_\sigma$, which is independent of z. Moreover, $k_{\mathcal{R}_l, \gamma_{\tilde{\mathcal{R}}_l}}$ is given by equation (4.4.117) from lemma 4.18. Recalling from lemma 6.3 that $\tilde{\gamma}_{\mathcal{R}_0}(z) = \frac{-z-2}{z+1}$, $\tilde{\gamma}_{\mathcal{R}_1}(z) = \frac{-z-1}{2z+1}$ and $\tilde{\gamma}_{\mathcal{R}_\infty}(z) = -\frac{1}{z}$, we compute

$$\partial_z \tilde{\gamma}_{\mathcal{R}_0}(z) = \frac{1}{(z+1)^2}, \tag{6.2.28}$$

$$\partial_z \tilde{\gamma}_{\mathcal{R}_1}(z) = \frac{1}{(2z+1)^2}, \tag{6.2.29}$$

$$\partial_z \tilde{\gamma}_{\mathcal{R}_\infty}(z) = \frac{1}{z^2}, \tag{6.2.30}$$

which implies

$$\frac{\partial_z \tilde{\gamma}_{\mathcal{R}_0}(z)}{\mid \partial_z \tilde{\gamma}_{\mathcal{R}_0}(z) \mid} = \frac{\mid z+1 \mid^2}{(z+1)^2} = \frac{\bar{z}+1}{z+1}, \tag{6.2.31}$$

$$\frac{\partial_z \tilde{\gamma}_{\mathcal{R}_1}(z)}{\mid \partial_z \tilde{\gamma}_{\mathcal{R}_1}(z) \mid} = \frac{\mid 2z+1 \mid^2}{(2z+1)^2} = \frac{2\bar{z}+1}{2z+1}, \tag{6.2.32}$$

$$\frac{\partial_z \tilde{\gamma}_{\mathcal{R}_\infty}(z)}{\mid \partial_z \tilde{\gamma}_{\mathcal{R}_\infty}(z) \mid} = \frac{\mid z \mid^2}{z^2} = \frac{\bar{z}}{z}. \tag{6.2.33}$$

Hence, we obtain from (4.4.117) the claimed explicit forms for $k_{\mathcal{R}_l, \gamma_{\tilde{\mathcal{R}}_l}}$, $l = 0, 1, \infty$. This finishes the proof. □

6.3 Monodromy matrices of trinoids with properly embedded annular ends, which are rotationally symmetric with respect to a trinoid axis

We now study the (unitary) monodromy matrices $\hat{M}_0, \hat{M}_1, \hat{M}_\infty$ associated with a trinoid with properly embedded annular ends with rotational symmetry with respect to the trinoid axis A_l. Our considerations are based on the relations between the biholomorphic mappings $\tilde{\gamma}_{\mathcal{R}_l}$ associated with the symmetries $\mathcal{R}_l$ and the covering transformations $\tilde{\gamma}_j$ on $\tilde{M}$ generating the monodromy matrices $\hat{M}_j$. Recall the latter ones from section 3.3:

$$\tilde{\gamma}_0(z) = \frac{z}{-2z+1}, \quad \tilde{\gamma}_1(z) = z+2, \quad \tilde{\gamma}_\infty(z) = \frac{-3z-2}{2z+1}. \tag{6.3.1}$$

The corresponding inverse functions are given by

$$\tilde{\gamma}_0^{-1}(z) = \frac{z}{2z+1}, \quad \tilde{\gamma}_1^{-1}(z) = z-2, \quad \tilde{\gamma}_\infty^{-1}(z) = \frac{z+2}{-2z-3}. \tag{6.3.2}$$

The relations mentioned above are stated in the following lemma.

Lemma 6.5. *Let $\tilde{M} = \mathbb{H}$ and $\tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\gamma}_\infty : \tilde{M} \to \tilde{M}$ be given as above.*

1. *For $\tilde{\gamma}_{\mathcal{R}_0} : \tilde{M} \to \tilde{M}, \tilde{\gamma}_{\mathcal{R}_0}(z) = \frac{-z-2}{z+1}$, the following identities hold:*

$$\tilde{\gamma}_{\mathcal{R}_0} \circ \tilde{\gamma}_0 = \tilde{\gamma}_1^{-1} \circ \tilde{\gamma}_\infty^{-1} \circ \tilde{\gamma}_{\mathcal{R}_0}, \quad \tilde{\gamma}_{\mathcal{R}_0} \circ \tilde{\gamma}_1 = \tilde{\gamma}_\infty \circ \tilde{\gamma}_{\mathcal{R}_0}, \quad \tilde{\gamma}_{\mathcal{R}_0} \circ \tilde{\gamma}_\infty = \tilde{\gamma}_1 \circ \tilde{\gamma}_{\mathcal{R}_0}. \tag{6.3.3}$$

2. *For $\tilde{\gamma}_{\mathcal{R}_1} : \tilde{M} \to \tilde{M}, \tilde{\gamma}_{\mathcal{R}_1}(z) = \frac{-z-1}{2z+1}$, the following identities hold:*

$$\tilde{\gamma}_{\mathcal{R}_1} \circ \tilde{\gamma}_0 = \tilde{\gamma}_\infty \circ \tilde{\gamma}_{\mathcal{R}_1}, \quad \tilde{\gamma}_{\mathcal{R}_1} \circ \tilde{\gamma}_1 = \tilde{\gamma}_\infty^{-1} \circ \tilde{\gamma}_0^{-1} \circ \tilde{\gamma}_{\mathcal{R}_1}, \quad \tilde{\gamma}_{\mathcal{R}_1} \circ \tilde{\gamma}_\infty = \tilde{\gamma}_0 \circ \tilde{\gamma}_{\mathcal{R}_1}. \tag{6.3.4}$$

3. *For $\tilde{\gamma}_{\mathcal{R}_\infty} : \tilde{M} \to \tilde{M}, \tilde{\gamma}_{\mathcal{R}_\infty}(z) = -\frac{1}{z}$, the following identities hold:*

$$\tilde{\gamma}_{\mathcal{R}_\infty} \circ \tilde{\gamma}_0 = \tilde{\gamma}_1 \circ \tilde{\gamma}_{\mathcal{R}_\infty}, \quad \tilde{\gamma}_{\mathcal{R}_\infty} \circ \tilde{\gamma}_1 = \tilde{\gamma}_0 \circ \tilde{\gamma}_{\mathcal{R}_\infty}, \quad \tilde{\gamma}_{\mathcal{R}_\infty} \circ \tilde{\gamma}_\infty = \tilde{\gamma}_0^{-1} \circ \tilde{\gamma}_1^{-1} \circ \tilde{\gamma}_{\mathcal{R}_\infty}. \tag{6.3.5}$$

Proof. For $z \in \tilde{M}$ we have by direct computation

$$\tilde{\gamma}_{\mathcal{R}_0} \circ \tilde{\gamma}_1(z) = \frac{-(z+2)-2}{(z+2)+1} = \frac{-z-4}{z+3} = \frac{-3\frac{-z-2}{z+1}-2}{2\frac{-z-2}{z+1}+1} = \tilde{\gamma}_\infty \circ \tilde{\gamma}_{\mathcal{R}_0}(z), \tag{6.3.6}$$

$$\tilde{\gamma}_{\mathcal{R}_0} \circ \tilde{\gamma}_\infty(z) = \frac{-\frac{-3z-2}{2z+1}-2}{\frac{-3z-2}{2z+1}+1} = \frac{z}{z+1} = \frac{-z-2}{z+1}+2 = \tilde{\gamma}_1 \circ \tilde{\gamma}_{\mathcal{R}_0}(z), \tag{6.3.7}$$

$$\tilde{\gamma}_{\mathcal{R}_1} \circ \tilde{\gamma}_0(z) = \frac{-\frac{z}{-2z+1}-1}{2\frac{z}{-2z+1}+1} = z-1 = \frac{-3\frac{-z-1}{2z+1}-2}{2\frac{-z-1}{2z+1}+1} = \tilde{\gamma}_\infty \circ \tilde{\gamma}_{\mathcal{R}_1}(z), \tag{6.3.8}$$

$$\tilde{\gamma}_{\mathcal{R}_1} \circ \tilde{\gamma}_\infty(z) = \frac{-\frac{-3z-2}{2z+1}-1}{2\frac{-3z-2}{2z+1}+1} = \frac{-z-1}{4z+3} = \frac{\frac{-z-1}{2z+1}}{-2\frac{-z-1}{2z+1}+1} = \tilde{\gamma}_0 \circ \tilde{\gamma}_{\mathcal{R}_1}(z), \tag{6.3.9}$$

$$\tilde{\gamma}_{\mathcal{R}_\infty} \circ \tilde{\gamma}_0(z) = -\frac{1}{\frac{z}{-2z+1}} = \frac{2z-1}{z} = -\frac{1}{z}+2 = \tilde{\gamma}_1 \circ \tilde{\gamma}_{\mathcal{R}_\infty}(z), \tag{6.3.10}$$

$$\tilde{\gamma}_{\mathcal{R}_\infty} \circ \tilde{\gamma}_1(z) = -\frac{1}{z+2} = \frac{-\frac{1}{z}}{2\frac{1}{z}+1} = \tilde{\gamma}_0 \circ \tilde{\gamma}_{\mathcal{R}_\infty}(z). \tag{6.3.11}$$

The remaining identities follow from the ones above by use of $\tilde{\gamma}_0 \circ \tilde{\gamma}_1 \circ \tilde{\gamma}_\infty = \mathrm{id}$ on $\tilde{M}$:

$$\tilde{\gamma}_{\mathcal{R}_0} \circ \tilde{\gamma}_0(z) = \tilde{\gamma}_{\mathcal{R}_0} \circ \tilde{\gamma}_\infty^{-1} \circ \tilde{\gamma}_1^{-1}(z) = \tilde{\gamma}_1^{-1} \circ \tilde{\gamma}_{\mathcal{R}_0} \circ \tilde{\gamma}_1^{-1}(z) = \tilde{\gamma}_1^{-1} \circ \tilde{\gamma}_\infty^{-1} \circ \tilde{\gamma}_{\mathcal{R}_0}(z), \tag{6.3.12}$$

$$\tilde{\gamma}_{\mathcal{R}_1} \circ \tilde{\gamma}_1(z) = \tilde{\gamma}_{\mathcal{R}_1} \circ \tilde{\gamma}_0^{-1} \circ \tilde{\gamma}_\infty^{-1}(z) = \tilde{\gamma}_\infty^{-1} \circ \tilde{\gamma}_{\mathcal{R}_1} \circ \tilde{\gamma}_\infty^{-1}(z) = \tilde{\gamma}_\infty^{-1} \circ \tilde{\gamma}_0^{-1} \circ \tilde{\gamma}_{\mathcal{R}_1}(z), \tag{6.3.13}$$

and

$$\tilde{\gamma}_{\mathcal{R}_\infty} \circ \tilde{\gamma}_\infty(z) = \tilde{\gamma}_{\mathcal{R}_\infty} \circ \tilde{\gamma}_1^{-1} \circ \tilde{\gamma}_0^{-1}(z) = \tilde{\gamma}_0^{-1} \circ \tilde{\gamma}_{\mathcal{R}_\infty} \circ \tilde{\gamma}_0^{-1}(z) = \tilde{\gamma}_0^{-1} \circ \tilde{\gamma}_1^{-1} \circ \tilde{\gamma}_{\mathcal{R}_\infty}(z). \tag{6.3.14}$$

□

In view of this, we are able to prove the following theorem:

Theorem 6.6. *Let $M = \mathbb{C}\setminus\{0,1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi\circ\pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ as defined in* (3.2.2)*. Let ϕ be rotationally symmetric with respect to the trinoid axis A_l. Denote the corresponding symmetry by $\mathcal{R}_l$. Furthermore, let $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ be the extended frame associated with ψ by theorem 4.5. Denote by $\hat{M}_0, \hat{M}_1, \hat{M}_\infty \in \Lambda\mathrm{SU}(2,\mathbb{C})_\sigma$ the unitary monodromy matrices*

$$\hat{M}_j = -\left[\cos(2\pi\mu_j)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i\sin(2\pi\mu_j)\begin{pmatrix}p_j & \overline{q_j}\\ q_j & -p_j\end{pmatrix}\right] \tag{6.3.15}$$

associated with F as in (4.5.13) *by*

$$F(\tilde{\gamma}_j(z),\lambda) = \alpha_j\hat{M}_j(\lambda)F(z,\lambda)k_j(z), \quad j = 0,1,\infty, \tag{6.3.16}$$

where $\alpha_j \in \{\pm 1\}$ and $\tilde{\gamma}_j$ denote the covering transformations on $\tilde{M}$ from section 3.3. Finally, let $\tilde{\gamma}_{\mathcal{R}_l}$, be the biholomorphic mapping $\tilde{M} \to \tilde{M}$ associated with $\mathcal{R}_l$ as in theorem 4.9 and explicitly given in lemma 6.3, and

$$M_{\mathcal{R}_l}(\lambda) := \begin{pmatrix} a_{\mathcal{R}_l} & b_{\mathcal{R}_l}\\ -\overline{b_{\mathcal{R}_l}} & \overline{a_{\mathcal{R}_l}}\end{pmatrix} \tag{6.3.17}$$

the corresponding monodromy matrix of F satisfying (6.2.23)*.*

1. *In the case $l = 0$, the monodromy matrices satisfy*

$$M_{\mathcal{R}_0}(\lambda)\hat{M}_1(\lambda) = \hat{M}_\infty(\lambda)M_{\mathcal{R}_0}(\lambda), \tag{6.3.18}$$
$$M_{\mathcal{R}_0}(\lambda)\hat{M}_\infty(\lambda) = \hat{M}_1(\lambda)M_{\mathcal{R}_0}(\lambda), \tag{6.3.19}$$
$$M_{\mathcal{R}_0}(\lambda)\hat{M}_0(\lambda) = (\hat{M}_1(\lambda))^{-1}(\hat{M}_\infty(\lambda))^{-1}M_{\mathcal{R}_0}(\lambda). \tag{6.3.20}$$

   In terms of the functions p_j and q_j occurring in $\hat{M}_j$, equations (6.3.18) *to* (6.3.20) *are equivalent to*

$$a_{\mathcal{R}_0}p_1 + b_{\mathcal{R}_0}q_1 = a_{\mathcal{R}_0}p_\infty - \overline{b_{\mathcal{R}_0}q_\infty}, \tag{6.3.21}$$
$$-b_{\mathcal{R}_0}p_1 + a_{\mathcal{R}_0}\overline{q_1} = b_{\mathcal{R}_0}p_\infty + \overline{a_{\mathcal{R}_0}q_\infty}, \tag{6.3.22}$$
$$a_{\mathcal{R}_0}p_\infty + b_{\mathcal{R}_0}q_\infty = a_{\mathcal{R}_0}p_1 - \overline{b_{\mathcal{R}_0}q_1}, \tag{6.3.23}$$
$$-b_{\mathcal{R}_0}p_\infty + a_{\mathcal{R}_0}\overline{q_\infty} = b_{\mathcal{R}_0}p_1 + \overline{a_{\mathcal{R}_0}q_1}. \tag{6.3.24}$$

2. *In the case $l = 1$, the monodromy matrices satisfy*

$$M_{\mathcal{R}_1}(\lambda)\hat{M}_\infty(\lambda) = \hat{M}_0(\lambda)M_{\mathcal{R}_1}(\lambda), \tag{6.3.25}$$
$$M_{\mathcal{R}_1}(\lambda)\hat{M}_0(\lambda) = \hat{M}_\infty(\lambda)M_{\mathcal{R}_1}(\lambda), \tag{6.3.26}$$
$$M_{\mathcal{R}_1}(\lambda)\hat{M}_1(\lambda) = (\hat{M}_\infty(\lambda))^{-1}(\hat{M}_0(\lambda))^{-1}M_{\mathcal{R}_1}(\lambda). \tag{6.3.27}$$

   In terms of the functions p_j and q_j occurring in $\hat{M}_j$, equations (6.3.25) *to* (6.3.27) *are equivalent to*

$$a_{\mathcal{R}_1}p_\infty + b_{\mathcal{R}_1}q_\infty = a_{\mathcal{R}_1}p_0 - \overline{b_{\mathcal{R}_1}q_0}, \tag{6.3.28}$$
$$-b_{\mathcal{R}_1}p_\infty + a_{\mathcal{R}_1}\overline{q_\infty} = b_{\mathcal{R}_1}p_0 + \overline{a_{\mathcal{R}_1}q_0}, \tag{6.3.29}$$
$$a_{\mathcal{R}_1}p_0 + b_{\mathcal{R}_1}q_0 = a_{\mathcal{R}_1}p_\infty - \overline{b_{\mathcal{R}_1}q_\infty}, \tag{6.3.30}$$
$$-b_{\mathcal{R}_1}p_0 + a_{\mathcal{R}_1}\overline{q_0} = b_{\mathcal{R}_1}p_\infty + \overline{a_{\mathcal{R}_1}q_\infty}. \tag{6.3.31}$$

3. *In the case $l = \infty$, the monodromy matrices satisfy*

$$M_{\mathcal{R}_\infty}(\lambda)\hat{M}_0(\lambda) = \hat{M}_1(\lambda)M_{\mathcal{R}_\infty}(\lambda), \tag{6.3.32}$$
$$M_{\mathcal{R}_\infty}(\lambda)\hat{M}_1(\lambda) = \hat{M}_0(\lambda)M_{\mathcal{R}_\infty}(\lambda), \tag{6.3.33}$$
$$M_{\mathcal{R}_\infty}(\lambda)\hat{M}_\infty(\lambda) = (\hat{M}_0(\lambda))^{-1}(\hat{M}_1(\lambda))^{-1}M_{\mathcal{R}_\infty}(\lambda). \tag{6.3.34}$$

In terms of the functions p_j and q_j occurring in $\hat{M}_j$, equations (6.3.32) *to* (6.3.34) *are equivalent to*

$$a_{\mathcal{R}_\infty} p_0 + b_{\mathcal{R}_\infty} q_0 = a_{\mathcal{R}_\infty} p_1 - \overline{b_{\mathcal{R}_\infty}}\,\overline{q_1}, \tag{6.3.35}$$

$$-b_{\mathcal{R}_\infty} p_0 + a_{\mathcal{R}_\infty} \overline{q_0} = b_{\mathcal{R}_\infty} p_1 + \overline{a_{\mathcal{R}_\infty} q_1}, \tag{6.3.36}$$

$$a_{\mathcal{R}_\infty} p_1 + b_{\mathcal{R}_\infty} q_1 = a_{\mathcal{R}_\infty} p_0 - \overline{b_{\mathcal{R}_\infty}}\,\overline{q_0}, \tag{6.3.37}$$

$$-b_{\mathcal{R}_\infty} p_1 + a_{\mathcal{R}_\infty} \overline{q_1} = b_{\mathcal{R}_\infty} p_0 + \overline{a_{\mathcal{R}_\infty} q_0}. \tag{6.3.38}$$

Proof. We start with the case $l = 0$, i.e. with a trinoid with properly embedded annular ends, that is symmetric with respect to the rotation $\mathcal{R}_0$ by the angle π around the trinoid axis A_0. Combining (6.2.23) from theorem 6.4, equation (6.3.16) and the identities (6.3.3) from the above lemma, we obtain

$$\begin{aligned} M_{\mathcal{R}_0}(\lambda)\alpha_\infty \hat{M}_\infty(\lambda) F(z,\lambda) k_\infty(z) k_{\mathcal{R}_0,\tilde{\gamma}_{\mathcal{R}_0}}(\tilde{\gamma}_\infty(z)) &= M_{\mathcal{R}_0}(\lambda) F(\tilde{\gamma}_\infty(z),\lambda) k_{\mathcal{R}_0,\tilde{\gamma}_{\mathcal{R}_0}}(\tilde{\gamma}_\infty(z)) \\ &= F(\tilde{\gamma}_{\mathcal{R}_0} \circ \tilde{\gamma}_\infty(z),\lambda) = F(\tilde{\gamma}_1 \circ \tilde{\gamma}_{\mathcal{R}_0}(z),\lambda) \\ = \alpha_1 \hat{M}_1(\lambda) F(\tilde{\gamma}_{\mathcal{R}_0}(z),\lambda) k_1(\tilde{\gamma}_{\mathcal{R}_0}(z)) &= \alpha_1 \hat{M}_1(\lambda) M_{\mathcal{R}_0}(\lambda) F(z,\lambda) k_{\mathcal{R}_0,\tilde{\gamma}_{\mathcal{R}_0}}(z) k_1(\tilde{\gamma}_{\mathcal{R}_0}(z)) \end{aligned} \tag{6.3.39}$$

and

$$\begin{aligned} M_{\mathcal{R}_0}(\lambda)\alpha_1 \hat{M}_1(\lambda) F(z,\lambda) k_1(z) k_{\mathcal{R}_0,\tilde{\gamma}_{\mathcal{R}_0}}(\tilde{\gamma}_1(z)) &= M_{\mathcal{R}_0}(\lambda) F(\tilde{\gamma}_1(z),\lambda) k_{\mathcal{R}_0,\tilde{\gamma}_{\mathcal{R}_0}}(\tilde{\gamma}_1(z)) \\ &= F(\tilde{\gamma}_{\mathcal{R}_0} \circ \tilde{\gamma}_1(z),\lambda) = F(\tilde{\gamma}_\infty \circ \tilde{\gamma}_{\mathcal{R}_0}(z),\lambda) \\ = \alpha_\infty \hat{M}_\infty(\lambda) F(\tilde{\gamma}_{\mathcal{R}_0}(z),\lambda) k_\infty(\tilde{\gamma}_{\mathcal{R}_0}(z)) &= \alpha_\infty \hat{M}_\infty(\lambda) M_{\mathcal{R}_0}(\lambda) F(z,\lambda) k_{\mathcal{R}_0,\tilde{\gamma}_{\mathcal{R}_0}}(z) k_\infty(\tilde{\gamma}_{\mathcal{R}_0}(z)). \end{aligned} \tag{6.3.40}$$

Computing (due to the occurring complex roots up to sign)

$$\begin{aligned} k_1(z) k_{\mathcal{R}_0,\tilde{\gamma}_{\mathcal{R}_0}}(\tilde{\gamma}_1(z)) &= \begin{pmatrix} \sqrt{\frac{\bar{z}+3}{z+3}} & 0 \\ 0 & \sqrt{\frac{\bar{z}+3}{z+3}} \end{pmatrix} \\ &= \pm \begin{pmatrix} \sqrt{\frac{\bar{z}+1}{z+1}} & 0 \\ 0 & \sqrt{\frac{\bar{z}+1}{z+1}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1+2\frac{-\bar{z}-2}{\bar{z}+1}}{1+2\frac{-z-2}{z+1}}} & 0 \\ 0 & \sqrt{\frac{1+2\frac{-\bar{z}-2}{\bar{z}+1}}{1+2\frac{-z-2}{z+1}}} \end{pmatrix} = \pm k_{\mathcal{R}_0,\tilde{\gamma}_{\mathcal{R}_0}}(z) k_\infty(\tilde{\gamma}_{\mathcal{R}_0}(z)) \end{aligned} \tag{6.3.41}$$

and

$$\begin{aligned} k_\infty(z) k_{\mathcal{R}_0,\tilde{\gamma}_{\mathcal{R}_0}}(\tilde{\gamma}_\infty(z)) &= \begin{pmatrix} \sqrt{\frac{1+2\bar{z}}{1+2z}} & 0 \\ 0 & \sqrt{\frac{1+2\bar{z}}{1+2z}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{\frac{-3\bar{z}-2}{2\bar{z}+1}+1}{\frac{-3z-2}{2z+1}+1}} & 0 \\ 0 & \sqrt{\frac{\frac{-3\bar{z}-2}{2\bar{z}+1}+1}{\frac{-3z-2}{2z+1}+1}} \end{pmatrix} \\ &= \pm \begin{pmatrix} \sqrt{\frac{\bar{z}+1}{z+1}} & 0 \\ 0 & \sqrt{\frac{\bar{z}+1}{z+1}} \end{pmatrix} = \pm k_{\mathcal{R}_0,\tilde{\gamma}_{\mathcal{R}_0}}(z) k_1(\tilde{\gamma}_{\mathcal{R}_0}(z)), \end{aligned} \tag{6.3.42}$$

we conclude that

$$M_{\mathcal{R}_0}(\lambda)\hat{M}_1(\lambda) = \pm\alpha_1\alpha_\infty \hat{M}_\infty(\lambda) M_{\mathcal{R}_0}(\lambda), \tag{6.3.43}$$

$$M_{\mathcal{R}_0}(\lambda)\hat{M}_\infty(\lambda) = \pm\alpha_\infty\alpha_1 \hat{M}_1(\lambda) M_{\mathcal{R}_0}(\lambda). \tag{6.3.44}$$

This can be reformulated as

$$\hat{M}_1(\lambda) = \pm\alpha_1\alpha_\infty M_{\mathcal{R}_0}(\lambda)^{-1} \hat{M}_\infty(\lambda) M_{\mathcal{R}_0}(\lambda), \tag{6.3.45}$$

$$\hat{M}_\infty(\lambda) = \pm\alpha_\infty\alpha_1 M_{\mathcal{R}_0}(\lambda)^{-1} \hat{M}_1(\lambda) M_{\mathcal{R}_0}(\lambda). \tag{6.3.46}$$

Comparing the upper left entries as well as the lower right entries of both sides in each of these equations, we obtain

$$\begin{aligned} &-\cos(2\pi\mu_1) - i\sin(2\pi\mu_1) p_1 \\ &= \pm\alpha_1\alpha_\infty \left[-\cos(2\pi\mu_\infty) - i\sin(2\pi\mu_\infty)(a_{\mathcal{R}_0}\overline{a_{\mathcal{R}_0}} p_\infty - a_{\mathcal{R}_0} b_{\mathcal{R}_0} q_\infty - \overline{a_{\mathcal{R}_0}}\,\overline{b_{\mathcal{R}_0}}\,\overline{q_\infty} - b_{\mathcal{R}_0}\overline{b_{\mathcal{R}_0}} p_\infty)\right], \end{aligned} \tag{6.3.47}$$

$$
\begin{aligned}
&-\cos(2\pi\mu_1) + i\sin(2\pi\mu_1)p_1\\
&= \pm\alpha_1\alpha_\infty\left[-\cos(2\pi\mu_\infty) + i\sin(2\pi\mu_\infty)(a_{\mathcal{R}_0}\overline{a_{\mathcal{R}_0}}p_\infty - a_{\mathcal{R}_0}b_{\mathcal{R}_0}q_\infty - \overline{a_{\mathcal{R}_0}}\overline{b_{\mathcal{R}_0}}\overline{q_\infty} - b_{\mathcal{R}_0}\overline{b_{\mathcal{R}_0}}p_\infty)\right]
\end{aligned} \tag{6.3.48}
$$

and

$$
\begin{aligned}
&-\cos(2\pi\mu_\infty) - i\sin(2\pi\mu_\infty)p_\infty\\
&\quad= \pm\alpha_\infty\alpha_1\left[-\cos(2\pi\mu_1) - i\sin(2\pi\mu_1)(a_{\mathcal{R}_0}\overline{a_{\mathcal{R}_0}}p_1 - a_{\mathcal{R}_0}b_{\mathcal{R}_0}q_1 - \overline{a_{\mathcal{R}_0}}\overline{b_{\mathcal{R}_0}}\overline{q_1} - b_{\mathcal{R}_0}\overline{b_{\mathcal{R}_0}}p_1)\right],
\end{aligned} \tag{6.3.49}
$$

$$
\begin{aligned}
&-\cos(2\pi\mu_\infty) + i\sin(2\pi\mu_\infty)p_\infty\\
&\quad= \pm\alpha_\infty\alpha_1\left[-\cos(2\pi\mu_1) + i\sin(2\pi\mu_1)(a_{\mathcal{R}_0}\overline{a_{\mathcal{R}_0}}p_1 - a_{\mathcal{R}_0}b_{\mathcal{R}_0}q_1 - \overline{a_{\mathcal{R}_0}}\overline{b_{\mathcal{R}_0}}\overline{q_1} - b_{\mathcal{R}_0}\overline{b_{\mathcal{R}_0}}p_1)\right],
\end{aligned} \tag{6.3.50}
$$

respectively. By summing up the first two equations (and recalling that $\mu_\infty = \mu_1$), we conclude that the factor $\pm\alpha_1\alpha_\infty$ necessarily equals $+1$. Analogously, by summing up the other two equations, we deduce as well $\pm\alpha_\infty\alpha_1 = +1$. Therefore,

$$
M_{\mathcal{R}_0}(\lambda)\hat{M}_1(\lambda) = \hat{M}_\infty(\lambda)M_{\mathcal{R}_0}(\lambda), \tag{6.3.51}
$$

$$
M_{\mathcal{R}_0}(\lambda)\hat{M}_\infty(\lambda) = \hat{M}_1(\lambda)M_{\mathcal{R}_0}(\lambda). \tag{6.3.52}
$$

as claimed.

What remains to prove is (6.3.20). But this equation follows in view of (3.9.32) directly from equations (6.3.19) and (6.3.18):

$$
\begin{aligned}
M_{\mathcal{R}_0}(\lambda)\hat{M}_0(\lambda) &= M_{\mathcal{R}_0}(\lambda)(\hat{M}_\infty(\lambda))^{-1}(\hat{M}_1(\lambda))^{-1}\\
&= (\hat{M}_1(\lambda))^{-1}M_{\mathcal{R}_0}(\lambda)(\hat{M}_1(\lambda))^{-1} = (\hat{M}_1(\lambda))^{-1}(\hat{M}_\infty(\lambda))^{-1}M_{\mathcal{R}_0}(\lambda).
\end{aligned} \tag{6.3.53}
$$

As equation (6.3.20) is implied by equations (6.3.18) and (6.3.19), these three equations are equivalent to the scalar reformulations of the equations (6.3.18) and (6.3.19), which read

$$
-\cos(2\pi\mu_1)a_{\mathcal{R}_0} - i\sin(2\pi\mu_1)(a_{\mathcal{R}_0}p_1 + b_{\mathcal{R}_0}q_1) = -\cos(2\pi\mu_\infty)a_{\mathcal{R}_0} - i\sin(2\pi\mu_\infty)(a_{\mathcal{R}_0}p_\infty - \overline{b_{\mathcal{R}_0}}\overline{q_\infty}), \tag{6.3.54}
$$

$$
-\cos(2\pi\mu_1)b_{\mathcal{R}_0} - i\sin(2\pi\mu_1)(a_{\mathcal{R}_0}\overline{q_1} - b_{\mathcal{R}_0}p_1) = -\cos(2\pi\mu_\infty)b_{\mathcal{R}_0} - i\sin(2\pi\mu_\infty)(b_{\mathcal{R}_0}p_\infty + \overline{a_{\mathcal{R}_0}}\overline{q_\infty}), \tag{6.3.55}
$$

and

$$
-\cos(2\pi\mu_\infty)a_{\mathcal{R}_0} - i\sin(2\pi\mu_\infty)(a_{\mathcal{R}_0}p_\infty + b_{\mathcal{R}_0}q_\infty) = -\cos(2\pi\mu_1)a_{\mathcal{R}_0} - i\sin(2\pi\mu_1)(a_{\mathcal{R}_0}p_1 - \overline{b_{\mathcal{R}_0}}\overline{q_1}), \tag{6.3.56}
$$

$$
-\cos(2\pi\mu_\infty)b_{\mathcal{R}_0} - i\sin(2\pi\mu_\infty)(a_{\mathcal{R}_0}\overline{q_\infty} - b_{\mathcal{R}_0}p_\infty) = -\cos(2\pi\mu_1)b_{\mathcal{R}_0} - i\sin(2\pi\mu_1)(b_{\mathcal{R}_0}p_1 + \overline{a_{\mathcal{R}_0}}\overline{q_1}), \tag{6.3.57}
$$

respectively. A straightforward simplification of these equations yields the claimed ones and finishes the proof of the first case, $l = 0$.

The cases $l = 1$ and $l = \infty$ are proved completely analogously by simply shifting labels. The only remaining identities, which still need to be checked, are

$$
k_\infty(z)k_{\mathcal{R}_1,\tilde{\gamma}_{\mathcal{R}_1}}(\tilde{\gamma}_\infty(z)) = \pm k_{\mathcal{R}_1,\tilde{\gamma}_{\mathcal{R}_1}}(z)k_0(\tilde{\gamma}_{\mathcal{R}_1}(z)), \tag{6.3.58}
$$

$$
k_0(z)k_{\mathcal{R}_1,\tilde{\gamma}_{\mathcal{R}_1}}(\tilde{\gamma}_0(z)) = \pm k_{\mathcal{R}_1,\tilde{\gamma}_{\mathcal{R}_1}}(z)k_\infty(\tilde{\gamma}_{\mathcal{R}_1}(z)), \tag{6.3.59}
$$

$$
k_0(z)k_{\mathcal{R}_\infty,\tilde{\gamma}_{\mathcal{R}_\infty}}(\tilde{\gamma}_0(z)) = \pm k_{\mathcal{R}_\infty,\tilde{\gamma}_{\mathcal{R}_\infty}}(z)k_1(\tilde{\gamma}_{\mathcal{R}_\infty}(z)), \tag{6.3.60}
$$

$$
k_1(z)k_{\mathcal{R}_\infty,\tilde{\gamma}_{\mathcal{R}_\infty}}(\tilde{\gamma}_1(z)) = \pm k_{\mathcal{R}_\infty,\tilde{\gamma}_{\mathcal{R}_\infty}}(z)k_0(\tilde{\gamma}_{\mathcal{R}_\infty}(z)). \tag{6.3.61}
$$

This is done by direct computations:

$$
\begin{aligned}
k_\infty(z)k_{\mathcal{R}_1,\tilde{\gamma}_{\mathcal{R}_1}}(\tilde{\gamma}_\infty(z)) &= \begin{pmatrix}\sqrt{\sqrt{\frac{1+2\bar{z}}{1+2z}}} & 0\\ 0 & \sqrt{\frac{1+2\bar{z}}{1+2z}}\end{pmatrix}\begin{pmatrix}\sqrt{\frac{2\frac{-3\bar{z}-2}{2\bar{z}+1}+1}{2\frac{-3z-2}{2z+1}+1}} & 0\\ 0 & \sqrt{\frac{2\frac{-3\bar{z}-2}{2\bar{z}+1}+1}{2\frac{-3z-2}{2z+1}+1}}\end{pmatrix} = \pm\begin{pmatrix}\sqrt{\sqrt{\frac{4\bar{z}+3}{4z+3}}} & 0\\ 0 & \sqrt{\frac{4\bar{z}+3}{4z+3}}\end{pmatrix}\\
&= \pm\begin{pmatrix}\sqrt{\sqrt{\frac{2\bar{z}+1}{2z+1}}} & 0\\ 0 & \sqrt{\frac{2\bar{z}+1}{2z+1}}\end{pmatrix}\begin{pmatrix}\sqrt{\sqrt{\frac{1-2\bar{z}}{1-2z}}} & 0\\ 0 & \sqrt{\frac{1-2\frac{-\bar{z}-1}{2\bar{z}+1}}{1-2\frac{-z-1}{2z+1}}}\end{pmatrix} = \pm k_{\mathcal{R}_1,\tilde{\gamma}_{\mathcal{R}_1}}(z)k_0(\tilde{\gamma}_{\mathcal{R}_1}(z)),
\end{aligned} \tag{6.3.62}
$$

$$k_0(z)k_{\mathcal{R}_1,\tilde{\gamma}_{\mathcal{R}_1}}(\tilde{\gamma}_0(z)) = \begin{pmatrix} \sqrt{\sqrt{\frac{1-2\bar{z}}{1-2z}}} & 0 \\ 0 & \sqrt{\frac{1-2\bar{z}}{1-2z}} \end{pmatrix} \begin{pmatrix} \sqrt{\sqrt{\frac{2\frac{\bar{z}}{-2\bar{z}+1}+1}{2\frac{z}{-2z+1}+1}}} & 0 \\ 0 & \sqrt{\frac{2\frac{\bar{z}}{-2\bar{z}+1}+1}{2\frac{z}{-2z+1}+1}} \end{pmatrix} = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \pm \begin{pmatrix} \sqrt{\sqrt{\frac{2\bar{z}+1}{2z+1}}} & 0 \\ 0 & \sqrt{\frac{2\bar{z}+1}{2z+1}} \end{pmatrix} \begin{pmatrix} \sqrt{\sqrt{\frac{1+2\frac{-\bar{z}-1}{2\bar{z}+1}}{1+2\frac{-z-1}{2z+1}}}} & 0 \\ 0 & \sqrt{\frac{1+2\frac{-\bar{z}-1}{2\bar{z}+1}}{1+2\frac{-z-1}{2z+1}}} \end{pmatrix} = \pm k_{\mathcal{R}_1,\tilde{\gamma}_{\mathcal{R}_1}}(z)k_\infty(\tilde{\gamma}_{\mathcal{R}_1}(z)), \quad (6.3.63)$$

$$k_0(z)k_{\mathcal{R}_\infty,\tilde{\gamma}_{\mathcal{R}_\infty}}(\tilde{\gamma}_0(z)) = \begin{pmatrix} \sqrt{\sqrt{\frac{1-2\bar{z}}{1-2z}}} & 0 \\ 0 & \sqrt{\frac{1-2\bar{z}}{1-2z}} \end{pmatrix} \begin{pmatrix} \sqrt{\sqrt{\frac{\frac{\bar{z}}{-2\bar{z}+1}}{\frac{z}{-2z+1}}}} & 0 \\ 0 & \sqrt{\frac{\frac{\bar{z}}{-2\bar{z}+1}}{\frac{z}{-2z+1}}} \end{pmatrix}$$

$$= \pm \begin{pmatrix} \sqrt{\sqrt{\frac{\bar{z}}{z}}} & 0 \\ 0 & \sqrt{\frac{\bar{z}}{z}} \end{pmatrix} = \pm k_{\mathcal{R}_\infty,\tilde{\gamma}_{\mathcal{R}_\infty}}(z)k_1(\tilde{\gamma}_{\mathcal{R}_\infty}(z)), \quad (6.3.64)$$

$$k_1(z)k_{\mathcal{R}_\infty,\tilde{\gamma}_{\mathcal{R}_\infty}}(\tilde{\gamma}_1(z)) = \begin{pmatrix} \sqrt{\sqrt{\frac{\bar{z}+2}{z+2}}} & 0 \\ 0 & \sqrt{\frac{\bar{z}+2}{z+2}} \end{pmatrix}$$

$$= \pm \begin{pmatrix} \sqrt{\sqrt{\frac{\bar{z}}{z}}} & 0 \\ 0 & \sqrt{\frac{\bar{z}}{z}} \end{pmatrix} \begin{pmatrix} \sqrt{\sqrt{\frac{1+\frac{2}{\bar{z}}}{1+\frac{2}{z}}}} & 0 \\ 0 & \sqrt{\frac{1+\frac{2}{\bar{z}}}{1+\frac{2}{z}}} \end{pmatrix} = \pm k_{\mathcal{R}_\infty,\tilde{\gamma}_{\mathcal{R}_\infty}}(z)k_0(\tilde{\gamma}_{\mathcal{R}_\infty}(z)). \quad (6.3.65)$$

□

6.4 Normalized trinoids with properly embedded annular ends, which are rotationally symmetric with respect to a trinoid axis

Let $l \in \{0, 1, \infty\}$ and $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends, which is rotationally symmetric with respect to the trinoid axis A_l. Moreover, let $\psi = \phi \circ \pi$ be the associated CMC-immersion $\tilde{M} \to \mathbb{R}^3$. Denote by $\mathcal{R}_\lambda$ the corresponding symmetry of ϕ (and ψ), i.e. the rotation by the angle π around the trinoid axis A_l.

We review the results of section 6.3 in the special case that the extended frame $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ associated with ψ as in section 4.2 is "normalized" at $z_{*,l} \in \tilde{M}$, which we choose depending on l as follows:

$$z_{*,0} = -1 + i \in \tilde{M}, \quad (6.4.1)$$

$$z_{*,1} = \frac{-1+i}{2} \in \tilde{M}, \quad (6.4.2)$$

$$z_{*,\infty} = i \in \tilde{M}. \quad (6.4.3)$$

The "normalization" of F is realized in form of the presumption that

$$F(z_{*,l}, \lambda) = \mathrm{I} \quad (6.4.4)$$

for all $\lambda \in S^1$. More precisely (cf. section 4.2), the normalization $F(z_{*,l}, \lambda) = \mathrm{I}$ of F is a consequence of normalizing the (conformal) CMC-immersion ψ, such that

$$\psi(z_{*,l}) = \frac{1}{2H} e_3, \quad \mathcal{U}(z_{*,l}) = \mathcal{G}(1), \quad (6.4.5)$$

where $\mathcal{U} \in \mathrm{SO}(3)$ represents the natural orthonormal frame corresponding to ψ, and $\mathcal{G}(1)$ is given in (4.2.5). Recall from section 4.2, that this normalization of ψ corresponds to rotating and shifting the (image of the) trinoid in $\mathbb{R}^3$, such that the conditions (6.4.5) are met. It turns out (cf. corollary 6.8), that the choice of $z_{*,l}$ as above (for a trinoid ϕ with properly embedded annular ends, which is rotationally symmetric with respect to the trinoid axis A_l) corresponds to arranging the (image of the) trinoid in $\mathbb{R}^3$, such that the rotation axis of $\mathcal{R}_l$ is the z-axis in $\mathbb{R}^3$.

A trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends, which is rotationally symmetric with respect to the trinoid axis A_l and, in addition, is "well positioned" in $\mathbb{R}^3$ in the sense that the associated conformal CMC-immersion $\psi : \tilde{M} \to M$ meets the normalization conditions (6.4.5), is called a *normalized* trinoid with properly embedded annular ends, which is rotationally symmetric with respect to the trinoid axis A_l.

We now formulate a more explicit version of theorem 6.4:

Theorem 6.7. *Let $M = \mathbb{C} \setminus \{0, 1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ as defined in* (3.2.2)*. Let $l \in \{0, 1, \infty\}$ and ϕ be rotationally symmetric with respect to the trinoid axis A_l. Moreover, let, according to l, $z_{*,l}$ be given in* (6.4.1), (6.4.2) *or* (6.4.3),

$$z_{*,0} = -1 + i, \quad z_{*,1} = \frac{-1 + i}{2}, \quad z_{*,\infty} = i, \tag{6.4.6}$$

and assume that ψ has been normalized at $z_{*,l}$, such that $\psi(z_{*,l}) = \frac{1}{2H} e_3$ and $F(z_{*,l}, \lambda) = \mathrm{I}$, where $F : \tilde{M} \to \Lambda \mathrm{SU}(2)_\sigma$ denotes the extended frame corresponding to ψ by theorem 4.5. Denote by $\mathcal{R}_l$ the corresponding symmetry of ϕ and by $\tilde{\gamma}_{\mathcal{R}_l}$ the biholomorphic mapping $\tilde{M} \to \tilde{M}$ associated with $\mathcal{R}_l$ as in theorem 4.9 and, according to l, explicitly given in (6.2.10), (6.2.13) *or* (6.2.16)*:*

$$\tilde{\gamma}_{\mathcal{R}_0}(z) = \frac{-z - 2}{z + 1}, \quad \tilde{\gamma}_{\mathcal{R}_1}(z) = \frac{-z - 1}{2z + 1}, \quad \tilde{\gamma}_{\mathcal{R}_\infty}(z) = -\frac{1}{z}. \tag{6.4.7}$$

Then, the extended frame F transforms under $\tilde{\gamma}_{\mathcal{R}_l}$ as

$$F(\tilde{\gamma}_{\mathcal{R}_l}(z), \lambda) = M_{\mathcal{R}_l}(\lambda) F(z, \lambda) k_{\mathcal{R}_l, \gamma_{\tilde{\mathcal{R}}_l}}(z) \tag{6.4.8}$$

where $k_{\mathcal{R}_l, \gamma_{\tilde{\mathcal{R}}_l}}(z)$ is, according to l, given in (6.2.24), (6.2.25) *or* (6.2.26) *and*

$$M_{\mathcal{R}_l}(\lambda) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{6.4.9}$$

In particular, $M_{\mathcal{R}_l}$ is actually independent of λ.

Proof. In view of theorem 6.4, we only have to prove the equation (6.4.9). To this end, we compute

$$\tilde{\gamma}_{\mathcal{R}_0}(z_{*,0}) = \frac{-(-1 + i) - 2}{(-1 + i) + 1} = z_{*,0}, \tag{6.4.10}$$

$$\tilde{\gamma}_{\mathcal{R}_1}(z_{*,1}) = \frac{-\frac{-1+i}{2} - 1}{2\frac{-1+i}{2} + 1} = z_{*,1}, \tag{6.4.11}$$

$$\tilde{\gamma}_{\mathcal{R}_\infty}(z_{*,\infty}) = -\frac{1}{i} = z_{*,\infty}, \tag{6.4.12}$$

which shows that we have for all $l \in \{0, 1, \infty\}$

$$\tilde{\gamma}_{\mathcal{R}_l}(z_{*,l}) = z_{*,l}. \tag{6.4.13}$$

Furthermore, $F(z_{*,l}, \lambda) = \mathrm{I}$. Thus, evaluating equation (6.4.8) at $z = z_{*,l}$ yields

$$\mathrm{I} = F(z_{*,l}, \lambda) = F(\tilde{\gamma}_{\mathcal{R}_l}(z_{*,l}), \lambda) = M_{\mathcal{R}_l}(\lambda) F(z_{*,l}, \lambda) k_{\mathcal{R}_l, \gamma_{\tilde{\mathcal{R}}_l}}(z_{*,l}), \tag{6.4.14}$$

i.e.

$$M_{\mathcal{R}_l}(\lambda) = \left(k_{\mathcal{R}_l, \gamma_{\tilde{\mathcal{R}}_l}}(z_{*,l}) \right)^{-1}. \tag{6.4.15}$$

In view of remark 4.14 (for our definition of the complex square root) and equations (6.2.24), (6.2.25) and (6.2.26), we have

$$k_{\mathcal{R}_0,\gamma_{\tilde{\mathcal{R}}_l}}(z_{*,0}) = \begin{pmatrix} \overline{\sqrt{\frac{\overline{(-1+i)}+1}{(-1+i)+1}}} & 0 \\ 0 & \sqrt{\frac{\overline{(-1+i)}+1}{(-1+i)+1}} \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{6.4.16}$$

$$k_{\mathcal{R}_1,\gamma_{\tilde{\mathcal{R}}_l}}(z_{*,1}) = \begin{pmatrix} \overline{\sqrt{\frac{2\frac{-1-i}{2}+1}{2\frac{-1+i}{2}+1}}} & 0 \\ 0 & \sqrt{\frac{2\frac{-1-i}{2}+1}{2\frac{-1+i}{2}+1}} \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{6.4.17}$$

$$k_{\mathcal{R}_\infty,\gamma_{\tilde{\mathcal{R}}_l}}(z_{*,\infty}) = \begin{pmatrix} \overline{\sqrt{\frac{-i}{i}}} & 0 \\ 0 & \sqrt{\frac{\overline{-i}}{i}} \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{6.4.18}$$

i.e.

$$k_{\mathcal{R}_l,\gamma_{\tilde{\mathcal{R}}_l}}(z_{*,l}) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{6.4.19}$$

for all $l \in \{0, 1, \infty\}$. Altogether, (6.4.9) follows. □

Corollary 6.8. *We retain the notation and the assumptions of theorem 6.7. The rotation axis of the symmetry $\mathcal{R}_l$ of the normalized trinoid ϕ is the z-axis in $\mathbb{R}^3$.*

Proof. Applying (the first part of) theorem 4.17, we know that the monodromy matrix $M_{\mathcal{R}_l}(\lambda)$ explicitly given in theorem 6.7 satisfies at $\lambda = 1$

$$M_{\mathcal{R}_l}(1) = \pm A_{\mathcal{R}_l}, \tag{6.4.20}$$

where $A_{\mathcal{R}_l} \in \mathrm{SU}(2)$ denotes the conjugation matrix realizing the orthogonal part $\mathcal{A}_{\mathcal{R}_l}$ of the symmetry $\mathcal{R}_l$ in the su(2)-model. In view of equation (6.4.9), this yields

$$A_{\mathcal{R}_l} = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{6.4.21}$$

Recalling that $\mathcal{A}_{\mathcal{R}_l}$ and $A_{\mathcal{R}_l}$ are linked via the Lie Algebra isomorphism $J : \mathbb{R}^3 \to \mathrm{su}(2)$ defined in (3.4.3) as in (3.4.7), i.e.

$$(J \circ \mathcal{A}_{\mathcal{R}_l} \circ J^{-1})(X) = A_{\mathcal{R}_l} X A_{\mathcal{R}_l}^{-1} \text{ for all } X \in \mathrm{su}(2), \tag{6.4.22}$$

we obtain by a direct computation that

$$\mathcal{A}_{\mathcal{R}_l} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.4.23}$$

Thus, $\mathcal{A}_{\mathcal{R}_l}$ defines the rotation (in $\mathbb{R}^3$) by the angles π around the z-axis in $\mathbb{R}^3$, $\mathbb{R}e_3$. Consequently, the symmetry $\mathcal{R}_l$ of the normalized trinoid ϕ is a rotation by the angle π around an axis in $\mathbb{R}^3$, which is parallel to the z-axis. As the point $\psi(z_{*,l}) \in \mathbb{R}^3$ (with $z_{*,l}$ given in (6.4.1), (6.4.2) or (6.4.3) according to l) satisfies

$$\mathcal{R}_l(\psi(z_{*,l})) = \psi(\tilde{\gamma}_{\mathcal{R}_l}(z_{*,l})) = \psi(z_{*,l}), \tag{6.4.24}$$

it lies on the rotation axis of $\mathcal{R}_l$. Since by assumption we have $\psi(z_{*,l}) = \frac{1}{2H} e_3$, we infer that the rotation axis of $\mathcal{R}_l$ is actually the z-axis in $\mathbb{R}^3$. □

Applying the theorems 6.6 and 6.7, we obtain the following result:

Theorem 6.9. *Let $M = \mathbb{C} \setminus \{0, 1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ as defined in (3.2.2). Let $l \in \{0, 1, \infty\}$ and ϕ be rotationally symmetric with respect to the trinoid axis A_l. Moreover, let, according to l, $z_{*,l}$ be given in (6.4.1), (6.4.2) or (6.4.3), and assume that ψ has been normalized at $z_{*,l}$, such that $\psi(z_{*,l}) = \frac{1}{2H} e_3$ and $F(z_{*,l}, \lambda) = \mathrm{I}$, where $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ denotes the extended frame corresponding to ψ by theorem 4.5.*

1. *In the case $l = 0$, the unitary monodromy matrices $\hat{M}_j \in \Lambda\mathrm{SU}(2,\mathbb{C})_\sigma$, $j = 0, 1, \infty$, associated with F as in (6.3.16) are of the form*

$$\hat{M}_0 = -\cos(2\pi\mu_0)\mathrm{I} - 2\alpha i\cos(\pi\mu_0)\begin{pmatrix}\cos(2\pi\mu_1) & -i\overline{\zeta_1} \\ i\zeta_1 & -\cos(2\pi\mu_1)\end{pmatrix}, \tag{6.4.25}$$

$$\hat{M}_1 = -\cos(2\pi\mu_1)\mathrm{I} - i\begin{pmatrix}\alpha\cos(\pi\mu_1) & \overline{\zeta_1} \\ \zeta_1 & -\alpha\cos(\pi\mu_1)\end{pmatrix}, \tag{6.4.26}$$

$$\hat{M}_\infty = -\cos(2\pi\mu_1)\mathrm{I} - i\begin{pmatrix}\alpha\cos(\pi\mu_1) & -\overline{\zeta_1} \\ -\zeta_1 & -\alpha\cos(\pi\mu_1)\end{pmatrix}, \tag{6.4.27}$$

where $\alpha \in \{\pm 1\}$ and ζ_1 is an odd function in λ and a solution to

$$\zeta_1\overline{\zeta_1} = \sin^2(2\pi\mu_1) - \cos^2(\pi\mu_0). \tag{6.4.28}$$

2. *In the case $l = 1$, the unitary monodromy matrices $\hat{M}_j \in \Lambda\mathrm{SU}(2,\mathbb{C})_\sigma$, $j = 0, 1, \infty$, associated with F as in (6.3.16) are of the form*

$$\hat{M}_0 = -\cos(2\pi\mu_\infty)\mathrm{I} - i\begin{pmatrix}\alpha\cos(\pi\mu_\infty) & -\overline{\zeta_\infty} \\ -\zeta_\infty & -\alpha\cos(\pi\mu_\infty)\end{pmatrix}, \tag{6.4.29}$$

$$\hat{M}_1 = -\cos(2\pi\mu_1)\mathrm{I} - 2\alpha i\cos(\pi\mu_1)\begin{pmatrix}\cos(2\pi\mu_\infty) & -i\overline{\zeta_\infty} \\ i\zeta_\infty & -\cos(2\pi\mu_\infty)\end{pmatrix}, \tag{6.4.30}$$

$$\hat{M}_\infty = -\cos(2\pi\mu_\infty)\mathrm{I} - i\begin{pmatrix}\alpha\cos(\pi\mu_\infty) & \overline{\zeta_\infty} \\ \zeta_\infty & -\alpha\cos(\pi\mu_\infty)\end{pmatrix}, \tag{6.4.31}$$

where $\alpha \in \{\pm 1\}$ and ζ_∞ is an odd function in λ and a solution to

$$\zeta_\infty\overline{\zeta_\infty} = \sin^2(2\pi\mu_\infty) - \cos^2(\pi\mu_1). \tag{6.4.32}$$

3. *In the case $l = \infty$, the unitary monodromy matrices $\hat{M}_j \in \Lambda\mathrm{SU}(2,\mathbb{C})_\sigma$, $j = 0, 1, \infty$, associated with F as in (6.3.16) are of the form*

$$\hat{M}_0 = -\cos(2\pi\mu_0)\mathrm{I} - i\begin{pmatrix}\alpha\cos(\pi\mu_0) & \overline{\zeta_0} \\ \zeta_0 & -\alpha\cos(\pi\mu_0)\end{pmatrix}, \tag{6.4.33}$$

$$\hat{M}_1 = -\cos(2\pi\mu_0)\mathrm{I} - i\begin{pmatrix}\alpha\cos(\pi\mu_0) & -\overline{\zeta_0} \\ -\zeta_0 & -\alpha\cos(\pi\mu_0)\end{pmatrix}, \tag{6.4.34}$$

$$\hat{M}_\infty = -\cos(2\pi\mu_\infty)\mathrm{I} - 2\alpha i\cos(\pi\mu_\infty)\begin{pmatrix}\cos(2\pi\mu_0) & -i\overline{\zeta_0} \\ i\zeta_0 & -\cos(2\pi\mu_0)\end{pmatrix}, \tag{6.4.35}$$

where $\alpha \in \{\pm 1\}$ and ζ_0 is an odd function in λ and a solution to

$$\zeta_0\overline{\zeta_0} = \sin^2(2\pi\mu_0) - \cos^2(\pi\mu_\infty). \tag{6.4.36}$$

Proof. We prove the case $l = \infty$. As before, we denote the symmetry of ϕ by $\mathcal{R}_l = \mathcal{R}_\infty$ and by $\tilde{\gamma}_{\mathcal{R}_\infty}$ the biholomorphic mapping $\tilde{M} \to \tilde{M}$ associated with $\mathcal{R}_\infty$ as in theorem 4.9 and explicitly given in lemma 6.3. Moreover, let $M_{\mathcal{R}_\infty}(\lambda)$ be the corresponding monodromy matrix of F as introduced in equation (6.2.23). In view of theorem 6.7, we have

$$M_{\mathcal{R}_\infty}(\lambda) = \begin{pmatrix} a_{\mathcal{R}_\infty} & b_{\mathcal{R}_\infty} \\ -\overline{b_{\mathcal{R}_\infty}} & \overline{a_{\mathcal{R}_\infty}}\end{pmatrix} \tag{6.4.37}$$

where

$$a_{\mathcal{R}_\infty} = i \quad\text{and}\quad b_{\mathcal{R}_\infty} = 0. \tag{6.4.38}$$

Moreover, by theorem 6.6, we obtain the following relations between the unitary monodromy matrices $\hat{M}_j$ of the extended frame F:

$$M_{\mathcal{R}_\infty}(\lambda)\hat{M}_0(\lambda) = \hat{M}_1(\lambda)M_{\mathcal{R}_\infty}(\lambda), \tag{6.4.39}$$

$$M_{\mathcal{R}_\infty}(\lambda)\hat{M}_1(\lambda) = \hat{M}_0(\lambda)M_{\mathcal{R}_\infty}(\lambda), \tag{6.4.40}$$

$$M_{\mathcal{R}_\infty}(\lambda)\hat{M}_\infty(\lambda) = (\hat{M}_0(\lambda))^{-1}(\hat{M}_1(\lambda))^{-1}M_{\mathcal{R}_\infty}(\lambda), \tag{6.4.41}$$

which translate into the following scalar equations involving the functions p_j and q_j occurring in $\hat{M}_j$ (cf. (3.9.26)):

$$p_1 = p_0, \quad q_1 = -q_0. \tag{6.4.42}$$

Thus, in the case of a normalized trinoid, which is rotationally symmetric with respect to the trinoid axis A_∞, we obtain the following equivalent reformulations of (3.9.50) and (3.9.51), characterizing the monodromy matrices $\hat{M}_j$ (recall from lemma 6.2 that $\mu_0 = \mu_1$):

$$p_0 = \overline{p_0} \text{ and } p_0^2 + q_0\overline{q_0} = 1, \tag{6.4.43}$$

$$p_0^2 - q_0\overline{q_0} = \frac{\cos^2(2\pi\mu_0) + \cos(2\pi\mu_\infty)}{\sin^2(2\pi\mu_0)}. \tag{6.4.44}$$

Here, the second equation follows in view of $q_0\overline{q_1} + \overline{q_0}q_1 = -2q_0\overline{q_0}$.

We derive directly from (6.4.43) that

$$q_0\overline{q_0} = 1 - p_0^2. \tag{6.4.45}$$

Inserting this into the second equation, we obtain

$$p_0^2 = \frac{\cos^2(2\pi\mu_0) + \cos(2\pi\mu_\infty) + \sin^2(2\pi\mu_0)}{2\sin^2(2\pi\mu_0)} = \frac{1 + \cos(2\pi\mu_\infty)}{2\sin^2(2\pi\mu_0)} = \frac{\cos^2(\pi\mu_\infty)}{\sin^2(2\pi\mu_0)}. \tag{6.4.46}$$

This in turn implies

$$q_0\overline{q_0} = 1 - p_0^2 = \frac{\sin^2(2\pi\mu_0) - \cos^2(\pi\mu_\infty)}{\sin^2(2\pi\mu_0)}. \tag{6.4.47}$$

So far, we conclude that

$$p_1 = p_0 = \alpha\frac{\cos(\pi\mu_\infty)}{\sin(2\pi\mu_0)}, \tag{6.4.48}$$

$$-q_1 = q_0 = \frac{\zeta_0}{\sin(2\pi\mu_0)}, \tag{6.4.49}$$

where $\alpha \in \{\pm 1\}$ and ζ_0 is an odd function in λ satisfying

$$\zeta_0\overline{\zeta_0} = \sin^2(2\pi\mu_0) - \cos^2(\pi\mu_\infty). \tag{6.4.50}$$

(Recall that, by remark 3.44, q_0 and thus also ζ_0 are necessarily odd functions in λ.)

Next, recall that the monodromy matrices $\hat{M}_j$ satisfy (3.9.32), i.e. $\hat{M}_0\hat{M}_1\hat{M}_\infty = \mathrm{I}$, which reads in scalar form as (3.9.33) and (3.9.34). Inserting the previous results together with the identity $\mu_0 = \mu_1$ from lemma 6.2 into (3.9.33), we obtain

$$\begin{aligned}\cos(2\pi\mu_\infty) + i\sin(2\pi\mu_\infty)p_\infty &= -\cos^2(2\pi\mu_0) + 2i\cos(2\pi\mu_0)\sin(2\pi\mu_0)p_0 + \sin^2(2\pi\mu_0)(p_0^2 - q_0\overline{q_0}) \\ &= -\cos^2(2\pi\mu_0) + 2\alpha i\cos(2\pi\mu_0)\cos(\pi\mu_\infty) + \cos^2(\pi\mu_\infty) - \sin^2(2\pi\mu_0) + \cos^2(\pi\mu_\infty)\end{aligned} \tag{6.4.51}$$

which in view of $\cos(2\pi\mu_\infty) = 2\cos^2(\pi\mu_\infty) - 1$ transforms into

$$\sin(2\pi\mu_\infty)p_\infty = 2\alpha\cos(2\pi\mu_0)\cos(\pi\mu_\infty), \tag{6.4.52}$$

i.e.

$$p_\infty = \alpha\frac{\cos(2\pi\mu_0)}{\sin(\pi\mu_\infty)}. \tag{6.4.53}$$

Similarly, (3.9.34) reads as

$$i\sin(2\pi\mu_\infty)q_\infty = -2\alpha\zeta_0\cos(\pi\mu_\infty), \tag{6.4.54}$$

which implies

$$q_\infty = \alpha i\frac{\zeta_0}{\sin(\pi\mu_\infty)}. \tag{6.4.55}$$

Applying our results to (3.9.26), we obtain the claimed forms for the monodromy matrices $\hat{M}_j$.

The cases $l = 0$ and $l = 1$ are proved analogously (by shifting indices) by using the equivalent reformulations (3.9.37), (3.9.38) and (3.9.35), (3.9.36), respectively, of (3.9.33), (3.9.34), and the according reformulation of (3.9.51) as given in remark 3.56. □

Theorem 6.9 describes the (unitary) monodromy matrices associated with the extended frame $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ of a trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends, which is rotationally symmetric with respect to the trinoid axis A_l for some $l \in \{0, 1, \infty\}$, and which has been normalized such that $F(z_{*,l}) = \mathrm{I}$ and $\psi(z_{*,l}) = \frac{1}{2H} e_3$, where $z_{*,l} \in \tilde{M}$ is given, according to l, in (6.4.1), (6.4.2) or (6.4.3), respectively, and ψ denotes the conformal CMC-immersion $\tilde{M} \to \mathbb{R}^3$ corresponding to ϕ. It turns out that, in this setting, we can also prove the converse result: A trinoid ϕ with properly embedded annular ends and with extended frame F satisfying (for some $l \in \{0, 1, \infty\}$) $F(z_{*,l}) = \mathrm{I}$ at $z_{*,l} \in \tilde{M}$ from, according to l, (6.4.1), (6.4.2) or (6.4.3), respectively, and corresponding monodromy matrices of the form given in theorem 6.9 is necessarily rotationally symmetric with respect to the trinoid axis A_l. This result is formulated in the following theorem.

Theorem 6.10. *Let η be a (standardized) trinoid potential associated with three off-diagonal Delaunay matrices D_0, D_1, D_∞ with eigenvalues $\pm\mu_0$, $\pm\mu_1$ and $\pm\mu_\infty$, respectively. Denote by $\phi : M \to \mathbb{R}^3$ a trinoid with properly embedded annular ends on $M = \mathbb{C} \setminus \{0, 1\}$ generated by η via the loop group method. Moreover, let $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ be the extended frame associated with the mapping $\psi = \phi \circ \pi$ by theorem 4.5.*

1. *Let $\mu_1 = \mu_\infty$, $z_{*,0} \in \tilde{M}$ given in (6.4.1) and $F(z_{*,0}) = \mathrm{I}$. Assume the unitary monodromy matrices $\hat{M}_0, \hat{M}_1, \hat{M}_\infty \in \Lambda\mathrm{SU}(2, \mathbb{C})_\sigma$ associated with F are given by*
$$\hat{M}_0 = -\cos(2\pi\mu_0)\mathrm{I} - 2\alpha i \cos(\pi\mu_0) \begin{pmatrix} \cos(2\pi\mu_1) & -i\overline{\zeta_1} \\ i\zeta_1 & -\cos(2\pi\mu_1) \end{pmatrix}, \tag{6.4.56}$$
$$\hat{M}_1 = -\cos(2\pi\mu_1)\mathrm{I} - i \begin{pmatrix} \alpha\cos(\pi\mu_1) & \overline{\zeta_1} \\ \zeta_1 & -\alpha\cos(\pi\mu_1) \end{pmatrix}, \tag{6.4.57}$$
$$\hat{M}_\infty = -\cos(2\pi\mu_1)\mathrm{I} - i \begin{pmatrix} \alpha\cos(\pi\mu_1) & -\overline{\zeta_1} \\ -\zeta_1 & -\alpha\cos(\pi\mu_1) \end{pmatrix}, \tag{6.4.58}$$
where $\alpha \in \{\pm 1\}$ and ζ_1 is an odd function in λ and a solution to
$$\zeta_1\overline{\zeta_1} = \sin^2(2\pi\mu_1) - \cos^2(\pi\mu_0). \tag{6.4.59}$$
Then, ϕ is rotationally symmetric with respect to the trinoid axis A_0.

2. *Let $\mu_0 = \mu_\infty$, $z_{*,1} \in \tilde{M}$ given in (6.4.2) and $F(z_{*,1}) = \mathrm{I}$. Assume the unitary monodromy matrices $\hat{M}_0, \hat{M}_1, \hat{M}_\infty \in \Lambda\mathrm{SU}(2, \mathbb{C})_\sigma$ associated with F are given by*
$$\hat{M}_0 = -\cos(2\pi\mu_\infty)\mathrm{I} - i \begin{pmatrix} \alpha\cos(\pi\mu_\infty) & -\overline{\zeta_\infty} \\ -\zeta_\infty & -\alpha\cos(\pi\mu_\infty) \end{pmatrix}, \tag{6.4.60}$$
$$\hat{M}_1 = -\cos(2\pi\mu_1)\mathrm{I} - 2\alpha i \cos(\pi\mu_1) \begin{pmatrix} \cos(2\pi\mu_\infty) & -i\overline{\zeta_\infty} \\ i\zeta_\infty & -\cos(2\pi\mu_\infty) \end{pmatrix}, \tag{6.4.61}$$
$$\hat{M}_\infty = -\cos(2\pi\mu_\infty)\mathrm{I} - i \begin{pmatrix} \alpha\cos(\pi\mu_\infty) & \overline{\zeta_\infty} \\ \zeta_\infty & -\alpha\cos(\pi\mu_\infty) \end{pmatrix}, \tag{6.4.62}$$
where $\alpha \in \{\pm 1\}$ and ζ_∞ is an odd function in λ and a solution to
$$\zeta_\infty\overline{\zeta_\infty} = \sin^2(2\pi\mu_\infty) - \cos^2(\pi\mu_1). \tag{6.4.63}$$
Then, ϕ is rotationally symmetric with respect to the trinoid axis A_1.

3. *Let $\mu_0 = \mu_1$, $z_{*,\infty} \in \tilde{M}$ given in (6.4.3) and $F(z_{*,\infty}) = \mathrm{I}$. Assume the unitary monodromy matrices $\hat{M}_0, \hat{M}_1, \hat{M}_\infty \in \Lambda\mathrm{SU}(2, \mathbb{C})_\sigma$ associated with F are given by*
$$\hat{M}_0 = -\cos(2\pi\mu_0)\mathrm{I} - i \begin{pmatrix} \alpha\cos(\pi\mu_0) & \overline{\zeta_0} \\ \zeta_0 & -\alpha\cos(\pi\mu_0) \end{pmatrix}, \tag{6.4.64}$$
$$\hat{M}_1 = -\cos(2\pi\mu_0)\mathrm{I} - i \begin{pmatrix} \alpha\cos(\pi\mu_0) & -\overline{\zeta_0} \\ -\zeta_0 & -\alpha\cos(\pi\mu_0) \end{pmatrix}, \tag{6.4.65}$$
$$\hat{M}_\infty = -\cos(2\pi\mu_\infty)\mathrm{I} - 2\alpha i \cos(\pi\mu_\infty) \begin{pmatrix} \cos(2\pi\mu_0) & -i\overline{\zeta_0} \\ i\zeta_0 & -\cos(2\pi\mu_0) \end{pmatrix}, \tag{6.4.66}$$
where $\alpha \in \{\pm 1\}$ and ζ_0 is an odd function in λ and a solution to
$$\zeta_0\overline{\zeta_0} = \sin^2(2\pi\mu_0) - \cos^2(\pi\mu_\infty). \tag{6.4.67}$$
Then, ϕ is rotationally symmetric with respect to the trinoid axis A_∞.

Proof. We start by considering the special form of the potential η in each of the three cases. We associate the first, second, third case with $l=0$, $l=1$, $l=\infty$, respectively, and denote the corresponding potential by η_0, η_1, η_∞, respectively.

In the first case ($l=0$) we have $\mu_1=\mu_\infty$ and thus (cf. section 3.6)

$$\eta_0 = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q_0(z,\lambda) & 0 \end{pmatrix} \mathrm{d}z, \tag{6.4.68}$$

where

$$\begin{aligned} Q_0(z,\lambda) &= \frac{b_0(\lambda)}{z^2} + \frac{b_1(\lambda)}{(z-1)^2} + \frac{b_0(\lambda)}{z} - \frac{b_0(\lambda)}{z-1} \\ &= \frac{b_0(\lambda)(z-1)^2 + b_1(\lambda)z^2 - b_0(\lambda)z(z-1)}{z^2(z-1)^2} = \frac{b_0(\lambda)(1-z)+b_1(\lambda)z^2}{z^2(z-1)^2} \end{aligned} \tag{6.4.69}$$

and $b_j(\lambda)=\frac{1}{4}-(\mu_j(\lambda))^2$ for $j=0,1$. Considering the biholomorphic mapping $\gamma_{\mathcal{R}_0}: M \to M$ defined by $z \mapsto \gamma_{\mathcal{R}_0}(z) := \frac{z}{z-1}$ and the function $h_0 : M \to \mathbb{C}\setminus\{0\}$, $z \mapsto h_0(z) = -i(z-1)$, we compute

$$Q_0(\gamma_{\mathcal{R}_0}(z),\lambda) = \frac{b_0(\lambda)(1-\frac{z}{z-1}) + b_1(\lambda)\frac{z^2}{(z-1)^2}}{\frac{z^2}{(z-1)^2}(\frac{z}{z-1}-1)^2} = \frac{b_0(\lambda)(1-z)+b_1(\lambda)z^2}{z^2\frac{1}{(z-1)^2}} = (h_0(z))^4 Q_0(z,\lambda). \tag{6.4.70}$$

Recalling from lemma 4.21 that $\gamma_{\mathcal{R}_0}$ corresponds to the permutation $\sigma=(1\ \infty)$ of the set $\{0,1,\infty\}$, we apply lemma 4.25 to infer that η_0 transforms under $\gamma_{\mathcal{R}_0}$ as

$$\gamma_{\mathcal{R}_0}^* \eta_0 = \eta_0 \# W_{+,0}, \tag{6.4.71}$$

where

$$W_{+,0} = W_{+,0}(z,\lambda) = \begin{pmatrix} h_0(z) & 0 \\ -\lambda \partial_z h_0(z) & (h_0(z))^{-1} \end{pmatrix}. \tag{6.4.72}$$

Analogously, in the second case ($l=1$) we have $\mu_0=\mu_\infty$ and thus (cf. section 3.6)

$$\eta_1 = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q_1(z,\lambda) & 0 \end{pmatrix} \mathrm{d}z, \tag{6.4.73}$$

where

$$\begin{aligned} Q_1(z,\lambda) &= \frac{b_0(\lambda)}{z^2} + \frac{b_1(\lambda)}{(z-1)^2} + \frac{b_1(\lambda)}{z} - \frac{b_1(\lambda)}{z-1} \\ &= \frac{b_0(\lambda)(z-1)^2 + b_1(\lambda)z^2 - b_1(\lambda)z(z-1)}{z^2(z-1)^2} = \frac{b_0(\lambda)(z-1)^2+b_1(\lambda)z}{z^2(z-1)^2} \end{aligned} \tag{6.4.74}$$

and $b_j(\lambda)=\frac{1}{4}-(\mu_j(\lambda))^2$ for $j=0,1$. Considering the biholomorphic mapping $\gamma_{\mathcal{R}_1}: M \to M$ defined by $z \mapsto \gamma_{\mathcal{R}_1}(z) := \frac{1}{z}$ and the function $h_1 : M \to \mathbb{C}\setminus\{0\}$, $z \mapsto h_1(z) = -iz$, we compute

$$Q_1(\gamma_{\mathcal{R}_1}(z),\lambda) = \frac{b_0(\lambda)(\frac{1}{z}-1)^2 + b_1(\lambda)\frac{1}{z}}{\frac{1}{z^2}(\frac{1}{z}-1)^2} = \frac{b_0(\lambda)(z-1)^2+b_1(\lambda)z}{\frac{1}{z^2}(z-1)^2} = (h_1(z))^4 Q_1(z,\lambda). \tag{6.4.75}$$

Recalling from lemma 4.21 that $\gamma_{\mathcal{R}_1}$ corresponds to the permutation $\sigma=(0\ \infty)$ of the set $\{0,1,\infty\}$, we apply lemma 4.25 to infer that η_1 transforms under $\gamma\mathcal{R}_1$ as

$$\gamma_{\mathcal{R}_1}^* \eta_1 = \eta_1 \# W_{+,1}, \tag{6.4.76}$$

where

$$W_{+,1} = W_{+,1}(z,\lambda) = \begin{pmatrix} h_1(z) & 0 \\ -\lambda \partial_z h_1(z) & (h_1(z))^{-1} \end{pmatrix}. \tag{6.4.77}$$

Finally, in the third case ($l=\infty$) we have $\mu_0=\mu_1$ and thus (cf. section 3.6)

$$\eta_\infty = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q_\infty(z,\lambda) & 0 \end{pmatrix} \mathrm{d}z, \tag{6.4.78}$$

where

$$Q_\infty(z,\lambda) = \frac{b_0(\lambda)}{z^2} + \frac{b_0(\lambda)}{(z-1)^2} + \frac{c_0(\lambda)}{z} - \frac{c_0(\lambda)}{z-1} = \frac{\beta_0(\lambda)(z-1)^2 + b_0(\lambda)z^2 - c_0(\lambda)z(z-1)}{z^2(z-1)^2}, \quad (6.4.79)$$

$b_j(\lambda) = \frac{1}{4} - (\mu_0(\lambda))^2$ for $j = 0, \infty$ and $c_0(\lambda) = 2b_0(\lambda) - b_\infty(\lambda)$. Considering the biholomorphic mapping $\gamma_{\mathcal{R}_\infty} : M \to M$ defined by $z \mapsto \gamma_{\mathcal{R}_\infty}(z) := 1 - z$ and the function $h_\infty : M \to \mathbb{C} \setminus \{0\}$, $z \mapsto h_\infty(z) = -i$, we compute

$$Q_\infty(\gamma_{\mathcal{R}_\infty}(z), \lambda) = \frac{\beta_0(\lambda)(-z)^2 + b_0(\lambda)(1-z)^2 - c_0(\lambda)(1-z)(-z)}{(1-z)^2(-z)^2} = (h_\infty(z))^4 Q_\infty(z,\lambda). \quad (6.4.80)$$

Recalling from lemma 4.21 that $\gamma_{\mathcal{R}_\infty}$ corresponds to the permutation $\sigma = (0\ 1)$ of the set $\{0, 1, \infty\}$, we apply lemma 4.25 to infer that η_∞ transforms under $\gamma\mathcal{R}_\infty$ as

$$\gamma_{\mathcal{R}_\infty}^* \eta_\infty = \eta_\infty \# W_{+,\infty}, \quad (6.4.81)$$

where

$$W_{+,\infty} = W_{+,\infty}(z,\lambda) = \begin{pmatrix} h_\infty(z) & 0 \\ -\lambda \partial_z h_\infty(z) & (h_\infty(z))^{-1} \end{pmatrix}. \quad (6.4.82)$$

Altogether, we have for all $l \in \{0, 1, \infty\}$ the relation

$$\gamma_{\mathcal{R}_l}^* \eta_l = \eta_l \# W_{+,l}. \quad (6.4.83)$$

Applying the pullback construction with respect to the covering mapping $\pi : \tilde{M} \to M$ to (6.4.83), we obtain

$$\pi^*(\gamma_{\mathcal{R}_l}^* \eta_l) = \pi^*(\eta_l \# W_{+,l}) = \tilde{\eta}_l \# \tilde{W}_{+,l}, \quad (6.4.84)$$

where $\tilde{\eta}_l = \pi^* \eta_l$ denotes the pullback potential of the trinoid potential η_l (cf. section 2.3) and $\tilde{W}_{+,l} = W_{+,l} \circ \pi$. Moreover, recall that the biholomorphic mappings $\tilde{\gamma}_{\mathcal{R}_l} : \tilde{M} \to \tilde{M}$,

$$\tilde{\gamma}_{\mathcal{R}_0} : z \mapsto \frac{-z-2}{z+1}, \quad \tilde{\gamma}_{\mathcal{R}_1} : z \mapsto \frac{-z-1}{2z+1}, \quad \tilde{\gamma}_{\mathcal{R}_\infty} : z \mapsto -\frac{1}{z} \quad (6.4.85)$$

from lemma 6.3 satisfy $\gamma_{\mathcal{R}_l} \circ \pi = \pi \circ \tilde{\gamma}_{\mathcal{R}_l}$. Thus, the left hand side of (6.4.84) can be transformed as follows:

$$\begin{aligned}
\pi^*(\gamma_{\mathcal{R}_l}^* \eta_l) &= \pi^* \left[\begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q_l(\gamma_{\mathcal{R}_l}(z), \lambda) & 0 \end{pmatrix} \mathrm{d}\gamma_{\mathcal{R}_l}(z) \right] \\
&= \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q_l((\gamma_{\mathcal{R}_l} \circ \pi)(z), \lambda) & 0 \end{pmatrix} \mathrm{d}(\gamma_{\mathcal{R}_l} \circ \pi)(z) = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q_l((\pi \circ \tilde{\gamma}_{\mathcal{R}_l})(z), \lambda) & 0 \end{pmatrix} \mathrm{d}(\pi \circ \tilde{\gamma}_{\mathcal{R}_l})(z) \\
&= \tilde{\gamma}_{\mathcal{R}_l}^* \left[\begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q_l(\pi(z), \lambda) & 0 \end{pmatrix} \mathrm{d}\pi(z) \right] = \tilde{\gamma}_{\mathcal{R}_l}^*(\pi^* \eta_l) = \tilde{\gamma}_{\mathcal{R}_l}^* \tilde{\eta}_l. \quad (6.4.86)
\end{aligned}$$

Altogether, (6.4.84) yields

$$\tilde{\gamma}_{\mathcal{R}_l}^* \tilde{\eta}_l = \tilde{\eta}_l \# \tilde{W}_{+,l}. \quad (6.4.87)$$

Considering the extended frame F associated with the trinoid ϕ, we obtain (for $l \in \{0, 1, \infty\}$) a solution $\Psi_l = F B_{+,l}$ to the differential equation $\mathrm{d}\Psi_l = \Psi_l \tilde{\eta}_l$. Note that Ψ_l possesses the same (unitary) monodromy matrices as F at the singularities of the potential η_l, namely $\hat{M}_0$, $\hat{M}_1$ and $\hat{M}_\infty$.

Naturally, the mapping $\tilde{\gamma}_{\mathcal{R}_l}^* \Psi_l = \Psi_l \circ \tilde{\gamma}_{\mathcal{R}_l}$ defines a solution to the differential equation $\mathrm{d}(\tilde{\gamma}_{\mathcal{R}_l}^* \Psi_l) = (\tilde{\gamma}_{\mathcal{R}_l}^* \Psi_l)(\tilde{\gamma}_{\mathcal{R}_l}^* \tilde{\eta}_l)$, which in view of (6.4.87) reads as

$$\mathrm{d}(\tilde{\gamma}_{\mathcal{R}_l}^* \Psi_l) = (\tilde{\gamma}_{\mathcal{R}_l}^* \Psi_l)(\tilde{\eta}_l \# \tilde{W}_{+,l}). \quad (6.4.88)$$

Since this differential equation is also solved by the mapping $\Psi_l \tilde{W}_{+,l}$, i.e.

$$\mathrm{d}(\Psi_l \tilde{W}_{+,l}) = (\Psi_l \tilde{W}_{+,l})(\tilde{\eta}_l \# \tilde{W}_{+,l}), \quad (6.4.89)$$

the mappings $\tilde{\gamma}_{\mathcal{R}_l}^* \Psi_l$ and $\Psi_l \tilde{W}_{+,l}$ only differ by a λ-dependent matrix $\rho_l = \rho_l(\lambda)$:

$$\tilde{\gamma}_{\mathcal{R}_l}^* \Psi_l = \rho_l \Psi_l \tilde{W}_{+,l}. \quad (6.4.90)$$

Consider the case $l=0$. Applying the relation $\tilde{\gamma}_{\mathcal{R}_0}\circ\tilde{\gamma}_1=\tilde{\gamma}_\infty\circ\tilde{\gamma}_{\mathcal{R}_0}$ from (6.3.3), involving the covering transformations $\tilde{\gamma}_1$ and $\tilde{\gamma}_\infty$ on $\tilde{M}$ as given in section 3.3, we compute

$$\begin{aligned}\rho_0(\lambda)\hat{M}_1(\lambda)\Psi_0(z,\lambda)\tilde{W}_{+,0}(\tilde{\gamma}_1(z),\lambda)&=\rho_0(\lambda)\Psi_0(\tilde{\gamma}_1(z),\lambda)\tilde{W}_{+,0}(\tilde{\gamma}_1(z),\lambda)=\tilde{\gamma_{\mathcal{R}_0}}^*\Psi_0(\tilde{\gamma}_1(z),\lambda)\\&=\Psi_0((\tilde{\gamma}_{\mathcal{R}_0}\circ\tilde{\gamma}_1)(z),\lambda)=\Psi_0((\tilde{\gamma}_\infty\circ\tilde{\gamma}_{\mathcal{R}_0})(z),\lambda)=\hat{M}_\infty(\lambda)\Psi_0(\tilde{\gamma}_{\mathcal{R}_0}(z),\lambda)\\&=\hat{M}_\infty(\lambda)(\tilde{\gamma}^*_{\mathcal{R}_0}\Psi_0(z,\lambda))=\hat{M}_\infty(\lambda)\rho_0(\lambda)\Psi_0(z,\lambda)\tilde{W}_{+,0}(z,\lambda).\end{aligned}\tag{6.4.91}$$

As $\tilde{W}_{+,0}$ defines the pullback of the mapping $W_{+,0}$, which is holomorphic on M (with respect to z), $\tilde{W}_{+,0}$ is holomorphic on $\tilde{M}$ and therefore does not pick up any monodromy under $\tilde{\gamma}_1$, i.e. $\tilde{W}_{+,0}(\tilde{\gamma}_1(z),\lambda)=\tilde{W}_{+,0}(z,\lambda)$. Thus, we conclude that

$$\rho_0(\lambda)\hat{M}_1(\lambda)=\hat{M}_\infty(\lambda)\rho_0(\lambda).\tag{6.4.92}$$

Analogously, applying $\tilde{\gamma}_{\mathcal{R}_0}\circ\tilde{\gamma}_\infty=\tilde{\gamma}_1\circ\tilde{\gamma}_{\mathcal{R}_0}$ from (6.3.3), we have

$$\begin{aligned}\rho_0(\lambda)\hat{M}_\infty(\lambda)\Psi_0(z,\lambda)\tilde{W}_{+,0}(\tilde{\gamma}_\infty(z),\lambda)&=\rho_0(\lambda)\Psi_0(\tilde{\gamma}_\infty(z),\lambda)\tilde{W}_{+,0}(\tilde{\gamma}_\infty(z),\lambda)=\tilde{\gamma_{\mathcal{R}_0}}^*\Psi_0(\tilde{\gamma}_\infty(z),\lambda)\\&=\Psi_0((\tilde{\gamma}_{\mathcal{R}_0}\circ\tilde{\gamma}_\infty)(z),\lambda)=\Psi_0((\tilde{\gamma}_1\circ\tilde{\gamma}_{\mathcal{R}_0})(z),\lambda)=\hat{M}_1(\lambda)\Psi_0(\tilde{\gamma}_{\mathcal{R}_0}(z),\lambda)\\&=\hat{M}_1(\lambda)(\tilde{\gamma}^*_{\mathcal{R}_0}\Psi_0(z,\lambda))=\hat{M}_1(\lambda)\rho_0(\lambda)\Psi_0(z,\lambda)\tilde{W}_{+,0}(z,\lambda).\end{aligned}\tag{6.4.93}$$

Using the holomorphicity of $\tilde{W}_{+,0}$ on $\tilde{M}$, we know that $\tilde{W}_{+,0}(\tilde{\gamma}_1(z),\lambda)=\tilde{W}_{+,0}(z,\lambda)$, which yields

$$\rho_0(\lambda)\hat{M}_\infty(\lambda)=\hat{M}_1(\lambda)\rho_0(\lambda).\tag{6.4.94}$$

We set

$$\rho_0(\lambda)=\begin{pmatrix}a_0(\lambda)&b_0(\lambda)\\c_0(\lambda)&d_0(\lambda)\end{pmatrix},\tag{6.4.95}$$

where a_0, b_0, c_0 and d_0 define complex valued functions of λ satisfying $a_0(\lambda)d_0(\lambda)-b_0(\lambda)c_0(\lambda)=1$. Comparing the upper left entries of $\rho_0(\lambda)\hat{M}_1(\lambda)$ and $\hat{M}_\infty(\lambda)\rho_0(\lambda)$, we obtain

$$b_0(\lambda)\zeta_1=-c_0(\lambda)\overline{\zeta_1}.\tag{6.4.96}$$

Then, by comparing the upper right entries of $\rho_0(\lambda)\hat{M}_1(\lambda)$ and $\hat{M}_\infty(\lambda)\rho_0(\lambda)$ (resp. of $\rho_0(\lambda)\hat{M}_\infty(\lambda)$ and $\hat{M}_1(\lambda)\rho_0(\lambda)$), we infer that

$$a_0(\lambda)\overline{\zeta_1}-b_0(\lambda)\alpha\cos(2\pi\mu_1)=b_0(\lambda)\alpha\cos(2\pi\mu_1)-d_0(\lambda)\overline{\zeta_1},\tag{6.4.97}$$
$$-a_0(\lambda)\overline{\zeta_1}-b_0(\lambda)\alpha\cos(2\pi\mu_1)=b_0(\lambda)\alpha\cos(2\pi\mu_1)+d_0(\lambda)\overline{\zeta_1},\tag{6.4.98}$$

which (by summing up these two equations) directly implies $b_0(\lambda)=0$ and thus, by (6.4.96), $c_0(\lambda)=0$. inserting this into (6.4.97), we infer that $a_0(\lambda)=-d_0(\lambda)$, which together with the relation $a_0(\lambda)d_0(\lambda)-b_0(\lambda)c_0(\lambda)=1$ implies $a_0(\lambda)=-d_0(\lambda)=\pm i$. Therefore, we have

$$\rho_0(\lambda)=\pm\begin{pmatrix}i&0\\0&-i\end{pmatrix},\tag{6.4.99}$$

in particular $\rho_0(\lambda)\in\Lambda\mathrm{SU}(2)_\sigma$.

Carrying out exactly the same computations as above for the cases $j=1$ and $j=\infty$ (only shifting indices appropriately), we obtain

$$\rho_1(\lambda)=\pm\begin{pmatrix}i&0\\0&-i\end{pmatrix},\tag{6.4.100}$$
$$\rho_\infty(\lambda)=\pm\begin{pmatrix}i&0\\0&-i\end{pmatrix}.\tag{6.4.101}$$

Consequently we have for all $l\in\{0,1,\infty\}$ in particular $\rho_l(\lambda)\in\Lambda\mathrm{SU}(2)_\sigma$. Thus, $(\rho_lF\rho_l^{-1})(\rho_lB_{+,l}\tilde{W}_{+,l})$ defines an Iwasawa-decomposition of $\rho_l\Psi_l\tilde{W}_{+,l}$ (pointwise for all $z\in\tilde{M}$) with $\rho_lF\rho_l^{-1}\in\Lambda\mathrm{SU}(2)_\sigma$, $\rho B_{+,l}\tilde{W}_{+,l}\in\Lambda^+\mathrm{SL}(2,\mathbb{C})_\sigma$ and $(\rho_lF\rho_l^{-1})(z_{*,l})=\mathrm{I}$. Therefore, we can write

$$(F\circ\tilde{\gamma}_{\mathcal{R}_l})(B_{+,l}\circ\tilde{\gamma}_{\mathcal{R}_l})=\tilde{\gamma}^*_{\mathcal{R}_l}\Psi_l=\rho_l\Psi_l\tilde{W}_{+,l}=(\rho_lF\rho_l^{-1})(\rho_lB_{+,l}\tilde{W}_{+,l}).\tag{6.4.102}$$

This implies that, using the loop group method, $\tilde{\gamma}_{\mathcal{R}_l}^* \Psi_l$ produces on the one hand the trinoid $J(\psi \circ \tilde{\gamma}_{\mathcal{R}_l}) = \mathrm{SymBob}(F \circ \gamma_{\mathcal{R}_l})|_{\lambda=1}$ and on the other hand the rotated trinoid $\rho_l J(\psi) \rho_l^{-1} = \mathrm{SymBob}(\rho_l F \rho_l^{-1})|_{\lambda=1}$. Consequently, these two surfaces coincide, i.e.

$$J(\psi \circ \tilde{y}_{\mathcal{R}_l})(\tilde{M}) = (\rho_l J(\psi) \rho_l^{-1})(\tilde{M}). \tag{6.4.103}$$

Using the identity $\rho_l J(\psi) \rho_l^{-1} = J \circ \mathcal{A}_{\mathcal{R}_l} \circ \psi$, where

$$\mathcal{A}_{\mathcal{R}_l} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{6.4.104}$$

from the proof of corollary 6.8, we switch into the $\mathbb{R}^3$ model and obtain $\psi \circ \tilde{\gamma}_{\mathcal{R}_l} = \mathcal{A}_{\mathcal{R}_l} \circ \psi$. As $\tilde{\gamma}_{\mathcal{R}_l}(\tilde{M}) = \tilde{M}$, this yields

$$\psi(\tilde{M}) = \mathcal{A}_{\mathcal{R}_l}(\psi(\tilde{M})). \tag{6.4.105}$$

This means that ψ (and thus also ϕ) is symmetric with respect to the Euclidean motion $\mathcal{A}_{\mathcal{R}_l} \in \mathrm{Iso}(\mathbb{R}^3)$ defining the rotation by the angle π around the z-axis in $\mathbb{R}^3$. Thus, ϕ is necessarily rotationally symmetric with respect to the trinoid axis A_l. (In view of theorem 4.31, which lists all possible trinoid symmetries, only the rotation by the angle π around the trinoid axis A_l shows the behaviour of $\mathcal{A}_{\mathcal{R}_l}$. In particular, $\mathcal{A}_{\mathcal{R}_l}$ is associated with the biholomorphic mapping $\tilde{\gamma}_{\mathcal{R}_l} : \tilde{M} \to \tilde{M}$ keeping the trinoid end corresponding to the singularity z_l fixed. Thus, we infer that the z-axis in $\mathbb{R}^3$ coincides with the trinoid axis A_l, and that ϕ is rotationally symmetric with respect to the trinoid axis A_l, coinciding with the z-axis in $\mathbb{R}^3$.) □

7 Reflectional symmetry with respect to the trinoid plane

7.1 Definition

In this section we turn to trinoids with properly embedded annular ends with another symmetry property, namely trinoids with properly embedded annular ends which are symmetric with respect to the (orientation reversing) reflection $\mathcal{S}$ in some trinoid plane E. In particular, $\mathcal{S}$ "permutes" the trinoid ends B_0, B_1 and B_∞ according to the permutation ( ). Recall that, though there exist a priori possibly several trinoid planes of ϕ, E is uniquely determined by the symmetry $\mathcal{S}$. Throughout this section, we will - by a slight abuse of notation - speak of the trinoid plane E, which is the plane of reflection of the trinoid symmetry $\mathcal{S}$, simply as of *the* trinoid plane of ϕ.

Definition 7.1. Let $M = \mathbb{C} \setminus \{0, 1\}$ and $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends. Let $\tilde{M} = \mathbb{H}$ and $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$ the conformal CMC-immersion associated with ϕ via the universal covering $\pi : \tilde{M} \to M$ given in (3.2.2). Then, if ϕ (or, equivalently, ψ) is symmetric with respect to the reflection $\mathcal{S}$ in the trinoid plane E,

$$\mathcal{S}(\phi(M)) = \phi(M), \tag{7.1.1}$$

ϕ (or ψ) is called *reflectionally symmetric with respect to the trinoid plane.*

Like in the case of trinoids with other symmetries, we are interested in translating the symmetry property (7.1.1) into further constraints on the monodromy matrices associated with the extended frame F of ψ.

7.2 Implications of reflectional symmetry with respect to the trinoid plane

Let $M = \mathbb{C} \setminus \{0, 1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ given in (3.2.2). Let ϕ (or, equivalently, ψ) be reflectionally symmetric with respect to the trinoid plane E and let $\mathcal{S}$ denote the corresponding symmetry. Since $\mathcal{S}$ reverses orientation on $\mathbb{R}^3$, we obtain by theorem 4.9 a pair of bi-antiholomorphic mappings, $\gamma_{\mathcal{S}} : M \to M$ and $\tilde{\gamma}_{\mathcal{S}} : \tilde{M} \to \tilde{M}$ satisfying

$$\mathcal{S} \circ \phi = \phi \circ \gamma_{\mathcal{S}}, \tag{7.2.1}$$

$$\mathcal{S} \circ \psi = \psi \circ \tilde{\gamma}_{\mathcal{S}}, \tag{7.2.2}$$

$$\pi \circ \tilde{\gamma}_{\mathcal{R}_l} = \gamma_{\mathcal{S}} \circ \pi. \tag{7.2.3}$$

The mapping $\gamma_{\mathcal{S}}$ can be explicitly computed, as done in lemma 4.21:

$$\gamma_{\mathcal{S}}(z) = \bar{z}. \tag{7.2.4}$$

The mapping $\tilde{\gamma}_{\mathcal{S}}$ is uniquely determined up to composition from the left with an element of the automorphism group $\mathrm{Aut}(\tilde{M}/M)$ of π. The following lemma explicitly states a valid choice for $\tilde{\gamma}_{\mathcal{S}}$:

Lemma 7.2. *Let $M = \mathbb{C} \setminus \{0, 1\}$, $\tilde{M} = \mathbb{H}$ and $\pi : \tilde{M} \to M$ be the universal covering as given in* (3.2.2)*. Let $\gamma_{\mathcal{S}} : M \to M$ be given by* (7.2.4)*. Then, the mapping*

$$\tilde{\gamma}_{\mathcal{S}} : \tilde{M} \to \tilde{M}, \tilde{\gamma}_{\mathcal{S}}(z) = -\bar{z} \tag{7.2.5}$$

is bi-antiholomorphic and satisfies

$$\pi \circ \tilde{\gamma}_{\mathcal{S}} = \gamma_{\mathcal{S}} \circ \pi, \tag{7.2.6}$$

$$\mathcal{S} \circ \psi = \psi \circ \tilde{\gamma}_{\mathcal{S}}. \tag{7.2.7}$$

Proof. Define $\tilde{\gamma}_{\mathcal{S}}$ as in (7.2.5). Obviously, $\tilde{\gamma}_{\mathcal{S}}$ is a bi-antiholomorphic function. By applying the relations (3.2.12) of lemma 3.4, we obtain for all $z \in \tilde{M}$

$$\pi \circ \tilde{\gamma}_{\mathcal{S}}(z) = \pi(-\bar{z}) = \overline{\pi(z)} = \gamma_{\mathcal{S}} \circ \pi(z). \tag{7.2.8}$$

and, consequently,

$$\mathcal{S} \circ \psi = \mathcal{S} \circ \phi \circ \pi = \phi \circ \gamma_{\mathcal{S}} \circ \pi = \phi \circ \pi \circ \tilde{\gamma}_{\mathcal{S}} = \psi \circ \tilde{\gamma}_{\mathcal{S}}. \tag{7.2.9}$$

□

By the above lemma, we have explicitly determined a mapping $\tilde{\gamma}_{\mathcal{S}}$ corresponding to the trinoid symmetry $\mathcal{S}$ in the sense of theorem 4.9. Thus, we can apply theorem 4.17 to obtain

Theorem 7.3. *Let $M = \mathbb{C} \setminus \{0, 1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ as defined in (3.2.2). Let ϕ be reflectionally symmetric with respect to the trinoid plane E. Denote the corresponding symmetry by $\mathcal{S}$ and by $\tilde{\gamma}_{\mathcal{S}}$ the bi-antiholomorphic mapping $\tilde{M} \to \tilde{M}$ associated with $\mathcal{S}$ as in theorem 4.9 and explicitly defined in lemma 7.2, $\tilde{\gamma}_{\mathcal{S}}(z) = -\bar{z}$. Then the extended frame $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ corresponding to ψ by theorem 4.5 transforms under $\tilde{\gamma}_{\mathcal{S}}(z)$ as*

$$F(\tilde{\gamma}_{\mathcal{S}}(z), \lambda^{-1}) = M_{\mathcal{S}}(\lambda)\overline{F(z, \lambda)}k_{\mathcal{S},\tilde{\gamma}_{\mathcal{S}}}(z), \tag{7.2.10}$$

where

$$k_{\mathcal{S},\tilde{\gamma}_{\mathcal{S}}}(z) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{7.2.11}$$

and $M_{\mathcal{S}}(\lambda)$ denotes an element of $\Lambda\mathrm{SU}(2)_\sigma$, which is independent of z.

Proof. We proceed as in the proof of theorem 5.6. As $\mathcal{S} \in \mathrm{O}(3) \setminus \mathrm{SO}(3)$ reverses orientation on $\mathbb{R}^3$, we apply the second part of theorem 4.17 to obtain

$$F(\tilde{\gamma}_{\mathcal{S}}(z), \lambda^{-1}) = M_{\mathcal{S}}(\lambda)\overline{F(z, \lambda)}k_{\mathcal{S},\tilde{\gamma}_{\mathcal{S}}}(z), \tag{7.2.12}$$

where $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ denotes the extended frame corresponding to ψ by theorem 4.5 and $M_{\mathcal{S}} := M_{\tilde{\gamma}_{\mathcal{S}}}(\lambda)$ denotes an element of $\Lambda\mathrm{SU}(2)_\sigma$, which is independent of z. $k_{\mathcal{S},\tilde{\gamma}_{\mathcal{S}}}(z)$ is given by equation (4.4.118) from lemma 4.18. By computing

$$\partial_{\bar{z}}\tilde{\gamma}(z) = -1 \tag{7.2.13}$$

we infer that

$$\frac{\partial_{\bar{z}}\tilde{\gamma}(z)}{\mid \partial_{\bar{z}}\tilde{\gamma}(z) \mid} = -1 \tag{7.2.14}$$

and thus obtain from (4.4.118) (in view of our definition of the complex square root on the z-plane given in remark 4.14)

$$k_{\mathcal{S},\tilde{\gamma}_{\mathcal{S}}}(z) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \tag{7.2.15}$$

□

7.3 Monodromy matrices of trinoids with properly embedded annular ends, which are reflectionally symmetric with respect to the trinoid plane

We now study the (unitary) monodromy matrices $\hat{M}_0, \hat{M}_1, \hat{M}_\infty$ associated with a trinoid with properly embedded annular ends with reflectional symmetry in the trinoid plane E. Our considerations are based on the relations between the bi-antiholomorphic mapping $\tilde{\gamma}_{\mathcal{S}}$ associated with the symmetry $\mathcal{S}$ and the covering transformations $\tilde{\gamma}_j$ on $\tilde{M}$ generating the monodromy matrices $\hat{M}_j$. Recall the latter ones from section 3.3:

$$\tilde{\gamma}_0(z) = \frac{z}{-2z+1}, \quad \tilde{\gamma}_1(z) = \frac{z-2}{2z-3}, \quad \tilde{\gamma}_\infty(z) = z + 2. \tag{7.3.1}$$

The corresponding inverse functions are given by

$$\tilde{\gamma}_0^{-1}(z) = \frac{z}{2z+1}, \quad \tilde{\gamma}_1^{-1}(z) = \frac{-3z+2}{-2z+1}, \quad \tilde{\gamma}_\infty^{-1}(z) = z - 2. \tag{7.3.2}$$

The relations mentioned above are stated in the following lemma.

Lemma 7.4. *Let $\tilde{M} = \mathbb{H}$ and $\tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\gamma}_\infty : \tilde{M} \to \tilde{M}$ be given as above. For $\tilde{\gamma}_{\mathcal{S}} : \tilde{M} \to \tilde{M}, \tilde{\gamma}_{\mathcal{S}}(z) = -\bar{z}$, the following identities hold:*

$$\tilde{\gamma}_{\mathcal{S}} \circ \tilde{\gamma}_0 = \tilde{\gamma}_0^{-1} \circ \tilde{\gamma}_{\mathcal{S}}, \quad \tilde{\gamma}_{\mathcal{S}} \circ \tilde{\gamma}_1 = \tilde{\gamma}_1^{-1} \circ \tilde{\gamma}_{\mathcal{S}}, \quad \tilde{\gamma}_{\mathcal{S}} \circ \tilde{\gamma}_\infty = \tilde{\gamma}_1 \circ \tilde{\gamma}_0 \circ \tilde{\gamma}_{\mathcal{S}}. \tag{7.3.3}$$

Proof. For $z \in \tilde{M}$ we have by direct computation

$$\tilde{\gamma}_{\mathcal{S}} \circ \tilde{\gamma}_0(z) = \frac{-\bar{z}}{-2\bar{z}+1} = \tilde{\gamma}_0^{-1} \circ \tilde{\gamma}_{\mathcal{S}}(z), \tag{7.3.4}$$

$$\tilde{\gamma}_{\mathcal{S}} \circ \tilde{\gamma}_1(z) = -\bar{z} - 2 = \tilde{\gamma}_1^{-1} \circ \tilde{\gamma}_{\mathcal{S}}(z), \tag{7.3.5}$$

and thus $\tilde{\gamma}_{\mathcal{S}} \circ \tilde{\gamma}_0 = \tilde{\gamma}_0^{-1} \circ \tilde{\gamma}_{\mathcal{S}}$ and $\tilde{\gamma}_{\mathcal{S}} \circ \tilde{\gamma}_1 = \tilde{\gamma}_1^{-1} \circ \tilde{\gamma}_{\mathcal{S}}$. Using this and recalling $\tilde{\gamma}_0 \circ \tilde{\gamma}_1 \circ \tilde{\gamma}_\infty = \mathrm{id}$ on $\tilde{M}$, we obtain

$$\tilde{\gamma}_{\mathcal{S}} \circ \tilde{\gamma}_\infty(z) = \tilde{\gamma}_{\mathcal{S}} \circ \tilde{\gamma}_1^{-1} \circ \tilde{\gamma}_0^{-1}(z) = \tilde{\gamma}_1 \circ \tilde{\gamma}_0 \circ \tilde{\gamma}_{\mathcal{S}}(z) \tag{7.3.6}$$

for all $z \in \tilde{M}$, which proves the remaining identity. □

In view of this, we are able to prove the following theorem:

Theorem 7.5. *Let $M = \mathbb{C} \setminus \{0,1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ as defined in* (3.2.2)*. Let ϕ be reflectionally symmetric with respect to the trinoid plane E. Denote the corresponding symmetry by $\mathcal{S}$. Furthermore, let $F : \tilde{M} \to \Lambda \mathrm{SU}(2)_\sigma$ be the extended frame associated with ψ by theorem 4.5. Denote by $\hat{M}_0, \hat{M}_1, \hat{M}_\infty \in \Lambda \mathrm{SU}(2,\mathbb{C})_\sigma$ the unitary monodromy matrices*

$$\hat{M}_j = -\left[\cos(2\pi\mu_j)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i\sin(2\pi\mu_j)\begin{pmatrix}p_j & \overline{q_j}\\ q_j & -p_j\end{pmatrix}\right] \tag{7.3.7}$$

associated with F as in (4.5.13) *by*

$$F(\tilde{\gamma}_j(z),\lambda) = \alpha_j \hat{M}_j(\lambda) F(z,\lambda) k_j(z), \quad j = 0,1,\infty, \tag{7.3.8}$$

where $\alpha_j \in \{\pm 1\}$ and $\tilde{\gamma}_j$ denote the covering transformations on $\tilde{M}$ from section 3.3. Finally, let $\tilde{\gamma}_{\mathcal{S}}$ be the bi-antiholomorphic mapping $\tilde{M} \to \tilde{M}$ associated with $\mathcal{S}$ as in theorem 4.9 and explicitly given in lemma 7.2, and

$$M_{\mathcal{S}}(\lambda) := \begin{pmatrix} a_{\mathcal{S}} & b_{\mathcal{S}} \\ -\overline{b_{\mathcal{S}}} & \overline{a_{\mathcal{S}}}\end{pmatrix} \tag{7.3.9}$$

the corresponding transformation matrix of F satisfying (7.2.10)*. Then, the monodromy matrices satisfy*

$$M_{\mathcal{S}}(\lambda)\overline{\hat{M}_0(\lambda)} = (\hat{M}_0(\lambda^{-1}))^{-1} M_{\mathcal{S}}(\lambda), \tag{7.3.10}$$

$$M_{\mathcal{S}}(\lambda)\overline{\hat{M}_1(\lambda)} = (\hat{M}_1(\lambda^{-1}))^{-1} M_{\mathcal{S}}(\lambda), \tag{7.3.11}$$

$$M_{\mathcal{S}}(\lambda)\overline{\hat{M}_\infty(\lambda)} = \hat{M}_1(\lambda^{-1})\hat{M}_0(\lambda^{-1}) M_{\mathcal{S}}(\lambda). \tag{7.3.12}$$

In terms of the functions p_j and q_j occurring in $\hat{M}_j$, equations (7.3.10) *to* (7.3.12) *are equivalent to*

$$a_{\mathcal{S}} p_j(\lambda) + b_{\mathcal{S}} \overline{q_j(\lambda)} = a_{\mathcal{S}} p_j(\lambda^{-1}) - \overline{b_{\mathcal{S}}}\,\overline{q_j(\lambda^{-1})}, \tag{7.3.13}$$

$$a_{\mathcal{S}} q_j(\lambda) - b_{\mathcal{S}} p_j(\lambda) = b_{\mathcal{S}} p_j(\lambda^{-1}) + \overline{a_{\mathcal{S}}}\,\overline{q_j(\lambda^{-1})}, \tag{7.3.14}$$

for $j \in \{0,1\}$.

Proof. We start with the following observation, which is implied by (7.3.8):

$$F(z,\lambda) = F(\tilde{\gamma}_j \circ \tilde{\gamma}_j^{-1}(z),\lambda) = \alpha_j \hat{M}_j(\lambda) F(\tilde{\gamma}_j^{-1}(z),\lambda) k_j(\tilde{\gamma}_j^{-1}(z)) \tag{7.3.15}$$

and thus we have

$$F(\tilde{\gamma}_j^{-1}(z),\lambda) = \alpha_j \hat{M}_j(\lambda)^{-1} F(z,\lambda)(k_j(\tilde{\gamma}_j^{-1}(z)))^{-1}. \tag{7.3.16}$$

Additionally, by equation (7.2.10) from theorem 7.3,

$$F(\tilde{\gamma}_{\mathcal{S}}(z),\lambda^{-1}) = M_{\mathcal{S}}(\lambda)\overline{F(z,\lambda)} k_{\mathcal{S},\tilde{\gamma}_{\mathcal{S}}}(z), \tag{7.3.17}$$

where

$$k_{\mathcal{S},\tilde{\gamma}_{\mathcal{S}}}(z) = \begin{pmatrix} -i & 0 \\ 0 & i\end{pmatrix}. \tag{7.3.18}$$

Combining these results with the identities (7.3.3) from the above lemma, we obtain for $j \in \{0, 1\}$:

$$\begin{aligned}
M_{\mathcal{S}}(\lambda)\alpha_j \overline{\hat{M}_j(\lambda)} \overline{F(z,\lambda) k_j(z)} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} &= M_{\mathcal{S}}(\lambda) \overline{F(\tilde{\gamma}_j(z), \lambda)} k_{\mathcal{S}, \tilde{\gamma}_{\mathcal{S}}}(\tilde{\gamma}_j(z)) \\
= F(\tilde{\gamma}_{\mathcal{S}} \circ \tilde{\gamma}_j(z), \lambda^{-1}) = F(\tilde{\gamma}_j^{-1} \circ \tilde{\gamma}_{\mathcal{S}}(z), \lambda^{-1}) &= \alpha_j \hat{M}_j(\lambda^{-1})^{-1} F(\tilde{\gamma}_{\mathcal{S}}(z), \lambda^{-1}) (k_j(\tilde{\gamma}_j^{-1}(\tilde{\gamma}_{\mathcal{S}}(z))))^{-1} \\
&= \alpha_j \hat{M}_j(\lambda^{-1})^{-1} M_{\mathcal{S}}(\lambda) \overline{F(z,\lambda)} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} (k_j(\tilde{\gamma}_j^{-1}(\tilde{\gamma}_{\mathcal{S}}(z))))^{-1}. \quad (7.3.19)
\end{aligned}$$

Computing

$$\begin{aligned}
\overline{k_0(z)} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} &= \begin{pmatrix} -i\sqrt{\frac{1-2\bar{z}}{1-2z}} & 0 \\ 0 & \frac{}{i\sqrt{\frac{1-2\bar{z}}{1-2z}}} \end{pmatrix} \\
&= \begin{pmatrix} -i\sqrt{\frac{1-2\frac{-z}{-2z+1}}{1-2\frac{-\bar{z}}{-2\bar{z}+1}}} & 0 \\ 0 & \frac{}{i\sqrt{\frac{1-2\frac{-z}{-2z+1}}{1-2\frac{-\bar{z}}{-2\bar{z}+1}}}} \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} (k_0(\tilde{\gamma}_0^{-1}(\tilde{\gamma}_{\mathcal{S}}(z))))^{-1} \quad (7.3.20)
\end{aligned}$$

and

$$\overline{k_1(z)} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} (k_1(\tilde{\gamma}_1^{-1}(\tilde{\gamma}_{\mathcal{S}}(z))))^{-1}, \quad (7.3.21)$$

we conclude for $j \in \{0, 1\}$ that

$$M_{\mathcal{S}}(\lambda) \overline{\hat{M}_j(\lambda)} = (\hat{M}_j(\lambda^{-1}))^{-1} M_{\mathcal{S}}(\lambda). \quad (7.3.22)$$

What remains to prove is (7.3.11). We show that this equation is a direct consequence of equations (7.3.10) and (7.3.12), which can be equivalently formulated as

$$M_{\mathcal{S}}(\lambda) \overline{\hat{M}_j(\lambda)}^{-1} = \hat{M}_j(\lambda^{-1}) M_{\mathcal{S}}(\lambda), \quad j = 0, 1. \quad (7.3.23)$$

Together with the identity (3.9.32) we obtain

$$M_{\mathcal{S}}(\lambda) \overline{\hat{M}_\infty(\lambda)} = M_{\mathcal{S}}(\lambda) \overline{\hat{M}_1(\lambda)}^{-1} \overline{\hat{M}_0(\lambda)}^{-1} = \hat{M}_1(\lambda^{-1}) \hat{M}_0(\lambda^{-1}) M_{\mathcal{S}}(\lambda), \quad (7.3.24)$$

as claimed.

As equation (7.3.12) is implied by equations (7.3.10) and (7.3.11), these three equations are equivalent to the scalar reformulations of the equations (7.3.10) and (7.3.11), which are obtained as follows: First recall that the monodromy matrices $\hat{M}_j(\lambda)$ are of the form

$$\hat{M}_j(\lambda) = -\left[\cos(2\pi\mu_j) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin(2\pi\mu_j) \begin{pmatrix} p_j(\lambda) & \overline{q_j(\lambda)} \\ q_j(\lambda) & -p_j(\lambda) \end{pmatrix}\right] \quad (7.3.25)$$

with

$$p_j^2 + q_j \overline{q_j} = 1 \text{ and } p_j = \overline{p_j}, \quad (7.3.26)$$

which implies that

$$(\hat{M}_j(\lambda^{-1}))^{-1} = -\left[\cos(2\pi\mu_j) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \sin(2\pi\mu_j) \begin{pmatrix} p_j(\lambda^{-1}) & \overline{q_j(\lambda^{-1})} \\ q_j(\lambda^{-1}) & -p_j(\lambda^{-1}) \end{pmatrix}\right] \quad (7.3.27)$$

and

$$\overline{\hat{M}_j(\lambda)} = -\left[\cos(2\pi\mu_j) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \sin(2\pi\mu_j) \begin{pmatrix} p_j(\lambda) & q_j(\lambda) \\ \overline{q_j(\lambda)} & -p_j(\lambda) \end{pmatrix}\right]. \quad (7.3.28)$$

The scalar equations associated with (7.3.10) and (7.3.11), respectively, are then (omitting redundant ones) given by ($j \in \{0, 1\}$)

$$-\cos(2\pi\mu_j) a_{\mathcal{S}} + i \sin(2\pi\mu_j)(a_{\mathcal{S}} p_j(\lambda) + b_{\mathcal{S}} \overline{q_j(\lambda)}) = -\cos(2\pi\mu_j) a_{\mathcal{S}} + i \sin(2\pi\mu_j)(a_{\mathcal{S}} p_j(\lambda^{-1}) - \overline{b_{\mathcal{S}}} \overline{q_j(\lambda^{-1})}), \quad (7.3.29)$$

$$-\cos(2\pi\mu_j) b_{\mathcal{S}} + i \sin(2\pi\mu_j)(a_{\mathcal{S}} q_j(\lambda) - b_{\mathcal{S}} p_j(\lambda)) = -\cos(2\pi\mu_j) b_{\mathcal{S}} + i \sin(2\pi\mu_j)(b_{\mathcal{S}} p_j(\lambda^{-1}) + \overline{a_{\mathcal{S}}} \overline{q_j(\lambda^{-1})}). \quad (7.3.30)$$

These equations simplify to ($j \in \{0,1\}$)

$$a_{\mathcal{S}} p_j(\lambda) + b_{\mathcal{S}} \overline{q_j(\lambda)} = a_{\mathcal{S}} p_j(\lambda^{-1}) - \overline{b_{\mathcal{S}}} \overline{q_j(\lambda^{-1})}, \tag{7.3.31}$$
$$a_{\mathcal{S}} q_j(\lambda) - b_{\mathcal{S}} p_j(\lambda) = b_{\mathcal{S}} p_j(\lambda^{-1}) + \overline{a_{\mathcal{S}}} \overline{q_j(\lambda^{-1})}, \tag{7.3.32}$$

which finishes the proof. □

7.4 Normalized trinoids with properly embedded annular ends, which are reflectionally symmetric with respect to the trinoid plane

Let $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends, which is reflectionally symmetric with respect to the trinoid plane. Moreover, let $\psi = \phi \circ \pi$ be the associated conformal CMC-immersion $\tilde{M} \to \mathbb{R}^3$. Denote by $\mathcal{S}$ the corresponding symmetry of ϕ (and ψ), i.e. the reflection in the trinoid plane.

Normalizing the extended frame $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ associated with ψ as in section 4.2, such that $F(z_{**}, \lambda) = \mathrm{I}$ at

$$z_{**} = i, \tag{7.4.1}$$

we can formulate a more explicit version of theorem 7.3 (see below). The normalization $F(z_{**}, \lambda) = \mathrm{I}$ of F is a consequence of normalizing the (conformal) CMC-immersion ψ, such that

$$\psi(z_{**}) = \frac{1}{2H} e_3, \quad \mathcal{U}(z_{**}) = \mathcal{G}(1), \tag{7.4.2}$$

where $\mathcal{U} \in \mathrm{SO}(3)$ represents the natural orthonormal frame corresponding to ψ, and $\mathcal{G}(1)$ is given in (4.2.5). Recall from section 4.2, that this normalization of ψ corresponds to rotating and shifting the (image of the) trinoid in $\mathbb{R}^3$, such that the conditions (7.4.2) are met. It turns out (cf. corollary 7.7), that the choice of z_{**} as above (for a trinoid ϕ with properly embedded annular ends, which is reflectionally symmetric with respect to the trinoid plane) corresponds to arranging the (image of the) trinoid in $\mathbb{R}^3$, such that the trinoid plane is the y-z-plane in $\mathbb{R}^3$.

A trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends, which is reflectionally symmetric with respect to the trinoid plane and, in addition, is "well positioned" in $\mathbb{R}^3$ in the sense that the associated conformal CMC-immersion $\psi : \tilde{M} \to M$ meets the normalization conditions (7.4.2), is called a *normalized* trinoid with properly embedded annular ends, which is reflectionally symmetric with respect to the trinoid plane.

We now formulate a more explicit version of theorem 7.3:

Theorem 7.6. *Let $M = \mathbb{C} \setminus \{0,1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ as defined in* (3.2.2). *Let ϕ be reflectionally symmetric with respect to the trinoid plane. Moreover, let z_{**} be given in* (7.4.1),

$$z_{**} = i, \tag{7.4.3}$$

and assume that ψ has been normalized at z_{**}, such that $\psi(z_{*,l}) = \frac{1}{2H} e_3$ and $F(z_{*,l}, \lambda) = \mathrm{I}$, where $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ denotes the extended frame corresponding to ψ by theorem 4.5. Denote by $\mathcal{S}$ the corresponding symmetry of ϕ and by $\tilde{\gamma}_{\mathcal{S}}$ the bi-antiholomorphic mapping $\tilde{M} \to \tilde{M}$ associated with $\mathcal{S}$ as in theorem 4.9 and explicitly given in (7.2.5)*:*

$$\tilde{\gamma}_{\mathcal{S}}(z) = -\bar{z}. \tag{7.4.4}$$

Then, the extended frame F transforms under $\tilde{\gamma}_{\mathcal{S}}$ as

$$F(\tilde{\gamma}_{\mathcal{S}}(z), \lambda^{-1}) = M_{\mathcal{S}}(\lambda) \overline{F(z,\lambda)} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{7.4.5}$$

where

$$M_{\mathcal{S}}(\lambda) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{7.4.6}$$

In particular, $M_{\mathcal{S}}$ is actually independent of λ.

Proof. In view of theorem 7.3, we only have to prove equation (7.4.6). Note that $\tilde{\gamma}_{\mathcal{S}}(z_*) = -\overline{z_*} = z_*$. Furthermore, $F(z_*, \lambda) = \mathrm{I}$ for all $\lambda \in S^1$. Keeping this in mind, we evaluate equation (7.4.5) at $z = z_*$ to obtain

$$\mathrm{I} = F(z_*, \lambda^{-1}) = F(\tilde{\gamma}_{\mathcal{S}}(z_*), \lambda^{-1}) = M_{\mathcal{S}}(\lambda)\overline{F(z_*, \lambda)}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = M_{\mathcal{S}}(\lambda)\mathrm{I}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{7.4.7}$$

and equation (7.4.6) follows. □

Corollary 7.7. *We retain the notation and the assumptions of theorem 7.6. The reflection plane of the symmetry $\mathcal{S}$ of the normalized trinoid ϕ, i.e. the trinoid plane, is the y-z-plane in $\mathbb{R}^3$.*

Proof. Applying (the second part of) theorem 4.17, we know that the monodromy matrix $M_{\mathcal{S}}(\lambda)$ explicitly given in theorem 7.6 satisfies at $\lambda = 1$

$$M_{\mathcal{S}}(1) = \pm A_{\mathcal{S}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{7.4.8}$$

where $A_{\mathcal{S}} \in \mathrm{SU}(2)$ denotes the conjugation matrix realizing the orthogonal part $\mathcal{A}_{\mathcal{S}}$ of the symmetry $\mathcal{S}$ in the su(2)-model. In view of equation (7.4.6), this yields

$$A_{\mathcal{S}} = \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{7.4.9}$$

Recalling that $\mathcal{A}_{\mathcal{S}}$ and $A_{\mathcal{S}}$ are linked via the Lie Algebra isomorphism $J : \mathbb{R}^3 \to \mathrm{su}(2)$ defined in (3.4.3) as in (3.4.8), i.e.

$$(J \circ \mathcal{A}_{\mathcal{S}} \circ J^{-1})(X) = -A_{\mathcal{S}} X A_{\mathcal{S}}^{-1} \text{ for all } X \in \mathrm{su}(2), \tag{7.4.10}$$

we obtain by a direct computation that

$$\mathcal{A}_{\mathcal{S}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.4.11}$$

Thus, $\mathcal{A}_{\mathcal{S}}$ defines the reflection (in $\mathbb{R}^3$) in the y-z-plane in $\mathbb{R}^3$. Consequently, the symmetry $\mathcal{S}$ of the normalized trinoid ϕ is a reflection in some plane in $\mathbb{R}^3$, which is parallel to the y-z-plane. Since the point $\psi(z_{**}) \in \mathbb{R}^3$ (with z_{**} given in (7.4.1)) satisfies

$$\mathcal{S}(\psi(z_{**})) = \psi(\tilde{\gamma}_{\mathcal{S}}(z_{**})) = \psi(z_{**}), \tag{7.4.12}$$

it lies in the reflection plane of $\mathcal{S}$. Since by assumption we have $\psi(z_{**}) = \frac{1}{2H} e_3$, we infer that the reflection plane of $\mathcal{S}$ (i.e. the trinoid plane) is actually the y-z-plane in $\mathbb{R}^3$. □

Applying the theorems 7.5 and 7.6, we obtain the following result:

Theorem 7.8. *Let $M = \mathbb{C} \setminus \{0, 1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ as defined in (3.2.2). Let ϕ be reflectionally symmetric with respect to the trinoid plane. Moreover, let z_{**} be given in (7.4.1) and assume that ψ has been normalized at z_{**}, such that $\psi(z_{**}) = \frac{1}{2H} e_3$ and $F(z_{**}, \lambda) = \mathrm{I}$, where $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ denotes the extended frame corresponding to ψ by theorem 4.5.*

Then, the unitary monodromy matrices $\hat{M}_j \in \Lambda\mathrm{SU}(2, \mathbb{C})_\sigma$, $j = 0, 1, \infty$, associated with F as in (7.3.8) satisfy equations (7.3.10) to (7.3.12) from theorem 7.5. In terms of the functions p_j and q_j occurring in $\hat{M}_j$, these equations are equivalent to

$$p_j(\lambda) = p_j(\lambda^{-1}), \tag{7.4.13}$$

$$q_j(\lambda) = -\overline{q_j(\lambda^{-1})}, \tag{7.4.14}$$

for $j \in \{0, 1\}$.

Proof. As before, we denote the symmetry of ϕ by $\mathcal{S}$ and by $\tilde{\gamma}_{\mathcal{S}}$ the bi-antiholomorphic mapping $\tilde{M} \to \tilde{M}$ associated with $\mathcal{S}$ as in theorem 4.9 and explicitly given in lemma 7.2. Moreover, let $M_{\mathcal{S}}(\lambda)$ be the corresponding monodromy matrix of F as introduced in equation (7.2.10). Keeping in mind that by theorem 7.6

$$M_{\mathcal{S}}(\lambda) = \begin{pmatrix} a_{\mathcal{S}} & b_{\mathcal{S}} \\ -\overline{b_{\mathcal{S}}} & \overline{a_{\mathcal{S}}} \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{7.4.15}$$

the given identities follow directly from theorem 7.5. □

Theorem 7.8 describes the (unitary) monodromy matrices associated with the extended frame $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ of a trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends, which is reflectionally symmetric with respect to the trinoid plane, and which has been normalized such that $F(z_{**}) = \mathrm{I}$ and $\psi(z_{**}) = \frac{1}{2H} e_3$, where $z_{**} \in \tilde{M}$ is given in (7.4.1) and ψ denotes the conformal CMC-immersion $\tilde{M} \to \mathbb{R}^3$ corresponding to ϕ. It turns out that, in this setting, we can also prove the converse result: A trinoid ϕ with properly embedded annular ends and with extended frame F satisfying $F(z_{**}) = \mathrm{I}$ at $z_{**} \in \tilde{M}$ from (7.4.1) and corresponding monodromy matrices of the form given in theorem 7.8 is necessarily reflectionally symmetric with respect to the trinoid plane. This result is formulated in the following theorem.

Theorem 7.9. *Let η be a (standardized) trinoid potential associated with three off-diagonal Delaunay matrices D_0, D_1, D_∞ with eigenvalues $\pm\mu_0$, $\pm\mu_1$ and $\pm\mu_\infty$, respectively. Denote by $\phi : M \to \mathbb{R}^3$ a trinoid with properly embedded annular ends on $M = \mathbb{C} \setminus \{0, 1\}$ generated by η via the loop group method. Moreover, let $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ be the extended frame associated with the mapping $\psi = \phi \circ \pi$ by theorem 4.5, satisfying $F(z_{**}) = \mathrm{I}$ at $z_{**} \in \tilde{M}$ given in* (7.4.1)*. Assume the unitary monodromy matrices $\hat{M}_j \in \Lambda\mathrm{SU}(2,\mathbb{C})_\sigma$, $j = 0, 1, \infty$, associated with F are of the form*

$$\hat{M}_j = -\cos(2\pi\mu_j)\mathrm{I} - i\sin(2\pi\mu_j) \begin{pmatrix} p_j(\lambda) & \overline{q_j(\lambda)} \\ q_j(\lambda) & -p_j(\lambda) \end{pmatrix}, \tag{7.4.16}$$

with functions p_j and q_j satisfying (3.9.51) *and* (3.9.50) *and, additionally, for $j \in \{0, 1\}$,*

$$p_j(\lambda) = p_j(\lambda^{-1}), \tag{7.4.17}$$

$$q_j(\lambda) = -\overline{q_j(\lambda^{-1})}. \tag{7.4.18}$$

Then, ϕ is reflectionally symmetric with respect to the trinoid plane.

Proof. Consider the standardized trinoid potential (cf. section 3.6)

$$\eta = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q(z,\lambda) & 0 \end{pmatrix} \mathrm{d}z, \tag{7.4.19}$$

where

$$Q(z,\lambda) = \frac{b_0(\lambda)}{z^2} + \frac{b_1(\lambda)}{(z-1)^2} + \frac{c_0(\lambda)}{z} + \frac{c_1(\lambda)}{z-1} = \frac{b_0(\lambda)(z-1)^2 + b_1(\lambda)z^2 - c_0(\lambda)z(z-1)}{z^2(z-1)^2} \tag{7.4.20}$$

and b_0, b_1, b_∞, c_0, c_1 are obtained from

$$b_j(\lambda) = \frac{1}{4} - \mu_j^2 \quad \text{for } j = 0, 1, \infty, \tag{7.4.21}$$

$$b_0(\lambda) + b_1(\lambda) + 0 \cdot c_0(\lambda) + 1 \cdot c_1(\lambda) = b_\infty(\lambda), \tag{7.4.22}$$

$$c_0(\lambda) + c_1(\lambda) = 0. \tag{7.4.23}$$

Then, for the bi-antiholomorphic mapping $\gamma_{\mathcal{S}} : M \to M$, $z \mapsto \gamma_{\mathcal{S}}(z) = \bar{z}$ and the function $h : M \to \mathbb{C}\setminus\{0\}$, $z \mapsto h_0(z) = 1$, we compute for $\lambda \in S^1$

$$Q(\gamma_{\mathcal{S}}(z),\lambda) = \frac{b_0(\lambda)(\bar{z}-1)^2 + b_1(\lambda)\bar{z}^2 - c_0(\lambda)\bar{z}(\bar{z}-1)}{\bar{z}^2(\bar{z}-1)^2} =$$

$$\frac{\overline{b_0(\lambda^{-1})}(\bar{z}-1)^2 + \overline{b_1(\lambda^{-1})}\bar{z}^2 - \overline{c_0(\lambda^{-1})}\bar{z}(\bar{z}-1)}{\bar{z}^2(\bar{z}-1)^2} = (h(z))^4 \overline{Q((z),\lambda^{-1})}, \tag{7.4.24}$$

where we used the fact that for $\lambda \in S^1$ the identities

$$\overline{b_j(\lambda^{-1})} = b_j(\lambda) \quad \text{for } j = 0, 1, \infty \quad \text{and} \tag{7.4.25}$$

$$\overline{c_0(\lambda^{-1})} = c_0(\lambda) \tag{7.4.26}$$

hold. Recalling from lemma 4.21 that $\gamma_{\mathcal{S}}$ corresponds to the permutation $\sigma = (\,)$ of the set $\{0, 1, \infty\}$, we apply lemma 4.25 to infer that η transforms under $\gamma_{\mathcal{S}}$ as

$$\gamma_{\mathcal{S}}^* \eta(z, \lambda) = \overline{\eta(z, \lambda^{-1})} \# W_+, \tag{7.4.27}$$

where

$$W_+ = W_+(z, \lambda) = \begin{pmatrix} h(z) & 0 \\ -\lambda \partial_{\bar{z}} h(z) & (h(z))^{-1} \end{pmatrix}. \tag{7.4.28}$$

Applying the pullback construction with respect to the covering mapping $\pi : \tilde{M} \to M$ to (7.4.27), we obtain

$$\pi^*(\gamma_{\mathcal{S}}^* \eta(z, \lambda)) = \pi^*(\overline{\eta(z, \lambda^{-1})} \# W_+) = \overline{\tilde{\eta}(z, \lambda^{-1})} \# \tilde{W}_+, \tag{7.4.29}$$

where $\tilde{\eta} = \pi^* \eta$ denotes the pullback potential of the trinoid potential η (cf. section 2.3) and $\tilde{W}_+ = W_+ \circ \pi$. Moreover, recall that the bi-antiholomorphic mapping $\tilde{\gamma}_{\mathcal{S}} : \tilde{M} \to \tilde{M}$, $z \mapsto -\bar{z}$, from lemma 7.2 satisfies $\gamma_{\mathcal{S}} \circ \pi = \pi \circ \tilde{\gamma}_{\mathcal{S}}$. Thus, the left hand side of (7.4.29) can be transformed as follows:

$$\begin{aligned}
\pi^*(\gamma_{\mathcal{S}}^* \eta) &= \pi^* \left[\begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q(\gamma_{\mathcal{S}}(z), \lambda) & 0 \end{pmatrix} \mathrm{d}\gamma_{\mathcal{S}}(z) \right] \\
&= \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q((\gamma_{\mathcal{S}} \circ \pi)(z), \lambda) & 0 \end{pmatrix} \mathrm{d}(\gamma_{\mathcal{S}} \circ \pi)(z) = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q((\pi \circ \tilde{\gamma}_{\mathcal{S}})(z), \lambda) & 0 \end{pmatrix} \mathrm{d}(\pi \circ \tilde{\gamma}_{\mathcal{S}})(z) \\
&= \tilde{\gamma}_{\mathcal{S}}^* \left[\begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q(\pi(z), \lambda) & 0 \end{pmatrix} \mathrm{d}\pi(z) \right] = \tilde{\gamma}_{\mathcal{S}}^*(\pi^* \eta) = \tilde{\gamma}_{\mathcal{S}}^* \tilde{\eta}.
\end{aligned} \tag{7.4.30}$$

Altogether, (7.4.29) yields

$$\tilde{\gamma}_{\mathcal{S}}^* \tilde{\eta}(z, \lambda) = \overline{\tilde{\eta}(z, \lambda^{-1})} \# \tilde{W}_+. \tag{7.4.31}$$

Considering the extended frame F associated with the trinoid ϕ, we obtain a solution $\Psi = F B_+$ to the differential equation $\mathrm{d}\Psi = \Psi \tilde{\eta}$. Note that Ψ possesses the same (unitary) monodromy matrices as F at the singularities of the potential η, namely $\hat{M}_0$, $\hat{M}_1$ and $\hat{M}_\infty$.

Naturally, the mapping $\tilde{\gamma}_{\mathcal{S}}^* \Psi = \Psi \circ \tilde{\gamma}_{\mathcal{S}}$ defines a solution to the differential equation $\mathrm{d}(\tilde{\gamma}_{\mathcal{S}}^* \Psi) = (\tilde{\gamma}_{\mathcal{S}}^* \Psi)(\tilde{\gamma}_{\mathcal{S}}^* \tilde{\eta})$, which in view of (7.4.31) reads as

$$\mathrm{d}(\tilde{\gamma}_{\mathcal{S}}^* \Psi(z, \lambda)) = (\tilde{\gamma}_{\mathcal{S}}^* \Psi(z, \lambda))(\overline{\tilde{\eta}(z, \lambda^{-1})} \# \tilde{W}_+). \tag{7.4.32}$$

Since this differential equation is also solved by the mapping $\overline{\Psi(z, \lambda^{-1})} \tilde{W}_+$, i.e.

$$\mathrm{d}(\overline{\Psi(z, \lambda^{-1})} \tilde{W}_+) = (\overline{\Psi(z, \lambda^{-1})} \tilde{W}_+)(\overline{\tilde{\eta}(z, \lambda^{-1})} \# \tilde{W}_+), \tag{7.4.33}$$

the mappings $\tilde{\gamma}_{\mathcal{S}}^* \Psi(z, \lambda)$ and $\overline{\Psi(z, \lambda^{-1})} \tilde{W}_+$ only differ by a λ-dependent matrix $\rho = \rho(\lambda)$:

$$\tilde{\gamma}_{\mathcal{S}}^* \Psi(z, \lambda) = \rho \overline{\Psi(z, \lambda^{-1})} \tilde{W}_+. \tag{7.4.34}$$

Applying (for $j = 0, 1$) the relation $\tilde{\gamma}_{\mathcal{S}} \circ \tilde{\gamma}_j = \tilde{\gamma}_j^{-1} \circ \tilde{\gamma}_{\mathcal{S}}$ from (7.3.3), involving the covering transformations $\tilde{\gamma}_j$, $j = 0, 1$, on $\tilde{M}$ as given in section 3.3, we compute

$$\begin{aligned}
\rho(\lambda) \overline{\hat{M}_j(\lambda^{-1})}\, \overline{\Psi(z, \lambda^{-1})} \tilde{W}_+(\tilde{\gamma}_j(z), \lambda) &= \rho(\lambda) \overline{\Psi(\tilde{\gamma}_j(z), \lambda^{-1})} \tilde{W}_+(\tilde{\gamma}_j(z), \lambda) = \tilde{\gamma}_{\mathcal{S}}{}^* \Psi(\tilde{\gamma}_j(z), \lambda) \\
&= \Psi((\tilde{\gamma}_{\mathcal{S}} \circ \tilde{\gamma}_j)(z), \lambda) = \Psi((\tilde{\gamma}_j^{-1} \circ \tilde{\gamma}_{\mathcal{S}})(z), \lambda) = (\hat{M}_j(\lambda))^{-1} \Psi(\tilde{\gamma}_{\mathcal{S}}(z), \lambda) \\
&= (\hat{M}_j(\lambda))^{-1} (\tilde{\gamma}_{\mathcal{S}}^* \Psi(z, \lambda)) = (\hat{M}_j(\lambda))^{-1} \rho(\lambda) \overline{\Psi(z, \lambda^{-1})} \tilde{W}_+(z, \lambda),
\end{aligned} \tag{7.4.35}$$

where we have made use of the identity

$$\Psi(\tilde{\gamma}_j^{-1}(z), \lambda) = (\hat{M}_j(\lambda))^{-1} \Psi(z, \lambda), \tag{7.4.36}$$

a direct consequence of the relation

$$\Psi(z, \lambda) = \Psi(\tilde{\gamma}_j(\tilde{\gamma}_j^{-1}(z)), \lambda) = \hat{M}_j(\lambda) \Psi(\tilde{\gamma}_j^{-1}(z), \lambda). \tag{7.4.37}$$

As $\tilde{W}_+$ defines the pullback of the mapping W_+, which is antiholomorphic on M (with respect to z), $\tilde{W}_+$ is antiholomorphic on $\tilde{M}$ and therefore does not pick up any monodromy under $\tilde{\gamma}_0$, i.e. $\tilde{W}_+(\tilde{\gamma}_0(z), \lambda) = \tilde{W}_+(z, \lambda)$. Thus, we conclude that

$$\rho(\lambda)\overline{\hat{M}_j(\lambda^{-1})} = (\hat{M}_j(\lambda))^{-1}\rho(\lambda). \tag{7.4.38}$$

Setting

$$\rho(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}, \tag{7.4.39}$$

where a, b, c and d define complex valued functions of λ satisfying $a(\lambda)d(\lambda)-b(\lambda)c(\lambda) = 1$, and comparing the upper left entries (resp. the upper right entries) of $\rho(\lambda)\overline{\hat{M}_j(\lambda^{-1})}$ and $(\hat{M}_j(\lambda))^{-1}\rho(\lambda)$, we obtain

$$a(\lambda)\overline{p_j(\lambda^{-1})} + b(\lambda)\overline{q_j(\lambda^{-1})} = a(\lambda)p_j(\lambda) + c(\lambda)\overline{q_j(\lambda)}, \tag{7.4.40}$$
$$a(\lambda)q_j(\lambda^{-1}) - b(\lambda)\overline{p_j(\lambda^{-1})} = b(\lambda)p_j(\lambda) + d(\lambda)\overline{q_j(\lambda)}. \tag{7.4.41}$$

In view of (3.9.50), (7.4.17) and (7.4.18), these equations simplify into

$$-b(\lambda)q_j(\lambda) = c(\lambda)\overline{q_j(\lambda)}, \tag{7.4.42}$$
$$-(a(\lambda) + d(\lambda))\overline{q_j(\lambda)} = 2b(\lambda)p_j(\lambda). \tag{7.4.43}$$

Since in general (i.e. for all λ in S^1 excluding a finite subset of S^1) $p_j, q_j \neq 0$, we can solve for $c(\lambda)$ and $b(\lambda)$, respectively:

$$c(\lambda) = -b(\lambda)q_j(\lambda)\overline{q_j(\lambda)}^{-1}, \tag{7.4.44}$$
$$b(\lambda) = -\frac{1}{2}(a(\lambda) + d(\lambda))\overline{q_j(\lambda)}(p_j(\lambda))^{-1}. \tag{7.4.45}$$

This yields (using (3.9.50) again)

$$\begin{aligned} 1 = a(\lambda)d(\lambda) - b(\lambda)c(\lambda) &= a(\lambda)d(\lambda) + \frac{1}{4}(a(\lambda) + d(\lambda))^2 q_j(\lambda)\overline{q_j(\lambda)}(p_j(\lambda))^{-2} \\ = a(\lambda)d(\lambda)+\frac{1}{4}(a(\lambda)+d(\lambda))^2(p_j(\lambda))^{-2}-\frac{1}{4}(a(\lambda)+d(\lambda))^2 &= \frac{1}{4}(a(\lambda)+d(\lambda))^2(p_j(\lambda))^{-2}-\frac{1}{4}(a(\lambda)-d(\lambda))^2, \end{aligned} \tag{7.4.46}$$

or, equivalently,

$$(p_j(\lambda))^2(4 + (a(\lambda) - d(\lambda))^2) = (a(\lambda) + d(\lambda))^2. \tag{7.4.47}$$

Assume now that, in general, $4 + (a(\lambda) - d(\lambda))^2 \neq 0$ (i.e. $4 + (a(\lambda) - d(\lambda))^2 = 0$ for at most finitely many $\lambda \in S^1$). We infer that

$$(p_0(\lambda))^2 = (p_1(\lambda))^2 = \frac{(a(\lambda) + d(\lambda))^2}{4 + (a(\lambda) - d(\lambda))^2} \tag{7.4.48}$$

for all but (at most) finitely many $\lambda \in S^1$ and thus

$$p_0(\lambda) = \alpha p_1(\lambda) \tag{7.4.49}$$

for some $\alpha \in \{\pm 1\}$ and all but (at most) finitely many $\lambda \in S^1$. Consequently, by (7.4.45), this implies

$$\overline{q_0(\lambda)} = \alpha\overline{q_1(\lambda)}, \quad q_0(\lambda) = \alpha q_1(\lambda) \tag{7.4.50}$$

and thus

$$p_0(\lambda)p_1(\lambda) + \frac{q_0(\lambda)\overline{q_1(\lambda)} + \overline{q_0(\lambda)}q_1(\lambda)}{2} = \alpha((p_0(\lambda))^2 + q_0(\lambda)\overline{q_0(\lambda)}) = \alpha \tag{7.4.51}$$

for all but (at most) finitely many $\lambda \in S^1$, which clearly is a contradiction to equation (3.9.51). Therefore, we conclude that $4+(a(\lambda)-d(\lambda))^2 = 0$ for all $\lambda \in S^1$ and (by (7.4.47)) $(a(\lambda)+d(\lambda))^2 = 0$ for all $\lambda \in S^1$. Together, these relations yield $a(\lambda) = -d(\lambda) = \pm i$ and (by (7.4.45) and (7.4.44)) $b(\lambda) = c(\lambda) = 0$. Thus,

$$\rho(\lambda) = \pm\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{7.4.52}$$

in particular $\rho(\lambda) \in \Lambda\mathrm{SU}(2)_\sigma \cap \Lambda^+\mathrm{SL}(2,\mathbb{C})_\sigma$.

Consequently, $(\rho\overline{F(z,\lambda^{-1})}\rho^{-1})(\rho\overline{B_+(z,\lambda^{-1})}\tilde{W}_+(z,\lambda))$ defines an Iwasawa-decomposition of

$$\rho\overline{\Psi(z,\lambda^{-1})}\tilde{W}_+(z,\lambda) \tag{7.4.53}$$

(pointwise for all $z \in \tilde{M}$) with $\rho\overline{F(z,\lambda^{-1})}\rho^{-1} \in \Lambda\mathrm{SU}(2)_\sigma$, $\rho\overline{B_+(z,\lambda-1)}\tilde{W}_+(z,\lambda) \in \Lambda^+\mathrm{SL}(2,\mathbb{C})_\sigma$ and $\rho\overline{F(z_{**},\lambda^{-1})}\rho^{-1} = \mathrm{I}$. Therefore, we can write

$$\begin{aligned} F(\tilde{\gamma}_{\mathcal{S}}(z),\lambda)B_+(\tilde{\gamma}_{\mathcal{S}}(z),\lambda) &= \tilde{\gamma}_{\mathcal{S}}^*\Psi(z,\lambda) = \rho(\lambda)\overline{\Psi(z,\lambda^{-1})}\tilde{W}_+(z,\lambda) \\ &= (\rho(\lambda)\overline{F(z,\lambda^{-1})}(\rho(\lambda))^{-1})(\rho(\lambda)\overline{B_+(z,\lambda^{-1})}\tilde{W}_+(z,\lambda)). \end{aligned} \tag{7.4.54}$$

Thus, $\tilde{\gamma}_{\mathcal{S}}^*\Psi$ produces (by the loop group method) on the one hand the trinoid $\mathrm{SymBob}(F(\tilde{\gamma}_{\mathcal{S}}(z),\lambda))|_{\lambda=1}$ and on the other hand the trinoid $\mathrm{SymBob}(\rho(\lambda)\overline{F(z,\lambda^{-1})}(\rho(\lambda))^{-1})|_{\lambda=1}$. Consequently, these two surfaces coincide, and, using the straightforward identities

$$\mathrm{SymBob}(\overline{F(z,\lambda^{-1})})|_{\lambda=1} = -\overline{\mathrm{SymBob}(F(z,\lambda))}|_{\lambda=1} \tag{7.4.55}$$

and

$$\overline{X} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ for all } X \in \mathrm{su}(2), \tag{7.4.56}$$

we compute

$$\begin{aligned} J(\psi \circ \tilde{\gamma}_{\mathcal{S}}) &= \mathrm{SymBob}(F(\tilde{\gamma}_{\mathcal{S}}(z),\lambda))|_{\lambda=1} = \mathrm{SymBob}(\rho(\lambda)\overline{F(z,\lambda^{-1})}(\rho(\lambda))^{-1})|_{\lambda=1} \\ &= \rho(\lambda)\mathrm{SymBob}(\overline{F(z,\lambda^{-1})})|_{\lambda=1}(\rho(\lambda))^{-1} = -\rho(\lambda)\overline{\mathrm{SymBob}(F(z,\lambda))}|_{\lambda=1}(\rho(\lambda))^{-1} = -\rho(\lambda)\overline{J(\psi)}(\rho(\lambda))^{-1} \\ &= -\rho(\lambda)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} J(\psi) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(\rho(\lambda))^{-1} = -\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} J(\psi) \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \end{aligned} \tag{7.4.57}$$

Using the identity

$$-\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} X \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = (J \circ \mathcal{A}_{\mathcal{S}} \circ J^{-1})(X) \text{ for all } X \in \mathrm{su}(2), \tag{7.4.58}$$

where

$$\mathcal{A}_{\mathcal{S}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.4.59}$$

from the proof of corollary 7.7, we switch into the $\mathbb{R}^3$ model and obtain $\psi \circ \tilde{\gamma}_{\mathcal{S}} = \mathcal{A}_{\mathcal{S}} \circ \psi$. As $\tilde{\gamma}_{\mathcal{S}}(\tilde{M}) = \tilde{M}$, this yields

$$\psi(\tilde{M}) = \mathcal{A}_{\mathcal{S}}(\psi(\tilde{M})). \tag{7.4.60}$$

This means that ψ (and thus also ϕ) is symmetric with respect to the Euclidean motion $\mathcal{A}_{\mathcal{S}} \in \mathrm{Iso}(\mathbb{R}^3)$ defining the reflection in the y-z-plane in $\mathbb{R}^3$. Thus, ϕ is necessarily reflectionally symmetric with respect to the trinoid plane. (In view of theorem 4.31, which lists all possible trinoid symmetries, only the reflection in the trinoid plane shows the behaviour of $\mathcal{A}_{\mathcal{S}}$. In particular, $\mathcal{A}_{\mathcal{S}}$ is associated with the bi-antiholomorphic mapping $\tilde{\gamma}_{\mathcal{S}} : \tilde{M} \to \tilde{M}$ keeping all three trinoid ends fixed. Thus, we infer that the y-z-plane in $\mathbb{R}^3$ coincides with the trinoid plane, and that ϕ is reflectionally symmetric with respect to the trinoid plane, coinciding with the y-z-plane in $\mathbb{R}^3$.) □

8 Reflectional symmetry with respect to a trinoid normal plane

8.1 Definition

In this section, we consider the possible trinoid symmetries with respect to the (orientation reversing) reflections $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{S}_\infty$ in $\mathbb{R}^3$ that fix one axis of the trinoid while interchanging the other two, i.e. with respect to the reflections in some trinoid normal planes E_0, E_1 and E_∞ along the trinoid axes A_0, A_1 and A_∞, respectively. More precisely (cf. theorem 4.31), we denote by $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{S}_\infty$ the orientation reversing Euclidean motions in $\mathbb{R}^3$ which permute the trioid ends according to the permutations $(1\ \infty)$, $(0\ \infty)$ and $(0\ 1)$ of the set $\{0, 1, \infty\}$, respectively.

Recall that, though there exist a priori possibly several trinoid normal planes of ϕ along each trinoid axis A_j, the trinoid normal planes E_j we consider are uniquely determined by the respective symmetry $\mathcal{S}_j$. Throughout this section, we will - by a slight abuse of notation - speak of the trinoid normal plane E_j, which is the plane of reflection of the trinoid symmetry $\mathcal{S}_j$, simply as of *the* trinoid normal plane of ϕ (along the trinoid axis A_j).

Definition 8.1. Let $M = \mathbb{C} \setminus \{0, 1\}$ and $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends. Let $\tilde{M} = \mathbb{H}$ and $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$ the conformal CMC-immersion associated with ϕ via the universal covering $\pi : \tilde{M} \to M$ given in (3.2.2). Then, if ϕ (or, equivalently, ψ) is symmetric with respect to the reflection $\mathcal{S}_l$ in the trinoid normal plane E_l, i.e. if

$$\mathcal{S}_l(\phi(M)) = \phi(M), \quad \mathcal{S}_l(\psi(\tilde{M})) = \psi(\tilde{M}), \tag{8.1.1}$$

ϕ is called *reflectionally symmetric with respect to the trinoid normal plane E_l.*

Again, we are interested in translating the symmetry property (8.1.1) into constraints on the monodromy matrices associated with the extended frame F of ψ.

8.2 Implications of reflectional symmetry with respect to a trinoid normal plane

As a direct consequence of definition 8.1, we state the following lemma:

Lemma 8.2. *Let $M = \mathbb{C} \setminus \{0, 1\}$ and $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends produced from a trinoid potential η as in theorem 3.14. Denote by D_0, D_1, D_∞ the corresponding Delaunay matrices with eigenvalues $\pm\mu_0$, $\pm\mu_1$, $\pm\mu_\infty$, respectively, where, for $j \in \{0, 1, \infty\}$, $\mu_j = \sqrt{X_j \overline{X_j}} = \sqrt{\frac{1}{4} + w_j(\lambda - \lambda^{-1})^2}$ and $w_j = s_j t_j$ as in section 3.5. Moreover, denote by B_0, B_1 and B_∞ the trinoid ends and by E_0, E_1 and E_∞ the trinoid normal planes (along the trinoid axes). Then, the following holds:*

1. *If ϕ is reflectionally symmetric with respect to the trinoid normal plane E_0, we have*
$$\mu_1 = \mu_\infty. \tag{8.2.1}$$
2. *If ϕ is reflectionally symmetric with respect to the trinoid normal plane E_1, we have*
$$\mu_0 = \mu_\infty. \tag{8.2.2}$$
3. *If ϕ is reflectionally symmetric with respect to the trinoid normal plane E_∞, we have*
$$\mu_0 = \mu_1. \tag{8.2.3}$$

Proof. We carry out the proof for the first case, i.e. suppose ϕ is reflectionally symmetric with respect to the trinoid normal plane E_0. By theorem 4.31, the corresponding symmetry $\mathcal{S}_0$ preserves the trinoid end B_0, while it maps the trinoid ends B_1 and B_∞ onto each other. This means that the asymptotic Delaunay surfaces associated with the ends at B_1 and B_∞ are mapped onto each other as well. Hence, these Delaunay surfaces only differ by a rigid motion on $\mathbb{R}^3$. In particular, this implies that the corresponding Delaunay matrices D_1 and D_∞ possess the same eigenvalues, i.e. $\mu_1 = \mu_\infty$.

The other two cases are proved analogously. □

Let $M = \mathbb{C} \setminus \{0, 1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ given in (3.2.2). Suppose ϕ (or, equivalently, ψ) is reflectionally symmetric with respect to the trinoid normal plane E_l, and denote the corresponding symmetry by $\mathcal{S}_l$. Since $\mathcal{S}_l$ reverses orientation on $\mathbb{R}^3$, we obtain by theorem 4.9 a pair of biholomorphic mappings, $\gamma_{\mathcal{S}_l} : M \to M$ and $\tilde{\gamma}_{\mathcal{S}_l} : \tilde{M} \to \tilde{M}$ satisfying

$$\mathcal{S}_l \circ \phi = \phi \circ \gamma_{\mathcal{S}_l}, \tag{8.2.4}$$
$$\mathcal{S}_l \circ \psi = \psi \circ \tilde{\gamma}_{\mathcal{S}_l}, \tag{8.2.5}$$
$$\pi \circ \tilde{\gamma}_{\mathcal{S}_l} = \gamma_{\mathcal{S}_l} \circ \pi. \tag{8.2.6}$$

The mappings $\gamma_{\mathcal{S}_l}$, $l = 0, 1, \infty$, are uniquely determined and explicitly given by lemma 4.21:

$$\gamma_{\mathcal{S}_0}(z) = \frac{\bar{z}}{\bar{z} - 1}, \tag{8.2.7}$$
$$\gamma_{\mathcal{S}_1}(z) = \frac{1}{\bar{z}}, \tag{8.2.8}$$
$$\gamma_{\mathcal{S}_\infty}(z) = 1 - \bar{z}. \tag{8.2.9}$$

The mappings $\tilde{\gamma}_{\mathcal{S}_l}$, $l = 0, 1, \infty$, are uniquely determined up to composition from the left with elements of the automorphism group $\mathrm{Aut}(\tilde{M}/M)$ of π. The following lemma explicitly states valid choices for $\tilde{\gamma}_{\mathcal{S}_l}$, $l = 0, 1, \infty$:

Lemma 8.3. *Let $M = \mathbb{C} \setminus \{0, 1\}$, $\tilde{M} = \mathbb{H}$ and $\pi : \tilde{M} \to M$ be the universal covering as given in* (3.2.2)*. Let $\gamma_{\mathcal{S}_l} : M \to M$, $l = 0, 1, \infty$, be given by* (8.2.7), (8.2.8) *and* (8.2.9)*, respectively. Then, the following holds:*

1. *The mapping $\tilde{\gamma}_{\mathcal{S}_0} : \tilde{M} \to \tilde{M}$,*
$$\tilde{\gamma}_{\mathcal{S}_0}(z) = \frac{\bar{z}}{-\bar{z} - 1}, \tag{8.2.10}$$
is bi-antiholomorphic and satisfies
$$\pi \circ \tilde{\gamma}_{\mathcal{S}_0} = \gamma_{\mathcal{S}_0} \circ \pi, \tag{8.2.11}$$
$$\mathcal{S}_0 \circ \psi = \psi \circ \tilde{\gamma}_{\mathcal{S}_0}. \tag{8.2.12}$$

2. *The mapping $\tilde{\gamma}_{\mathcal{S}_1} : \tilde{M} \to \tilde{M}$,*
$$\tilde{\gamma}_{\mathcal{S}_1}(z) = -\bar{z} - 1, \tag{8.2.13}$$
is bi-antiholomorphic and satisfies
$$\pi \circ \tilde{\gamma}_{\mathcal{S}_1} = \gamma_{\mathcal{S}_1} \circ \pi, \tag{8.2.14}$$
$$\mathcal{S}_1 \circ \psi = \psi \circ \tilde{\gamma}_{\mathcal{S}_1}. \tag{8.2.15}$$

3. *The mapping $\tilde{\gamma}_{\mathcal{S}_\infty} : \tilde{M} \to \tilde{M}$,*
$$\tilde{\gamma}_{\mathcal{S}_\infty}(z) = \frac{1}{\bar{z}}, \tag{8.2.16}$$
is bi-antiholomorphic and satisfies
$$\pi \circ \tilde{\gamma}_{\mathcal{S}_\infty} = \gamma_{\mathcal{S}_\infty} \circ \pi, \tag{8.2.17}$$
$$\mathcal{S}_\infty \circ \psi = \psi \circ \tilde{\gamma}_{\mathcal{S}_\infty}. \tag{8.2.18}$$

Proof. Direct computations show that $\tilde{\gamma}_{\mathcal{S}_l}$, $l = 0, 1, \infty$ define bi-antiholomorphic mappings $\tilde{M} \to \tilde{M}$. Moreover, by applying the relations (3.2.10), (3.2.11) and (3.2.12) of lemma 3.4, we obtain for all $z \in \tilde{M}$

$$\pi \circ \tilde{\gamma}_{\mathcal{S}_0}(z) = \pi\left(\frac{\bar{z}}{-\bar{z} - 1}\right) = \overline{\pi\left(\frac{z}{z + 1}\right)} = \overline{\pi\left(1 - \frac{1}{z + 1}\right)} = \frac{1}{\overline{\pi\left(-\frac{1}{z+1}\right)}}$$
$$= \frac{1}{1 - \overline{\pi(z + 1)}} = \frac{1}{1 - \frac{1}{\overline{\pi(z)}}} = \frac{\overline{\pi(z)}}{\overline{\pi(z)} - 1} = \gamma_{\mathcal{S}_0} \circ \pi(z), \tag{8.2.19}$$

$$\pi \circ \tilde{\gamma}_{\mathcal{S}_1}(z) = \pi(-\bar{z}-1) = \overline{\pi(z+1)} = \frac{1}{\overline{\pi(z)}} = \gamma_{\mathcal{S}_1} \circ \pi(z), \tag{8.2.20}$$

$$\pi \circ \tilde{\gamma}_{\mathcal{S}_\infty}(z) = \pi\left(\frac{1}{\bar{z}}\right) = \overline{\pi\left(\frac{-1}{z}\right)} = 1 - \overline{\pi(z)} = \gamma_{\mathcal{S}_\infty} \circ \pi(z), \tag{8.2.21}$$

i.e. $\pi \circ \tilde{\gamma}_{\mathcal{S}_l} = \gamma_{\mathcal{S}_l} \circ \pi$ for $l = 0, 1, \infty$. Consequently,

$$\mathcal{S}_l \circ \psi = \mathcal{S}_l \circ \phi \circ \pi = \phi \circ \gamma_{\mathcal{S}_l} \circ \pi = \phi \circ \pi \circ \tilde{\gamma}_{\mathcal{S}_l} = \psi \circ \tilde{\gamma}_{\mathcal{S}_l}, \tag{8.2.22}$$

i.e. $\mathcal{S}_l \circ \psi = \psi \circ \tilde{\gamma}_{\mathcal{S}_l}$ for $l = 0, 1, \infty$. □

By the above lemma, we have explicitly determined mappings $\tilde{\gamma}_{\mathcal{S}_l}$, $l = 0, 1, \infty$, corresponding to the trinoid symmetries $\mathcal{S}_l$, $l = 0, 1, \infty$, respectively, in the sense of theorem 4.9. Thus, we can apply theorem 4.17 to obtain

Theorem 8.4. *Let $M = \mathbb{C} \setminus \{0, 1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ as defined in (3.2.2). Let ϕ be reflectionally symmetric with respect to the trinoid normal plane E_l. Denote the corresponding symmetry by $\mathcal{S}_l$ and by $\tilde{\gamma}_{\mathcal{S}_l}$ the bi-antiholomorphic mapping $\tilde{M} \to \tilde{M}$ associated with $\mathcal{S}_l$ as in theorem 4.9 and explicitly defined in lemma 8.3. Then, the extended frame $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ corresponding to ψ by theorem 4.5 transforms under $\tilde{\gamma}_{\mathcal{S}_l}$ as*

$$F(\tilde{\gamma}_{\mathcal{S}_l}(z), \lambda^{-1}) = M_{\mathcal{S}_l}(\lambda)\overline{F(z,\lambda)}k_{\mathcal{S}_l, \tilde{\gamma}_{\mathcal{S}_l}}(z), \tag{8.2.23}$$

where $M_{\mathcal{S}_l}(\lambda)$ denotes an element of $\Lambda\mathrm{SU}(2)_\sigma$, which is independent of z, and

$$k_{\mathcal{S}_0, \tilde{\gamma}_{\mathcal{S}_0}}(z) = \begin{pmatrix} \sqrt{\frac{-z-1}{\bar{z}+1}} & 0 \\ 0 & \sqrt{\frac{-z-1}{\bar{z}+1}} \end{pmatrix} \quad \text{in the case } l = 0, \tag{8.2.24}$$

$$k_{\mathcal{S}_1, \tilde{\gamma}_{\mathcal{S}_1}}(z) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad \text{in the case } l = 1, \tag{8.2.25}$$

$$k_{\mathcal{S}_\infty, \tilde{\gamma}_{\mathcal{S}_\infty}}(z) = \begin{pmatrix} \sqrt{\frac{-z}{\bar{z}}} & 0 \\ 0 & \sqrt{\frac{-z}{\bar{z}}} \end{pmatrix} \quad \text{in the case } l = \infty. \tag{8.2.26}$$

Proof. As $\mathcal{S}_l$ reverses orientation, we apply the second part of theorem 4.17 to obtain

$$F(\tilde{\gamma}_{\mathcal{S}_l}(z), \lambda^{-1}) = M_{\mathcal{S}_l}(\lambda)\overline{F(z,\lambda)}k_{\mathcal{S}_l, \gamma_{\tilde{\mathcal{S}}_l}}(z), \tag{8.2.27}$$

where $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ denotes the extended frame corresponding to ψ by theorem 4.5 and $M_{\mathcal{S}_l} := M_{\tilde{\gamma}_{\mathcal{S}_l}}$ denotes an element of $\Lambda\mathrm{SU}(2)_\sigma$, which is independent of z. Moreover, $k_{\mathcal{S}_l, \gamma_{\tilde{\mathcal{S}}_l}}$ is given by equation (4.4.118) from lemma 4.18. Recalling from lemma 8.3 that $\tilde{\gamma}_{\mathcal{S}_0}(z) = \frac{\bar{z}}{-\bar{z}-1}$, $\tilde{\gamma}_{\mathcal{R}_1}(z) = -\bar{z} - 1$ and $\tilde{\gamma}_{\mathcal{R}_\infty}(z) = \frac{1}{\bar{z}}$, we compute

$$\partial_{\bar{z}}\tilde{\gamma}_{\mathcal{S}_0}(z) = \frac{-1}{(\bar{z}+1)^2}, \tag{8.2.28}$$

$$\partial_{\bar{z}}\tilde{\gamma}_{\mathcal{S}_1}(z) = -1, \tag{8.2.29}$$

$$\partial_{\bar{z}}\tilde{\gamma}_{\mathcal{S}_\infty}(z) = \frac{-1}{\bar{z}^2}. \tag{8.2.30}$$

This implies

$$\frac{\partial_{\bar{z}}\tilde{\gamma}_{\mathcal{S}_0}(z)}{\mid \partial_{\bar{z}}\tilde{\gamma}_{\mathcal{S}_0}(z) \mid} = -\frac{\mid \bar{z}+1 \mid^2}{(\bar{z}+1)^2} = \frac{-z-1}{\bar{z}+1}, \tag{8.2.31}$$

$$\frac{\partial_{\bar{z}}\tilde{\gamma}_{\mathcal{S}_1}(z)}{\mid \partial_{\bar{z}}\tilde{\gamma}_{\mathcal{S}_1}(z) \mid} = -1, \tag{8.2.32}$$

$$\frac{\partial_{\bar{z}}\tilde{\gamma}_{\mathcal{S}_\infty}(z)}{\mid \partial_{\bar{z}}\tilde{\gamma}_{\mathcal{S}_\infty}(z) \mid} = -\frac{\mid \bar{z} \mid^2}{\bar{z}^2} = \frac{-z}{\bar{z}}. \tag{8.2.33}$$

and hence we obtain from (4.4.118) the claimed explicit forms for $k_{\mathcal{S}_l}$, $l \in \{0, 1, \infty\}$. □

8.3 Monodromy matrices of trinoids with properly embedded annular ends, which are reflectionally symmetric with respect to a trinoid normal plane

We now study the (unitary) monodromy matrices $\hat{M}_0, \hat{M}_1, \hat{M}_\infty$ associated with a trinoid with properly embedded annular ends and with reflectional symmetry with respect to one of the trinoid normal planes. Our considerations are based on the relations between the bi-antiholomorphic mappings $\tilde{\gamma}_{\mathcal{S}_l}$ associated with the symmetries $\mathcal{S}_l$ and the covering transformations $\tilde{\gamma}_j$ on $\tilde{M}$ generating the monodromy matrices $\hat{M}_j$. Recall the latter ones from section 3.3:

$$\tilde{\gamma}_0(z) = \frac{z}{-2z+1}, \quad \tilde{\gamma}_1(z) = z+2, \quad \tilde{\gamma}_\infty(z) = \frac{-3z-2}{2z+1}. \tag{8.3.1}$$

The corresponding inverse functions are given by

$$\tilde{\gamma}_0^{-1}(z) = \frac{z}{2z+1}, \quad \tilde{\gamma}_1^{-1}(z) = z-2, \quad \tilde{\gamma}_\infty^{-1}(z) = \frac{z+2}{-2z-3}. \tag{8.3.2}$$

The relations mentioned above are stated in the following lemma.

Lemma 8.5. *Let $\tilde{M} = \mathbb{H}$ and $\tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\gamma}_\infty : \tilde{M} \to \tilde{M}$ be given as above.*

1. *For $\tilde{\gamma}_{\mathcal{S}_0} : \tilde{M} \to \tilde{M}, \tilde{\gamma}_{\mathcal{S}_0}(z) = \frac{\bar{z}}{-\bar{z}-1}$, the following identities hold:*

$$\tilde{\gamma}_{\mathcal{S}_0} \circ \tilde{\gamma}_0 = \tilde{\gamma}_0^{-1} \circ \tilde{\gamma}_{\mathcal{S}_0}, \quad \tilde{\gamma}_{\mathcal{S}_0} \circ \tilde{\gamma}_1 = \tilde{\gamma}_\infty^{-1} \circ \tilde{\gamma}_{\mathcal{S}_0}, \quad \tilde{\gamma}_{\mathcal{S}_0} \circ \tilde{\gamma}_\infty = \tilde{\gamma}_1^{-1} \circ \tilde{\gamma}_{\mathcal{S}_0}. \tag{8.3.3}$$

2. *For $\tilde{\gamma}_{\mathcal{S}_1} : \tilde{M} \to \tilde{M}, \tilde{\gamma}_{\mathcal{S}_1}(z) = -\bar{z}-1$, the following identities hold:*

$$\tilde{\gamma}_{\mathcal{S}_1} \circ \tilde{\gamma}_0 = \tilde{\gamma}_\infty^{-1} \circ \tilde{\gamma}_{\mathcal{S}_1}, \quad \tilde{\gamma}_{\mathcal{S}_1} \circ \tilde{\gamma}_1 = \tilde{\gamma}_1^{-1} \circ \tilde{\gamma}_{\mathcal{S}_1}, \quad \tilde{\gamma}_{\mathcal{S}_1} \circ \tilde{\gamma}_\infty = \tilde{\gamma}_0^{-1} \circ \tilde{\gamma}_{\mathcal{S}_1}. \tag{8.3.4}$$

3. *For $\tilde{\gamma}_{\mathcal{S}_\infty} : \tilde{M} \to \tilde{M}, \tilde{\gamma}_{\mathcal{S}_\infty}(z) = \frac{1}{\bar{z}}$, the following identities hold:*

$$\tilde{\gamma}_{\mathcal{S}_\infty} \circ \tilde{\gamma}_0 = \tilde{\gamma}_1^{-1} \circ \tilde{\gamma}_{\mathcal{S}_\infty}, \quad \tilde{\gamma}_{\mathcal{S}_\infty} \circ \tilde{\gamma}_1 = \tilde{\gamma}_0^{-1} \circ \tilde{\gamma}_{\mathcal{S}_\infty}, \quad \tilde{\gamma}_{\mathcal{S}_\infty} \circ \tilde{\gamma}_\infty = \tilde{\gamma}_\infty^{-1} \circ \tilde{\gamma}_{\mathcal{S}_\infty}. \tag{8.3.5}$$

Proof. This is proved by direct computation: Let $z \in \tilde{M}$, then

$$\tilde{\gamma}_{\mathcal{S}_0} \circ \tilde{\gamma}_0(z) = \frac{\frac{\bar{z}}{-2\bar{z}+1}}{-\frac{\bar{z}}{-2\bar{z}+1} - 1} = \frac{\bar{z}}{\bar{z}-1} = \frac{\frac{\bar{z}}{-\bar{z}-1}}{2\frac{\bar{z}}{-\bar{z}-1}+1} = \tilde{\gamma}_0^{-1} \circ \tilde{\gamma}_{\mathcal{S}_0}(z), \tag{8.3.6}$$

$$\tilde{\gamma}_{\mathcal{S}_0} \circ \tilde{\gamma}_1(z) = \frac{\bar{z}+2}{-\bar{z}-2-1} = \frac{\bar{z}+2}{-\bar{z}-3} = \frac{\frac{\bar{z}}{-\bar{z}-1}+2}{-2\frac{\bar{z}}{-\bar{z}-1}-3} = \tilde{\gamma}_\infty^{-1} \circ \tilde{\gamma}_{\mathcal{S}_0}(z), \tag{8.3.7}$$

$$\tilde{\gamma}_{\mathcal{S}_0} \circ \tilde{\gamma}_\infty(z) = \frac{\frac{-3\bar{z}-2}{2\bar{z}+1}}{\frac{3\bar{z}+2}{2\bar{z}+1}-1} = \frac{-3\bar{z}-2}{\bar{z}+1} = \frac{\bar{z}}{-\bar{z}-1} - 2 = \tilde{\gamma}_1^{-1} \circ \tilde{\gamma}_{\mathcal{S}_0}(z), \tag{8.3.8}$$

$$\tilde{\gamma}_{\mathcal{S}_1} \circ \tilde{\gamma}_0(z) = -\frac{\bar{z}}{-2\bar{z}+1} - 1 = \frac{-\bar{z}+2\bar{z}-1}{-2\bar{z}+1} = \frac{-\bar{z}-1+2}{2\bar{z}+2-3} = \tilde{\gamma}_\infty^{-1} \circ \tilde{\gamma}_{\mathcal{S}_1}(z), \tag{8.3.9}$$

$$\tilde{\gamma}_{\mathcal{S}_1} \circ \tilde{\gamma}_1(z) = -(\bar{z}+2) - 1 = -\bar{z}-3 = -\bar{z}-1-2 = \tilde{\gamma}_1^{-1} \circ \tilde{\gamma}_{\mathcal{S}_1}(z), \tag{8.3.10}$$

$$\tilde{\gamma}_{\mathcal{S}_1} \circ \tilde{\gamma}_\infty(z) = \frac{3\bar{z}+2}{2\bar{z}+1} - 1 = \frac{3\bar{z}+2-2\bar{z}-1}{2\bar{z}+1} = \frac{-\bar{z}-1}{-2\bar{z}-2+1} = \tilde{\gamma}_0^{-1} \circ \tilde{\gamma}_{\mathcal{S}_1}(z), \tag{8.3.11}$$

$$\tilde{\gamma}_{\mathcal{S}_\infty} \circ \tilde{\gamma}_0(z) = \frac{-2\bar{z}+1}{\bar{z}} = \frac{1}{\bar{z}} - 2 = \tilde{\gamma}_1^{-1} \circ \tilde{\gamma}_{\mathcal{S}_\infty}(z), \tag{8.3.12}$$

$$\tilde{\gamma}_{\mathcal{S}_\infty} \circ \tilde{\gamma}_1(z) = \frac{1}{\bar{z}+2} = \frac{\frac{1}{\bar{z}}}{2\frac{1}{\bar{z}}+1} = \tilde{\gamma}_0^{-1} \circ \tilde{\gamma}_{\mathcal{S}_\infty}(z), \tag{8.3.13}$$

$$\tilde{\gamma}_{\mathcal{S}_\infty} \circ \tilde{\gamma}_\infty(z) = \frac{2\bar{z}+1}{-3\bar{z}-2} = \frac{\frac{1}{\bar{z}}+2}{-2\frac{1}{\bar{z}}-3} = \tilde{\gamma}_\infty^{-1} \circ \tilde{\gamma}_{\mathcal{S}_\infty}(z). \tag{8.3.14}$$

□

In view of this, we are able to prove the following theorem:

Theorem 8.6. *Let $M = \mathbb{C} \setminus \{0,1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ as defined in* (3.2.2)*. Let ϕ be reflectionally symmetric with respect to the trinoid normal plane E_l. Denote the corresponding symmetry by $\mathcal{S}_l$. Furthermore, let $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ be the extended frame associated with ψ by theorem 4.5. Denote by $\hat{M}_0, \hat{M}_1, \hat{M}_\infty \in \Lambda\mathrm{SU}(2,\mathbb{C})_\sigma$ the unitary monodromy matrices*

$$\hat{M}_j = -\left[\cos(2\pi\mu_j)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\sin(2\pi\mu_j)\begin{pmatrix} p_j & \overline{q_j} \\ q_j & -p_j \end{pmatrix}\right] \tag{8.3.15}$$

associated with F as in (4.5.13) *by*

$$F(\tilde{\gamma}_j(z),\lambda) = \alpha_j \hat{M}_j(\lambda) F(z,\lambda) k_j(z), \quad j = 0, 1, \infty, \tag{8.3.16}$$

where $\alpha_j \in \{\pm 1\}$ and $\tilde{\gamma}_j$ denote the covering transformations on $\tilde{M}$ from section 3.3. Finally, let $\tilde{\gamma}_{\mathcal{S}_l}$, be the bi-antiholomorphic mapping $\tilde{M} \to \tilde{M}$ associated with $\mathcal{S}_l$ as in theorem 4.9 and explicitly given in lemma 8.3, and

$$M_{\mathcal{S}_l}(\lambda) := \begin{pmatrix} a_{\mathcal{S}_l} & b_{\mathcal{S}_l} \\ -\overline{b_{\mathcal{S}_l}} & \overline{a_{\mathcal{S}_l}} \end{pmatrix} \tag{8.3.17}$$

the corresponding monodromy matrix of F satisfying (8.2.23)*.*

1. *In the case $l = 0$, the monodromy matrices satisfy*

$$M_{\mathcal{S}_0}(\lambda)\overline{\hat{M}_0(\lambda)} = (\hat{M}_0(\lambda^{-1}))^{-1} M_{\mathcal{S}_0}(\lambda), \tag{8.3.18}$$
$$M_{\mathcal{S}_0}(\lambda)\overline{\hat{M}_1(\lambda)} = (\hat{M}_\infty(\lambda^{-1}))^{-1} M_{\mathcal{S}_0}(\lambda), \tag{8.3.19}$$
$$M_{\mathcal{S}_0}(\lambda)\overline{\hat{M}_\infty(\lambda)} = (\hat{M}_1(\lambda^{-1}))^{-1} M_{\mathcal{S}_0}(\lambda). \tag{8.3.20}$$

In terms of the functions p_j and q_j occurring in $\hat{M}_j$, equations (8.3.18) *to* (8.3.20) *are equivalent to*

$$a_{\mathcal{S}_0} p_0(\lambda) + b_{\mathcal{S}_0}\overline{q_0(\lambda)} = a_{\mathcal{S}_0} p_0(\lambda^{-1}) - \overline{b_{\mathcal{S}_0} q_0(\lambda^{-1})}, \tag{8.3.21}$$
$$a_{\mathcal{S}_0} q_0(\lambda) - b_{\mathcal{S}_0}\overline{p_0(\lambda)} = b_{\mathcal{S}_0} p_0(\lambda^{-1}) + \overline{a_{\mathcal{S}_0} q_0(\lambda^{-1})}, \tag{8.3.22}$$
$$a_{\mathcal{S}_0} p_1(\lambda) + b_{\mathcal{S}_0}\overline{q_1(\lambda)} = a_{\mathcal{S}_0} p_\infty(\lambda^{-1}) - \overline{b_{\mathcal{S}_0} q_\infty(\lambda^{-1})}, \tag{8.3.23}$$
$$a_{\mathcal{S}_0} q_1(\lambda) - b_{\mathcal{S}_0}\overline{p_1(\lambda)} = b_{\mathcal{S}_0} p_\infty(\lambda^{-1}) + \overline{a_{\mathcal{S}_0} q_\infty(\lambda^{-1})}. \tag{8.3.24}$$

2. *In the case $l = 1$, the monodromy matrices satisfy*

$$M_{\mathcal{S}_1}(\lambda)\overline{\hat{M}_0(\lambda)} = (\hat{M}_\infty(\lambda^{-1}))^{-1} M_{\mathcal{S}_1}(\lambda), \tag{8.3.25}$$
$$M_{\mathcal{S}_1}(\lambda)\overline{\hat{M}_1(\lambda)} = (\hat{M}_1(\lambda^{-1}))^{-1} M_{\mathcal{S}_1}(\lambda), \tag{8.3.26}$$
$$M_{\mathcal{S}_1}(\lambda)\overline{\hat{M}_\infty(\lambda)} = (\hat{M}_0(\lambda^{-1}))^{-1} M_{\mathcal{S}_1}(\lambda). \tag{8.3.27}$$

In terms of the functions p_j and q_j occurring in $\hat{M}_j$, equations (8.3.25) *to* (8.3.27) *are equivalent to*

$$a_{\mathcal{S}_1} p_0(\lambda) + b_{\mathcal{S}_1}\overline{q_0(\lambda)} = a_{\mathcal{S}_1} p_\infty(\lambda^{-1}) - \overline{b_{\mathcal{S}_1} q_\infty(\lambda^{-1})}, \tag{8.3.28}$$
$$a_{\mathcal{S}_1} q_0(\lambda) - b_{\mathcal{S}_1}\overline{p_0(\lambda)} = b_{\mathcal{S}_1} p_\infty(\lambda^{-1}) + \overline{a_{\mathcal{S}_1} q_\infty(\lambda^{-1})}, \tag{8.3.29}$$
$$a_{\mathcal{S}_1} p_1(\lambda) + b_{\mathcal{S}_1}\overline{q_1(\lambda)} = a_{\mathcal{S}_1} p_1(\lambda^{-1}) - \overline{b_{\mathcal{S}_1} q_1(\lambda^{-1})}, \tag{8.3.30}$$
$$a_{\mathcal{S}_1} q_1(\lambda) - b_{\mathcal{S}_1}\overline{p_1(\lambda)} = b_{\mathcal{S}_1} p_1(\lambda^{-1}) + \overline{a_{\mathcal{S}_1} q_1(\lambda^{-1})}. \tag{8.3.31}$$

3. *In the case $l = \infty$, the monodromy matrices satisfy*

$$M_{\mathcal{S}_\infty}(\lambda)\overline{\hat{M}_0(\lambda)} = (\hat{M}_1(\lambda^{-1}))^{-1} M_{\mathcal{S}_\infty}(\lambda), \tag{8.3.32}$$
$$M_{\mathcal{S}_\infty}(\lambda)\overline{\hat{M}_1(\lambda)} = (\hat{M}_0(\lambda^{-1}))^{-1} M_{\mathcal{S}_\infty}(\lambda), \tag{8.3.33}$$
$$M_{\mathcal{S}_\infty}(\lambda)\overline{\hat{M}_\infty(\lambda)} = (\hat{M}_\infty(\lambda^{-1}))^{-1} M_{\mathcal{S}_\infty}(\lambda). \tag{8.3.34}$$

In terms of the functions p_j *and* q_j *occurring in* $\hat{M}_j$*, equations* (8.3.32) *to* (8.3.34) *are equivalent to*

$$a_{\mathcal{S}_\infty} p_0(\lambda) + b_{\mathcal{S}_\infty} \overline{q_0(\lambda)} = a_{\mathcal{S}_\infty} p_1(\lambda^{-1}) - \overline{b_{\mathcal{S}_\infty}}\,\overline{q_1(\lambda^{-1})}, \tag{8.3.35}$$

$$a_{\mathcal{S}_\infty} q_0(\lambda) - b_{\mathcal{S}_\infty} p_0(\lambda) = b_{\mathcal{S}_\infty} p_1(\lambda^{-1}) + \overline{a_{\mathcal{S}_\infty}}\,\overline{q_1(\lambda^{-1})}, \tag{8.3.36}$$

$$a_{\mathcal{S}_\infty} p_\infty(\lambda) + b_{\mathcal{S}_\infty} \overline{q_\infty(\lambda)} = a_{\mathcal{S}_\infty} p_\infty(\lambda^{-1}) - \overline{b_{\mathcal{S}_\infty}}\,\overline{q_\infty(\lambda^{-1})}, \tag{8.3.37}$$

$$a_{\mathcal{S}_\infty} q_\infty(\lambda) - b_{\mathcal{S}_\infty} p_\infty(\lambda) = b_{\mathcal{S}_\infty} p_\infty(\lambda^{-1}) + \overline{a_{\mathcal{S}_\infty}}\,\overline{q_\infty(\lambda^{-1})}. \tag{8.3.38}$$

Proof. Like in the proof of theorem 7.5, we make use of the following fact, a direct consequence of (8.3.16):

$$F(\tilde{\gamma}_j^{-1}(z), \lambda) = \alpha_j \hat{M}_j(\lambda)^{-1} F(z, \lambda)(k_j(\tilde{\gamma}_j^{-1}(z)))^{-1}. \tag{8.3.39}$$

We consider the proof of the first case: $l = 0$. Combining (8.2.23) from theorem 8.4, equations (8.3.16) and (8.3.39) and the identities (8.3.3) from the above lemma, we obtain

$$\begin{aligned}
&M_{\mathcal{S}_0}(\lambda)\alpha_0 \overline{\hat{M}_0(\lambda)}\,\overline{F(z,\lambda)k_0(z)}\,k_{\mathcal{S}_0,\tilde{\gamma}_{\mathcal{S}_0}}(\tilde{\gamma}_0(z)) = M_{\mathcal{S}_0}(\lambda)\overline{F(\tilde{\gamma}_0(z),\lambda)}\,k_{\mathcal{S}_0,\tilde{\gamma}_{\mathcal{S}_0}}(\tilde{\gamma}_0(z)) \\
&= F(\tilde{\gamma}_{\mathcal{S}_0}(\tilde{\gamma}_0(z)), \lambda^{-1}) = F(\tilde{\gamma}_0^{-1}(\tilde{\gamma}_{\mathcal{S}_0}(z)), \lambda^{-1}) = \alpha_0(\hat{M}_0(\lambda^{-1}))^{-1} F(\tilde{\gamma}_{\mathcal{S}_0}(z), \lambda^{-1})(k_0(\tilde{\gamma}_0^{-1}(\tilde{\gamma}_{\mathcal{S}_0}(z))))^{-1} \\
&= \alpha_0(\hat{M}_0(\lambda^{-1}))^{-1} M_{\mathcal{S}_0}(\lambda)\overline{F(z,\lambda)}\,k_{\mathcal{S}_0,\tilde{\gamma}_{\mathcal{S}_0}}(z)(k_0(\tilde{\gamma}_0^{-1}(\tilde{\gamma}_{\mathcal{S}_0}(z))))^{-1},
\end{aligned} \tag{8.3.40}$$

$$\begin{aligned}
&M_{\mathcal{S}_0}(\lambda)\alpha_1 \overline{\hat{M}_1(\lambda)}\,\overline{F(z,\lambda)k_1(z)}\,k_{\mathcal{S}_0,\tilde{\gamma}_{\mathcal{S}_0}}(\tilde{\gamma}_1(z)) = M_{\mathcal{S}_0}(\lambda)\overline{F(\tilde{\gamma}_1(z),\lambda)}\,k_{\mathcal{S}_0,\tilde{\gamma}_{\mathcal{S}_0}}(\tilde{\gamma}_1(z)) \\
&= F(\tilde{\gamma}_{\mathcal{S}_0}(\tilde{\gamma}_1(z)), \lambda^{-1}) = F(\tilde{\gamma}_\infty^{-1}(\tilde{\gamma}_{\mathcal{S}_0}(z)), \lambda^{-1}) = \alpha_\infty(\hat{M}_\infty(\lambda^{-1}))^{-1} F(\tilde{\gamma}_{\mathcal{S}_0}(z), \lambda^{-1})(k_\infty(\tilde{\gamma}_\infty^{-1}(\tilde{\gamma}_{\mathcal{S}_0}(z))))^{-1} \\
&= \alpha_\infty(\hat{M}_\infty(\lambda^{-1}))^{-1} M_{\mathcal{S}_0}(\lambda)\overline{F(z,\lambda)}\,k_{\mathcal{S}_0,\tilde{\gamma}_{\mathcal{S}_0}}(z)(k_\infty(\tilde{\gamma}_\infty^{-1}(\tilde{\gamma}_{\mathcal{S}_0}(z))))^{-1},
\end{aligned} \tag{8.3.41}$$

and

$$\begin{aligned}
&M_{\mathcal{S}_0}(\lambda)\alpha_\infty \overline{\hat{M}_\infty(\lambda)}\,\overline{F(z,\lambda)k_\infty(z)}\,k_{\mathcal{S}_0,\tilde{\gamma}_{\mathcal{S}_0}}(\tilde{\gamma}_\infty(z)) = M_{\mathcal{S}_0}(\lambda)\overline{F(\tilde{\gamma}_\infty(z),\lambda)}\,k_{\mathcal{S}_0,\tilde{\gamma}_{\mathcal{S}_0}}(\tilde{\gamma}_\infty(z)) \\
&= F(\tilde{\gamma}_{\mathcal{S}_0}(\tilde{\gamma}_\infty(z)), \lambda^{-1}) = F(\tilde{\gamma}_1^{-1}(\tilde{\gamma}_{\mathcal{S}_0}(z)), \lambda^{-1}) = \alpha_1(\hat{M}_1(\lambda^{-1}))^{-1} F(\tilde{\gamma}_{\mathcal{S}_0}(z), \lambda^{-1})(k_1(\tilde{\gamma}_1^{-1}(\tilde{\gamma}_{\mathcal{S}_0}(z))))^{-1} \\
&= \alpha_1(\hat{M}_1(\lambda^{-1}))^{-1} M_{\mathcal{S}_0}(\lambda)\overline{F(z,\lambda)}\,k_{\mathcal{S}_0,\tilde{\gamma}_{\mathcal{S}_0}}(z)(k_1(\tilde{\gamma}_1^{-1}(\tilde{\gamma}_{\mathcal{S}_0}(z))))^{-1}.
\end{aligned} \tag{8.3.42}$$

We continue by computing (due to the occurring complex roots up to sign)

$$\begin{aligned}
\overline{k_0(z)}\,k_{\mathcal{S}_0,\tilde{\gamma}_{\mathcal{S}_0}}(\tilde{\gamma}_0(z)) &= \begin{pmatrix} \sqrt{\frac{1-2\bar{z}}{1-2z}} & 0 \\ 0 & \frac{}{\sqrt{\frac{1-2\bar{z}}{1-2z}}} \end{pmatrix} \begin{pmatrix} \overline{\sqrt{\frac{-\frac{z}{-2z+1}-1}{\frac{\bar{z}}{-2\bar{z}+1}+1}}} & 0 \\ 0 & \frac{}{\sqrt{\frac{-\frac{z}{-2z+1}-1}{\frac{\bar{z}}{-2\bar{z}+1}+1}}} \end{pmatrix} \\
&= \pm \begin{pmatrix} \sqrt{\frac{(1-2\bar{z})(-\frac{\bar{z}}{-2\bar{z}+1}-1)}{(1-2z)(\frac{z}{-2z+1}+1)}} & 0 \\ 0 & \frac{}{\sqrt{\frac{(1-2\bar{z})(-\frac{\bar{z}}{-2\bar{z}+1}-1)}{(1-2z)(\frac{z}{-2z+1}+1)}}} \end{pmatrix} = \pm \begin{pmatrix} \sqrt{\frac{\bar{z}-1}{-z+1}} & 0 \\ = & \frac{}{\sqrt{\frac{\bar{z}-1}{-z+1}}} \end{pmatrix} \\
&= \pm \begin{pmatrix} \sqrt{\frac{(-\bar{z}-1)(z-1-2z)(\bar{z}-1)}{(z+1)(2\bar{z}-1-2\bar{z})(z-1)}} & 0 & \\ 0 & \frac{}{\sqrt{\frac{(-\bar{z}-1)(z-1-2z)(\bar{z}-1)}{(z+1)(2\bar{z}-1-2\bar{z})(z-1)}}} & 0 \end{pmatrix} \\
&= \pm \begin{pmatrix} \overline{\sqrt{\frac{-z-1}{\bar{z}+1}}} & 0 \\ 0 & \sqrt{\frac{-z-1}{\bar{z}+1}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1-2\frac{z}{z-1}}{1-2\frac{\bar{z}}{\bar{z}-1}}} & 0 \\ 0 & \frac{}{\sqrt{\frac{1-2\frac{z}{z-1}}{1-2\frac{\bar{z}}{\bar{z}-1}}}} \end{pmatrix} = \pm k_{\mathcal{S}_0,\tilde{\gamma}_{\mathcal{S}_0}}(z)(k_0(\tilde{\gamma}_0^{-1}(\tilde{\gamma}_{\mathcal{S}_0}(z))))^{-1},
\end{aligned} \tag{8.3.43}$$

$$\overline{k_1(z)}k_{\mathcal{S}_0,\tilde{\gamma}_{\mathcal{S}_0}}(\tilde{\gamma}_1(z)) = \begin{pmatrix} \sqrt{\frac{-z-3}{\bar{z}+3}} & 0 \\ 0 & \sqrt{\frac{-z-3}{\bar{z}+3}} \end{pmatrix}$$

$$= \pm \begin{pmatrix} \sqrt{\frac{(-z-1)(\bar{z}+3-2\bar{z}-4)(z+3)}{(z+1)(z+3-2z-4)(\bar{z}+3)}} & 0 \\ 0 & \sqrt{\frac{(-z-1)(\bar{z}+3-2\bar{z}-4)(z+3)}{(z+1)(z+3-2z-4)(\bar{z}+3)}} \end{pmatrix}$$

$$= \pm \begin{pmatrix} \sqrt{\frac{-z-1}{\bar{z}+1}} & 0 \\ 0 & \sqrt{\frac{-z-1}{\bar{z}+1}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1+2\frac{-z-2}{z+3}}{1+2\frac{-\bar{z}-2}{\bar{z}+3}}} & 0 \\ 0 & \frac{1}{\sqrt{\frac{1+2\frac{-z-2}{z+3}}{1+2\frac{-\bar{z}-2}{\bar{z}+3}}}} \end{pmatrix} = \pm k_{\mathcal{S}_0,\tilde{\gamma}_{\mathcal{S}_0}}(z)(k_\infty(\tilde{\gamma}_\infty^{-1}(\tilde{\gamma}_{\mathcal{S}_0}(z))))^{-1} \quad (8.3.44)$$

and

$$\overline{k_\infty(z)}k_{\mathcal{S}_0,\tilde{\gamma}_{\mathcal{S}_0}}(\tilde{\gamma}_\infty(z)) = \begin{pmatrix} \sqrt{\frac{1+2\bar{z}}{1+2z}} & 0 \\ 0 & \sqrt{\frac{1+2\bar{z}}{1+2z}} \end{pmatrix} \begin{pmatrix} \sqrt{-\frac{\frac{-3z-2}{2z+1}-1}{\frac{-3\bar{z}-2}{2\bar{z}+1}+1}} & 0 \\ 0 & \sqrt{-\frac{\frac{-3z-2}{2z+1}-1}{\frac{-3\bar{z}-2}{2\bar{z}+1}+1}} \end{pmatrix}$$

$$= \pm \begin{pmatrix} \sqrt{\frac{(1+2z)(3z+2-2z-1)(2\bar{z}+1)}{(1+2\bar{z})(-3\bar{z}-2+2\bar{z}+1)(2z+1)}} & 0 \\ 0 & \sqrt{\frac{(1+2z)(3z+2-2z-1)(2\bar{z}+1)}{(1+2\bar{z})(-3\bar{z}-2+2\bar{z}+1)(2z+1)}} \end{pmatrix}$$

$$= \pm \begin{pmatrix} \sqrt{\frac{-z-1}{\bar{z}+1}} & 0 \\ 0 & \sqrt{\frac{-z-1}{\bar{z}+1}} \end{pmatrix} = \pm k_{\mathcal{S}_0,\tilde{\gamma}_{\mathcal{S}_0}}(z)(k_1(\tilde{\gamma}_1^{-1}(\tilde{\gamma}_{\mathcal{S}_0}(z))))^{-1}. \quad (8.3.45)$$

Combining these results with the equations above, we obtain

$$M_{\mathcal{S}_0}(\lambda)\overline{\hat{M}_0(\lambda)} = \beta_0(\hat{M}_0(\lambda^{-1}))^{-1}M_{\mathcal{S}_0}(\lambda), \quad (8.3.46)$$

$$M_{\mathcal{S}_0}(\lambda)\overline{\hat{M}_1(\lambda)} = \beta_1\alpha_1\alpha_\infty(\hat{M}_\infty(\lambda^{-1}))^{-1}M_{\mathcal{S}_0}(\lambda), \quad (8.3.47)$$

$$M_{\mathcal{S}_0}(\lambda)\overline{\hat{M}_\infty(\lambda)} = \beta_\infty\alpha_\infty\alpha_1(\hat{M}_1(\lambda^{-1}))^{-1}M_{\mathcal{S}_0}(\lambda). \quad (8.3.48)$$

with $\beta_0, \beta_1, \beta_\infty \in \{\pm 1\}$. This can be reformulated as

$$(\hat{M}_0(\lambda^{-1}))^{-1} = \beta_0 M_{\mathcal{S}_0}(\lambda)\overline{\hat{M}_0(\lambda)}(M_{\mathcal{S}_0}(\lambda))^{-1}, \quad (8.3.49)$$

$$(\hat{M}_\infty(\lambda^{-1}))^{-1} = \beta_1\alpha_1\alpha_\infty M_{\mathcal{S}_0}(\lambda)\overline{\hat{M}_1(\lambda)}(M_{\mathcal{S}_0}(\lambda))^{-1}, \quad (8.3.50)$$

$$(\hat{M}_1(\lambda^{-1}))^{-1} = \beta_\infty\alpha_\infty\alpha_1 M_{\mathcal{S}_0}(\lambda)\overline{\hat{M}_\infty(\lambda)}(M_{\mathcal{S}_0}(\lambda))^{-1}. \quad (8.3.51)$$

Comparing the upper left entries as well as the lower right entries of both sides in each of these equations, we obtain

$$-\cos(2\pi\mu_0) + i\sin(2\pi\mu_0)p_0(\lambda^{-1}) = \\ \beta_0\left[-\cos(2\pi\mu_0) + i\sin(2\pi\mu_0)(a_{\mathcal{S}_0}\overline{a_{\mathcal{S}_0}}p_0(\lambda) + \overline{a_{\mathcal{S}_0}}b_{\mathcal{S}_0}\overline{q_0(\lambda)} + a_{\mathcal{S}_0}\overline{b_{\mathcal{S}_0}}q_0(\lambda) - b_{\mathcal{S}_0}\overline{b_{\mathcal{S}_0}}p_0(\lambda))\right], \quad (8.3.52)$$

$$-\cos(2\pi\mu_0) - i\sin(2\pi\mu_0)p_0(\lambda^{-1}) = \\ \beta_0\left[-\cos(2\pi\mu_0) - i\sin(2\pi\mu_0)(a_{\mathcal{S}_0}\overline{a_{\mathcal{S}_0}}p_0(\lambda) + \overline{a_{\mathcal{S}_0}}b_{\mathcal{S}_0}\overline{q_0(\lambda)} + a_{\mathcal{S}_0}\overline{b_{\mathcal{S}_0}}q_0(\lambda) - b_{\mathcal{S}_0}\overline{b_{\mathcal{S}_0}}p_0(\lambda))\right], \quad (8.3.53)$$

$$-\cos(2\pi\mu_\infty) + i\sin(2\pi\mu_\infty)p_\infty(\lambda^{-1}) = \\ \beta_1\alpha_1\alpha_\infty\left[-\cos(2\pi\mu_1) + i\sin(2\pi\mu_1)(a_{\mathcal{S}_0}\overline{a_{\mathcal{S}_0}}p_1(\lambda) + \overline{a_{\mathcal{S}_0}}b_{\mathcal{S}_0}\overline{q_1(\lambda)} + a_{\mathcal{S}_0}\overline{b_{\mathcal{S}_0}}q_1(\lambda) - b_{\mathcal{S}_0}\overline{b_{\mathcal{S}_0}}p_1(\lambda))\right], \quad (8.3.54)$$

$$-\cos(2\pi\mu_\infty) - i\sin(2\pi\mu_\infty)p_\infty(\lambda^{-1}) = \\ \beta_1\alpha_1\alpha_\infty\left[-\cos(2\pi\mu_1) - i\sin(2\pi\mu_1)(a_{\mathcal{S}_0}\overline{a_{\mathcal{S}_0}}p_1(\lambda) + \overline{a_{\mathcal{S}_0}}b_{\mathcal{S}_0}\overline{q_1(\lambda)} + a_{\mathcal{S}_0}\overline{b_{\mathcal{S}_0}}q_1(\lambda) - b_{\mathcal{S}_0}\overline{b_{\mathcal{S}_0}}p_1(\lambda))\right], \quad (8.3.55)$$

$$-\cos(2\pi\mu_1)+i\sin(2\pi\mu_1)p_1(\lambda^{-1}) =$$
$$\beta_\infty\alpha_\infty\alpha_1\left[-\cos(2\pi\mu_\infty)+i\sin(2\pi\mu_\infty)(a_{\mathcal{S}_0}\overline{a_{\mathcal{S}_0}}p_\infty(\lambda)+\overline{a_{\mathcal{S}_0}}b_{\mathcal{S}_0}\overline{q_\infty(\lambda)}+a_{\mathcal{S}_0}\overline{b_{\mathcal{S}_0}}q_\infty(\lambda)-b_{\mathcal{S}_0}\overline{b_{\mathcal{S}_0}}p_\infty(\lambda))\right], \tag{8.3.56}$$

$$-\cos(2\pi\mu_1)-i\sin(2\pi\mu_1)p_1(\lambda^{-1}) =$$
$$\beta_\infty\alpha_\infty\alpha_1\left[-\cos(2\pi\mu_\infty)-i\sin(2\pi\mu_\infty)(a_{\mathcal{S}_0}\overline{a_{\mathcal{S}_0}}p_\infty(\lambda)+\overline{a_{\mathcal{S}_0}}b_{\mathcal{S}_0}\overline{q_\infty(\lambda)}+a_{\mathcal{S}_0}\overline{b_{\mathcal{S}_0}}q_\infty(\lambda)-b_{\mathcal{S}_0}\overline{b_{\mathcal{S}_0}}p_\infty(\lambda))\right], \tag{8.3.57}$$

respectively. By summing up the first two equations, we conclude that β_0 necessarily equals $+1$. Analogously, by summing up the next two (resp. the last two) equations and recalling that $\mu_\infty = \mu_1$, we deduce $\beta_1\alpha_1\alpha_\infty = +1$ (resp. $\beta_\infty\alpha_\infty\alpha_1 = +1$). Therefore,

$$M_{\mathcal{S}_0}(\lambda)\overline{\hat{M}_0(\lambda)} = (\hat{M}_0(\lambda^{-1}))^{-1}M_{\mathcal{S}_0}(\lambda), \tag{8.3.58}$$
$$M_{\mathcal{S}_0}(\lambda)\overline{\hat{M}_1(\lambda)} = (\hat{M}_\infty(\lambda^{-1}))^{-1}M_{\mathcal{S}_0}(\lambda), \tag{8.3.59}$$
$$M_{\mathcal{S}_0}(\lambda)\overline{\hat{M}_\infty(\lambda)} = (\hat{M}_1(\lambda^{-1}))^{-1}M_{\mathcal{S}_0}(\lambda), \tag{8.3.60}$$

as claimed. Note that in view of (3.9.32) equation (8.3.20) is implied by equations (8.3.18) and (8.3.19). Thus, all three equations are equivalent to the scalar reformulations of the equations (8.3.18) and (8.3.19), which read

$$-\cos(2\pi\mu_0)a_{\mathcal{S}_0}+i\sin(2\pi\mu_0)(a_{\mathcal{S}_0}p_0(\lambda)+b_{\mathcal{S}_0}\overline{q_0(\lambda)})$$
$$=-\cos(2\pi\mu_0)a_{\mathcal{S}_0}+i\sin(2\pi\mu_0)(a_{\mathcal{S}_0}p_0(\lambda^{-1})-\overline{b_{\mathcal{S}_0}}\overline{q_0(\lambda^{-1})}), \tag{8.3.61}$$

$$-\cos(2\pi\mu_0)b_{\mathcal{S}_0}+i\sin(2\pi\mu_0)(a_{\mathcal{S}_0}q_0(\lambda)-b_{\mathcal{S}_0}p_0(\lambda))$$
$$=-\cos(2\pi\mu_0)b_{\mathcal{S}_0}+i\sin(2\pi\mu_0)(b_{\mathcal{S}_0}p_0(\lambda^{-1})+\overline{a_{\mathcal{S}_0}}\overline{q_0(\lambda^{-1})} \tag{8.3.62}$$

and

$$-\cos(2\pi\mu_1)a_{\mathcal{S}_0}+i\sin(2\pi\mu_1)(a_{\mathcal{S}_0}p_1(\lambda)+b_{\mathcal{S}_0}\overline{q_1(\lambda)})$$
$$=-\cos(2\pi\mu_\infty)a_{\mathcal{S}_0}+i\sin(2\pi\mu_\infty)(a_{\mathcal{S}_0}p_\infty(\lambda^{-1})-\overline{b_{\mathcal{S}_0}}\overline{q_\infty(\lambda^{-1})}), \tag{8.3.63}$$

$$-\cos(2\pi\mu_1)b_{\mathcal{S}_0}+i\sin(2\pi\mu_1)(a_{\mathcal{S}_0}q_1(\lambda)-b_{\mathcal{S}_0}p_1(\lambda))$$
$$=-\cos(2\pi\mu_\infty)b_{\mathcal{S}_0}+i\sin(2\pi\mu_\infty)(b_{\mathcal{S}_0}p_\infty(\lambda^{-1})+\overline{a_{\mathcal{S}_0}}\overline{q_\infty(\lambda^{-1})}), \tag{8.3.64}$$

respectively. A straightforward simplification of these equations yields the claimed ones and finishes the proof for $l = 0$.

The claims in the cases $l = 1$ and $l = \infty$ are proved analogously. □

8.4 Normalized trinoids with properly embedded annular ends, which are refletionally symmetric with respect to a trinoid normal plane

Let $l \in \{0, 1, \infty\}$ and $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends, which is reflectionally symmetric with respect to the trinoid normal plane E_l. Moreover, let $\psi = \phi \circ \pi$ be the associated CMC-immersion $\tilde{M} \to \mathbb{R}^3$. Denote by $\mathcal{S}_\lambda$ the corresponding symmetry of ϕ (and ψ), i.e. the reflection in the trinoid normal plane E_l.

We review the results of section 8.3 in the special case that the extended frame $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ associated with ψ as in section 4.2 is "normalized" at $z_* \in \tilde{M}$, which we choose independent of l as follows:

$$z_* = \frac{-1+i\sqrt{3}}{2} \in \tilde{M}. \tag{8.4.1}$$

The "normalization" of F is realized in form of the presumption that

$$F(z_*, \lambda) = \mathrm{I} \tag{8.4.2}$$

for all $\lambda \in S^1$. More precisely (cf. section 4.2), the normalization $F(z_*, \lambda) = \mathrm{I}$ of F is a consequence of normalizing the (conformal) CMC-immersion ψ, such that

$$\psi(z_*) = \frac{1}{2H} e_3, \quad \mathcal{U}(z_*) = \mathcal{G}(1), \tag{8.4.3}$$

where $\mathcal{U} \in \mathrm{SO}(3)$ represents the natural orthonormal frame corresponding to ψ, and $\mathcal{G}(1)$ is given in (4.2.5). Recall from section 4.2, that this normalization of ψ corresponds to rotating and shifting the (image of the) trinoid in $\mathbb{R}^3$, such that the conditions (8.4.3) are met. It turns out (cf. corollary 8.8), that the choice of z_* as above (for a trinoid ϕ with properly embedded annular ends, which is reflectionally symmetric with respect to the trinoid normal plane E_l) corresponds to arranging the (image of the) trinoid in $\mathbb{R}^3$, such that the reflection plane E_l of $\mathcal{S}_l$ contains the z-axis in $\mathbb{R}^3$.

A trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends, which is reflectionally symmetric with respect to the trinoid normal plane E_l and, in addition, is "well positioned" in $\mathbb{R}^3$ in the sense that the associated conformal CMC-immersion $\psi : \tilde{M} \to M$ meets the normalization conditions (8.4.3), is called a *normalized* trinoid with properly embedded annular ends, which is reflectionally symmetric with respect to the trinoid normal plane E_l.

We now formulate a more explicit version of theorem 8.4:

Theorem 8.7. *Let $M = \mathbb{C} \setminus \{0, 1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ as defined in* (3.2.2). *Let $l \in \{0, 1, \infty\}$ and ϕ be reflectionally symmetric with respect to the trinoid normal plane E_l. Moreover, let z_* be given in* (8.4.1)

$$z_* = \frac{-1 + i\sqrt{3}}{2} \in \tilde{M}, \tag{8.4.4}$$

and assume that ψ has been normalized at z_*, such that $\psi(z_*) = \frac{1}{2H} e_3$ and $F(z_*, \lambda) = \mathrm{I}$, where $F : \tilde{M} \to \Lambda \mathrm{SU}(2)_\sigma$ denotes the extended frame corresponding to ψ by theorem 4.5. Denote by $\mathcal{S}_l$ the corresponding symmetry of ϕ and by $\tilde{\gamma}_{\mathcal{S}_l}$ the bi-antiholomorphic mapping $\tilde{M} \to \tilde{M}$ associated with $\mathcal{S}_l$ as in theorem 4.9 and, according to l, explicitly given in (8.2.10), (8.2.13) *or* (8.2.16)*:*

$$\tilde{\gamma}_{\mathcal{S}_0}(z) = \frac{\bar{z}}{-\bar{z} - 1}, \quad \tilde{\gamma}_{\mathcal{S}_1}(z) = -\bar{z} - 1, \quad \tilde{\gamma}_{\mathcal{S}_\infty}(z) = \frac{1}{\bar{z}}. \tag{8.4.5}$$

Then, the extended frame F transforms under $\tilde{\gamma}_{\mathcal{S}_l}$ as

$$F(\tilde{\gamma}_{\mathcal{S}_l}(z), \lambda^{-1}) = M_{\mathcal{S}_l}(\lambda) \overline{F(z, \lambda)} k_{\mathcal{S}_l, \tilde{\gamma_{\mathcal{S}_l}}}(z) \tag{8.4.6}$$

where $k_{\mathcal{S}_l, \tilde{\gamma_{\mathcal{S}_l}}}(z)$ is, according to l, given in (8.2.24), (8.2.25) *or* (8.2.26) *and*

$$M_{\mathcal{S}_0}(\lambda) = \begin{pmatrix} e^{-\frac{\pi i}{6}} & 0 \\ 0 & e^{\frac{\pi i}{6}} \end{pmatrix}, \tag{8.4.7}$$

$$M_{\mathcal{S}_1}(\lambda) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{8.4.8}$$

$$M_{\mathcal{S}_\infty}(\lambda) = \begin{pmatrix} e^{\frac{\pi i}{6}} & 0 \\ 0 & e^{-\frac{\pi i}{6}} \end{pmatrix}. \tag{8.4.9}$$

In particular, the matrices $M_{\mathcal{S}_l}$, $l = 0, 1, \infty$, are actually independent of λ.

Proof. In view of theorem 8.4, we only have to prove the equations (8.4.7), (8.4.8) and (8.4.9). To this end, we compute

$$\tilde{\gamma}_{\mathcal{S}_0}(z_*) = \frac{-1 - i\sqrt{3}}{1 + i\sqrt{3} - 2} = z_*, \tag{8.4.10}$$

$$\tilde{\gamma}_{\mathcal{S}_1}(z_*) = -\frac{-1 - i\sqrt{3}}{2} - 1 = z_*, \tag{8.4.11}$$

$$\tilde{\gamma}_{\mathcal{S}_\infty}(z_*) = \frac{2}{-1 - i\sqrt{3}} = z_*, \tag{8.4.12}$$

which shows that we have for all $l \in \{0, 1, \infty\}$

$$\tilde{\gamma}_{\mathcal{S}_l}(z_*) = z_*. \tag{8.4.13}$$

Furthermore, $F(z_*, \lambda) = \mathrm{I}$. Thus, evaluating equation (8.4.6) at $z = z_*$ yields

$$\mathrm{I} = F(z_*, \lambda^{-1}) = F(\tilde{\gamma}_{\mathcal{S}_l}(z_*), \lambda^{-1}) = M_{\mathcal{S}_l}(\lambda)\overline{F(z_*, \lambda)}k_{\mathcal{S}_l, \tilde{\gamma_{\mathcal{S}_l}}}(z_*), \tag{8.4.14}$$

i.e.

$$M_{\mathcal{S}_l}(\lambda) = \left(k_{\mathcal{S}_l, \tilde{\gamma_{\mathcal{S}_l}}}(z_*)\right)^{-1}. \tag{8.4.15}$$

In view of remark 4.14 (for our definition of the complex square root) and equations (8.2.24), (8.2.25) and (8.2.26), we have

$$k_{\mathcal{S}_0, \tilde{\gamma_{\mathcal{S}_0}}}(z_*) = \begin{pmatrix} \sqrt{\frac{1-i\sqrt{3}-2}{-1-i\sqrt{3}+2}} & 0 \\ 0 & \sqrt{\frac{1-i\sqrt{3}-2}{-1-i\sqrt{3}+2}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2-2i\sqrt{3}}{4}} & 0 \\ 0 & \sqrt{\frac{2-2i\sqrt{3}}{4}} \end{pmatrix} = \begin{pmatrix} e^{\frac{\pi i}{6}} & 0 \\ 0 & e^{-\frac{\pi i}{6}} \end{pmatrix}, \tag{8.4.16}$$

$$k_{\mathcal{S}_1, \tilde{\gamma_{\mathcal{S}_1}}}(z_*) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{8.4.17}$$

$$k_{\mathcal{S}_\infty, \tilde{\gamma_{\mathcal{S}_\infty}}}(z_*) = \begin{pmatrix} \sqrt{\frac{1-i\sqrt{3}}{-1-i\sqrt{3}}} & 0 \\ 0 & \sqrt{\frac{1-i\sqrt{3}}{-1-i\sqrt{3}}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2+2i\sqrt{3}}{4}} & 0 \\ 0 & \sqrt{\frac{2+2i\sqrt{3}}{4}} \end{pmatrix} = \begin{pmatrix} e^{-\frac{\pi i}{6}} & 0 \\ 0 & e^{\frac{\pi i}{6}} \end{pmatrix}. \tag{8.4.18}$$

In view of equation (8.4.15), the claimed identities (8.4.7), (8.4.8) and (8.4.9) follow. □

Corollary 8.8. *We retain the notation and the assumptions of theorem 8.7. The reflection plane of the symmetry $\mathcal{S}_l$ of the normalized trinoid ϕ contains the z-axis in $\mathbb{R}^3$.*

Proof. Applying (the second part of) theorem 4.17, we know that the monodromy $M_{\mathcal{S}_l}(\lambda)$ explicitly given in theorem 8.7 satisfies at $\lambda = 1$

$$M_{\mathcal{S}_l}(1) = \pm A_{\mathcal{S}_l}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{8.4.19}$$

where $A_{\mathcal{S}_l} \in \mathrm{SU}(2)$ denotes the conjugation matrix realizing the orthogonal part $\mathcal{A}_{\mathcal{S}_l}$ of the symmetry $\mathcal{S}_l$ in the su(2)-model. In view of the equations (8.4.7), (8.4.8) and (8.4.9), this yields

$$A_{\mathcal{S}_0} = \pm\begin{pmatrix} 0 & -e^{-\frac{\pi i}{6}} \\ e^{\frac{\pi i}{6}} & 0 \end{pmatrix}, \tag{8.4.20}$$

$$A_{\mathcal{S}_1} = \pm\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \tag{8.4.21}$$

$$A_{\mathcal{S}_\infty} = \pm\begin{pmatrix} 0 & -e^{\frac{\pi i}{6}} \\ e^{-\frac{\pi i}{6}} & 0 \end{pmatrix}. \tag{8.4.22}$$

Recalling that $\mathcal{A}_{\mathcal{S}_l}$ and $A_{\mathcal{S}_l}$ are linked via the Lie Algebra isomorphism $J : \mathbb{R}^3 \to \mathrm{su}(2)$ defined in (3.4.3) as in (3.4.8), i.e.

$$(J \circ \mathcal{A}_{\mathcal{S}_l} \circ J^{-1})(X) = -A_{\mathcal{S}_l} X A_{\mathcal{S}_l}^{-1} \text{ for all } X \in \mathrm{su}(2), \tag{8.4.23}$$

we obtain by a direct computation that

$$\mathcal{A}_{\mathcal{S}_0} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{8.4.24}$$

$$\mathcal{A}_{\mathcal{S}_1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{8.4.25}$$

$$\mathcal{A}_{\mathcal{S}_\infty} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8.4.26}$$

Thus, for all $l \in \{0, 1, \infty\}$, $\mathcal{A}_{\mathcal{S}_l}$ defines a reflection (in $\mathbb{R}^3$), whose reflection plane contains the z-axis in $\mathbb{R}^3$, $\mathbb{R}e_3$. Consequently, for all $l \in \{0, 1, \infty\}$, the symmetry $\mathcal{S}_l$ of the normalized trinoid ϕ is a reflection in some plane in $\mathbb{R}^3$, which is parallel to the z-axis. As the point $\psi(z_*) \in \mathbb{R}^3$ (with z_* given in (8.4.1)) satisfies

$$\mathcal{S}_l(\psi(z_*)) = \psi(\tilde{\gamma}_{\mathcal{S}_l}(z_*)) = \psi(z_*), \tag{8.4.27}$$

it lies in the reflection plane of $\mathcal{S}_l$. Since by assumption we have $\psi(z_*) = \frac{1}{2H} e_3$, we infer that the reflection plane of $\mathcal{S}_l$ actually contains the z-axis in $\mathbb{R}^3$. □

Applying the theorems 8.6 and 8.7, we obtain the following result:

Theorem 8.9. *Let $M = \mathbb{C} \setminus \{0, 1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ as defined in* (3.2.2)*. Let $l \in \{0, 1, \infty\}$ and ϕ be reflectionally symmetric with respect to the trinoid normal plane E_l. Moreover, let z_* be given in* (8.4.1) *and assume that ψ has been normalized at z_*, such that $\psi(z_*) = \frac{1}{2H} e_3$ and $F(z_*, \lambda) = \mathrm{I}$, where $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ denotes the extended frame corresponding to ψ by theorem 4.5.*

1. *In the case $l = 0$, the unitary monodromy matrices $\hat{M}_j \in \Lambda\mathrm{SU}(2, \mathbb{C})_\sigma$, $j = 0, 1, \infty$, associated with F as in* (8.3.16) *satisfy equations* (8.3.18) *to* (8.3.20) *from theorem 8.6. In terms of the functions p_j and q_j occurring in $\hat{M}_j$, these equations are equivalent to*

$$p_0(\lambda) = p_0(\lambda^{-1}), \tag{8.4.28}$$
$$q_0(\lambda) = e^{\frac{\pi i}{3}} \overline{q_0(\lambda^{-1})}, \tag{8.4.29}$$
$$p_1(\lambda) = p_\infty(\lambda^{-1}), \tag{8.4.30}$$
$$q_1(\lambda) = e^{\frac{\pi i}{3}} \overline{q_\infty(\lambda^{-1})}. \tag{8.4.31}$$

2. *In the case $l = 1$, the unitary monodromy matrices $\hat{M}_j \in \Lambda\mathrm{SU}(2, \mathbb{C})_\sigma$, $j = 0, 1, \infty$, associated with F as in* (8.3.16) *satisfy equations* (8.3.25) *to* (8.3.27) *from theorem 8.6. In terms of the functions p_j and q_j occurring in $\hat{M}_j$, these equations are equivalent to*

$$p_0(\lambda) = p_\infty(\lambda^{-1}), \tag{8.4.32}$$
$$q_0(\lambda) = -\overline{q_\infty(\lambda^{-1})}, \tag{8.4.33}$$
$$p_1(\lambda) = p_1(\lambda^{-1}), \tag{8.4.34}$$
$$q_1(\lambda) = -\overline{q_1(\lambda^{-1})}. \tag{8.4.35}$$

3. *In the case $l = \infty$, the unitary monodromy matrices $\hat{M}_j \in \Lambda\mathrm{SU}(2, \mathbb{C})_\sigma$, $j = 0, 1, \infty$, associated with F as in* (8.3.16) *satisfy equations* (8.3.32) *to* (8.3.34) *from theorem 8.6. In terms of the functions p_j and q_j occurring in $\hat{M}_j$, these equations are equivalent to*

$$p_0(\lambda) = p_1(\lambda^{-1}), \tag{8.4.36}$$
$$q_0(\lambda) = e^{-\frac{\pi i}{3}} \overline{q_1(\lambda^{-1})}, \tag{8.4.37}$$
$$p_\infty(\lambda) = p_\infty(\lambda^{-1}), \tag{8.4.38}$$
$$q_\infty(\lambda) = e^{-\frac{\pi i}{3}} \overline{q_\infty(\lambda^{-1})}. \tag{8.4.39}$$

Proof. Keeping in mind that by theorem 8.7

$$M_{\mathcal{S}_0}(\lambda) = \begin{pmatrix} a_{\mathcal{S}_0} & b_{\mathcal{S}_0} \\ -\overline{b_{\mathcal{S}_0}} & \overline{a_{\mathcal{S}_0}} \end{pmatrix} = \begin{pmatrix} e^{-\frac{\pi i}{6}} & 0 \\ 0 & e^{\frac{\pi i}{6}} \end{pmatrix}, \tag{8.4.40}$$
$$M_{\mathcal{S}_1}(\lambda) = \begin{pmatrix} a_{\mathcal{S}_1} & b_{\mathcal{S}_1} \\ -\overline{b_{\mathcal{S}_1}} & \overline{a_{\mathcal{S}_1}} \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{8.4.41}$$
$$M_{\mathcal{S}_\infty}(\lambda) = \begin{pmatrix} a_{\mathcal{S}_\infty} & b_{\mathcal{S}_\infty} \\ -\overline{b_{\mathcal{S}_\infty}} & \overline{a_{\mathcal{S}_\infty}} \end{pmatrix} = \begin{pmatrix} e^{\frac{\pi i}{6}} & 0 \\ 0 & e^{-\frac{\pi i}{6}} \end{pmatrix}, \tag{8.4.42}$$

the claimed identities follow directly from theorem 8.6. □

Theorem 8.9 describes the (unitary) monodromy matrices associated with the extended frame $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ of a trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends, which is reflectionally symmetric with respect to the trinoid normal plane E_l for some $l \in \{0, 1, \infty\}$, and which has been normalized such that $F(z_*) = \mathrm{I}$ and $\psi(z_*) = \frac{1}{2H} e_3$, where $z_* \in \tilde{M}$ is given in (8.4.1) and ψ denotes the conformal CMC-immersion $\tilde{M} \to \mathbb{R}^3$ corresponding to ϕ. It turns out that, in this setting, we can also prove the converse result: A trinoid ϕ with properly embedded annular ends and with extended frame F satisfying $F(z_*) = \mathrm{I}$ at $z_* \in \tilde{M}$ from (8.4.1) and corresponding monodromy matrices of the form given in theorem 8.9 is necessarily reflectionally symmetric with respect to the trinoid normal plane E_l. This result is formulated in the following theorem.

Theorem 8.10. *Let η be a (standardized) trinoid potential associated with three off-diagonal Delaunay matrices D_0, D_1, D_∞ with eigenvalues $\pm\mu_0$, $\pm\mu_1$ and $\pm\mu_\infty$, respectively. Denote by $\phi : M \to \mathbb{R}^3$ a trinoid with properly embedded annular ends on $M = \mathbb{C} \setminus \{0, 1\}$ generated by η via the loop group method. Moreover, let $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ be the extended frame associated with the mapping $\psi = \phi \circ \pi$ by theorem 4.5, satisfying $F(z_*) = \mathrm{I}$ at $z_* \in \tilde{M}$ given in* (8.4.1).

1. *Let $\mu_1 = \mu_\infty$. Assume the unitary monodromy matrices $\hat{M}_j \in \Lambda\mathrm{SU}(2, \mathbb{C})_\sigma$, $j = 0, 1, \infty$, associated with F are of the form*

$$\hat{M}_j = -\cos(2\pi\mu_j)\mathrm{I} - i\sin(2\pi\mu_j)\begin{pmatrix} p_j(\lambda) & \overline{q_j(\lambda)} \\ q_j(\lambda) & -p_j(\lambda) \end{pmatrix}, \tag{8.4.43}$$

with functions p_j and q_j satisfying (3.9.51) *and* (3.9.50) *and, additionally,*

$$p_0(\lambda) = p_0(\lambda^{-1}), \tag{8.4.44}$$
$$q_0(\lambda) = e^{\frac{\pi i}{3}}\overline{q_0(\lambda^{-1})}, \tag{8.4.45}$$
$$p_1(\lambda) = p_\infty(\lambda^{-1}), \tag{8.4.46}$$
$$q_1(\lambda) = e^{\frac{\pi i}{3}}\overline{q_\infty(\lambda^{-1})}. \tag{8.4.47}$$

Then, ϕ is reflectionally symmetric with respect to the trinoid normal plane E_0.

2. *Let $\mu_0 = \mu_\infty$. Assume the unitary monodromy matrices $\hat{M}_j \in \Lambda\mathrm{SU}(2, \mathbb{C})_\sigma$, $j = 0, 1, \infty$, associated with F are of the form*

$$\hat{M}_j = -\cos(2\pi\mu_j)\mathrm{I} - i\sin(2\pi\mu_j)\begin{pmatrix} p_j(\lambda) & \overline{q_j(\lambda)} \\ q_j(\lambda) & -p_j(\lambda) \end{pmatrix}, \tag{8.4.48}$$

with functions p_j and q_j satisfying (3.9.51) *and* (3.9.50) *and, additionally,*

$$p_0(\lambda) = p_\infty(\lambda^{-1}), \tag{8.4.49}$$
$$q_0(\lambda) = -\overline{q_\infty(\lambda^{-1})}, \tag{8.4.50}$$
$$p_1(\lambda) = p_1(\lambda^{-1}), \tag{8.4.51}$$
$$q_1(\lambda) = -\overline{q_1(\lambda^{-1})}. \tag{8.4.52}$$

Then, ϕ is reflectionally symmetric with respect to the trinoid normal plane E_1.

3. *Let $\mu_0 = \mu_1$. Assume the unitary monodromy matrices $\hat{M}_j \in \Lambda\mathrm{SU}(2, \mathbb{C})_\sigma$, $j = 0, 1, \infty$, associated with F are of the form*

$$\hat{M}_j = -\cos(2\pi\mu_j)\mathrm{I} - i\sin(2\pi\mu_j)\begin{pmatrix} p_j(\lambda) & \overline{q_j(\lambda)} \\ q_j(\lambda) & -p_j(\lambda) \end{pmatrix}, \tag{8.4.53}$$

with functions p_j and q_j satisfying (3.9.51) *and* (3.9.50) *and, additionally,*

$$p_0(\lambda) = p_1(\lambda^{-1}), \tag{8.4.54}$$
$$q_0(\lambda) = e^{-\frac{\pi i}{3}}\overline{q_1(\lambda^{-1})}, \tag{8.4.55}$$
$$p_\infty(\lambda) = p_\infty(\lambda^{-1}), \tag{8.4.56}$$
$$q_\infty(\lambda) = e^{-\frac{\pi i}{3}}\overline{q_\infty(\lambda^{-1})}. \tag{8.4.57}$$

Then, ϕ is reflectionally symmetric with respect to the trinoid normal plane E_∞.

Proof. We start by considering the special form of the potential η in each of the three cases. We associate the first, second, third case with $l = 0$, $l = 1$, $l = \infty$, respectively, and denote the corresponding potential by η_0, η_1, η_∞, respectively.

In the first case ($l = 0$) we have $\mu_1 = \mu_\infty$ and thus (cf. section 3.6)

$$\eta_0 = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q_0(z,\lambda) & 0 \end{pmatrix} \mathrm{d}z, \tag{8.4.58}$$

where

$$\begin{aligned} Q_0(z,\lambda) &= \frac{b_0(\lambda)}{z^2} + \frac{b_1(\lambda)}{(z-1)^2} + \frac{b_0(\lambda)}{z} - \frac{b_0(\lambda)}{z-1} \\ &= \frac{b_0(\lambda)(z-1)^2 + b_1(\lambda)z^2 - b_0(\lambda)z(z-1)}{z^2(z-1)^2} = \frac{b_0(\lambda)(1-z) + b_1(\lambda)z^2}{z^2(z-1)^2} \end{aligned} \tag{8.4.59}$$

and $b_j(\lambda) = \frac{1}{4} - (\mu_j(\lambda))^2$ for $j = 0, 1$. Considering the bi-antiholomorphic mapping $\gamma_{\mathcal{S}_0} : M \to M$ defined by $z \mapsto \gamma_{\mathcal{S}_0}(z) := \frac{\bar z}{\bar z - 1}$ and the function $h_0 : M \to \mathbb{C} \setminus \{0\}$, $z \mapsto h_0(z) = -i(\bar z - 1)$, we compute

$$Q_0(\gamma_{\mathcal{S}_0}(z),\lambda) = \frac{b_0(\lambda)(\frac{-1}{\bar z - 1}) + b_1(\lambda)\frac{\bar z^2}{(\bar z - 1)^2}}{\frac{\bar z^2}{(\bar z-1)^2}\frac{1}{(\bar z - 1)^2}} = (\bar z - 1)^4 \frac{b_0(\lambda)(1-\bar z) + b_1(\lambda)\bar z^2}{\bar z^2(\bar z - 1)^2} = (h_0(z))^4 \overline{Q_0(z,\lambda^{-1})}, \tag{8.4.60}$$

where we used the fact that for $\lambda \in S^1$ the identity

$$\overline{b_j(\lambda^{-1})} = b_j(\lambda) \quad \text{for } j = 0, 1, \infty \tag{8.4.61}$$

holds. Recalling from lemma 4.21 that $\gamma_{\mathcal{S}_0}$ corresponds to the permutation $\sigma = (1\ \infty)$ of the set $\{0, 1, \infty\}$, we apply lemma 4.25 to infer that η_0 transforms under $\gamma_{\mathcal{S}_0}$ as

$$\gamma_{\mathcal{S}_0}^* \eta_0(z,\lambda) = \overline{\eta_0(z,\lambda^{-1})} \# W_{+,0}, \tag{8.4.62}$$

where

$$W_{+,0} = W_{+,0}(z,\lambda) = \begin{pmatrix} h_0(z) & 0 \\ -\lambda \partial_{\bar z} h_0(z) & (h_0(z))^{-1} \end{pmatrix}. \tag{8.4.63}$$

Analogously, in the second case ($l = 1$) we have $\mu_0 = \mu_\infty$ and thus (cf. section 3.6)

$$\eta_1 = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q_1(z,\lambda) & 0 \end{pmatrix} \mathrm{d}z, \tag{8.4.64}$$

where

$$\begin{aligned} Q_1(z,\lambda) &= \frac{b_0(\lambda)}{z^2} + \frac{b_1(\lambda)}{(z-1)^2} + \frac{b_1(\lambda)}{z} - \frac{b_1(\lambda)}{z-1} \\ &= \frac{b_0(\lambda)(z-1)^2 + b_1(\lambda)z^2 - b_1(\lambda)z(z-1)}{z^2(z-1)^2} = \frac{b_0(\lambda)(z-1)^2 + b_1(\lambda)z}{z^2(z-1)^2} \end{aligned} \tag{8.4.65}$$

and $b_j(\lambda) = \frac{1}{4} - (\mu_j(\lambda))^2$ for $j = 0, 1$. Considering the bi-antiholomorphic mapping $\gamma_{\mathcal{S}_1} : M \to M$ defined by $z \mapsto \gamma_{\mathcal{S}_1}(z) := \frac{1}{\bar z}$ and the function $h_1 : M \to \mathbb{C} \setminus \{0\}$, $z \mapsto h_1(z) = -i\bar z$, we compute

$$Q_1(\gamma_{\mathcal{S}_1}(z),\lambda) = \frac{b_0(\lambda)\frac{(1-\bar z)^2}{\bar z^2} + b_1(\lambda)\frac{1}{\bar z}}{\frac{1}{\bar z^2}\frac{(1-\bar z)^2}{\bar z^2}} = \bar z^4 \frac{b_0(\lambda)(\bar z - 1)^2 + b_1(\lambda)\bar z}{\bar z^2(\bar z - 1)^2} = (h_1(z))^4 \overline{Q_1(z,\lambda^{-1})}, \tag{8.4.66}$$

where we have again used the identity (8.4.61). Recalling from lemma 4.21 that $\gamma_{\mathcal{S}_1}$ corresponds to the permutation $\sigma = (0\ \infty)$ of the set $\{0, 1, \infty\}$, we apply lemma 4.25 to infer that η_1 transforms under $\gamma_{\mathcal{S}_1}$ as

$$\gamma_{\mathcal{S}_1}^* \eta_1(z,\lambda) = \overline{\eta_1(z,\lambda^{-1})} \# W_{+,1}, \tag{8.4.67}$$

where

$$W_{+,1} = W_{+,1}(z,\lambda) = \begin{pmatrix} h_1(z) & 0 \\ -\lambda \partial_{\bar z} h_1(z) & (h_1(z))^{-1} \end{pmatrix}. \tag{8.4.68}$$

Finally, in the third case ($l = \infty$) we have $\mu_0 = \mu_1$ and thus (cf. section 3.6)

$$\eta_\infty = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q_\infty(z,\lambda) & 0 \end{pmatrix} \mathrm{d}z, \tag{8.4.69}$$

where

$$Q_\infty(z,\lambda) = \frac{b_0(\lambda)}{z^2} + \frac{b_0(\lambda)}{(z-1)^2} + \frac{c_0(\lambda)}{z} - \frac{c_0(\lambda)}{z-1} = \frac{\beta_0(\lambda)(z-1)^2 + b_0(\lambda)z^2 - c_0(\lambda)z(z-1)}{z^2(z-1)^2}, \tag{8.4.70}$$

$b_j(\lambda) = \frac{1}{4} - (\mu_0(\lambda))^2$ for $j = 0, \infty$ and $c_0(\lambda) = 2b_0(\lambda) - b\infty(\lambda)$. Considering the bi-antiholomorphic mapping $\gamma_{\mathcal{S}_\infty} : M \to M$ defined by $z \mapsto \gamma_{\mathcal{S}_\infty}(z) := 1 - \bar{z}$ and the function $h_\infty : M \to \mathbb{C} \setminus \{0\}$, $z \mapsto h_\infty(z) = -i$, we compute

$$Q_\infty(\gamma_{\mathcal{S}_\infty}(z),\lambda) = \frac{\beta_0(\lambda)\bar{z}^2 + b_0(\lambda)(\bar{z}-1)^2 - c_0(\lambda)(1-\bar{z})(-\bar{z})}{(\bar{z}-1)^2(\bar{z})^2} = (h_\infty(z))^4 \overline{Q_\infty(z,\lambda^{-1})}, \tag{8.4.71}$$

where we have used (8.4.61) together with the identity

$$\overline{c_0(\lambda^{-1})} = c_0(\lambda) \quad \text{for all } \lambda \in S^1. \tag{8.4.72}$$

Recalling from lemma 4.21 that $\gamma_{\mathcal{S}_\infty}$ corresponds to the permutation $\sigma = (0\ 1)$ of the set $\{0, 1, \infty\}$, we apply lemma 4.25 to infer that η_∞ transforms under $\gamma_{\mathcal{S}_\infty}$ as

$$\gamma^*_{\mathcal{S}_\infty} \eta_\infty(z,\lambda) = \overline{\eta_\infty(z,\lambda^{-1})} \# W_{+,\infty}, \tag{8.4.73}$$

where

$$W_{+,\infty} = W_{+,\infty}(z,\lambda) = \begin{pmatrix} h_\infty(z) & 0 \\ -\lambda \partial_{\bar{z}} h_\infty(z) & (h_\infty(z))^{-1} \end{pmatrix}. \tag{8.4.74}$$

Altogether, we have for all $l \in \{0, 1, \infty\}$ the relation

$$\gamma^*_{\mathcal{S}_l} \eta_l(z,\lambda) = \overline{\eta_l(z,\lambda^{-1})} \# W_{+,l}(z,\lambda). \tag{8.4.75}$$

Applying the pullback construction with respect to the covering mapping $\pi : \tilde{M} \to M$ to (8.4.75), we obtain

$$\pi^*(\gamma^*_{\mathcal{S}_l} \eta_l(z,\lambda)) = \pi^*(\overline{\eta_l(z,\lambda^{-1})} \# W_{+,l}) = \overline{\tilde{\eta}_l(z,\lambda^{-1})} \# \tilde{W}_{+,l}, \tag{8.4.76}$$

where $\tilde{\eta}_l = \pi^* \eta_l$ denotes the pullback potential of the trinoid potential η_l (cf. section 2.3) and $\tilde{W}_{+,l} = W_{+,l} \circ \pi$. Moreover, recall that the bi-antiholomorphic mappings $\tilde{\gamma}_{\mathcal{S}_l} : \tilde{M} \to \tilde{M}$,

$$\tilde{\gamma}_{\mathcal{S}_0} : z \mapsto \frac{\bar{z}}{-\bar{z}-1}, \quad \tilde{\gamma}_{\mathcal{S}_1} : z \mapsto -\bar{z} - 1, \quad \tilde{\gamma}_{\mathcal{S}_\infty} : z \mapsto \frac{1}{\bar{z}} \tag{8.4.77}$$

from lemma 8.3 satisfy $\gamma_{\mathcal{S}_l} \circ \pi = \pi \circ \tilde{\gamma}_{\mathcal{S}_l}$. Thus, the left hand side of (8.4.76) can be transformed as follows:

$$\begin{aligned} \pi^*(\gamma^*_{\mathcal{S}_l} \eta_l) &= \pi^* \left[\begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q_l(\gamma_{\mathcal{S}_l}(z),\lambda) & 0 \end{pmatrix} \mathrm{d}\gamma_{\mathcal{S}_l}(z) \right] \\ &= \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q_l((\gamma_{\mathcal{S}_l} \circ \pi)(z),\lambda) & 0 \end{pmatrix} \mathrm{d}(\gamma_{\mathcal{S}_l} \circ \pi)(z) = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q_l((\pi \circ \tilde{\gamma}_{\mathcal{S}_l})(z),\lambda) & 0 \end{pmatrix} \mathrm{d}(\pi \circ \tilde{\gamma}_{\mathcal{S}_l})(z) \\ &= \tilde{\gamma}^*_{\mathcal{S}_l} \left[\begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda Q_l(\pi(z),\lambda) & 0 \end{pmatrix} \mathrm{d}\pi(z) \right] = \tilde{\gamma}^*_{\mathcal{S}_l}(\pi^* \eta_l) = \tilde{\gamma}^*_{\mathcal{S}_l} \tilde{\eta}_l. \end{aligned} \tag{8.4.78}$$

Altogether, (8.4.76) yields

$$\tilde{\gamma}^*_{\mathcal{S}_l} \tilde{\eta}_l(z,\lambda) = \overline{\tilde{\eta}_l(z,\lambda^{-1})} \# \tilde{W}_{+,l}. \tag{8.4.79}$$

Considering the extended frame F associated with the trinoid ϕ, we obtain (for $l \in \{0, 1, \infty\}$) a solution $\Psi_l = F B_{+,l}$ to the differential equation $\mathrm{d}\Psi_l = \Psi_l \tilde{\eta}_l$. Note that Ψ_l possesses the same (unitary) monodromy matrices as F at the singularities of the potential η_l, namely $\hat{M}_0$, $\hat{M}_1$ and $\hat{M}_\infty$.

Naturally, the mapping $\tilde{\gamma}^*_{\mathcal{S}_l} \Psi_l = \Psi_l \circ \tilde{\gamma}_{\mathcal{S}_l}$ defines a solution to the differential equation $\mathrm{d}(\tilde{\gamma}^*_{\mathcal{S}_l} \Psi_l) = (\tilde{\gamma}^*_{\mathcal{S}_l} \Psi_l)(\tilde{\gamma}^*_{\mathcal{S}_l} \tilde{\eta}_l)$, which in view of (8.4.79) reads as

$$\mathrm{d}(\tilde{\gamma}^*_{\mathcal{S}_l} \Psi_l(z,\lambda)) = (\tilde{\gamma}^*_{\mathcal{S}_l} \Psi_l(z,\lambda))(\overline{\tilde{\eta}_l(z,\lambda^{-1})} \# \tilde{W}_{+,l}). \tag{8.4.80}$$

Since this differential equation is also solved by the mapping $\overline{\Psi_l(z,\lambda^{-1})}\tilde{W}_{+,l}$, i.e.

$$\mathrm{d}(\overline{\Psi_l(z,\lambda^{-1})}\tilde{W}_{+,l}) = (\overline{\Psi_l(z,\lambda^{-1})}\tilde{W}_{+,l})(\overline{\tilde{\eta}_l(z,\lambda^{-1})}\#\tilde{W}_{+,l}), \tag{8.4.81}$$

the mappings $\tilde{\gamma}^*_{\mathcal{S}_l}\Psi(z,\lambda)$ and $\overline{\Psi(z,\lambda^{-1})}\tilde{W}_{+,l}$ only differ by a λ-dependent matrix $\rho_l = \rho_l(\lambda)$:

$$\tilde{\gamma}^*_{\mathcal{S}_l}\Psi_l(z,\lambda) = \rho_l\overline{\Psi_l(z,\lambda^{-1})}\tilde{W}_{+,l}. \tag{8.4.82}$$

Consider the case $l = 0$. Applying the relation $\tilde{\gamma}_{\mathcal{S}_0}\circ\tilde{\gamma}_0 = \tilde{\gamma}_0^{-1}\circ\tilde{\gamma}_{\mathcal{S}_0}$ from (8.3.3), involving the covering transformation $\tilde{\gamma}_0$ on $\tilde{M}$ as given in section 3.3, we compute

$$\begin{aligned}\rho_0(\lambda)\overline{\hat{M}_0(\lambda^{-1})}\,\overline{\Psi_0(z,\lambda^{-1})}\tilde{W}_{+,0}(\tilde{\gamma}_0(z),\lambda) &= \rho_0(\lambda)\overline{\Psi_0(\tilde{\gamma}_0(z),\lambda^{-1})}\tilde{W}_{+,0}(\tilde{\gamma}_0(z),\lambda) = \tilde{\gamma_{\mathcal{S}_0}}^*\Psi_0(\tilde{\gamma}_0(z),\lambda)\\ &= \Psi_0((\tilde{\gamma}_{\mathcal{S}_0}\circ\tilde{\gamma}_0)(z),\lambda) = \Psi_0((\tilde{\gamma}_0^{-1}\circ\tilde{\gamma}_{\mathcal{S}_0})(z),\lambda) = (\hat{M}_0(\lambda))^{-1}\Psi_0(\tilde{\gamma}_{\mathcal{S}_0}(z),\lambda)\\ &= (\hat{M}_0(\lambda))^{-1}(\tilde{\gamma}^*_{\mathcal{S}_0}\Psi_0(z,\lambda)) = (\hat{M}_0(\lambda))^{-1}\rho_0(\lambda)\overline{\Psi_0(z,\lambda^{-1})}\tilde{W}_{+,0}(z,\lambda),\end{aligned} \tag{8.4.83}$$

where we have made use of the identity

$$\Psi_0(\tilde{\gamma}_0^{-1}(z),\lambda) = (\hat{M}_0(\lambda))^{-1}\Psi_0(z,\lambda), \tag{8.4.84}$$

which is a direct consequence of the relation

$$\Psi_0(z,\lambda) = \Psi_0(\tilde{\gamma}_0(\tilde{\gamma}_0^{-1}(z)),\lambda) = \hat{M}_0(\lambda)\Psi_0(\tilde{\gamma}_0^{-1}(z),\lambda). \tag{8.4.85}$$

As $\tilde{W}_{+,0}$ defines the pullback of the mapping $W_{+,0}$, which is antiholomorphic on M (with respect to z), $\tilde{W}_{+,0}$ is antiholomorphic on $\tilde{M}$ and therefore does not pick up any monodromy under $\tilde{\gamma}_0$, i.e. $\tilde{W}_{+,0}(\tilde{\gamma}_0(z),\lambda) = \tilde{W}_{+,0}(z,\lambda)$. Thus, we conclude that

$$\rho_0(\lambda)\overline{\hat{M}_0(\lambda^{-1})} = (\hat{M}_0(\lambda))^{-1}\rho_0(\lambda). \tag{8.4.86}$$

Analogously, applying $\tilde{\gamma}_{\mathcal{S}_0}\circ\tilde{\gamma}_1 = \tilde{\gamma}_\infty^{-1}\circ\tilde{\gamma}_{\mathcal{S}_0}$ from (8.3.3), we have

$$\begin{aligned}\rho_0(\lambda)\overline{\hat{M}_1(\lambda^{-1})}\,\overline{\Psi_0(z,\lambda^{-1})}\tilde{W}_{+,0}(\tilde{\gamma}_1(z),\lambda) &= \rho_0(\lambda)\overline{\Psi_0(\tilde{\gamma}_1(z),\lambda^{-1})}\tilde{W}_{+,0}(\tilde{\gamma}_1(z),\lambda) = \tilde{\gamma_{\mathcal{S}_0}}^*\Psi_0(\tilde{\gamma}_1(z),\lambda)\\ &= \Psi_0((\tilde{\gamma}_{\mathcal{S}_0}\circ\tilde{\gamma}_1)(z),\lambda) = \Psi_0((\tilde{\gamma}_\infty^{-1}\circ\tilde{\gamma}_{\mathcal{S}_0})(z),\lambda) = (\hat{M}_\infty(\lambda))^{-1}\Psi_0(\tilde{\gamma}_{\mathcal{S}_0}(z),\lambda)\\ &= (\hat{M}_\infty(\lambda))^{-1}(\tilde{\gamma}^*_{\mathcal{S}_0}\Psi_0(z,\lambda)) = (\hat{M}_\infty(\lambda))^{-1}\rho_0(\lambda)\overline{\Psi_0(z,\lambda^{-1})}\tilde{W}_{+,0}(z,\lambda).\end{aligned} \tag{8.4.87}$$

Using the anti-holomorphicity of $\tilde{W}_{+,0}$ on $\tilde{M}$, we know that $\tilde{W}_{+,0}(\tilde{\gamma}_1(z),\lambda) = \tilde{W}_{+,0}(z,\lambda)$, which yields

$$\rho_0(\lambda)\overline{\hat{M}_1(\lambda^{-1})} = (\hat{M}_\infty(\lambda))^{-1}\rho_0(\lambda). \tag{8.4.88}$$

Setting

$$\rho_0(\lambda) = \begin{pmatrix} a_0(\lambda) & b_0(\lambda)\\ c_0(\lambda) & d_0(\lambda)\end{pmatrix}, \tag{8.4.89}$$

where a_0, b_0, c_0 and d_0 define complex valued functions of λ satisfying $a_0(\lambda)d_0(\lambda) - b_0(\lambda)c_0(\lambda) = 1$, and comparing the upper left entries (resp. the upper right entries) of $\rho_0(\lambda)\overline{\hat{M}_0(\lambda^{-1})}$ and $(\hat{M}_0(\lambda))^{-1}\rho_0(\lambda)$, we obtain

$$a_0(\lambda)\overline{p_0(\lambda^{-1})} + b_0(\lambda)\overline{q_0(\lambda^{-1})} = a_0(\lambda)p_0(\lambda) + c_0(\lambda)\overline{q_0(\lambda)}, \tag{8.4.90}$$

$$a_0(\lambda)q_0(\lambda^{-1}) - b_0(\lambda)\overline{p_0(\lambda^{-1})} = b_0(\lambda)p_0(\lambda) + d_0(\lambda)\overline{q_0(\lambda)}. \tag{8.4.91}$$

In view of (3.9.50) and the assumption, these equations simplify into

$$b_0(\lambda)e^{-\frac{\pi i}{3}}q_0(\lambda) = c_0(\lambda)\overline{q_0(\lambda)}, \tag{8.4.92}$$

$$(a_0(\lambda)e^{\frac{\pi i}{3}} - d_0(\lambda))\overline{q_0(\lambda)} = 2b_0(\lambda)p_0(\lambda). \tag{8.4.93}$$

Similarly, comparing the upper left entries (resp. the upper right entries) of $\rho_0(\lambda)\overline{\hat{M}_1(\lambda^{-1})}$ and $(\hat{M}_\infty(\lambda))^{-1}\rho_0(\lambda)$, we infer (by using $\mu_1 = \mu_\infty$) that

$$a_0(\lambda)\overline{p_1(\lambda^{-1})} + b_0(\lambda)\overline{q_1(\lambda^{-1})} = a_0(\lambda)p_\infty(\lambda) + c_0(\lambda)\overline{q_\infty(\lambda)}, \tag{8.4.94}$$

$$a_0(\lambda)q_1(\lambda^{-1}) - b_0(\lambda)\overline{p_1(\lambda^{-1})} = b_0(\lambda)p_\infty(\lambda) + d_0(\lambda)\overline{q_\infty(\lambda)}. \tag{8.4.95}$$

In view of (3.9.50) and the assumption, these equations simplify into

$$b_0(\lambda)e^{-\frac{\pi i}{3}}q_\infty(\lambda) = c_0(\lambda)\overline{q_\infty(\lambda)}, \tag{8.4.96}$$

$$(a_0(\lambda)e^{\frac{\pi i}{3}} - d_0(\lambda))\overline{q_\infty(\lambda)} = 2b_0(\lambda)p_\infty(\lambda). \tag{8.4.97}$$

Together, we have for $j = 0, \infty$:

$$b_0(\lambda)e^{-\frac{\pi i}{3}}q_j(\lambda) = c_0(\lambda)\overline{q_j(\lambda)}, \tag{8.4.98}$$

$$(a_0(\lambda)e^{\frac{\pi i}{3}} - d_0(\lambda))\overline{q_j(\lambda)} = 2b_0(\lambda)p_j(\lambda). \tag{8.4.99}$$

Since in general (i.e. for all λ in S^1 excluding a finite subset of S^1) $p_j, q_j \neq 0$, we can solve for $c_0(\lambda)$ and $b_0(\lambda)$, respectively:

$$c_0(\lambda) = e^{-\frac{pii}{3}}b_0(\lambda)q_j(\lambda)\overline{q_j(\lambda)}^{-1}, \tag{8.4.100}$$

$$b_0(\lambda) = \frac{1}{2}(e^{\frac{pii}{3}}a_0(\lambda) + d_0(\lambda))\overline{q_j(\lambda)}(p_j(\lambda))^{-1}. \tag{8.4.101}$$

This yields (using (3.9.50) again)

$$\begin{aligned} 1 = a_0(\lambda)d_0(\lambda) - b_0(\lambda)c_0(\lambda) &= a_0(\lambda)d_0(\lambda) - \frac{1}{4}q_j(\lambda)\overline{q_j(\lambda)}(p_j(\lambda))^{-2}(e^{\frac{pii}{3}}a_0(\lambda) - d_0(\lambda))(a_0(\lambda) - e^{-\frac{pii}{3}}d_0(\lambda)) \\ &= a_0(\lambda)d_0(\lambda) - \frac{1}{4}(p_j(\lambda))^{-2}(e^{\frac{pii}{6}}a_0(\lambda) - e^{-\frac{pii}{6}}d_0(\lambda))^2 + \frac{1}{4}(e^{\frac{pii}{6}}a_0(\lambda) - e^{-\frac{pii}{6}}d_0(\lambda))^2 \\ &= -\frac{1}{4}(p_j(\lambda))^{-2}(e^{\frac{pii}{6}}a_0(\lambda) - e^{-\frac{pii}{6}}d_0(\lambda))^2 + \frac{1}{4}(e^{\frac{pii}{6}}a_0(\lambda) + e^{-\frac{pii}{6}}d_0(\lambda))^2, \end{aligned} \tag{8.4.102}$$

or, equivalently,

$$(p_j(\lambda))^2(4 - (e^{\frac{pii}{6}}a_0(\lambda) + e^{-\frac{pii}{6}}d_0(\lambda))^2) = -(e^{\frac{pii}{6}}a_0(\lambda) - e^{-\frac{pii}{6}}d_0(\lambda))^2. \tag{8.4.103}$$

Assume now that, in general, $4 - (e^{\frac{pii}{6}}a_0(\lambda) + e^{-\frac{pii}{6}}d_0(\lambda))^2 \neq 0$ (i.e. $4 - (e^{\frac{pii}{6}}a_0(\lambda) + e^{-\frac{pii}{6}}d_0(\lambda))^2 = 0$ for at most finitely many $\lambda \in S^1$). We infer that

$$(p_0(\lambda))^2 = (p_\infty(\lambda))^2 = -\frac{(e^{\frac{pii}{6}}a_0(\lambda) - e^{-\frac{pii}{6}}d_0(\lambda))^2}{4 - (e^{\frac{pii}{6}}a_0(\lambda) + e^{-\frac{pii}{6}}d_0(\lambda))^2} \tag{8.4.104}$$

for all but (at most) finitely many $\lambda \in S^1$ and thus

$$p_0(\lambda) = \alpha p_\infty(\lambda) \tag{8.4.105}$$

for some $\alpha \in \{\pm 1\}$ and all but (at most) finitely many $\lambda \in S^1$. Consequently, by (8.4.101), this implies

$$\overline{q_0(\lambda)} = \alpha\overline{q_\infty(\lambda)}, \quad q_0(\lambda) = \alpha q_\infty(\lambda) \tag{8.4.106}$$

and thus

$$p_0(\lambda)p_\infty(\lambda) + \frac{q_0(\lambda)\overline{q_\infty(\lambda)} + \overline{q_0(\lambda)}q_\infty(\lambda)}{2} = \alpha((p_0(\lambda))^2 + q_0(\lambda)\overline{q_0(\lambda)}) = \alpha \tag{8.4.107}$$

for all but (at most) finitely many $\lambda \in S^1$, which in view of remark 3.56 clearly is a contradiction to equation (3.9.51). Therefore, we conclude that $4 - (e^{\frac{pii}{6}}a_0(\lambda) + e^{-\frac{pii}{6}}d_0(\lambda))^2 = 0$ for all $\lambda \in S^1$ and (by (8.4.103)) $(e^{\frac{pii}{6}}a_0(\lambda) - e^{-\frac{pii}{6}}d_0(\lambda))^2 = 0$ for all $\lambda \in S^1$. Together, these relations yield $a_0(\lambda) = e^{-\frac{pii}{3}}d_0(\lambda) = \pm e^{-\frac{pii}{6}}$ and (by (8.4.101) and (8.4.100)) $b_0(\lambda) = c_0(\lambda) = 0$. Thus,

$$\rho_0(\lambda) = \pm\begin{pmatrix} e^{-\frac{pii}{6}} & 0 \\ 0 & e^{\frac{pii}{6}} \end{pmatrix}, \tag{8.4.108}$$

in particular $\rho_0(\lambda) \in \Lambda\mathrm{SU}(2)_\sigma \cap \Lambda^+\mathrm{SL}(2,\mathbb{C})_\sigma$.

We proceed analogously in the cases $l = 1$ and $l = \infty$: Applying the relations $\tilde{\gamma}_{\mathcal{S}_1} \circ \tilde{\gamma}_0 = \tilde{\gamma}_\infty^{-1} \circ \tilde{\gamma}_{\mathcal{S}_1}$ and $\tilde{\gamma}_{\mathcal{S}_1} \circ \tilde{\gamma}_1 = \tilde{\gamma}_1^{-1} \circ \tilde{\gamma}_{\mathcal{S}_1}$ from (8.3.4), we deduce in the case $l = 1$ the identities

$$\rho_1(\lambda)\overline{\hat{M}_0(\lambda^{-1})} = (\hat{M}_\infty(\lambda))^{-1}\rho_1(\lambda), \tag{8.4.109}$$

$$\rho_1(\lambda)\overline{\hat{M}_1(\lambda^{-1})} = (\hat{M}_1(\lambda))^{-1}\rho_1(\lambda), \tag{8.4.110}$$

which (analogously as above) in view of (3.9.50) and the assumption translate into

$$-b_1(\lambda)q_1(\lambda) = c_1(\lambda)\overline{q_1(\lambda)}, \tag{8.4.111}$$

$$(-a_1(\lambda) - d_1(\lambda))\overline{q_1(\lambda)} = 2b_1(\lambda)p_1(\lambda), \tag{8.4.112}$$

$$-b_1(\lambda)q_\infty(\lambda) = c_1(\lambda)\overline{q_\infty(\lambda)}, \tag{8.4.113}$$

$$(-a_1(\lambda) - d_1(\lambda))\overline{q_\infty(\lambda)} = 2b_1(\lambda)p_\infty(\lambda), \tag{8.4.114}$$

where a_1, b_1, c_1 and d_1 define the complex valued functions of λ satisfying $a_1(\lambda)d_1(\lambda) - b_1(\lambda)c_1(\lambda) = 1$ and occurring in

$$\rho_1(\lambda) = \begin{pmatrix} a_1(\lambda) & b_1(\lambda) \\ c_1(\lambda) & d_1(\lambda) \end{pmatrix}. \tag{8.4.115}$$

Applying the argument as in the case $l = 0$ (basically only adjusting indices) we conclude that

$$\rho_1(\lambda) = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{8.4.116}$$

in particular $\rho_1(\lambda) \in \Lambda\mathrm{SU}(2)_\sigma \cap \Lambda^+\mathrm{SL}(2, \mathbb{C})_\sigma$.

Applying the relations $\tilde{\gamma}_{\mathcal{S}_\infty} \circ \tilde{\gamma}_0 = \tilde{\gamma}_1^{-1} \circ \tilde{\gamma}_{\mathcal{S}_\infty}$ and $\tilde{\gamma}_{\mathcal{S}_\infty} \circ \tilde{\gamma}_\infty = \tilde{\gamma}_\infty^{-1} \circ \tilde{\gamma}_{\mathcal{S}_\infty}$ from (8.3.5), we deduce in the case $l = \infty$ the identities

$$\rho_\infty(\lambda)\overline{\hat{M}_0(\lambda^{-1})} = (\hat{M}_1(\lambda))^{-1}\rho_\infty(\lambda), \tag{8.4.117}$$

$$\rho_\infty(\lambda)\overline{\hat{M}_\infty(\lambda^{-1})} = (\hat{M}_\infty(\lambda))^{-1}\rho_\infty(\lambda), \tag{8.4.118}$$

which (analogously as above) in view of (3.9.50) and the assumption translate into

$$b_\infty(\lambda)e^{\frac{\pi i}{3}}q_\infty(\lambda) = c_\infty(\lambda)\overline{q_\infty(\lambda)}, \tag{8.4.119}$$

$$(a_\infty(\lambda)e^{-\frac{\pi i}{3}} - d_\infty(\lambda))\overline{q_\infty(\lambda)} = 2b_\infty(\lambda)p_\infty(\lambda), \tag{8.4.120}$$

$$b_\infty(\lambda)e^{\frac{\pi i}{3}}q_1(\lambda) = c_\infty(\lambda)\overline{q_1(\lambda)}, \tag{8.4.121}$$

$$(a_\infty(\lambda)e^{-\frac{\pi i}{3}} - d_\infty(\lambda))\overline{q_1(\lambda)} = 2b_\infty(\lambda)p_1(\lambda), \tag{8.4.122}$$

where a_∞, b_∞, c_∞ and d_∞ define the λ-dependent, complex valued functions satisfying $a_\infty(\lambda)d_\infty(\lambda) - b_\infty(\lambda)c_\infty(\lambda) = 1$ and occurring in

$$\rho_\infty(\lambda) = \begin{pmatrix} a_\infty(\lambda) & b_\infty(\lambda) \\ c_\infty(\lambda) & d_\infty(\lambda) \end{pmatrix}. \tag{8.4.123}$$

Applying the same argument as in the case $l = 0$ (basically only adjusting indices) we conclude that

$$\rho_\infty(\lambda) = \pm \begin{pmatrix} e^{\frac{pii}{6}} & 0 \\ 0 & e^{-\frac{pii}{6}} \end{pmatrix}, \tag{8.4.124}$$

in particular $\rho_\infty(\lambda) \in \Lambda\mathrm{SU}(2)_\sigma \cap \Lambda^+\mathrm{SL}(2, \mathbb{C})_\sigma$.

Altogether, for all $l \in \{0, 1, \infty\}$, we have $\rho_l \in \Lambda\mathrm{SU}(2)_\sigma \cap \Lambda^+\mathrm{SL}(2, \mathbb{C})_\sigma$. Consequently,

$$(\rho_l\overline{F(z, \lambda^{-1})}\rho_l^{-1})(\rho_l\overline{B_{+,l}(z, \lambda^{-1})}\tilde{W}_{+,l}(z, \lambda)) \tag{8.4.125}$$

defines an Iwasawa-decomposition of $\rho_l\overline{\Psi_l(z, \lambda^{-1})}\tilde{W}_{+,l}(z, \lambda)$ (pointwise for all $z \in \tilde{M}$) with

$$\rho_l\overline{F(z, \lambda^{-1})}\rho_l^{-1} \in \Lambda\mathrm{SU}(2)_\sigma, \quad \rho_l\overline{B_{+,l}(z, \lambda-1)}\tilde{W}_{+,l}(z, \lambda) \in \Lambda^+\mathrm{SL}(2, \mathbb{C})_\sigma \tag{8.4.126}$$

and $\rho_l \overline{F(z_*, \lambda^{-1})} \rho_l^{-1} = \mathrm{I}$. Therefore, we can write

$$\begin{aligned} F(\tilde{\gamma}_{\mathcal{S}_l}(z), \lambda) B_{+,l}(\tilde{\gamma}_{\mathcal{S}_l}(z), \lambda) &= \tilde{\gamma}_{\mathcal{S}_l}^* \Psi(z, \lambda) = \rho_l(\lambda) \overline{\Psi_l(z, \lambda^{-1})} \tilde{W}_{+,l}(z, \lambda) \\ &= (\rho_l(\lambda) \overline{F(z, \lambda^{-1})} (\rho_l(\lambda))^{-1})(\rho(\lambda) \overline{B_+(z, \lambda^{-1})} \tilde{W}_+(z, \lambda)). \end{aligned} \tag{8.4.127}$$

Thus, $\tilde{\gamma}_{\mathcal{S}_l}^* \Psi_l$ produces (by the loop group method) on the one hand the trinoid $\mathrm{SymBob}(F(\tilde{\gamma}_{\mathcal{S}_l}(z), \lambda))|_{\lambda=1}$ and on the other hand the trinoid $\mathrm{SymBob}(\rho_l(\lambda) \overline{F(z, \lambda^{-1})} (\rho_l(\lambda))^{-1})|_{\lambda=1}$. Consequently, these two surfaces coincide, and, using the straightforward identities

$$\mathrm{SymBob}(\overline{F(z, \lambda^{-1})})|_{\lambda=1} = -\overline{\mathrm{SymBob}(F(z, \lambda))}|_{\lambda=1} \tag{8.4.128}$$

and

$$\overline{X} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ for all } X \in \mathrm{su}(2), \tag{8.4.129}$$

we compute

$$\begin{aligned} J(\psi \circ \tilde{\gamma}_{\mathcal{S}_l}) &= \mathrm{SymBob}(F(\tilde{\gamma}_{\mathcal{S}_l}(z), \lambda))|_{\lambda=1} = \mathrm{SymBob}(\rho_l(\lambda) \overline{F(z, \lambda^{-1})} (\rho_l(\lambda))^{-1})|_{\lambda=1} \\ &= \rho_l(\lambda) \mathrm{SymBob}(\overline{F(z, \lambda^{-1})})|_{\lambda=1} (\rho_l(\lambda))^{-1} = -\rho_l(\lambda) \overline{\mathrm{SymBob}(F(z, \lambda))}|_{\lambda=1} (\rho_l(\lambda))^{-1} \\ &= -\rho_l(\lambda) \overline{J(\psi)} (\rho_l(\lambda))^{-1} = -\rho_l(\lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} J(\psi) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\rho_l(\lambda))^{-1}. \end{aligned} \tag{8.4.130}$$

We obtain

$$J(\psi \circ \tilde{\gamma}_{\mathcal{S}_0}) = -\begin{pmatrix} 0 & e^{-\frac{pii}{6}} \\ -e^{\frac{pii}{6}} & 0 \end{pmatrix} J(\psi) \begin{pmatrix} 0 & -e^{-\frac{pii}{6}} \\ e^{\frac{pii}{6}} & 0 \end{pmatrix} \tag{8.4.131}$$

in the case $l = 0$,

$$J(\psi \circ \tilde{\gamma}_{\mathcal{S}_0}) = -\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} J(\psi) \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{8.4.132}$$

in the case $l = 1$ and

$$J(\psi \circ \tilde{\gamma}_{\mathcal{S}_\infty}) = -\begin{pmatrix} 0 & e^{\frac{pii}{6}} \\ -e^{-\frac{pii}{6}} & 0 \end{pmatrix} J(\psi) \begin{pmatrix} 0 & -e^{\frac{pii}{6}} \\ e^{-\frac{pii}{6}} & 0 \end{pmatrix} \tag{8.4.133}$$

in the case $l = \infty$.

Using the identities

$$-\begin{pmatrix} 0 & e^{-\frac{pii}{6}} \\ -e^{\frac{pii}{6}} & 0 \end{pmatrix} X \begin{pmatrix} 0 & -e^{-\frac{pii}{6}} \\ e^{\frac{pii}{6}} & 0 \end{pmatrix} = (J \circ \mathcal{A}_{\mathcal{S}_0} \circ J^{-1})(X) \text{ for all } X \in \mathrm{su}(2), \tag{8.4.134}$$

$$-\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} X \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = (J \circ \mathcal{A}_{\mathcal{S}_1} \circ J^{-1})(X) \text{ for all } X \in \mathrm{su}(2), \tag{8.4.135}$$

$$-\begin{pmatrix} 0 & e^{\frac{pii}{6}} \\ -e^{-\frac{pii}{6}} & 0 \end{pmatrix} X \begin{pmatrix} 0 & -e^{\frac{pii}{6}} \\ e^{-\frac{pii}{6}} & 0 \end{pmatrix} = (J \circ \mathcal{A}_{\mathcal{S}_\infty} \circ J^{-1})(X) \text{ for all } X \in \mathrm{su}(2), \tag{8.4.136}$$

where

$$\mathcal{A}_{\mathcal{S}_0} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{8.4.137}$$

$$\mathcal{A}_{\mathcal{S}_1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{8.4.138}$$

$$\mathcal{A}_{\mathcal{S}_\infty} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{8.4.139}$$

from the proof of corollary 8.8, we switch into the $\mathbb{R}^3$ model and obtain $\psi \circ \tilde{\gamma}_{\mathcal{S}_l} = \mathcal{A}_{\mathcal{S}_l} \circ \psi$. As $\tilde{\gamma}_{\mathcal{S}_l}(\tilde{M}) = \tilde{M}$, this yields in each case ($l \in \{0, 1, \infty\}$)

$$\psi(\tilde{M}) = \mathcal{A}_{\mathcal{S}_l}(\psi(\tilde{M})). \tag{8.4.140}$$

This means that, for a given $l \in \{0, 1, \infty\}$, ψ (and thus also ϕ) is symmetric with respect to the Euclidean motion $\mathcal{A}_{\mathcal{S}_l} \in \mathrm{Iso}(\mathbb{R}^3)$ defining a reflection in $\mathbb{R}^3$. In view of theorem 4.31, which lists all possible trinoid symmetries, we observe that only the reflection in the trinoid normal plane E_l shows the behaviour of the reflection $\mathcal{A}_{\mathcal{S}_l}$ (concerning the permutation of the trinoid ends), which can be read off the associated bi-antiholomorphic mapping $\tilde{\gamma}_{\mathcal{S}_l} : \tilde{M} \to \tilde{M}$, or, more precisely, its permutation behaviour of the trinoid singularities. Thus, ϕ is necessarily reflectionally symmetric with respect to the trinoid normal plane E_l, as claimed. □

9 Rotoreflectional symmetry with respect to the trinoid normal

9.1 Definition

Finally, in this section we discuss trinoids $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends on $M = \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ which are symmetric in the sense of definition 4.2 with respect to the rotoreflection $\hat{\mathcal{S}}$ composed of the rotation $\mathcal{R}$ by the angle $\pm\frac{2\pi}{3}$ around the trinoid normal (as studied in section 5) and the reflection $\mathcal{S}$ in the trinoid plane (as studied in section 7), $\hat{\mathcal{S}} = \mathcal{S} \circ \mathcal{R}$. Recall that, in the case that a trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends is symmetric with respect to the given Euclidean motion $\hat{\mathcal{S}}$, there exists a unique trinoid plane and a unique trinoid normal of ϕ, which enables us to speak of *the* trinoid plane and *the* trinoid normal of ϕ, respectively.)

$\hat{\mathcal{S}}$ reverses orientation and permutes the trinoid ends according to the permutation $(0\ 1\ \infty)$ of the set $\{0, 1, \infty\}$. Moreover, since we have

$$\hat{\mathcal{S}}(\phi(M)) = \phi(M) \quad \Longleftrightarrow \quad \hat{\mathcal{S}}^{-1}(\phi(M)) = \phi(M), \tag{9.1.1}$$

we note right away that a given trinoid $\phi : M \to \mathbb{R}^3$ with properly embedded annular ends is symmetric with respect to $\hat{\mathcal{S}}$ if and only if it is symmetric with respect to $\hat{\mathcal{S}}^{-1} = \mathcal{R}^{-1} \circ \mathcal{S}^{-1} = \mathcal{S} \circ \mathcal{R}^{-1}$, defining the rotoreflection composed of the rotation $\mathcal{R}^{-1}$ by the angle $\mp\frac{2\pi}{3}$ around the trinoid normal (as studied in section 5) and the reflection $\mathcal{S}$ in the trinoid plane (as studied in section 7). (Note the fact that $\mathcal{S}^{-1} = \mathcal{S}$ and $\mathcal{R}^{-1} \circ \mathcal{S} = \mathcal{S} \circ \mathcal{R}^{-1}$.) $\hat{\mathcal{S}}^{-1}$ reverses orientation and permutes the trinoid ends according to the permutation $(0\ \infty\ 1)$ of the set $\{0, 1, \infty\}$.

Definition 9.1. Let $M = \mathbb{C} \setminus \{0, 1\}$ and $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends. Let $\tilde{M} = \mathbb{H}$ and $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$ the conformal CMC-immersion associated with ϕ via the universal covering $\pi : \tilde{M} \to M$ given in (3.2.2). Let $A_n = \{C + \lambda n; \lambda \in \mathbb{R}\}$, where C denotes the trinoid center and n a normal vector of the trinoid plane E, be the trinoid normal. Then, if ϕ (or, equivalently, ψ) is symmetric with respect to the rotoreflection $\hat{\mathcal{S}}$, composed of the rotation $\mathcal{R}$ by the angle $\pm\frac{2\pi}{3}$ around the trinoid normal and the reflection $\mathcal{S}$ in the trinoid plane, $\hat{\mathcal{S}} = \mathcal{S} \circ \mathcal{R}$, and permuting the trinoid ends according to the permutation $\sigma = (0\ 1\ \infty)$ of the set $\{0, 1, \infty\}$,

$$\hat{\mathcal{S}}(\phi(M)) = \phi(M), \quad \hat{\mathcal{S}}(\psi(\tilde{M})) = \psi(\tilde{M}), \tag{9.1.2}$$

or, equivalently, if ϕ (or, equivalently, ψ) is symmetric with respect to the inverse rotoreflection $\hat{\mathcal{S}}^{-1}$,

$$\hat{\mathcal{S}}^{-1}(\phi(M)) = \phi(M), \quad \hat{\mathcal{S}}^{-1}(\psi(\tilde{M})) = \psi(\tilde{M}), \tag{9.1.3}$$

ϕ (or ψ) is called *rotoreflectionally symmetric with respect to the trinoid normal.*

Again, we are interested in translating this symmetry property into further constraints on the monodromy matrices associated with the extended frame F of ψ.

9.2 Implications of rotoreflectional symmetry with respect to the trinoid normal

The following result is an immediate consequence of definition 9.1:

Lemma 9.2. *Let $M = \mathbb{C} \setminus \{0, 1\}$ and $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends produced from a trinoid potential η as in theorem 3.14. Denote by D_0, D_1, D_∞ the corresponding Delaunay matrices with eigenvalues $\pm\mu_0$, $\pm\mu_1$, $\pm\mu_\infty$, respectively, where, for $j \in \{0, 1, \infty\}$, $\mu_j = \sqrt{X_j \overline{X_j}} = \sqrt{\frac{1}{4} + w_j(\lambda - \lambda^{-1})^2}$ and $w_j = s_j t_j$ as in section 3.5. Then, if ϕ is rotoreflectionally symmetric with respect to the trinoid normal, we have*

$$\mu := \mu_0 = \mu_1 = \mu_\infty = \sqrt{\frac{1}{4} + w(\lambda - \lambda^{-1})^2}, \tag{9.2.1}$$

where

$$w := w_0 = w_1 = w_\infty. \tag{9.2.2}$$

Proof. By definition 9.1, the ends of a trinoid with properly embedded annular ends, which is rotoreflectionally symmetric with respect to the trinoid normal, are mapped by the corresponding symmetry $\hat{\mathcal{S}}$ (resp. $\hat{\mathcal{S}}^{-1}$) into each other according to the permutation $\sigma = (0\ 1\ \infty)$ (resp. $\sigma^{-1} = (0\ \infty\ 1)$). This means that the asymptotic Delaunay surfaces associated with the ends are mapped onto each other as well. Hence, these Delaunay surfaces only differ by a rigid motion on $\mathbb{R}^3$. In particular, this implies that the corresponding Delaunay matrices D_j, $j = 0, 1, \infty$, (see section 3.5 for more details) all possess the same eigenvalues. This yields $\mu_0 = \mu_1 = \mu_\infty$ and allows for defining $\mu := \mu_0 = \mu_1 = \mu_\infty$. Using lemma B.6, we infer that $w_0 = w_1 = w_\infty$, whence w given in (9.2.2) is well defined. Consequently, $\mu = \sqrt{\frac{1}{4} + w(\lambda - \lambda^{-1})^2}$ holds. This finishes the proof. □

Let $M = \mathbb{C} \setminus \{0, 1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ given in (3.2.2). Suppose ϕ (or, equivalently, ψ) is rotoreflectionally symmetric with respect to the trinoid normal, and denote the corresponding symmetry by $\hat{\mathcal{S}}$. Since $\hat{\mathcal{S}}$ reverses orientation on $\mathbb{R}^3$, we obtain by theorem 4.9 a pair of bi-antiholomorphic mappings, $\gamma_{\hat{\mathcal{S}}} : M \to M$ and $\tilde{\gamma}_{\hat{\mathcal{S}}} : \tilde{M} \to \tilde{M}$ satisfying

$$\hat{\mathcal{S}} \circ \phi = \phi \circ \gamma_{\hat{\mathcal{S}}}, \tag{9.2.3}$$

$$\hat{\mathcal{S}} \circ \psi = \psi \circ \tilde{\gamma}_{\hat{\mathcal{S}}}, \tag{9.2.4}$$

$$\pi \circ \tilde{\gamma}_{\hat{\mathcal{S}}} = \gamma_{\hat{\mathcal{S}}} \circ \pi. \tag{9.2.5}$$

Analogously, we obtain for $\hat{\mathcal{S}}^{-1}$ a pair of bi-antiholomorphic mappings, $\gamma_{\hat{\mathcal{S}}^{-1}} : M \to M$ and $\tilde{\gamma}_{\hat{\mathcal{S}}^{-1}} : \tilde{M} \to \tilde{M}$ satisfying

$$\hat{\mathcal{S}}^{-1} \circ \phi = \phi \circ \gamma_{\hat{\mathcal{S}}^{-1}}, \tag{9.2.6}$$

$$\hat{\mathcal{S}}^{-1} \circ \psi = \psi \circ \tilde{\gamma}_{\hat{\mathcal{S}}^{-1}}, \tag{9.2.7}$$

$$\pi \circ \tilde{\gamma}_{\hat{\mathcal{S}}^{-1}} = \gamma_{\hat{\mathcal{S}}^{-1}} \circ \pi. \tag{9.2.8}$$

The mappings $\gamma_{\hat{\mathcal{S}}}$ and $\gamma_{\hat{\mathcal{S}}^{-1}}$ are uniquely determined and explicitly given by lemma 4.21:

$$\gamma_{\hat{\mathcal{S}}}(z) = \frac{1}{1 - \bar{z}}, \tag{9.2.9}$$

$$\gamma_{\hat{\mathcal{S}}^{-1}}(z) = \frac{\bar{z} - 1}{\bar{z}}. \tag{9.2.10}$$

The mappings $\tilde{\gamma}_{\hat{\mathcal{S}}}$ and $\tilde{\gamma}_{\hat{\mathcal{S}}^{-1}}$ are uniquely determined up to composition from the left with an element of the automorphism group $\mathrm{Aut}(\tilde{M}/M)$ of π. The following lemma explicitly states a pair of valid choices for $\tilde{\gamma}_{\hat{\mathcal{S}}}$ and $\tilde{\gamma}_{\hat{\mathcal{S}}^{-1}}$:

Lemma 9.3. *Let $M = \mathbb{C} \setminus \{0, 1\}$, $\tilde{M} = \mathbb{H}$ and $\pi : \tilde{M} \to M$ be the universal covering as given in* (3.2.2)*. Let $\gamma_{\hat{\mathcal{S}}} : M \to M$ and $\gamma_{\hat{\mathcal{S}}^{-1}} : M \to M$ be given by* (9.2.9) *and* (9.2.10)*, respectively. Then, the following holds:*

1. *The mapping $\tilde{\gamma}_{\hat{\mathcal{S}}} : \tilde{M} \to \tilde{M}$,*

$$\tilde{\gamma}_{\hat{\mathcal{S}}}(z) = \frac{\bar{z} + 1}{\bar{z}}, \tag{9.2.11}$$

is bi-antiholomorphic and satisfies

$$\pi \circ \tilde{\gamma}_{\hat{\mathcal{S}}} = \gamma_{\hat{\mathcal{S}}} \circ \pi, \tag{9.2.12}$$

$$\hat{\mathcal{S}} \circ \psi = \psi \circ \tilde{\gamma}_{\hat{\mathcal{S}}}. \tag{9.2.13}$$

2. *The mapping $\tilde{\gamma}_{\hat{\mathcal{S}}^{-1}} : \tilde{M} \to \tilde{M}$,*

$$\tilde{\gamma}_{\hat{\mathcal{S}}^{-1}}(z) = \frac{1}{\bar{z} - 1}, \tag{9.2.14}$$

is bi-antiholomorphic and satisfies

$$\pi \circ \tilde{\gamma}_{\hat{\mathcal{S}}^{-1}} = \gamma_{\hat{\mathcal{S}}^{-1}} \circ \pi, \tag{9.2.15}$$

$$\hat{\mathcal{S}}^{-1} \circ \psi = \psi \circ \tilde{\gamma}_{\hat{\mathcal{S}}^{-1}}. \tag{9.2.16}$$

Proof. We start with proving the first claim. A direct computation shows that $\tilde{\gamma}_{\hat{\mathcal{S}}}$ is a bi-antiholomorphic mapping $\tilde{M} \to \tilde{M}$. Moreover, by applying the relations (3.2.10), (3.2.11) and (3.2.12) of lemma 3.4, we obtain for all $z \in \tilde{M}$

$$\pi \circ \tilde{\gamma}_{\hat{\mathcal{S}}}(z) = \pi\left(\frac{\bar{z}+1}{\bar{z}}\right) = \pi\left(1+\frac{1}{\bar{z}}\right) = \frac{1}{\pi\left(\frac{1}{\bar{z}}\right)} = \frac{1}{1-\pi(-\bar{z})} = \frac{1}{1-\overline{\pi(z)}} = \gamma_{\hat{\mathcal{S}}} \circ \pi(z), \tag{9.2.17}$$

i.e. $\pi \circ \tilde{\gamma}_{\hat{\mathcal{S}}} = \gamma_{\hat{\mathcal{S}}} \circ \pi$. Consequently,

$$\hat{\mathcal{S}} \circ \psi = \hat{\mathcal{S}} \circ \phi \circ \pi = \phi \circ \gamma_{\hat{\mathcal{S}}} \circ \pi = \phi \circ \pi \circ \tilde{\gamma}_{\hat{\mathcal{S}}} = \psi \circ \tilde{\gamma}_{\hat{\mathcal{S}}}, \tag{9.2.18}$$

i.e. $\hat{\mathcal{S}} \circ \psi = \psi \circ \tilde{\gamma}_{\hat{\mathcal{S}}}$.

Now we turn to the second claim. A direct computation shows that $\tilde{\gamma}_{\hat{\mathcal{S}}^{-1}}$ is a bi-antiholomorphic mapping $\tilde{M} \to \tilde{M}$. Moreover, by applying the relations (3.2.10), (3.2.11) and (3.2.12) of lemma 3.4, we obtain for all $z \in \tilde{M}$

$$\pi \circ \tilde{\gamma}_{\hat{\mathcal{S}}^{-1}}(z) = \pi\left(\frac{1}{\bar{z}-1}\right) = 1-\pi(1-\bar{z}) = 1-\frac{1}{\pi(-\bar{z})} = 1-\frac{1}{\overline{\pi(z)}} = \frac{\overline{\pi(z)}-1}{\overline{\pi(z)}} = \gamma_{\hat{\mathcal{S}}^{-1}} \circ \pi(z), \tag{9.2.19}$$

i.e. $\pi \circ \tilde{\gamma}_{\hat{\mathcal{S}}^{-1}} = \gamma_{\hat{\mathcal{S}}^{-1}} \circ \pi$. Consequently,

$$\hat{\mathcal{S}}^{-1} \circ \psi = \hat{\mathcal{S}}^{-1} \circ \phi \circ \pi = \phi \circ \gamma_{\hat{\mathcal{S}}^{-1}} \circ \pi = \phi \circ \pi \circ \tilde{\gamma}_{\hat{\mathcal{S}}^{-1}} = \psi \circ \tilde{\gamma}_{\hat{\mathcal{S}}^{-1}}, \tag{9.2.20}$$

i.e. $\hat{\mathcal{S}}^{-1} \circ \psi = \psi \circ \tilde{\gamma}_{\hat{\mathcal{S}}^{-1}}$. □

Remark 9.4. Note that, since $\tilde{\gamma}_{\hat{\mathcal{S}}} \circ \tilde{\gamma}_{\hat{\mathcal{S}}^{-1}} = \tilde{\gamma}_{\hat{\mathcal{S}}^{-1}} \circ \tilde{\gamma}_{\hat{\mathcal{S}}} = \mathrm{id}$ for the mappings $\tilde{\gamma}_{\hat{\mathcal{S}}}$ and $\tilde{\gamma}_{\hat{\mathcal{S}}^{-1}}$ defined in (9.2.11) and (9.2.14), respectively, we have

$$\tilde{\gamma}_{\hat{\mathcal{S}}^{-1}} = \tilde{\gamma}_{\hat{\mathcal{S}}}^{-1}. \tag{9.2.21}$$

By the above lemma, we have explicitly determined mappings $\tilde{\gamma}_{\hat{\mathcal{S}}}$ and $\tilde{\gamma}_{\hat{\mathcal{S}}^{-1}}$ corresponding to the trinoid symmetries $\hat{\mathcal{S}}$ and $\hat{\mathcal{S}}^{-1}$, respectively, in the sense of theorem 4.9. Thus, we can apply theorem 4.17 to obtain

Theorem 9.5. *Let $M = \mathbb{C} \setminus \{0,1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ as defined in (3.2.2). Let ϕ be rotoreflectionally symmetric with respect to the trinoid normal. Denote by $\hat{\mathcal{S}}$ and $\hat{\mathcal{S}}^{-1}$ the corresponding symmetries permuting the trinoid ends according to the permutations $\sigma = (0\ 1\ \infty)$ and $\sigma^{-1} = (0\ \infty\ 1)$, respectively. Moreover, denote by $\tilde{\gamma}_{\hat{\mathcal{S}}}$ and by $\tilde{\gamma}_{\hat{\mathcal{S}}^{-1}}$ the bi-antiholomorphic mappings $\tilde{M} \to \tilde{M}$ associated with $\hat{\mathcal{S}}$ and $\hat{\mathcal{S}}^{-1}$, respectively, as in theorem 4.9 and explicitly given in lemma 9.3. Then, the following holds:*

1. *The extended frame $F : \tilde{M} \to \Lambda \mathrm{SU}(2)_\sigma$ corresponding to ψ by theorem 4.5 transforms under $\tilde{\gamma}_{\hat{\mathcal{S}}}$ as*

$$F(\tilde{\gamma}_{\hat{\mathcal{S}}}(z), \lambda^{-1}) = M_{\hat{\mathcal{S}}}(\lambda)\overline{F(z,\lambda)}k_{\hat{\mathcal{S}},\tilde{\gamma}_{\hat{\mathcal{S}}}}(z), \tag{9.2.22}$$

where

$$k_{\hat{\mathcal{S}},\tilde{\gamma}_{\hat{\mathcal{S}}}}(z) = \begin{pmatrix} \overline{\sqrt{-\frac{z}{\bar{z}}}} & 0 \\ 0 & \sqrt{-\frac{z}{\bar{z}}} \end{pmatrix}. \tag{9.2.23}$$

and $M_{\hat{\mathcal{S}}}(\lambda)$ denotes an element of $\Lambda \mathrm{SU}(2)_\sigma$, which is independent of z.

2. *The extended frame $F : \tilde{M} \to \Lambda \mathrm{SU}(2)_\sigma$ corresponding to ψ by theorem 4.5 transforms under $\tilde{\gamma}_{\hat{\mathcal{S}}^{-1}}$ as*

$$F(\tilde{\gamma}_{\hat{\mathcal{S}}^{-1}}(z), \lambda^{-1}) = M_{\hat{\mathcal{S}}^{-1}}(\lambda)\overline{F(z,\lambda)}k_{\hat{\mathcal{S}}^{-1},\tilde{\gamma}_{\hat{\mathcal{S}}^{-1}}}(z), \tag{9.2.24}$$

where

$$k_{\hat{\mathcal{S}}^{-1},\tilde{\gamma}_{\hat{\mathcal{S}}^{-1}}}(z) = \begin{pmatrix} \overline{\sqrt{\frac{z-1}{\bar{z}-1}}} & 0 \\ 0 & \sqrt{\frac{z-1}{\bar{z}-1}} \end{pmatrix}, \tag{9.2.25}$$

and $M_{\hat{\mathcal{S}}^{-1}}$ denotes an element of $\Lambda \mathrm{SU}(2)_\sigma$, which is independent of z.

Proof. We start with the proof of the first part. Let $\tilde{\gamma}(z) = \tilde{\gamma}_{\hat{\mathcal{S}}}(z) = \frac{\bar{z}+1}{\bar{z}}$ for all $z \in \tilde{M} = \mathbb{H}$. (For convenience we omit the index $\hat{\mathcal{S}}$ throughout this proof.) As $\hat{\mathcal{S}}$ reverses orientation on $\mathbb{R}^3$, we apply the second part of theorem 4.17 to obtain

$$F(\tilde{\gamma}(z), \lambda^{-1}) = M_{\tilde{\gamma}}(\lambda)\overline{F(z, \lambda)}k_{\hat{\mathcal{S}},\tilde{\gamma}}(z), \tag{9.2.26}$$

where $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ denotes the extended frame corresponding to ψ by theorem 4.5 and $M_{\tilde{\gamma}}$ denotes an element of $\Lambda\mathrm{SU}(2)_\sigma$, which is independent of z. $k_{\hat{\mathcal{S}},\tilde{\gamma}}(z)$ is given by equation (4.4.118) from lemma 4.18. By computing

$$\partial_{\bar{z}}\tilde{\gamma}(z) = -\frac{1}{\bar{z}^2} \tag{9.2.27}$$

we infer that

$$\frac{\partial_{\bar{z}}\tilde{\gamma}(z)}{\mid \partial_{\bar{z}}\tilde{\gamma}(z) \mid} = -\frac{\mid \bar{z} \mid^2}{\bar{z}^2} = -\frac{z}{\bar{z}} \tag{9.2.28}$$

and thus obtain from (4.4.118)

$$k_{\hat{\mathcal{S}},\tilde{\gamma}}(z) = \begin{pmatrix} \overline{\sqrt{-\frac{z}{\bar{z}}}} & 0 \\ 0 & \sqrt{-\frac{z}{\bar{z}}} \end{pmatrix}. \tag{9.2.29}$$

As $\tilde{\gamma} = \tilde{\gamma}_{\hat{\mathcal{S}}}$, we denote $M_{\tilde{\gamma}}$ by $M_{\hat{\mathcal{S}}}$. This finishes the proof of equation (9.2.22).

To prove the second part of the theorem, we define $\tilde{\gamma}(z) = \tilde{\gamma}_{\hat{\mathcal{S}}^{-1}}(z) = \frac{1}{\bar{z}-1}$ on $\tilde{M} = \mathbb{H}$. Everything is then done analogously. We have

$$\partial_{\bar{z}}\tilde{\gamma}(z) = \frac{1}{(\bar{z}-1)^2} \tag{9.2.30}$$

and thus

$$\frac{\partial_{\bar{z}}\tilde{\gamma}(z)}{\mid \partial_{\bar{z}}\tilde{\gamma}(z) \mid} = \frac{\mid \bar{z}-1 \mid^2}{(\bar{z}-1)^2} = \frac{z-1}{\bar{z}-1}. \tag{9.2.31}$$

Formula (4.4.118) from lemma 4.18 then yields

$$k_{\hat{\mathcal{S}}^{-1},\tilde{\gamma}}(z) = \begin{pmatrix} \overline{\sqrt{\frac{z-1}{\bar{z}-1}}} & 0 \\ 0 & \sqrt{\frac{z-1}{\bar{z}-1}} \end{pmatrix}, \tag{9.2.32}$$

and by setting $M_{\hat{\mathcal{S}}^{-1}}(\lambda) := M_{\tilde{\gamma}}(\lambda)$, the second part of theorem 4.17 implies (9.2.24). □

9.3 Monodromy matrices of trinoids with properly embedded annular ends, which are rotoreflectionally symmetric with respect to the trinoid normal

With the results of the previous section we are now able to describe the (unitary) monodromy matrices $\hat{M}_0, \hat{M}_1, \hat{M}_\infty$ associated with a trinoid with properly embedded annular ends, which is rotoreflectionally symmetric with respect to the trinoid normal. As a start, recall from section 3.3 the covering transformations $\tilde{\gamma}_j$, $j = 0, 1, \infty$, on $\tilde{M}$ generating the monodromy matrices $\hat{M}_j$, $j = 0, 1, \infty$, respectively:

$$\tilde{\gamma}_0(z) = \frac{z}{-2z+1}, \quad \tilde{\gamma}_1(z) = z+2, \quad \tilde{\gamma}_\infty(z) = \frac{-3z-2}{2z+1}. \tag{9.3.1}$$

The corresponding inverse functions are given by

$$\tilde{\gamma}_0^{-1}(z) = \frac{z}{2z+1}, \quad \tilde{\gamma}_1^{-1}(z) = z-2, \quad \tilde{\gamma}_\infty^{-1}(z) = \frac{z+2}{-2z-3}. \tag{9.3.2}$$

Lemma 9.6. *Let $\tilde{M} = \mathbb{H}$ and $\tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\gamma}_\infty : \tilde{M} \to \tilde{M}$ be given as above. Then, for the bi-antiholomorphic mapping $\tilde{\gamma}_{\hat{\mathcal{S}}} : \tilde{M} \to \tilde{M}, \tilde{\gamma}_{\hat{\mathcal{S}}}(z) = \frac{\bar{z}+1}{\bar{z}}$ the following identities hold:*

$$\tilde{\gamma}_{\hat{\mathcal{S}}} \circ \tilde{\gamma}_0 = \tilde{\gamma}_1^{-1} \circ \tilde{\gamma}_{\hat{\mathcal{S}}}, \quad \tilde{\gamma}_{\hat{\mathcal{S}}} \circ \tilde{\gamma}_1 = \tilde{\gamma}_1 \circ \tilde{\gamma}_0 \circ \tilde{\gamma}_{\hat{\mathcal{S}}}, \quad \tilde{\gamma}_{\hat{\mathcal{S}}} \circ \tilde{\gamma}_\infty = \tilde{\gamma}_0^{-1} \circ \tilde{\gamma}_{\hat{\mathcal{S}}}. \tag{9.3.3}$$

Proof. The claim is proved by straightforward computation: For $z \in \tilde{M}$ we have

$$\tilde{\gamma}_{\hat{\mathcal{S}}} \circ \tilde{\gamma}_0(z) = \frac{\frac{\bar{z}}{-2\bar{z}+1} + 1}{\frac{\bar{z}}{-2\bar{z}+1}} = \frac{\bar{z} - 2\bar{z} + 1}{\bar{z}} = \frac{\bar{z}+1}{\bar{z}} - 2\tilde{\gamma}_1^{-1} \circ \tilde{\gamma}_{\hat{\mathcal{S}}}(z) \tag{9.3.4}$$

$$\tilde{\gamma}_{\hat{\mathcal{S}}} \circ \tilde{\gamma}_1(z) = \frac{\bar{z}+3}{\bar{z}+2} = \frac{-\bar{z}-3}{-\bar{z}-2} = \tilde{\gamma}_1\left(\frac{\bar{z}+1}{-\bar{z}-2}\right) = \tilde{\gamma}_1\left(\frac{\frac{\bar{z}+1}{\bar{z}}}{-2\frac{\bar{z}+1}{\bar{z}}+1}\right) = \tilde{\gamma}_1 \circ \tilde{\gamma}_0 \circ \tilde{\gamma}_{\hat{\mathcal{S}}}(z) \tag{9.3.5}$$

$$\tilde{\gamma}_{\hat{\mathcal{S}}} \circ \tilde{\gamma}_\infty(z) = \frac{\frac{-3\bar{z}-2}{2\bar{z}+1} + 1}{\frac{-3\bar{z}-2}{2\bar{z}+1}} = \frac{3\bar{z}+2-2\bar{z}-1}{3\bar{z}+2} = \frac{\frac{\bar{z}+1}{\bar{z}}}{2\frac{\bar{z}+1}{\bar{z}}+1}\tilde{\gamma}_0^{-1} \circ \tilde{\gamma}_{\hat{\mathcal{S}}}(z). \tag{9.3.6}$$

□

The above lemma is needed to prove the following theorem, which states further necessary conditions on the monodromy matrices of the extended frame F associated with a trinoid with properly embedded annular ends, which is rotoreflectionally symmetric with respect to the trinoid normal.

Theorem 9.7. *Let $M = \mathbb{C} \setminus \{0,1\}$, $\phi : M \to \mathbb{R}^3$ be a trinoid with properly embedded annular ends and ψ the associated conformal CMC-immersion on $\tilde{M} = \mathbb{H}$, $\psi = \phi \circ \pi : \tilde{M} \to \mathbb{R}^3$, where π denotes the universal covering $\tilde{M} \to M$ as defined in* (3.2.2)*. Let ϕ be rotoreflectionally symmetric with respect to the trinoid normal. Denote by $\hat{\mathcal{S}}$ the corresponding symmetry permuting the trinoid ends according to the permutation $\sigma = (0\ 1\ \infty)$. Furthermore, let $F : \tilde{M} \to \Lambda\mathrm{SU}(2)_\sigma$ be the extended frame associated with ψ by theorem 4.5. Denote by $\hat{M}_0, \hat{M}_1, \hat{M}_\infty \in \Lambda\mathrm{SU}(2,\mathbb{C})_\sigma$ the unitary monodromy matrices*

$$\hat{M}_j = -\left[\cos(2\pi\mu_j)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i\sin(2\pi\mu_j)\begin{pmatrix}p_j & \overline{q_j}\\ q_j & -p_j\end{pmatrix}\right] \tag{9.3.7}$$

associated with F as in (4.5.13) *by*

$$F(\tilde{\gamma}_j(z),\lambda) = \alpha_j \hat{M}_j(\lambda) F(z,\lambda) k_j(z), \quad j = 0,1,\infty, \tag{9.3.8}$$

where $\alpha_j \in \{\pm 1\}$ and $\tilde{\gamma}_j$ denote the covering transformations on $\tilde{M}$ from section 3.3. Finally, let $\tilde{\gamma}_{\hat{\mathcal{S}}}$ be the bi-antiholomorphic mapping $\tilde{M} \to \tilde{M}$ associated with $\hat{\mathcal{S}}$ as in theorem 4.9 and explicitly given in lemma 9.3, and let $M_{\hat{\mathcal{S}}}(\lambda)$ be the corresponding monodromy matrix of F as given in equation (9.2.22)*. We set*

$$M_{\hat{\mathcal{S}}}(\lambda) = \pm(M_{\hat{\mathcal{S}}^{-1}}(\lambda))^{-1} =: \begin{pmatrix} a_{\hat{\mathcal{S}}} & b_{\hat{\mathcal{S}}} \\ -\overline{b_{\hat{\mathcal{S}}}} & \overline{a_{\hat{\mathcal{S}}}}\end{pmatrix}. \tag{9.3.9}$$

Then, the monodromy matrices satisfy

$$M_{\hat{\mathcal{S}}}(\lambda)\overline{\hat{M}_0(\lambda)} = (\hat{M}_1(\lambda^{-1}))^{-1} M_{\hat{\mathcal{S}}}(\lambda), \tag{9.3.10}$$

$$M_{\hat{\mathcal{S}}}(\lambda)\overline{\hat{M}_\infty(\lambda)} = (\hat{M}_0(\lambda^{-1}))^{-1} M_{\hat{\mathcal{S}}}(\lambda), \tag{9.3.11}$$

$$M_{\hat{\mathcal{S}}}(\lambda)\overline{\hat{M}_1(\lambda)} = (\hat{M}_1(\lambda^{-1}))(\hat{M}_0(\lambda^{-1})) M_{\hat{\mathcal{S}}}(\lambda). \tag{9.3.12}$$

In terms of the functions p_j and q_j occurring in $\hat{M}_j$, equations (9.3.10) *to* (9.3.12) *are equivalent to*

$$a_{\hat{\mathcal{S}}} p_0(\lambda) + b_{\hat{\mathcal{S}}}\overline{q_0(\lambda)} = a_{\hat{\mathcal{S}}} p_1(\lambda^{-1}) - \overline{b_{\hat{\mathcal{S}}}}\,\overline{q_1(\lambda^{-1})}, \tag{9.3.13}$$

$$a_{\hat{\mathcal{S}}} q_0(\lambda) - b_{\hat{\mathcal{S}}} p_0(\lambda) = b_{\hat{\mathcal{S}}} p_1(\lambda^{-1}) + \overline{a_{\hat{\mathcal{S}}}}\,\overline{q_1(\lambda^{-1})}, \tag{9.3.14}$$

$$a_{\hat{\mathcal{S}}} p_\infty(\lambda) + b_{\hat{\mathcal{S}}}\overline{q_\infty(\lambda)} = a_{\hat{\mathcal{S}}} p_0(\lambda^{-1}) - \overline{b_{\hat{\mathcal{S}}}}\,\overline{q_0(\lambda^{-1})}, \tag{9.3.15}$$

$$a_{\hat{\mathcal{S}}} q_\infty(\lambda) - b_{\hat{\mathcal{S}}} p_\infty(\lambda) = b_{\hat{\mathcal{S}}} p_0(\lambda^{-1}) + \overline{a_{\hat{\mathcal{S}}}}\,\overline{q_0(\lambda^{-1})}. \tag{9.3.16}$$

Proof. Like in the proof of theorem 7.5, we make use of the following fact, a direct consequence of (9.3.8):

$$F(\tilde{\gamma}_j^{-1}(z),\lambda) = \alpha_j \hat{M}_j(\lambda)^{-1} F(z,\lambda)(k_j(\tilde{\gamma}_j^{-1}(z)))^{-1}. \tag{9.3.17}$$

Consider the bi-antiholomorphic mapping $\tilde{\gamma}_{\hat{\mathcal{S}}} : \tilde{M} \to \tilde{M}$ given in (9.2.11): $\tilde{\gamma}_{\hat{\mathcal{S}}}(z) = \frac{-z-1}{z}$. Applying theorem 9.5, we obtain

$$F(\tilde{\gamma}_{\hat{\mathcal{S}}}(z),\lambda^{-1}) = M_{\hat{\mathcal{S}}}(\lambda)\overline{F(z,\lambda)} k_{\hat{\mathcal{S}},\tilde{\gamma}_{\hat{\mathcal{S}}}}(z), \tag{9.3.18}$$

where

$$k_{\hat{\mathcal{S}},\tilde{\gamma}_{\hat{\mathcal{S}}}}(z) = \begin{pmatrix} \sqrt{-\frac{z}{\bar{z}}} & 0 \\ 0 & \sqrt{-\frac{z}{\bar{z}}} \end{pmatrix}. \tag{9.3.19}$$

Combining this with the monodromy equations (9.3.8) and (9.3.17) and applying the identities (9.3.3) from the above lemma, we deduce

$$\begin{aligned} M_{\hat{\mathcal{S}}}(\lambda)\alpha_0\overline{\hat{M}_0(\lambda)}\overline{F(z,\lambda)k_0(z)}k_{\hat{\mathcal{S}},\tilde{\gamma}_{\hat{\mathcal{S}}}}(\tilde{\gamma}_0(z)) &= M_{\hat{\mathcal{S}}}(\lambda)\overline{F(\tilde{\gamma}_0(z),\lambda)}k_{\hat{\mathcal{S}},\tilde{\gamma}_{\hat{\mathcal{S}}}}(\tilde{\gamma}_0(z)) \\ = F(\tilde{\gamma}_{\hat{\mathcal{S}}}(\tilde{\gamma}_0(z)),\lambda^{-1}) = F(\tilde{\gamma}_1^{-1}(\tilde{\gamma}_{\hat{\mathcal{S}}}(z)),\lambda^{-1}) &= \alpha_1(\hat{M}_1(\lambda^{-1}))^{-1}F(\tilde{\gamma}_{\hat{\mathcal{S}}}(z),\lambda^{-1})(k_1(\tilde{\gamma}_1^{-1}(\tilde{\gamma}_{\hat{\mathcal{S}}}(z))))^{-1} \\ &= \alpha_1(\hat{M}_1(\lambda^{-1}))^{-1}M_{\hat{\mathcal{S}}}(\lambda)\overline{F(z,\lambda)}k_{\hat{\mathcal{S}},\tilde{\gamma}_{\hat{\mathcal{S}}}}(z)(k_1(\tilde{\gamma}_1^{-1}(\tilde{\gamma}_{\hat{\mathcal{S}}}(z))))^{-1}, \end{aligned} \tag{9.3.20}$$

$$\begin{aligned} M_{\hat{\mathcal{S}}}(\lambda)\alpha_\infty\overline{\hat{M}_\infty(\lambda)}\overline{F(z,\lambda)k_\infty(z)}k_{\hat{\mathcal{S}},\tilde{\gamma}_{\hat{\mathcal{S}}}}(\tilde{\gamma}_\infty(z)) &= M_{\hat{\mathcal{S}}}(\lambda)\overline{F(\tilde{\gamma}_\infty(z),\lambda)}k_{\hat{\mathcal{S}},\tilde{\gamma}_{\hat{\mathcal{S}}}}(\tilde{\gamma}_\infty(z)) \\ = F(\tilde{\gamma}_{\hat{\mathcal{S}}}(\tilde{\gamma}_\infty(z)),\lambda^{-1}) = F(\tilde{\gamma}_0^{-1}(\tilde{\gamma}_{\hat{\mathcal{S}}}(z)),\lambda^{-1}) &= \alpha_0(\hat{M}_0(\lambda^{-1}))^{-1}F(\tilde{\gamma}_{\hat{\mathcal{S}}}(z),\lambda^{-1})(k_0(\tilde{\gamma}_0^{-1}(\tilde{\gamma}_{\hat{\mathcal{S}}}(z))))^{-1} \\ &= \alpha_0(\hat{M}_0(\lambda^{-1}))^{-1}M_{\hat{\mathcal{S}}}(\lambda)\overline{F(z,\lambda)}k_{\hat{\mathcal{S}},\tilde{\gamma}_{\hat{\mathcal{S}}}}(z)(k_0(\tilde{\gamma}_0^{-1}(\tilde{\gamma}_{\hat{\mathcal{S}}}(z))))^{-1}. \end{aligned} \tag{9.3.21}$$

We continue by computing (due to the occurring complex roots up to sign)

$$\begin{aligned} \overline{k_0(z)}k_{\hat{\mathcal{S}},\tilde{\gamma}_{\hat{\mathcal{S}}}}(\tilde{\gamma}_0(z)) &= \begin{pmatrix} \sqrt{\frac{1-2\bar{z}}{1-2z}} & 0 \\ 0 & \frac{1}{\sqrt{\frac{1-2\bar{z}}{1-2z}}} \end{pmatrix} \begin{pmatrix} \sqrt{-\frac{\frac{z}{-2z+1}}{\frac{\bar{z}}{-2\bar{z}+1}}} & 0 \\ 0 & \sqrt{-\frac{\frac{z}{-2z+1}}{\frac{\bar{z}}{-2\bar{z}+1}}} \end{pmatrix} \\ &= \pm \begin{pmatrix} \sqrt{-\frac{(1-2z)\frac{z}{1-2z}}{(1-2\bar{z})\frac{\bar{z}}{1-2\bar{z}}}} & 0 \\ 0 & \sqrt{-\frac{(1-2z)\frac{z}{1-2z}}{(1-2\bar{z})\frac{\bar{z}}{1-2\bar{z}}}} \end{pmatrix} = \pm \begin{pmatrix} \sqrt{-\frac{z}{\bar{z}}} & 0 \\ 0 & \sqrt{-\frac{z}{\bar{z}}} \end{pmatrix} = \pm k_{\hat{\mathcal{S}},\tilde{\gamma}_{\hat{\mathcal{S}}}}(z)(k_1(\tilde{\gamma}_1^{-1}(\tilde{\gamma}_{\hat{\mathcal{S}}}(z))))^{-1} \end{aligned} \tag{9.3.22}$$

and

$$\begin{aligned} \overline{k_\infty(z)}k_{\hat{\mathcal{S}},\tilde{\gamma}_{\hat{\mathcal{S}}}}(\tilde{\gamma}_\infty(z)) &= \begin{pmatrix} \sqrt{\frac{1+2\bar{z}}{1+2z}} & 0 \\ 0 & \frac{1}{\sqrt{\frac{1+2\bar{z}}{1+2z}}} \end{pmatrix} \begin{pmatrix} \sqrt{-\frac{\frac{-3z-2}{2z+1}}{\frac{-3\bar{z}-2}{2\bar{z}+1}}} & 0 \\ 0 & \sqrt{-\frac{\frac{-3z-2}{2z+1}}{\frac{-3\bar{z}-2}{2\bar{z}+1}}} \end{pmatrix} \\ &= \pm \begin{pmatrix} \sqrt{-\frac{(1+2z)\frac{-3z-2}{1+2z}}{(1+2\bar{z})\frac{-3\bar{z}-2}{1+2\bar{z}}}} & 0 \\ 0 & \sqrt{-\frac{(1+2z)\frac{-3z-2}{1+2z}}{(1+2\bar{z})\frac{-3\bar{z}-2}{1+2\bar{z}}}} \end{pmatrix} = \pm \begin{pmatrix} \sqrt{\frac{-3z-2}{3\bar{z}+2}} & 0 \\ 0 & \sqrt{\frac{-3z-2}{3\bar{z}+2}} \end{pmatrix} \\ &= \pm \begin{pmatrix} \sqrt{-\frac{z\frac{3\bar{z}+2-2\bar{z}-2}{3\bar{z}+2}}{\bar{z}\frac{3z+2-2z-2}{3z+2}}} & 0 \\ 0 & \sqrt{-\frac{z\frac{3\bar{z}+2-2\bar{z}-2}{3\bar{z}+2}}{\bar{z}\frac{3z+2-2z-2}{3z+2}}} \end{pmatrix} = \pm \begin{pmatrix} \sqrt{-\frac{z}{\bar{z}}} & 0 \\ 0 & \sqrt{-\frac{z}{\bar{z}}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1-2\frac{z+1}{3z+2}}{1-2\frac{\bar{z}+1}{3\bar{z}+2}}} & 0 \\ 0 & \frac{1}{\sqrt{\frac{1-2\frac{z+1}{3z+2}}{1-2\frac{\bar{z}+1}{3\bar{z}+2}}}} \end{pmatrix} \\ &= \pm k_{\hat{\mathcal{S}},\tilde{\gamma}_{\hat{\mathcal{S}}}}(z)(k_0(\tilde{\gamma}_0^{-1}(\tilde{\gamma}_{\hat{\mathcal{S}}}(z))))^{-1}. \end{aligned} \tag{9.3.23}$$

Combining these results with the equations above, we obtain

$$M_{\hat{\mathcal{S}}}(\lambda)\overline{\hat{M}_0(\lambda)} = \beta_0(\hat{M}_1(\lambda^{-1}))^{-1}M_{\hat{\mathcal{S}}}(\lambda), \tag{9.3.24}$$

$$M_{\hat{\mathcal{S}}}(\lambda)\overline{\hat{M}_\infty(\lambda)} = \beta_1(\hat{M}_0(\lambda^{-1}))^{-1}M_{\hat{\mathcal{S}}}(\lambda) \tag{9.3.25}$$

with $\beta_0, \beta_1 \in \{\pm 1\}$. This can be reformulated as

$$(\hat{M}_1(\lambda^{-1}))^{-1} = \beta_0 M_{\hat{\mathcal{S}}}(\lambda)\overline{\hat{M}_0(\lambda)}(M_{\hat{\mathcal{S}}}(\lambda))^{-1}, \tag{9.3.26}$$

$$(\hat{M}_0(\lambda^{-1}))^{-1} = \beta_1 M_{\hat{\mathcal{S}}}(\lambda)\overline{\hat{M}_\infty(\lambda)}(M_{\hat{\mathcal{S}}}(\lambda))^{-1}. \tag{9.3.27}$$

Comparing the upper left entries as well as the lower right entries of both sides in each of these equations, we obtain

$$
\begin{aligned}
&-\cos(2\pi\mu_1) + i\sin(2\pi\mu_1)p_1(\lambda^{-1}) = \\
&\qquad \beta_0 \Big[-\cos(2\pi\mu_0) + i\sin(2\pi\mu_0)(a_{\hat{\mathcal{S}}}\overline{a_{\hat{\mathcal{S}}}}p_0(\lambda) + \overline{a_{\hat{\mathcal{S}}}}b_{\hat{\mathcal{S}}}\overline{q_0(\lambda)} + a_{\hat{\mathcal{S}}}\overline{b_{\hat{\mathcal{S}}}}q_0(\lambda) - b_{\hat{\mathcal{S}}}\overline{b_{\hat{\mathcal{S}}}}p_0(\lambda))\Big], \quad (9.3.28)
\end{aligned}
$$

$$
\begin{aligned}
&-\cos(2\pi\mu_1) - i\sin(2\pi\mu_1)p_1(\lambda^{-1}) = \\
&\qquad \beta_0 \Big[-\cos(2\pi\mu_0) - i\sin(2\pi\mu_0)(a_{\hat{\mathcal{S}}}\overline{a_{\hat{\mathcal{S}}}}p_0(\lambda) + \overline{a_{\hat{\mathcal{S}}}}b_{\hat{\mathcal{S}}}\overline{q_0(\lambda)} + a_{\hat{\mathcal{S}}}\overline{b_{\hat{\mathcal{S}}}}q_0(\lambda) - b_{\hat{\mathcal{S}}}\overline{b_{\hat{\mathcal{S}}}}p_0(\lambda))\Big], \quad (9.3.29)
\end{aligned}
$$

$$
\begin{aligned}
&-\cos(2\pi\mu_0) + i\sin(2\pi\mu_0)p_0(\lambda^{-1}) = \\
&\qquad \beta_1 \Big[-\cos(2\pi\mu_\infty) + i\sin(2\pi\mu_\infty)(a_{\hat{\mathcal{S}}}\overline{a_{\hat{\mathcal{S}}}}p_\infty(\lambda) + \overline{a_{\hat{\mathcal{S}}}}b_{\hat{\mathcal{S}}}\overline{q_\infty(\lambda)} + a_{\hat{\mathcal{S}}}\overline{b_{\hat{\mathcal{S}}}}q_\infty(\lambda) - b_{\hat{\mathcal{S}}}\overline{b_{\hat{\mathcal{S}}}}p_\infty(\lambda))\Big], \quad (9.3.30)
\end{aligned}
$$

$$
\begin{aligned}
&-\cos(2\pi\mu_0) - i\sin(2\pi\mu_0)p_0(\lambda^{-1}) = \\
&\qquad \beta_1 \Big[-\cos(2\pi\mu_\infty) - i\sin(2\pi\mu_\infty)(a_{\hat{\mathcal{S}}}\overline{a_{\hat{\mathcal{S}}}}p_\infty(\lambda) + \overline{a_{\hat{\mathcal{S}}}}b_{\hat{\mathcal{S}}}\overline{q_\infty(\lambda)} + a_{\hat{\mathcal{S}}}\overline{b_{\hat{\mathcal{S}}}}q_\infty(\lambda) - b_{\hat{\mathcal{S}}}\overline{b_{\hat{\mathcal{S}}}}p_\infty(\lambda))\Big], \quad (9.3.31)
\end{aligned}
$$

respectively. By summing up the first two equations and recalling $\mu_1 = \mu_0$, we conclude that β_0 necessarily equals $+1$. Analogously, by summing up the other two equations and recalling that $\mu_0 = \mu_\infty$, we deduce $\beta_1 = +1$. Therefore,

$$M_{\hat{\mathcal{S}}}(\lambda)\overline{\hat{M}_0(\lambda)} = (\hat{M}_1(\lambda^{-1}))^{-1}M_{\hat{\mathcal{S}}}(\lambda), \quad (9.3.32)$$

$$M_{\hat{\mathcal{S}}}(\lambda)\overline{\hat{M}_\infty(\lambda)} = (\hat{M}_0(\lambda^{-1}))^{-1}M_{\hat{\mathcal{S}}}(\lambda), \quad (9.3.33)$$

proving (9.3.10) and (9.3.11). Finally, we compute in view of (3.9.32)

$$M_{\hat{\mathcal{S}}}(\lambda)\overline{\hat{M}_1(\lambda)} = M_{\hat{\mathcal{S}}}(\lambda)\overline{(\hat{M}_0(\lambda))^{-1}(\hat{M}_\infty(\lambda))^{-1}} = \hat{M}_1(\lambda^{-1})\hat{M}_0(\lambda^{-1})M_{\hat{\mathcal{S}}}(\lambda), \quad (9.3.34)$$

proving (9.3.12). Since by use of (3.9.32) equation (9.3.12) is implied by equations (9.3.10) and (9.3.11), all three equations are equivalent to the scalar reformulations of the equations (9.3.10) and (9.3.11), which read

$$
\begin{aligned}
&-\cos(2\pi\mu_0)a_{\hat{\mathcal{S}}} + i\sin(2\pi\mu_0)(a_{\hat{\mathcal{S}}}p_0(\lambda) + b_{\hat{\mathcal{S}}}\overline{q_0(\lambda)}) \\
&\qquad\qquad = -\cos(2\pi\mu_1)a_{\hat{\mathcal{S}}} + i\sin(2\pi\mu_1)(a_{\hat{\mathcal{S}}}p_1(\lambda^{-1}) - \overline{b_{\hat{\mathcal{S}}}}\overline{q_1(\lambda^{-1})}), \quad (9.3.35)
\end{aligned}
$$

$$
\begin{aligned}
&-\cos(2\pi\mu_0)b_{\hat{\mathcal{S}}} + i\sin(2\pi\mu_0)(a_{\hat{\mathcal{S}}}q_0(\lambda) - b_{\hat{\mathcal{S}}}p_0(\lambda)) \\
&\qquad\qquad = -\cos(2\pi\mu_1)b_{\hat{\mathcal{S}}} + i\sin(2\pi\mu_1)(b_{\hat{\mathcal{S}}}p_1(\lambda^{-1}) + \overline{a_{\hat{\mathcal{S}}}}\overline{q_1(\lambda^{-1}})) \quad (9.3.36)
\end{aligned}
$$

and

$$
\begin{aligned}
&-\cos(2\pi\mu_\infty)a_{\hat{\mathcal{S}}} + i\sin(2\pi\mu_\infty)(a_{\hat{\mathcal{S}}}p_\infty(\lambda) + b_{\hat{\mathcal{S}}}\overline{q_\infty(\lambda)}) \\
&\qquad\qquad = -\cos(2\pi\mu_0)a_{\hat{\mathcal{S}}} + i\sin(2\pi\mu_0)(a_{\hat{\mathcal{S}}}p_0(\lambda^{-1}) - \overline{b_{\hat{\mathcal{S}}}}\overline{q_0(\lambda^{-1})}), \quad (9.3.37)
\end{aligned}
$$

$$
\begin{aligned}
&-\cos(2\pi\mu_\infty)b_{\hat{\mathcal{S}}} + i\sin(2\pi\mu_\infty)(a_{\hat{\mathcal{S}}}q_\infty(\lambda) - b_{\hat{\mathcal{S}}}p_\infty(\lambda)) \\
&\qquad\qquad = -\cos(2\pi\mu_0)b_{\hat{\mathcal{S}}} + i\sin(2\pi\mu_0)(b_{\hat{\mathcal{S}}}p_0(\lambda^{-1}) + \overline{a_{\hat{\mathcal{S}}}}\overline{q_0(\lambda^{-1}}), \quad (9.3.38)
\end{aligned}
$$

respectively. A straightforward simplification of these equations (using again $\mu_0 = \mu_1 = \mu_\infty$) yields the claimed ones and finishes the proof. □

Remark 9.8. Since the bi-antiholomorphic mapping $\tilde{\gamma}_{\hat{\mathcal{S}}} : \tilde{M} \to \tilde{M}$, $z \mapsto \tilde{\gamma}_{\hat{\mathcal{S}}}(z) = \frac{\bar{z}+1}{\bar{z}}$ does not possess any fixed points in $\tilde{M} = \mathbb{H}$, there exists – unlike in the preceding sections dealing with the other possible trinoid symmetries – no similar choice for a special "basepoint" for the extended frame of a given trinoid with properly embedded annular ends, which is rotoreflectionally symmetric with respect to the trinoid normal, which would lead to more explicit constraints on the functions p_j, q_j occurring in the monodromy matrices associated with such a trinoid. Consequently, there will be no further section dealing with "normalized" trinoids with properly embedded annular ends, which are rotoreflectionally symmetric with respect to the trinoid normal.

A Appendix: Basic Topology

The main goal of this appendix is to explicate the relation between the fundamental group $\pi_1(X, x)$ of a (path-connected) topological space X and the automorphism group $\mathrm{Aut}(Y/X)$ of the universal covering $\pi : Y \to X$ of X. We briefly review some basic topology in section A.1, before introducing the concepts of the fundamental group and the automorphism group in sections A.2 and A.3, respectively. Both concepts are then related to each other in section A.4.

Throughout this appendix, we follow the book of Fulton ([20]), to which we refer the reader for more details.

A.1 Topological spaces, continuous mappings and paths

A *topological space* is a set X together with a collection $\mathcal{M}$ of subsets of X with the following properties: $\mathcal{M}$ contains the empty set and X itself, as well as any union and any finite intersection of elements of $\mathcal{M}$. $\mathcal{M}$ is called a *topology* (on X) and its elements are called *open sets* (in X). A subset N of X is called a *neighborhood* of a point x in X if it includes an open set M containing x.

A topological space X is called *Hausdorff (space)* if any two different points in X can be seperated by two disjoint open sets, one containing one of the points, the other one containing the other.

A mapping $f : X \to Y$ between two topological spaces X and Y is called *continuous* if the preimage under f of any open set in Y is open in X. Obviously, the composition $g \circ f$ of two continuous mappings $f : X \to Y$ and $g : y \to Z$ is again continuous. An equivalent characterization of a continuous mapping, which it is often more convenient to work with, is the following: A mapping $f : X \to Y$ between two topological spaces X and Y is continuous if and only if there exists for every $x \in X$ and every neighborhood U' of $f(x)$ in Y a neighborhood U of x in X, such that $f(U) \subseteq U'$.

A continuous mapping $\gamma : [0,1] \to X$ defines a *path* from $\gamma(0)$ to $\gamma(1)$ (in X). A path $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1)$ is called a *loop* (based at $\gamma(0)$). If $\gamma : [0,1] \to X$ satifies $\gamma(t) = x$ for some fixed $x \in X$ and for all $t \in [0,1]$, we speak of a *constant loop* (based at x), which we often denote by ϵ_x. Moreover, for a path $\gamma : [0,1] \to X$ we denote by γ^{-1} the *inverse path* which traverses γ "backwards" from $\gamma(1)$ to $\gamma(0)$. Strictly speaking, we define $\gamma^{-1} : [0,1] \to X$ by $\gamma^{-1}(t) := \gamma(1-t)$. Finally, given two paths $\gamma_1, \gamma_2 : [0,1] \to X$ satisfying $\gamma_1(1) = \gamma_2(0)$, it is appropriate to define by $\gamma_1 \cdot \gamma_2 : [0,1] \to X$, $(\gamma_1 \cdot \gamma_2)(t) = \gamma_1(2t)$ for $t \in [0, \frac{1}{2}]$ and $(\gamma_1 \cdot \gamma_2)(t) = \gamma_2(2t-1)$ for $t \in [\frac{1}{2}, 1]$ the *product path* $\gamma_1 \cdot \gamma_2$ traversing first γ_1 from $\gamma_1(0)$ to $\gamma_1(1)$ and γ_2 from $\gamma_2(0) = \gamma_1(1)$ to $\gamma_2(1)$ afterwards, both with twice the original "speed".

A.2 The fundamental group

For two paths $\gamma_1, \gamma_2 : [0,1] \to X$ in a topological space X with common endpoints $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$ a continuous mapping $H : [0,1] \times [0,1] \to X$ satisfying $H(t,0) = \gamma_1(t)$ for all $t \in [0,1]$, $H(t,1) = \gamma_2(t)$ for all $t \in [0,1]$, $H(0,s) = \gamma_1(0) = \gamma_2(0)$ for all $s \in [0,1]$ and $H(1,s) = \gamma_1(1) = \gamma_2(1)$ for all $s \in [0,1]$ is called a *homotopy from γ_1 to γ_2* (with fixed endpoints). If such a homotopy exists, γ_1 and γ_2 are called *homotopic.*

Lemma A.1. *Let X be a topological space and $x, x' \in X$. The homotopy relation on $\{\gamma : [0,1] \to X \text{ path}; \gamma(0) = x, \gamma(1) = x'\}$ given by*

$$\gamma_1 \sim \gamma_2 :\Longleftrightarrow \quad \gamma_1 \text{ and } \gamma_2 \text{ are homotopic} \tag{A.2.1}$$

is an equivalence relation.

Proof. For any path $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = x'$, the mapping $H : [0,1] \times [0,1] \to X$, $H(t,s) := \gamma(t)$ defines a homotopy from γ to γ itself, which proves the reflexivity of $\sim$. Suppose now $\gamma_1 \sim \gamma_2$ for two paths $\gamma_1, \gamma_2 : [0,1] \to X$ from x to x'. Then there exists a homotopy $H : [0,1] \times [0,1] \to X$ from γ_1 to γ_2. By defining $\hat{H} : [0,1] \times [0,1] \to X$, $\hat{H}(t,s) := H(t, 1-s)$ we obtain a homotopy from γ_2 to γ_1. Thus we have $\gamma_2 \sim \gamma_1$, which proves the symmetry of the relation $\sim$. Finally, let $\gamma_1 \sim \gamma_2$ and $\gamma_2 \sim \gamma_3$ for paths γ_1,γ_2,γ_3 from x to x' and let $H_1 : [0,1] \times [0,1] \to X$ and $H_2 : [0,1] \times [0,1] \to X$ denote the homotopies from γ_1 to γ_2 and from γ_2 to γ_3, respectively. Then $H : [0,1] \times [0,1] \to X$, $H(t,s) := H_1(t, 2s)$ for $s \in [0, \frac{1}{2}]$ and $H(t,s) := H_2(t, 2s-1)$ for $s \in [\frac{1}{2}, 1]$ defines a homotopy from γ_1 to γ_3, which means $\gamma_1 \sim \gamma_3$ and proves the transitivity of the relation. Altogether, $\sim$ defines an equivalence relation on $\{\gamma : [0,1] \to X \text{ path}; \gamma(0) = x, \gamma(1) = x'\}$. $\square$

Definition A.2. Let X be a topological space and $x \in X$. Furthermore, denote for any loop γ based at x by $[\gamma]$ the equivalence class of γ with respect to the homotopy relation on the set of all loops based at x. Then, the set

$$\pi_1(X, x) := \{[\gamma]; \gamma \text{ loop based at } x\} \tag{A.2.2}$$

is called the *fundamental group of X with base point x.*

The use of the term "group" for the fundamental group is justified by the following result:

Lemma A.3. *Let X be a topological space and $x \in X$. The group operation of the fundamental group $\pi_1(X, x)$ is given by*

$$[\gamma_1] \cdot [\gamma_2] := [\gamma_1 \cdot \gamma_2], \tag{A.2.3}$$

where $\gamma_1 \cdot \gamma_2$ denotes the path product of the loops γ_1 and γ_2.

Proof. This is explicated in detail in section 12a of [20]. □

A topological space X is called *connected* if it cannot be written as a union of two disjoint non-empty open sets (in X). Furthermore, X is called *path-connected* if any two points x and x' in X can be joined by a path $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = x'$. As we will need them soon, we give some more definitions concerning the 'connectedness" of a topological space: A topological space X is called *locally path-connected* if any neighborhood of any point x in X contains a path-connected neighborhood of x. Furthermore, a path-connected topological space X is called *simply connected* if its fundamental group is trivial, i.e. if any loop based at a point x in X is homotopic to the constant loop based at x. X is called *semilocally simply connected* if any point x in X possesses a neighborhood N, such that all loops in N are homotopic to a constant loop.

In the case of a path-connected topological space X, we have the following result:

Lemma A.4. *Let X be a path-connected topological space and $x, x' \in X$. Then, the fundamental groups of X with base points x and x', respectively, are isomorphic via*

$$\pi_1(X, x) \to \pi_1(X, x'), \quad [\gamma] \mapsto [\delta^{-1} \cdot \gamma \cdot \delta], \tag{A.2.4}$$

where δ denotes a path from x to x'.

Proof. Also this proof can be found in section 12a of [20]. □

As a consequence of the above lemma, it is convenient to speak of "the fundamental group" of a path-connected topological space X without specifying a basepoint in X.

A.3 The automorphism group

A bijective mapping $f : X \to Y$ between topological spaces X and Y, such that f and its inverse function f^{-1} are continuous, is called a *homeomorphism.* If such a mapping exists, X and Y are called *homeomorphic.*

Definition A.5. Let X and Y be topological spaces. A continuous mapping $\pi : Y \to X$ is called a *covering of* X, if there exists for any point in X a neighborhood N, such that the preimage $\pi^{-1}(N)$ of N can be written as a (possibly infinite) disjoint union of open sets M_i in Y with the property that the restriction of π to any of the M_i defines a homeomorphism $M_i \to N$. In this case Y is called the *covering space* of X (with regard to π). The neighborhood N involved is called *evenly covered* by π. If $\pi : Y \to X$ is a covering of X and Y is simply connected, π is called the *universal covering* of X and Y is called the *universal cover* of X.

We state the following result concerning the uniqueness and existence of the universal cover of a connected and locally path-connected topological space.

Theorem A.6. *(Corollary 13.6 and Theorem 13.20 of [20]) Let X be a connected and locally path-connected topological space. The universal covering $\pi : Y \to X$ of X, if it exists, is uniquely determined (up to an isomorphic change of the covering space). Furthermore, a universal covering does exist if and only if X is semilocally simply connected.*

Now we are set to define the so-called automorphism group associated with a covering of a topological space X.

Definition A.7. Given a topological space X and a covering $\pi : Y \to X$, the set

$$\mathrm{Aut}(Y/X) := \{\tilde{\sigma} : Y \to Y; \tilde{\sigma} \text{ homeomorphism}, \pi \circ \tilde{\sigma} = \pi\} \tag{A.3.1}$$

is called the *automorphism group of* π. (The group operation is given by the composition of functions.) The elements of $\mathrm{Aut}(Y/X)$ are called *covering transformations*.

We end this section by recording another useful result:

Theorem A.8. *(Propostion 11.38 of [20]) Let X be a topological space and $\pi : Y \to X$ a covering. If Y is connected, the automorphism group $\mathrm{Aut}(Y/X)$ of π acts properly discontinuously on Y, i.e. any point y in Y possesses a neighborhood N in Y, such that $\tilde{\sigma}_1(N) \cap \tilde{\sigma}_2(N) = \emptyset$ for any two covering transformations $\tilde{\sigma}_1, \tilde{\sigma}_2 \in \mathrm{Aut}(Y/X)$ with $\tilde{\sigma}_1 \neq \tilde{\sigma}_2$.*

A.4 The monodromy action of the fundamental group

We are now interested in relating the fundamental group $\pi_1(X, x)$ of a (connected, locally path-connected and semilocally simply connected) topological space X to the automorphism group $\mathrm{Aut}(Y/X)$ of the universal covering $\pi : Y \to X$ of X. More precisely, we want to associate with any homotopy class $[\gamma] \in \pi_1(X, x)$ a corresponding covering transformation on Y denoted by $z \mapsto [\gamma] \cdot z$.

We start by collecting some results from [20]:

Theorem A.9. *(Proposition 11.6 of [20]) Let X be a topological space and $\pi : Y \to X$ a covering. Furthermore, let $\gamma : [0, 1] \to X$ be a path and y a point in Y with $\pi(y) = \gamma(0)$. Then there exists a unique path $\tilde{\gamma} : [0, 1] \to Y$, such that $\pi \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = y$. $\tilde{\gamma}$ is called the (path) lift of γ.*

Theorem A.10. *(Proposition 11.8 of [20]) Let X be a topological space and $\pi : Y \to X$ a covering. Moreover, let $\gamma : [0, 1] \to X$ be a path with lift $\tilde{\gamma} : [0, 1] \to Y$. Suppose $H : [0, 1] \times [0, 1] \to X$ is a homotopy from γ to another path in X. Then, there exists a unique continuous mapping $\tilde{H} : [0, 1] \times [0, 1] \to Y$, such that $\pi \circ \tilde{H} = H$ and $\tilde{H}(t, 0) = \tilde{\gamma}(t)$ for all $t \in [0, 1]$. $\tilde{H}$ is called the (homotopy) lift of H.*

For a path $\gamma : [0, 1] \to X$ in a topological space X with covering $\pi : Y \to X$ we introduce the following notation: If y is a point in Y with $\pi(y) = \gamma(0)$, the endpoint of the (unique) lift of γ starting at y is denoted by $y * \gamma$.

Given $[\gamma] \in \pi_1(X, x)$ and a point z in Y, we construct a point $[\gamma] \cdot z$ as follows: First, we fix $y \in Y$, such that $\pi(y) = x$ and choose a path δ in Y connecting y and z. (As Y is simply connected and thus path-connected, such a δ does always exist.) Next, we consider the path product $\gamma \cdot (\pi \circ \delta)$ in X. By theorem A.9, this path can be (uniquely) lifted to a path in Y starting at y and ending at $y * (\gamma \cdot (\pi \circ \delta))$. We define the desired point $[\gamma] \cdot z$ to be this endpoint, i.e. we set

$$[\gamma] \cdot z := y * (\gamma \cdot (\pi \circ \delta)). \tag{A.4.1}$$

Note that replacing γ by any other loop $\gamma' \in [\gamma]$ will not change the right hand side in the above definition, i.e. $y * (\gamma \cdot (\pi \circ \delta)) = y * (\gamma' \cdot (\pi \circ \delta))$. The reason for this is that for the construction of $y * (\gamma \cdot (\pi \circ \delta))$ – as far as γ is concerned – only the endpoint of the lift of γ is important. But, by theorem A.10, the lifts of the homotopic loops γ and γ' are homotopic paths in Y and thus have the same endpoint.

In order to obtain for fixed $y \in Y$ and $[\gamma] \in \pi_1(X, x)$ a well defined mapping $z \mapsto [\gamma] \cdot z$, we need to show in addition that the point $y * (\gamma \cdot (\pi \circ \delta))$ as defined above is actually independent of the choice of the path δ connecting y and z. This is proved over the course of the next three lemmas.

Given a topological space X, a covering $\pi : Y \to X$ and a loop $\tilde{\gamma} : [0, 1] \to Y$ based at $y \in Y$, the composition $\pi \circ \tilde{\gamma}$ defines a loop in X based at $x = \pi(y)$. In fact, the following holds:

Lemma A.11. *(cf. proposition 13.1 of [20]) Let $\pi : Y \to X$ be a covering of a topological space X with $\pi(y) = x$. Then π induces a group homomorphism*

$$\pi_* : \pi_1(Y, y) \to \pi_1(X, x), \quad [\tilde{\gamma}] \mapsto \pi_*([\tilde{\gamma}]) := [\pi \circ \tilde{\gamma}] \tag{A.4.2}$$

between the fundamental groups $\pi_1(Y, y)$ and $\pi_1(X, x)$. Furthermore, π_* is injective.

Proof. First we show that the mapping π_* is well defined. To this end, let $[\tilde{\gamma}_1] = [\tilde{\gamma}_2]$, i.e. $\tilde{\gamma}_1 \sim \tilde{\gamma}_2$ for two loops $\tilde{\gamma}_1, \tilde{\gamma}_2$ based at y. Denoting by $\tilde{H} : [0, 1] \times [0, 1] \to Y$ the homotopy from $\tilde{\gamma}_1$ to $\tilde{\gamma}_2$, we prove that $H := \pi \circ \tilde{H}$ defines a homotopy from $\pi \circ \tilde{\gamma}_1$ to $\pi \circ \tilde{\gamma}_2$: As π and $\tilde{H}$ are continuous, H defines a continuous mapping $[0, 1] \times [0, 1] \to X$. Furthermore, we have $H(t, 0) = (\pi \circ \tilde{\gamma}_1)(t)$ and $H(t, 1) = (\pi \circ \tilde{\gamma}_2)(t)$ for all

$t \in [0,1]$, as well as $H(0,s) = \pi(\tilde{\gamma}_1(0)) = \pi(\tilde{\gamma}_2(0))$ and $H(1,s) = \pi(\tilde{\gamma}_1(1)) = \pi(\tilde{\gamma}_2(1))$ for all $s \in [0,1]$. This proves $(\pi \circ \tilde{\gamma}_1) \sim (\pi \circ \tilde{\gamma}_2)$ and thus $\pi_*([\tilde{\gamma}_1]) = \pi_*([\tilde{\gamma}_2])$, i.e. π_* is well defined. Next, we prove that π_* defines a group homomorphism: The neutral elements (with respect to the corresponding group operations) of the fundamental groups involved are given by the equivalence classes of the constant loops ϵ_y based at y and ϵ_x based at x, respectively. For these, we have $\pi_*([\epsilon_y]) = [\pi \circ \epsilon_y] = [\epsilon_{\pi(y)}] = [\epsilon_x]$. Furthermore, for $[\tilde{\gamma}_1]$ and $[\tilde{\gamma}_2]$ in $\pi_1(Y,y)$, we have $\pi_*([\tilde{\gamma}_1] \cdot [\tilde{\gamma}_2]) = \pi_*([\tilde{\gamma}_1 \cdot \tilde{\gamma}_2]) = [\pi \circ (\tilde{\gamma}_1 \cdot \tilde{\gamma}_2)] = [(\pi \circ \tilde{\gamma}_1) \cdot (\pi \circ \tilde{\gamma}_2)] = [\pi \circ \tilde{\gamma}_1] \cdot [\pi \circ \tilde{\gamma}_2] = \pi_*([\tilde{\gamma}_1]) \cdot \pi_*([\tilde{\gamma}_2])$. Together, this proves that π_* defines a group homomorphism. It remains to show that π_* is injective. To this end, let $[\tilde{\gamma}] \in \pi_1(Y,y)$ with $\pi_*([\tilde{\gamma}]) = [\epsilon_x]$, where ϵ_x denotes the constant loop based at $x \in X$. We need to prove $[\tilde{\gamma}] = [\epsilon_y]$, where ϵ_y denotes the constant loop based at $y \in Y$. By assumption, we have $(\pi \circ \tilde{\gamma}) \sim \epsilon_x$. Denote the corresponding homotopy by H and consider the (by lemma A.10) unique homotopy lift $\tilde{H}$, which in turn defines a homotopy from $\tilde{\gamma}$ (the unique lift of $\pi \circ \tilde{\gamma}$ based at y) to ϵ_y (the unique lift of ϵ_x based at y). But this implies $\tilde{\gamma} \sim \epsilon_y$ and therefore $[\tilde{\gamma}] = [\epsilon_y]$, which proves the claim and completes the proof of the lemma. □

The mapping π_* provides us with some useful results concerning paths in the covering space Y of a topological space X:

Lemma A.12. *Let X be a (connected, locally path-connected and semilocally simply connected) topological space and $\pi : Y \to X$ the universal covering of X.*

1. *Let γ be a loop based at $x \in X$ and $\tilde{\gamma}$ the lift of γ starting at $y \in Y$. Then:*

$$\tilde{\gamma} \text{ ends at } y \iff [\gamma] \in \pi_*(\pi_1(Y,y)). \tag{A.4.3}$$

2. *Let $x, x' \in X$ and γ_1, γ_2 be two paths in X from x to x'. Furthermore, let $y \in Y$ and $\tilde{\gamma}_1, \tilde{\gamma}_2$ be the unique lifts of γ_1, γ_2, respectively, to paths in Y starting at y. Noting that $\gamma_2 \cdot \gamma_1^{-1}$ defines a loop based at x, the following holds:*

$$\tilde{\gamma}_1, \tilde{\gamma}_2 \text{ have the same endpoint} \iff [\gamma_2 \cdot \gamma_1^{-1}] \in \pi_*(\pi_1(Y,y)). \tag{A.4.4}$$

Proof. We start with the proof of (A.4.3). There are two directions to show. On the one hand, if $\tilde{\gamma}$ ends at y, we have $[\tilde{\gamma}] \in \pi_1(Y,y)$ and thus $[\gamma] = [\pi \circ \tilde{\gamma}] = \pi_*([\tilde{\gamma}]) \in \pi_*(\pi_1(Y,y))$. On the other hand, if $[\gamma] \in \pi_*(\pi_1(Y,y))$, there exists a loop $\tilde{\gamma}'$ based at y satisfying $[\pi \circ \tilde{\gamma}] = [\gamma] = \pi_*([\tilde{\gamma}']) = [\pi \circ \tilde{\gamma}']$. This implies the existence of a homotopy H from $\pi \circ \tilde{\gamma}$ to $\pi \circ \tilde{\gamma}'$, which can be lifted to a homotopy $\tilde{H}$ from $\tilde{\gamma}$ to a lift of $\pi \circ \tilde{\gamma}'$ starting at y, which, by the uniqueness property of path lifts, has to be $\tilde{\gamma}'$ itself. Thus, $\tilde{\gamma}$ is homotopic to $\tilde{\gamma}'$ and therefore necessarily a loop based at y, i.e ending at y. Altogether, we have proved relation (A.4.3).

Applying (A.4.3) to $\gamma = \gamma_2 \cdot \gamma_1^{-1}$, the claimed relation (A.4.4) reduces to

$$\tilde{\gamma}_1, \tilde{\gamma}_2 \text{ have the same endpoint} \iff \tilde{\gamma} \text{ ends at } y, \tag{A.4.5}$$

where $\tilde{\gamma}$ now denotes the (unique) lift of $\gamma_2 \cdot \gamma_1^{-1}$ starting at y. We proof (A.4.5) in two steps. First, suppose $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ have the same endpoint, say y'. In this case, the (unique) lift $\tilde{\gamma}$ of $\gamma_2 \cdot \gamma_1^{-1}$ starting at y is given by the path product of the lift of γ_2 starting at y (i.e. $\tilde{\gamma}_2$) and the lift of γ_1^{-1} starting at y' (i.e. $\tilde{\gamma}_1^{-1}$). So, $\tilde{\gamma} = \tilde{\gamma}_2 \cdot \tilde{\gamma}_1^{-1}$, which means that $\tilde{\gamma}$ ends at y. This proves the first direction of (A.4.5). Now, suppose that the (unique) lift $\tilde{\gamma}$ of $\gamma_2 \cdot \gamma_1^{-1}$ starting at y also ends there. Furthermore, denote the endpoint of $\tilde{\gamma}_2$ by y'. Note that $\tilde{\gamma}$ can be written as the path product of the lift of γ_2 starting at y (i.e. $\tilde{\gamma}_2$) and the lift of γ_1^{-1} starting at y' and (by assumption) ending at y. Consequently, the inverse path of this second lift is a lift of γ_1 starting at y (and ending at y'). Actually, by the uniqueness property of path lifts, this has to be $\tilde{\gamma}_1$, which in turn necessarily has to end at y'. Thus, $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ have the same endpoint, which proves the second direction of (A.4.5) and finishes the proof of the lemma. □

Finally, we can prove the following result.

Lemma A.13. *Let X be a (connected, locally path-connected and semilocally simply connected) topological space and $\pi : Y \to X$ the universal covering of X. Furthermore let $[\gamma] \in \pi_1(X,x)$, $x \in X$ and $y \in Y$ with $\pi(y) = x$. Then, given a point $z \in Y$, for any two paths δ_1 and δ_2 connecting y and z in Y we have*

$$y * (\gamma \cdot (\pi \circ \delta_1)) = y * (\gamma \cdot (\pi \circ \delta_2)). \tag{A.4.6}$$

Proof. We need to show that the lifts of the paths $\gamma \cdot (\pi \circ \delta_1)$ and $\gamma \cdot (\pi \circ \delta_2)$ have the same endpoints. By making use of lemma A.12 and keeping in mind that Y is simply connected (and thus $\pi_1(Y, y) = \{[\epsilon_y]\}$, where ϵ_y denotes the constant loop based at y), this is equivalent to showing

$$[(\gamma \cdot (\pi \circ \delta_2)) \cdot (\gamma \cdot (\pi \circ \delta_1))^{-1}] \in \{[\epsilon_x]\}. \tag{A.4.7}$$

As $\delta_2 \cdot \delta_1^{-1}$ defines a loop based at y, which – as $\pi_1(Y, y) = \{[\epsilon_y]\}$ – is homotopic to the constant loop ϵ_y based at y, we have

$$\begin{aligned}[(\gamma \cdot (\pi \circ \delta_2)) \cdot (\gamma \cdot (\pi \circ \delta_1))^{-1}] &= [(\gamma \cdot (\pi \circ \delta_2)) \cdot (\pi \circ \delta_1)^{-1} \cdot \gamma^{-1}] = [\gamma \cdot (\pi \circ (\delta_2 \cdot \delta_1^{-1})) \cdot \gamma^{-1}] \\ &= [\gamma] \cdot \pi_*([\delta_2 \cdot \delta_1^{-1}]) \cdot [\gamma^{-1}] = [\gamma] \cdot \pi_*([\epsilon_y]) \cdot [\gamma^{-1}] = [\gamma] \cdot [\epsilon_x] \cdot [\gamma^{-1}] = [\epsilon_x] \in \{[\epsilon_x]\},\end{aligned} \tag{A.4.8}$$

where we made use of the fact that the path $\gamma \cdot \epsilon_x \cdot \gamma^{-1}$ is homotopic to the constant path ϵ_x based at x. So the claim is proved. □

We now resume our considerations concerning the relation between the fundamental group $\pi_1(X, x)$ and the automorphism group $\mathrm{Aut}(Y/X)$ of the universal covering $\pi : Y \to X$ of X. As explicated earlier, once we have chosen $y \in Y$ with $\pi(y) = x$, any element $[\gamma] \in \pi_1(X, x)$ allows for associating with any point $z \in Y$ the point $y * (\gamma \cdot (\pi \circ \delta))$, which depends on y, z, $[\gamma]$ and a path δ from y to z. However, by lemma A.13, the choice of δ will actually not affect the resulting point. Thus, keeping $y \in Y$ fixed, we can relate to any $[\gamma] \in \pi_1(X, x)$ the mapping $Y \to Y$, $z \mapsto [\gamma] \cdot z := y * (\gamma \cdot (\pi \circ \delta))$, only depending on $[\gamma]$. More precisely, one can prove the following theorem:

Theorem A.14. *(Corollary 13.15 of [20]) Let X be a connected, locally path-connected and semilocally simply connected topological space and $\pi : Y \to X$ the universal covering of X. Then, for a fixed $y \in Y$ with $\pi(y) = x$ and any $[\gamma] \in \pi_1(X, x)$ the mapping $Y \to Y$, $z \mapsto [\gamma] \cdot z := y * (\gamma \cdot (\pi \circ \delta))$, where δ denotes an arbitrary path in Y from y to z, defines a covering transformation on Y. Furthermore, the mapping*

$$\pi_1(X, x) \to \mathrm{Aut}(Y/X), \quad [\gamma] \mapsto (z \mapsto [\gamma] \cdot z), \tag{A.4.9}$$

defines an isomorphism between the fundamental group $\pi_1(X, x)$ and the automorphism group $\mathrm{Aut}(Y/X)$ *of π, i.e.*

$$\pi_1(X, x) \cong \mathrm{Aut}(Y/X). \tag{A.4.10}$$

Remark A.15. Assuming the premises of theorem A.14, by evaluating the correspondence between $\pi_1(X, x)$ and $\mathrm{Aut}(Y/X)$, it is easy to see that the automorphism group $\mathrm{Aut}(Y/X)$ of π acts transitively on $\pi^{-1}(x)$. Thus, $\pi : Y \to X$, defines a so called *G-covering*, i.e. the space X can be identified with the set Y/G of orbits under a given action of a group G on Y, whereby, in our case, $G = \mathrm{Aut}(Y/X)$. In particular, this implies

$$X \cong Y/\mathrm{Aut}(Y/X) \cong Y/\pi_1(X, x). \tag{A.4.11}$$

In view of this, we think of Y as of the disjoint union of so called *sheets* $\mathcal{F}_{\tilde{\sigma}}$, which are "indexed" by the elements $\tilde{\sigma}$ of the automorphism group of Y:

$$Y = \bigcup_{\tilde{\sigma} \in \mathrm{Aut}(Y/X)} \mathcal{F}_{\tilde{\sigma}}. \tag{A.4.12}$$

We define the "starting" sheet $\mathcal{F}_{\mathrm{id}}$ by choosing a subset $\mathcal{F}$ of Y containing y, such that $\pi|_{\mathcal{F}}$ defines a bijection $\mathcal{F} \to X$, and setting $\mathcal{F}_{\mathrm{id}} := \mathcal{F}$. All other sheets are then given by the images of $\mathcal{F}_{\mathrm{id}}$ under the elements of $\mathrm{Aut}(Y/X)$,

$$\mathcal{F}_{\tilde{\sigma}} = \tilde{\sigma}(\mathcal{F}_{\mathrm{id}}). \tag{A.4.13}$$

Furthermore, we note that each sheet $\mathcal{F}_{\tilde{\sigma}}$ can be identified bijectively with the covered space X via the restriction of the universal covering π to the particular sheet, $\pi|_{\mathcal{F}_{\tilde{\sigma}}} : \mathcal{F}_{\tilde{\sigma}} \to X$. The decomposition of Y into sheets as introduced above is referred to as a *tesselation* (of Y).

The proof of theorem A.14 is based on the following lemma (cf. section 13b of [20]), which additionally implies that the mapping $\pi_1(X, x) \times Y \to Y$, $([\gamma], z) \mapsto [\gamma] \cdot z$ defines an action of the fundamental group $\pi_1(X, x)$ of X on Y. This action is called the *monodromy action (of $\pi_1(X, x)$ on Y).*

Lemma A.16. *Let X be a connected, locally path-connected and semilocally simply connected topological space and $\pi : Y \to X$ the universal covering of X. For fixed y in Y and $[\gamma] \in \pi_1(X, x)$ define the mapping $z \mapsto [\gamma] \cdot z$ as in* (A.4.1).

1. *Let* $[\gamma_1], [\gamma_2] \in \pi_1(X, x)$*. Then for all* $z \in Y$ *we have*

$$([\gamma_1] \cdot [\gamma_2]) \cdot z = [\gamma_1] \cdot ([\gamma_2] \cdot z). \tag{A.4.14}$$

2. *Consider* $[\epsilon_x] \in \pi_1(X, x)$*, where* ϵ_x *denotes the constant loop based at* x *in* X*. For all* $z \in Y$ *we have*

$$[\epsilon_x] \cdot z = z. \tag{A.4.15}$$

Proof. We start with the proof of equation (A.4.14). For $[\gamma_1], [\gamma_2] \in \pi_1(X, x)$ and z in Y we have by definition

$$([\gamma_1] \cdot [\gamma_2]) \cdot z = ([\gamma_1 \cdot \gamma_2]) \cdot z = y * ((\gamma_1 \cdot \gamma_2) \cdot (\pi \circ \delta)), \tag{A.4.16}$$

where δ denotes a path in Y connecting y and z. As $(\gamma_1 \cdot \gamma_2) \cdot (\pi \circ \delta)$ is homotopic to $\gamma_1 \cdot (\gamma_2 \cdot (\pi \circ \delta))$, the corresponding lifts starting at y have the same endpoints, and hence we infer that

$$y * ((\gamma_1 \cdot \gamma_2) \cdot (\pi \circ \delta)) = y * (\gamma_1 \cdot (\gamma_2 \cdot (\pi \circ \delta))), \tag{A.4.17}$$

which implies

$$([\gamma_1] \cdot [\gamma_2]) \cdot z = y * (\gamma_1 \cdot (\gamma_2 \cdot (\pi \circ \delta))). \tag{A.4.18}$$

On the other hand we have

$$[\gamma_1] \cdot ([\gamma_2] \cdot z) = y * (\gamma_1 \cdot (\pi \circ \delta')) \tag{A.4.19}$$

for an arbitrary path δ' in Y connecting y and $[\gamma_2] \cdot z$. Choosing for δ' the lift of $\gamma_2 \cdot (\pi \circ \delta)$ starting at y, we obtain

$$y * (\gamma_1 \cdot (\pi \circ \delta')) = y * (\gamma_1 \cdot (\gamma_2 \cdot (\pi \circ \delta))) \tag{A.4.20}$$

and thus

$$[\gamma_1] \cdot ([\gamma_2] \cdot z) = y * (\gamma_1 \cdot (\gamma_2 \cdot (\pi \circ \delta))). \tag{A.4.21}$$

Altogether, equation (A.4.14) is proved.

Equation (A.4.15) can be proved by direct computation: With δ again denoting a path in Y from y to z we have

$$[\epsilon_x] \cdot z = y * (\epsilon_x \cdot (\pi \circ \delta)) = y * (\pi \circ \delta) = \delta(1) = z, \tag{A.4.22}$$

where we made use of the fact that the paths $\epsilon_x \cdot (\pi \circ \delta)$ and $\pi \circ \delta$ are homotopic and thus induce lifts with the same endpoint. This finishes the proof of the lemma. □

Proof of theorem A.14. To prove the fact that the mapping $z \mapsto [\gamma] \cdot z := y * (\gamma \cdot (\pi \circ \delta))$ defines a covering transformation on Y, we have to show that it is a continuous bijection with continuous inverse mapping satisfying $\pi([\gamma] \cdot z) = \pi(z)$. We start with the proof of the bijection property: Using the preceding lemma we observe that the mapping $z \mapsto [\gamma^{-1}] \cdot z$ associated with the homotopy class of the inverse path γ^{-1} of γ defines the inverse mapping of the studied mapping $z \mapsto [\gamma] \cdot z$:

$$[\gamma] \cdot ([\gamma^{-1}] \cdot z) = ([\gamma] \cdot [\gamma^{-1}]) \cdot z = [\gamma \cdot \gamma^{-1}] \cdot z = [\epsilon_x] \cdot z = z, \tag{A.4.23}$$

$$[\gamma^{-1}] \cdot ([\gamma] \cdot z) = ([\gamma^{-1}] \cdot [\gamma]) \cdot z = [\gamma^{-1} \cdot \gamma] \cdot z = [\epsilon_x] \cdot z = z. \tag{A.4.24}$$

This already proves that $z \mapsto [\gamma] \cdot z$ is a bijection. Next, we prove for $[\gamma] \in \pi_1(X, x)$ the continuity of the mapping $z \mapsto [\gamma] \cdot z$. To this end, let $z \in Y$ and U' be a neighborhood of $z' := [\gamma] \cdot z$ in Y. We need to find a neighborhood U of z which is mapped by $u \mapsto [\gamma] \cdot u$ into U'. Let N be an evenly covered neighborhood of $\pi(z) = \pi(z')$ in X. As X is locally path-connected, we can assume that N is path-connected. Let V and V' be the (disjoint) open sets in Y containing z and z', respectively, which are mapped by π homeomorphically onto N. Defining the open set $W' := U' \cap V'$ in Y, the path-connected neighborhood $\tilde{N} := \pi(W')$ of $\pi(z)$ in X and the open set $W := V \cap \pi^{-1}(\tilde{N})$ in Y, the restricted mappings $\pi|_W : W \to \tilde{N}$ and $\pi|_{W'} : W' \to \tilde{N}$ are again homeomorphisms. Now, set $U := W$ and let $u \in U$. We need to prove $[\gamma] \cdot u \in U'$. To this end, choose a path δ' in $\tilde{N}$ from $\pi(z)$ to $\pi(u)$ and denote its lift connecting z and u in U by $\tilde{\delta}'$. Then, we have

$$[\gamma] \cdot u = y * (\gamma \cdot (\pi \circ (\tilde{\delta} \cdot \tilde{\delta}'))), \tag{A.4.25}$$

where $\tilde{\delta}$ denotes a path connecting y and z in Y. (Note that thus $\tilde{\delta} \cdot \tilde{\delta}'$ defines a path connecting y and u in Y.) This can be transformed into

$$[\gamma] \cdot u = y * (\gamma \cdot (\pi \circ \tilde{\delta}) \cdot (\pi \circ \tilde{\delta}')) = y * ((\gamma \cdot (\pi \circ \tilde{\delta})) \cdot \delta') = ([\gamma] \cdot z) * \delta' = z' * \delta', \tag{A.4.26}$$

which means that $[\gamma] \cdot u$ is the endpoint of the lift of δ' to a path starting at $z' \in W'$. As δ' is a path in $\tilde{N}$, which is homeomorphic to W', this lift and in particular its endpoint $[\gamma] \cdot u$ lie in W'. Therefore, $[\gamma] \cdot u \in W' \subseteq U'$, which completes the proof of the continuity of the mapping $z \mapsto [\gamma] \cdot z$. Note that, as the arguments involved hold for arbitrary homotopy classes in $\pi_1(X, x)$, in particular also for $[\gamma^{-1}]$, we have also already proved the continuity of the inverse mapping $z \mapsto [\gamma^{-1}] \cdot z$. Finally, keeping in mind that for any z in Y the path $\gamma \cdot (\pi \circ \delta)$ runs from x to $\pi(z)$ and thus the point $y * (\gamma \cdot (\pi \circ \delta))$ is in $\pi^{-1}(\pi(z))$, we have

$$\pi([\gamma] \cdot z) = \pi(y * (\gamma \cdot (\pi \circ \delta))) = \pi(z), \tag{A.4.27}$$

which proves the last desired property. Altogether, the mapping $z \mapsto [\gamma] \cdot z := y * (\gamma \cdot (\pi \circ \delta))$ defines a covering transformation on Y.

It remains to prove that the mapping $\pi_1(X, x) \to \mathrm{Aut}(Y/X)$, $[\gamma] \mapsto (z \mapsto [\gamma] \cdot z)$ defines an isomorphism between $\pi_1(X, x)$ and $\mathrm{Aut}(Y/X)$. The fact that it defines a homomorphism between the groups involved is proved by the identity A.4.14, so we only need to show that the mapping is bijective. To this end, let $\tilde{\sigma} : Y \to Y$ be a covering transformation of π. Denote $\tilde{\sigma}(y)$ by y'. Choosing a path $\tilde{\gamma}_0$ in Y from y to y' and defining $\gamma_0 := \pi \circ \tilde{\gamma}$, we have

$$[\gamma_0] \cdot y = y * (\gamma_0 \cdot (\pi \circ \epsilon_y)) = y * (\gamma_0 \cdot \epsilon_x) = y * \gamma_0 = y', \tag{A.4.28}$$

where ϵ_y and ϵ_x denote the constant loops in Y (resp. X) based at y (resp. x). This means that the mapping $z \mapsto [\gamma_0] \cdot z$ also maps the point y to the point y'. By theorem A.8 this implies that the mappings $z \mapsto [\gamma_0] \cdot z$ and $\tilde{\sigma}$ coincide, i.e. $\tilde{\sigma}(z) = [\gamma_0] \cdot z$ for all z in Y, which proves that the homomorphism $\pi_1(X, x) \to \mathrm{Aut}(Y/X)$ is surjective. To prove that it is also injective, suppose $[\gamma_1] \in \pi_1(X, x)$ with $[\gamma_1] \cdot z = z$ for all z in Y, in particular $[\gamma_1] \cdot y = y$, which means

$$y = [\gamma_1] \cdot y = y * (\gamma_1 \cdot (\pi \circ \epsilon_y)) = y * (\gamma_1 \cdot \epsilon_x) = y * \gamma_1. \tag{A.4.29}$$

But this implies that the lift $\tilde{\gamma}_1$ of γ_1 to a path starting at y also ends there, which by lemma A.12 is equivalent to $[\gamma_1] \in \pi_*(\pi_1(Y, y))$. As Y is simply connected, we have $\pi_*(\pi_1(Y, y)) = \{[\epsilon_x]\}$ and therefore $[\gamma_1] = [\epsilon_x]$, which completes the proof of the injectivity. Altogether, the mapping $\pi_1(X, x) \to \mathrm{Aut}(Y/X)$, $[\gamma] \mapsto (z \mapsto [\gamma] \cdot z)$ defines an isomorphism between the fundamental group $\pi_1(X, x)$ and the automorphism group $\mathrm{Aut}(Y/X)$ of π. □

Remark A.17. In order to express the correspondence of the covering transformation $z \mapsto [\gamma] \cdot z$ to the loop γ in M involved, we denote this transformation by $\tilde{\gamma}$. Note that we also use "$\tilde{\gamma}$" to denote the lift of the loop γ in M to $\tilde{M}$. However, it should be clear from the particular context which notion we are referring to when writing "$\tilde{\gamma}$".

The construction of the covering transformation $\tilde{\gamma}$ from a given element $[\gamma]$ of the fundamental group $\pi_1(X, x)$ involves the choice of a point $y \in Y$ with $\pi(y) = x$. As this choice is not unique, we are interested in the effect of replacing y by another point in Y, which is mapped by π onto x.

Corollary A.18. *Let X be a connected, locally path-connected and semilocally simply connected topological space and $\pi : Y \to X$ the universal covering of X. Let $y \in Y$ with $\pi(y) = x$ and consider for any $[\gamma] \in \pi_1(X, x)$ the covering transformation $\tilde{\gamma} : Y \to Y$, $\tilde{\gamma}(z) := [\gamma] \cdot z$ given by theorem A.14. Then, replacing $y \in Y$ in the construction of the mapping $\tilde{\gamma}$ by a point $\tilde{\sigma}(y) \in Y$, where $\tilde{\sigma}$ denotes a covering transformation on Y, we obtain another covering transformation $\tilde{\gamma}_{new} : z \mapsto [\gamma] \cdot_{new} z$ on Y, which is related to $\tilde{\gamma}$ as follows:*

$$\tilde{\gamma}_{new}(z) = (\tilde{\sigma} \circ \tilde{\gamma} \circ \tilde{\sigma}^{-1})(z). \tag{A.4.30}$$

Proof. By theorem A.14, there exists $[\sigma] \in \pi_1(X, x)$, such that

$$\tilde{\sigma}(z) = [\sigma] \cdot z := y * (\sigma \cdot (\pi \circ \delta)), \tag{A.4.31}$$

where δ denotes an arbitrary path in Y from y to z. Furthermore, denoting by $\tilde{\sigma}$ also the (unique) lift of the loop σ to a path in Y starting at y (cf. remark A.17), the path product $\tilde{\sigma}^{-1} \cdot \delta$ defines a path in Y starting at $y * \sigma = \tilde{\sigma}(y)$ and ending at z. Putting these results together, we obtain

$$\begin{aligned} \tilde{\gamma}_{new}(z) = [\gamma] \cdot_{new} z &= (\tilde{\sigma}(y)) * (\gamma \cdot (\pi \circ (\tilde{\sigma}^{-1} \cdot \delta))) = (y * \sigma) * (\gamma \cdot \sigma^{-1} \cdot (\pi \circ \delta)) = y * (\sigma \cdot \gamma \cdot \sigma^{-1} \cdot (\pi \circ \delta)) \\ &= [\sigma \cdot \gamma \cdot \sigma^{-1}] \cdot z = [\sigma] \cdot ([\gamma] \cdot ([\sigma^{-1}] \cdot z)) = \tilde{\sigma}(\tilde{\gamma}(\tilde{\sigma}^{-1}(z))) = (\tilde{\sigma} \circ \tilde{\gamma} \circ \tilde{\sigma}^{-1})(z), \end{aligned} \tag{A.4.32}$$

which proves the claim. □

B Appendix: The function $\mu_j = \sqrt{X_j\overline{X_j}}$

In this appendix we study the λ-dependent function

$$\mu_j = \sqrt{X_j\overline{X_j}} \tag{B.1}$$

as introduced in section 3.5 in more detail.

Recall that X_j and $\overline{X_j}$ are defined by

$$X_j = s_j\lambda^{-1} + t_j\lambda, \quad \overline{X_j} = s_j\lambda + t_j\lambda^{-1} \tag{B.2}$$

for all $\lambda \in \mathbb{C}^*$. Moreover, the parameters s_j, t_j satisfy $s_j \in [\frac{1}{4}, \frac{1}{2})$ and $s_j + t_j = \frac{1}{2}$. Using $s_j^2 + t_j^2 = \frac{1}{4} - 2s_jt_j$ and setting

$$w_j = s_jt_j, \tag{B.3}$$

we can rewrite μ_j as

$$\mu_j = \sqrt{X_j\overline{X_j}} = \sqrt{s_j^2 + t_j^2 + s_jt_j(\lambda^{-2} + \lambda^2)} = \sqrt{\frac{1}{4} + w_j(\lambda^{-2} - 2 + \lambda^2)} = \sqrt{\frac{1}{4} + w_j(\lambda - \lambda^{-1})^2}, \tag{B.4}$$

as stated earlier in (3.5.24).

We now turn to the question for which $\lambda \in \mathbb{C}^*$ the mapping μ_j is well defined. To this end, we first need to define the complex square root involved in the definition of μ_j explicitly. Here, we distinguish between the case that $s_j \neq t_j$ and the case that $s_j = t_j = \frac{1}{4}$.

In the case $s_j \neq t_j$, we will use the following holomorphic extension of the usual real square root to the cut complex plane $\mathbb{C}_{>0} = \mathbb{C} \setminus \{x \in \mathbb{R}; x \leq 0\}$:

$$\sqrt{} : \mathbb{C}_{>0} \to \mathbb{C}^*, \ \lambda = re^{i\theta} \mapsto \sqrt{\lambda} := \sqrt{r}e^{i\frac{\theta}{2}}, \tag{B.5}$$

where we write $\lambda \in \mathbb{C}^*$ in the form $\lambda = re^{i\theta}$ with $r \in \mathbb{R}^+$ and $\theta \in (-\pi, \pi)$, and $\sqrt{r}$ defines the value of the positive real square root of r.

Naturally, for $\theta = 0$, we obtain the usual real square root $\mathbb{R}^+ \to \mathbb{R}^+$. Moreover, note that $\frac{\theta}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and thus $\sqrt{}$ actually maps $\mathbb{C}_{>0}$ to $\{z \in \mathbb{C}^*; \Re(z) > 0\}$. We extend $\sqrt{}$ continuously (but *not* holomorphically) to $\mathbb{C}_{\geq 0} = \mathbb{C}_{>0} \cup \{0\}$ by setting

$$\sqrt{0} := 0. \tag{B.6}$$

In view of (B.5) and (B.6), the function μ_j is in the case $s_j \neq t_j$ defined and continuous for all $\lambda \in \mathbb{C}^*$, for which the expression

$$\chi_j(\lambda) = \frac{1}{4} + w_j(\lambda - \lambda^{-1})^2 \tag{B.7}$$

takes values in $\mathbb{C}_{\geq 0}$. Moreover, since χ is holomorphic in $\lambda \in \mathbb{C}^*$, μ_j is holomorphic in λ for all $\lambda \in \mathbb{C}^*$, for which χ_j takes values in $\mathbb{C}_{>0}$ (as a composition of holomorphic functions).

Let $\lambda \in \mathbb{C}^*$. Using $w_j = s_jt_j$ and $\frac{1}{4} - 2w_j = s_j^2 + t_j^2$, we compute

$$\begin{aligned}\chi_j(\lambda) = 0 \iff & w_j\lambda^4 + (\frac{1}{4} - 2w_j)\lambda^2 + w_j = 0 \\ \iff & \lambda^2 = \frac{-(\frac{1}{4} - 2w_j) \pm \sqrt{(\frac{1}{4} - 2w_j)^2 - 4w_j^2}}{2w_j} = \frac{-s_j^2 - t_j^2 \pm |s_j^2 - t_j^2|}{2s_jt_j} \\ \iff & \lambda \in \{\pm i\sqrt{\frac{t_j}{s_j}}, \pm i\sqrt{\frac{s_j}{t_j}}\}.\end{aligned} \tag{B.8}$$

In particular, this implies

$$\mu_j(\lambda) = 0 \iff \lambda \in \{\pm i\sqrt{\frac{t_j}{s_j}}, \pm i\sqrt{\frac{s_j}{t_j}}\}. \tag{B.9}$$

Moreover, writing $\lambda \in \mathbb{C}^*$ as $\lambda = re^{i\theta}$ with $r \in \mathbb{R}^+$ and $\theta \in [-\pi, \pi)$ and recalling $s_j \geq t_j$, standard analysis of $\chi_j(\lambda)$ yields

$$\chi_j(\lambda) \in \mathbb{C}_{>0} \iff \lambda \in \mathbb{C}^* \setminus W_{1,j}, \tag{B.10}$$

where

$$W_{1,j} = \{z \in \mathbb{C}; \Re(z) = 0 \text{ and } \Im(z) \in (-\infty, -\sqrt{\frac{s_j}{t_j}}] \cup [-\sqrt{\frac{t_j}{s_j}}, \sqrt{\frac{t_j}{s_j}}] \cup [\sqrt{\frac{s_j}{t_j}}, +\infty)\}. \tag{B.11}$$

We now turn to the case $s_j = t_j = \frac{1}{4}$. Here, since $w_j = s_j t_j = \frac{1}{16}$, we have

$$\mu_j = \sqrt{\frac{1}{4} + \frac{1}{16}(\lambda - \lambda^{-1})^2} = \sqrt{\frac{1}{16}(\lambda + \lambda^{-1})^2}, \tag{B.12}$$

which allows for resolving the occurring complex square explicitly as:

$$\mu_j = \frac{1}{4}(\lambda + \lambda^{-1}). \tag{B.13}$$

We will use this definition of μ_j in the special case that $s_j = t_j = \frac{1}{4}$. Obviously, in this case, μ_j defines a continuous and holomorphic function in $\lambda \in \mathbb{C}^*$. We note that

$$\mu_j(\lambda) = 0 \iff \lambda \in \{\pm i\}. \tag{B.14}$$

Based on our considerations above, we obtain the following result:

Lemma B.1. *Consider the mapping μ_j given in* (B.1)*.*

1. *In the case $s_j \neq t_j$, μ_j defines a continuous mapping*

$$\mu_j : \mathbb{C}^* \setminus \tilde{W}_{1,j} \to \mathbb{C}, \tag{B.15}$$

where

$$\tilde{W}_{1,j} = W_{1,j} \setminus \{\pm i\sqrt{\frac{t_j}{s_j}}, \pm i\sqrt{\frac{s_j}{t_j}}\} \tag{B.16}$$

and $W_{1,j}$ is given in (B.11)*. Moreover, the restriction of μ_j to $\mathbb{C}^* \setminus W_{1,j}$ is holomorphic.*

2. *In the case $s_j = t_j = \frac{1}{4}$, μ_j defines a continuous mapping*

$$\mu_j : \mathbb{C}^* \to \mathbb{C}. \tag{B.17}$$

Moreover, μ_j is holomorphic in $\lambda \in \mathbb{C}^*$.

Remark B.2. Consider the case $s_j \neq t_j$ in lemma B.1. Excluding the set $W_{1,j}$ (resp. $\tilde{W}_{1,j}$) from the λ-domain corresponds to "cutting" the punctured λ-plane $\mathbb{C}^*$ along the imaginary axis thrice: once from $-i\infty$ to $-i\sqrt{\frac{s_j}{t_j}}$, once from $-i\sqrt{\frac{t_j}{s_j}}$ to $i\sqrt{\frac{t_j}{s_j}}$ and once from $i\sqrt{\frac{s_j}{t_j}}$ to $+i\infty$. The finite endpoints $\pm i\sqrt{\frac{t_j}{s_j}}$, $\pm i\sqrt{\frac{s_j}{t_j}}$ of the cuts are also removed in the case that we exclude $W_{1,j}$ from $\mathbb{C}^*$, while they are retained in the case that we exclude $\tilde{W}_{1,j}$ from $\mathbb{C}^*$. We would like to point out that this implies in particular that, in the case $s_j \neq t_j$, the mapping μ_j is continuous and holomorphic on (a sufficiently small neighborhood of) the unit circle S^1. (This is of course also true for the mapping μ_j in the case $s_j = t_j = \frac{1}{4}$.)

The following lemma summarizes some basic properties of the mappings X_j, $\overline{X_j}$ and μ_j.

Lemma B.3. *Consider the mapping μ_j given in* (B.1)*.*

1. *In the case $s_j \neq t_j$ we have:*

$$\frac{\overline{X_j}}{\mu_j} = \frac{\mu_j}{X_j} \text{ for all } \lambda \in \mathbb{C}^* \setminus W_{1,j} \tag{B.18}$$

$$X_j(\lambda = 1) = \overline{X_j}(\lambda = 1) = \frac{1}{2}, \tag{B.19}$$

$$\mu_j(\lambda = 1) = \frac{1}{2}, \quad (\partial_\lambda \mu_j)(\lambda = 1) = 0. \tag{B.20}$$

2. In the case $s_j = t_j = \frac{1}{4}$, we have:

$$\frac{\overline{X_j}}{\mu_j} = 1 = \frac{\mu_j}{X_j} \text{ for all } \lambda \in \mathbb{C}^* \setminus \{\pm i\} \tag{B.21}$$

$$X_j(\lambda = 1) = \overline{X_j}(\lambda = 1) = \frac{1}{2}, \tag{B.22}$$

$$\mu_j(\lambda = 1) = \frac{1}{2}, \quad (\partial_\lambda \mu_j)(\lambda = 1) = 0. \tag{B.23}$$

Proof. Let first $s_j \neq t_j$. For $\lambda \in \mathbb{C}^* \setminus W_{1,j}$ we have $\mu_j(\lambda) \neq 0$ and $X_j(\lambda) \neq 0$, which allows for writing

$$\frac{\overline{X_j}}{\mu_j} = \frac{X_j \overline{X_j}}{X_j \mu_j} = \frac{\mu_j^2}{X_j \mu_j} = \frac{\mu_j}{X_j}. \tag{B.24}$$

Moreover, by direct computations we have

$$X_j(\lambda = 1) = s_j + t_j = \frac{1}{2} = s_j + t_j = \overline{X_j}(\lambda = 1), \tag{B.25}$$

and

$$\mu_j(\lambda = 1) = \sqrt{\frac{1}{4} + w_j(1-1)^2} = \frac{1}{2}, \tag{B.26}$$

$$(\partial_\lambda \mu_j)(\lambda = 1) = \left(\frac{2w_j(\lambda - \lambda^{-1})(1 - \lambda^{-2})}{2\sqrt{\frac{1}{4} + w_j(\lambda - \lambda^{-1})^2}} \right)_{\lambda = 1} = 0. \tag{B.27}$$

Let now $s_j = t_j = \frac{1}{4}$. Then, we have for all $\lambda \in \mathbb{C}^* \setminus \{\pm i\}$ the relation $\mu_j(\lambda) = X_j(\lambda) = \overline{X_j}(\lambda) \neq 0$, which implies the first claim. The other claims follow by direct computations. □

Next, we turn to the parameter $w_j = s_j t_j$:

Lemma B.4. *Interpreting $w_j = s_j t_j = s_j(\frac{1}{2} - s_j)$ as a function of s_j,*

$$w_j : [\frac{1}{4}, \frac{1}{2}) \to \mathbb{R}, \tag{B.28}$$

has the following properties:

$$w_j([\frac{1}{4}, \frac{1}{2})) = (0, \frac{1}{16}], \tag{B.29}$$

$$w_j(s_j) \in (0, \frac{1}{18}) \iff s_j \in (\frac{1}{3}, \frac{1}{2}), \quad w_j(s_j) \in (\frac{1}{18}, \frac{1}{16}) \iff s_j \in (\frac{1}{4}, \frac{1}{3}), \tag{B.30}$$

$$w_j(s_j) = \frac{1}{18} \iff s_j = \frac{1}{3}, \quad w_j(s_j) = \frac{1}{16} \iff s_j = \frac{1}{4}. \tag{B.31}$$

Proof. Elementary analysis of $w_j : [\frac{1}{4}, \frac{1}{2}) \to \mathbb{R}, s_j \mapsto s_j(\frac{1}{2} - s_j)$. □

Based on lemma B.4 we study the behaviour of μ_j for $\lambda \in S^1$:

Lemma B.5. *Let the mapping μ_j be given in* (B.1).

1. In the case $s_j \neq t_j$, μ_j takes positive real values for $\lambda \in S^1$. More precisely, we have

$$\mu_j(S^1) = [\sqrt{\frac{1}{4} - 4w_j}, \frac{1}{2}]. \tag{B.32}$$

2. In the case $s_j = t_j = \frac{1}{4}$, μ_j takes real values for $\lambda \in S^1$. More precisely, we have

$$\mu_j(S^1) = [-\frac{1}{2}, \frac{1}{2}]. \tag{B.33}$$

Proof. Let first $s_j \neq t_j$. Writing $\lambda \in S^1$ as $e^{i\theta}$ with $\theta \in (-\pi, \pi]$, we compute

$$\mu_j(\lambda) = \sqrt{\frac{1}{4} + w_j(e^{i\theta} - e^{-i\theta})^2} = \sqrt{\frac{1}{4} + w_j(2i\sin(\theta))^2} = \sqrt{\frac{1}{4} - 4w_j\sin^2(\theta)}. \tag{B.34}$$

Using $w_j < \frac{1}{16}$ from lemma B.4, we obtain $\frac{1}{4} - 4w_j\sin^2(\theta) \geq \frac{1}{4} - 4w_j > 0$. Consequently, we have $\mu_j(\lambda) > 0$ for all $\lambda \in S^1$. Moreover, we infer that, for $\lambda \in S^1$,

$$\sqrt{\frac{1}{4} - 4w_j} \leq \mu_j(\lambda) \leq \sqrt{\frac{1}{4}} = \frac{1}{2}. \tag{B.35}$$

Since we have $\mu_j(\lambda = 1) = \frac{1}{2}$ and $\mu_j(\lambda = i) = \sqrt{\frac{1}{4} - 4w_j}$, we conclude by continuity arguments that actually $\mu_j(S^1) = [\sqrt{\frac{1}{4} - 4w_j}, \frac{1}{2}]$.

We now turn to the case $s_j = t_j = \frac{1}{4}$. Writing again $\lambda \in S^1$ as $e^{i\theta}$ with $\theta \in (-\pi, \pi]$, we obtain

$$\mu_j(\lambda) = \frac{1}{4}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\cos(\theta), \tag{B.36}$$

which directly implies $-\frac{1}{2} \leq \mu_j(\lambda) \leq \frac{1}{2}$ for all $\lambda \in S^1$. Since $\mu_j(1) = \frac{1}{2}$, $\mu_j(-1) = -\frac{1}{2}$ and μ_j is continuous on S^1, we infer that $\mu_j(S^1) = [-\frac{1}{2}, \frac{1}{2}]$. □

We close this appendix with the following lemma, dealing with the case $\mu_0 = \mu_1 = \mu_1$. (Note that, in the framework of this thesis, this case only occurs for $s_j \neq t_j$ for all $j \in \{0, 1, \infty\}$.)

Lemma B.6. *Let, for $j = 0, 1, \infty$, $\mu_j = \sqrt{\frac{1}{4} + w_j(\lambda - \lambda^{-1})^2}$, where $w_j = s_j t_j$ for $s_j \in (\frac{1}{4}, \frac{1}{2})$ and $s_j + t_j = \frac{1}{2}$. (In particular, this implies $s_j \neq t_j$.) Then, the following holds:*

1.

$$\mu_0 = \mu_1 = \mu_\infty \iff w_0 = w_1 = w_\infty \iff s_0 = s_1 = s_\infty \tag{B.37}$$

2. Let $\mu := \mu_0 = \mu_1 = \mu_\infty$, $w := w_0 = w_1 = w_\infty$ and $s := s_0 = s_1 = s_\infty$. Then

$$\mu \textit{ satisfies } (3.5.28) \textit{ for all } \lambda \in S^1 \iff w \in (0, \frac{1}{18}] \iff s \in [\frac{1}{3}, \frac{1}{2}). \tag{B.38}$$

Proof. First, we give the proof of (B.37): Assume $\mu_0 = \mu_1 = \mu_\infty$. This yields $w_0(\lambda - \lambda^{-1})^2 = w_1(\lambda - \lambda^{-1})^2 = w_\infty(\lambda - \lambda^{-1})^2$ for all $\lambda \in \mathbb{C}^* \setminus (\tilde{W}_{1,0} \cap \tilde{W}_{1,1} \cap \tilde{W}_{1,\infty})$ and thus $w_0 = w_1 = w_\infty$. As $\omega_j = s_j(\frac{1}{2} - s_j)$ is injective for $s_j \in [\frac{1}{4}, \frac{1}{2})$, we infer that $s_0 = s_1 = s_\infty$. The other way round, assume $s_0 = s_1 = s_\infty$. This implies directly $w_0 = w_1 = w_\infty$ and $\mu_0 = \mu_1 = \mu_\infty$.

We now turn to the second claim. As

$$w \in (0, \frac{1}{18}] \iff s \in [\frac{1}{3}, \frac{1}{2}) \tag{B.39}$$

is a direct consequence of lemma B.4, it remains only to prove

$$\mu \text{ satisfies } (3.5.28) \text{ for all } \lambda \in S^1 \iff w \in (0, \frac{1}{18}]. \tag{B.40}$$

In view of the assumption, the "unitarizability condition" (3.5.28) reads

$$0 \leq \frac{\cos^2(\pi\mu)}{\sin^2(2\pi\mu)} \leq 1 \tag{B.41}$$

for all $\lambda \in S^1$. This is equivalent to

$$0 \leq \frac{1}{4\sin^2(\pi\mu)} \leq 1 \tag{B.42}$$

for all $\lambda \in S^1$, which in turn holds if and only if

$$\mu(\lambda) \in [\frac{1}{6}, \frac{1}{2}] \tag{B.43}$$

for all $\lambda \in S^1$. (Recall that, by lemma B.5, μ takes for $\lambda \in S^1$ only values in $(0, \frac{1}{2}]$.) Since, again by lemma B.5, $\mu_j(S^1) = [\sqrt{\frac{1}{4} - 4w_j}, \frac{1}{2}]$, (B.43) holds for all $\lambda \in S^1$ if and only if $\sqrt{\frac{1}{4} - 4w_j} \geq \frac{1}{6}$, or, equivalently, $w \leq \frac{1}{18}$. In view of the general relation $w \in (0, \frac{1}{16}]$ from lemma B.4, this finishes the proof. □

C Appendix: Proof of lemma 3.37

In this appendix, we give the proof of lemma 3.37, which explicitly states the connection coefficients α_j, β_j, δ_j, ϵ_j relating (by lemma 3.12) the local solution Φ_j to (3.8.9) around z_j to the fundamental system y_{j1}, y_{j2} (around z_j) given in equations (3.7.23) to (3.7.26):

$$\Phi_j = \begin{pmatrix} \frac{\alpha_j y_{j1}' + \beta_j y_{j2}'}{\nu} & \alpha_j y_{j1} + \beta_j y_{j2} \\ \frac{\delta_j y_{j1}' + \epsilon_j y_{j2}'}{\nu} & \delta_j y_{j1} + \epsilon_j y_{j2} \end{pmatrix}. \tag{C.1}$$

Lemma 3.37. The connection coefficients α_j, β_j, δ_j, ϵ_j occuring in (C.1) are given by

$$\alpha_j = -\beta_j = (i)^j \frac{X_j}{2\mu_j\sqrt{\lambda X_j}}, \tag{C.2}$$

$$\delta_j = \epsilon_j = (i)^j \frac{1}{2\sqrt{\lambda X_j}}. \tag{C.3}$$

Proof. As earlier in this work (cf. (2.6.21)), we write

$$D_j = \begin{pmatrix} 0 & X_j \\ \overline{X_j} & 0 \end{pmatrix} = \mu_j R_j S \sigma_3 S^{-1} R_j^{-1}, \tag{C.4}$$

where $R_j = \begin{pmatrix} \frac{\sqrt{\lambda X_j}}{\sqrt{\mu_j}} & 0 \\ 0 & \frac{\sqrt{\lambda^{-1}\overline{X_j}}}{\sqrt{\mu_j}} \end{pmatrix}$ and $S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\lambda^{-1} \\ \lambda & 1 \end{pmatrix}$. (Note that (2.6.21) applies since we consider only the cases $j = 0, 1$ and since in these cases, by assumption, $s_j > t_j$.) From (C.4), we infer that

$$e^{\ln(z)D_0} = R_0 S e^{\ln(z)\mu_0\sigma_3} S^{-1} R_0^{-1} = R_0 S \begin{pmatrix} z^{\mu_0} & 0 \\ 0 & z^{-\mu_0} \end{pmatrix} S^{-1} R_0^{-1} = \frac{1}{2}\begin{pmatrix} s_{0,+} & \frac{X_0}{\mu_0} s_{0,-} \\ \frac{\overline{X_0}}{\mu_0} s_{0,-} & s_{0,+} \end{pmatrix}, \tag{C.5}$$

where

$$s_{0,+} = z^{\mu_0} + z^{-\mu_0}, \quad s_{0,-} = z^{\mu_0} - z^{-\mu_0}, \tag{C.6}$$

and, similarly,

$$\begin{aligned} e^{\ln(1-z)D_1} &= R_1 S e^{\ln(1-z)\mu_1\sigma_3} S^{-1} R_1^{-1} \\ &= R_1 S \begin{pmatrix} (1-z)^{\mu_1} & 0 \\ 0 & (1-z)^{-\mu_1} \end{pmatrix} S^{-1} R_1^{-1} = \frac{1}{2}\begin{pmatrix} s_{1,+} & \frac{X_1}{\mu_1} s_{1,-} \\ \frac{\overline{X_1}}{\mu_1} s_{1,-} & s_{1,+} \end{pmatrix}, \end{aligned} \tag{C.7}$$

where

$$s_{1,+} = (1-z)^{\mu_1} + (1-z)^{-\mu_1}, \quad s_{1,-} = (1-z)^{\mu_1} - (1-z)^{-\mu_1}. \tag{C.8}$$

By equations (3.8.33) and (3.8.34), we have

$$P_0 = \left(e^{\ln(z)D_0}\right)^{-1} \Phi_0 V_{+,0}, \tag{C.9}$$

$$P_1 = \left(e^{\ln(1-z)D_1}\right)^{-1} \Phi_1 V_{+,1}. \tag{C.10}$$

Thus, by our considerations above, we obtain

$$\begin{aligned} P_j &= \frac{1}{2}\begin{pmatrix} s_{j,+} & -\frac{X_j}{\mu_j} s_{j,-} \\ -\frac{\overline{X_j}}{\mu_j} s_{j,-} & s_{j,+} \end{pmatrix} \begin{pmatrix} \frac{\alpha_j y_{j1}' + \beta_j y_{j2}'}{\nu} & \alpha_j y_{j1} + \beta_j y_{j2} \\ \frac{\delta_j y_{j1}' + \epsilon_j y_{j2}'}{\nu} & \delta_j y_{j1} + \epsilon_j y_{j2} \end{pmatrix} V_{+,j} \\ &= \frac{1}{2}\begin{pmatrix} * & s_{j,+}(\alpha_j y_{j1} + \beta_j y_{j2}) - \frac{X_j}{\mu_j} s_{j,-}(\delta_j y_{j1} + \epsilon_j y_{j2}) \\ * & -\frac{\overline{X_j}}{\mu_j} s_{j,-}(\alpha_j y_{j1} + \beta_j y_{j2}) + s_{j,+}(\delta_j y_{j1} + \epsilon_j y_{j2}) \end{pmatrix} V_{+,j}. \end{aligned} \tag{C.11}$$

In particular, we obtain for the upper right and the lower right entry of P_j

$$\begin{aligned} P_{j,12} &= \frac{1}{2}(V_{+,j})_{22}\left(s_{j,+}(\alpha_j y_{j1} + \beta_j y_{j2}) - \frac{X_j}{\mu_j} s_{j,-}(\delta_j y_{j1} + \epsilon_j y_{j2})\right) \\ &= \frac{1}{2}(V_{+,j})_{22}\left(y_{j1}(\alpha_j s_{j,+} - \frac{X_j}{\mu_j}\delta_j s_{j,-}) + y_{j2}(\beta_j s_{j,+} - \frac{X_j}{\mu_j}\epsilon_j s_{j,-})\right), \end{aligned} \tag{C.12}$$

$$P_{j,22} = \frac{1}{2}(V_{+,j})_{22}\left(-\frac{\overline{X_j}}{\mu_j}s_{j,-}(\alpha_j y_{j1} + \beta_j y_{j2}) + s_{j,+}(\delta_j y_{j1} + \epsilon_j y_{j2})\right)$$
$$= \frac{1}{2}(V_{+,j})_{22}\left(y_{j1}(-\frac{\overline{X_j}}{\mu_j}\alpha_j s_{j,-} + \delta_j s_{j,+}) + y_{j2}(-\frac{\overline{X_j}}{\mu_j}\beta_j s_{j,-} + \epsilon_j s_{j,+})\right), \quad \text{(C.13)}$$

where $(V_{+,j})_{22}$ denotes the lower right entry of $V_{+,j}$.

Next, recall from (3.7.23) to (3.7.26) the fundamental systems y_{j1}, y_{j2} around z_j ($j = 0, 1$) solving the Fuchsian equation (3.7.1):

$$y_{01} = z^{r_0}(1-z)^{r_1}F(\alpha, \beta, \gamma; z), \quad \text{(C.14)}$$
$$y_{02} = z^{r_0+1-\gamma}(1-z)^{r_1}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z), \quad \text{(C.15)}$$

$$y_{11} = z^{r_0}(1-z)^{r_1}F(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z), \quad \text{(C.16)}$$
$$y_{12} = z^{r_0}(1-z)^{r_1+\gamma-\alpha-\beta}F(\gamma-\beta, \gamma-\alpha, \gamma-\alpha-\beta+1; 1-z). \quad \text{(C.17)}$$

Observing that $r_0 + 1 - \gamma = \frac{1}{2} - \mu_0$ and $r_1 + \gamma - \alpha - \beta = \frac{1}{2} - \mu_1$, we can rewrite these equations as

$$y_{01} = z^{\frac{1}{2}+\mu_0}h_0, \quad \text{(C.18)}$$
$$y_{02} = z^{\frac{1}{2}-\mu_0}\tilde{h}_0, \quad \text{(C.19)}$$

$$y_{11} = (1-z)^{\frac{1}{2}+\mu_1}h_1, \quad \text{(C.20)}$$
$$y_{12} = (1-z)^{\frac{1}{2}-\mu_1}\tilde{h}_1 \quad \text{(C.21)}$$

with mappings h_j, $\tilde{h}_j$, which are holomorphic around z_j, given by

$$h_0 = (1-z)^{\frac{1}{2}+\mu_1}F(\alpha, \beta, \gamma; z), \quad \text{(C.22)}$$
$$\tilde{h}_0 = (1-z)^{\frac{1}{2}+\mu_1}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z), \quad \text{(C.23)}$$

$$h_1 = z^{\frac{1}{2}+\mu_0}F(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z), \quad \text{(C.24)}$$
$$\tilde{h}_1 = z^{\frac{1}{2}+\mu_0}F(\gamma-\beta, \gamma-\alpha, \gamma-\alpha-\beta+1; 1-z). \quad \text{(C.25)}$$

We proceed by inserting (C.18) to (C.21) into equations (C.12) and (C.13). First, we consider the case $j = 0$:

$$P_{0,12} = \frac{1}{2}z^{-\frac{1}{2}}\sqrt{\lambda X_0}\left\{z^{\frac{1}{2}+\mu_0}h_0(z)\left(\alpha_0(z^{\mu_0} + z^{-\mu_0}) - \frac{X_0}{\mu_0}\delta_0(z^{\mu_0} - z^{-\mu_0})\right)\right.$$
$$\left. + z^{\frac{1}{2}-\mu_0}\tilde{h}_0(z)\left(\beta_0(z^{\mu_0} + z^{-\mu_0}) - \frac{X_0}{\mu_0}\epsilon_0(z^{\mu_0} - z^{-\mu_0})\right)\right\}$$
$$= \frac{1}{2}\sqrt{\lambda X_0}\left\{z^{2\mu_0}\left(\alpha_0 h_0(z) - \frac{X_0}{\mu_0}\delta_0 h_0(z)\right) + \left(\alpha_0 h_0(z) + \frac{X_0}{\mu_0}\delta_0 h_0(z) + \beta_0\tilde{h}_0(z) - \frac{X_0}{\mu_0}\epsilon_0\tilde{h}_0(z)\right)\right.$$
$$\left. + z^{-2\mu_0}\left(\beta_0\tilde{h}_0(z) + \frac{X_0}{\mu_0}\epsilon_0\tilde{h}_0(z)\right)\right\}, \quad \text{(C.26)}$$

$$P_{0,22} = \frac{1}{2}z^{-\frac{1}{2}}\sqrt{\lambda X_0}\left\{z^{\frac{1}{2}+\mu_0}h_0(z)\left(-\frac{\overline{X_0}}{\mu_0}\alpha_0(z^{\mu_0} - z^{-\mu_0}) + \delta_0(z^{\mu_0} + z^{-\mu_0})\right)\right.$$
$$\left. + z^{\frac{1}{2}-\mu_0}\tilde{h}_0(z)\left(-\frac{\overline{X_0}}{\mu_0}\beta_0(z^{\mu_0} - z^{-\mu_0}) + \epsilon_0(z^{\mu_0} + z^{-\mu_0})\right)\right\}$$
$$= \frac{1}{2}\sqrt{\lambda X_0}\left\{z^{2\mu_0}\left(-\frac{\overline{X_0}}{\mu_0}\alpha_0 h_0(z) + \delta_0 h_0(z)\right) + \left(\frac{\overline{X_0}}{\mu_0}\alpha_0 h_0(z) + \delta_0 h_0(z) - \frac{\overline{X_0}}{\mu_0}\beta_0\tilde{h}_0(z) + \epsilon_0\tilde{h}_0(z)\right)\right.$$
$$\left. + z^{-2\mu_0}\left(\frac{\overline{X_0}}{\mu_0}\beta_0\tilde{h}_0(z) + \epsilon_0\tilde{h}_0(z)\right)\right\}. \quad \text{(C.27)}$$

As P_0 shall be holomorphic at $z = z_0$, also the matrix coefficient $P_{0,12}$ shall be holomorphic at $z = z_0$. Since μ_0 is not an integer except for some λ from a discrete subset of $\mathbb{C}^*$, this is only possible if the coefficients of $z^{2\mu_0}$ and $z^{-2\mu_0}$ vanish. Furthermore $P_0(z_0) = I$ and thus $P_{0,12}(z_0) = 0$. Altogether, we infer the following conditions for the coefficient $P_{0,12}$:

$$\delta_0 = \frac{\mu_0}{X_0}\alpha_0, \tag{C.28}$$

$$\epsilon_0 = -\frac{\mu_0}{X_0}\beta_0, \tag{C.29}$$

$$\alpha_0 h_0(0) + \frac{X_0}{\mu_0}\delta_0 h_0(0) + \beta_0\tilde{h}_0(0) - \frac{X_0}{\mu_0}\epsilon_0\tilde{h}_0(0) = 0. \tag{C.30}$$

Similarly, as $P_{0,22}$ shall be holomorphic at $z = z_0$ and $P_{0,22}(z_0) = 1$, we obtain:

$$\delta_0 = \frac{\overline{X_0}}{\mu_0}\alpha_0, \tag{C.31}$$

$$\epsilon_0 = -\frac{\overline{X_0}}{\mu_0}\beta_0, \tag{C.32}$$

$$\frac{1}{2}\sqrt{\lambda X_0}\left(\frac{\overline{X_0}}{\mu_0}\alpha_0 h_0(0) + \delta_0 h_0(0) - \frac{\overline{X_0}}{\mu_0}\beta_0\tilde{h}_0(0) + \epsilon_0\tilde{h}_0(0)\right) = 1. \tag{C.33}$$

Since, by lemma B.3 given in appendix B, $\frac{\overline{X_0}}{\mu_0} = \frac{\mu_0}{X_0}$, conditions (C.31) and (C.32) are equivalent to (C.28) and (C.29). Moreover, $h_0(0) = \tilde{h}_0(0) = 1$ (cf. (3.7.16)). Altogether, conditions (C.28) to (C.33) are equivalent to

$$\delta_0 = \frac{\mu_0}{X_0}\alpha_0, \tag{C.34}$$

$$\epsilon_0 = -\frac{\mu_0}{X_0}\beta_0, \tag{C.35}$$

$$2\alpha_0 + 2\beta_0 = 0, \tag{C.36}$$

$$\frac{1}{2}\sqrt{\lambda X_0}\left(2\frac{\mu_0}{X_0}\alpha_0 - 2\frac{\mu_0}{X_0}\beta_0\right) = 1. \tag{C.37}$$

By an easy computation, these equations yield

$$\alpha_0 = -\beta_0 = \frac{X_0}{2\mu_0\sqrt{\lambda X_0}}, \tag{C.38}$$

$$\delta_0 = \epsilon_0 = \frac{1}{2\sqrt{\lambda X_0}}. \tag{C.39}$$

We turn to the case $j = 1$.

$$\begin{aligned}
P_{1,12} = &-\frac{1}{2}i(1-z)^{-\frac{1}{2}}\sqrt{\lambda X_1}\\
&\cdot\Bigg\{(1-z)^{\frac{1}{2}+\mu_1}h_1(z)\left(\alpha_1((1-z)^{\mu_1}+(1-z)^{-\mu_1}) - \frac{X_1}{\mu_1}\delta_1((1-z)^{\mu_1}-(1-z)^{-\mu_1})\right)\\
&+(1-z)^{\frac{1}{2}-\mu_1}\tilde{h}_1(z)\left(\beta_1((1-z)^{\mu_1}+(1-z)^{-\mu_1}) - \frac{X_1}{\mu_1}\epsilon_1((1-z)^{\mu_1}-(1-z)^{-\mu_1})\right)\Bigg\}\\
=&-\frac{1}{2}i\sqrt{\lambda X_1}\Bigg\{(1-z)^{2\mu_1}\left(\alpha_1 h_1(z) - \frac{X_1}{\mu_1}\delta_1 h_1(z)\right)\\
&+\left(\alpha_1 h_1(z) + \frac{X_1}{\mu_1}\delta_1 h_1(z) + \beta_1\tilde{h}_1(z) - \frac{X_1}{\mu_1}\epsilon_1\tilde{h}_1(z)\right)\\
&+(1-z)^{-2\mu_1}\left(\beta_1\tilde{h}_1(z) + \frac{X_1}{\mu_1}\epsilon_1\tilde{h}_1(z)\right)\Bigg\},
\end{aligned} \tag{C.40}$$

$$
\begin{aligned}
P_{1,22} = &-\frac{1}{2}i(1-z)^{-\frac{1}{2}}\sqrt{\lambda X_1}\cdot \\
&\cdot\Big\{(1-z)^{\frac{1}{2}+\mu_1}h_1(z)\left(-\frac{\overline{X_1}}{\mu_1}\alpha_1((1-z)^{\mu_1}-(1-z)^{-\mu_1})+\delta_1((1-z)^{\mu_1}+(1-z)^{-\mu_1})\right) \\
&+(1-z)^{\frac{1}{2}-\mu_1}\tilde{h}_1(z)\left(-\frac{\overline{X_1}}{\mu_1}\beta_1((1-z)^{\mu_1}-(1-z)^{-\mu_1})+\epsilon_1((1-z)^{\mu_1}+(1-z)^{-\mu_1})\right)\Big\} \\
=&-\frac{1}{2}i\sqrt{\lambda X_1}\Big\{(1-z)^{2\mu_1}\left(-\frac{\overline{X_1}}{\mu_1}\alpha_1 h_1(z)+\delta_1 h_1(z)\right) \\
&+\left(\frac{\overline{X_1}}{\mu_1}\alpha_1 h_1(z)+\delta_1 h_1(z)-\frac{\overline{X_1}}{\mu_1}\beta_1\tilde{h}_1(z)+\epsilon_1\tilde{h}_1(z)\right) \\
&+(1-z)^{-2\mu_1}\left(\frac{\overline{X_1}}{\mu_1}\beta_1\tilde{h}_1(z)+\epsilon_1\tilde{h}_1(z)\right)\Big\}.
\end{aligned}
\tag{C.41}
$$

As P_1 shall be holomorphic at $z=z_1$, also the matrix coefficients $P_{1,12}$ and $P_{1,22}$ shall be holomorphic at $z=z_1$. Since μ_1 is not an integer except for some λ from a discrete subset of $\mathbb{C}^*$, this is only possible if the coefficients of $(1-z)^{2\mu_1}$ and $(1-z)^{-2\mu_1}$ vanish. Furthermore $P_1(z_1)=I$ and thus $P_{1,12}(z_1)=0$, $P_{1,22}(z_1)=1$. Altogether, we infer the following conditions for the coefficients $P_{1,12}$ and $P_{1,22}$, whereby we have already incorporated $\frac{\overline{X_1}}{\mu_1}=\frac{\mu_1}{X_1}$ and $h_1(1)=\tilde{h}_1(1)=1$:

$$\delta_1 = \frac{\mu_1}{X_1}\alpha_1, \tag{C.42}$$

$$\epsilon_1 = -\frac{\mu_1}{X_1}\beta_1, \tag{C.43}$$

$$2\alpha_1+2\beta_1=0 \tag{C.44}$$

$$-\frac{1}{2}i\sqrt{\lambda X_1}\left(2\frac{\mu_1}{X_1}\alpha_1-2\frac{\mu_1}{X_1}\beta_1\right)=1. \tag{C.45}$$

This yields

$$\alpha_1=-\beta_1=i\frac{X_1}{2\mu_1\sqrt{\lambda X_1}}, \tag{C.46}$$

$$\delta_1=\epsilon_1=i\frac{1}{2\sqrt{\lambda X_1}}. \tag{C.47}$$

□

D Appendix: On the necessity of the unitarizing matrix T

In section 3.9, we present a matrix T simultaneously unitarizing the monodromy matrices $M_0(\lambda)$ and $M_1(\lambda)$ given in (3.9.3) and (3.9.4), respectively, for all $\lambda \in S^1$. This is motivated by the claim, that $M_1(\lambda)$ is in general not already unitary for $\lambda \in S^1$. In this appendix, we prove the mentioned claim by constructing a counterexample, in which actually $M_1(\lambda_0)$ is *not* unitary for an appropriate $\lambda_0 \in S^1$.

We consider the matrix $M_1(\lambda)$. Assuming both s_0 and s_1 do not equal $\frac{1}{4}$ and using equations (2.6.23) and (3.8.56) as well as $\det(A) = 1$ from lemma 3.39, we compute

$$\begin{aligned} M_1(\lambda) &= -Ae^{2\pi i D_1}A^{-1} \\ &= \frac{\mu_1}{\mu_0} R_0 S \begin{pmatrix} \kappa_{11}^{01} & \lambda^{-1}\kappa_{12}^{01} \\ \lambda\kappa_{11}^{02} & \kappa_{12}^{02} \end{pmatrix} S^{-1}R_1^{-1}R_1 S \begin{pmatrix} e^{2\pi i\mu_1} & 0 \\ 0 & e^{-2\pi i\mu_1} \end{pmatrix} S^{-1}R_1^{-1}R_1 S \begin{pmatrix} \kappa_{12}^{02} & -\lambda^{-1}\kappa_{12}^{01} \\ -\lambda\kappa_{11}^{02} & \kappa_{11}^{01} \end{pmatrix} S^{-1}R_0^{-1} \\ &= \frac{\mu_1}{\mu_0} R_0 S \begin{pmatrix} e^{2\pi i\mu_1}\kappa_{11}^{01}\kappa_{12}^{02} - e^{-2\pi i\mu_1}\kappa_{12}^{01}\kappa_{11}^{02} & \lambda^{-1}\kappa_{11}^{01}\kappa_{12}^{01}(e^{-2\pi i\mu_1} - e^{2\pi i\mu_1}) \\ \lambda\kappa_{11}^{02}\kappa_{12}^{02}(e^{2\pi i\mu_1} - e^{-2\pi i\mu_1}) & e^{-2\pi i\mu_1}\kappa_{11}^{01}\kappa_{12}^{02} - e^{2\pi i\mu_1}\kappa_{12}^{01}\kappa_{11}^{02} \end{pmatrix} S^{-1}R_0^{-1} \\ &= \frac{1}{2}\frac{\mu_1}{\mu_0} \begin{pmatrix} A - C - B + \overline{A} & \frac{X_0}{\mu_0}(A - C + B - \overline{A}) \\ \frac{\overline{X_0}}{\mu_0}(A + C - B - \overline{A}) & A + C + B + \overline{A} \end{pmatrix}, \end{aligned} \tag{D.1}$$

where

$$A = A(\lambda) = e^{2\pi i\mu_1}\kappa_{11}^{01}\kappa_{12}^{02} - e^{-2\pi i\mu_1}\kappa_{12}^{01}\kappa_{11}^{02}, \tag{D.2}$$

$$B = B(\lambda) = \kappa_{11}^{01}\kappa_{12}^{01}(e^{-2\pi i\mu_1} - e^{2\pi i\mu_1}), \tag{D.3}$$

$$C = C(\lambda) = \kappa_{11}^{02}\kappa_{12}^{02}(e^{2\pi i\mu_1} - e^{-2\pi i\mu_1}). \tag{D.4}$$

Now, an easy calculation shows that M_1 is unitary for all $\lambda \in S^1$, i.e. of the form $\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$ for all $\lambda \in S^1$ if and only if $C(\lambda) = -\overline{B(\lambda)}$ for all $\lambda \in S^1$, i.e. if and only if

$$-\kappa_{11}^{01}\kappa_{12}^{01}\sin(2\pi\mu_1) = \kappa_{11}^{02}\kappa_{12}^{02}\sin(2\pi\mu_1) \tag{D.5}$$

for all $\lambda \in S^1$. (Note that μ_1 as well as the connection coefficients κ_{kl}^{ij} are real valued on S^1, as long as they are defined at all, cf. lemma B.5 and equations (3.7.19) to (3.7.22).)

To explicitly construct a counterexample, in which $M_1(\lambda_0)$ is *not* unitary for an appropriate $\lambda_0 \in S^1$, we set $s := s_0 = s_1 = s_\infty := \frac{1}{8}$. By lemma B.6, we obtain $\mu := \mu_0 = \mu_1 = \mu_\infty = \sqrt{\frac{1}{4} + w(\lambda - \lambda^{-1})^2}$ with $w := w_0 = w_1 = w_\infty = \frac{3}{64}$. Moreover, also by lemma B.6, μ_0, μ_1 and μ_∞ satisfy the unitarizability condition (3.5.28) and therefore give rise to a trinoid potential η. Thus, the given choice of s_0, s_1 and s_∞ is valid for our considerations.

In view of equations (3.7.19) to (3.7.22) as well as equations (3.7.8) to (3.7.10), (D.5) reads under the given presumptions as

$$-\sin(2\pi\mu)\frac{\Gamma(1+2\mu)\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu)\Gamma(\frac{1}{2}+\mu)}\frac{\Gamma(1+2\mu)\Gamma(2\mu)}{\Gamma(\frac{1}{2}+3\mu)\Gamma(\frac{1}{2}+\mu)} = \sin(2\pi\mu)\frac{\Gamma(-2\mu)\Gamma(1-2\mu)}{\Gamma(\frac{1}{2}-3\mu)\Gamma(\frac{1}{2}-\mu)}\frac{\Gamma(2\mu)\Gamma(1-2\mu)}{\Gamma(\frac{1}{2}+\mu)\Gamma(\frac{1}{2}-\mu)}. \tag{D.6}$$

Consequently, $M_1(\lambda)$ is unitary for all $\lambda \in S^1$ if and only if (D.6) holds for all $\lambda \in S^1$.

However, setting $\lambda_0 := i \in S^1$, we have $\mu(\lambda_0) = \frac{1}{4}$ and thus obtain for the left hand side of (D.6):

$$-\frac{\Gamma(1+\frac{1}{2})\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{4})\Gamma(1-\frac{1}{4})}\frac{\Gamma(1+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1+\frac{1}{4})\Gamma(1-\frac{1}{4})} = -\frac{\frac{1}{2}\Gamma(\frac{1}{2})\Gamma(-\frac{1}{2})\frac{1}{2}\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4})(-\frac{1}{4})\Gamma(-\frac{1}{4})\frac{1}{4}\Gamma(\frac{1}{4})(-\frac{1}{4})\Gamma(-\frac{1}{4})} = -16\frac{(\Gamma(\frac{1}{2}))^3\Gamma(-\frac{1}{2})}{(\Gamma(\frac{1}{4}))^2(\Gamma(-\frac{1}{4}))^2}. \tag{D.7}$$

(Here, we have used the relation $\Gamma(z+1) = z\Gamma(z)$, which holds for z in $\mathbb{C}$ excluding the non-positive integers (cf. [33], chapter 2, 2.1). Similarly, we compute the right hand side of (D.6):

$$\frac{\Gamma(-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(-\frac{1}{4})\Gamma(\frac{1}{4})}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1-\frac{1}{4})\Gamma(\frac{1}{4})} = \frac{\Gamma(-\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(-\frac{1}{4})\Gamma(\frac{1}{4})(-\frac{1}{4})\Gamma(-\frac{1}{4})\Gamma(\frac{1}{4})} = -4\frac{(\Gamma(\frac{1}{2}))^3\Gamma(-\frac{1}{2})}{(\Gamma(\frac{1}{4}))^2(\Gamma(-\frac{1}{4}))^2}. \tag{D.8}$$

Comparing (D.7) and (D.8), we infer that (D.6) does not hold for $\lambda = \lambda_0$, and thus that $M_1(\lambda_0)$ is not unitary.

E Appendix: Amendments to the proof of theorem 3.53

In this appendix, we provide the outstanding computations excluded from the proof of theorem 3.53. I.e., we prove that the matrix equation (3.9.56), reading

$$\begin{aligned} &-i\frac{\sqrt{\mu_1}}{\sqrt{\mu_0}}\begin{pmatrix}\kappa_{11}^{01} & \lambda^{-1}\kappa_{12}^{01}\\ \lambda\kappa_{11}^{02} & \kappa_{12}^{02}\end{pmatrix}\\ &=\frac{1}{2\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}}\begin{pmatrix}\omega_0^{-1}\omega_1\lambda^{-1}(q_0+q_1-p_0q_1+p_1q_0) & \omega_0^{-1}\omega_1^{-1}\lambda^{-1}(-q_0+q_1-p_0q_1+p_1q_0)\\ \omega_0\omega_1(-q_0+q_1+p_0q_1-p_1q_0) & \omega_0\omega_1^{-1}\lambda^{-1}(q_0+q_1+p_0q_1-p_1q_0)\end{pmatrix},\end{aligned} \tag{E.1}$$

is equivalent to the scalar equations

$$\omega_0=\delta\frac{\sqrt{\kappa_{12}^{02}}}{\sqrt{\kappa_{12}^{01}}}\frac{\sqrt{-q_0+q_1-p_0q_1+p_1q_0}}{\sqrt{q_0+q_1+p_0q_1-p_1q_0}}, \tag{E.2}$$

$$\omega_1=\tilde{\delta}\frac{\sqrt{\kappa_{11}^{01}}}{\sqrt{\kappa_{12}^{01}}}\frac{\sqrt{-q_0+q_1-p_0q_1+p_1q_0}}{\sqrt{q_0+q_1-p_0q_1+p_1q_0}}, \tag{E.3}$$

$$p_0p_1+\frac{q_0\overline{q_1}+\overline{q_0}q_1}{2}=\frac{\cos(2\pi\mu_0)\cos(2\pi\mu_1)+\cos(2\pi\mu_\infty)}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)}, \tag{E.4}$$

where $\delta,\tilde{\delta}\in\{\pm1\}$, such that

$$\delta\tilde{\delta}=\frac{\sqrt{\mu_0}}{\sqrt{\mu_1}}\frac{\sqrt{q_0+q_1+p_0q_1-p_1q_0}\sqrt{q_0+q_1-p_0q_1+p_1q_0}}{-2i\lambda\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}\sqrt{\kappa_{11}^{01}}\sqrt{\kappa_{12}^{02}}}. \tag{E.5}$$

Of course, this equivalence holds only for those $\lambda\in S^1$, for which all occurring terms are well defined. In view of remark 3.43, we will tacitly assume this in the following. In particular, we will ignore the cases, in which certain λ-dependent terms we divide by should vanish for certain isolated values of λ.

Throughout this section, we make use of the following relations, holding for $j=0,1$:

$$p_j^2+q_j\overline{q_j}=1\quad\text{and}\quad p_j=\overline{p_j}. \tag{E.6}$$

Moreover, we will use the following lemma:

Lemma E.1.

$$\begin{aligned} &-4q_0q_1\frac{\mu_1}{\mu_0}\kappa_{11}^{01}\kappa_{12}^{02}=(q_0+q_1)^2-(p_0q_1-p_1q_0)^2\\ &\iff p_0p_1+\frac{q_0\overline{q_1}+\overline{q_0}q_1}{2}=\frac{\cos(2\pi\mu_0)\cos(2\pi\mu_1)+\cos(2\pi\mu_\infty)}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)}\\ &\iff -4q_0q_1\frac{\mu_1}{\mu_0}\kappa_{11}^{02}\kappa_{12}^{01}=(q_0-q_1)^2-(p_0q_1-p_1q_0)^2.\end{aligned} \tag{E.7}$$

Proof. Recall equations (3.7.19) to (3.7.22)

$$\kappa_{11}^{01}=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \tag{E.8}$$

$$\kappa_{12}^{01}=\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}, \tag{E.9}$$

$$\kappa_{11}^{02}=\frac{\Gamma(\gamma-\alpha-\beta)\Gamma(2-\gamma)}{\Gamma(1-\alpha)\Gamma(1-\beta)}, \tag{E.10}$$

$$\kappa_{12}^{02}=\frac{\Gamma(\alpha+\beta-\gamma)\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)}, \tag{E.11}$$

as well as equations (3.7.8) to (3.7.10)

$$\alpha=\frac{1}{2}+\mu_0+\mu_1+\mu_\infty, \tag{E.12}$$

$$\beta=\frac{1}{2}+\mu_0+\mu_1-\mu_\infty, \tag{E.13}$$

$$\gamma=1+2\mu_0. \tag{E.14}$$

The Gamma function Γ satifies the well known relations (cf. [33], chapter 2, 2.1)

$$\Gamma(\frac{1}{2}+z)\Gamma(\frac{1}{2}-z) = \frac{\pi}{\cos(\pi z)}, \quad z \notin \{\frac{1}{2}+w; w \in \mathbb{Z}\}, \tag{E.15}$$

$$\Gamma(z)\Gamma(-z) = \frac{-\pi}{z\sin(\pi z)}, \quad z \notin \mathbb{Z}, \tag{E.16}$$

$$\Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin(\pi z)}, \quad z \notin \mathbb{Z}. \tag{E.17}$$

Using these relations, we obtain

$$\begin{aligned}\kappa_{11}^{01}\kappa_{12}^{02} &= \frac{\Gamma(1+2\mu_0)\Gamma(-2\mu_1)}{\Gamma(\frac{1}{2}+\mu_0-\mu_1-\mu_\infty)\Gamma(\frac{1}{2}+\mu_0-\mu_1+\mu_\infty)}\frac{\Gamma(2\mu_1)\Gamma(1-2\mu_0)}{\Gamma(\frac{1}{2}-\mu_0+\mu_1+\mu_\infty)\Gamma(\frac{1}{2}-\mu_0+\mu_1-\mu_\infty)} \\ &= \frac{\frac{2\pi\mu_0}{\sin(2\pi\mu_0)}\frac{-\pi}{2\mu_1\sin(2\pi\mu_1)}}{\frac{\pi}{\cos(\pi(\mu_0-\mu_1-\mu_\infty))}\frac{\pi}{\cos(\pi(\mu_0-\mu_1+\mu_\infty))}} = -\frac{\mu_0}{\mu_1}\frac{\cos(\pi(\mu_0-\mu_1-\mu_\infty))\cos(\pi(\mu_0-\mu_1+\mu_\infty))}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)},\end{aligned} \tag{E.18}$$

and

$$\begin{aligned}\kappa_{12}^{01}\kappa_{11}^{02} &= \frac{\Gamma(1+2\mu_0)\Gamma(2\mu_1)}{\Gamma(\frac{1}{2}+\mu_0+\mu_1+\mu_\infty)\Gamma(\frac{1}{2}+\mu_0+\mu_1-\mu_\infty)}\frac{\Gamma(-2\mu_1)\Gamma(1-2\mu_0)}{\Gamma(\frac{1}{2}-\mu_0-\mu_1-\mu_\infty)\Gamma(\frac{1}{2}-\mu_0-\mu_1+\mu_\infty)} \\ &= \frac{\frac{2\pi\mu_0}{\sin(2\pi\mu_0)}\frac{-\pi}{2\mu_1\sin(2\pi\mu_1)}}{\frac{\pi}{\cos(\pi(\mu_0+\mu_1+\mu_\infty))}\frac{\pi}{\cos(\pi(\mu_0+\mu_1-\mu_\infty))}} = -\frac{\mu_0}{\mu_1}\frac{\cos(\pi(\mu_0+\mu_1+\mu_\infty))\cos(\pi(\mu_0+\mu_1-\mu_\infty))}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)}.\end{aligned} \tag{E.19}$$

This yields

$$-4q_0q_1\frac{\mu_1}{\mu_0}\kappa_{11}^{01}\kappa_{12}^{02} = 4q_0q_1\frac{\cos(\pi(\mu_0-\mu_1-\mu_\infty))\cos(\pi(\mu_0-\mu_1+\mu_\infty))}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)}, \tag{E.20}$$

$$-4q_0q_1\frac{\mu_1}{\mu_0}\kappa_{11}^{02}\kappa_{12}^{01} = 4q_0q_1\frac{\cos(\pi(\mu_0+\mu_1+\mu_\infty))\cos(\pi(\mu_0+\mu_1-\mu_\infty))}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)}. \tag{E.21}$$

Moreover, in view of (E.6), we compute

$$\begin{aligned}(q_0 \pm q_1)^2 - (p_0q_1 - p_1q_0)^2 &= q_0^2(1-p_1^2) + q_1^2(1-p_0^2) + 2q_0q_1(p_0p_1 \pm 1) \\ &= q_0^2q_1\overline{q_1} + q_1^2q_0\overline{q_0} + 2q_0q_1(p_0p_1 \pm 1) = 2q_0q_1(\frac{q_0\overline{q_1}+\overline{q_0}q_1}{2} + p_0p_1 \pm 1).\end{aligned} \tag{E.22}$$

Altogether, using equations (E.20), (E.21) and (E.22) and the fact that, in general, $q_0, q_1 \neq 0$, the claimed equivalence can be rewritten as

$$\begin{aligned} 2\frac{\cos(\pi(\mu_0-\mu_1-\mu_\infty))\cos(\pi(\mu_0-\mu_1+\mu_\infty))}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)} &= \frac{q_0\overline{q_1}+\overline{q_0}q_1}{2} + p_0p_1 + 1 \\ \iff p_0p_1 + \frac{q_0\overline{q_1}+\overline{q_0}q_1}{2} &= \frac{\cos(2\pi\mu_0)\cos(2\pi\mu_1)+\cos(2\pi\mu_\infty)}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)} \\ \iff 2\frac{\cos(\pi(\mu_0+\mu_1+\mu_\infty))\cos(\pi(\mu_0+\mu_1-\mu_\infty))}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)} &= \frac{q_0\overline{q_1}+\overline{q_0}q_1}{2} + p_0p_1 - 1.\end{aligned} \tag{E.23}$$

Finally, using the formulas $\cos(x)\cos(y) = \frac{1}{2}(\cos(x-y)+\cos(x+y))$ and $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$, the following computations finish the proof:

$$\begin{aligned} 2\frac{\cos(\pi(\mu_0-\mu_1-\mu_\infty))\cos(\pi(\mu_0-\mu_1+\mu_\infty))}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)} - 1 &= \frac{\cos(2\pi\mu_\infty)+\cos(2\pi(\mu_0-\mu_1))}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)} - 1 \\ &= \frac{\cos(2\pi\mu_\infty)+\cos(2\pi\mu_0)\cos(2\pi(\mu_1)+\sin(2\pi\mu_0)\sin(2\pi(\mu_1)}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)} - 1 \\ &= \frac{\cos(2\pi\mu_\infty)+\cos(2\pi\mu_0)\cos(2\pi(\mu_1)}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)},\end{aligned} \tag{E.24}$$

$$2\frac{\cos(\pi(\mu_0+\mu_1+\mu_\infty))\cos(\pi(\mu_0+\mu_1-\mu_\infty))}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)}+1=\frac{\cos(2\pi\mu_\infty)+\cos(2\pi(\mu_0+\mu_1))}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)}+1$$
$$=\frac{\cos(2\pi\mu_\infty)+\cos(2\pi\mu_0)\cos(2\pi(\mu_1)-\sin(2\pi\mu_0)\sin(2\pi(\mu_1)}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)}+1$$
$$=\frac{\cos(2\pi\mu_\infty)+\cos(2\pi\mu_0)\cos(2\pi(\mu_1)}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)}. \quad (E.25)$$

□

With these preparations made, we can finally prove the claimed equivalence of the matrix equation (E.1) on the one side and the three scalar equations (E.2), (E.3) and (E.4) on the other side. Since (E.1) is equivalent to the four scalar equations

$$-i\lambda\frac{\sqrt{\mu_1}}{\sqrt{\mu_0}}\kappa_{11}^{01}=\frac{1}{2\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}}\omega_0^{-1}\omega_1(q_0+q_1-p_0q_1+p_1q_0), \quad (E.26)$$
$$-i\lambda\frac{\sqrt{\mu_1}}{\sqrt{\mu_0}}\kappa_{12}^{01}=\frac{1}{2\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}}\omega_0^{-1}\omega_1^{-1}(-q_0+q_1-p_0q_1+p_1q_0), \quad (E.27)$$
$$-i\lambda\frac{\sqrt{\mu_1}}{\sqrt{\mu_0}}\kappa_{11}^{02}=\frac{1}{2\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}}\omega_0\omega_1(-q_0+q_1+p_0q_1-p_1q_0), \quad (E.28)$$
$$-i\lambda\frac{\sqrt{\mu_1}}{\sqrt{\mu_0}}\kappa_{12}^{02}=\frac{1}{2\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}}\omega_0\omega_1^{-1}(q_0+q_1+p_0q_1-p_1q_0), \quad (E.29)$$

it remains to show that

$$\left.\begin{matrix}(E.26)\\(E.27)\\(E.28)\\(E.29)\end{matrix}\right\}\iff\left\{\begin{matrix}(E.2)\\(E.3)\\(E.4)\end{matrix}\right. \quad (E.30)$$

As remarked before, we exclude values of $\lambda\in S^1$ from our considerations, for which any of the terms involved in the following computations are not well defined.

Proof of "⇒" in (E.30). Dividing equation (E.29) by equation (E.27), we obtain

$$\frac{\kappa_{12}^{02}}{\kappa_{12}^{01}}=\omega_0^2\frac{q_0+q_1+p_0q_1-p_1q_0}{-q_0+q_1-p_0q_1+p_1q_0}, \quad (E.31)$$

which implies

$$\omega_0=\delta\frac{\sqrt{\kappa_{12}^{02}}}{\sqrt{\kappa_{12}^{01}}}\frac{\sqrt{-q_0+q_1-p_0q_1+p_1q_0}}{\sqrt{q_0+q_1+p_0q_1-p_1q_0}}, \quad (E.32)$$

where $\delta\in\{\pm1\}$. Similarly, dividing equation (E.26) by equation (E.27) yields

$$\frac{\kappa_{11}^{01}}{\kappa_{12}^{01}}=\omega_1^2\frac{q_0+q_1-p_0q_1+p_1q_0}{-q_0+q_1-p_0q_1+p_1q_0}, \quad (E.33)$$

and thus

$$\omega_1=\tilde\delta\frac{\sqrt{\kappa_{11}^{01}}}{\sqrt{\kappa_{12}^{01}}}\frac{\sqrt{-q_0+q_1-p_0q_1+p_1q_0}}{\sqrt{q_0+q_1-p_0q_1+p_1q_0}}, \quad (E.34)$$

where $\tilde\delta\in\{\pm1\}$. By multiplying (E.26) and (E.29), we infer that

$$-\frac{\mu_1}{\mu_0}\kappa_{11}^{01}\kappa_{12}^{02}=\frac{1}{4}\frac{1}{q_0q_1}((q_0+q_1)^2-(p_0q_1-p_1q_0)^2), \quad (E.35)$$

which, by lemma E.1, implies (E.4). As a further consequence of (E.35) we have

$$\epsilon(-i)\frac{\sqrt{\mu_1}}{\sqrt{\mu_0}}\sqrt{\kappa_{11}^{01}}\sqrt{\kappa_{12}^{02}}=\frac{1}{2\lambda\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}}\sqrt{q_0+q_1+p_0q_1-p_1q_0}\sqrt{q_0+q_1-p_0q_1+p_1q_0}, \quad (E.36)$$

where

$$\epsilon=\frac{\sqrt{\mu_0}}{\sqrt{\mu_1}}\frac{\sqrt{q_0+q_1+p_0q_1-p_1q_0}\sqrt{q_0+q_1-p_0q_1+p_1q_0}}{-2i\lambda\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}\sqrt{\kappa_{11}^{01}}\sqrt{\kappa_{12}^{02}}}\in\{\pm1\}. \quad (E.37)$$

Thus, using (E.35) once again, we infer that

$$\begin{aligned}\omega_0\omega_1 &= \delta\tilde{\delta}\frac{\sqrt{\kappa_{12}^{02}}\sqrt{\kappa_{11}^{01}}}{\kappa_{12}^{01}}\frac{-q_0+q_1-p_0q_1+p_1q_0}{\sqrt{q_0+q_1+p_0q_1-p_1q_0}\sqrt{q_0+q_1-p_0q_1+p_1q_0}}\\ &= \delta\tilde{\delta}\frac{\sqrt{\kappa_{12}^{02}}\sqrt{\kappa_{11}^{01}}}{\kappa_{12}^{01}}\frac{(-q_0+q_1-p_0q_1+p_1q_0)\sqrt{q_0+q_1+p_0q_1-p_1q_0}\sqrt{q_0+q_1-p_0q_1+p_1q_0}}{-4q_0q_1\frac{\mu_1}{\mu_0}\kappa_{11}^{01}\kappa_{12}^{02}}\\ &= \delta\tilde{\delta}\frac{-q_0+q_1-p_0q_1+p_1q_0}{-2i\lambda\frac{\sqrt{\mu_1}}{\sqrt{\mu_0}}\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}\kappa_{12}^{01}}\cdot\epsilon. \quad (E.38)\end{aligned}$$

In view of (E.27), this implies $\omega_0\omega_1 = \delta\tilde{\delta}\omega_0\omega_1\epsilon$, i.e.

$$\delta\tilde{\delta} = \epsilon, \quad (E.39)$$

which completes the proof. □

Proof of "⇐" in (E.30). By lemma E.1, we have the relations

$$-4q_0q_1\frac{\mu_1}{\mu_0}\kappa_{11}^{01}\kappa_{12}^{02} = (q_0+q_1)^2-(p_0q_1-p_1q_0)^2, \quad (E.40)$$

$$-4q_0q_1\frac{\mu_1}{\mu_0}\kappa_{11}^{02}\kappa_{12}^{01} = (q_0-q_1)^2-(p_0q_1-p_1q_0)^2. \quad (E.41)$$

Using these together with the assumptions, we obtain by direct computation

$$\begin{aligned}\omega_0^{-1}\omega_1(q_0+q_1-p_0q_1+p_1q_0) &= \delta\tilde{\delta}\frac{\sqrt{\kappa_{11}^{01}}}{\sqrt{\kappa_{12}^{02}}}\sqrt{q_0+q_1+p_0q_1-p_1q_0}\sqrt{q_0+q_1-p_0q_1+p_1q_0}\\ = \frac{\sqrt{\mu_0}}{\sqrt{\mu_1}}\frac{(q_0+q_1+p_0q_1-p_1q_0)(q_0+q_1-p_0q_1+p_1q_0)}{-2i\lambda\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}\kappa_{12}^{02}} &\overset{(E.40)}{=} -2i\lambda\frac{\sqrt{\mu_1}}{\sqrt{\mu_0}}\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}\kappa_{11}^{01}, \quad (E.42)\end{aligned}$$

$$\begin{aligned}\omega_0^{-1}\omega_1^{-1}(-q_0+q_1-p_0q_1+p_1q_0) &= \delta\tilde{\delta}\frac{\kappa_{12}^{01}}{\sqrt{\kappa_{12}^{02}}\sqrt{\kappa_{11}^{01}}}\sqrt{q_0+q_1+p_0q_1-p_1q_0}\sqrt{q_0+q_1-p_0q_1+p_1q_0}\\ = \frac{\sqrt{\mu_0}}{\sqrt{\mu_1}}\frac{(q_0+q_1+p_0q_1-p_1q_0)(q_0+q_1-p_0q_1+p_1q_0)\kappa_{12}^{01}}{-2i\lambda\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}\kappa_{12}^{02}\kappa_{11}^{01}} &\overset{(E.40)}{=} -2i\lambda\frac{\sqrt{\mu_1}}{\sqrt{\mu_0}}\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}\kappa_{12}^{01}, \quad (E.43)\end{aligned}$$

$$\begin{aligned}\omega_0\omega_1(-q_0+q_1+p_0q_1-p_1q_0) &= \delta\tilde{\delta}\frac{\sqrt{\kappa_{12}^{02}}\sqrt{\kappa_{11}^{01}}}{\kappa_{12}^{01}}\frac{(-q_0+q_1-p_0q_1+p_1q_0)(-q_0+q_1+p_0q_1-p_1q_0)}{\sqrt{q_0+q_1+p_0q_1-p_1q_0}\sqrt{q_0+q_1-p_0q_1+p_1q_0}}\\ = \frac{\sqrt{\mu_0}}{\sqrt{\mu_1}}\frac{(-q_0+q_1-p_0q_1+p_1q_0)(-q_0+q_1+p_0q_1-p_1q_0)}{-2i\lambda\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}\kappa_{12}^{01}} &\overset{(E.41)}{=} -2i\lambda\frac{\sqrt{\mu_1}}{\sqrt{\mu_0}}\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}\kappa_{11}^{02} \quad (E.44)\end{aligned}$$

and

$$\begin{aligned}\omega_0\omega_1^{-1}(q_0+q_1+p_0q_1-p_1q_0) &= \delta\tilde{\delta}\frac{\sqrt{\kappa_{12}^{02}}}{\sqrt{\kappa_{11}^{01}}}\sqrt{q_0+q_1+p_0q_1-p_1q_0}\sqrt{q_0+q_1-p_0q_1+p_1q_0}\\ = \frac{\sqrt{\mu_0}}{\sqrt{\mu_1}}\frac{(q_0+q_1-p_0q_1+p_1q_0)(q_0+q_1+p_0q_1-p_1q_0)}{-2i\lambda\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}\kappa_{11}^{01}} &\overset{(E.40)}{=} -2i\lambda\frac{\sqrt{\mu_1}}{\sqrt{\mu_0}}\sqrt{\lambda^{-1}q_0}\sqrt{\lambda^{-1}q_1}\kappa_{12}^{02}. \quad (E.45)\end{aligned}$$

These relations imply equations (E.26) to (E.29). □

F Appendix: Proof of remark 3.55

We prove the statement of remark 3.55:

Lemma F.1. *Equation* (3.9.51), *i.e.*

$$p_0p_1 + \frac{q_0\overline{q_1} + \overline{q_0}q_1}{2} = \frac{\cos(2\pi\mu_0)\cos(2\pi\mu_1) + \cos(2\pi\mu_\infty)}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)}, \tag{F.1}$$

is solvable for functions p_0, q_0, p_1, q_1 *in* $\lambda \in S^1$ *satisfying* (3.9.50), *i.e.*

$$p_j^2 + q_j\overline{q_j} = 1 \quad \text{and} \quad p_j = \overline{p_j}, \tag{F.2}$$

if and only if the eigenvalues μ_j *of the Delaunay matrices* D_j *inducing the potential* η *meet the unitarizability condition* (3.5.28), *i.e.*

$$0 \leq \frac{cos(\pi(\mu_0 - \mu_1 - \mu_\infty))cos(\pi(\mu_0 - \mu_1 + \mu_\infty))}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)} \leq 1, \tag{F.3}$$

for all $\lambda \in S^1$.

Proof. On the one hand, presume the eigenvalues μ_j of the Delaunay matrices D_j meet the unitarizability condition (F.3) for $\lambda \in S^1$. Then, by theorem 3.14, there exists a matrix T simultaneously untarizing the monodromy matrices M_0 and M_1. By theorem 3.53, T is of the form (3.9.49) involving functions p_0, q_0, p_1, q_1 in $\lambda \in S^1$ satisfying (F.2) and (F.1). I.e., (F.1) is solvable.

On the other hand, suppose there exist functions p_0, q_0, p_1, q_1 in $\lambda \in S^1$ satisfying (F.2) and solving (F.1). Decomposing $q_j = u_j + iv_j$ with real valued functions u_j and v_j, we note that

$$p_j^2 = 1 - q_j\overline{q_j} = 1 - u_j^2 - v_j^2, \tag{F.4}$$

$$\frac{q_0\overline{q_1} + \overline{q_0}q_1}{2} = \frac{1}{2}(u_0u_1 + v_0v_1 + i(v_0u_1 - u_0v_1) + u_0u_1 + v_0v_1 - i(v_0u_1 - u_0v_1)) = u_0u_1 + v_0v_1. \tag{F.5}$$

Using these relations and applying elementary estimates, we obtain

$$\begin{aligned} |p_0p_1 + \frac{q_0\overline{q_1} + \overline{q_0}q_1}{2}| &= |p_0p_1 + u_0u_1 + v_0v_1| \\ &\leq |p_0p_1| + |u_0u_1| + |v_0v_1| \leq \frac{1}{2}(p_0^2 + p_1^2) + \frac{1}{2}(u_0^2 + u_1^2) + \frac{1}{2}(v_0^2 + v_1^2) = 1. \end{aligned} \tag{F.6}$$

As, by assumption, p_0, q_0, p_1, q_1 solve (F.1) for $\lambda \in S^1$, we conclude that, for $\lambda \in S^1$,

$$-1 \leq \frac{\cos(2\pi\mu_0)\cos(2\pi\mu_1) + \cos(2\pi\mu_\infty)}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)} \leq 1. \tag{F.7}$$

Recall now that, by remark 3.16, $s_0 \neq t_0$ and $s_1 \neq t_1$. Thus, by lemma B.5 of appendix B, we have $0 < \mu_j \leq \frac{1}{2}$ for $j = 0, 1$ and for all $\lambda \in S^1$ and therefore $\sin(2\pi\mu_0)\sin(2\pi\mu_1) \geq 0$ for all $\lambda \in S^1$. Consequently, (F.7) is equivalent to

$$-\sin(2\pi\mu_0)\sin(2\pi\mu_1) \leq \cos(2\pi\mu_0)\cos(2\pi\mu_1) + \cos(2\pi\mu_\infty) \leq \sin(2\pi\mu_0)\sin(2\pi\mu_1) \tag{F.8}$$

and, by further transformations, equivalent to

$$0 \leq \cos(2\pi\mu_0)\cos(2\pi\mu_1) + \sin(2\pi\mu_0)\sin(2\pi\mu_1) + \cos(2\pi\mu_\infty) \leq 2\sin(2\pi\mu_0)\sin(2\pi\mu_1) \tag{F.9}$$

and

$$0 \leq \cos(2\pi\mu_0 - 2\pi\mu_1) + \cos(2\pi\mu_\infty) \leq 2\sin(2\pi\mu_0)\sin(2\pi\mu_1) \tag{F.10}$$

and

$$\begin{aligned} 0 \leq \frac{1}{2}\left(\cos(\pi(\mu_0 - \mu_1 - \mu_\infty) + \pi(\mu_0 - \mu_1 + \mu_\infty)) + \cos(\pi(\mu_0 - \mu_1 - \mu_\infty) - \pi(\mu_0 - \mu_1 + \mu_\infty))\right) \\ \leq \sin(2\pi\mu_0)\sin(2\pi\mu_1). \end{aligned} \tag{F.11}$$

This yields, using the trigonometric identity $\frac{1}{2}(\cos(x+y) + \cos(x-y)) = \cos(x)\cos(y)$,

$$0 \leq \cos(\pi(\mu_0 - \mu_1 - \mu_\infty))\cos(\pi(\mu_0 - \mu_1 + \mu_\infty)) \leq \sin(2\pi\mu_0)\sin(2\pi\mu_1). \tag{F.12}$$

Since, as stated before, $\sin(2\pi\mu_0)\sin(2\pi\mu_1) \geq 0$ for $\lambda \in S^1$, we end up with

$$0 \leq \frac{cos(\pi(\mu_0 - \mu_1 - \mu_\infty))cos(\pi(\mu_0 - \mu_1 + \mu_\infty))}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)} \leq 1. \tag{F.13}$$

□

G Appendix: Proof of remark 3.56

We prove the statement of remark 3.56:

Lemma G.1. *For $j = 0, 1, \infty$, let p_j, q_j be the functions occurring in the unitary monodromy matrix $\hat{M}_j$ as in* (3.9.26)*, satisfying* (3.9.27)*. The following holds:*

$$\begin{aligned}
&p_0, q_0, p_1, q_1 \ \ \textit{solve} \ \ p_0 p_1 + \frac{q_0 \overline{q_1} + \overline{q_0} q_1}{2} = \frac{\cos(2\pi\mu_0)\cos(2\pi\mu_1) + \cos(2\pi\mu_\infty)}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)} \\
&\qquad \textit{and} \ \ p_\infty, q_\infty \ \ \textit{are given by} \ (3.9.33) \ \textit{and} \ (3.9.34) \\
\Longleftrightarrow \quad &p_0, q_0, p_\infty, q_\infty \ \ \textit{solve} \ \ p_0 p_\infty + \frac{q_0 \overline{q_\infty} + \overline{q_0} q_\infty}{2} = \frac{\cos(2\pi\mu_0)\cos(2\pi\mu_\infty) + \cos(2\pi\mu_1)}{\sin(2\pi\mu_0)\sin(2\pi\mu_\infty)} \\
&\qquad \textit{and} \ \ p_1, q_1 \ \ \textit{are given by} \ (3.9.35) \ \textit{and} \ (3.9.36) \\
\Longleftrightarrow \quad &p_1, q_1, p_\infty, q_\infty \ \ \textit{solve} \ \ p_1 p_\infty + \frac{q_1 \overline{q_\infty} + \overline{q_1} q_\infty}{2} = \frac{\cos(2\pi\mu_1)\cos(2\pi\mu_\infty) + \cos(2\pi\mu_0)}{\sin(2\pi\mu_1)\sin(2\pi\mu_\infty)} \\
&\qquad \textit{and} \ \ p_0, q_0 \ \ \textit{are given by} \ (3.9.37) \ \textit{and} \ (3.9.38).
\end{aligned} \tag{G.1}$$

Proof. Recall the identity (3.9.32), $\hat{M}_0 \hat{M}_1 \hat{M}_\infty = \mathrm{I}$, which in view of remark 3.48 implies the relations (3.9.33), (3.9.34), (3.9.35), (3.9.36) (3.9.37) and (3.9.38).

Using (3.9.33) and (3.9.34) (and (3.9.27)), we prove the implication

$$\begin{aligned}
&p_0, q_0, p_1, q_1 \ \ \text{solve} \ \ p_0 p_1 + \frac{q_0 \overline{q_1} + \overline{q_0} q_1}{2} = \frac{\cos(2\pi\mu_0)\cos(2\pi\mu_1) + \cos(2\pi\mu_\infty)}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)} \\
\Longrightarrow \quad &p_0, q_0, p_\infty, q_\infty \ \ \text{solve} \ \ p_0 p_\infty + \frac{q_0 \overline{q_\infty} + \overline{q_0} q_\infty}{2} = \frac{\cos(2\pi\mu_0)\cos(2\pi\mu_\infty) + \cos(2\pi\mu_1)}{\sin(2\pi\mu_0)\sin(2\pi\mu_\infty)}.
\end{aligned} \tag{G.2}$$

To this end, we compute

$$\begin{aligned}
& i\sin(2\pi\mu_\infty)\Big(p_0 p_\infty + \frac{q_0 \overline{q_\infty} + \overline{q_0} q_\infty}{2}\Big) = \ p_0 (i\sin(2\pi\mu_\infty) p_\infty) + \frac{q_0}{2}(i\sin(2\pi\mu_\infty)\overline{q_\infty}) + \frac{\overline{q_0}}{2}(i\sin(2\pi\mu_\infty) q_\infty) \\
= \ & p_0[-\cos(2\pi\mu_\infty) - \cos(2\pi\mu_0)\cos(2\pi\mu_1) + i\cos(2\pi\mu_0)\sin(2\pi\mu_1) p_1 + i\sin(2\pi\mu_0)\cos(2\pi\mu_1) p_0 \\
& + \sin(2\pi\mu_0)\sin(2\pi\mu_1)(p_0 p_1 + q_0 \overline{q_1})] \\
& + \frac{q_0}{2}[i\cos(2\pi\mu_0)\sin(2\pi\mu_1)\overline{q_1} + i\sin(2\pi\mu_0)\cos(2\pi\mu_1)\overline{q_0} - \sin(2\pi\mu_0)\sin(2\pi\mu_1)(p_0\overline{q_1} - p_1\overline{q_0})] \\
& + \frac{\overline{q_0}}{2}[i\cos(2\pi\mu_0)\sin(2\pi\mu_1) q_1 + i\sin(2\pi\mu_0)\cos(2\pi\mu_1) q_0 + \sin(2\pi\mu_0)\sin(2\pi\mu_1)(p_0 q_1 - p_1 q_0)] \\
= \ & p_0\left(-\cos(2\pi\mu_\infty) - \cos(2\pi\mu_0)\cos(2\pi\mu_1)\right) + p_0 p_1\left(i\cos(2\pi\mu_0)\sin(2\pi\mu_1)\right) + p_0^2\left(i\sin(2\pi\mu_0)\cos(2\pi\mu_1)\right) \\
& + p_0^2 p_1\left(\sin(2\pi\mu_0)\sin(2\pi\mu_1)\right) + p_0 q_0 \overline{q_1}\left(\sin(2\pi\mu_0)\sin(2\pi\mu_1)\right) \\
& + \frac{q_0\overline{q_1}}{2}\left(i\cos(2\pi\mu_0)\sin(2\pi\mu_1)\right) + \frac{q_0\overline{q_0}}{2}\left(i\sin(2\pi\mu_0)\cos(2\pi\mu_1)\right) \\
& - \frac{p_0 q_0\overline{q_1}}{2}\left(\sin(2\pi\mu_0)\sin(2\pi\mu_1)\right) + \frac{p_1 q_0\overline{q_0}}{2}\left(\sin(2\pi\mu_0)\sin(2\pi\mu_1)\right) \\
& + \frac{\overline{q_0} q_1}{2}\left(i\cos(2\pi\mu_0)\sin(2\pi\mu_1)\right) + \frac{q_0\overline{q_0}}{2}\left(i\sin(2\pi\mu_0)\cos(2\pi\mu_1)\right) \\
& + \frac{p_0\overline{q_0} q_1}{2}\left(\sin(2\pi\mu_0)\sin(2\pi\mu_1)\right) - \frac{p_1 q_0\overline{q_0}}{2}\left(\sin(2\pi\mu_0)\sin(2\pi\mu_1)\right) \\
= \ & \left(p_0 p_1 + \frac{q_0\overline{q_1} + \overline{q_0} q_1}{2}\right)\left(i\cos(2\pi\mu_0)\sin(2\pi\mu_1)\right) + \left(p_0^2 + q_0\overline{q_0}\right)\left(i\sin(2\pi\mu_0)\cos(2\pi\mu_1)\right) \\
& + p_o\left(p_0 p_1 + \frac{q_0\overline{q_1} + \overline{q_0} q_1}{2}\right)\left(\sin(2\pi\mu_0)\sin(2\pi\mu_1)\right) + p_0\left(-\cos(2\pi\mu_\infty) - \cos(2\pi\mu_0)\cos(2\pi\mu_1)\right) \\
= \ & i\left(\cos(2\pi\mu_0)\cos(2\pi\mu_1) + \cos(2\pi\mu_\infty)\right)\frac{\cos(2\pi\mu_0)}{\sin(2\pi\mu_0)} + i\sin(2\pi\mu_0)\cos(2\pi\mu_1).
\end{aligned} \tag{G.3}$$

This implies

$$\begin{aligned}
&p_0 p_\infty + \frac{q_0\overline{q_\infty} + \overline{q_0} q_\infty}{2} \\
&\qquad = (\cos(2\pi\mu_0)\cos(2\pi\mu_1) + \cos(2\pi\mu_\infty))\frac{\cos(2\pi\mu_0)}{\sin(2\pi\mu_0)\sin(2\pi\mu_\infty)} + \frac{\sin(2\pi\mu_0)\cos(2\pi\mu_1)}{\sin(2\pi\mu_\infty)} \\
&\qquad = \frac{\cos(2\pi\mu_0)\cos(2\pi\mu_\infty) + \cos(2\pi\mu_1)}{\sin(2\pi\mu_0)\sin(2\pi\mu_i nfty)},
\end{aligned} \tag{G.4}$$

which proves the claimed implication.

Simply shifting indices, we prove completely analogously

$$\begin{aligned}
& p_0, q_0, p_\infty, q_\infty \ \text{ solve } \ p_0 p_\infty + \frac{q_0\overline{q_\infty} + \overline{q_0} q_\infty}{2} = \frac{\cos(2\pi\mu_0)\cos(2\pi\mu_\infty) + \cos(2\pi\mu_1)}{\sin(2\pi\mu_0)\sin(2\pi\mu_\infty)} \\
\Longrightarrow \quad & p_1, q_1, p_\infty, q_\infty \ \text{ solve } \ p_1 p_\infty + \frac{q_1\overline{q_\infty} + \overline{q_1} q_\infty}{2} = \frac{\cos(2\pi\mu_1)\cos(2\pi\mu_\infty) + \cos(2\pi\mu_0)}{\sin(2\pi\mu_1)\sin(2\pi\mu_\infty)}
\end{aligned} \tag{G.5}$$

by using (3.9.35) and (3.9.36), and, by using (3.9.37) and (3.9.38)

$$\begin{aligned}
& p_1, q_1, p_\infty, q_\infty \ \text{ solve } \ p_1 p_\infty + \frac{q_1\overline{q_\infty} + \overline{q_1} q_\infty}{2} = \frac{\cos(2\pi\mu_1)\cos(2\pi\mu_\infty) + \cos(2\pi\mu_0)}{\sin(2\pi\mu_1)\sin(2\pi\mu_\infty)} \\
\Longrightarrow \quad & p_0, q_0, p_1, q_1 \ \text{ solve } \ p_0 p_1 + \frac{q_0\overline{q_1} + \overline{q_0} q_1}{2} = \frac{\cos(2\pi\mu_0)\cos(2\pi\mu_1) + \cos(2\pi\mu_\infty)}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)}.
\end{aligned} \tag{G.6}$$

Altogether, the statement of the lemma follows. □

H Appendix: Proof of theorem 5.16

In this appendix, we give the proof of

Theorem 5.16. For all $\lambda \in \mathbb{C}^*$ the following holds:

$$4\sin^2(\pi\mu(\lambda)) - 1 = 4\pi^2 \prod_{k=-\infty}^{\infty} C_k(1-\frac{\lambda}{\lambda_k})(1+\frac{\lambda}{\lambda_k})(1-\frac{\lambda^{-1}}{\lambda_k})(1+\frac{\lambda^{-1}}{\lambda_k}), \tag{H.1}$$

where

$$C_k := \begin{cases} -\lambda_k^2 w & \text{for } k = 0 \\ \lambda_k^2 w & \text{for } k = -1 \\ \frac{\lambda_k^2 w}{k^2(1+\frac{1}{k})^{\frac{1}{3}}} & \text{for } k \in \mathbb{Z} \setminus \{-1, 0\} \end{cases} \tag{H.2}$$

and, for $k \in \mathbb{Z}$,

$$\lambda_k := \sqrt{\frac{1}{2w}\left[d_k + \sqrt{d_k^2 - 4w^2}\right]} \tag{H.3}$$

with

$$d_k := (\frac{1}{6}+k)^2 - \frac{1}{4} + 2w. \tag{H.4}$$

Recall from lemma 5.3 that

$$\mu(\lambda) = \sqrt{\frac{1}{4} + w(\lambda - \lambda^{-1})^2}, \tag{H.5}$$

where $w = s_0t_0 = s_1t_1 = s_\infty t_\infty$ and s_j, t_j denote the parameters occurring in the Delaunay matrices D_j defined in (3.5.7). (Note that remark 3.16 implies that $w \neq \frac{1}{16}$.)

The proof of theorem 5.16 is prepared in the following three lemmas.

Lemma H.1. *For all $k \in \mathbb{Z}$ let $\mathcal{I}_k := \{\pm\lambda_k, \pm\lambda_k^{-1}\}$ as in lemma 5.15. Then, for all $\lambda \in \mathbb{C}^* \setminus \bigcup_{k\in\mathbb{Z}} \mathcal{I}_k$ the following holds:*

$$4\sin^2(\pi\mu(\lambda)) - 1 = \frac{-4\pi^2}{\Gamma(\frac{1}{6}-\mu(\lambda))\Gamma(\frac{1}{6}+\mu(\lambda))\Gamma(\frac{5}{6}-\mu(\lambda))\Gamma(\frac{5}{6}+\mu(\lambda))}, \tag{H.6}$$

where Γ denotes the Gamma function $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$.

Remark H.2. Writing $\sin(\pi\mu)$ in its power series representation, we observe that the expression $\sin^2(\pi\mu)$ only involves even powers of μ. Since, by remark 3.13, μ^2 defines a holomorphic function on $\mathbb{C}^*$, we interpret $\sin^2(\pi\mu)$ as a holomorphic function of $\lambda \in \mathbb{C}^*$.

Proof of lemma H.1. We use the following well known formula for the Gamma function:

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \quad z \notin \mathbb{Z}. \tag{H.7}$$

We consider the product of Gamma functions

$$\Gamma(\frac{1}{6}-\mu(\lambda))\Gamma(\frac{1}{6}+\mu(\lambda))\Gamma(\frac{5}{6}-\mu(\lambda))\Gamma(\frac{5}{6}+\mu(\lambda)). \tag{H.8}$$

This product is well defined (and non-zero) for all $\lambda \in \mathbb{C}^*$, for which none of the occurring arguments $\frac{1}{6} \pm \mu(\lambda)$, $\frac{5}{6} \pm \mu(\lambda)$ takes a non-positive integer value. I.e., the product is well defined (and non-zero) for all $\lambda \in \mathbb{C}^*$, such that $\mu(\lambda) \neq \pm(\frac{1}{6}+k)$ for all $k \in \mathbb{Z}$, or, equivalently due to lemma 5.15, such that $\lambda \notin \bigcup_{k\in\mathbb{Z}} \mathcal{I}_k$. Thus, using (H.7), we can compute for all $\lambda \in \mathbb{C}^* \setminus \bigcup_{k\in\mathbb{Z}} \mathcal{I}_k$:

$$\begin{aligned} &\frac{1}{\Gamma(\frac{1}{6}-\mu(\lambda))\Gamma(\frac{1}{6}+\mu(\lambda))\Gamma(\frac{5}{6}-\mu(\lambda))\Gamma(\frac{5}{6}+\mu(\lambda))} \\ &= \frac{1}{\Gamma(\frac{1}{6}-\mu(\lambda))\Gamma(1-(\frac{1}{6}-\mu(\lambda)))\Gamma(\frac{1}{6}+\mu(\lambda)))\Gamma(1-(\frac{1}{6}+\mu(\lambda)))} = \frac{\sin(\pi(\frac{1}{6}-\mu(\lambda)))}{\pi}\frac{\sin(\pi(\frac{1}{6}+\mu(\lambda)))}{\pi} \\ &= \frac{[\sin(\frac{\pi}{6})\cos(-\pi\mu(\lambda)) + \cos(\frac{\pi}{6})\sin(-\pi\mu(\lambda))][\sin(\frac{\pi}{6})\cos(\pi\mu(\lambda)) + \cos(\frac{\pi}{6})\sin(\pi\mu(\lambda))]}{\pi^2} \\ &= \frac{\sin^2(\frac{\pi}{6})\cos^2(\pi\mu(\lambda)) - \cos^2(\frac{\pi}{6})\sin^2(\pi\mu(\lambda))}{\pi^2} = \frac{\cos^2(\pi\mu(\lambda)) - 3\sin^2(\pi\mu(\lambda))}{4\pi^2} = \frac{1-4\sin^2(\pi\mu(\lambda))}{4\pi^2}. \end{aligned} \tag{H.9}$$

This implies the claim. ☐

Lemma H.3. *Let Γ denote the Gamma function $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$ and for all $k \in \mathbb{Z}$ let $\mathcal{I}_k := \{\pm\lambda_k, \pm\lambda_k^{-1}\}$ as in lemma 5.15. Moreover, let*

$$\tilde{C}_k = \begin{cases} 1 & \text{for } k \in \{-1,0\} \\ \frac{1}{k^2(1+\frac{1}{k})^{\frac{1}{3}}} & \text{for } k \in \mathbb{Z} \setminus \{-1,0\} \end{cases} \tag{H.10}$$

Then, for all $\lambda \in \mathbb{C}^* \setminus \bigcup_{k\in\mathbb{Z}} \mathcal{I}_k$ we have

1.
$$\frac{1}{\Gamma(\frac{1}{6}-\mu(\lambda))\Gamma(\frac{1}{6}+\mu(\lambda))} = \prod_{k=0}^{\infty} \tilde{C}_k \left[(\frac{1}{6}+k)^2 - (\mu(\lambda))^2\right], \tag{H.11}$$

2.
$$\frac{1}{\Gamma(\frac{5}{6}-\mu(\lambda))\Gamma(\frac{5}{6}+\mu(\lambda))} = \prod_{k=-\infty}^{-1} \tilde{C}_k \left[(\frac{1}{6}+k)^2 - (\mu(\lambda))^2\right]. \tag{H.12}$$

Proof. We apply the following formula, which allows to represent $\Gamma(z)$ as an infinite product. For all $z \in \mathbb{C}$, excepting the non-positive integers, we have (cf. [33], chapter 2, 2.2)

$$\Gamma(z) = \frac{1}{z}\prod_{k=1}^{\infty} \frac{(1+\frac{1}{k})^z}{1+\frac{z}{k}}. \tag{H.13}$$

The expressions

$$\frac{1}{\Gamma(\frac{1}{6}-\mu(\lambda))\Gamma(\frac{1}{6}+\mu(\lambda))} \quad \text{and} \quad \frac{1}{\Gamma(\frac{5}{6}-\mu(\lambda))\Gamma(\frac{5}{6}+\mu(\lambda))} \tag{H.14}$$

are well defined for all $\lambda \in \mathbb{C}^* \setminus \bigcup_{k\in\mathbb{Z}} \mathcal{I}_k$. This has already been explained in the proof of the previous lemma. Thus, applying (H.13), we obtain for all $\lambda \in \mathbb{C}^* \setminus \bigcup_{k\in\mathbb{Z}} \mathcal{I}_k$

$$\begin{aligned}\frac{1}{\Gamma(\frac{1}{6}-\mu(\lambda))\Gamma(\frac{1}{6}+\mu(\lambda))} &= (\frac{1}{6}-\mu(\lambda))(\frac{1}{6}+\mu(\lambda))\prod_{k=1}^{\infty}\frac{(k+\frac{1}{6}-\mu(\lambda))(k+\frac{1}{6}+\mu(\lambda))}{k^2(1+\frac{1}{k})^{\frac{1}{3}}} \\ &= \prod_{k=0}^{\infty} \tilde{C}_k \left[(\frac{1}{6}+k)^2 - (\mu(\lambda))^2\right], \end{aligned} \tag{H.15}$$

where $\tilde{C}_k$ is defined in (H.10).

In view of the formula ($k \neq -1$)

$$(-k-1)^2(1+\frac{1}{-k-1})^{\frac{5}{3}} = (k+1)^2(\frac{k}{k+1})^{\frac{5}{3}} = (k+1)^{\frac{1}{3}}k^{\frac{5}{3}} = (1+\frac{1}{k})^{\frac{1}{3}}k^2 \tag{H.16}$$

we furthermore compute for all $\lambda \in \mathbb{C}^* \setminus \bigcup_{k\in\mathbb{Z}} \mathcal{I}_k$

$$\begin{aligned}\frac{1}{\Gamma(\frac{5}{6}-\mu(\lambda))\Gamma(\frac{5}{6}+\mu(\lambda))} &= (\frac{5}{6}-\mu(\lambda))(\frac{5}{6}+\mu(\lambda))\prod_{l=1}^{\infty}\frac{(l+\frac{5}{6}-\mu(\lambda))(l+\frac{5}{6}+\mu(\lambda))}{l^2(1+\frac{1}{l})^{\frac{5}{3}}} \\ = \left[(\frac{5}{6})^2-(\mu(\lambda))^2\right]\prod_{l=1}^{\infty}\frac{(\frac{5}{6}+l)^2-(\mu(\lambda))^2}{l^2(1+\frac{1}{l})^{\frac{5}{3}}} &= \left[(\frac{1}{6}-1)^2-(\mu(\lambda))^2\right]\prod_{k=-\infty}^{-2}\frac{(\frac{1}{6}+k)^2-(\mu(\lambda))^2}{(-k-1)^2(1+\frac{1}{-k-1})^{\frac{5}{3}}} \\ = \left[(\frac{1}{6}-1)^2-(\mu(\lambda))^2\right]\prod_{k=-\infty}^{-2}\frac{(\frac{1}{6}+k)^2-(\mu(\lambda))^2}{(1+\frac{1}{k})^{\frac{1}{3}}k^2} &= \prod_{k=-\infty}^{-1}\tilde{C}_k\left[(\frac{1}{6}+k)^2-(\mu(\lambda))^2\right], \end{aligned} \tag{H.17}$$

where we have substituted $k = -l-1$ and $\tilde{C}_k$ is defined in (H.10). □

Remark H.4. Note that the infinite product (H.13) represents the Gamma function Γ and thus takes finite values in $\mathbb{C}$ on the complex plane excluding the non-positive integers. Consequently, also the product of two (or, more generally, of finitely many) such infinite products of the form (H.13) is well defined on the complex plane excluding the non-positive integers. This justifies the calculations involving infinite products occurring in the proof above as well as in the proof of theorem 5.16 below.

Lemma H.5. *For all* $\lambda \in \mathbb{C}^*$ *and all* $k \in \mathbb{Z}$ *the following holds:*

$$(\frac{1}{6}+k)^2 - (\mu(\lambda))^2 = \lambda_k^2 w(1-\frac{\lambda}{\lambda_k})(1+\frac{\lambda}{\lambda_k})(1-\frac{\lambda^{-1}}{\lambda_k})(1+\frac{\lambda^{-1}}{\lambda_k}) \tag{H.18}$$

where λ_k *is defined by* (5.5.4) *in lemma 5.15.*

Proof. Recalling that $\mu(\lambda) = \sqrt{\frac{1}{4}+w(\lambda-\lambda^{-1})^2}$, this is proved by a direct computation. In view of lemma 5.15 we have

$$\lambda_0 = \sqrt{\frac{1}{2w}[d_0+\sqrt{d_0^2-4w^2}]} \text{ and } \lambda_0^{-1} = -\sqrt{\frac{1}{2w}[d_0-\sqrt{d_0^2-4w^2}]}, \tag{H.19}$$

$$\lambda_k = \sqrt{\frac{1}{2w}[d_k+\sqrt{d_k^2-4w^2}]} \text{ and } \lambda_k^{-1} = \sqrt{\frac{1}{2w}[d_k-\sqrt{d_k^2-4w^2}]} \text{ for } k \neq 0, \tag{H.20}$$

which in any case ($k \in \mathbb{Z}$) implies

$$\lambda_k^2 + \lambda_k^{-2} = \frac{d_k}{w}. \tag{H.21}$$

Hence, we compute for all $\lambda \in \mathbb{C}^*$

$$\begin{aligned}(\frac{1}{6}+k)^2-(\mu(\lambda))^2 &= d_k-2w-w(\lambda-\lambda^{-1})^2 = w(\lambda_k^2+\lambda_k^{-2})-w(\lambda^2+\lambda^{-2}) = \lambda_k^2 w(1+\lambda_k^{-4}-\lambda^2\lambda_k^{-2}-\lambda^{-2}\lambda_k^{-2}) \\ &= \lambda_k^2 w(1-\lambda_k^{-2}\lambda^2)(1-\lambda_k^{-2}\lambda^{-2}) = \lambda_k^2 w(1-\frac{\lambda}{\lambda_k})(1+\frac{\lambda}{\lambda_k})(1-\frac{\lambda^{-1}}{\lambda_k})(1+\frac{\lambda^{-1}}{\lambda_k}),\end{aligned} \tag{H.22}$$

which proves the claim. □

Proof of theorem 5.16. In view of the lemmas H.1, H.3 and H.5, we obtain for all $\lambda \in \mathbb{C}^* \setminus \bigcup_{k\in\mathbb{Z}} \mathcal{I}_k$, where $\mathcal{I}_k := \{\pm\lambda_k, \pm\lambda_k^{-1}\}$, that

$$\begin{aligned}4\sin^2(\pi\mu(\lambda)) - 1 &= \frac{-4\pi^2}{\Gamma(\frac{1}{6}-\mu(\lambda))\Gamma(\frac{1}{6}+\mu(\lambda))\Gamma(\frac{5}{6}-\mu(\lambda))\Gamma(\frac{5}{6}+\mu(\lambda))} \\ &= -4\pi^2 \prod_{k=-\infty}^{\infty} \tilde{C}_k \left[(\frac{1}{6}+k)^2-(\mu(\lambda))^2\right] = 4\pi^2 \prod_{k=-\infty}^{\infty} C_k(1-\frac{\lambda}{\lambda_k})(1+\frac{\lambda}{\lambda_k})(1-\frac{\lambda^{-1}}{\lambda_k})(1+\frac{\lambda^{-1}}{\lambda_k}).\end{aligned} \tag{H.23}$$

Here, $\tilde{C}_k$ is given in (H.10).

We infer that

$$4\sin^2(\pi\mu(\lambda)) - 1 = 4\pi^2 \prod_{k=-\infty}^{\infty} C_k(1-\frac{\lambda}{\lambda_k})(1+\frac{\lambda}{\lambda_k})(1-\frac{\lambda^{-1}}{\lambda_k})(1+\frac{\lambda^{-1}}{\lambda_k}) \tag{H.24}$$

for all $\lambda \in \mathbb{C}^* \setminus \bigcup_{k\in\mathbb{Z}} \mathcal{I}_k$. But, naturally, this equation also holds for all $\lambda \in \bigcup_{k\in\mathbb{Z}} \mathcal{I}_k$, as both sides of the equation are zero for $\lambda \in \bigcup_{k\in\mathbb{Z}} \mathcal{I}_k$ (cf. lemma 5.15). Altogether, the claim follows. □

I Appendix: Proofs of lemma 5.21 and lemma 5.24

First, we give the proof of

Lemma 5.21. For all $k \in \mathbb{Z}$, let λ_k and C_k be given by (5.5.4) and (5.5.32), respectively. Moreover, let the λ-dependent functions $p_k^{(\nu)}(\lambda)$, $\nu \in \{1,2,3,4\}$ be defined by (5.5.44), (5.5.45), (5.5.46) and (5.5.47), respectively. Then, we have:

1. The infinite product $\prod_{k=-\infty}^{\infty} \sqrt{C_k}$ converges.
2. The infinite product $\prod_{k=-\infty}^{\infty} p_k^{(1)}$ is normally convergent on $\mathbb{C}^*$.
3. The infinite product $\prod_{k=-\infty}^{\infty} p_k^{(2)}$ is normally convergent on $\mathbb{C}^*$.
4. The infinite product $\prod_{k=-\infty}^{\infty} p_k^{(3)}$ is divergent on $\mathbb{C}^*$.
5. The infinite product $\prod_{k=-\infty}^{\infty} p_k^{(4)}$ is divergent on $\mathbb{C}^*$.

For the convenience of the reader, we recall the definitions of λ_k, of C_k and of the functions $p_k^{(\nu)}(\lambda)$, $\nu \in \{1,2,3,4\}$: For $k \in \mathbb{Z}$ we have

$$\lambda_k = \sqrt{\frac{1}{2w}\left[d_k + \sqrt{d_k^2 - 4w^2}\right]}, \tag{I.1}$$

where

$$d_k := (\frac{1}{6}+k)^2 - \frac{1}{4} + 2w. \tag{I.2}$$

Moreover, also for $k \in \mathbb{Z}$,

$$C_k := \begin{cases} -\lambda_k^2 w & \text{for } k = 0 \\ \lambda_k^2 w & \text{for } k = -1 \\ \frac{\lambda_k^2 w}{k^2(1+\frac{1}{k})^{\frac{1}{3}}} & \text{for } k \in \mathbb{Z}\setminus\{-1,0\} \end{cases} \tag{I.3}$$

Finally, for $k \in \mathbb{Z}$,

$$p_k^{(1)}(\lambda) = (1-\frac{\lambda}{\lambda_k})(1+\frac{\lambda}{\lambda_k}) \tag{I.4}$$

$$p_k^{(2)}(\lambda) = (1-\frac{\lambda^{-1}}{\lambda_k})(1+\frac{\lambda^{-1}}{\lambda_k}) \tag{I.5}$$

$$p_k^{(3)}(\lambda) = \begin{cases} (1-\frac{\lambda}{\lambda_0})(1-\frac{\lambda^{-1}}{\lambda_0}) & \text{for } k=0 \\ (1-\frac{\lambda}{\lambda_k})(1+\frac{\lambda^{-1}}{\lambda_k}) & \text{for } k \in \mathbb{Z}\setminus\{0\} \end{cases} \tag{I.6}$$

$$p_k^{(4)}(\lambda) = \begin{cases} (1+\frac{\lambda}{\lambda_0})(1+\frac{\lambda^{-1}}{\lambda_0}) & \text{for } k=0 \\ (1+\frac{\lambda}{\lambda_k})(1-\frac{\lambda^{-1}}{\lambda_k}) & \text{for } k \in \mathbb{Z}\setminus\{0\} \end{cases} \tag{I.7}$$

Proof of lemma 5.21. We start with the proof of the first claim: The infinite product $\prod_{k=-\infty}^{\infty} \sqrt{C_k}$ converges. Referring to remark 5.19, we prove that the infinite products $\prod_{n=0}^{\infty} \sqrt{C_k}$ and $\prod_{n=1}^{\infty} \sqrt{C_{-k}}$ converge. To this end, we apply lemma 5.20, i.e. we prove the convergence of the series $\sum_{k=0}^{\infty}(\sqrt{C_k}-1)$ and $\sum_{k=1}^{\infty}(\sqrt{C_{-k}}-1)$, respectively.

We will use the well known formula ("binomial theorem")

$$(x+y)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n y^{\alpha-n}, \tag{I.8}$$

which is valid for $x, y \in \mathbb{C}$ satisfying $|\frac{x}{y}| < 1$ and $\alpha \in \mathbb{C}$. Here, the generalized binomial coefficient is defined by

$$\binom{\alpha}{0} = 1, \quad \binom{\alpha}{n} = \frac{\alpha\cdot(\alpha-1)\cdots(\alpha-n+1)}{n!} \quad \text{for } n \in \mathbb{N}. \tag{I.9}$$

Consider (for $k \in \mathbb{Z}$)

$$\lambda_k = \sqrt{\frac{1}{2w}\left[(\frac{1}{6}+k)^2 - \frac{1}{4} + 2w + \sqrt{((\frac{1}{6}+k)^2 - \frac{1}{4} + 2w)^2 - 4w^2}\right]}$$
$$= k\sqrt{\frac{1}{2w}\left[1 + \frac{1}{3k} - \frac{2}{9k^2} + \frac{2w}{k^2} + \sqrt{(1 + \frac{1}{3k} - \frac{2}{9k^2} + \frac{2w}{k^2})^2 - \frac{4w^2}{k^2}}\right]}$$
$$= k\sqrt{\frac{1}{2w}\left[1 + \frac{1}{3k} - \frac{2}{9k^2} + \frac{2w}{k^2} + \sqrt{1 + \frac{2}{3k} - \frac{1}{3k^2} + \frac{4w}{k^2} - \frac{4}{27k^3} + \frac{4w}{3k^3} + \frac{4}{81k^4} - \frac{8w}{9k^4}}\right]}. \quad \text{(I.10)}$$

By a direct computation one verifies that the expression

$$x_k := \frac{2}{3k} - \frac{1}{3k^2} + \frac{4w}{k^2} - \frac{4}{27k^3} + \frac{4w}{3k^3} + \frac{4}{81k^4} - \frac{8w}{9k^4} \quad \text{(I.11)}$$

satisfies $-1 < x_k < 1$ for at least all k greater or equal some $k_0 \in \mathbb{N}$. Thus, we can apply (I.8) (with $x = x_k$, $y = 1$ and $\alpha = \frac{1}{2}$) for all $k \geq k_0$ to obtain

$$\lambda_k = k\sqrt{\frac{1}{2w}\left[1 + \frac{1}{3k} - \frac{2}{9k^2} + \frac{2w}{k^2} + \sqrt{1 + x_k}\right]}$$
$$= k\sqrt{\frac{1}{2w}\left[1 + \frac{1}{3k} + \mathcal{O}(\frac{1}{k^2}) + 1 + \frac{x_k}{2} + \mathcal{O}(\frac{1}{k^2})\right]} = k\sqrt{\frac{1}{w}\left[1 + \frac{1}{3k} + \mathcal{O}(\frac{1}{k^2})\right]}, \quad \text{(I.12)}$$

where we have used the notation $\mathcal{O}(f)$ for a function $\mathbb{Z} \to \mathbb{R}$ of $k \in \mathbb{Z}$, which asymptotically (i.e. for $k \to \infty$) behaves as the function $f : \mathbb{Z} \to \mathbb{R}$, $k \mapsto f(k)$. Again, we can apply (I.8) for all $k \geq k_0$ (adjusting k_0 if necessary), to continue our computation:

$$\lambda_k = \frac{k}{\sqrt{w}}(1 + \frac{1}{6k} + \mathcal{O}(\frac{1}{k^2})). \quad \text{(I.13)}$$

In particular, this shows that λ_k is of the form $\mathcal{O}(k)$.

Based on (I.13), we compute (once more using (I.8))

$$\sqrt{C_k} - 1 = \frac{\sqrt{w}\lambda_k - k(1+\frac{1}{k})^{\frac{1}{6}}}{k(1+\frac{1}{k})^{\frac{1}{6}}} = \frac{k(1 + \frac{1}{6k} + \mathcal{O}(\frac{1}{k^2})) - k(1 + \frac{1}{6k} + \mathcal{O}(\frac{1}{k^2}))}{k(1 + \frac{1}{6k} + \mathcal{O}(\frac{1}{k^2}))} = \mathcal{O}(\frac{1}{k^2}), \quad \text{(I.14)}$$

which means that $\sqrt{C_k} - 1$ asymptotically behaves like $\frac{1}{k^2}$. The same holds for $\sqrt{C_{-k}} - 1$. Consequently, the series $\sum_{k=0}^{\infty}(\sqrt{C_k} - 1)$ and $\sum_{k=1}^{\infty}(\sqrt{C_{-k}} - 1)$ converge, which implies the first claim of lemma 5.21.

The remaining claims of lemma 5.21 are now proved quite easily: Since, by (I.13), λ_k is of the form $\mathcal{O}(k)$, we infer that λ_k^{-1} is of the form $\mathcal{O}(\frac{1}{k})$. Consequently, the series

$$\sum_{k=0}^{\infty} -\frac{\lambda^2}{\lambda_k^2}, \quad \sum_{k=1}^{\infty} -\frac{\lambda^2}{\lambda_{-k}^2}, \quad \sum_{k=0}^{\infty} -\frac{\lambda^{-2}}{\lambda_k^2}, \quad \sum_{k=1}^{\infty} -\frac{\lambda^{-2}}{\lambda_{-k}^2} \quad \text{(I.15)}$$

converge normally on $\mathbb{C}^*$, while the series

$$\sum_{k=1}^{\infty} -\frac{\lambda}{\lambda_k} + \frac{\lambda^{-1}}{\lambda_k} - \frac{1}{\lambda_k^2}, \quad \sum_{k=1}^{\infty} -\frac{\lambda}{\lambda_{-k}} + \frac{\lambda^{-1}}{\lambda_{-k}} - \frac{1}{\lambda_{-k}^2}, \quad \sum_{k=1}^{\infty} \frac{\lambda}{\lambda_k} - \frac{\lambda^{-1}}{\lambda_k} - \frac{1}{\lambda_k^2}, \quad \sum_{k=1}^{\infty} \frac{\lambda}{\lambda_{-k}} - \frac{\lambda^{-1}}{\lambda_{-k}} - \frac{1}{\lambda_{-k}^2} \quad \text{(I.16)}$$

diverge on $\mathbb{C}^*$. By definition 5.18 (in conjunction with remark 5.19), we conclude that the infinite products

$$\prod_{k=-\infty}^{\infty} p_k^{(1)}(\lambda) = \prod_{k=-\infty}^{\infty} 1 - \frac{\lambda^2}{\lambda_k^2}, \quad \prod_{k=-\infty}^{\infty} p_k^{(2)}(\lambda) = \prod_{k=-\infty}^{\infty} 1 - \frac{\lambda^{-2}}{\lambda_k^2} \quad \text{(I.17)}$$

are normally convergent on $\mathbb{C}^*$, while the infinite products

$$\prod_{k=-\infty}^{\infty} p_k^{(3)}(\lambda), \quad \prod_{k=-\infty}^{\infty} p_k^{(4)}(\lambda) \quad \text{(I.18)}$$

are divergent on $\mathbb{C}^*$. □

Next, we prove

Lemma 5.24. The infinite products (of functions)

$$\prod_{j\in\mathbb{N}} g_{j,1}(\lambda) \quad \text{and} \quad \prod_{j\in\mathbb{N}} g_{j,2}(\lambda), \tag{I.19}$$

where, for all $j \in \mathbb{N}$, $g_{j,1}$ is given in (5.5.94) and $g_{j,2}$ is given in (5.5.95), are normally convergent on $\mathbb{C}^* \setminus \bigcup_{j\in\mathbb{N}}\{\pm\lambda_j\}$ and $\mathbb{C}^* \setminus \bigcup_{j\in\mathbb{N}}\{\pm\lambda_j^{-1}\}$, respectively.

The expressions occurring in lemma 5.24 are given by

$$g_{j,1}(\lambda) = \frac{(1-\frac{\lambda^{-1}}{\lambda_j})(1+\frac{\lambda^{-1}}{\lambda_j})}{(1-\frac{\lambda}{\lambda_j})(1+\frac{\lambda}{\lambda_j})}, \tag{I.20}$$

$$g_{j,2}(\lambda) = \frac{(1-\frac{\lambda}{\lambda_j})(1+\frac{\lambda}{\lambda_j})}{(1-\frac{\lambda^{-1}}{\lambda_j})(1+\frac{\lambda^{-1}}{\lambda_j})}, \tag{I.21}$$

where

$$\lambda_j := \sqrt{\frac{1}{2w}\left[d_j + \sqrt{d_j^2 - 4w^2}\right]} \tag{I.22}$$

with

$$d_j := (\frac{1}{2}+j)^2 - \frac{1}{4} + 2w. \tag{I.23}$$

Proof of lemma 5.24. Analogously as in the proof of lemma 5.21 for λ_k (given in (I.1)), one shows for λ_j given in (I.22) that

$$\lambda_j = \frac{j}{\sqrt{w}}(1+\frac{1}{2j}+\mathcal{O}(\frac{1}{j^2})). \tag{I.24}$$

This shows in particular that λ_j is of the form $\mathcal{O}(j)$. Consequently, the series

$$\sum_{j=1}^{\infty} \frac{\lambda^2-\lambda^{-2}}{\lambda_j^2-\lambda^2} = \sum_{j=1}^{\infty} \frac{\frac{\lambda^2}{\lambda_j^2}-\frac{\lambda^{-2}}{\lambda_j^2}}{1-\frac{\lambda^2}{\lambda_j^2}}, \tag{I.25}$$

$$\sum_{j=1}^{\infty} \frac{\lambda^{-2}-\lambda^2}{\lambda_j^2-\lambda^{-2}} = \sum_{j=1}^{\infty} \frac{\frac{\lambda^{-2}}{\lambda_j^2}-\frac{\lambda^2}{\lambda_j^2}}{1-\frac{\lambda^{-2}}{\lambda_j^2}} \tag{I.26}$$

converge normally on $\mathbb{C}^*\setminus\bigcup_{j\in\mathbb{N}}\{\pm\lambda_j\}$ and $\mathbb{C}^*\setminus\bigcup_{j\in\mathbb{N}}\{\pm\lambda_j^{-1}\}$, respectively. By definition 5.18, we conclude that the infinite products

$$\prod_{j=1}^{\infty} g_{j,1}(\lambda) = \prod_{j=1}^{\infty} \frac{1-\frac{\lambda^{-2}}{\lambda_j^2}}{1-\frac{\lambda^2}{\lambda_j^2}} = \prod_{j=1}^{\infty}(1+\frac{\frac{\lambda^2}{\lambda_j^2}-\frac{\lambda^{-2}}{\lambda_j^2}}{1-\frac{\lambda^2}{\lambda_j^2}}), \tag{I.27}$$

$$\prod_{j=1}^{\infty} g_{j,2}(\lambda) == \prod_{j=1}^{\infty} \frac{1-\frac{\lambda^2}{\lambda_j^2}}{1-\frac{\lambda^{-2}}{\lambda_j^2}} = \prod_{j=1}^{\infty}(1+\frac{\frac{\lambda^{-2}}{\lambda_j^2}-\frac{\lambda^2}{\lambda_j^2}}{1-\frac{\lambda^{-2}}{\lambda_j^2}}) \tag{I.28}$$

are normally convergent on $\mathbb{C}^* \setminus \bigcup_{j\in\mathbb{N}}\{\pm\lambda_j\}$ and $\mathbb{C}^* \setminus \bigcup_{j\in\mathbb{N}}\{\pm\lambda_j^{-1}\}$, respectively. □

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